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THERMAL STRESSES IN A SEMI-INFINITE HOLLOW CYLINDER

by

Hwa-Ping Lee

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ABSTRACT

A general method is presented to determine two dimensional thermal stresses in a thick-walled semi-infinite circular hollow cylinder subjected to an axisymmetric temperature distribution.

In the analysis, the problem was reduced to finding a solution due to the end effect by applying over the end section a normal loading which is equal to but opposite in direction to the self-equilibrating axial stresses produced in an infinitely long circular hollow cylinder under a given radial temperature distribution.

The method of solution was developed within the scope of Horvay's general analysis of variational approach. Modified Sadowsky-Sternberg type stress functions involving radial and axial functions in bi-product form were used in formulating the stress expressions. The radial functions were obtained by means of a set of differential equations in conjunction with equilibrium equations and prescribed lateral boundary conditions. Upon application of the variational method to the complementary energy principle, the Euler equations were obtained. As a results, the general form of the axial functions was determined with arbitrary constants to be specified by auxiliary conditions. The state of stress due to the end effect was obtained easily

in terms of these bi-product radial and axial functions. By superimposing the solution for the end effect on the corresponding stress components in an infinitely long circular hollow cylinder, the resultant stresses in a semi-infinite circular hollow cylinder were obtained.

The utilization of the developed method is advantageous from both conceptual and practical points of view. Several essential conclusions can be drawn as follows:

(1) The difficulties which would be encountered by the classical thermo-elasticity treatment of a semi-infinite hollow cylinder are overcome by this method. The end effect problem can be treated by this method separately with no simplifying assumptions.

(2) This analysis has developed a unique method of formulating the radial functions which are solutions to a set of differential equations obtained from the equilibrium conditions.

(3) The application of the variational approach which eliminates the preliminary restriction that the chosen stress functions satisfy the compatibility equations, reduces the problem partly to an eigenvalue problem. The resulting Euler equations are two fourth-order, linear, constant-coefficient differential equations. The solution of the Euler equations give the axial functions in the form of exponentially decaying expressions and are conceivable

on account of the Saint-Venant's principle, i.e., the self-equilibrating stress distribution applied to the end gives rise only to local effects.

(4) The numerical results show good qualitative agreement with the available experimental data.

(5) In addition to the thermal stress problems in semi-infinite hollow cylinders with any given radial temperature distribution, the method developed above can be applied to other axisymmetric problems (such as shrink-fit and punch problems) with non-uniform end tractions.

SYMBOLS

a_{ij} ($i, j = 1, 2, \dots, 6$)	Constant coefficients in the strain energy expression
b_n, c_n	Abbreviated constant coefficients
$C_{1,2,3,4}$	Undetermined constants associated with radial functions
E	Young's modulus of elasticity
$f(r), F(r)$	Radial functions of the bi-product variational stress functions
G	Shear modulus of elasticity
$g(r), G(r)$	Axial functions of the bi-product variational stress functions
$h_{0,1,2,3}$	Abbreviated constant coefficients of a quartic equation
$I(g, \dots, G'')$	Integrand of the strain energy expression subjected to variational operation
$k_{1,2,3,4}$	Constants of integration associated with radial functions
$K_1, z(r)$	Tentative radial functions
m, n, p, q M, N, P, Q	Undetermined constants associated with axial functions
r_o	Outer radius of a hollow cylinder
r, θ, z	Cylindrical coordinates
$S(r), S_f(r), S_F(r)$	Radial functions
$t(r), t_f(r), t_F(r)$	Radial functions
$T(r)$	Axisymmetric temperature function
ΔT	Temperature difference between inner and outer radii of a concentric hollow cylinder

U	Strain energy
U^*	$\frac{E}{2\pi} U$
$W(r), X(r)$ $Y(r), Z(r)$	Abbreviated radial functions of stress expressions
y	γ^2
α	Coefficient of linear thermal expansion
α_k	Real part of complex roots
β_k	Immaginary part of complex roots
γ_k	Complex roots ($\alpha_k \pm i\beta_k$)
δ	Symbol of the variational operation
ϵ_{ij} ($i, j = r, \theta, z$)	Strain components in cylindrical coordinates
ζ, η	Arbitrary constants
λ	Ratio of inner radius to outer radius of a concentric hollow cylinder
ν	Poisson's ratio
σ_{ij} ($i, j = r, \theta, z$)	Stress components in cylindrical coordinates *
$\varphi(r, z), \phi(r, z)$	Bi-product variational stress functions

