A Theoretical Study of Skew Plates.

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UNIVERSITY OF TORONTO

A THEORETICAL STUDY OF SKEW PLATES

by

Francis Louis Ehasz

A DISSERTATION

Presented to the Graduate Faculty

of Lehigh University

in Candidacy for the Degree of

Doctor of Philosophy

Lehigh University

1941

211656

Approved and recommended for acceptance as a dissertation in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Hale Sutherland

Professor in Charge.

Accepted, June 3 1941

Special committee directing the doctoral work of Mr. Y. L. Chang.

Hale Sutherland Chairman

Bruce Johnston

Jos. B. Reynolds

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ACKNOWLEDGMENTS

This dissertation was prepared under the general supervision of Professor Hale Sutherland, Head of the Department of Civil Engineering and Director of the Fritz Engineering Laboratory of Lehigh University. The writer wishes to thank Professor Sutherland for his continued advice and encouragement. He is indebted to Dr. Joseph B. Reynolds, Professor of Mathematics and Theoretical Mechanics, and Dr. Bruce G. Johnston, Assistant Professor of Civil Engineering and Assistant Director of the Fritz Engineering Laboratory, both of Lehigh University, for their helpful criticisms regarding the scope of this work.

The investigation reported herein was conducted while the writer served as Instructor in Theoretical and Applied Mechanics at the University of Illinois. Its progress and completion were facilitated as a result of courtesies extended by Professor F. B. Seely, Head of the Department of Theoretical and Applied Mechanics, and F. E. Richart, Research Professor of Engineering Materials, under whose general supervision an extensive investigation of reinforced concrete bridge slabs is being carried out in the Engineering Experiment Station of the University of Illinois. The suggestions and criticisms of Dr. N. M. Newmark, Research Assistant Professor of Civil Engineering, and Dr. V. P. Jensen, Special Research Assistant Professor of Theoretical and Applied Mechanics, are greatly appreciated. Acknowledgment is also made to Mr. C. R. Chambers, senior student in mechanical engineering at the University of Illinois, who assisted with some of the detailed numerical computations.

A THEORETICAL STUDY OF SKEW PLATES

I. INTRODUCTION

1. Objects and Scope of Investigation. - During the past few years some consideration has been given by various investigators in this country and abroad to the analysis and design of skew plates and slabs. In the field of reinforced concrete these studies have been of considerable interest and practical importance, especially in the case of highway bridge slabs which cross streams, railways, or other highways below at an oblique angle. It is common knowledge that modern traffic and speed requirements in the United States stipulate the construction of many skew structures. Whereas bridge slabs usually are provided with integral curbs and rails, which tend to make the structural behavior more complex, it is nevertheless of interest to ascertain the manner in which the more simple slabs perform. Structural engineers have been studying the skew plate simply supported on all four sides, the clamped skew plate, and skew slabs simply supported on two sides with two free edges, in addition to the more practical bridge slabs with integral curbs mentioned previously.

A skew plate or slab may be defined as one having the shape of a parallelogram which is not a rectangle or a square. Some engineers designate the rectangular or square plate as one having zero skew. In any case, by the term degree of skew, in so far as this investigation is concerned, there will be meant the angle of distortion from the rectangular or square shape. This designation corresponds in general with the practice and the technical literature pertaining to other types of skew structures. As in the case of skew arches,[1] the skew effect in plates depends upon the ratio of the lengths of sides as well as on the angle of skew.

It is the purpose of this investigation to study theoretically the structural action of skew plates simply supported on all four sides, subjected to a uniformly distributed lateral load, and a clamped skew plate similarly loaded. In the course of these studies various systems of coordinates were employed to determine whether the skew boundaries would lend themselves to treatment more readily by one system than by the others. It is to be observed, moreover, that although the results reported herein pertain directly to metal plates (Poisson's ratio approximately 0.3), they can be easily modified to include other materials, such as concrete in which case Poisson's ratio may be varied from 0.15 to 0.20.

In this work simply supported skew plates were investigated by the method of finite differences, by trigonometric series satisfying the plate equation throughout and the boundary conditions only at certain points, and by a few power-series approaches. A clamped skew plate was also treated by these methods and other possible lines of attack were pointed out. A preliminary review was made of solutions for triangular plates with a view to ascertaining the effect of the density of difference-equation networks and to

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[1] The Design and Construction of a Skew Arch, S. C. Hollister, Proc. A.C.I., V. XXIV, 1928, p. 371.

determining some measure of the accuracy to be expected for similar meshes in the case of skew plates.

2. Previous Studies of Skew Plates and Slabs. - In recent years the structural behavior of skew plates and slabs has been investigated mainly by means of the method of finite differences. The first publication on skew plates that has come to the writer's attention is a paper by Cecilia Vittoria Brigatti.[1] Results were obtained by difference equations for uniformly loaded skew plates simply supported and clamped, the sides being of the same length. The fundamental considerations in this work appear to be open to question.

Adolf Anzelius[2] has given some attention to a uniformly loaded skew plate, simply supported on two opposite sides and free on the other two. His solution is in the form of series involving hyperbolic and trigonometric functions, which yield an infinite system of linear equations. The method employed is approximate and appears to be quite cumbersome. As in other similar approaches, the accuracy of the method depends upon the number of coefficients taken in the series. Anzelius gave, qualitatively, only twisting moments for a 45-degree skew slab.

A summary of results obtained by means of difference equations for uniformly loaded skew slabs, in particular, with two opposite sides simply supported and the other two edges free, has been

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[1] Applicazione del metodo di H. Marcus al calcolo della piastra parallelogrammica, Cecilia Vittoria Brigatti, Ricerche di Ingegneria, Vol. XVI, March-April 1938, No. 2, p. 42.

[2] Über die elastische Deformation parallelogrammförmiger Platten, Adolf Anzelius, Der Bauingenieur, Vol. 20, Sept. 1939, No. 35/36, p. 478.

abstracted from the doctoral work of Helmut Vogt[1] and promulgated in a paper[2] by the same author. Vogt has given particular attention to the arrangement of the reinforcement in bridge slabs. Brief mention was also made of skew slabs simply supported on four sides and uniformly loaded.

Studies of skew slabs by means of difference equations have also been made during recent months by V. P. Jensen, and his results are scheduled to appear in a University of Illinois Bulletin.[3] Uniformly loaded skew slabs simply supported on four sides, and similarly loaded skew slabs simply supported on two opposite edges and free on the other two sides, were analyzed by use of difference equations developed in a form readily applicable to networks made up of lines parallel to the sides, regardless of the degree of skew. In these slabs the ratio of the length of the long side to that of the short span was kept nearly constant and equal to a value of 2.0. Poisson's ratio was taken to be 0.2. Particular attention was given by Jensen to a simple span slab-bridge with curbs and a 45-degree skew. The Newmark method[3] of obtaining influence surfaces by means of difference equations proved to be useful in these studies. Also

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[1]Beitrag zur Berechnung schiefwinkliger Platten, nebst Anwendung bei der Berechnung und Anordnung der Bewehrung schiefwinkliger Brückenbauwerke, Helmut Vogt, Dissertation, Technische Hochschule, Hanover.

[2]Die Berechnung schiefwinkliger Platten und plattenartiger Brückensysteme, Helmut Vogt, Beton und Eisen, 39, No. 17, Sept. 1940, pp. 243-245.

[3]Analyses of Skew Slabs, V. P. Jensen, Bulletin in Press, Reinforced Concrete Slabs Investigation conducted in the Engineering Experiment Station of the University of Illinois in cooperation with the Public Roads Administration of the Federal Works Agency and the Illinois Division of Highways.

there was considered by Jensen the effectiveness of reinforcement for various arrangements of the main and transverse steel.

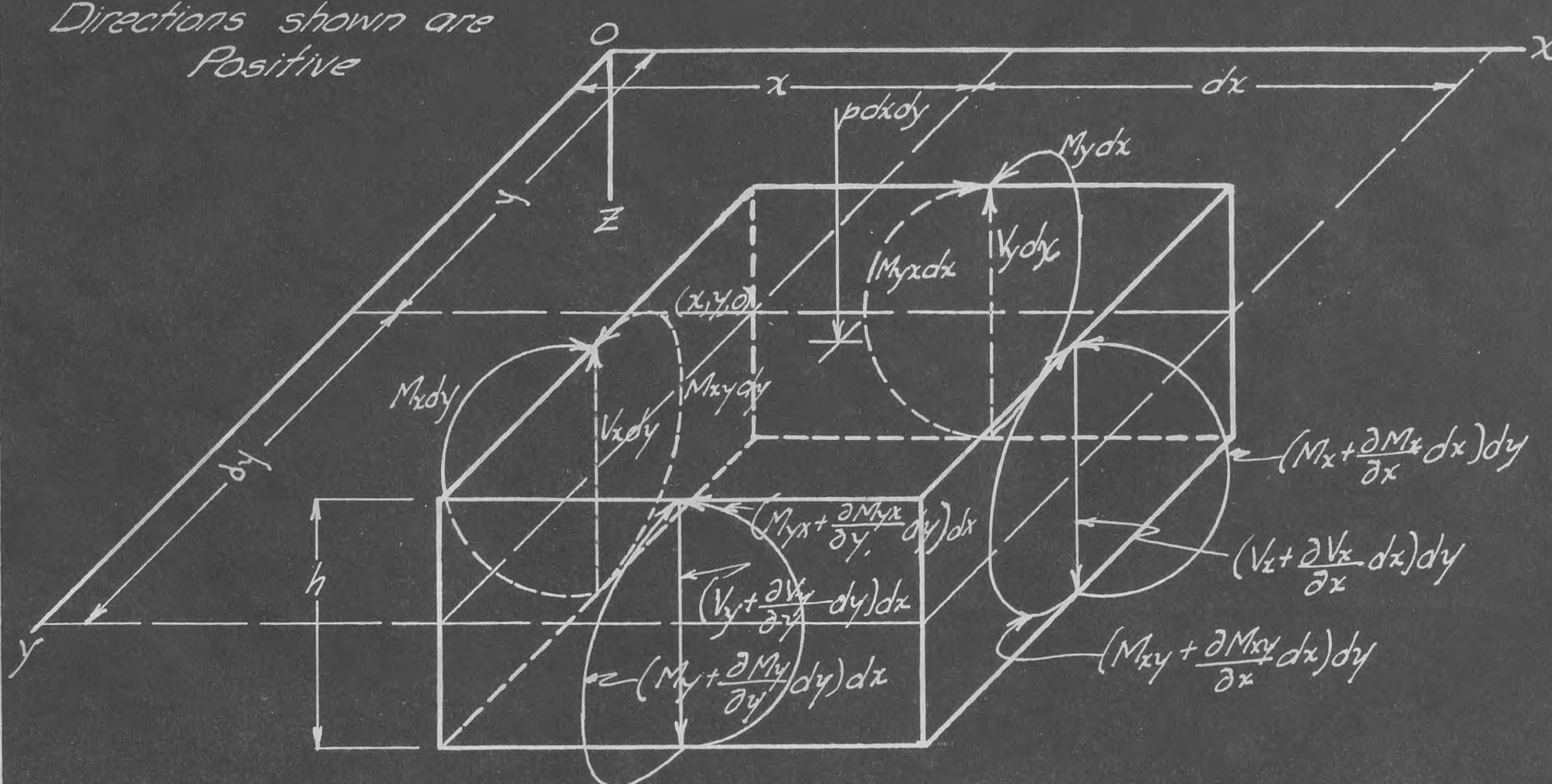
3. Notation. - The following designations were selected for this dissertation:

x, y, z	rectangular coordinates
r, θ	polar coordinates
t_1, t_2, t_3	trilinear coordinates
q_1, q_2, q_3, q_4	quadrilinear coordinates
u, v, w	displacements in x, y, z directions, respectively
a, c	horizontal dimensions indicated for each solution
p	intensity of uniformly distributed load
h	thickness of plate
E	modulus of elasticity of material in tension and compression
G	modulus of elasticity of material in shear
μ	Poisson's ratio (lateral contraction to longitudinal elongation)
$N = \dfrac{Eh^3}{12(1 - \mu^2)}$	flexural rigidity of plate
M_x, M_y	bending moments per unit of length in sections perpendicular to x- and y-axes respectively; positive directions for moments, shears and stresses shown in Figs. 1 and 2.
M_{xy}	twisting moment per unit of length in sections perpendicular to x-axis
M_r, M_t	bending moments per unit of length in sections perpendicular to radial and tangential directions; positive when producing compression on top of plate

M_{rt}	twisting moment per unit of length in sections perpendicular to radial directions; positive when producing compression in line parallel to direction $r = t$ on top of plate
M_1, M_2, M_3	bending moments per unit of length in sections perpendicular to directions 1, 2, and 3, respectively
σ_x, σ_y	normal components of stress in x- and y-directions, respectively
τ_{xy}, τ_{xz}, τ_{yz}	shearing components of stress; first subscript indicates the axis to which plane (in which stress is acting) is normal, and second subscript indicates direction of shearing stress
V_x, V_y	vertical shear per unit of length in sections normal to x- and y-axes, respectively
ϵ_x, ϵ_y, ϵ_z	unit elongations in x- , y- and z-directions, respectively
r_x, r_y	radii of curvatures of middle surface of plate in planes parallel to the xz- and yz-planes, respectively
γ_{xy}, γ_{xz}, γ_{yz}	shearing strain components
$C_{m,n}$ etc.	coefficients in power series
λ, λ_x, λ_y	intervals in difference equations networks indicated for each solution

II. ORDINARY THEORY OF PLATES

4. Assumptions and Limitations. – In the ordinary theory of plates several simplifying assumptions are made, resulting in concomitant limitations. As a matter of convenience the plate is considered to be horizontal, homogeneous, elastic, isotropic, uniform in thickness, and subjected to lateral loads only. In addition to being uniform, the thickness must be small compared with the lateral dimensions. The deflections and the energy of deformation in the medium-thick plate are not affected by the vertical stresses (tensions, compressions, and shears) as they are in the case of thick plates. Likewise, the energy due to shortening and stretching of the middle surface may be neglected, which is tantamount to stating that the deflections are small relative to the thickness of the plate. This work of deformation, naturally, must be considered in the theory of thin plates. It is taken for granted, furthermore, that Hooke's law applies to the horizontal strains. In the theory of beams, it will be recalled, plane cross-sections before bending are assumed to remain plane after bending. Similarly, in the ordinary theory of plates, straight lines normal to the middle surface of the plate before bending remain straight and normal after bending. Analogous to the concept of straight-line variation of tensile and compressive stresses in the cross-section of the beam, there follows in the case of the plate the postulation that horizontal unit-stresses (tensions, compressions and shears) in vertical sections are distributed linearly.



RECTANGULAR ELEMENT OF PLATE

Fig. 1

5. Fundamental Differential Equations. - The derivation of the differential equations for the ordinary theory of plates in Cartesian coordinates has been given by various writers in the technical literature,[1] and it will, therefore, be given briefly only in appendix form (see Appendix A). It is pertinent, however, to outline here the basic equations for later reference.

The bending and twisting moments in the distorted plate (see Fig. 1) are given by the expressions:

$$M_x = -N\left(\frac{\partial^2 w}{\partial x^2} + \mu\,\frac{\partial^2 w}{\partial y^2}\right)$$
$$M_y = -N\left(\frac{\partial^2 w}{\partial y^2} + \mu\,\frac{\partial^2 w}{\partial x^2}\right) \qquad (1)$$
$$M_{xy} = -(1-\mu)\,N\,\frac{\partial^2 w}{\partial x \partial y}$$

in which w designates the deflection, μ is Poisson's ratio for the material, and N, the flexural rigidity, is a function of the modulus of elasticity, E, of the material, the plate thickness, h, and Poisson's ratio, as is manifested from the equation:

$$N = \frac{Eh^3}{12(1-\mu^2)} \qquad (2)$$

[1]Theory of Plates and Shells, S. Timoshenko, 1940, p. 85;
Theory of Elasticity, R. V. Southwell, 1936, p. 228;
Die elastischen Platten, A. Nadai, 1925, p. 18;
Die Theorie elastischer Gewebe und ihre Anwendung auf die Berechnung biegsamer Platten, H. Marcus, 1932, p. 1, etc.

Lagrange's equation for the flexure of plates produced by a uniformly distributed lateral load p may be shown to be

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{N} \tag{3}$$

or

$$\nabla^4 w = \nabla^2 \nabla^2 w = \frac{p}{N} \tag{3a}$$

in which

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is Laplace's operator.

From the equations (1) it is clear that the moment sum is expressible in the form

$$M_x + M_y = -(1+\mu) N \nabla^2 w \tag{4}$$

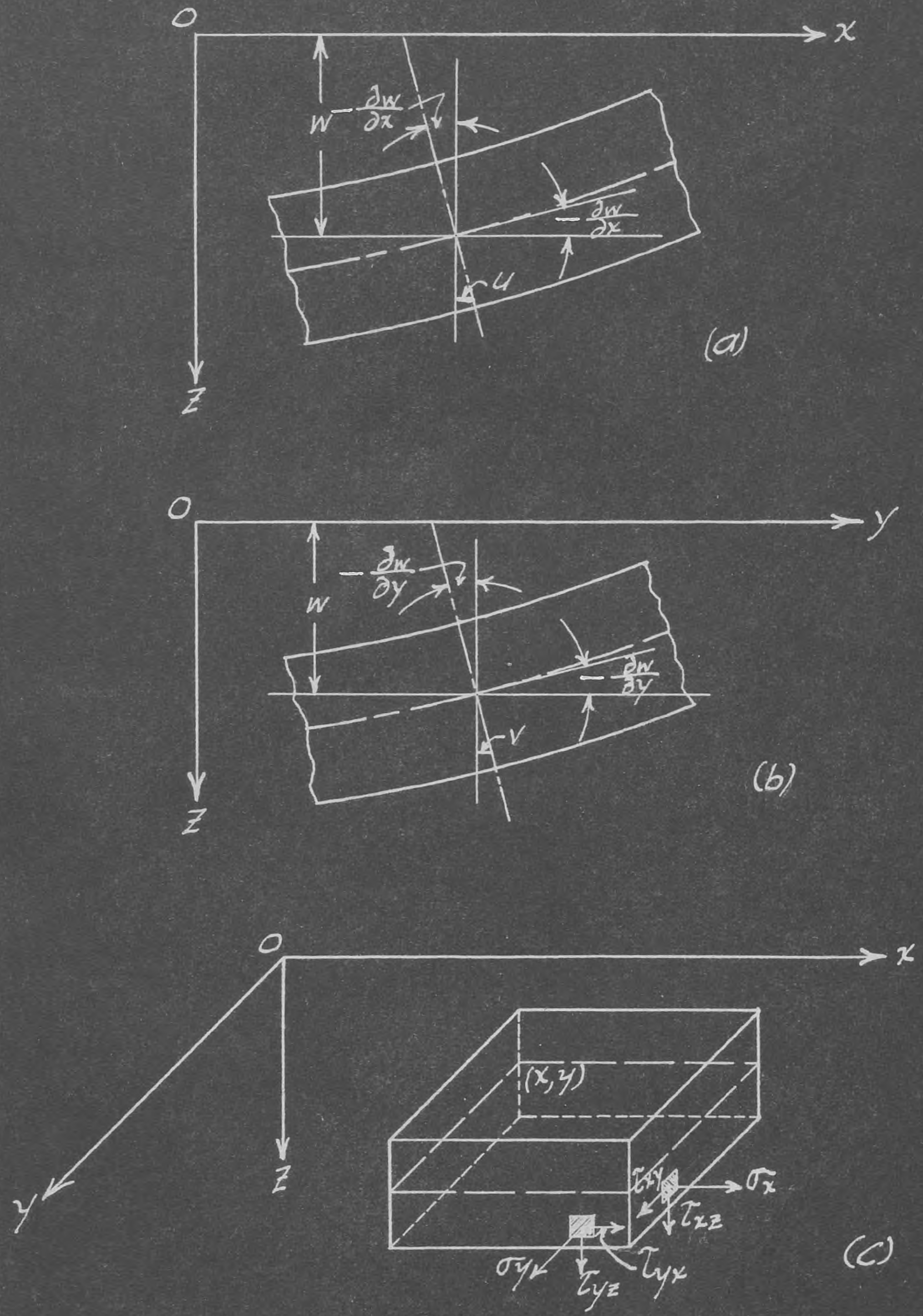
The vertical shears are given by the equations:

$$\begin{aligned} V_x &= -N \frac{\partial}{\partial x} \nabla^2 w \\ V_y &= -N \frac{\partial}{\partial y} \nabla^2 w \end{aligned} \tag{5}$$

Maximum stresses, occurring at the surface, are

$$\sigma_x = \pm \frac{6M_x}{h^2}, \quad \sigma_y = \pm \frac{6M_x}{h^2}, \quad \tau_{xy} = \pm \frac{6M_{xy}}{h^2}. \tag{6}$$

From the shears and the twisting moments it is possible to deduce the vertical reactions:



DEFORMATIONS AND STRESSES IN ELASTIC PLATE

Fig. 2

$$R_x = V_x + \frac{\partial M_{xy}}{\partial y}$$

$$= - N\left[\frac{\partial^3 w}{\partial x^3} + (2 - \mu)\frac{\partial^3 w}{\partial x \partial y^2}\right]$$

(7)

$$R_y = V_y + \frac{\partial M_{xy}}{\partial x}$$

$$= - N\left[\frac{\partial^3 w}{\partial y^3} + (2 - \mu)\frac{\partial^3 w}{\partial x^2 \partial y}\right]$$

These reactions are defined in the same manner as the vertical shears.

6. Polar Coordinates. - The differential equations for the bending of plates may be readily placed into polar form by some simple transformations. It will be necessary later to use the concept of average curvature at a point of the distorted plate and, consequently, it is introduced here. Curvature in the xz- and yz-planes are clearly

$$\frac{1}{r_x} = \frac{\partial}{\partial x}\frac{\partial w}{\partial x} = \frac{\partial^2 w}{\partial x^2} \tag{8}$$

and

$$\frac{1}{r_y} = \frac{\partial}{\partial y}\frac{\partial w}{\partial y} = \frac{\partial^2 w}{\partial y^2} \tag{9}$$

respectively (see Fig. 2).

It can be shown that at the same point the sum of the curvatures in any two perpendicular directions n and t is equal to the sum of equations (8) and (9), or symbolically,

$$\frac{1}{r_n} + \frac{1}{r_t} = \frac{\partial^2 w}{\partial n^2} + \frac{\partial^2 w}{\partial t^2} = \frac{1}{r_x} + \frac{1}{r_y} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \tag{10}$$

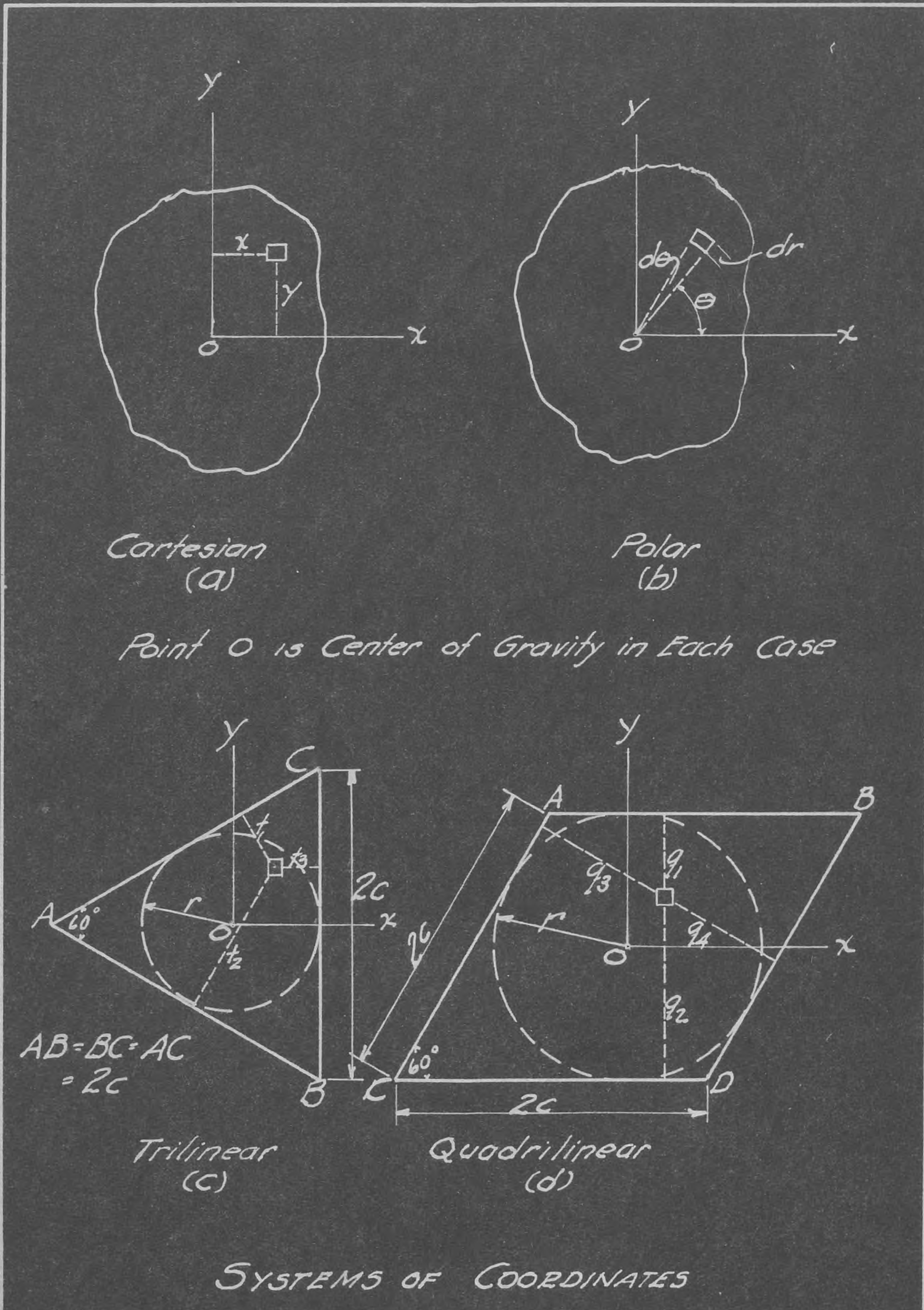

y
x
y
o
x
Cartesian
(a)
y
dθ
dr
θ
o
x
Polar
(b)
Point O is Center of Gravity in Each Case
y
C
t1
t3
2c
A
60°
r
o
x
t2
AB=BC=AC
=2c
B
Trilinear
(c)
y
A
B
q1
q3
2c
r
q4
o
x
q2
60°
C
D
2c
Quadrilinear
(d)
SYSTEMS OF COORDINATES

Fig. 3

where r_n and r_t are defined in a manner similar to r_x and r_y. This invariant sum has been designated as the average curvature of the plate's surface at a point.

Transformation from Cartesian to polar coordinates (see Fig. 3) yields the result

$$\nabla^2 w = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} \tag{11}$$

By repetition of this operation it is possible to obtain Lagrange's plate equation in the form

$$\nabla^2\nabla^2 w = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) = \frac{p}{N} \tag{12}$$

Equations (11) and (12) will be employed later in the approximate trigonometric-series solution for a skew plate.

From simple transformation equations, expressions for moments in polar form may be deduced by stipulating that the x-axis coincides with the radius r. If t is the direction (tangential) perpendicular to the radial direction r, M_r and M_t are the moments per unit length acting in sections perpendicular to r and t, respectively. These moments and the corresponding twisting moments, M_{rt}, may be shown to be

$$\begin{aligned}
M_r &= -N\left[\frac{\partial^2 w}{\partial r^2} + \mu\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right] \\
M_t &= -N\left(\mu\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) \\
M_{rt} &= -(1-\mu)\,N\left(\frac{1}{r}\frac{\partial^2 w}{\partial r\partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}\right) \\
&= -(1-\mu)\,N\frac{\partial}{\partial r}\,\frac{1}{r}\frac{\partial w}{\partial \theta}
\end{aligned} \tag{13}$$

7. Trilinear Coordinates. - In the solution of problems involving triangular shapes, for example, in the torsion of a bar having an equilateral, triangular section, in the flexure of a plate having the same superficial dimensions and subjected to lateral loads, and in the buckling of a similar plate due to forces acting in the plane of the original middle surface, it would not appear to be particularly fortuitous to attempt a solution by means of trilinear coordinates.

As may be seen from Fig. 3(c), elements in the triangular plate are located by means of the three coordinates t_1, t_2, and t_3, measured perpendicularly from the sides AC, AB and BC, respectively. For the equilateral triangular plate the following equations relate the trilinear coordinates to the Cartesian:[1]

$$t_1 = r + \frac{x}{2} - \frac{\sqrt{3}}{2} y$$

$$t_2 = r + \frac{x}{2} + \frac{\sqrt{3}}{2} y \qquad (14)$$

$$t_3 = r - x \, ,$$

where r is the radius of the inscribed radius.