* Note: The single subscripts are used in the un-abridged notations for stresses, thus $\sigma_r = \sigma_{rr}, \sigma_\theta = \sigma_{\theta\theta}, \sigma_z = \sigma_{zz}$, and $\tau = \sigma_{rz}$. Due to the axisymmetric nature of the problem $\sigma_{r\theta} = \sigma_{\theta z} = 0$.

I INTRODUCTION

Thermal stresses in a long concentric circular hollow cylinder subjected to an axisymmetric temperature field were first investigated by Lorenz (Ref. 1). The solution, however, does not apply to cylinders with free ends. Far away from the ends in the cylinder, the axial stress distribution is similar to the temperature profile; but at the ends the axial stresses must be zero so that the traction free boundary conditions over the ends are fulfilled. For the axial stress to change along the cylinder, an additional shear stress must be present. These conditions require the solution of a two-dimensional stress problem with a one-dimensional temperature distribution.

A rigorous solution of the end effect problem for a solid cylinder can be formulated, if one relates the stresses and displacements to the well-known Love function in the manner given by Love (Ref. 2). This technique gives rise to a new set of radially distributed end tractions which prevents the use of conventional Fourier expansion techniques. For the hollow cylinder a further complication is caused due to the additional boundary requirements arising from the inner curved boundary.

The end effect problem, therefore, has long withstood all attempts to develop an adequate approximate analysis.

A variational approach to the end effect problem of a solid cylinder was first employed by Horvay and his associates (Ref. 3, 4). Their solution was obtained in anticipation of a set of special self-equilibrating end tractions which, in turn, imply some special temperature profiles. These temperature profiles, however, do not commonly exist in the thermal stress problems, except for the heat generating case, and were chosen primarily to accommodate the special technique developed by the authors.

Timoshenko and Goodier (Ref. 5) treated a thin-walled concentric hollow cylinder, free from surface tractions and having a linear radial temperature distribution, as a bar on an elastic foundation. It was concluded that the maximum tensile stress which occurs on the outer boundary at the free end of the cylinder is twenty-five percent greater than that which occurs at points remote from the ends. The solution can not, however, be extended to include thick-walled cylinders.

Recently, Bellamy (Ref. 6) obtained an approximate solution of a hollow cylinder of finite length for the case of a linear temperature field. However, Bellamy's assumption that the change in the radial stress due to the end effect is zero, which leads to the result that the circumferential stresses due to the end effect on both lateral curved boundaries also become zero, is unjustified

from stress equilibrium considerations.

Although, the end effect problem has broad applications, such as in thermal stress problems, shrink-fit problems, punch problems, etc., up to the present time no satisfactory work has been done on the thick-walled hollow cylinders because of the analytical difficulties involved.

It is the purpose of this analytic investigation to develop a method of solution for determining the thermal stresses in a semi-infinite circular hollow cylinder with a given temperature distribution. This method, however, can be extended readily to include other axisymmetric problems with z-dependent boundary conditions.

The solution that follows will be based on the usual assumption of linearly elastic, homogeneous and isotropic material with temperature-independent thermo-elastic properties.

The problem of determining thermal stresses in a semi-infinite circular hollow cylinder subjected to an axisymmetric temperature field will be treated by applying over the end section a normal loading which is equal to but opposite in direction to the self-equilibrating axial stresses produced in an infinitely long circular hollow cylinder under a given radial temperature distribution. This procedure reduces the thermal stress problem in a

thick-walled semi-infinite circular hollow cylinder to an isothermal problem in which the end tractions are the only external forces.

The method of solution will be developed within the scope of Horvay's general analysis of variational approach in which the selected stress functions do not have to satisfy the compatibility equations. Modified Sadowsky-Sternberg type stress functions (Ref. 7) in bi-product functions will be used in formulating the stress expressions. A unique procedure will be developed to determine the radial functions by means of a set of differential equations in conjunction with equilibrium equations and prescribed boundary conditions. Upon application of the variational method to the complementary energy principle, the Euler equations, consisting of two fourth-order linear differential equations with constant coefficients, will be obtained. These simultaneous differential equations determine the axial functions in a general form where the arbitrary integration constants are determined by using the auxiliary conditions. The stress components can be obtained easily in terms of these bi-product radial and axial functions. By superimposing the solution of the end effect on the corresponding stress components in an infinitely long circular hollow cylinder, the resultant stresses in a semi-infinite circular

hollow cylinder can be obtained.

As a numerical example a second degree polynomial is chosen as the temperature profile and the resulting stresses are plotted as functions of r and z .

II THEORETICAL ANALYSIS

1. Statement of Problem

Thermal stresses in an infinitely long circular hollow cylinder subjected to an axisymmetric temperature field, but free from surface tractions on its lateral curved boundaries and also having zero body force are given by following expressions, (Ref. 1, 5):

$$(1.1) \quad \left\{ \begin{array}{l} \sigma_r^\infty = \frac{\alpha E}{(1-\nu)r^2} \left[\frac{r^2 - \lambda^2}{1 - \lambda^2} \int_\lambda^1 T(r) r dr - \int_\lambda^r T(r) dr \right] \\ \sigma_\theta^\infty = \frac{\alpha E}{(1-\nu)r^2} \left[\frac{r^2 + \lambda^2}{1 - \lambda^2} \int_\lambda^1 T(r) r dr + \int_\lambda^r T(r) dr - T(r) r^2 \right] \\ \sigma_z^\infty = \frac{\alpha E}{(1-\nu)} \left[\frac{2}{1 - \lambda^2} \int_\lambda^1 T(r) r dr - T(r) \right] \end{array} \right.$$

where σ_r^∞ , σ_θ^∞ , σ_z^∞ denote the normal stresses in the radial, circumferential and axial directions, respectively. Figure 1 gives the notation used in this representation. The fractional parameter λ is the ratio of the inner radius of the cylinder to the outer radius. The unit length is taken as the outer radius. The superscript ∞ denotes the state of stress without the influence of the free end effect, i.e., in the portion far away from the end of the hollow cylinder.

These equations fail, however, to satisfy the boundary condition in normal stress over the end face of a semi-infinite circular hollow cylinder. These boundary conditions can be written as

$$\begin{cases} \sigma_z^\infty = 0 \\ \tau_z^\infty = 0 \end{cases} \quad \text{over } z = 0$$

To satisfy these boundary conditions, the total stresses in a semi-infinite hollow cylinder will be represented as the sum of two stress states given by

$$(1.2) \quad \sigma_{ij}^T = \sigma_{ij}^\infty + \sigma_{ij}$$

where $i, j = r, \theta, z$; σ_{ij}^∞ are the stresses in an infinitely long hollow cylinder subjected to a radial temperature distribution $T(r)$; and σ_{ij}^T are the stresses in the semi-infinite hollow cylinder obtained by using the following boundary conditions

$$(1.3) \quad \begin{cases} \sigma_z(r, 0) = -\sigma_z^\infty, & \tau(r, 0) = 0 \\ \sigma_r(1, z) = \sigma_r(\lambda, z) = \tau(1, z) = \tau(\lambda, z) = 0 \end{cases}$$

The stresses σ_{ij}^∞ are given by (1.1), thus the solution of the problem is reduced to finding the stresses, σ_{ij} , solely due to the end effect.