Laplace's operator for trilinear coordinates is

$$\nabla^2 = \frac{\partial^2}{\partial t_1^2} + \frac{\partial^2}{\partial t_2^2} + \frac{\partial^2}{\partial t_3^2} - \frac{\partial^2}{\partial t_1 \partial t_2} - \frac{\partial^2}{\partial t_2 \partial t_3} - \frac{\partial^2}{\partial t_3 \partial t_1} \qquad (15)$$

Repetition of this operation twice gives the plate equation in trilinear form

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[1]Note on Some Two Dimensional Problems of Elasticity Connected with Plates having Triangular Boundaries, B. Sen, Calcutta Mathematical Society Bulletin, Vol. XXVI, 1934, pp. 65-72.

$$\nabla^2\nabla^2 w = \left(\frac{\partial^2}{\partial t_1^2} + \frac{\partial^2}{\partial t_2^2} + \frac{\partial^2}{\partial t_3^2} - \frac{\partial^2}{\partial t_1 \partial t_2} - \frac{\partial^2}{\partial t_2 \partial t_3} - \frac{\partial^2}{\partial t_3 \partial t_1}\right) \left(\frac{\partial^2 w}{\partial t_1^2} + \frac{\partial^2 w}{\partial t_2^2} + \frac{\partial^2 w}{\partial t_3^2} - \frac{\partial^2 w}{\partial t_1 \partial t_2} - \frac{\partial^2 w}{\partial t_2 \partial t_3} - \frac{\partial^2 w}{\partial t_3 \partial t_1}\right) = \frac{p}{N} \tag{16}$$

It should be noted, however, that solutions for the problems concerning the equilateral triangular shape have also been obtained by the use of Cartesian coordinates.[1] Moreover, it is to be observed that there appears to be no particular advantage of the trilinear coordinate approach over the other.

8. Quadrilinear Coordinates. - The expression for average curvature in a thirty-degree skew plate with equal sides may be put into quadrilinear-coordinate form by means of simple transformations. Reference to Fig. 3(d) enables one to relate the quadrilinear coordinates, measured perpendicularly from the sides to the element in question, to the Cartesian coordinates by means of the equations

$$\begin{aligned} q_1 &= r - y \\ q_2 &= r + y \\ q_3 &= r + \frac{\sqrt{3}}{2} x - \frac{y}{2} \\ q_4 &= r - \frac{\sqrt{3}}{2} x + \frac{y}{2} \quad , \end{aligned} \tag{17}$$

in which r is the radius of the inscribed circle, and q_1, q_2, q_3, and q_4 are the coordinates of the element from AB, CD, AC, and BD, respectively.

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[1]Berechnung der ringsum frei aufliegenden gleichseitigen Dreiecksplatte, S. Woinowsky-Krieger, Ingenieur-Archiv, IV Band 1933, pp. 254-262.

For the curvature in the x-direction there results the relation

$$\frac{\partial^2 w}{\partial x^2} = \frac{3}{4}\frac{\partial^2 w}{\partial q_3^2} - \frac{3}{2}\frac{\partial^2 w}{\partial q_3 \partial q_4} + \frac{3}{4}\frac{\partial^2 w}{\partial q_4^2} \tag{18}$$

and the corresponding relation for curvature in the y-direction is given by the equation

$$\begin{aligned}\frac{\partial^2 w}{\partial y^2} = {} & \frac{\partial^2 w}{\partial q_1^2} + \frac{\partial^2 w}{\partial q_2^2} + \frac{1}{4}\frac{\partial^2 w}{\partial q_3^2} + \frac{1}{4}\frac{\partial^2 w}{\partial q_4^2} \\ & - 2\frac{\partial^2 w}{\partial q_1 \partial q_2} + \frac{\partial^2 w}{\partial q_1 \partial q_3} - \frac{\partial^2 w}{\partial q_1 \partial q_4} \\ & - \frac{\partial^2 w}{\partial q_2 \partial q_3} + \frac{\partial^2 w}{\partial q_2 \partial q_4} - \frac{1}{2}\frac{\partial^2 w}{\partial q_3 \partial q_4}\end{aligned} \tag{19}$$

The average curvature for the equal-sided, thirty-degree skew plate in quadrilinear form is, therefore,

$$\begin{aligned}\nabla^2 w = {} & \frac{\partial^2 w}{\partial q_1^2} + \frac{\partial^2 w}{\partial q_2^2} + \frac{\partial^2 w}{\partial q_3^2} + \frac{\partial^2 w}{\partial q_4^2} - 2\left(\frac{\partial^2 w}{\partial q_1 \partial q_2} + \frac{\partial^2 w}{\partial q_3 \partial q_4}\right) \\ & + \frac{\partial^2 w}{\partial q_1 \partial q_3} + \frac{\partial^2 w}{\partial q_2 \partial q_4} - \left(\frac{\partial^2 w}{\partial q_1 \partial q_4} + \frac{\partial^2 w}{\partial q_2 \partial q_3}\right).\end{aligned} \tag{20}$$

Repetition of this operation twice, with $\frac{p}{N}$ on the right side, results in the plate equation in quadrilinear form.

A few expressions involving polynomials in quadrilinear-coordinate form were assumed for the deflection function, but it did not appear possible with these to satisfy simultaneously the plate equation and the boundary conditions for the equal-sided, thirty-degree skew plates. Since simple polynomial forms in Cartesian coordinates do not appear to be satisfactory for expressing the deflection function, there is little wonder that the same should be found to be true for polynomials expressed in quadrilinear-coordinate form.

It is possible to obtain a power series in quadrilinear form which would yield an approximate solution of skew plate problems. Since this procedure would have entailed considerably more work than a similar series approach in Cartesian form, solutions were not sought by the method employing quadrilinear coordinates. In this dissertation the power-series approach, useful for attacking a wide variety of plate problems, was dealt with only in Cartesian coordinates and was applied to a few plate problems which were treated also by other methods.

9. Difference Equations. - By means of difference equations it is possible to obtain approximate solutions for many plate problems which cannot be treated easily by exact methods. Skew plates and slabs fall into this category. Many writers have applied this method to plate problems.[1] Since finite squares, rectangles and triangles (hexagons)

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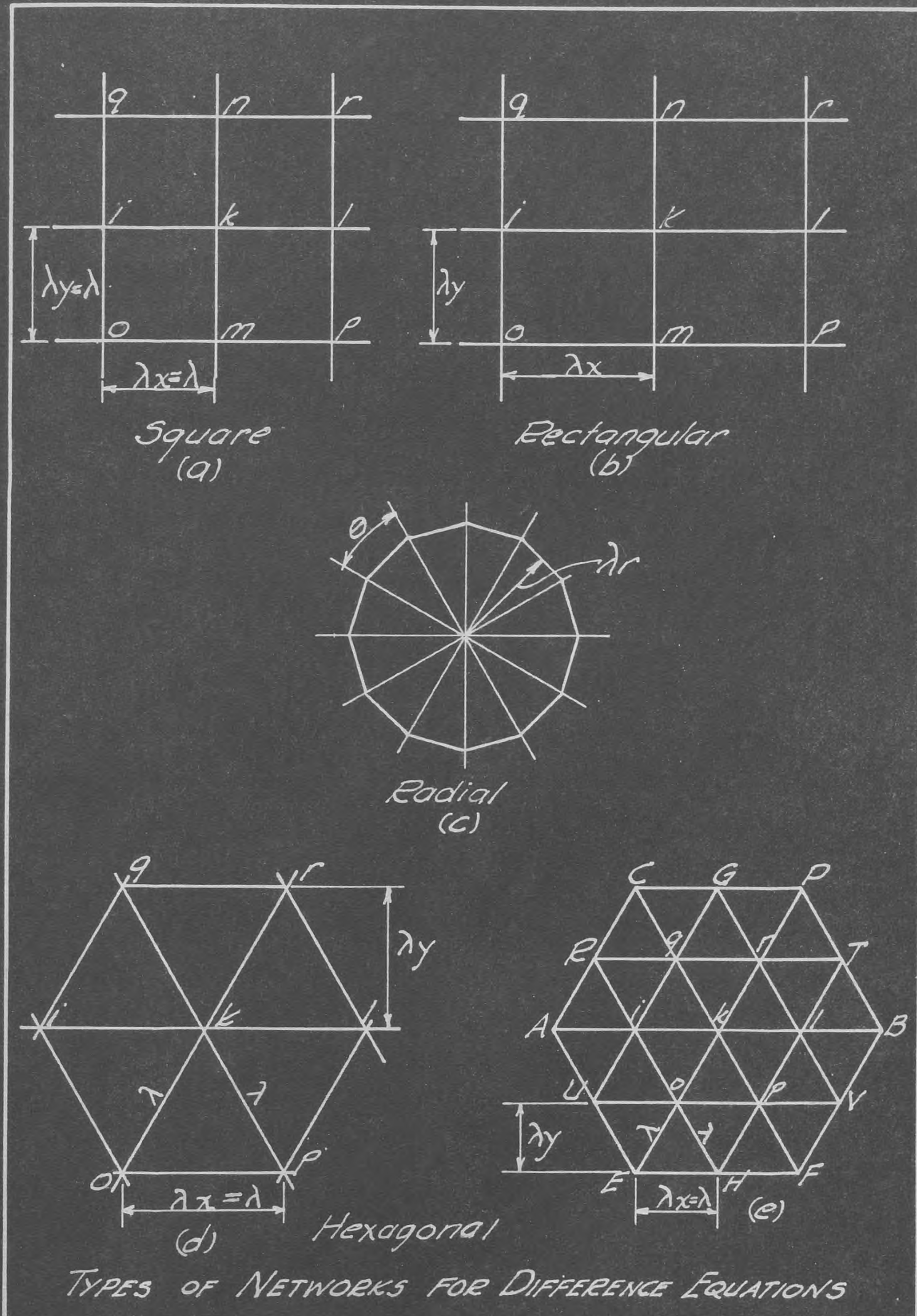
[1]In addition to those mentioned previously, the following references are given:
Bestemmelse af Spoendinger i Plader ved anvendelse af Differensligninger, N. S. Nielsen, Kopenhagen, 1920; Die Theorie elastischer Gewebe und ihre Anwendung auf die Berechnung biegsamer Platten, H. Marcus, 2nd Ed., Julius Springer, Berlin, 1932; Die elastischen Platten, A. Nadai, Julius Springer, Berlin, 1925, p. 205; The Calculation of Flat Plates by the Elastic Web Method, Joseph A. Wise, Proc. A.C.I., Vol. XXIV, 1928, p. 408; Design of Reinforced Concrete Slabs, Joseph A. Wise, Proc. A.C.I., Vol. XXV, 1929, p. 712; Analysis of Plate Examples by Difference Methods and the Superposition Principle, D. L. Holl, Journal of Appl. Mech., A.S.M.E., Vol. 3, No. 3, Sept. 1936, p. A-81; Cantilever Plate with Concentrated Edge Load, D. L. Holl, Journal of Appl. Mech., A.S.M.E., Vol. 4, No. 1, March 1937, p. A-8.


q
n
r
i
k
l
λy=λ
o
m
p
λx=λ
Square
(a)
q
n
r
i
k
l
λy
o
m
p
λx
Rectangular
(b)
θ
λr
Radial
(c)
q
r
λy
i
k
l
λ
λ
o
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λx=λ
(d)
Hexagonal
C
G
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q
r
T
A
i
k
l
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o
p
V
λy
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λ
E
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λx=λ
(e)
TYPES OF NETWORKS FOR DIFFERENCE EQUATIONS

Fig. 4

(see Fig.4) are used in the formation of difference-equations networks, rather than the corresponding infinitesimal elements associated with differential equations, only approximate results are obtained. The approximations become better, the smaller the elements. An attempt was made in this investigation to ascertain the effect of the density of the networks for triangular as well as skew plates.

The procedure for the method of finite differences as presented by H. Marcus has been followed throughout this dissertation. According to Marcus, the elastic web is said to consist of a network of elastic strings attached to the edges of the plate in such a way as to conform with the boundary conditions of the actual plate or slab. In the case of a plate loaded by a concentrated load, it is desirous to arrange a network which has a point of intersection at the position of the load. If the load is uniformly distributed, it is replaced by a statically equipollent system of concentrated forces applied at the intersection points contiguous to the loaded area. The difference equations relating the deflections of the web usually yield a system of simultaneous linear equations in normal form.

From the equations

$$\nabla^4 w = \frac{p}{N} \tag{3a}$$

and

$$\bar{M} = - N \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \tag{21}$$

there may be deduced the equation of the elastic surface:

$$\nabla^2 \bar{M} = - p \tag{22}$$

where $\bar{M}$ is a function of the moment sum.

The sum of the moments in the x- and y-directions is

$$M_x + M_y = - N\,(1 + \mu)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = (1 + \mu)\,\bar{M} \quad . \qquad (23)$$

The deflections, $\bar{w}$, of a membrane subjected only to fiber stresses parallel to the surface, are related to the horizontal component, S, of the surface stresses by the differential equation

$$\nabla^2 \bar{w} = \frac{\partial^2 \bar{w}}{\partial x^2} + \frac{\partial^2 \bar{w}}{\partial y^2} = - \frac{p}{S} \quad . \qquad (24)$$

Manifestly, S must be constant over the entire membrane because of conditions of static equilibrium. If $S = 1$, comparison of equations (22) and (24) suggests the theorem:[1]

> "The deflection of a membrane loaded with loads proportional to those on a given plate may be considered as the sum of the principal moments of the actual plate."

Similar to the procedure in the case of beams it is possible to arrive at another theorem:

> "A second membrane may be loaded with elastic weights proportional to these moment sums and, subject to appropriate boundary conditions, the deflections of the latter membrane will be proportional to the deflections of the actual plate under the given loading system."

The elastic weights in the second theorem are actually $\frac{\bar{M}}{N}$.

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[1]Analysis of Plate Examples by Difference Methods and the Superposition Principle, D. L. Holl, Journal of Appl. Mech., A. S. M. E., Vol. 3, No. 3, Sept. 1936, p. A-81.

Differentiation of equation (21) results in the equations:[1]

$$\frac{\partial^2\bar{M}}{\partial x^2} + \mu\,\frac{\partial^2\bar{M}}{\partial y^2} = -N\left[\frac{\partial^4 w}{\partial x^4} + (1+\mu)\,\frac{\partial^4 w}{\partial x^2\partial y^2} + \mu\,\frac{\partial^4 w}{\partial y^4}\right]$$

$$\frac{\partial^2\vec{M}}{\partial y^2} + \mu\,\frac{\partial^2\vec{M}}{\partial x^2} = -N\left[\frac{\partial^4 w}{\partial y^4} + (1+\mu)\,\frac{\partial^4 w}{\partial x^2\partial y^2} + \mu\,\frac{\partial^4 w}{\partial x^4}\right] \quad . \qquad (25)$$

Likewise, by differentiation of equations (1), there are deduced the expressions:

$$\nabla^2 M_x = \frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_x}{\partial y^2} = -N\left[\frac{\partial^4 w}{\partial x^4} + (1+\mu)\,\frac{\partial^4 w}{\partial x^2\partial y^2} + \mu\,\frac{\partial^4 w}{\partial y^4}\right]$$

$$\nabla^2 M_y = \frac{\partial^2 M_y}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} = -N\left[\frac{\partial^4 w}{\partial y^4} + (1+\mu)\,\frac{\partial^4 w}{\partial x^2\partial y^2} + \mu\,\frac{\partial^4 w}{\partial x^4}\right] \qquad (26)$$

The designations

$$-\phi_x = \frac{\partial^2\bar{M}}{\partial x^2} + \mu\,\frac{\partial^2\bar{M}}{\partial y^2}$$

$$-\phi_y = \frac{\partial^2\vec{M}}{\partial y^2} + \mu\,\frac{\partial^2\vec{M}}{\partial x^2} \qquad (27)$$

$$-\phi_{xy} = (1-\mu)\,\frac{\partial^2\overleftarrow{M}}{\partial x\partial y}$$

result in the identities

$$\nabla^2 M_x = -\phi_x$$

$$\nabla^2 M_y = -\phi_y \qquad (28)$$

$$\nabla^2 M_{xy} = -\phi_{xy}$$

[1] Die Theorie elastischer Gewebe, H. Marcus, p. 11.

The third theorem resulting from this discussion may be stated as follows:[1]

"The membrane carrying loads

$$-\phi_x = \frac{\partial^2 \bar{M}}{\partial x^2} + \mu \frac{\partial^2 \bar{M}}{\partial y^2}$$

$$-\phi_y = \frac{\partial^2 \bar{M}}{\partial y^2} + \mu \frac{\partial^2 \bar{M}}{\partial x^2}$$

$$-\phi_{xy} = (1 - \mu) \frac{\partial^2 \bar{M}}{\partial x \partial y}$$

and having $S = 1$, forms a moment diagram for the M_x, M_y, and M_{xy} moments of the elastic plate."

Equation (5), giving the shearing forces, may be put in the form

$$V_x = -N \frac{\partial}{\partial x} \nabla^2 w = \frac{\partial \bar{M}}{\partial x}$$

$$V_y = -N \frac{\partial}{\partial y} \nabla^2 w = \frac{\partial \bar{M}}{\partial y} \qquad (29)$$

It is obvious from this discussion that the solution of the plate problem is manifested by the equations

$$\nabla^2 \bar{M} = -p$$

$$\nabla^2 w = -\frac{\bar{M}}{N} \quad .$$

In accordance with the Marcus procedure, the membrane is considered to be replaced by an elastic web and the relations between the elastic plate and the elastic web are to be briefly indicated. The complete derivation is given by Marcus and Wise, and it will therefore not be given here.

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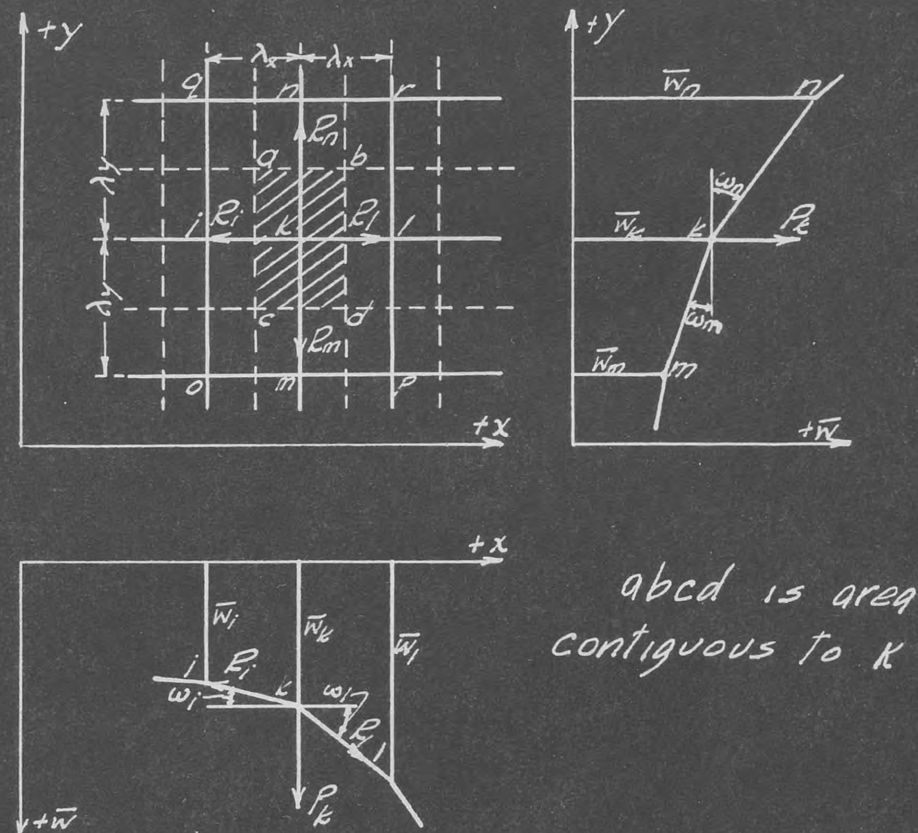
[1] Die Theorie elastischer Gewebe, H. Marcus, p. 11;
The Calculation of Flat Plates by the Elastic Web Method, Joseph A. Wise, Proc. A.C.I., Vol. XXIV, 1928, p. 416.



ELASTIC WEB WITH RECTANGULAR MESH

Fig. 5

The mesh widths of the web are λ_x in the x-direction and λ_y in the y-direction, and the force P_k is the load contiguous to the intersection point k. If H_x and H_y are taken to be the horizontal components of stress in the wires of the elastic web in the x- and y-directions, respectively, and $\bar{\bar{w}}_k$ is the deflection of the point k, it is possible to deduce the relation[1] (see Fig. 5)

$$\frac{H_x}{\lambda_x}(\Delta^2 \bar{\bar{w}}_k)_x + \frac{H_y}{\lambda_y}(\Delta^2 \bar{\bar{w}}_k)_y = -P_k \qquad (30)$$

where $(\Delta^2 \bar{\bar{w}}_k)_x$ is the second derivative between the $\bar{\bar{w}}$-ordinates of the nodal points, spaced at intervals λ_x, and $(\Delta^2 \bar{\bar{w}}_k)_y$ is similarly defined for the y-direction.

The horizontal components of surface stress in the membrane are related to the horizontal components of stress in the wires of the web by the defining equations

$$H_x = \lambda_y S_x \quad , \qquad H_y = \lambda_x S_y \quad .$$

The load P_k is given by the equation

$$P_k = p_k \lambda_x \lambda_y \quad .$$

Therefore,

$$\frac{S_x}{\lambda_x^2}(\Delta^2 \bar{\bar{w}}_k)_x + \frac{S_y}{\lambda_y^2}(\Delta^2 \bar{\bar{w}}_k)_y = -p_k \quad . \qquad (31)$$

Should $H_x = H_y = H$, and $S_x = S_y = S$, then there result the equations

$$\frac{1}{\lambda_x}(\Delta^2 \bar{\bar{w}}_k)_x + \frac{1}{\lambda_y}(\Delta^2 \bar{\bar{w}}_k)_y = -\frac{P_k}{H} \qquad (32)$$

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[1] Die Theorie elastischer Gewebe, H. Marcus, p. 16.

and

$$\frac{(\Delta^2 \bar{w}_k)_x}{\lambda_x^2} + \frac{(\Delta^2 \bar{w}_k)_y}{\lambda_y^2} = -\frac{P_k}{S} \qquad (33)$$

which, for $\lambda_x = \lambda_y = \lambda$, give

$$(\Delta^2 \bar{w}_k)_x + (\Delta^2 \bar{w}_k)_y = -P_k \cdot \frac{\lambda}{H} = -p_k \cdot \frac{\lambda^2}{S} . \qquad (34)$$

This may be put in the form generally used in the case of square meshes:

$$4\bar{w}_k - (\bar{w}_i + \bar{w}_l + \bar{w}_m + \bar{w}_n) = P_k \frac{\lambda}{H} = \frac{p_k \lambda^2}{S} \qquad (35)$$

The analogy inherent in equations (28) and (33) gives the relation

$$\bar{M}_k = S_1 \bar{w}_k \qquad (36)$$

where S_1 is the quantity S referred to the first membrane.

Now, if the elastic loads $p_k = \bar{w}_k$ are applied at the intersection points of the web and S is set equal to S_2, the deflections, $\bar{z}$, of the nodal points are related by the equations[1]

$$\frac{\bar{z}_i - 2\bar{z}_k + \bar{z}_l}{\lambda_x^2} + \frac{\bar{z}_m - 2\bar{z}_k + \bar{z}_n}{\lambda_y^2} =$$

$$\frac{(\Delta^2 \bar{z}_k)_x}{\lambda_x^2} \quad \frac{(\Delta^2 \bar{z}_k)_y}{\lambda_y^2} = -\frac{\bar{w}_k}{S_2} = -\frac{\bar{M}}{S_1 S_2}$$

For the deflections of the plate itself, there holds the expression

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{\bar{M}}{N} \qquad (37)$$

and, by analogy, it becomes apparent that

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[1] Die Theorie elastischer Gewebe, H. Marcus, p. 17.

$$w_k = \bar{\bar{z}}_k \frac{S_1 S_2}{N} \tag{38}$$

Equations similar to (35) may be written for determining the values of $\bar{\bar{z}}_k$.

Bending moments and twisting moments in the case of rectangular meshes (see Fig. 4(b)) are given by the expressions[1]

$$M_x = S_1 S_2 \left[\frac{2\bar{\bar{z}}_k - \bar{\bar{z}}_i - \bar{\bar{z}}_l}{\lambda_x^2} + \mu \frac{2\bar{\bar{z}}_k - \bar{\bar{z}}_m - \bar{\bar{z}}_n}{\lambda_y^2} \right]$$

$$M_y = S_1 S_2 \left[\frac{2\bar{\bar{z}}_k - \bar{\bar{z}}_m - \bar{\bar{z}}_n}{\lambda_y^2} + \mu \frac{2\bar{\bar{z}}_k - \bar{\bar{z}}_i - \bar{\bar{z}}_l}{\lambda_x^2} \right] \tag{39}$$

$$M_{xy} = (1 - \mu)\, S_1 S_2 \left[\frac{(\bar{\bar{z}}_p + \bar{\bar{z}}_q) - (\bar{\bar{z}}_o + \bar{\bar{z}}_r)}{4\lambda_x \lambda_y} \right].$$

The vertical shearing forces are to be found from the equations

$$V_x = S_1 \frac{\partial \bar{\bar{w}}}{\partial x} = \frac{S_1}{2\lambda_x} (\bar{\bar{w}}_l - \bar{\bar{w}}_i)$$

$$V_y = S_1 \frac{\partial \bar{\bar{w}}}{\partial y} = \frac{S_1}{2\lambda_y} (\bar{\bar{w}}_n - \bar{\bar{w}}_m) \tag{40}$$

A similar study of the web with an equilateral triangular (hexagonal) network (see Fig. 4(d)) yields the simple relations[2]

$$4\bar{\bar{w}}_k - \frac{2}{3} (\bar{\bar{w}}_i + \bar{\bar{w}}_l + \bar{\bar{w}}_o + \bar{\bar{w}}_p + \bar{\bar{w}}_q + \bar{\bar{w}}_r) = \frac{p\lambda^2}{S_1}$$

$$4\bar{\bar{z}}_k - \frac{2}{3} (\bar{\bar{z}}_i + \bar{\bar{z}}_l + \bar{\bar{z}}_o + \bar{\bar{z}}_p + \bar{\bar{z}}_q + \bar{\bar{z}}_r) = \bar{\bar{w}}_k \frac{\lambda^2}{S_2} = \bar{\bar{M}}_k \frac{\lambda^2}{S_1 S_2}. \tag{41}$$

From the $\bar{\bar{w}}$- and $\bar{\bar{z}}$-values, as before, it is possible to find the moment sums $\bar{\bar{M}} = \bar{\bar{w}} S_1$ and the deflections $w = S_1 S_2 \frac{\bar{\bar{z}}}{N}$.

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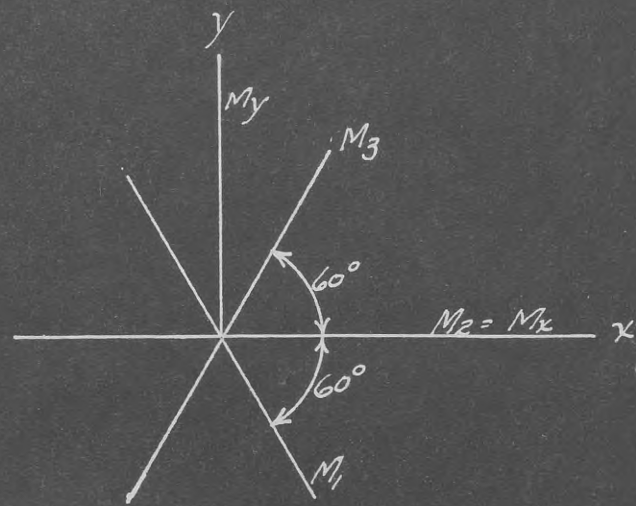
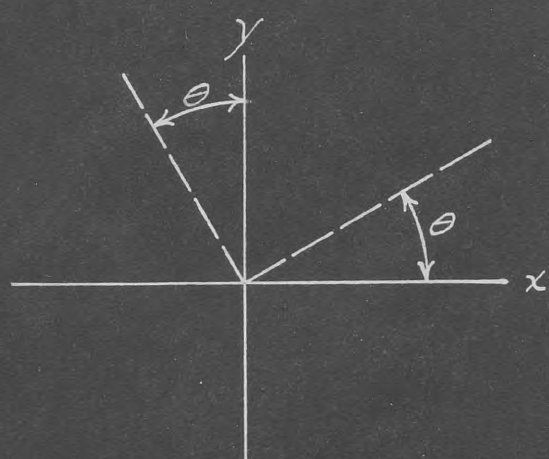
[1]Die Theorie elastischer Gewebe, H. Marcus, p. 18.