2. Boundary Conditions and Variational Stress Functions

The problem of determining stress distribution in a semi-infinite circular hollow cylinder has been reduced to obtaining the stresses σ_{ij} in (1.2) due to the end effect of an elastic hollow cylinder which is free from tractions on its curved boundaries. The stress components σ_{ij} , therefore, have to satisfy the following boundary conditions:

$$(2.1) \quad \begin{cases} \sigma_r(1, z) = 0 \\ \sigma_r(\lambda, z) = 0 \end{cases}$$

$$(2.2) \quad \begin{cases} \tau(1, z) = 0 \\ \tau(\lambda, z) = 0 \end{cases}$$

They must also satisfy equations of equilibrium which, for an axisymmetric body and in the absence of body force, are of the form

$$(2.3) \quad \begin{cases} \frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \\ \frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau}{\partial r} + \frac{\tau}{r} = 0 \end{cases}$$

At the end, $z = 0$, the cylinder is subjected to a radially symmetric normal end traction $\sigma_z(r, 0)$ which is characterized as self-equilibrating, i.e., which satisfies

$$(2.4) \quad \int_{\lambda}^1 \sigma_z(r, 0) r dr = 0$$

Let us assume that the problem can be formulated in terms of two stress functions, $\varphi(r, z)$ and $\phi(r, z)$. While the reason and derivation of the desired stress expressions will be cited in Appendix A, it may be verified by direct substitution that the equilibrium equations (2.3) are satisfied identically, if we write

$$(2.5) \quad \left\{ \begin{array}{l} \sigma_r = 5(\varphi + r^4 \varphi_r) - \nu(2\phi_{zz} + r^4 \phi_{rzz}) \\ \sigma_\theta = 5[r^5 \varphi_{rr} + (1+5r^3)r\varphi_r + \varphi] + \\ \quad - r(\varphi_{rrzz} + r\phi_{rrzz} + 2\phi_{rzz}) + \\ \quad - \nu[r^5 \phi_{rrzz} + (2+5r^3)r\phi_{rzz} + 2\phi_{zz}] \\ \sigma_z = \varphi_{rrr} + \frac{1}{r}\varphi_{rr} + r\phi_{rrr} + 4\phi_{rr} + \frac{2}{r}\phi_r \\ \tau = -\varphi_{rrz} - r\phi_{rrz} - 2\phi_{rz} \end{array} \right.$$

where the conventional subscript notation for partial derivatives is used for brevity; so that, for example, φ_r is written for $\frac{\partial \varphi}{\partial r}$. Let us seek a product solution

$$(2.6) \quad \left\{ \begin{array}{l} \varphi(r, z) = f(r) g(z) \\ \phi(r, z) = F(r) G(z) \end{array} \right.$$

where $f(r)$ and $F(r)$ will be selected so that the radial

boundary conditions (2.1) are automatically satisfied. By introducing the notation:

$$(2.7) \quad \left\{ \begin{array}{l} f' = \frac{df}{dr}, \quad g' = \frac{dg}{dz} \\ F' = \frac{dF}{dr}, \quad G' = \frac{dG}{dz} \\ t_f = f'' \\ t_F = rF'' + 2F' \\ s_f = t_f' + \frac{1}{r} t_f = f''' + \frac{1}{r} f'' \\ s_F = t_F' + \frac{1}{r} t_F = rF''' + 4F'' + \frac{2}{r} F' \end{array} \right.$$

the stress expressions (2.5) give

$$(2.8) \quad \left\{ \begin{array}{l} \sigma_r = 5(f + r^4 f')g - \nu(2F + r^4 F')G'' \\ \sigma_\theta = 5[r^5 f'' + (1+5r^3)rf' + f]g - rf''g'' + \\ \quad - \left\{ r(rF'' + 2F') + \nu[rF'' + (5r^3 + 2)rF' + 2F] \right\} G'' \\ \sigma_z = s_f g + s_F G \\ \tau = -t_f g' - t_F G' \end{array} \right.$$

3. Determination of Radial Functions

Stress expressions (2.8) will be fruitful and available for numerical treatment only if the basic functions in r and z , which have formed the bi-product functions, can be obtained appropriately. For radial functions $f(r)$ and $F(r)$, it is clear that a necessary condition for the success of this procedure is the existence of all the derivatives which are interrelated in accordance with (2.7). It is necessary that these radial functions with their derivatives forming the stress expressions (2.8) satisfy the prescribed radial boundary condition (2.1), and the equilibrium equations (2.3).