[2]Die Theorie elastischer Gewebe, H. Marcus, p. 30.



(a)

Directions of Bending Moments



(b)

Directions of Principal Moments

Fig. 6

Bending moments in the directions 1, 2, and 3 (see Fig. 6(a)) may be obtained from the following:

$$M_{1k} = (1 - \mu)\ S_1 S_2 \frac{(2\bar{z}_k - \bar{z}_p - \bar{z}_q)}{(\lambda_y \sec \alpha)^2} + \mu S_1 \bar{w}_k$$

$$M_{2k} = (1 - \mu)\ S_1 S_2 \frac{(2\bar{z}_k - \bar{z}_i - \bar{z}_l)}{\lambda_k^2} + \mu S_1 \bar{w}_k \qquad (42)$$

$$M_{3k} = (1 - \mu)\ S_1 S_2 \frac{(2\bar{z}_k - \bar{z}_o - \bar{z}_r)}{(\lambda_y \sec \alpha)^2} + \mu S_1 \bar{w}_k$$

wherein 2α equals 60 degrees.

Vertical shearing forces are given by the terms

$$V_{1k} = S_1 \frac{(\bar{w}_p - \bar{w}_q)}{2\lambda_y \sec \alpha}$$

$$V_{2k} = S_1 \frac{(\bar{w}_l - \bar{w}_i)}{2\lambda_x} \qquad (43)$$

$$V_{3k} = S_1 \frac{(\bar{w}_r - \bar{w}_o)}{2\lambda_y \sec \alpha}$$

The twisting moments result from the expression (see Fig. 4(d)):

$$M_{xy} = -(1 - \mu)\ N \frac{\partial^2 w}{\partial x \partial y}$$
$$= (1 - \mu) \frac{S_1 S_2}{2\lambda_x \lambda_y} \left[(\bar{z}_p + \bar{z}_q) - (\bar{z}_o + \bar{z}_r) \right] . \qquad (44)$$

From Fig. 6(a) it is clear that $M_x = M_2$. In order to find M_y, it is possible to use the simple equation

$$M_y = (1 + \mu)\ \bar{M} - M_x \quad . \qquad (23a)$$

The maximum and minimum bending moments may be shown to be

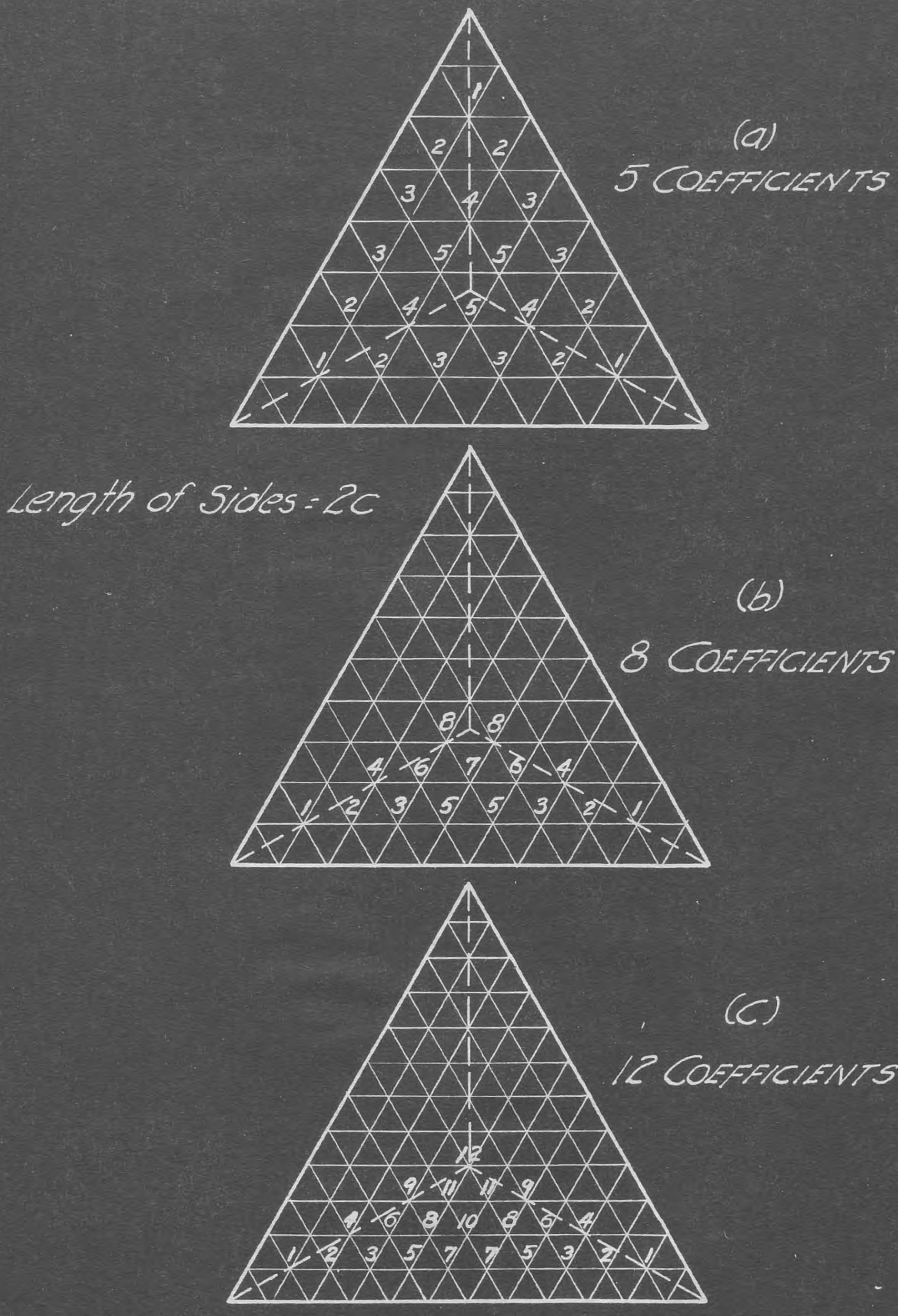
$$M_{\substack{max\\min}} = \frac{M_x + M_y}{2} \pm \frac{1}{2}\sqrt{(M_x - M_y)^2 + 4M_{xy}^2} \ . \qquad (45)$$

The maximum value of the twisting moment is

$$\frac{1}{2}\sqrt{(M_x - M_y)^2 + 4M_{xy}^2} \ . \qquad (46)$$

To obtain the trajectories of the principal moments (see Fig. 6(b)), use is made of the equation

$$\tan 2\theta = \frac{2M_{xy}}{M_x - M_y} \qquad (47)$$



NETWORKS FOR EQUILATERAL TRIANGULAR PLATES

Fig. 7

III. PRELIMINARY STUDY OF TRIANGULAR PLATES

10. Simply Supported Equilateral, Triangular Plate Subjected to a Uniformly Distributed Load. - A brief preliminary study of the effect of the density of difference-equations networks on certain deflections and moments was made for some triangular plates in order to obtain a rough measure of the reliability to be expected in the case of skew plates.

Considerable work has already been published on the simply supported equilateral, triangular plate subjected to a uniformly distributed load,[1,2] and some of this is to be mentioned here together with additional results that have been found by the writer.

Difference-equations networks for the equilateral, triangular plate are shown in Fig. 7. The five-coefficient case was analyzed by Marcus.[1] The eight-coefficient network was worked up by the writer and the maximum deflection was compared with that of the former case as well as with the exact value given by Woinowsky-Krieger.[2] This comparison is shown in Table I.

It is very likely that the twelve-coefficient value would be still closer to the exact deflection. Clearly, however, the agreement even in the five-coefficient solution is good.

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[1]Die Theorie elastischer Gewebe, H. Marcus, p. 135.

[2]Berechnung der ringsum frei aufliegenden gleichseitigen Dreiecksplatte, S. Woinowsky-Krieger, Ingenieur-Archiv, IV, Band 1933, p. 255.

TABLE I

COMPARISON OF MAXIMUM DEFLECTIONS

FOR SIMPLY SUPPORTED EQUILATERAL, TRIANGULAR PLATE

SUBJECTED TO A UNIFORMLY DISTRIBUTED LOAD

Solution	Maximum Deflection	Approximation
5 Coefficients	$0.00951 \frac{pc^4}{N}$	2.7 %
8 Coefficients	$0.00940 \frac{pc^4}{N}$	1.2 %
Exact	$0.00926 \frac{pc^4}{N}$	-

A comparison of the maximum values of the bending moments along the vertical angle-bisector has been made by Woinowsky-Krieger in the paper previously alluded to, and the agreement between his values and those of Marcus is remarkably good (see Table II).

TABLE II

COMPARISON OF MAXIMUM AND MINIMUM MOMENTS

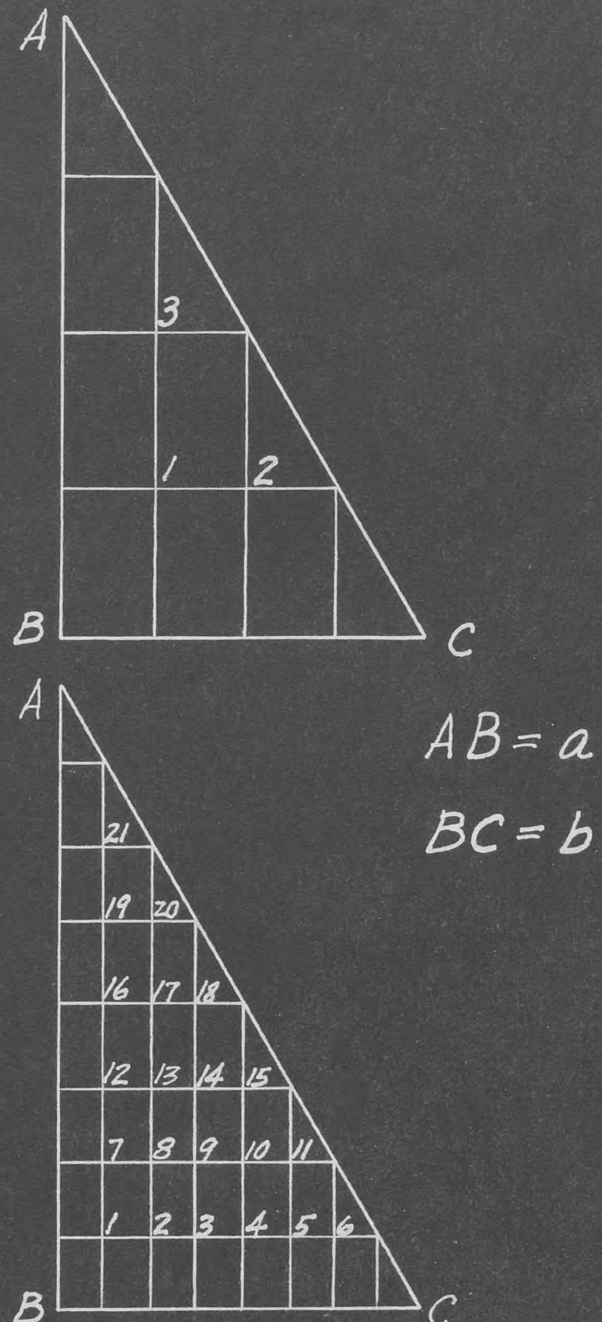
ALONG ANGLE-BISECTOR OF SIMPLY SUPPORTED EQUILATERAL,

TRIANGULAR PLATE SUBJECTED TO A UNIFORMLY DISTRIBUTED LOAD

($\mu = 0.3$)

Solution	Max. M_x $(y = -0.1074c)$[1]	Min. M_x $(y = 0.9180c)$[1]	Max. M_y $(y = 0.2234c)$[1]
5 Coefficients	$0.0753\ pc^2$	$-0.01785\ pc^2$	$0.0774\ pc^2$
Exact	$0.0744\ pc^2$	$-0.0168\ pc^2$	$0.0777\ pc^2$

[1]Distance measured vertically from center of gravity of plate (see Fig. 7).



NETWORKS FOR 30°-60°-90° TRIANGULAR PLATE

Fig. 8

It is evident from this discussion that in the case of equilateral, triangular plates, subjected to uniformly distributed loads, reasonably good results are obtainable by the method of finite differences even for a relatively small number of intersection points in the network.

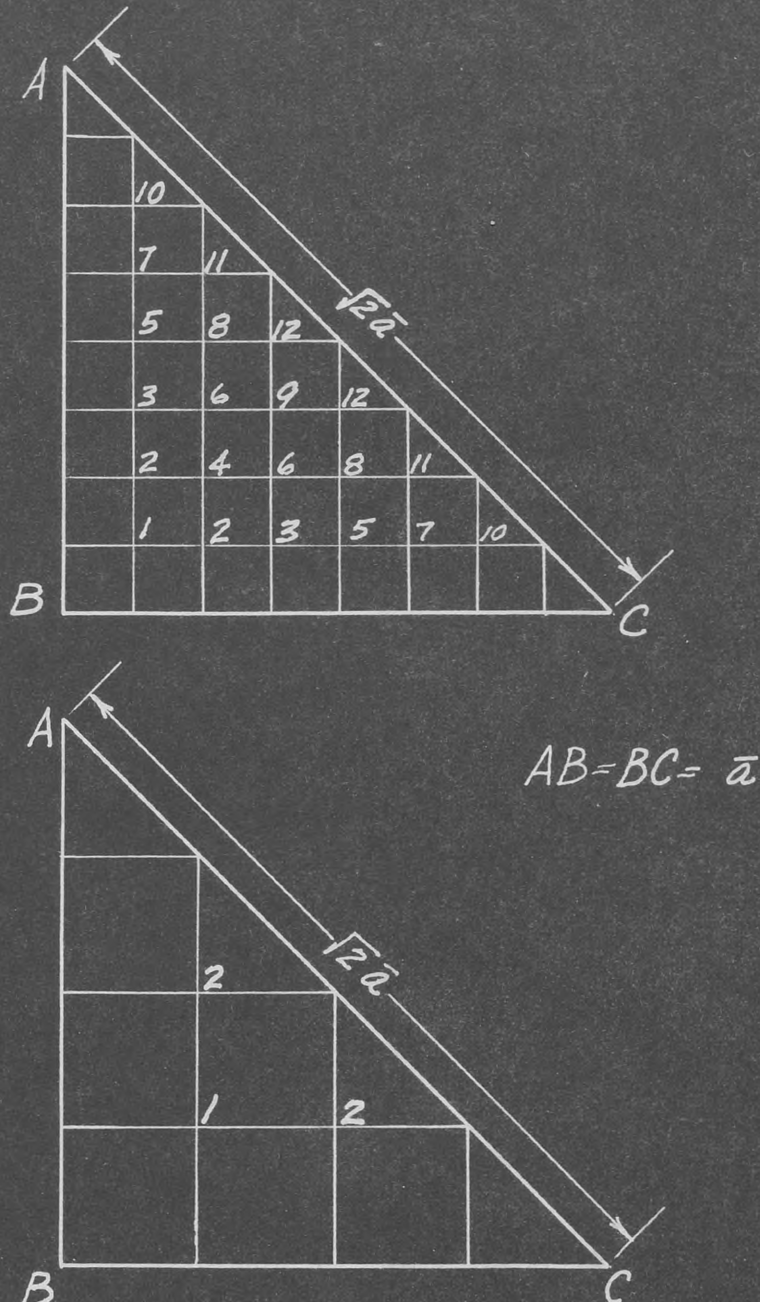
11. Simply Supported Thirty-Sixty-Ninety- Degree Triangular Plate Subjected to a Uniformly Distributed Load. - Only the deflections at certain points of the plate are compared for this case. From Fig. 8 it may be seen that two networks, having 3 and 21 nodal points, were considered. It is to be observed that the 21-coefficient analysis requires considerably more time than the other and yet, in so far as the deflections are concerned, the difference between the results does not appear to be too great (see Table III). The percentage difference is seen to be around 7.

TABLE III

COMPARISON OF DEFLECTIONS AT CERTAIN POINTS OF SIMPLY SUPPORTED THIRTY-SIXTY-NINETY- DEGREE PLATE SUBJECTED TO A UNIFORMLY DISTRIBUTED LOAD

Solution	Deflection 1[1]	2[1]	3[1]
3 Coefficients	$0.001664 \frac{pb^4}{N}$	$0.001539 \frac{pb^4}{N}$	$0.000879 \frac{pb^4}{N}$
21 Coefficients	$0.001566 \frac{pb^4}{N}$	$0.001437 \frac{pb^4}{N}$	$0.000818 \frac{pb^4}{N}$

[1]Points in 3-coefficient case.
Corresponding points in 21-coefficient solution are 8, 10, and 17 (see Fig. 8).



NETWORKS FOR ISOSCELES TRIANGULAR PLATE HAVING A RIGHT ANGLE

Fig. 9

12. Simply Supported Isosceles Right Triangular Plate Subjected to a Uniformly Distributed Load. - The deflections, for this case, as obtained by Nadai's approximate method[1] and the method of finite differences (12 coefficients) are compared in Table IV (see Fig. 9). It is to be observed that near the center of gravity of the plate (points 4, 6 and 9 in the twelve-coefficient network) there is good agreement, whereas nearer the boundaries there is considerable disagreement. Both methods in this comparison, however, are known to be approximate.

TABLE IV

COMPARISON OF DEFLECTIONS OBTAINED BY NADAI'S APPROXIMATION AND DIFFERENCE-EQUATION NETWORK HAVING 12 COEFFICIENTS FOR UNIFORMLY LOADED SIMPLY SUPPORTED ISOSCELES RIGHT TRIANGULAR PLATE

Point	Nadai $\frac{wN}{pa^4}$	Difference Equations $\frac{wN}{pa^4}$
1	0.000240	0.000253
2	0.000392	0.000401
3	0.000410	0.000425
4	0.000628	0.000625
5	0.000314	0.000348
6	0.000632	0.000635
7	0.000170	0.000216
8	0.000444	0.000471
9	0.000580	0.000582
10	0.000052	0.000082
11	0.000188	0.000222
12	0.000314	0.000326

[1]Die elastische Platten, A. Nadai, 1925, p. 177.

TABLE V

COMPARISON OF DEFLECTIONS AT CERTAIN POINTS

OF UNIFORMLY LOADED SIMPLY SUPPORTED

ISOSCELES RIGHT TRIANGULAR PLATE

Solution	Deflection 1^1	Deflection 2^1
2 Coefficients	$0.000678 \frac{p\bar{a}^4}{N}$	$0.000518 \frac{p\bar{a}^4}{N}$
12 Coefficients	$0.000625 \frac{p\bar{a}^4}{N}$	$0.000471 \frac{p\bar{a}^4}{N}$
Nadai	$0.000628 \frac{p\bar{a}^4}{N}$	$0.000444 \frac{p\bar{a}^4}{N}$

[1]Points refer to 2-coefficients case.

In Table V the deflections obtained for the two-coefficient and twelve-coefficient networks are compared for certain corresponding points with those obtained by Nadai's approximate method.

It is realized that the preliminary work on triangular plates discussed here is by no means complete. Only a rough measure of the effect of network density has been obtained for certain deflections and moments. In the case of skew plates the investigation of this effect has been more extensive.

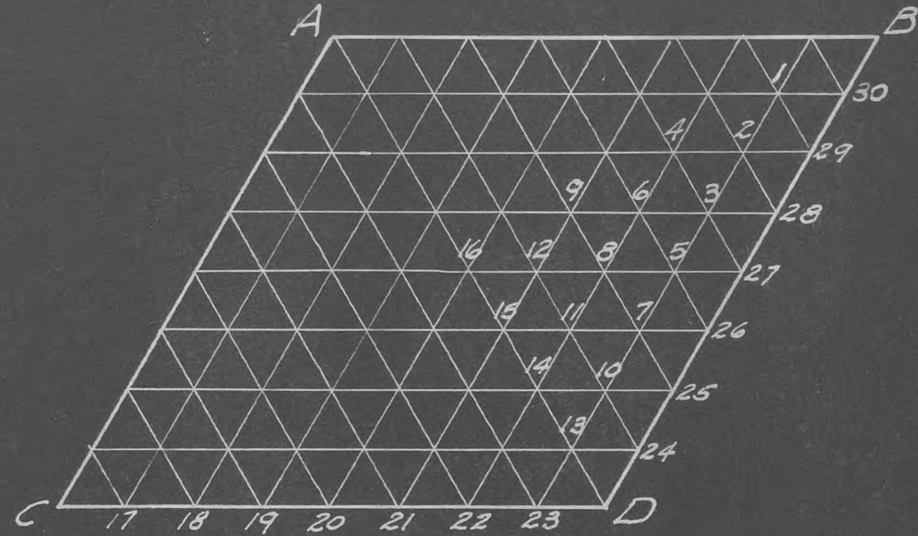
IV. SIMPLY SUPPORTED THIRTY-DEGREE SKEW PLATE WITH EQUAL SIDES SUBJECTED TO A UNIFORMLY DISTRIBUTED LOAD

13. Analysis by Method of Finite Differences. - The most convenient network for the analysis of a 30-degree skew plate by the difference-equations method is a triangular (hexagonal) arrangement formed by drawing lines parallel to the sides and the shorter diagonal of the plate (see Figs. 4(d), 10 and 11).

In order to determine the effect of the mesh intervals on the results for deflections and moments in the case of the uniformly loaded simply supported 30-degree skew plate having equal sides, (length = 2c), three networks of varying density ($\lambda = \frac{c}{2}, \frac{c}{4}, \frac{c}{5}$) were analyzed by this method.

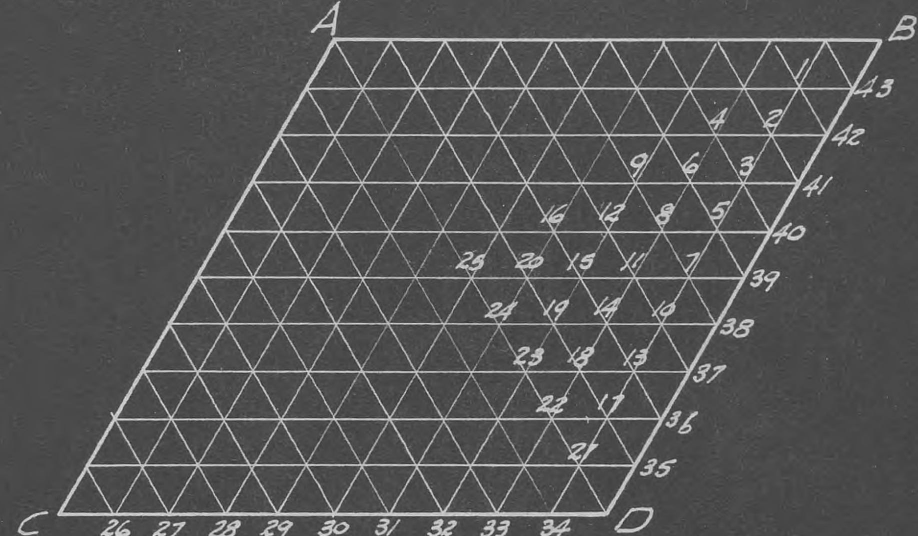
The difference equations relating the $\bar{w}$- and $\bar{z}$- values at the intersections of the lines forming the networks result in a system of simultaneous linear equations. The number of these equations depends upon the number of intersection or nodal points. In the particular plate now being considered, it is to be observed that there is symmetry about both diagonals and, consequently, only one quarter of the plate need be analyzed.

If the length of the sides is divided into four parts ($\lambda = \frac{c}{2}$), it is apparent that four nodal points result (some of them are repeated) and, therefore, four simultaneous linear equations must be written and four coefficients are to be found. For $\lambda = \frac{c}{4}$ and $\lambda = \frac{c}{5}$ there are to be written sixteen and twenty-five



16 Coefficients
(a)

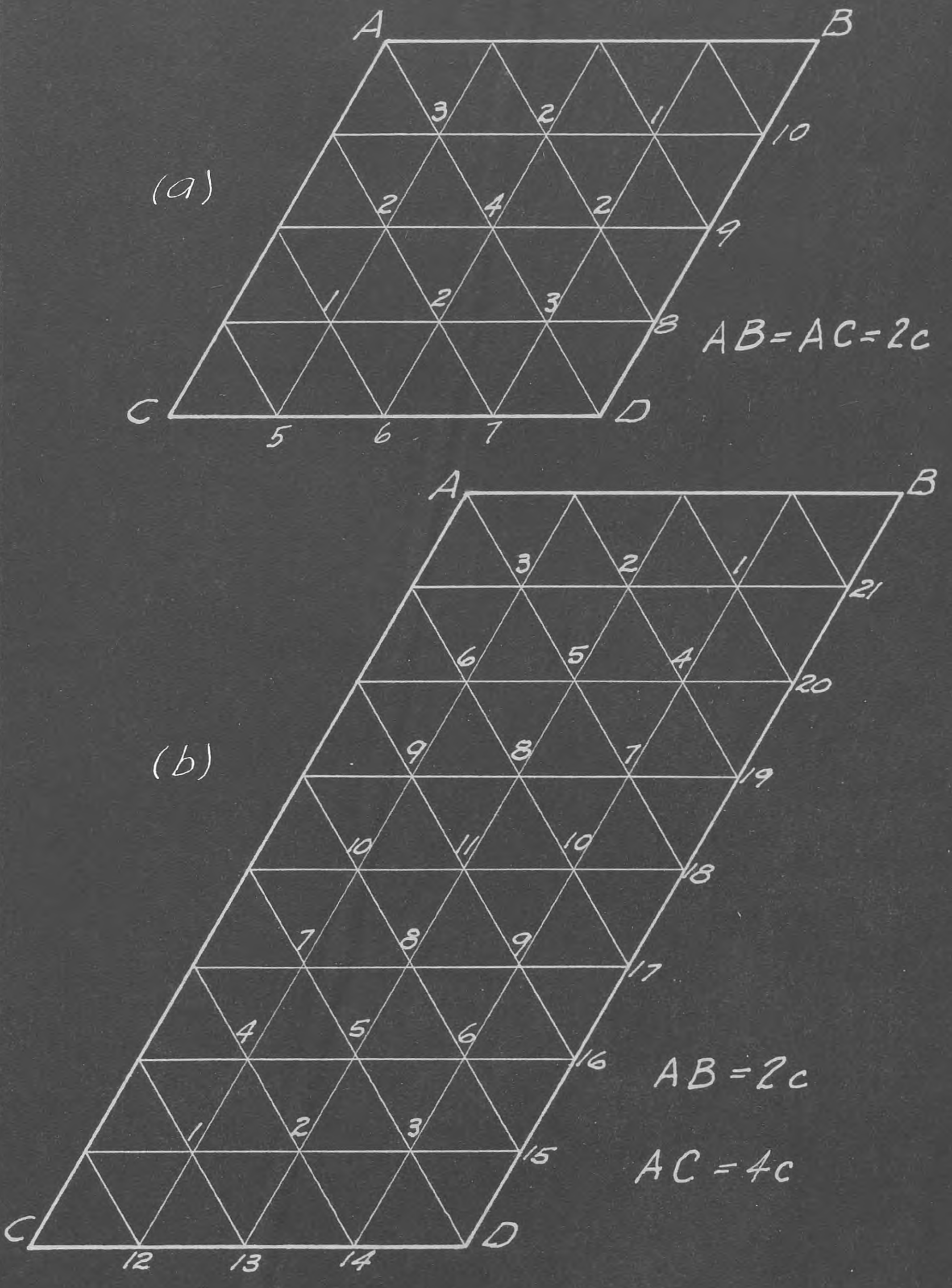
AB = AC = 2c



25 Coefficients
(b)

NETWORKS FOR 30° SKEW PLATES

Fig. 10



NETWORKS FOR 30° SKEW PLATES

Fig. 11

equations, respectively. As it was pointed out previously, the equations relating the deflections of the membranes involved in this method are usually in normal form, and they may be solved conveniently by the " Doolittle Method " of solving simultaneous equations.[1]

The boundary conditions for the membranes associated with simply supported plates[2] are indicated in Fig. 12. It is to be noted that the relations given for $\bar{z}$ in this figure are also valid for $\bar{w}$.

Substitution in equations (41) gives the following equations for the sixteen-coefficient solution:[3]

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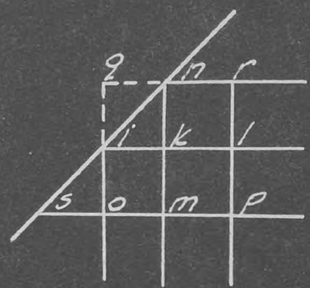
[1]Application of the Theory of Least Squares to the Adjustment of Triangulation, Oscar S. Adams, Special Publication No. 28, U. S. Coast and Geodetic Survey, 1915. See also: Analyses of Skew Slabs, V. P. Jensen, Concrete Slabs Investigation Progress Report - February 1941, Part I. Another method of substitution is given in the paper: Der abgekurtzte Gauss'sche Algorithmus als eine einheitliche Grundlage in der Baustatik, Peter Pasternak, 1926.