An examination of (1.3) and the third equation in (2.8) suggests the choice of S_f and S_F in the following form

$$(3.1) \quad S(r) = S_f(r) = S_F(r) = \sigma_z(r, 0)$$

which gives

$$(3.2) \quad \begin{aligned} S(r) &= -\sigma_z^\infty(r) \\ &= -\frac{2E}{1-\nu} \left[\frac{2}{1-\lambda^2} \int_\lambda^1 T(r) r dr - T(r) \right] \end{aligned}$$

where $T(r)$, the radial temperature distribution, may be entered with the exact solution which is obtainable from the classical method with the given temperature boundary conditions of the system. However, there are times that a

temperature approximation in the form of a polynomial is much more convenient than the direct use of an accurate but complicated expression of an exact solution, such as in the case of a transient temperature distribution. The number of terms of the chosen polynomial depends upon the degree of accuracy required.

The self-equilibrating condition (2.4) being inserted from (3.1) yields

$$(3.3) \quad \int_{\lambda}^{\prime} S(r) r dr = 0$$

An examination of the last two expressions of (2.7) suggests the choice of $t_f(r)$ and $t_F(r)$ as

$$(3.4) \quad t(r) = t_f(r) = t_F(r)$$

hence the axial and shear stresses in (2.8) can be re-written as

$$(3.5) \quad \begin{cases} \sigma_z = S (g + G) \\ \tau = -t(g' + G') \end{cases}$$

The substitution from σ_z in the second equation of (2.3) leads to a partial differential equation of the form

$$\frac{\partial \tau}{\partial r} + \frac{\tau}{r} = -S (g' + G')$$

which yields

$$(3.6) \quad \mathcal{C} = -(g' + G') \frac{1}{r} \int_{\lambda}^r S(r) r dr$$

where the limits of the integral are so chosen that $\mathcal{C}$ satisfies both of the radial boundary conditions given by (2.2).

A comparison of the second equation of (3.5) with (3.6) leads to the conclusion

$$(3.7) \quad t(r) = \frac{1}{r} \int_{\lambda}^r S(r) r dr$$

On the other hand, from (2.7) and (3.4) we have

$$t(r) = f''$$

which gives

$$(3.8) \quad \begin{aligned} f'(r) &= \int t(r) dr + C_1 \\ &= \int \frac{1}{r} \int_{\lambda}^r S(r) r dr dr + C_1 \end{aligned}$$

and

$$(3.9) \quad f(r) = \iint \frac{1}{r} \int_{\lambda}^r S(r) r dr dr dr + C_1 r + C_2$$

where C_1 and C_2 are constants of integration.

The function $F(r)$ can be obtained by solving a second order linear non-homogeneous differential equation from (2.7), which is of the form

$$(3.10) \quad r F'' + 2 F' = t(r)$$

Let F_c and F_p be the complementary and particular solutions of the differential equation, respectively; so that the complementary solution is

$$(3.11) \quad F_c = k_1 r^{-1} + k_2$$

where k_1 and k_2 are constants of integration.

By using the method of variation of parameters, the particular solution can be written as

$$(3.12) \quad F_p = K_1(r) r^{-1} + K_2(r)$$

where $K_1(r)$ and $K_2(r)$ are tentative functions in r being substituted for k_1 and k_2 in (3.11) and obtained as

$$(3.13) \quad \begin{aligned} K_1(r) &= - \int t(r) r dr + k_3 \\ &= - \iint_{\lambda}^r S(r) r dr dr + k_3 \end{aligned}$$

$$(3.14) \quad \begin{aligned} K_2(r) &= \int t(r) dr + k_4 \\ &= \int \frac{1}{r} \int_{\lambda}^r S(r) r dr dr + k_4 \end{aligned}$$

where k_3 and k_4 are constants of integration. Hence

$$(3.15) \quad \begin{aligned} F(r) &= F_c + F_p \\ &= \frac{-1}{r} \iint_{\lambda}^r S(r) r dr dr + \int \frac{1}{r} \int_{\lambda}^r S(r) r dr dr + \\ &\quad + C_3 + \frac{C_4}{r} \end{aligned}$$

where C_3 and C_4 are integration constants.