[2]Die Theorie elastischer Gewebe, H. Marcus, 1932, p. 37.

[3]The sixteen-coefficient case for the uniformly loaded, simply supported 30-degree skew plate is given as a sample illustration of the procedure associated with the difference-equation method of solving slab problems.



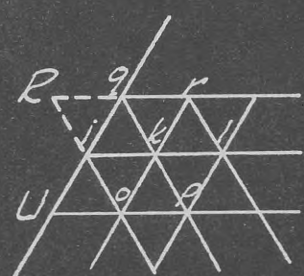
(a) Square Mesh

Simply Supported Edge

$$Z_s = Z_i = Z_n = 0$$
$$Z_q = -Z_k$$

Clamped Edge

$$Z_s = Z_i = Z_n = 0$$
$$Z_q = Z_k$$



(b) Hexagonal Mesh

Simply Supported Edge

$$Z_u = Z_i = Z_q = 0$$
$$Z_R = -Z_k$$

Clamped Edge

$$Z_u = Z_i = Z_q = 0$$
$$Z_R = Z_k$$

BOUNDRY CONDITIONS FOR SKEW PLATES

Fig. 12

$$
\begin{aligned}
4\bar{w}_1 - \tfrac{2}{3}(2\bar{w}_2) &= \frac{p\lambda^2}{S_1} \\
4\bar{w}_2 - \tfrac{2}{3}(\bar{w}_1 + \bar{w}_2 + \bar{w}_3 + \bar{w}_4) &= \text{"} \\
4\bar{w}_3 - \tfrac{2}{3}(\bar{w}_2 + \bar{w}_4 + \bar{w}_5 + \bar{w}_6) &= \text{"} \\
4\bar{w}_4 - \tfrac{2}{3}(2\bar{w}_2 + 2\bar{w}_3 + 2\bar{w}_6) &= \text{"} \\
4\bar{w}_5 - \tfrac{2}{3}(\bar{w}_3 + \bar{w}_6 + \bar{w}_7 + \bar{w}_8) &= \text{"} \\
4\bar{w}_6 - \tfrac{2}{3}(\bar{w}_3 + \bar{w}_4 + \bar{w}_5 + \bar{w}_6 + \bar{w}_8 + \bar{w}_9) &= \text{"} \\
4\bar{w}_7 - \tfrac{2}{3}(\bar{w}_5 + \bar{w}_8 + \bar{w}_{10} + \bar{w}_{11}) &= \text{"} \\
4\bar{w}_8 - \tfrac{2}{3}(\bar{w}_5 + \bar{w}_6 + \bar{w}_7 + \bar{w}_9 + \bar{w}_{11} + \bar{w}_{12}) &= \text{"} \qquad (48) \\
4\bar{w}_9 - \tfrac{2}{3}(2\bar{w}_6 + 2\bar{w}_8 + 2\bar{w}_{12}) &= \text{"} \\
4\bar{w}_{10} - \tfrac{2}{3}(\bar{w}_7 + \bar{w}_{11} + \bar{w}_{13} + \bar{w}_{14}) &= \text{"} \\
4\bar{w}_{11} - \tfrac{2}{3}(\bar{w}_7 + \bar{w}_8 + \bar{w}_{10} + \bar{w}_{12} + \bar{w}_{14} + \bar{w}_{15}) &= \text{"} \\
4\bar{w}_{12} - \tfrac{2}{3}(\bar{w}_8 + \bar{w}_9 + \bar{w}_{11} + \bar{w}_{12} + \bar{w}_{15} + \bar{w}_{16}) &= \text{"} \\
4\bar{w}_{13} - \tfrac{2}{3}(2\bar{w}_{10} + \bar{w}_{14}) &= \text{"} \\
4\bar{w}_{14} - \tfrac{2}{3}(2\bar{w}_{10} + 2\bar{w}_{11} + \bar{w}_{13} + \bar{w}_{15}) &= \text{"} \\
4\bar{w}_{15} - \tfrac{2}{3}(2\bar{w}_{11} + 2\bar{w}_{12} + \bar{w}_{14} + \bar{w}_{16}) &= \text{"} \\
4\bar{w}_{16} - \tfrac{2}{3}(4\bar{w}_{12} + 2\bar{w}_{15}) &= \text{"}
\end{aligned}
$$

$$
\begin{aligned}
4\bar{z}_1 &- \tfrac{2}{3}(2\bar{z}_2) &&= \bar{w}_1 \frac{\lambda^2}{S_2} \\
4\bar{z}_2 &- \tfrac{2}{3}(\bar{z}_1 + \bar{z}_2 + \bar{z}_3 + \bar{z}_4) &&= \bar{w}_2 \;\;'' \\
4\bar{z}_3 &- \tfrac{2}{3}(\bar{z}_2 + \bar{z}_4 + \bar{z}_5 + \bar{z}_6) &&= \bar{w}_3 \;\;'' \\
4\bar{z}_4 &- \tfrac{2}{3}(2\bar{z}_2 + 2\bar{z}_3 + 2\bar{z}_6) &&= \bar{w}_4 \;\;'' \\
4\bar{z}_5 &- \tfrac{2}{3}(\bar{z}_3 + \bar{z}_6 + \bar{z}_7 + \bar{z}_8) &&= \bar{w}_5 \;\;'' \\
4\bar{z}_6 &- \tfrac{2}{3}(\bar{z}_3 + \bar{z}_4 + \bar{z}_5 + \bar{z}_6 + \bar{z}_8 + \bar{z}_9) &&= \bar{w}_6 \;\;'' \\
4\bar{z}_7 &- \tfrac{2}{3}(\bar{z}_5 + \bar{z}_8 + \bar{z}_{10} + \bar{z}_{11}) &&= \bar{w}_7 \;\;'' \\
4\bar{z}_8 &- \tfrac{2}{3}(\bar{z}_5 + \bar{z}_6 + \bar{z}_7 + \bar{z}_9 + \bar{z}_{11} + \bar{z}_{12}) &&= \bar{w}_8 \;\;'' \qquad (49) \\
4\bar{z}_9 &- \tfrac{2}{3}(2\bar{z}_6 + 2\bar{z}_8 + 2\bar{z}_{12}) &&= \bar{w}_9 \;\;'' \\
4\bar{z}_{10} &- \tfrac{2}{3}(\bar{z}_7 + \bar{z}_{11} + \bar{z}_{13} + \bar{z}_{14}) &&= \bar{w}_{10} \;\;'' \\
4\bar{z}_{11} &- \tfrac{2}{3}(\bar{z}_7 + \bar{z}_8 + \bar{z}_{10} + \bar{z}_{12} + \bar{z}_{14} + \bar{z}_{15}) &&= \bar{w}_{11} \;\;'' \\
4\bar{z}_{12} &- \tfrac{2}{3}(\bar{z}_8 + \bar{z}_9 + \bar{z}_{11} + \bar{z}_{12} + \bar{z}_{15} + \bar{z}_{16}) &&= \bar{w}_{12} \;\;'' \\
4\bar{z}_{13} &- \tfrac{2}{3}(2\bar{z}_{10} + \bar{z}_{14}) &&= \bar{w}_{13} \;\;'' \\
4\bar{z}_{14} &- \tfrac{2}{3}(2\bar{z}_{10} + 2\bar{z}_{11} + \bar{z}_{13} + \bar{z}_{15}) &&= \bar{w}_{14} \;\;'' \\
4\bar{z}_{15} &- \tfrac{2}{3}(2\bar{z}_{11} + 2\bar{z}_{12} + \bar{z}_{14} + \bar{z}_{16}) &&= \bar{w}_{15} \;\;'' \\
4\bar{z}_{16} &- \tfrac{2}{3}(4\bar{z}_{12} + 2\bar{z}_{15}) &&= \bar{w}_{16} \;\;''
\end{aligned}
$$

TABLE VI

EQUAL-SIDED THIRTY-DEGREE SKEW SLAB

Uniformly Distributed Load - Simply Supported

Normal Form of Difference Equations

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\frac{p\lambda^2}{S_1}$	$\frac{p\lambda^4}{S_1S_2}$
3	-1															-0.750000	-0.463973
-1	5	-1	-1													-1.500000	-1.658843
	-1	6	-1	-1	-1											-1.500000	-2.192174
	-1	-1	3		-1											-0.750000	-1.462041
		-1		6	-1	-1	-1									-1.500000	-2.523863
		-1	-1	-1	5		-1	-1								-1.500000	-3.796233
				-1		6	-1		-1	-1						-1.500000	-2.636190
				-1	-1	-1	6	-1		-1	-1					-1.500000	-4.268592
					-1		-1	3			-1					-0.750000	-2.411222
						-1			6	-1		-1	-1			-1.500000	-2.469393
						-1	-1		-1	6	-1		-1	-1		-1.500000	-4.305309
							-1	-1		-1	5			-1	-1	-1.500000	-5.277498
									-1			3	-0.5			-0.750000	-0.915260
									-1	-1		-0.5	3	-0.5		-0.750000	-1.897150
										-1	-1		-0.5	3	-0.5	-0.750000	-2.567936
											-1			-0.5	1.50	-0.375000	-1.401323

TABLE VII

Solution of Difference Equations

1	2	3	4	5	6	$\frac{p\lambda^2}{S_1}$	$\sum$
3	-1					-0.750000	1.250000
$(-)\bar{w}_1$	0.333333					0.250000	-0.416667
	5	-1	-1			-1.500000	1.500000
	-0.333333					-0.250000	-0.583333
	4.666667	-1	-1			-1.750000	0.916667
	$(-)\bar{w}_2$	0.214286	0.214286			0.375000	-0.196428
		6	-1	-1	-1	-1.500000	1.500000
		-0.214286	-0.214286			-0.375000	-0.803572
		5.785714	-1.214286	-1	-1	-1.875000	0.696428
		$(-)\bar{w}_3$	0.209877	0.172840	0.172840	0.324074	-0.120369
			3		-1	-0.750000	1.250000
			-0.254851	-0.209877	-0.209877	-0.393519	-1.068124
			-0.214286			-0.375000	-0.589286
			2.530863	-0.209877	-1.209877	-1.518519	-0.407410
			$(-)\bar{w}_4$	0.082927	0.478049	0.600000	0.160976

TABLE VII (Cont'd)

Solution of Difference Equations

5	6	7	8	9	10	11	$\frac{p\lambda^2}{S_1}$	$\sum$
6	-1	-1	-1				-1.500000	1.500000
-0.017404	-0.100331						-0.125926	-0.243661
-0.172840	-0.172840						-0.324074	-0.669754
5.809756	-1.273171	-1	-1				-1.950000	0.586585
$(-)\bar{\bar{w}}_5$	0.219144	0.172124	0.172124				0.335642	-0.100966
	5		-1	-1			-1.500000	1.500000
	-0.279008	-0.219143	-0.219143				-0.427330	-1.144624
	-0.578380						-0.725926	-1.304306
	-0.172840						-0.324074	-0.496914
	3.969772	-0.219143	-1.219143	-1			-2.977330	-1.445844
	$(-)\bar{\bar{w}}_6$	0.055203	0.307107	0.251904			0.750000	0.364214
		6	-1		-1	-1	-1.500000	1.500000
		-0.012097	-0.067300	-0.055203			-0.164357	-0.298957
		-0.172124	-0.172124				-0.335642	-0.679890
		5.815779	-1.239424	-0.055203	-1	-1	-1.999999	0.521153
		$(-)\bar{\bar{w}}_7$	0.213114	0.009492	0.171946	0.171946	0.343892	-0.089610

TABLE VII (Cont'd)

Solution of Difference Equations

8	9	10	11	12	13	14	$\frac{p\lambda^2}{s_1}$	$\sum$
6	-1		-1	-1			-1.500000	1.500000
-0.264139	-0.011765	-0.213114	-0.213114				-0.426228	-1.128360
-0.374407	-0.307107						-0.914357	-1.595871
-0.172124							-0.335642	-0.507766
5.189330	-1.318872	-0.213114	-1.213114	-1			-3.176227	-1.731997
$(-)\bar{w}_8$	0.254151	0.041068	0.233771	0.192703			0.612069	0.333762
	3			-1			-0.750000	1.250000
	-0.335193	-0.054163	-0.308314	-0.254151			-0.807241	-1.759062
	-0.000524	-0.009492	-0.009492				-0.018984	-0.038492
	-0.251904						-0.750000	-1.001904
	2.412379	-0.063655	-0.317806	-1.254151			-2.326225	-1.549458
	$(-)\bar{w}_9$	0.026387	0.131740	0.519881			0.964287	0.642295
		6	-1		-1	-1	-1.500000	1.500000
		-0.001680	-0.008386	-0.033093			-0.061382	-0.104541
		-0.008752	-0.049820	-0.041068			-0.130440	-0.230080
		-0.171946	-0.171946				0.343892	-0.687784
		5.817622	-1.230152	-0.074161	-1	-1	-2.035714	0.477595
		$(-)\bar{w}_{10}$	0.211453	0.012748	0.171892	0.171892	0.349922	-0.082093

TABLE VII (Cont'd)

Solution of Difference Equations

11	12	13	14	15	16	$\frac{p\lambda^2}{s_1}$	$\sum$
6	-1		-1	-1		-1.500000	1.500000
-0.260119	-0.015682	-0.211453	-0.211453			-0.430457	-1.129164
-0.041868	-0.165221					-0.306456	-0.513545
-0.283591	-0.233771					-0.742509	-1.259871
-0.171946						-0.343892	-0.515838
5.242476	-1.414674	-0.211453	-1.211453	-1		-3.323314	-1.918418
$(-)\bar{w}_{11}$	0.269848	0.040335	0.231084	0.190750		0.633921	0.365938
	5			-1	-1	-1.500000	1.500000
	-0.381747	-0.057061	-0.326909	-0.269849		-0.896792	-1.932358
	-0.000945	-0.012748	-0.012748			-0.025951	-0.052392
	-0.652009					-1.209362	-1.861371
	-0.192703					-0.612069	-0.804772
	3.772596	-0.069809	-0.339657	-1.269849	-1	-4.244174	-3.150893
	$(-)\bar{w}_{12}$	0.018504	0.090033	0.336598	0.265069	1.125001	0.835205

TABLE VII (Cont'd)

Solution of Difference Equations

13	14	15	16	$\frac{p\lambda^2}{S_1}$	$\sum$
3	-0.5			-0.750000	1.750000
-0.001292	-0.006285	-0.023498	-0.018504	-0.078535	-0.128114
-0.008529	-0.048863	-0.040335		-0.134044	-0.231771
-0.171892	-0.171892			-0.349922	-0.693706
2.818287	-0.727040	-0.063833	-0.018504	-1.312501	0.696409
$(-)\bar{w}_{13}$	0.257972	0.022650	0.006566	0.465709	-0.247103
	3	-0.5		-0.750000	1.750000
	-0.187556	-0.016467	-0.004774	-0.338589	-0.547386
	-0.030580	-0.114328	-0.090033	-0.382114	-0.617055
	-0.279947	-0.231085		-0.767965	-1.278997
	-0.171892			-0.349922	-0.521814
	2.330025	-0.861880	-0.094807	-2.588590	-1.215252
	$(-)\bar{w}_{14}$	0.369902	0.040689	1.110971	0.521562

TABLE VII (Cont'd)

Solution of Difference Equations

15	16	$\frac{p\lambda^2}{S_1}$	$\sum$
3	-0.5	-0.750000	1.750000
-0.318811	-0.035069	-0.957524	-1.311404
-0.001446	-0.000419	-0.029728	-0.031593
-0.427429	-0.336598	-1.428581	-2.192608
-0.190750		-0.633921	-0.824671
2.061564	-0.872086	-3.799754	-2.610276
(-)$\bar{w}_{15}$	0.423022	1.843141	1.266163
	1.50	-0.375000	1.125000
	-0.368912	-1.607377	-1.976289
	-0.003858	-0.105328	-0.109186
	-0.000121	-0.008617	-0.008738
	-0.265069	-1.125001	-1.390070
	0.862040	-3.221323	-2.359283
	(-)$\bar{w}_{16}$	3.736860	2.736860

TABLE VII (Cont'd)

Solution of Difference Equations

1	2	3	4	5	6	$\frac{p\lambda^4}{s_1 s_2}$	$\sum$
3	-1					-0.463973	1.536027
$(-)\bar{z}_1$	0.333333					0.154658	-0.512009
	5	-1	-1			-1.658843	1.341157
	-0.333333					-0.154658	-0.487991
	4.666667	-1	-1			-1.813501	0.853166
	$(-)\bar{z}_2$	0.214286	0.214286			0.388607	-0.182821
		6	-1	-1	-1	-2.192174	0.807826
		-0.214286	-0.214286			-0.388607	-0.817179
		5.785714	-1.214286	-1	-1	-2.580781	-0.009353
		$(-)\bar{z}_3$	0.209877	0.172840	0.172840	0.446061	0.001618
			3	-1		-1.462041	0.537959
			-0.254851	-0.209877	-0.209877	-0.541646	-1.216251
			-0.214286			-0.388607	-0.602893
			2.530863	-0.209877	-1.209877	-2.392294	-1.281185
			$(-)\bar{z}_4$	0.082927	0.478049	0.945248	0.506224

TABLE VII (Cont'd)

Solution of Difference Equations

5	6	7	8	9	10	11	$\frac{p\lambda^4}{s_1 s_2}$	$\sum$
6	-1	-1	-1				-2.523863	0.476137
-0.017404	-0.100331						-0.198386	-0.316121
-0.172840	-0.172840						-0.446061	-0.791741
5.809756	-1.273171	-1	-1				-3.168310	-0.631725
$(-)\bar{z}_5$	0.219144	0.172124	0.172124				0.545343	0.108735
	5		-1	-1			-3.796233	-0.796233
	-0.279008	-0.219143	-0.219143				-0.694315	-1.411609
	-0.578380						-1.143634	-1.722014
	-0.172840						-0.446061	-0.618901
	3.969772	-0.219143	-1.219143	-1			-6.080243	-4.548757
	$(-)\bar{z}_6$	0.055203	0.307107	0.251904			1.531635	1.145849
		6	-1		-1	-1	-2.636190	0.363810
		-0.012097	-0.067300	-0.055203			-0.335647	-0.470247
		-0.172124	-0.172124				-0.545343	-0.889591
		5.815779	-1.239424	-0.055203	-1	-1	-3.517180	-0.996028
		$(-)\bar{z}_7$	0.213114	0.009492	0.171946	0.171946	0.604765	0.171263

TABLE VII (Cont'd)

Solution of Difference Equations

8	9	10	11	12	13	14	$\frac{p\lambda^4}{S_1 S_2}$	$\sum$
6	-1		-1	-1			-4.268592	-1.268592
-0.264139	-0.011765	-0.213114	-0.213114				-0.749560	-1.451692
-0.374407	-0.307107						-1.867282	-2.548796
-0.172124							-0.545343	-0.717467
5.189330	-1.318872	-0.213114	-1.213114	-1			-7.430777	-5.986547
$(-)\bar{z}_8$	0.254151	0.041068	0.233771	0.192703			1.431934	1.153627
	3			-1			-2.411222	-0.411222
	-0.335193	-0.054163	-0.308314	-0.254151			-1.888538	-2.840359
	-0.000524	-0.009492	-0.009492				-0.033385	-0.052893
	-0.251904						-1.531635	-1.783539
	2.412379	-0.063655	-0.317806	-1.254151			-5.864780	-5.088013
	$(-)\bar{z}_9$	0.026387	0.131740	0.519881			2.431119	2.109127
		6	-1		-1	-1	-2.469393	0.530607
		-0.001680	-0.008386	-0.033093			-0.154753	-0.197912
		-0.008752	-0.049820	-0.041068			-0.305165	-0.404805
		-0.171946	-0.171946				-0.604765	-0.948657
		5.817622	-1.230152	-0.074161	-1	-1	-3.534076	-1.020767
		$(-)\bar{z}_{10}$	0.211453	0.012748	0.171892	0.171892	0.607478	0.175463

TABLE VII (Cont'd)

Solution of Difference Equations

11	12	13	14	15	16	$\frac{p\lambda^4}{S_1 S_2}$	$\sum$
6	-1		-1	-1		-4.305309	-1.305309
-0.260119	-0.015682	-0.211453	-0.211453			-0.747290	-1.445997
-0.041868	-0.165221					-0.772624	-0.979713
-0.283591	-0.233771					-1.737099	-2.254461
-0.171946						-0.604765	-0.776711
5.242476	-1.414674	-0.211453	-1.211453	-1		-8.167087	-6.762191
$(-)\bar{z}_{11}$	0.269848	0.040335	0.231084	0.190750		1.557868	1.289885
	5			-1	-1	-5.277498	-2.277498
	-0.381747	-0.057061	-0.326909	-0.269849		-2.203875	-3.239441
	-0.000945	-0.012748	-0.012748			-0.045051	-0.071492
	-0.652009					-3.048990	-3.700999
	-0.192703					-1.431934	-1.624637
	3.772596	-0.069809	-0.339657	-1.269849	-1	-12.007348	-10.914067
	$(-)\bar{z}_{12}$	0.018504	0.090033	0.336598	0.265069	3.182781	2.892985

TABLE VII (Cont'd)

Solution of Difference Equations

13	14	15	16	$\frac{p\lambda^4}{S_1 S_2}$	$\sum$
3	-0.5			- 0.915260	1.584740
-0.001292	-0.006285	-0.023498	-0.018504	- 0.222187	-0.271766
-0.008529	-0.048863	-0.040335		- 0.329416	-0.427143
-0.171892	-0.171892			- 0.607478	-0.951262
2.818287	-0.727040	-0.063833	-0.018504	- 2.074341	-0.065431
$(-)\bar{z}_{13}$	0.257972	0.022650	0.006566	0.736029	0.023217
	3	-0.5		- 1.897150	0.602850
	-0.187556	-0.016467	-0.004774	- 0.535123	-0.743920
	-0.030580	-0.114328	-0.090033	- 1.081054	-1.315995
	-0.279947	-0.231085		- 1.887284	-2.398316
	-0.171892			- 0.607478	-0.779370
	2.330025	-0.861880	-0.094807	- 6.008089	-4.634751
	$(-)\bar{z}_{14}$	0.369902	0.040689	2.578551	1.989142

TABLE VII (Cont'd)

Solution of Difference Equations

15	16	$\frac{p\lambda^4}{S_1 S_2}$	$\sum$
3	-0.5	- 2.567936	- 0.067936
-0.318811	-0.035069	- 2.222402	- 2.576282
-0.001446	-0.000419	- 0.046938	- 0.048848
-0.427429	-0.336598	- 4.041651	- 4.805678
-0.190750		- 1.557868	- 1.748618
2.061564	-0.872086	-10.436840	- 9.247362
$(-)\bar{z}_{15}$	0.423022	5.062584	4.485606
	1.50	- 1.401323	0.098677
	-0.368912	- 4.415009	- 4.783921
	-0.003858	- 0.244465	- 0.248323
	-0.000121	- 0.013619	- 0.013740
	-0.265069	- 3.182781	- 3.447850
	0.862040	- 9.257197	- 8.395157
	$(-)\bar{z}_{16}$	10.738709	9.738709

These equations may be put into the form given in Table VI. The last two columns in this table are the terms which are on the right side of equations (48) and (49). Hence, it is to be remembered that after each of these two columns (Table VI) there are actually an equal sign and zero thereafter, since a minus sign precedes each of the items in the columns headed $\frac{p\lambda^2}{S_1}$ and $\frac{p\lambda^4}{S_1S_2}$.

The manner of solving the equations in Table VI is indicated in Table VII, without explanation. As it was pointed out previously, the " Doolittle Method " was adopted for simultaneous equations which happended to arise in normal form.[1] The details of the steps are omitted here, but they are given for a brief solution in Appendix B.

The solution resulting from Table VII yields the following values for $\bar{w}$ and $\bar{z}$:

$$\bar{w}_1 = 0.618631 \frac{p\lambda^2}{S_1} \qquad \bar{w}_9 = 3.214962 \frac{p\lambda^2}{S_1}$$

$\bar{w}_2$ = 1.105895	"	$\bar{w}_{10}$ = 1.646262	"
$\bar{w}_3$ = 1.461449	"	$\bar{w}_{11}$ = 2.870206	"
$\bar{w}_4$ = 1.949388	"	$\bar{w}_{12}$ = 3.518332	"
$\bar{w}_5$ = 1.682575	"	$\bar{w}_{13}$ = 1.220346	"
$\bar{w}_6$ = 2.530822	"	$\bar{w}_{14}$ = 2.529533	"
$\bar{w}_7$ = 1.757560	"	$\bar{w}_{15}$ = 3.423915	"
$\bar{w}_8$ = 2.845728	"	$\bar{w}_{16}$ = 3.736860	"

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[1]See paper by Oscar S. Adams, mentioned earlier in this section.

TABLE VIII

UNIFORMLY LOADED SIMPLY SUPPORTED

30-DEGREE SKEW PLATE HAVING EQUAL SIDES

16 COEFFICIENTS

$$\lambda = \frac{c}{4} \; : \; \mu = 0.3$$

Point	Deflection	Moments		
	$\frac{wN}{pc^4}$	$\frac{M_1}{p\lambda^2}$	$\frac{M_2}{p\lambda^2}$	$\frac{M_3}{p\lambda^2}$
1	0.003168	1.321059	-0.057366	-0.057366
2	0.007692	1.710194	0.100776	0.345535
3	0.012139	1.801289	0.402802	0.645765
4	0.016673	2.209804	0.795747	0.795747
5	0.015429	1.648328	0.776816	0.855873
6	0.024477	2.380634	1.268258	1.286222
7	0.016760	1.276270	1.150963	0.999806
8	0.029341	2.277665	1.671594	1.599924
9	0.033975	2.625404	1.821893	1.821893
10	0.015454	0.649457	1.436043	1.126153
11	0.030040	1.925130	1.900214	1.771555
12	0.038688	2.605217	2.146290	2.109223
13	0.010620	-0.425828	1.402770	1.402770
14	0.025658	1.328061	1.802272	1.802272
15	0.037521	2.359530	2.158576	2.158576
16	0.041948	2.707839	2.289523	2.289523

TABLE VIII (Cont'd)

UNIFORMLY LOADED SIMPLY SUPPORTED

30-DEGREE SKEW PLATE HAVING EQUAL SIDES

16 COEFFICIENTS

$\lambda = \frac{c}{4}$; $\mu = 0.3$

Point	Moments $\frac{M_x}{p\lambda^2}$	$\frac{M_y}{p\lambda^2}$	$\frac{M_{xy}}{p\lambda^2}$	$\frac{M_{max.}}{p\lambda^2}$	$\frac{M_{min.}}{p\lambda^2}$	θ Degrees
1	-0.057366	0.861586	-0.795830	1.321057	-0.516837	30
2	0.100776	1.336888	-0.787882	1.720206	-0.282542	26
3	0.402802	1.497082	-0.667139	1.812749	0.087135	25.5
4	0.795747	1.738457	-0.816402	2.209804	0.324400	30
5	0.776816	1.410532	-0.457522	1.650204	0.537145	27.5
6	1.268258	2.021811	-0.631856	2.380698	0.909369	29.5
7	1.150963	1.133735	-0.159615	1.302199	0.982499	-43.5
8	1.671594	2.027852	-0.391292	2.279652	1.419794	33
9	1.821893	2.357588	-0.463905	2.625396	1.554056	30
10	1.436043	0.704098	0.275219	1.527981	0.612163	18
11	1.900214	1.831054	-0.088666	1.960806	1.770462	-34.5
12	2.146290	2.427542	-0.286360	2.605942	1.967889	32
13	1.402770	0.183680	1.055736	2.012292	-0.425842	30
14	1.802272	1.487121	0.273784	1.960338	1.328054	30
15	2.158576	2.292514	-0.116020	2.359507	2.091583	30
16	2.289523	2.568395	-0.241513	2.707833	2.150086	30

TABLE VIII (Cont'd)

UNIFORMLY LOADED SIMPLY SUPPORTED

30-DEGREE SKEW PLATE HAVING EQUAL SIDES

16 COEFFICIENTS

$\lambda = \frac{c}{4}$; $\mu = 0.3$

Point	Deflection	Moments					
	$\frac{wN}{pc^4}$	$\frac{M_x}{p\lambda^2}$	$\frac{M_y}{p\lambda^2}$	$\frac{M_{xy}}{p\lambda^2}$	$\frac{M_{max.}}{p\lambda^2}$	$\frac{M_{min.}}{p\lambda^2}$	θ Degrees
17	0	0	0	-0.655560	0.655560	-0.655560	45
18	0	0	0	-0.936099	0.936099	-0.936099	45
19	0	0	0	-0.920203	0.920203	-0.920203	45
20	0	0	0	-0.680801	0.680801	-0.680801	45
21	0	0	0	-0.275388	0.275388	-0.275388	45
22	0	0	0	0.270284	0.270284	-0.270284	45
23	0	0	0	1.000367	1.000367	-1.000367	45
24	0	0.866348	-0.866348	0.500184	1.000367	-1.000367	15
25	0	0.234074	-0.234074	0.135142	0.270284	-0.270284	15
26	0	-0.238494	0.238494	-0.137694	0.275388	-0.275388	15
27	0	-0.589594	0.589594	-0.340401	0.680801	-0.680801	15
28	0	-0.796923	0.796923	-0.460101	0.920203	-0.920203	15
29	0	-0.810690	0.810690	-0.468049	0.936099	-0.936099	15
30	0	-0.567735	0.567735	-0.327780	0.655560	-0.655560	15

TABLE IX

UNIFORMLY LOADED SIMPLY SUPPORTED

30-DEGREE SKEW PLATE HAVING EQUAL SIDES

25 COEFFICIENTS

$\lambda = \frac{c}{5}$; $\mu = 0.3$

Point	Deflection $\frac{wN}{pc^4}$	Moments $\frac{M_1}{p\lambda^2}$	$\frac{M_2}{p\lambda^2}$	$\frac{M_3}{p\lambda^2}$
1	0.001722	1.700381	-0.221443	-0.221443
2	0.004393	2.277210	-0.191353	0.224161
3	0.007363	2.538187	0.034146	0.585009
4	0.010035	2.985662	0.612237	0.612237
5	0.010107	2.533434	0.584979	0.865899
6	0.015759	3.349697	1.104122	1.246712
7	0.012207	2.302882	0.849406	1.076280
8	0.020637	3.419632	1.635887	1.723842
9	0.023635	3.797361	1.917582	1.917582
10	0.013334	1.866957	1.333830	1.231947
11	0.023959	3.231963	2.139650	2.064339
12	0.029812	3.938352	2.521841	2.494873
13	0.013204	1.210377	1.798577	1.361199
14	0.025178	2.800533	2.539352	2.296099
15	0.033436	3.814297	2.977627	2.867122
16	0.036372	4.160562	3.091562	3.091562
17	0.011499	0.240260	2.155240	1.540837
18	0.023834	2.112634	2.730220	2.448525
19	0.033903	3.447857	3.194083	3.060257
20	0.039503	4.135562	3.430034	3.389648
21	0.007657	-1.376602	2.104669	2.104669
22	0.019456	1.158988	2.519951	2.519951
23	0.030787	2.859359	3.060090	3.060090
24	0.038739	3.891260	3.437674	3.437674
25	0.041583	4.239162	3.570311	3.570311

TABLE IX (Cont'd)

UNIFORMLY LOADED SIMPLY SUPPORTED

30-DEGREE SKEW PLATE HAVING EQUAL SIDES

25 COEFFICIENTS

$\lambda = \frac{c}{5}$; $\mu = 0.3$

Point	Moments $\frac{M_x}{p\lambda^2}$	$\frac{M_y}{p\lambda^2}$	$\frac{M_{xy}}{p\lambda^2}$	$\frac{M_{max.}}{p\lambda^2}$	$\frac{M_{min.}}{p\lambda^2}$	θ Degrees
1	-0.221443	1.059778	-1.109560	1.700379	-0.862044	30
2	-0.191353	1.731358	-1.185322	2.296172	-0.756167	25.5
3	0.034146	2.070763	-1.127662	2.571853	-0.466944	24
4	0.612237	2.194524	-1.370290	2.985658	-0.178897	30
5	0.584979	1.947334	-0.962747	2.445514	0.086798	27
6	1.104122	2.696254	-1.214152	3.352044	0.448332	28.5
7	0.849406	1.969639	-0.708175	2.312430	0.506614	26
8	1.635887	2.883688	-0.979059	3.420739	1.098836	29
9	1.917582	3.170767	-1.085285	3.797351	1.290992	30
10	1.333830	1.621261	-0.366621	1.871329	1.083762	34.5
11	2.139650	2.817641	-0.674125	3.233201	1.724079	31.5
12	2.521841	3.448240	-0.833389	3.938502	2.031577	30.5
13	1.798577	1.114906	0.087077	1.809493	1.103990	7
14	2.539352	2.551327	-0.291234	2.836634	2.254044	44.5
15	2.977627	3.461646	-0.546849	3.817643	2.621629	33.5
16	3.091562	3.804384	-0.617185	4.160676	2.735270	30
17	2.155240	0.468989	0.750884	2.441134	0.183094	21
18	2.730220	2.130674	0.193926	2.787478	2.073417	16.5
19	3.194083	3.273989	-0.223780	3.461354	3.006718	18.5
20	3.430034	3.873473	-0.430651	4.136128	3.167377	31.5
21	2.104669	-0.216173	2.009902	3.265084	-1.376588	30
22	2.519951	1.612574	0.785748	2.973585	1.158941	30
23	3.060090	2.926268	0.115892	3.126999	2.859358	30
24	3.437674	3.740012	-0.261877	3.891220	3.286466	30
25	3.570311	4.016252	-0.386159	4.239190	3.347372	30

TABLE IX (Cont'd

UNIFORMLY LOADED SIMPLY SUPPORTED

30-DEGREE SKEW PLATE HAVING EQUAL SIDES

25 COEFFICIENTS

$\lambda = \frac{c}{5}$; $\mu = 0.3$

Point	Deflection	Moments					
	$\frac{wN}{pc^4}$	$\frac{M_x}{p\lambda^2}$	$\frac{M_y}{p\lambda^2}$	$\frac{M_{xy}}{p\lambda^2}$	$\frac{M_{max.}}{p\lambda^2}$	$\frac{M_{min.}}{p\lambda^2}$	θ Degrees
26	0	0	0	-0.870016	0.870016	-0.870016	45
27	0	0	0	-1.349105	1.349105	-1.349105	45
28	0	0	0	-1.500628	1.500628	-1.500628	45
29	0	0	0	-1.386012	1.386012	-1.386012	45
30	0	0	0	-1.060943	1.060943	-1.060943	45
31	0	0	0	-0.569354	0.569354	-0.569354	45
32	0	0	0	0.065731	0.065731	-0.065731	45
33	0	0	0	0.861151	0.861151	-0.861151	45
34	0	0	0	1.941075	1.941075	-1.941075	45

TABLE IX (Cont'd)

UNIFORMLY LOADED SIMPLY SUPPORTED

30-DEGREE SKEW PLATE HAVING EQUAL SIDES

25 COEFFICIENTS

$\lambda = \frac{c}{5}$; $\mu = 0.3$

Point	Deflection	Moments					
	$\frac{wN}{pc^4}$	$\frac{M_x}{p\lambda^2}$	$\frac{M_y}{p\lambda^2}$	$\frac{M_{xy}}{p\lambda^2}$	$\frac{M_{max.}}{p\lambda^2}$	$\frac{M_{min.}}{p\lambda^2}$	θ Degrees
35	0	1.681029	-1.681029	0.970538	1.941075	-1.941075	15
36	0	0.745783	-0.745783	0.430576	0.861151	-0.861151	15
37	0	0.056925	-0.056925	0.032865	0.065731	-0.065731	15
38	0	-0.493078	0.493078	-0.284677	0.569354	-0.569354	15
39	0	-0.918809	0.918809	-0.530472	1.060943	-1.060943	15
40	0	-1.200328	1.200328	-0.693006	1.386012	-1.386012	15
41	0	-1.299589	1.299589	-0.750314	1.500628	-1.500628	15
42	0	-1.168364	1.168364	-0.674552	1.349105	-1.349105	15
43	0	-0.753460	0.753460	-0.435008	0.870016	-0.870016	15

TABLE X

UNIFORMLY LOADED SIMPLY SUPPORTED

30-DEGREE SKEW PLATE HAVING EQUAL SIDES

4 COEFFICIENTS

$\lambda = \frac{c}{2}$; $\mu = 0.3$

Point	Deflection	Moments		
	$\frac{wN}{pc^4}$	$\frac{M_1}{p\lambda^2}$	$\frac{M_2}{p\lambda^2}$	$\frac{M_3}{p\lambda^2}$
1	0.018215	0.554529	0.198897	0.198897
2	0.031753	0.570167	0.405769	0.418545
3	0.028216	0.318395	0.470017	0.470017
4	0.045291	0.665023	0.585802	0.585802

Point	Moments					
	$\frac{M_x}{p\lambda^2}$	$\frac{M_y}{p\lambda^2}$	$\frac{M_{xy}}{p\lambda^2}$	$\frac{M_{max.}}{p\lambda^2}$	$\frac{M_{min.}}{p\lambda^2}$	θ Degrees
1	0.198897	0.435987	-0.205323	0.554530	0.080353	30
2	0.405769	0.583883	-0.087539	0.570423	0.359229	28
3	0.470017	0.368937	0.087539	0.520558	0.318397	30
4	0.585802	0.638617	-0.045738	0.665022	0.559395	30

TABLE X (Cont'd)

UNIFORMLY LOADED SIMPLY SUPPORTED

30-DEGREE SKEW PLATE HAVING EQUAL SIDES

4 COEFFICIENTS

$\lambda = \frac{c}{2}$; $\mu = 0.3$

Point	Deflection	Moments					
	$\frac{wN}{pc^4}$	$\frac{M_x}{p\lambda^2}$	$\frac{M_y}{p\lambda^2}$	$\frac{M_{xy}}{p\lambda^2}$	$\frac{M_{max.}}{p\lambda^2}$	$\frac{M_{min.}}{p\lambda^2}$	θ Degrees
5	0	0	0	-0.235568	0.235568	-0.235568	45
6	0	0	0	-0.175079	0.175079	-0.175079	45
7	0	0	0	0.045738	0.045738	-0.045738	45
8	0	0.039610	-0.039610	0.022869	0.045738	-0.045738	15
9	0	-0.151625	0.151625	-0.087539	0.175079	-0.175079	15
10	0	-0.204009	0.204009	-0.117784	0.235568	-0.235568	15

TABLE XI

COMPARISON OF MOMENTS AND DEFLECTIONS

AT THE CENTER OF THE UNIFORMLY LOADED SIMPLY SUPPORTED

30-DEGREE SKEW PLATE FOR DIFFERENT NETWORKS

Moments	25 Coefficients $\lambda = \frac{c}{5}$	16 Coefficients $\lambda = \frac{c}{4}$	4 Coefficients $\lambda = \frac{c}{2}$
M_1	4.239162 $p\lambda^2$ 0.169566 pc^2	2.707839 $p\lambda^2$ 0.169240 pc^2	0.665023 $p\lambda^2$ 0.166256 pc^2
M_2	3.570311 $p\lambda^2$ 0.142812 pc^2	2.289523 $p\lambda^2$ 0.143095 pc^2	0.585802 $p\lambda^2$ 0.146451 pc^2
M_3	3.570311 $p\lambda^2$ 0.142812 pc^2	2.289523 $p\lambda^2$ 0.143095 pc^2	0.585802 $p\lambda^2$ 0.146451 pc^2
M_x	3.570311 $p\lambda^2$ 0.142812 pc^2	2.289523 $p\lambda^2$ 0.143095 pc^2	0.585802 $p\lambda^2$ 0.146451 pc^2
M_y	4.016252 $p\lambda^2$ 0.160650 pc^2	2.568395 $p\lambda^2$ 0.160524 pc^2	0.638617 $p\lambda^2$ 0.159654 pc^2
M_{xy}	-0.386159 $p\lambda^2$ -0.015446 pc^2	-0.241513 $p\lambda^2$ -0.015095 pc^2	-0.045738 $p\lambda^2$ -0.011434 pc^2
Deflections			
w	0.041583 $\frac{pc^4}{N}$	0.041948 $\frac{pc^4}{N}$	0.045291 $\frac{pc^4}{N}$

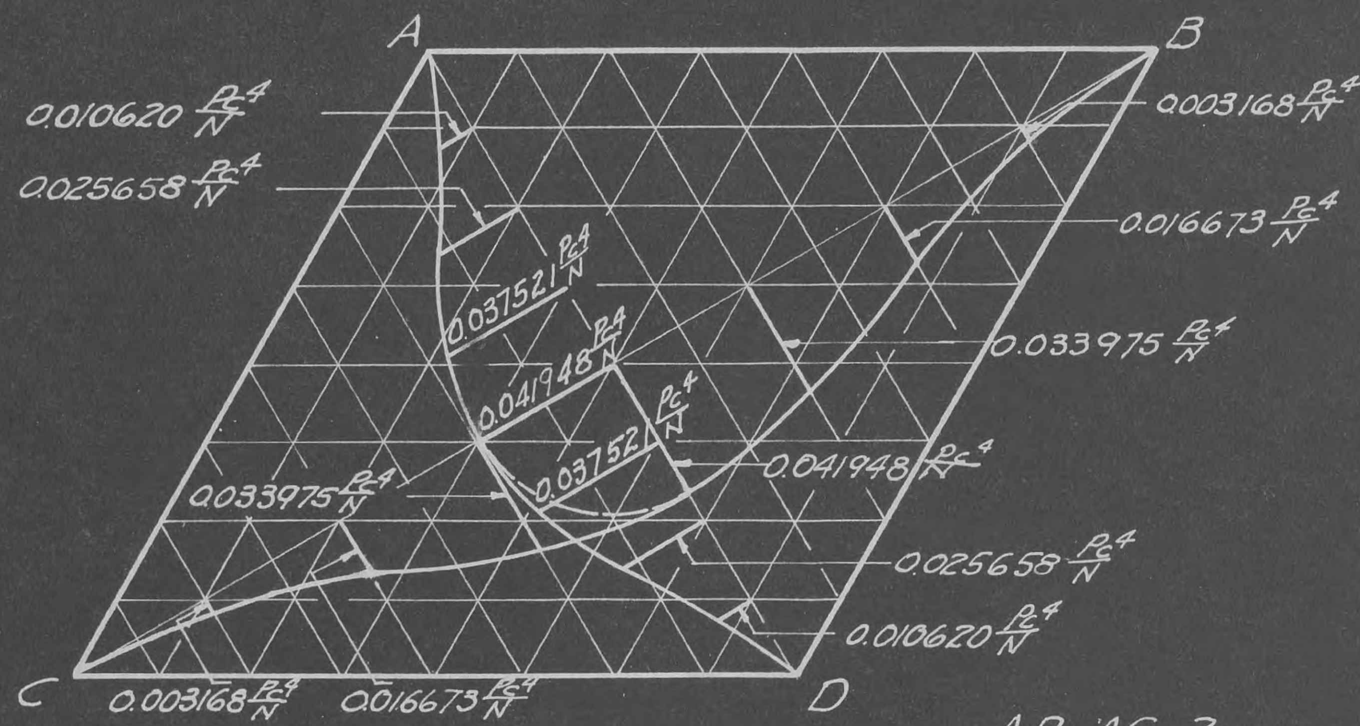
$\bar{z}_1 = 0.811050 \frac{p\lambda^4}{S_1 S_2}$	$\bar{z}_9 = 8.697569 \frac{p\lambda^4}{S_1 S_2}$
$\bar{z}_2 = 1.969178$ "	$\bar{z}_{10} = 3.956232$ "
$\bar{z}_3 = 3.107640$ "	$\bar{z}_{11} = 7.690210$ "
$\bar{z}_4 = 4.268346$ "	$\bar{z}_{12} = 9.904091$ "
$\bar{z}_5 = 3.949917$ "	$\bar{z}_{13} = 2.718592$ "
$\bar{z}_6 = 6.266184$ "	$\bar{z}_{14} = 6.568515$ "
$\bar{z}_7 = 4.290623$ "	$\bar{z}_{15} = 9.605294$ "
$\bar{z}_8 = 7.511201$ "	$\bar{z}_{16} = 10.738709$ "

Once all the $\bar{w}$- and $\bar{z}$- coefficients are found, it is an easy matter to substitute in the equations of Section 9 in order to obtain the deflections and moments. These are indicated for the 16-coefficient solution in Table VIII. It is necessary to refer to Figs. 4(d), 6 and 10(a) in the preparation and interpretation of these results.

Similar analyses were made for the uniformly loaded, simply supported 30-degree skew plate or slab, having equal sides, by utilizing 4-coefficient and 25-coefficient difference-equation networks (see Figs. 11(a) and 10(b)).

In Tables IX and X there are summarized the deflections and moments obtained for the 25- and 4- coefficient solutions. In the preparation and interpretation of these results it is necessary to refer to Figs. 4(d), 6, 10(b) and 11(a).

In Table XI there is indicated a comparison of the moments and deflections at the center of the equal-sided, uniformly

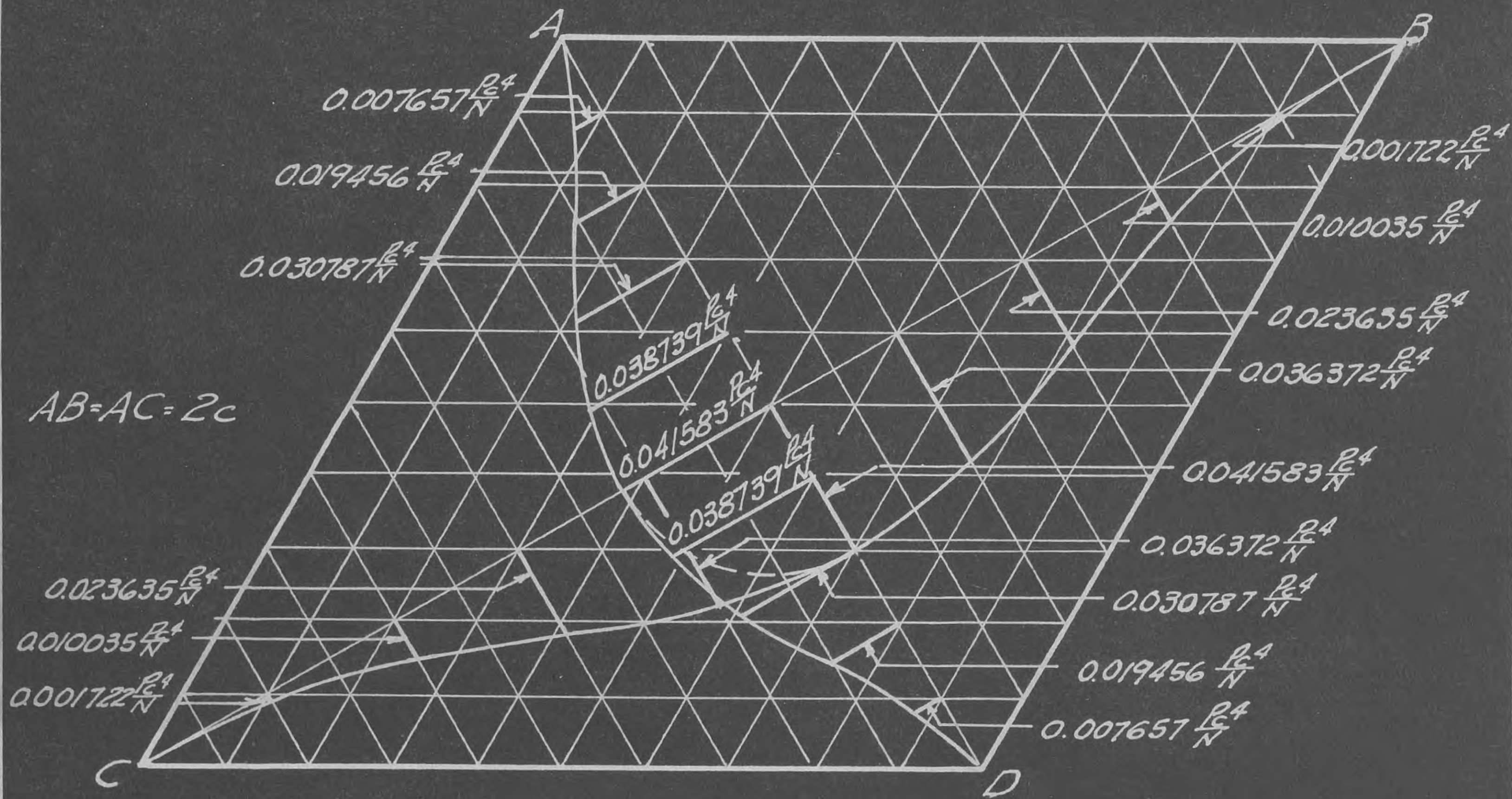


AB = AC = 2c

DEFLECTIONS ALONG DIAGONALS OF UNIFORMLY LOADED SIMPLY SUPPORTED 30° SKEW PLATE – 16 COEFFICIENTS

Scale :- 0.2" = 0.01 $\frac{Pc^4}{N}$

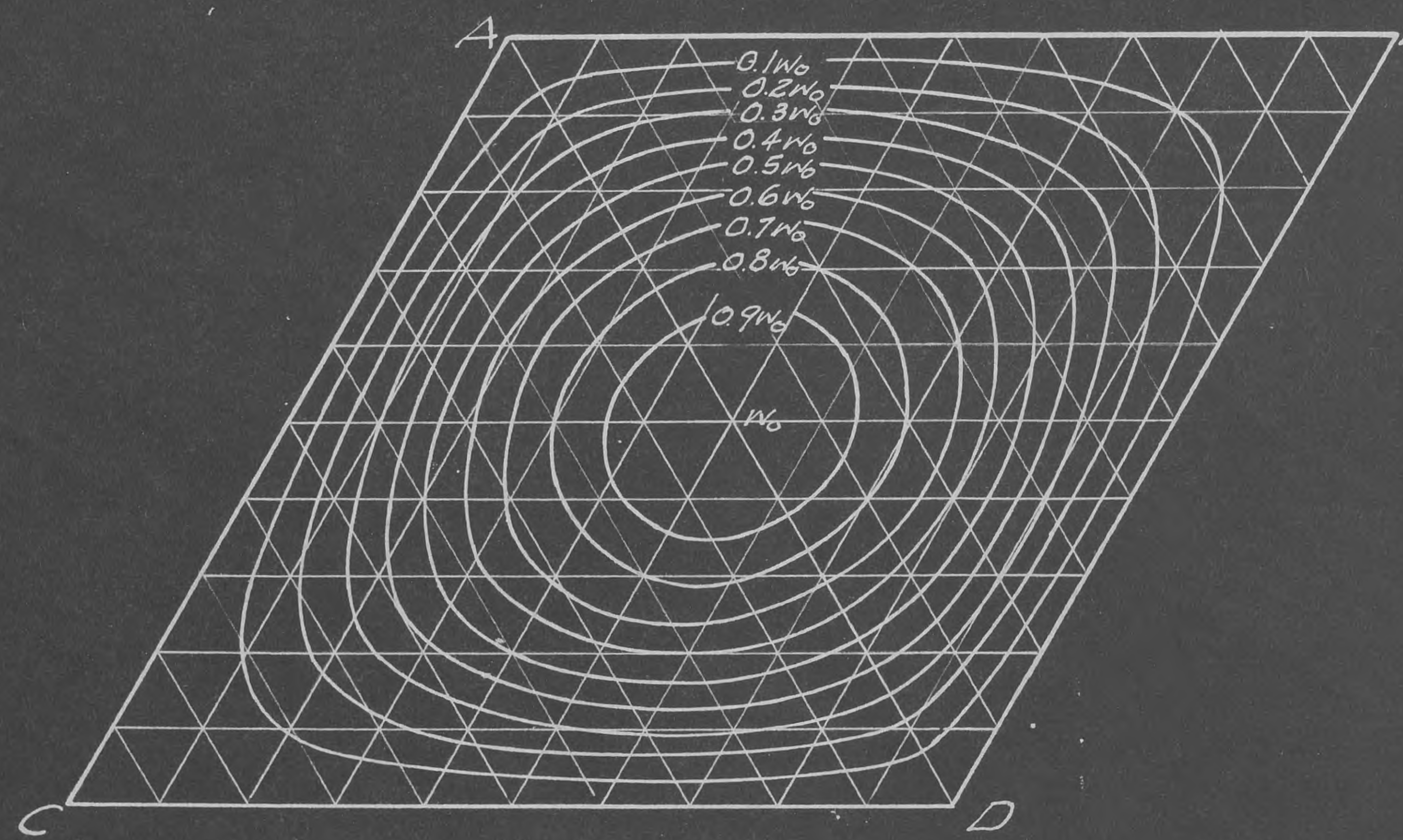
Fig. 13



DEFLECTIONS ALONG DIAGONALS OF UNIFORMLY LOADED SIMPLY SUPPORTED 30° SKEW PLATE – 25 COEFFICIENTS

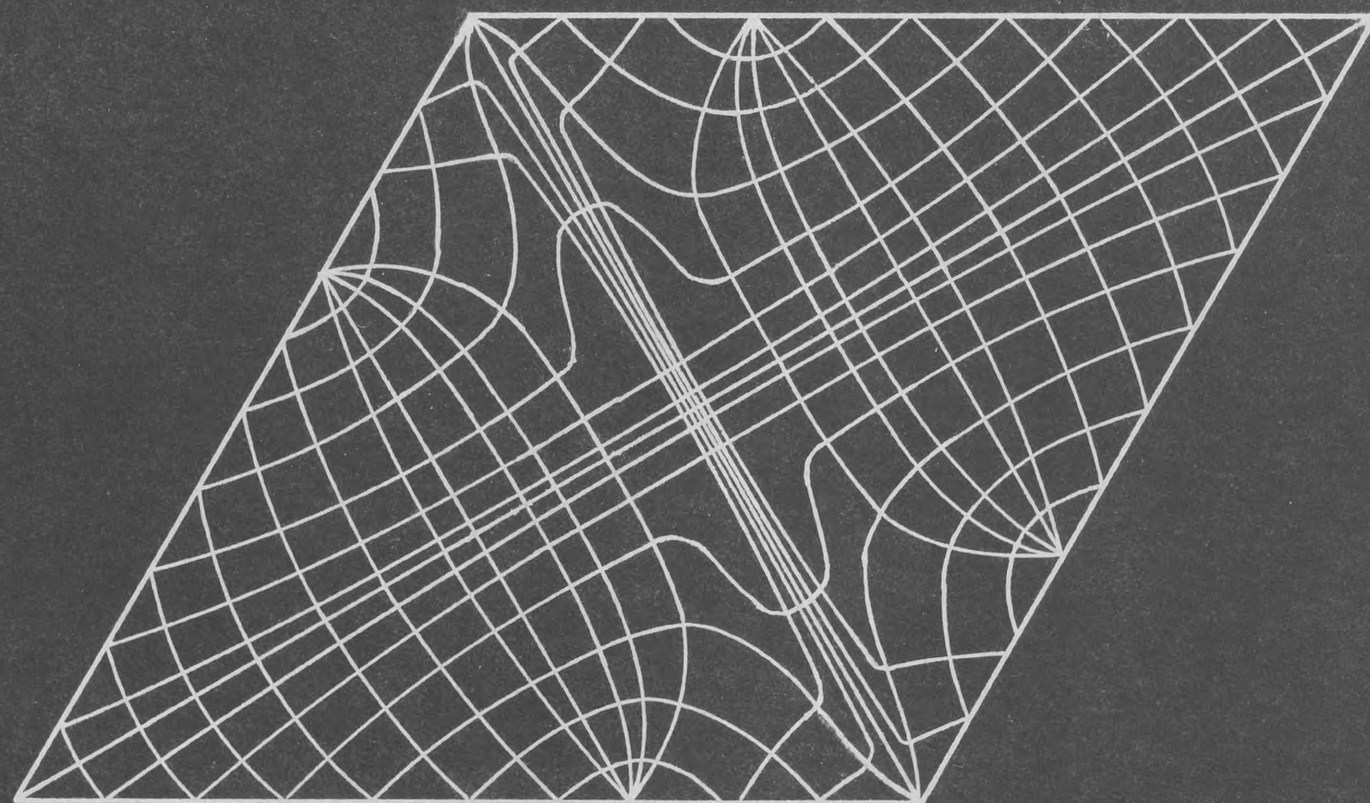
Scale: 0.25" = 0.01 $\frac{pc^4}{N}$

Fig. 14



CONTOURS FOR UNIFORMLY LOADED SIMPLY SUPPORTED 30° SKEW PLATE – 25 COEFFICIENTS.

Fig. 15



TRAJECTORIES OF PRINCIPAL MOMENTS FOR UNIFORMLY LOADED SIMPLY SUPPORTED 30° SKEW PLATE

Fig. 16

TABLE XII

COMPARISON OF MOMENTS AND DEFLECTIONS

AT SIMILAR POINTS OF THE UNIFORMLY LOADED SIMPLY SUPPORTED

30-DEGREE SKEW PLATE FOR TWO NETWORKS

Moments	16 Coefficients $\lambda = \frac{c}{4}$ Point 4	4 Coefficients $\lambda = \frac{c}{2}$ Point 1	16 Coefficients $\lambda = \frac{c}{4}$ Point 8	4 Coefficients $\lambda = \frac{c}{2}$ Point 2
M_1	$2.209804\ p\lambda^2$ $0.138113\ pc^2$	$0.554529\ p\lambda^2$ $0.138632\ pc^2$	$2.277665\ p\lambda^2$ $0.142354\ pc^2$	$0.570167\ p\lambda^2$ $0.142542\ pc^2$
M_2	$0.795747\ p\lambda^2$ $0.049734\ pc^2$	$0.198897\ p\lambda^2$ $0.049724\ pc^2$	$1.671594\ p\lambda^2$ $0.104474\ pc^2$	$0.405769\ p\lambda^2$ $0.101412\ pc^2$
M_3	$0.795747\ p\lambda^2$ $0.049734\ pc^2$	$0.198897\ p\lambda^2$ $0.049724\ pc^2$	$1.599924\ p\lambda^2$ $0.099995\ pc^2$	$0.418545\ p\lambda^2$ $0.104636\ pc^2$
M_x	$0.795747\ p\lambda^2$ $0.049734\ pc^2$	$0.198897\ p\lambda^2$ $0.049724\ pc^2$	$1.671594\ p\lambda^2$ $0.104474\ pc^2$	$0.405769\ p\lambda^2$ $0.101412\ pc^2$
M_y	$1.738457\ p\lambda^2$ $0.108653\ pc^2$	$0.435987\ p\lambda^2$ $0.108997\ pc^2$	$2.027852\ p\lambda^2$ $0.126741\ pc^2$	$0.523883\ p\lambda^2$ $0.130971\ pc^2$
M_{xy}	$-0.816402\ p\lambda^2$ $-0.051025\ pc^2$	$-0.205323\ p\lambda^2$ $-0.051441\ pc^2$	$-0.391292\ p\lambda^2$ $-0.024456\ pc^2$	$-0.087539\ p\lambda^2$ $-0.021885\ pc^2$
Deflections				
w	$0.016673\ \frac{pc^4}{N}$	$0.018215\ \frac{pc^4}{N}$	$0.029341\ \frac{pc^4}{N}$	$0.031753\ \frac{pc^4}{N}$

TABLE XII (Cont'd)

COMPARISON OF MOMENTS AND DEFLECTIONS

AT SIMILAR POINTS OF THE UNIFORMLY LOADED SIMPLY SUPPORTED

30-DEGREE PLATE FOR TWO NETWORKS

Moments	16 Coefficients $\lambda = \frac{c}{4}$ Point 14	4 Coefficients $\lambda = \frac{c}{4}$ Point 3
M_1	1.328061 $p\lambda^2$ 0.083004 pc^2	0.318395 $p\lambda^2$ 0.079599 pc^2
M_2	1.802272 $p\lambda^2$ 0.112642 pc^2	0.470017 $p\lambda^2$ 0.117504 pc^2
M_3	1.802272 $p\lambda^2$ 0.112642 pc^2	0.470017 $p\lambda^2$ 0.117504 pc^2
M_x	1.802272 $p\lambda^2$ 0.112642 pc^2	0.470017 $p\lambda^2$ 0.117504 pc^2
M_y	1.486121 $p\lambda^2$ 0.092882 pc^2	0.368937 $p\lambda^2$ 0.092234 pc^2
M_{xy}	0.273784 $p\lambda^2$ 0.017112 pc^2	0.087539 $p\lambda^2$ 0.021885 pc^2
Deflections		
w	0.025658 $\frac{pc^4}{N}$	0.028216 $\frac{pc^4}{N}$

loaded, simply supported 30-degree skew plate for three different networks. It is seen that the moments M_1, M_2, M_3, M_x, and M_y are in fairly good agreement even in the 4- coefficient case.

The twisting moments, M_{xy}, are in fair agreement only for the 16- and 25- coefficient solutions.

The deflections at the center of the plate agree fairly well again only for the 16- and 25- coefficient cases.

In Table XII may be found a comparison of the moments and deflections at corresponding points of the uniformly loaded, simply supported 30-degree skew plate for the 16- and 4- coefficient cases. Again, the bending moments compare favorably, and the larger differences occur for the deflections and twisting moments.

Figs. 13 and 14 give the deflections along the diagonals of the uniformly loaded simply supported 30-degree skew plate.

Fig. 15 represents the contours for this case.

The trajectories of principal moments for the same plate, similarly loaded, are given in Fig. 16.

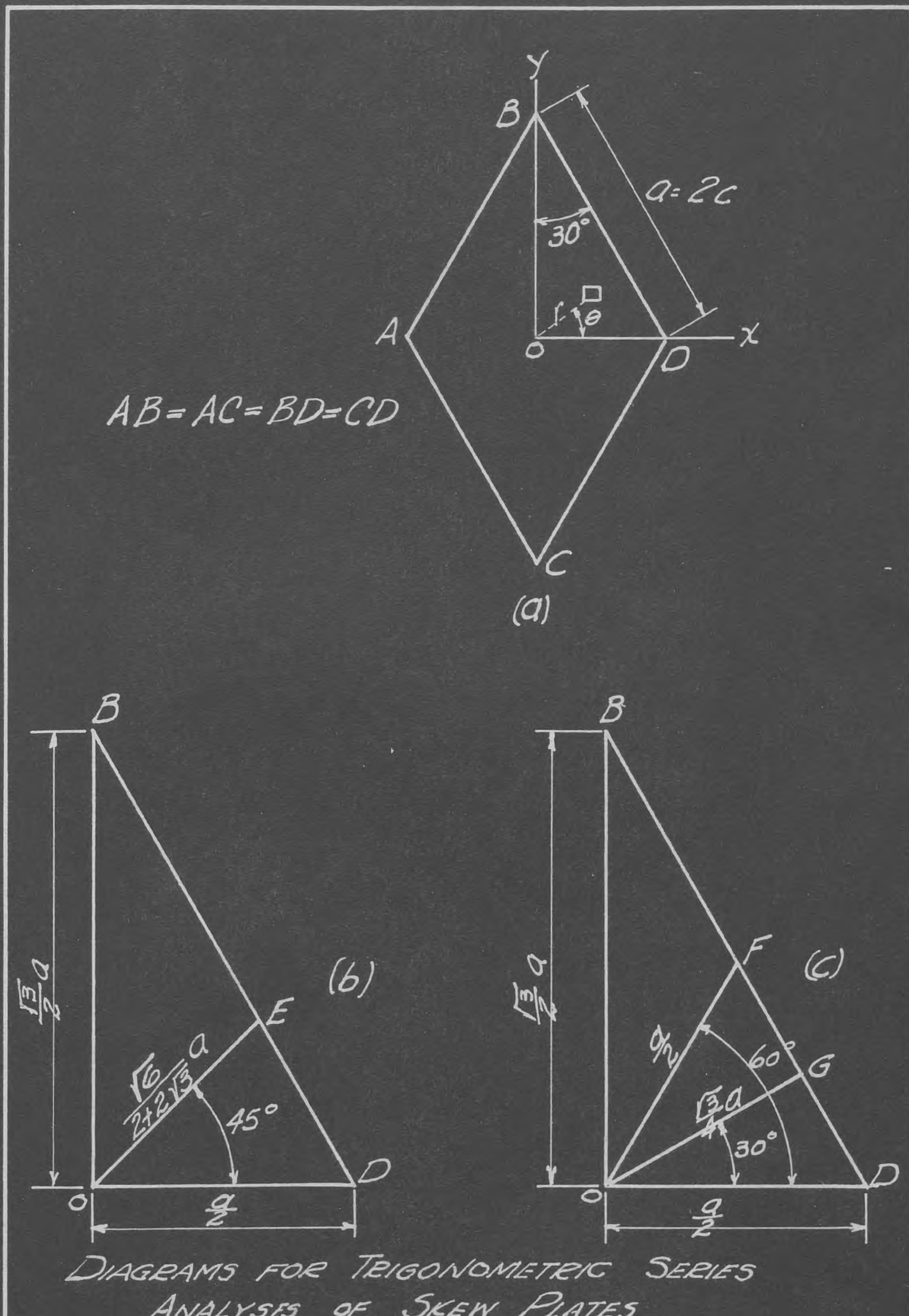

Y
B
a=2c
30°
A
o
D
x
r
θ
AB=AC=BD=CD
C
(a)
B
E
(b)
45°
o
D
a/2
B
F
(c)
60°
G
30°
o
D
a/2
DIAGRAMS FOR TRIGONOMETRIC SERIES
ANALYSES OF SKEW PLATES

Fig. 17

14. Solutions in Trigonometric Series. - Another approximate method for determining deflections and moments in a plate is one in which the deflection function is set up in the form of a trigonometric series, satisfying the plate equation. This series contains a set of arbitrary constants which can be solved for in such a manner as to satisfy the boundary conditions at an arbitrary number of points. The degree of approximation, naturally, depends upon the number of points selected on the boundary, but it has been found that in many cases a relatively small number of points is sufficient to give fairly reliable results. There is reason to believe that the use of approximate methods is justified, since the assumptions made in theories are very seldom entirely true in practice.

The approximate determination of the stress function for the prismatic bar in torsion and the deflection function in a clamped square plate has been indicated by J. Barta.[1] These problems had been worked out previously by more exact methods, and the agreement found by Barta was fairly good.

The trigonometric series approach may be used quite readily in the solution of both simply supported and clamped skew plates. Both of these cases are treated for a 30-degree skew in this dissertation. It is to be noted (see Fig. 17) that a new set of Cartesian-coordinate axes is chosen in order to take advantage of symmetry.

In Fig. 17(c) it is seen that the number of points chosen along the entire boundary is 12. It is sufficient, therefore, to

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[1]Uber die naherungsweise Losung uniger zweidimensionaler Elastizitatsaufgaben, J. Barta, ZAMM, Vol. 17, No. 3, 1937, pp. 184-185.

satisfy the boundary conditions at points B, D, F, and G.

Because of symmetry, the assumed deflection function is taken to be

$$w = \frac{pr^4}{64N} + a_0 + r^2 a_2 \cos 2\theta + r^2 c_2$$
$$+ r^4 a_4 \cos 4\theta + r^4 c_4 \cos 2\theta + r^6 a_6 \cos 6\theta \qquad (50)$$
$$+ r^6 c_6 \cos 4\theta + r^8 c_8 \cos 6\theta \; .$$

The boundary conditions for a simply supported edge (length of side = a) are

$$w = 0 \qquad (51)$$

and

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 \quad , \qquad (52)$$

which in polar form is given by

$$\nabla^2 w = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} = 0 \quad . \qquad (52a)$$

From equations (50) and (52a) it is clear that at the boundary points the following relations must be valid:

$$w = \frac{pr^4}{64N} + a_0 + r^2 a_2 \cos 2\theta + r^2 c_2$$
$$+ r^4 a_4 \cos 4\theta + r^4 c_4 \cos 2\theta + r^6 a_6 \cos 6\theta \qquad (53)$$
$$+ r^6 c_6 \cos 4\theta + r^8 c_8 \cos 6\theta = 0$$

$$\nabla^2 w = \frac{pr^2}{4N} + 4c_2 + 12r^2 c_4 \cos 2\theta$$
$$+ 20r^4 c_6 \cos 4\theta + 28r^6 c_8 \cos 6\theta = 0 \quad . \qquad (54)$$

It may be easily shown that the assumed deflection function (Equation 50) satisfies the plate equation

$$\nabla^2\nabla^2 w = \frac{p}{N} \quad . \tag{3a}$$

By substituting the values of r and θ for points B, D, F, and G in equations (53) and (54), and allowing a and its powers to equal unity, it is possible to obtain the following eight simultaneous equations:

(1) $$a_0 + \frac{1}{4}a_2 + \frac{1}{4}c_2 + \frac{1}{16}a_4 + \frac{1}{16}c_4 + \frac{1}{64}a_6 + \frac{1}{64}c_6 + \frac{1}{256}c_8 + \frac{p}{1024N} = 0$$

(2) $$4c_2 + 3c_4 + \frac{5}{4}c_6 + \frac{7}{16}c_8 + \frac{p}{16N} = 0$$

(3) $$a_0 - \frac{3}{4}a_2 + \frac{3}{4}c_2 + \frac{9}{16}a_4 - \frac{9}{16}c_4 - \frac{27}{64}a_6 + \frac{27}{64}c_6 - \frac{81}{256}c_8 + \frac{9p}{1024N} = 0$$

(4) $$4c_2 - 9c_4 + \frac{45}{4}c_6 - \frac{189}{16}c_8 + \frac{3p}{16N} = 0$$

(5) $$a_0 + \frac{3}{32}a_2 + \frac{3}{16}c_2 - \frac{9}{512}a_4 + \frac{9}{512}c_4 - \frac{27}{4096}a_6 - \frac{27}{8192}c_6 - \frac{81}{65536}c_8 + \frac{9p}{16384N} = 0 \tag{55}$$

(6) $$4c_2 + \frac{9}{8}c_4 - \frac{90}{256}c_6 - \frac{189}{1024}c_8 + \frac{3p}{64N} = 0$$

(7) $$a_0 - \frac{1}{8}a_2 + \frac{1}{4}c_2 - \frac{1}{32}a_4 - \frac{1}{32}c_4 + \frac{1}{64}a_6 - \frac{1}{128}c_6 + \frac{1}{256}c_8 + \frac{p}{1024N} = 0$$

(8) $$4c_2 - \frac{3}{2}c_4 - \frac{5}{8}c_6 + \frac{7}{16}c_8 + \frac{p}{16N} = 0$$

The solution of these equations yields the values of the arbitrary constants:

$a_0 = 0.002\ 705\ 9329\ \frac{p}{N}$ $\qquad$ $c_4 = 0.005\ 787\ 3004\ \frac{p}{N}$

$a_2 = -0.003\ 867\ 8560\ \frac{p}{N}$ $\qquad$ $a_6 = 0.004\ 115\ 0544\ \frac{p}{N}$

$c_2 = -0.014\ 880\ 8677\ \frac{p}{N}$ $\qquad$ $c_6 = -0.013\ 889\ 5211\ \frac{p}{N}$

$a_4 = 0.013\ 156\ 5067\ \frac{p}{N}$ $\qquad$ $c_8 = -0.006\ 803\ 4953\ \frac{p}{N}$

These values, multiplied by the corresponding powers of a (length of side), may be inserted in equation (50) to give the approximate solution for the deflection function.

From the resulting expression there may be obtained the maximum value of deflection by letting $r = 0$:

$$w_{max} = 0.002\ 705\ 9329\ \frac{pa^4}{N}$$

or

$$w_{max} = 0.043\ 294\ 9264\ \frac{pc^4}{N}$$

where $a = 2c$.

The corresponding value obtained by the method of finite differences is

$$w_{max} = 0.041\ 583\ \frac{pc^4}{N}$$

which indicates a difference of about 4 per cent, based upon the difference-equation value.

From equations (13) and (50), there may be deduced the expressions for the radial and tangential moments:

$$M_r = - N\left[\frac{3pr^2}{16N} + 2a_2 \cos 2\theta + 2c_2 + 12r^2a_4 \cos 4\theta \right.$$

$$+ 12r^2c_4 \cos 2\theta + 30r^4a_6 \cos 6\theta + 30r^4c_6 \cos 4\theta + 56r^6c_8 \cos 6\theta$$

$$+ \mu\left(\frac{pr^2}{16N} - 2a_2 \cos 2\theta + 2c_2 - 12r^2a_4 \cos 4\theta\right.$$

$$\left.\left. - 30r^4a_6 \cos 6\theta - 10r^4c_6 \cos 4\theta - 28r^6c_8 \cos 6\theta\right)\right]$$

(56)

$$M_t = - N\left[\mu\left(\frac{3pr^2}{16N} + 2a_2 \cos 2\theta + 2c_2 + 12r^2a_4 \cos 4\theta + 12r^2c_4 \cos 2\theta\right.\right.$$

$$\left. + 30r^4a_6 \cos 6\theta + 30r^4c_6 \cos 4\theta + 56r^6c_8 \cos 6\theta\right)$$

$$+ \frac{pr^2}{16N} - 2a_2 \cos 2\theta + 2c_2 - 12r^2a_4 \cos 4\theta$$

$$\left. - 30r^4a_6 \cos 6\theta - 10r^4c_6 \cos 4\theta - 28r^6c_8 \cos 6\theta\right]$$

For $r = 0$ and $\theta = 0$, at the center of the plate, these expressions reduce to

$$M_r = - 2N\left[a_2 + c_2 + \mu\,(c_2 - a_2)\right]$$

$$M \;\; = - 2N\left[\mu\,(a_2 + c_2) - a_2 + c_2\right] \qquad (57)$$

At the center of the plate these moments are found to be

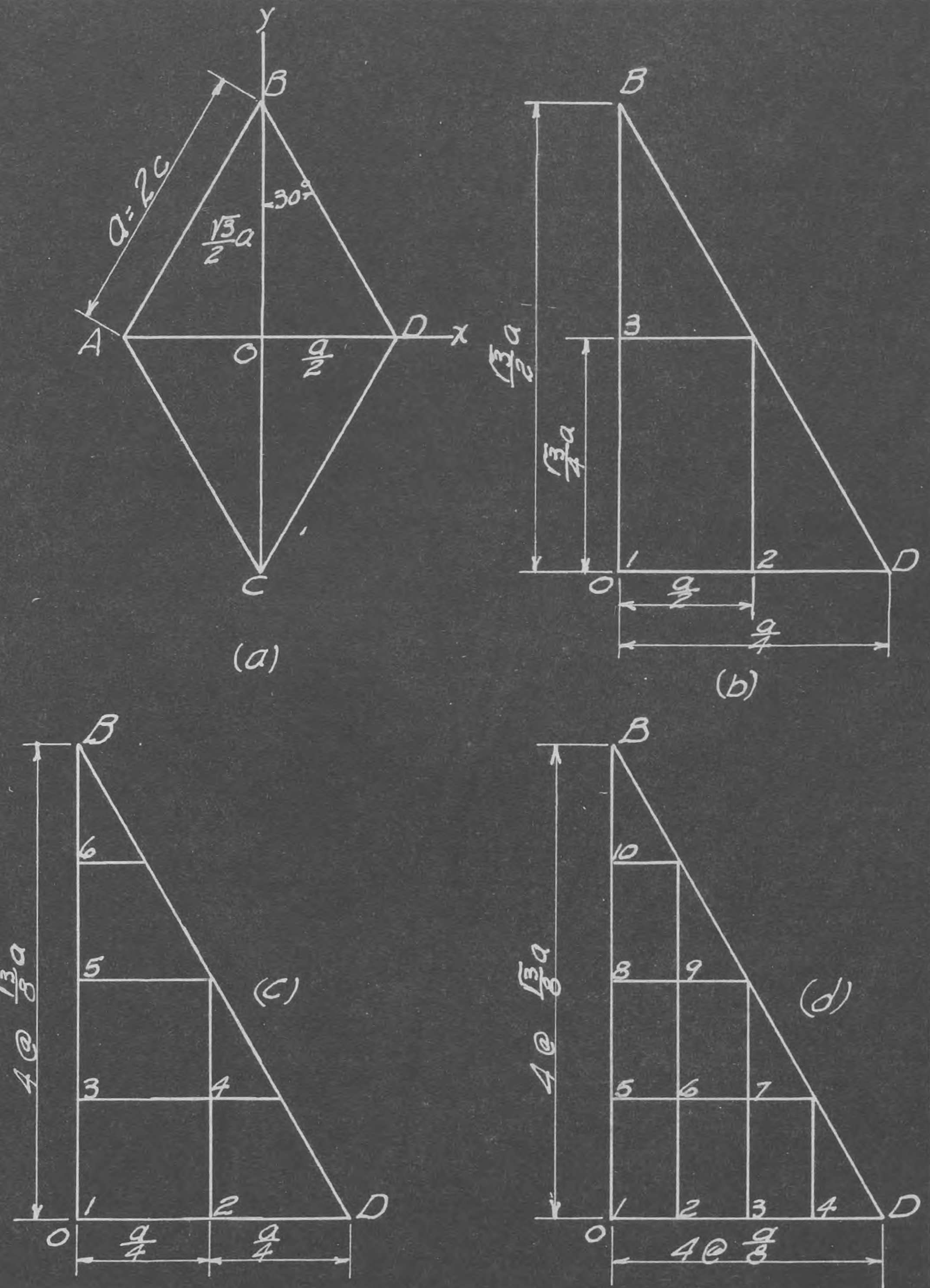
$$M_r = 0.044\ 105\ 2544\ pa^2$$

$$= 0.176\ 421\ 0176\ pc^2$$

$$M_t = 0.033\ 275\ 2576\ pa^2$$

$$= 0.133\ 101\ 0304\ pc^2 \quad ,$$

which compare fairly well with the corresponding difference-equation moments



POWER-SERIES NETWORKS FOR SKEW PLATES

Fig. 18

$$M_r = 0.169\ 566\ pc^2$$

$$M_t = 0.133\ 894\ pc^2\ .$$

The percentage differences based upon the difference-equation values, for these radial and tangential moments are 4 and 0.6, respectively. The agreement for the deflections and moments in this case is fairly good.

15. Power-Series Methods. - Several power-series approaches are to be indicated for the approximate solution of plate problems. In the case of the uniformly loaded, simply supported 30-degree skew plate having equal sides, it is desirable to set up a set of Cartesian-coordinate axes as shown in Fig. 18(a). This enables one to take advantage of the symmetry prevailing about both axes.

The deflection function may be written in the form

$$w = \frac{p}{N}\left(y + \sqrt{3}\,x + \frac{\sqrt{3}}{2}a\right)\left(y - \sqrt{3}\,x - \frac{\sqrt{3}}{2}a\right)\left(y + \sqrt{3}\,x - \frac{\sqrt{3}}{2}a\right)\left(y - \sqrt{3}\,x + \frac{\sqrt{3}}{2}a\right)\sum_{m=0,2..}^{\infty}\ \sum_{n=0,2..}^{\infty} c_{mn}\left(\frac{x}{a}\right)^m\left(\frac{y}{a}\right)^n \qquad (58)$$

Since this equation involves the product of expressions for the sides, it is apparent that one of the boundary conditions

$$w = 0 \qquad (59)$$

is already satisfied.

Other conditions to be imposed on equation (58) are that

$$\nabla^4 w = \frac{p}{N} \qquad (3a)$$

throughout the entire plate, and that

$$\nabla^2 w = 0 \tag{60}$$

along the boundary.

It is to be expected that an approximate solution of the deflection function should result, if the equations (3a) and (60) are satisfied only at certain points within the plate and certain other points along the boundary, respectively.

From equations (58) and (60), it is feasible to set up one of the condition equations in the form

$$\begin{aligned}\frac{N}{p}\nabla^2 w = {} & a^2\left(-12c_{0,0} + \frac{9}{8}c_{2,0} + \frac{9}{8}c_{0,2}\right) \\ & + x^2\left(96c_{0,0} - 57c_{2,0} - 9c_{0,2}\right) \\ & + y^2\left(-3c_{2,0} - 27c_{0,2}\right) \\ & + \frac{x^4}{a^2}\left(258c_{2,0} + 18c_{0,2}\right) \\ & + \frac{y^4}{a^2}\left(2c_{2,0} + 18c_{0,2}\right) \\ & + \frac{x^2y^2}{a^2}\left(-60c_{2,0} + 36c_{0,2}\right) = 0,\end{aligned} \tag{60a}$$

if only three terms are taken in the power series. This condition is to be satisfied for points B, D, and E along the boundary (see Fig. 18(c)).

Substitution of the values of x and y for these three points gives the equations

$$(1)\ -12c_{0,0} - 9c_{0,2} = 0 \ . \ . \ . \ . \ . \ w' = 2$$

$$(2)\ -6c_{0,0} - 2.625c_{2,0} - 3.375c_{0,2} = 0 \ . \ . \ . \ . \ . \ w' = 4$$

$$(3)\ 12c_{0,0} - 3c_{2,0} = 0, \ . \ . \ . \ . \ . \ w' = 2$$

if the side a and its powers are temporarily set equal to unity.

From equations (3a) and (58) there may be deduced the expression

$$\frac{N}{p}\nabla^4 w = 192c_{0,0} - 120c_{2,0} - 72c_{0,2} + \frac{x^2}{a^2}(2976c_{2,0} + 288c_{0,2}) + \frac{y^2}{a^2}(-96c_{2,0} + 288c_{0,2}) = 1. \qquad (61)$$

This equation is to be satisfied only for points 1, 2, 4 and 5 (see Fig. 18(c)). The resulting equations may be shown to be

$$(4)\ 192c_{0,0} - 120c_{2,0} - 72c_{0,2} = 1 \ . \ . \ . \ . \ . \ w' = 4$$

$$(5)\ 192c_{0,0} + 66c_{2,0} - 54c_{0,2} = 1 \ . \ . \ . \ . \ . \ w' = 2$$

$$(6)\ 192c_{0,0} + 61.5c_{2,0} - 40.5c_{0,2} = 1 \ . \ . \ . \ . w' = 2$$

$$(7)\ 192c_{0,0} - 138c_{2,0} - 126c_{0,2} = 1 \ . \ . \ . \ . \ . w' = 4$$

There are, therefore, seven equations in three unknowns which are to be solved. This may be done by observing the following rule:[1]

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[1]Precise Surveying and Geodesy, M. Merriman, 1908, p. 22.

> " For each of the unknown quantities form a normal equation by multiplying each observation equation by the coefficient of that unknown quantity in that equation, and also by its weight, and adding the results. The solution of these normal equations will furnish the most probable values of the unknown quantities."

The weights, w', of the equations are given after each one. They are arbitrarily selected as equal to the number of times the points occur in the four quadrants.

The solution of the resulting normal equations gives the following values:

$$c_{0,0} = 0.0050008205$$

$$c_{2,0} = 0.0001787776$$

$$c_{0,2} = -0.0006745046.$$

From equations (1) in Section 5 there may be obtained the bending moments, at the center,

$$\begin{aligned} M_x &= -N\left[-9c_{0,0} + \frac{9}{8} c_{2,0} + \mu\left(-3c_{0,0} + \frac{9}{8} c_{0,2}\right)\right] \\ M_y &= -N\left[-3c_{0,0} + \frac{9}{8} c_{0,2} + \mu\left(-9c_{0,0} + \frac{9}{8} c_{2,0}\right)\right] \end{aligned} \qquad (62)$$

At the center of the plate there result the values ($a = 2c$)

$$w = 0.045007 \quad \frac{pc^4}{N}$$

$$M_x = 0.196316 \quad pc^2$$

$$M_y = 0.110744 \quad pc^2$$

which are about 8, 16, and 17 percent different from the corresponding difference-equation values.

These differences may be reduced by satisfying the conditions for a greater number of points and by including more terms in the power series.

Another power-series method is suggested in which the following integrals are minimized:

$$I_1 = \int_S (\nabla^2 w)^2 \, ds \cong \text{min.} \qquad (63)$$

$$I_2 = \int\int (\nabla^4 w - \frac{p}{N})^2 \, dx \, dy \cong \text{min.} \qquad (64)$$

The constants may be determined from the equations

$$\frac{\partial I_1}{\partial c_{mn}} \cong 0 \qquad (65)$$

$$\frac{\partial I_2}{\partial c_{mn}} \cong 0 \qquad (66)$$

The same initial form for the deflection function is used in this method as in the one worked out previously in detail.

Still another power-series approach is suggested.[1] There is assumed the function

$$f = (y + \sqrt{3}\, x + \frac{\sqrt{3}}{2} a)(y - \sqrt{3}\, x - \frac{\sqrt{3}}{2} a)(y + \sqrt{3}\, x - \frac{\sqrt{3}}{2} a)$$

$$(y - \sqrt{3}\, x + \frac{\sqrt{3}}{2} a) \sum \sum c_{mn} (\frac{x}{a})^m (\frac{y}{a})^n \qquad (67)$$

upon which there is imposed the condition

[1]This method arose out of discussions with Dr. N. M. Newmark.

$$\iint \left(\nabla^2 f - \frac{p}{N} \right)^2 dx\, dy \cong \text{min.} \tag{68}$$

From (67) and (68) it is possible to obtain the coefficients c_{mn} and, subsequently, the function f.

Then the deflection function is assumed to be

$$w = \left(y + \sqrt{3}\, x + \frac{\sqrt{3}}{2} a\right) \left(y - \sqrt{3}\, x - \frac{\sqrt{3}}{2} a\right) \left(y + \sqrt{3}\, x - \frac{\sqrt{3}}{2} a\right) \left(y - \sqrt{3}\, x + \frac{\sqrt{3}}{2} a\right) \sum \sum k_{mn} \left(\frac{x}{a}\right)^m \left(\frac{y}{a}\right)^n \tag{69}$$

upon which we impose the condition

$$\iint (\nabla^2 w - f)^2 \, dA \cong \text{min.} \tag{70}$$

This method was modified to satisfy the conditions only at a limited number of points. The results, however, were not sufficiently in agreement with those obtained by the other approximate methods.

It should be observed that considerably more difficulty is encountered in the use of the power-series approaches suggested in this dissertation for the case of simply supported edges than is the case for clamped plates.

V. SIMPLY SUPPORTED THIRTY-DEGREE SKEW PLATE WITH RATIO OF SIDES TWO-TO-ONE SUBJECTED TO A UNIFORMLY DISTRIBUTED LOAD

16. Difference-Equations Solutions. - In Fig. 11(b) there is shown the network for the uniformly, simply supported 30-degree skew plate with the lengths of the sides in the ratio of two-to-one. The same procedure is followed here as in the plate with equal sides and, consequently, little need be said for this case.

Table XIII gives the deflections and moments at interior points and the boundary, excepting those at the corner points.

Fig. 19 shows the contours for this plate.

17. Other Methods. - The trigonometric-series and power-series approaches may be pursued here also, but the propitious symmetry conditions that prevail for the equal-sided plate are not to be found for the plate with unequal sides.

In the trigonometric-series solution the odd powers of r and sine terms must also be considered.

The power-series, likewise, will contain other than even powers of x and y.

TABLE XIII

UNIFORMLY LOADED SIMPLY SUPPORTED

30-DEGREE SKEW PLATE HAVING SIDES

IN THE RATIO OF 2 TO 1

11 COEFFICIENTS

$\lambda = \frac{c}{2}$; $\mu = 0.3$

Point	Deflection	Moments		
	$\frac{wN}{pc^4}$	$\frac{M_1}{p\lambda^2}$	$\frac{M_2}{p\lambda^2}$	$\frac{M_3}{p\lambda^2}$
1	0.023776	0.689485	0.185564	0.144880
2	0.044993	0.703715	0.499752	0.359635
3	0.043061	0.299954	0.684717	0.471828
4	0.048625	0.836282	0.451520	0.343665
5	0.079347	1.406879	1.357997	1.024324
6	0.064001	0.635564	0.838045	0.431601
7	0.065201	0.869524	0.667043	0.398508
8	0.099300	1.059317	1.048123	0.550359
9	0.072576	0.788324	0.849338	0.408598
10	0.072776	0.854491	0.793477	0.402557
11	0.102873	1.100053	1.095556	0.543417

TABLE XIII (Cont'd)

UNIFORMLY LOADED SIMPLY SUPPORTED

30-DEGREE SKEW PLATE HAVING SIDES

IN THE RATIO OF 2 TO 1

11 COEFFICIENTS

$\lambda = \frac{c}{2}$; $\mu = 0.3$

Point	Moments					
	$\frac{M_x}{p\lambda^2}$	$\frac{M_y}{p\lambda^2}$	$\frac{M_{xy}}{p\lambda^2}$	$\frac{M_{max.}}{p\lambda^2}$	$\frac{M_{min.}}{p\lambda^2}$	θ Degrees
1	0.185564	0.494388	-0.314426	0.690271	-0.010319	32
2	0.499752	0.542314	-0.198653	0.720823	0.321243	42
3	0.684717	0.286284	0.099231	0.708064	0.262938	13
4	0.451520	0.636124	-0.284411	0.842836	0.244807	36
5	1.357997	0.205799	-0.220866	1.398884	0.164912	-10.5
6	0.838045	0.432094	-0.117757	0.869730	0.400407	-15
7	0.667043	0.623009	-0.271940	0.917855	0.372197	-42.5
8	1.048123	0.723743	-0.293845	1.221568	0.550299	-30.5
9	0.849338	0.514835	-0.219234	0.957834	0.406338	-26.5
10	0.793477	0.573541	-0.260923	0.966659	0.400359	-33.5
11	1.095556	0.730463	-0.321372	1.282608	0.543410	-30

TABLE XIII (Cont'd)

UNIFORMLY LOADED SIMPLY SUPPORTED

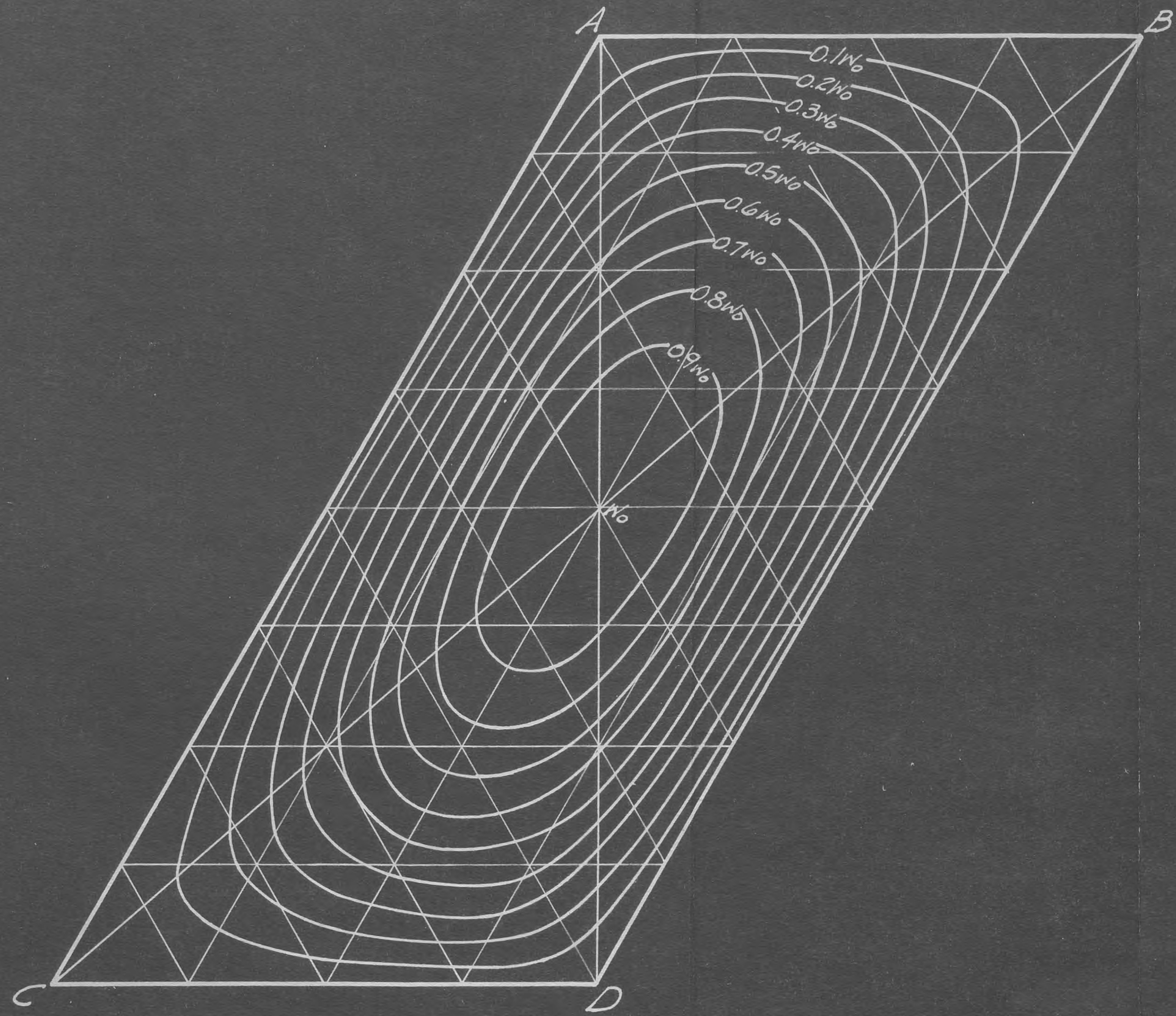
30-DEGREE SKEW PLATE HAVING SIDES

IN THE RATIO OF 2 TO 1

11 COEFFICIENTS

$\lambda = \frac{c}{2}$; $\mu = 0.3$

Point	Deflection	Moments					
	$\frac{wN}{pc^4}$	$\frac{M_x}{p\lambda^2}$	$\frac{M_y}{p\lambda^2}$	$\frac{M_{xy}}{p\lambda^2}$	$\frac{M_{max.}}{p\lambda^2}$	$\frac{M_{min.}}{p\lambda^2}$	θ Degrees
12	0	0	0	-0.307479	0.307479	-0.307479	45
13	0	0	0	-0.274395	0.274395	-0.274395	45
14	0	0	0	0.024988	0.024988	-0.024988	45
15	0	0.234529	-0.234529	0.135405	0.270812	-0.270812	15
16	0	0.096037	-0.096037	0.055447	0.110892	-0.110892	15
17	0	0.002248	-0.002248	0.001298	0.002646	-0.002646	15
18	0	-0.084844	0.084844	-0.048984	0.097967	-0.097967	15
19	0	-0.185648	0.185648	-0.107183	0.214367	-0.214367	15
20	0	-0.278318	0.278318	-0.160686	0.321374	-0.321374	15
21	0	-0.266286	0.266286	-0.153740	0.307481	-0.307481	15



CONTOURS FOR UNIFORMLY LOADED SIMPLY SUPPORTED 30° SKEW PLATE
RATIO OF SIDES - 2:1 11 COEFFICIENTS

Fig. 19

VI. CLAMPED THIRTY-DEGREE SKEW PLATE WITH EQUAL SIDES SUBJECTED TO A UNIFORMLY DISTRIBUTED LOAD

18. Analysis by Method of Finite Differences. - In solving the problem of the clamped plate, it is convenient to relate the membrane deflections in such a manner that only one set of simultaneous equations need be solved. This may be done in the following way.

From the second of equations (41) there may be written the relations (see Fig. 4(e))

$$
\begin{aligned}
4\,\bar{z}_k - \frac{2}{3}\left(\bar{z}_i + \bar{z}_l + \bar{z}_o + \bar{z}_p + \bar{z}_q + \bar{z}_r\right) &= \bar{w}_k\ \frac{\lambda^2}{S_2} \\
4\,\bar{z}_i - \frac{2}{3}\left(\bar{z}_A + \bar{z}_k + \bar{z}_R + \bar{z}_q + \bar{z}_U + \bar{z}_o\right) &= \bar{w}_i\ \text{"} \\
4\,\bar{z}_l - \frac{2}{3}\left(\bar{z}_k + \bar{z}_B + \bar{z}_r + \bar{z}_T + \bar{z}_p + \bar{z}_V\right) &= \bar{w}_l\ \text{"} \\
4\,\bar{z}_q - \frac{2}{3}\left(\bar{z}_R + \bar{z}_r + \bar{z}_C + \bar{z}_G + \bar{z}_i + \bar{z}_k\right) &= \bar{w}_q\ \text{"} \\
4\,\bar{z}_r - \frac{2}{3}\left(\bar{z}_q + \bar{z}_T + \bar{z}_G + \bar{z}_D + \bar{z}_k + \bar{z}_l\right) &= \bar{w}_r\ \text{"} \\
4\,\bar{z}_o - \frac{2}{3}\left(\bar{z}_U + \bar{z}_p + \bar{z}_i + \bar{z}_k + \bar{z}_E + \bar{z}_H\right) &= \bar{w}_o\ \text{"} \\
4\,\bar{z}_p - \frac{2}{3}\left(\bar{z}_o + \bar{z}_V + \bar{z}_k + \bar{z}_l + \bar{z}_H + \bar{z}_F\right) &= \bar{w}_p\ \text{"}
\end{aligned}
\qquad (71)
$$

From equations (71) and the first of equations (41) there is found the more convenient form

$$
\begin{aligned}
42\,\bar{z} &- 10\left(\bar{z}_i + \bar{z}_l + \bar{z}_o + \bar{z}_p + \bar{z}_q + \bar{z}_r\right) \\
&+ \left(\bar{z}_A + \bar{z}_B + \bar{z}_C + \bar{z}_D + \bar{z}_E + \bar{z}_F\right) \\
&+ 2\left(\bar{z}_R + \bar{z}_T + \bar{z}_U + \bar{z}_V + \bar{z}_G + \bar{z}_H\right) \\
&= \frac{9}{4}\ \frac{p\,\lambda^4}{S_1 S_2}\ .
\end{aligned}
\qquad (72)
$$

TABLE XIV

UNIFORMLY LOADED CLAMPED 30-DEGREE

SKEW PLATE HAVING EQUAL SIDES

16 COEFFICIENTS

$\lambda = \frac{c}{4}$; $\mu = 0.3$

Point	Deflection		Moments	
	$\frac{wN}{pc^4}$	$\frac{M_x}{p\lambda^2}$	$\frac{M_y}{p\lambda^2}$	$\frac{M_{xy}}{p\lambda^2}$
1	0.000440	-0.084288	0.075654	-0.138513
2	0.001339	-0.226305	0.209767	-0.156285
3	0.002410	-0.369557	0.201683	-0.073520
4	0.003919	0.107912	0.488398	-0.329509
5	0.003291	-0.431873	0.089358	0.061223
6	0.006665	0.277831	0.720474	-0.260080
7	0.003664	-0.362506	-0.060125	0.208976
8	0.008551	0.458341	0.744322	-0.119157
9	0.010656	0.939737	1.699625	-0.244719
10	0.003240	-0.177311	-0.216376	0.347719
11	0.008760	0.585401	0.611740	0.054672
12	0.012802	0.958844	1.123920	-0.131579
13	0.001782	-0.082983	-0.527702	0.385134
14	0.006963	0.524883	0.390833	0.202693
15	0.012224	0.958416	0.989663	-0.027060
16	0.014384	1.112409	1.250479	-0.119571

TABLE XIV (Cont'd)

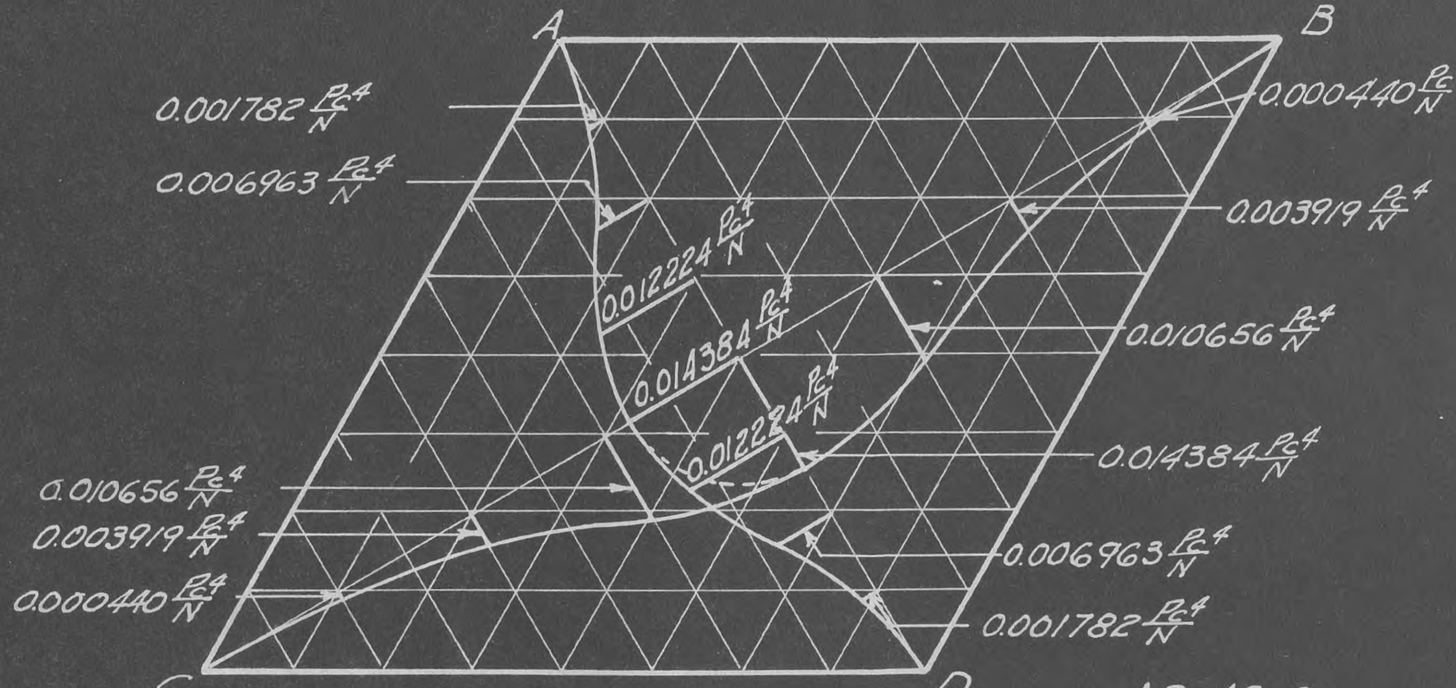
UNIFORMLY LOADED CLAMPED 30-DEGREE

SKEW PLATE HAVING EQUAL SIDES

16 COEFFICIENTS

$\lambda = \frac{c}{4}$; $\mu = 0.3$

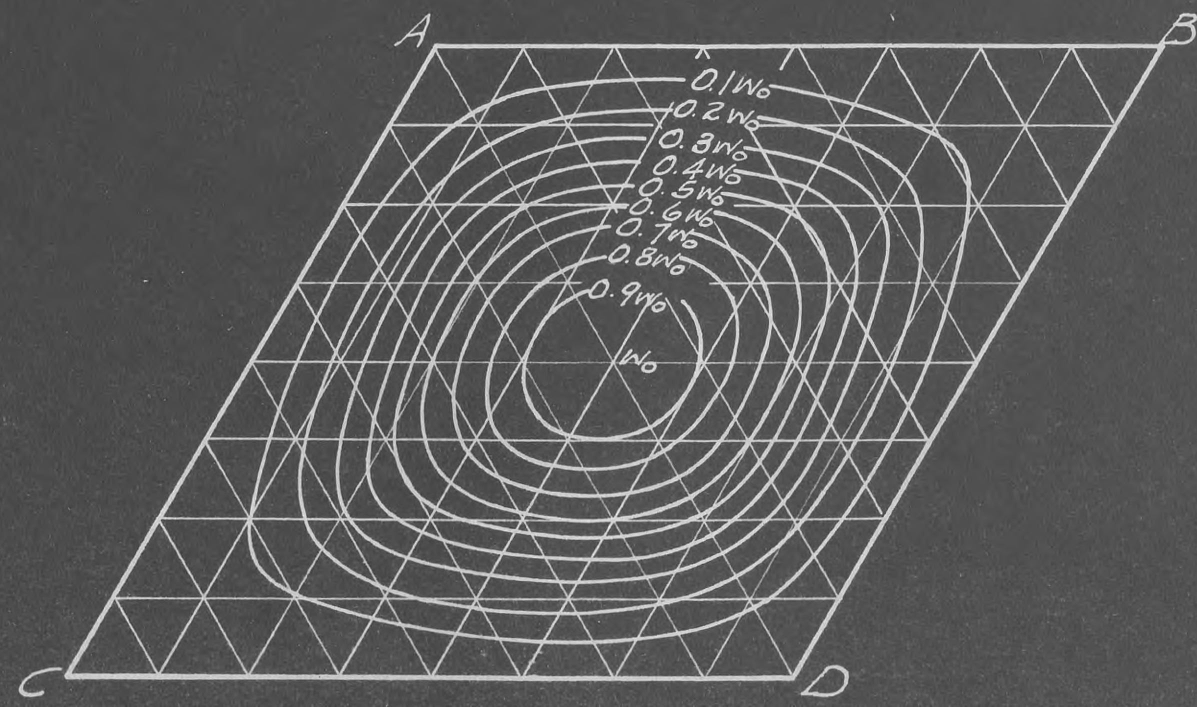
Point	Deflection	Moments		
	$\frac{wN}{pc^4}$	$\frac{M_x}{p\lambda^2}$	$\frac{M_y}{p\lambda^2}$	$\frac{M_{xy}}{p\lambda^2}$
17	0	-0.045034	-0.150112	0
18	0	-0.182127	-0.607089	0
19	0	-0.383836	-1.279452	0
20	0	-0.583738	-1.945795	0
21	0	-0.712181	-2.373603	0
22	0	-0.706964	-2.356548	0
23	0	-0.514224	-1.714081	0
24	0	-1.414117	-0.814189	0.519550
25	0	-1.944152	-1.119360	0.714287
26	0	-1.958497	-1.127620	0.719557
27	0	-1.605281	-0.924252	0.589784
28	0	-1.055548	-0.607740	0.387811
29	0	-0.500849	-0.288367	0.184013
30	0	-0.123842	-0.071303	0.045500



DEFLECTIONS ALONG DIAGONALS OF UNIFORMLY LOADED CLAMPED 30° SKEW PLATE - 16 COEFFICIENTS

Scale:- 0.5" = $0.01 \frac{Pc^4}{N}$

Fig. 20



CONTOURS FOR UNIFORMLY LOADED
CLAMPED 30° SKEW PLATE - 16 COEFFICIENTS

Fig. 21

For this particular clamped plate 16 coefficients were used (see Fig. 10(a). The procedure related previously gave the deflections along the diagonals, and these are represented in Fig. 20. The contours are shown in Fig. 21.

Table XIV contains the deflections and moments at the intersection points of the network (see Fig. 10(a)) and at certain boundary points.

19. Solutions in Trigonometric Series. - The trigonometric-series approach, as mentioned previously, applies to the clamped skew plate as well as to the simply supported case. Again, a plate having a thirty-degree skew is selected, the sides being clamped and the lengths equal to a.

The assumed function for the deflections must satisfy the plate equation

$$\nabla^2\nabla^2 w = \frac{p}{N} \tag{3a}$$

throughout the entire plate and the boundary conditions, as before are to be met at an arbitrary number of points (12 in this case).

Boundary conditions at points B, D, F, and G (see Fig. 17(c)), which must be satisfied, are clearly

$$\begin{aligned} w &= 0 \\ \frac{\partial w}{\partial r} &= 0 \\ \frac{\partial w}{\partial \theta} &= 0 \quad . \end{aligned} \tag{73}$$

Because of symmetry the deflection function may be put in the form

$$\begin{aligned} w = \frac{pr^4}{64N} &+ a_0 + r^2 a_2 \cos 2\theta + r^2 c_2 \\ &+ r^4 a_4 \cos 4\theta + r^4 c_4 \cos 2\theta + r^6 a_6 \cos 6\theta \\ &+ r^6 c_6 \cos 4\theta + r^8 a_8 \cos 8\theta + r^8 c_8 \cos 6\theta \\ &+ r^{10} c_{10} \cos 8\theta \quad . \end{aligned} \tag{74}$$

Differentiation of equation (74) results in

$$\frac{\partial w}{\partial r} = \frac{pr^3}{16N} + 2ra_2 \cos 2\theta + 2rc_2$$

$$+ 4r^3a_4 \cos 4\theta + 4r^3c_4 \cos 2\theta + 6r^5a_6 \cos 6\theta$$

$$+ 6r^5c_6 \cos 4\theta + 8r^7a_8 \cos 8\theta + 8r^7c_8 \cos 6\theta$$

$$+ 10r^9c_{10} \cos 8\theta$$

(75)

$$\frac{\partial w}{\partial \theta} = - 2r^2a_2 \sin 2\theta - 4r^4a_4 \sin 4\theta$$

$$- 2r^4c_4 \sin 2\theta - 6r^6a_6 \sin 6\theta - 4r^6c_6 \sin 4\theta$$

$$- 8r^8a_8 \sin 8\theta - 6r^8c_8 \sin 6\theta - 8r^{10}c_{10} \sin 8\theta$$

By substituting values of r and θ for points B, D, F and G (see Fig. 17(c)) and setting a equal to unity, it is feasible to arrive at the following ten simultaneous equations:

(1) $$a_0 + \frac{1}{4} a_2 + \frac{1}{4} c_2 + \frac{1}{16} a_4 + \frac{1}{16} c_4 + \frac{1}{64} a_6 + \frac{1}{64} c_6$$

$$+ \frac{1}{256} a_8 + \frac{1}{256} c_8 + \frac{1}{1024} c_{10} + \frac{p}{1024N} = 0$$

(2) $$a_2 + c_2 + \frac{1}{2} a_4 + \frac{1}{2} c_4 + \frac{6}{32} a_6 + \frac{6}{32} c_6$$

$$+ \frac{8}{128} a_8 + \frac{8}{128} c_8 + \frac{10}{512} c_{10} + \frac{p}{128N} = 0$$ (76)

(3) $$a_0 - \frac{3}{4} a_2 + \frac{3}{4} c_2 + \frac{9}{16} a_4 - \frac{9}{16} c_4 - \frac{27}{64} a_6 + \frac{27}{64} c_6$$

$$+ \frac{81}{256} a_8 - \frac{81}{256} c_8 + \frac{243}{1024} c_{10} + \frac{9p}{1024N} = 0$$

(4) $$a_2 - c_2 - \frac{3}{2} a_4 + \frac{3}{2} c_4 + \frac{54}{32} a_6 - \frac{54}{32} c_6$$
$$- \frac{27}{16} a_8 + \frac{27}{16} c_8 - \frac{810}{512} c_{10} - \frac{3p}{128N} = 0$$

(5) $$a_0 + \frac{3}{32} a_2 + \frac{3}{16} c_2 - \frac{9}{512} a_4 + \frac{9}{512} c_4 - \frac{27}{4096} a_6$$
$$- \frac{27}{8192} c_6 - \frac{81}{131072} a_8 - \frac{81}{65536} c_8$$
$$- \frac{243}{2097152} c_{10} + \frac{9p}{16384N} = 0$$

(6) $$\frac{1}{4} a_2 + \frac{1}{2} c_2 - \frac{3}{32} a_4 + \frac{3}{32} c_4 - \frac{54}{1024} a_6 - \frac{27}{1024} c_6$$
$$- \frac{27}{4096} a_8 - \frac{27}{2048} c_8 - \frac{810}{524288} c_{10} + \frac{3p}{1024N} = 0$$

(7) $$\frac{3}{16} a_2 + \frac{9}{128} a_4 + \frac{9}{256} c_4 + \frac{27}{2048} c_6$$ (76)
$$- \frac{81}{16384} a_8 - \frac{243}{262144} c_{10} = 0$$

(8) $$a_0 - \frac{1}{8} a_2 + \frac{1}{4} c_2 - \frac{1}{32} a_4 - \frac{1}{32} c_4 + \frac{1}{64} a_6$$
$$- \frac{1}{128} c_6 - \frac{1}{512} a_8 + \frac{1}{256} c_8 - \frac{1}{2048} c_{10} + \frac{p}{1024N} = 0$$

(9) $$\frac{1}{2} a_2 - c_2 + \frac{1}{4} a_4 + \frac{1}{4} c_4 - \frac{3}{16} a_6 + \frac{3}{32} c_6$$
$$+ \frac{1}{32} a_8 - \frac{1}{16} c_8 + \frac{5}{512} c_{10} - \frac{p}{128N} = 0$$

(10) $$- \frac{1}{4} a_2 + \frac{1}{8} a_4 - \frac{1}{16} c_4 + \frac{1}{32} c_6 - \frac{1}{64} a_8 - \frac{1}{256} c_{10} = 0$$

The solution of equations (76) gives the values

$$a_0 = 0.000\ 797\ 9061\ \frac{p}{N} \qquad a_6 = -0.016\ 888\ 5960\ \frac{p}{N}$$

$$a_2 = -0.002\ 205\ 7884\ \frac{p}{N} \qquad c_6 = -0.049\ 365\ 7788\ \frac{p}{N}$$

$$c_2 = -0.006\ 507\ 8196\ \frac{p}{N} \qquad a_8 = 0.017\ 786\ 7736\ \frac{p}{N}$$

$$a_4 = 0.012\ 884\ 7660\ \frac{p}{N} \qquad c_8 = 0.029\ 790\ 9016\ \frac{p}{N}$$

$$c_4 = 0.005\ 736\ 5111\ \frac{p}{N} \qquad c_{10} = 0.027\ 625\ 4724\ \frac{p}{N}$$

These values, it must be remembered, are to be multiplied by appropriate powers of a before they are inserted in equation (74) to give the approximate solution for the deflection function.

The maximum deflection for this plate occurs at the center ($r = 0$), and its value is found to be

$$w_{max} = 0.000\ 797\ 9061\ \frac{pa^4}{N}$$

or

$$w_{max} = 0.012\ 766\ 4976\ \frac{pc^4}{N}\ .$$

The corresponding value obtained by means of difference equations is

$$w_{max} \doteq 0.014\ 384\ \frac{pc^4}{N}$$

which indicates a difference of about 11 per cent based on the latter value.

Referring to equations (13), it is possible to deduce the following expressions for the radial and tangential moments:

$$M_r = -N\Bigg[\frac{3pr^2}{16N} + 2a_2 \cos 2\theta + 2c_2 + 12r^2 a_4 \cos 4\theta$$
$$+ 12r^2c_4 \cos 2\theta + 30r^4a_6 \cos 6\theta + 30r^4c_6 \cos 4\theta$$
$$+ 56r^6a_8 \cos 8\theta + 56r^6c_8 \cos 6\theta + 90r^8c_{10} \cos 8\theta$$
$$+ \mu\Bigg(\frac{pr^2}{16N} - 2a_2 \cos 2\theta + 2c_2 - 12r^2a_4 \cos 4\theta$$
$$- 30r^4a_6 \cos 6\theta - 10r^4c_6 \cos 4\theta - 56r^6a_8 \cos 8\theta$$
$$- 28r^6c_8 \cos 6\theta - 54r^8c_{10} \cos 8\theta\Bigg)\Bigg]$$

(77)

$$M_t = -N\Bigg[\mu\Bigg(\frac{3pr^2}{16N} + 2a_2 \cos 2\theta + 2c_2 + 12r^2a_4 \cos 4\theta$$
$$+ 12r^2c_4 \cos 2\theta + 30r^4a_6 \cos 6\theta + 30r^4c_6 \cos 4\theta$$
$$+ 56r^6a_8 \cos 8\theta + 56r^6c_8 \cos 6\theta + 90r^8c_{10} \cos 8\theta\Bigg)$$
$$+ \frac{pr^2}{16N} - 2a_2 \cos 2\theta + 2c_2 - 12r^2a_4 \cos 4\theta$$
$$- 30r^4a_6 \cos 6\theta - 10r^4c_6 \cos 4\theta - 56r^6a_8 \cos 8\theta$$
$$- 28r^6c_8 \cos 6\theta - 54r^8c_{10} \cos 8\theta\Bigg] .$$

For $r = 0$ and $\theta = 0$, at the center of the plate, these expressions reduce to

$$M_r = -2N\left[a_2 + c_2 + \mu\,(c_2 - a_2)\right]$$
$$M_t = -2N\left[\mu\,(a_2 + c_2) - a_2 + c_2\right] \qquad (78)$$

which are the same as found for the simply supported plate (constants are different).

Substitution of the values obtained for a_2 and c_2 gives

$$M_r = 0.020\ 008\ 4347\ pa^2$$
$$= 0.080\ 033\ 7388\ pc^2$$

and

$$M_t = 0.013\ 832\ 2272\ pa^2$$
$$= 0.055\ 328\ 9088\ pc^2$$

The corresponding difference-equations values are

$$M_r = 0.082\ 470\ pc^2$$
$$M_t = 0.065\ 211\ pc^2 \quad ,$$

indicating for the radial and tangential moments at the center a difference of about 3 per cent and 15 per cent, respectively, based on the difference-equation moments. Obviously, in the case of the clamped plate, the boundary conditions should be satisfied for a greater number of points (for the same degree of approximation) than in the case of the simply supported skew plate.

For the equal-sided clamped thirty-degree skew plate with the boundary conditions being satisfied at only eight points along the entire boundary (see Fig. 17(b)), the results are naturally less reliable than those obtained when twelve points are taken into consideration.

The maximum deflection of the plate obtained by the trigonometric-series approach for this case (eight boundary points considered) is approximately 17 per cent too low based on the difference-equation result. The difference between the radial and tangential moments as obtained by the trigonometric and difference-equation methods average about 20 per cent. Clearly, the degree of approximation in this case is not satisfactory.

20. <u>Power-Series Methods.</u> – The clamped 30-degree skew plate having equal sides and subjected to a uniformly distributed load may also be analyzed by power-series methods.

The deflection function may be written in the form (see Fig. 18)

$$w = \frac{p}{a^4N}\left(y + \sqrt{3}\,x + \frac{\sqrt{3}}{2}\,a\right)^2\left(y - \sqrt{3}\,x - \frac{\sqrt{3}}{2}\,a\right)^2\left(y + \sqrt{3}\,x - \frac{\sqrt{3}}{2}\,a\right)^2 \left(y - \sqrt{3}\,x + \frac{\sqrt{3}}{2}\,a\right)^2 \sum_{m=0,2,4\ldots}^{\infty} \; \sum_{n=0,2,4\ldots}^{\infty} c_{mn}\left(\frac{x}{a}\right)^m\left(\frac{y}{a}\right)^n , \tag{79}$$

from which it is apparent that the boundary conditions

$$w = 0 \tag{80}$$

$$\frac{\partial w}{\partial x} = 0 \; ; \qquad \frac{\partial w}{\partial y} = 0 \tag{81}$$

are manifestly valid.

The plate equation

$$\nabla^4 w = \frac{p}{N} \tag{3a}$$

must also be satisfied. By doing this for only a certain number of interior points, it is possible to arrive at an approximate solution of the problem. It is convenient to select the number of points to correspond to the number of constants, c_{mn}, chosen for the series; in this manner one obtains as many equations as there are unknowns in the series. The plate equation may be satisfied for a larger number of points by normalizing the resulting equations. This process yields a system of linear equations in normal form (symmetry about the diagonal) which may easily be solved by the method[1] mentioned previously.

---

[1]Doolittle method.

Six points are now selected within the plate (see Fig. 18(c)) and, correspondingly, six coefficients are taken in the power series. Obviously, from symmetry, these are $c_{0,0}$, $c_{2,0}$, $c_{0,2}$, $c_{4,0}$, $c_{2,2}$, $c_{0,4}$.

From equation (79), by a few simple operations, one may arrive at the differential equation

$$
\begin{aligned}
\frac{N}{p}\nabla^4 w = {} & 864c_{0,0} - 135c_{2,0} - 81c_{0,2} + \frac{243}{32}c_{4,0} + \frac{81}{32}c_{2,2} + \frac{243}{32}c_{0,4} \\
& + x^2\left(-27648c_{0,0} + 11340c_{2,0} + 1620c_{0,2} - \frac{3807}{2}c_{4,0} - \frac{567}{2}c_{2,2} - \frac{243}{2}c_{0,4}\right) \\
& + x^4\left(124416c_{0,0} - 132624c_{2,0} - 9072c_{0,2} + 51921c_{4,0} + 3807c_{2,2} + 729c_{0,4}\right) \\
& + x^6\left(385344c_{2,0} + 15552c_{0,2} - 401976c_{4,0} - 17496c_{2,2} - 1944c_{0,4}\right) \\
& + x^8\left(924696c_{4,0} + 26568c_{2,2} + 1944c_{0,4}\right) \\
& + y^2\left(324c_{2,0} + 2268c_{0,2} - \frac{81}{2}c_{4,0} - \frac{405}{2}c_{2,2} - \frac{1701}{2}c_{0,4}\right) \\
& + y^4\left(1536c_{0,0} - 144c_{2,0} - 3312c_{0,2} + 81c_{4,0} + 567c_{2,2} + 7209c_{0,4}\right) \\
& + y^6\left(-64c_{2,0} + 3648c_{0,2} - 72c_{4,0} - 456c_{2,2} - 12456c_{0,4}\right) \\
& + y^8\left(24c_{4,0} + 72c_{2,2} + 8856c_{0,4}\right) \\
& + x^2y^2\left(-27648c_{0,0} + 11232c_{2,0} - 18144c_{0,2} + 3402c_{4,0} + 14094c_{2,2} + 11178c_{0,4}\right) \\
& + x^2y^4\left(12480c_{2,0} - 20160c_{0,2} + 1080c_{4,0} + 11160c_{2,2} + 5400c_{0,4}\right) \\
& + x^2y^6\left(-2976c_{4,0} + 8352c_{2,2} - 26784c_{0,4}\right) \\
& + x^4y^2\left(-146880c_{2,0} + 77760c_{0,2} + 50760c_{4,0} - 113400c_{2,2} - 48600c_{0,4}\right) \\
& + x^4y^4\left(70800c_{4,0} - 104400c_{2,2} + 32400c_{0,4}\right) \\
& + x^6y^2\left(-476064c_{4,0} + 282528c_{2,2} + 69984c_{0,4}\right) = 1 \quad (\text{a and its powers equal } 1)
\end{aligned}
\tag{82}
$$

Substitution of values of x and y in equations (82) for the six points gives rise to the following system of equations

(1) $$864c_{0,0} - 135c_{2,0} - 81c_{0,2} + 7.59375c_{4,0} + 2.53125c_{2,2} + 7.59375c_{0,4} = 1$$

(2) $$-378c_{0,0} + 149.765625c_{2,0} - 11.390625c_{0,2} + 7.4124755859c_{4,0} - 4.1824951172c_{2,2} + 2.4027099609c_{0,4} = 1$$

(3) $$867.375c_{0,0} - 120.1354980469c_{2,0} + 18.4108886719c_{0,2} + 5.8659911155c_{4,0} - 5.7617068290c_{2,2} - 17.6735215187c_{0,4} = 1$$

(4) $$-455.625c_{0,0} + 172.3557128906c_{2,0} + 46.3337402344c_{0,2} + 20.2346663474c_{4,0} + 11.9749646187c_{2,2} + 5.1355376242c_{0,4} = 1$$

(5) $$918c_{0,0} - 79.734375c_{2,0} + 251.859375c_{0,2} + 2.4027099609c_{4,0} - 18.4207763672c_{2,2} + 30.4046630859c_{0,4} = 1$$

(6) $$1137.375c_{0,0} - 28.7468261719c_{2,0} + 560.2565917969c_{0,2} + 0.2782073021c_{4,0} - 13.9425387383c_{2,2} + 277.1071028709c_{0,4} = 1$$

Solution of this set gives the values

$$c_{0,0} = 2.6184138565 \times 10^{-3}$$

$$c_{2,0} = 11.8165756479 \times 10^{-3}$$

$$c_{0,2} = -2.6453532509 \times 10^{-3}$$

$$c_{4,0} = 20.1633240889 \times 10^{-3}$$

$$c_{2,2} = -10.3051787620 \times 10^{-3}$$

$$c_{0,4} = -1.1029769807 \times 10^{-3}$$

The maximum deflection of the plate is found to be

$$w_{max} = 0.0008284825 \frac{pa^4}{N}$$

which is about 7.8 per cent below the difference-equation value

$$w_{max} = 0.000899 \frac{pa^4}{N} \quad .$$

From equations (1) and (79) it is possible to determine the bending moments at the center of the plate

$$M_x = 0.084732 \ pc^2$$

$$M_y = 0.051492 \ pc^2 \quad .$$

These are about 3 and 21 per cent different from the corresponding difference-equations values.

Another power-series method is one which involves the Ritz method.[1] This method, however, will not be given here.

---

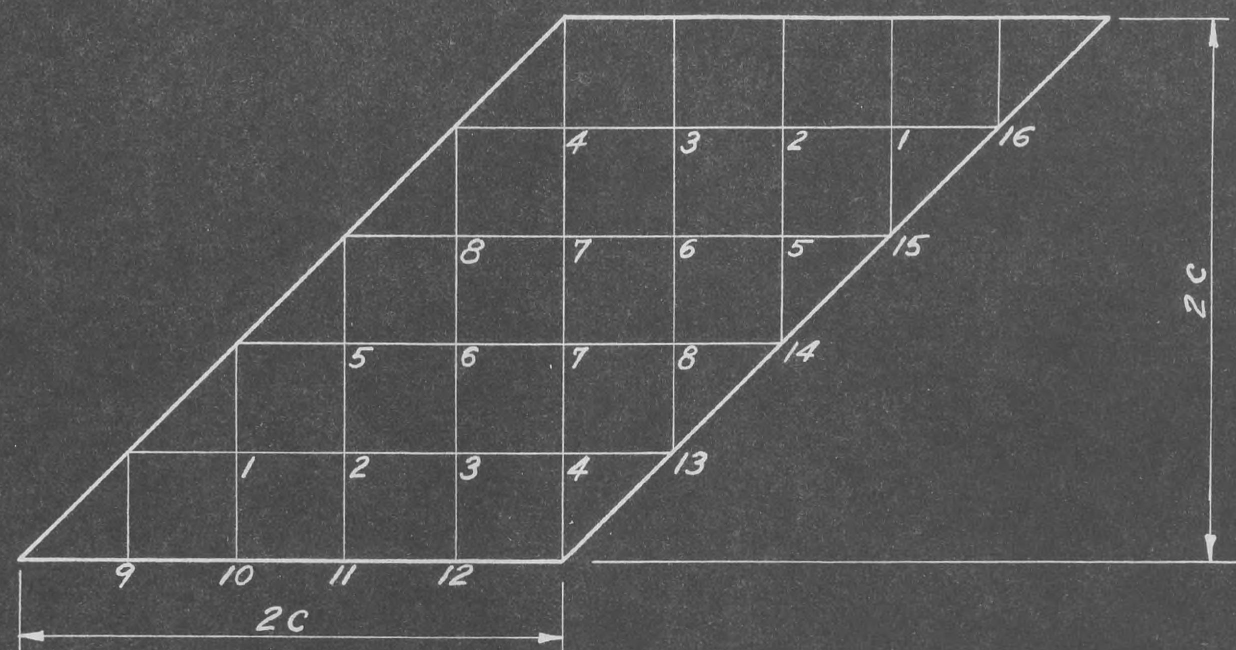
[1]A Laterally Loaded Clamped Square Plate with Large Deflections, Stewart Way, Stephen Timoshenko - 60th Anniversary Volume, 1938, pp. 240-256.

VII. SIMPLY SUPPORTED FORTY-FIVE DEGREE TOTAL SKEW PLATE

21. Analysis by Method of Finite Differences. - The network for a 45-degree total skew plate with eight coefficients is shown in Fig. 22. In this case a square mesh happens to be most convenient. The manner of treating this type of problem has been indicated previously (see Section 9).

Table XV contains the deflections and moments found for this plate.

Other approximate methods which have been indicated for previous cases may also be used in the solution of this problem.



NETWORK FOR 45° TOTAL SKEW PLATE – 8 COEFFICIENTS

Fig. 22

TABLE XV

UNIFORMLY LOADED SIMPLY SUPPORTED

45-DEGREE TOTAL SKEW PLATE

8 COEFFICIENTS

$$\lambda = \frac{2c}{5} \quad ; \quad \mu = 0.3$$

Point	Deflection	Moments					
	$\frac{wN}{pc^4}$	$\frac{M_x}{p\lambda^2}$	$\frac{M_y}{p\lambda^2}$	$\frac{M_{xy}}{p\lambda^2}$	$\frac{M_{max}}{p\lambda^2}$	$\frac{M_{min}}{p\lambda^2}$	θ Degrees
1	0.279143	0.056688	0.525047	-0.167638	0.578864	0.002871	17.5
2	0.669084	0.328387	0.698549	-0.226104	0.805663	0.221273	25
3	0.928451	0.562842	0.682892	-0.131899	0.767780	0.477953	33
4	0.794439	0.707340	0.354505	-0.070787	0.721013	0.340833	-11
5	0.678791	0.272107	0.708165	-0.163024	0.762375	0.217898	18
6	1.292024	0.698810	0.895341	-0.228751	1.046039	0.548111	33
7	1.432501	0.862314	0.839331	-0.230719	1.081827	0.619818	-44
8	0.902083	0.525308	0.623642	-0.164992	0.746637	0.402314	37

TABLE XV (Cont'd)

UNIFORMLY LOADED SIMPLY SUPPORTED

45-DEGREE TOTAL SKEW PLATE

8 COEFFICIENTS

$\lambda = \frac{2c}{5}$; $\mu = 0.3$

Point	Deflection	Moments					
	$\frac{wN}{pc^4}$	$\frac{M_x}{p\lambda^2}$	$\frac{M_y}{p\lambda^2}$	$\frac{M_{xy}}{p\lambda^2}$	$\frac{M_{max.}}{p\lambda^2}$	$\frac{M_{min.}}{p\lambda^2}$	θ Degrees
9	0	0	0	-0.097700	0.097700	-0.097700	45
10	0	0	0	-0.234179	0.234179	-0.234179	45
11	0	0	0	-0.227258	0.227258	-0.227258	45
12	0	0	0	-0.043874	0.043874	-0.043874	45
13	0	0.075351	-0.075351	0	0.075351	-0.075351	0
14	0	-0.156304	0.156304	0	0.156304	-0.156304	0
15	0	-0.279754	0.279754	0	0.279754	-0.279754	0
16	0	-0.195400	0.195400	0	0.195400	-0.195400	0

VIII. SUMMARY AND CONCLUSIONS

22. Recapitulation. - In this dissertation a theoretical study of uniformly loaded, simply supported and clamped skew plates has been made by means of approximate methods. No exact solution of skew plate problems has, as yet, come to the attention of the writer. Comparisons were, consequently, made between approximate methods.

A survey was made of a few systems of coordinates in order to ascertain their convenience in the treatment of triangular and skew plates. The trilinear and quadrilinear systems appear to have no particular advantages, except possibly as regards the question of symmetry. Both seem to require more effort than the Cartesian-coordinate system in the solution of the same problems. The trilinear system has been suggested by an investigator who solved plate problems that have also been analyzed by Cartesian coordinates. The quadrilinear system developed in this dissertation, in the case of power-series methods, would definitely involve more work. The polar-coordinate equations, on the other hand, have been very useful in the trigonometric-series solutions.

From a brief preliminary survey of triangular plates and a more extensive investigation of the uniformly loaded, simply supported 30-degree skew plate having equal sides, a measure of the reliability of difference-equations results for various densities of networks was obtained. In general, it may be stated that a relatively small number of nodal points gives values which are in fair agreement with those of more dense and, hence, more accurate networks. This is especially true in the case of bending moments. More specifically, in this connection, the following observations are noteworthy:

(1) In the case of simply supported equilateral, triangular plates, subjected to uniformly distributed loads, reasonably good results are obtainable by the method of finite differences even for a relatively small number (5) of coefficients. The maximum deflection given by the 5- coefficient and 8- coefficient solutions varied from the exact solution by 2.7 per cent and 1.2 per cent, respectively.

(2) A comparison of deflections at certain points of the simply supported thirty-sixty-ninety- degree plate subjected to a uniformly distributed load obtained by networks having three and twenty-one nodal points showed differences of about 6 or 7 per cent.

(3) For the simply supported isosceles right triangular plate subjected to a uniformly distributed load, the deflections as obtained by Nadai's approximation and a difference-equation network having twelve coefficients are in fair agreement at points near the center of gravity of the plate, but nearer the boundaries there is considerable disagreement.

(4) A comparison of bending moments and twisting moments at similar points of the uniformly loaded simply supported thirty-degree skew plate (equal sides) obtained by the 4- and16- coefficient solutions indicated reasonably good agreement. At one pair of corresponding points the results agreedwell within 1 per cent. Two other pairs, for which a comparison was made, showed a maximum difference of

about 5 per cent for bending moments, whereas the twisting moments varied by about 10 per cent in one case and by about 22 per cent in the other. The deflections differed by about 8 or 9 per cent for the three points investigated.

(5) At the center of the uniformly loaded simply supported thirty-degree skew plates having equal sides, the bending moments and deflections resulting from the networks having 25 and 16 coefficients agreed well within 1 per cent. The corresponding twisting moments differed by only 2 per cent.

(6) A similar comparison (see (5)) between these results as obtained by the 25- and 4- coefficient solutions showed good agreement (within 3 per cent) for the bending moments. The twisting moments and deflections at the center differed by 19 and 9 per cent, respectively.

The uniformly loaded, simply supported 30-degree skew plate with equal sides was analyzed by the trigonometric-series and power-series methods, as well as by the difference-equations method, and certain deflections and moments were compared. The following comparisons may be enumerated.

(1) The maximum deflection given by the trigonometric-series method for this case (8 coefficients) differs by about 4 per cent from the difference-equation value. At the center of the plate the radial and tangential moments differ by 4 per cent and 0.6 per cent, respectively, for these same two methods. The agreement here is seen to be good.

(2) At the center of the plate, the deflection, w, and the moments M_x and M_y, as obtained by the method using power series (only

three terms and seven points considered), differed by 8, 16, and 17 per cent, respectively, from the corresponding difference-equation values. By increasing the number of terms in the series and the number of points for which the conditions are satisfied, it is possible to increase the accuracy.

The clamped thirty-degree skew plate with equal sides, subjected to a uniformly distributed load, was studied by similar methods, and the following observations were made:

(1) A difference of 11 per cent was found between the maximum deflections as obtained by trigonometric series (10 coefficients) and by difference equations. The radial and tangential moments at the center differed (for these two methods) by about 3 per cent and 15 per cent, respectively.

(2) The values for the maximum deflection as obtained by power series (6 coefficients) and difference equations differed by 7.8 per cent. The moments M_x and M_y differed by 3 and 21 per cent, respectively.

Moments and deflections obtained by the method of finite differences were indicated also for the uniformly loaded, simply supported 30-degree skew plate with the ratio of sides two-to-one and for the 45-degree total skew plate similarly loaded and supported.

The structural action of skew plates and slabs in the region of corners is a matter which was not treated. Singularities prevailing in these sections probably necessitate the use of more rigorous, mathematical methods.

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X. APPENDICES

APPENDIX A

DERIVATION OF DIFFERENTIAL EQUATIONS OF ELASTIC SURFACE OF PLATE

The derivation of the fundamental differential equations of the elastic plate is given here only in skeleton form. Reference to the Notation (Section 3) and Figures 1 and 2 should clarify the designations in the steps of the derivation.[1]

$$u = - z \frac{\partial w}{\partial x} \qquad v = - z \frac{\partial w}{\partial y} \tag{a}$$

$$\epsilon_x = \frac{\partial u}{\partial x} = - z \frac{\partial^2 w}{\partial x^2} \qquad \epsilon_y = \frac{\partial v}{\partial y} = - z \frac{\partial^2 w}{\partial y^2} \tag{b}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = - 2z \frac{\partial^2 w}{\partial x \partial y} \tag{c}$$

$$\begin{aligned} \epsilon_x &= \frac{1}{E} (\sigma_x - \mu \sigma_y) \\ \epsilon_y &= \frac{1}{E} (\sigma_y - \mu \sigma_x) \end{aligned} \tag{d}$$

$$\begin{aligned} \sigma_x &= \frac{E}{1 - \mu^2} (\epsilon_x + \mu \epsilon_y) \\ &= - \frac{Ez}{1 - \mu^2} \left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} \right) \\ \sigma_y &= \frac{E}{1 - \mu^2} (\epsilon_y + \mu \epsilon_x) \\ &= - \frac{Ez}{1 - \mu^2} \left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2} \right) \end{aligned} \tag{e}$$

$$\tau_{xy} = G \gamma_{xy} = - 2Gz \frac{\partial^2 w}{\partial x \partial y} \tag{f}$$

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[1] Die elastische Platten, A. Nadai, 1925, p. 18.

$$M_x = \int \sigma_z \, zdz$$
$$M_y = \int \sigma_y \, zdz \qquad (g)$$
$$M_{xy} = \int \tau_{xy} \, zdz$$

$$M_x = - N \left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} \right)$$
$$M_y = - N \left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2} \right) \qquad (h)$$
$$M_{xy} = - (1 - \mu) N \frac{\partial^2 w}{\partial x \partial y}$$

$$V_x = \int \tau_{xz} \, dz \; ; \quad V_y = \int \tau_{yz} \, dz \qquad (i)$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - V_x = 0$$
$$\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - V_y = 0 \qquad (j)$$

$$V_x = - N \frac{\partial \nabla^2 w}{\partial x} \; ; \quad V_y = - N \frac{\partial \nabla^2 w}{\partial y} \qquad (k)$$

$$M_x + M_y = - (1 + \mu) N \nabla^2 w \qquad (l)$$

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + p = 0 \qquad (m)$$

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \nabla^4 w = \frac{p}{N} \qquad (n)$$

The last expression is Lagrange's plate equation.

APPENDIX B

SOLUTION OF SIMULTANEOUS EQUATIONS

The solution of simultaneous equations may be undertaken in the manner[1] used by most engineers or, if the equations can conveniently be put into normal form, the " Doolittle Method " is very satisfactory, especially in the case of a large number of unknowns.

In order to illustrate the " Doolittle Method " with an adequate explanation, the brief problem associated with the 4-coefficient solution of the uniformly loaded, simply supported plate having equal sides will now be worked out.

The normal form relating the deflections of the first membrane for this case may be written as follows:

$$\begin{aligned}
6\bar{w}_1 - 2\bar{w}_2 \qquad\qquad\qquad &= 1.5\,\frac{p\lambda^2}{S_1} \\
-2\bar{w}_1 + 10\bar{w}_2 - 2\bar{w}_3 - 2\bar{w}_4 &= 3\,\frac{p\lambda^2}{S_1} \\
-2\bar{w}_2 + 6\bar{w}_3 - \bar{w}_4 &= 1.5\,\frac{p\lambda^2}{S_1} \\
-2\bar{w}_2 - \bar{w}_3 + 3\bar{w}_4 &= 0.75\,\frac{p\lambda^2}{S_1}
\end{aligned} \qquad (a)$$

In equations (a) it is apparent that symmetry prevails about the diagonal terms on the left-hand side of the equations. It may be shown[2] that, for this reason, only the diagonal terms and

[1]Structural Theory, Sutherland and Bowman, Second Edit., pp. 282-283.

[2]Adjustment of Observations, Wright and Hayford, Chapt. IV, pp. 93-148.

those above and to the right need be considered.

SOLUTION OF DIFFERENCE EQUATIONS

	1	2	3	4	$\frac{p\lambda^2}{S_1}$	
(1)	6	−2			−1.5	2.5
(2)	$(-)\bar{w}_1$	0.333333			0.250000	−0.416667
(3)		10	−2	−2	−3	3
(4)		−0.666667			−0.500000	−1.166667
(5)		9.333333	−2	−2	−3.500000	1.833333
(6)		$(-)\bar{w}_2$	0.214286	0.214286	0.375000	−0.196428
(7)			6	−1	−1.5	3.5
(8)			−0.428572	−0.428572	−0.750000	−1.607144
(9)			5.571428	−1.428572	−2.250000	1.892856
(10)			$(-)\bar{w}_3$	0.256410	0.403846	−0.339744
(11)				3	−0.75	2.25
(12)				−0.366300	−0.576932	−0.943223
(13)				−0.428572	−0.750000	−1.178572
(14)				2.205128	−2.076923	0.128205
(15)				$(-)\bar{w}_4$	0.941861	−0.058139
(16)				$\bar{w}_4$ =	0.941861	
(17)			$(-)\bar{w}_3$	0.241503	0.403846	
(18)			$\bar{w}_3$ =		0.645349	
(19)		$(-)\bar{w}_2$	0.138289	0.201828	0.375000	
(20)		$\bar{w}_2$ =			0.715117	
(21)	$(-)\bar{w}_1$	0.238372			0.250000	
(22)	$\bar{w}_1$ =				0.488372	

The first step in this method is to write the first equation, with the sum of the terms placed in the check or summation column. Next, divide the entire line by the first number and change signs. Check this equation by adding the terms in the second line. This addition should give nearly the same value as the division gave. Underline this second line, since it is essentially the solution of the first coefficient in terms of the others.

Then, insert the second of equations (a), omitting the term to the left of the diagonal. The next step consists of multiplying the quantity 0.333333 and the other terms to the right in the second line by the factor - 2 which is above the 10 in line (3). Then, add lines (3) and (4) to obtain line (5). Again, divide by the first term in the last line mentioned and change signs. This equation is the solution of the second unknown in terms of the quantities to the right. Meanwhile, the check column is continued as the work progresses. It is to be observed that several plans of checking may be worked out, but these will not be enumerated here.

It should be noted that lines (12) and (13) are both required to be written. Line (12) is obtained by multiplying every number to the right of and including number 0.256510 in line (10) by the factor -1.428572. Likewise line (13) results from the multiplication of every number to the right of and including 0.214286 in line (6) by the factor -2 directly above the 0.214286. If lines (1) and (2) were to have numbers in them also in the column 4, a similar operation would be performed. This process is continued until the last unknown, $\bar{w}_4$, is found.

Finally, the known values are substituted successively back in the underscored lines to give the other unknowns. These values are to be substituted in the original equations to obtain a check.

A similar procedure is followed in connection with the $\bar{z}$- equations.

Considerable time is saved by this method, the saving being increased as the number of unknowns becomes greater.

XI. VITA

On June 12, 1911, Francis Louis Ehasz was born in New York City, the son of Frank Ehasz and Elizabeth Sima Ehasz. In his senior year at DeWitt Clinton High School in the City of New York, he was awarded a scholarship to the College of Engineering of New York University. After receiving the degree, Bachelor of Science in Civil Engineering, June 1933, he accepted for one academic year a fellowship, tendered by the Institute of International Education, to the University of Technical Sciences in Budapest, Hungary.

During 1934-1936, as Lawrence Calvin Brink Fellow in Civil Engineering, he continued his studies in structural engineering at Lehigh University and obtained the Master of Science degree in civil engineering, June 1936. For two years thereafter the writer served at the same institution as instructor in civil engineering, simultaneously completing his resident graduate study, language requirements, and preliminary examination towards the doctorate in structural engineering. Two summer sessions in engineering mechanics at the University of Michigan and one in mathematics and physics at Columbia University constituted a small part of his program of work.

Work for five summers with the Schiller Construction Corp. and Carroll-McCreary Co., Inc., both of New York City, made up the practical experience he has had in construction. Since September 1938 he has held the position of instructor in theoretical and