The arbitrary constants C_1 , C_2 , C_3 and C_4 involved in the radial functions $f(r)$ and $F(r)$ have yet to be determined. It is readily seen from the prescribed boundary conditions

$$\begin{cases} \sigma_r(1, z) = 0 \\ \sigma_r(\lambda, z) = 0 \end{cases}$$

that the radial stresses must be zero on both lateral boundaries for all values of z . This can be achieved by writing the coefficients of the axial functions $g(z)$ and $G''(z)$ for $r = 1$ and $r = \lambda$ in the first equation of (2.8). Thus, the arbitrary constants C_1 , C_2 , C_3 and C_4 are determined by solving two sets of simultaneous equations which are given by

$$(3.16) \quad r^4 f' + f = 0 \quad \text{at } r = 1, \lambda$$

and

$$(3.17) \quad r^4 F' + 2F = 0 \quad \text{at } r = 1, \lambda$$

4. Determination of Axial Functions by Variational Method — An Application of the Theorem of Minimum Complementary Energy

The method of our variational approach is an application of the Theorem of Minimum Complementary Energy, which states "of all the stress states satisfying the conditions of equilibrium in the interior and on that portion of the surface where the surface forces are prescribed, the actual state of stress is such as to minimize the complementary energy of the elastic system ". Here, the admissible set of stresses, σ_{ij} , satisfy the equilibrium equations as well as the prescribed boundary conditions, the variational equation is then equivalent to the conditions of compatibility or the stress-displacement relations. Thus, expressing the strain energy expression in terms of the stresses as given by (2.8), and taking integration over the variable r , one obtains an expression which contains the axial functions $g(z)$ and $G(z)$ and their derivatives but does not contain the variable z explicitly. By taking variation of the resulted strain energy and equating to zero, i.e., $\delta U = 0$, the Euler equations which are two fourth-order differential equations in $g(z)$ and $G(z)$ with constant coefficients are obtained.

To start the variational approach, we write the strain energy expression which, in cylindrical coordinates, reads

$$\begin{aligned}
 (4.1) \quad U &= \int_0^{2\pi} \int_0^\infty \int_\lambda^1 \frac{1}{2G} \left[\frac{1}{2(1+\nu)} (\sigma_r + \sigma_\theta + \sigma_z)^2 + \right. \\
 &\quad \left. - \sigma_r \sigma_\theta - (\sigma_r + \sigma_\theta) \sigma_z + \tau^2 \right] r dr dz d\theta \\
 &= \frac{2\pi}{E} \int_0^\infty \int_\lambda^1 \left[\frac{1}{2} (\sigma_r + \sigma_\theta + \sigma_z)^2 + (1+\nu) \left\{ -\sigma_r \sigma_\theta + \right. \right. \\
 &\quad \left. \left. - (\sigma_r + \sigma_\theta) \sigma_z + \tau^2 \right\} \right] r dr dz
 \end{aligned}$$

Substitution of the stress components (2.8) into (4.1) and simplifying gives an expression in the form

$$(4.2) \quad U = \frac{2\pi}{E} \int_0^\infty I(g, g', g'', G, G', G'') dz$$

where

$$\begin{aligned}
 (4.3) \quad I(g, g', g'', G, G', G'') &= \int_\lambda^1 \left[\frac{1}{2} (\sigma_r + \sigma_\theta + \sigma_z)^2 + \right. \\
 &\quad \left. + (1+\nu) \left\{ -\sigma_r \sigma_\theta - (\sigma_r + \sigma_\theta) \sigma_z + \tau^2 \right\} \right] r dr \\
 &= a_{11}g^2 + a_{13}gg'' + a_{14}gG + a_{16}gG'' + a_{22}g'^2 + \\
 &\quad + a_{25}g'G' + a_{33}g''^2 + a_{34}g''G + a_{36}g''G'' + \\
 &\quad + a_{44}G^2 + a_{46}GG'' + a_{55}G'^2 + a_{66}G''^2
 \end{aligned}$$

and the coefficients a_{ij} ($i, j = 1, 2, \dots, 6$) are defined as follows:

$$\begin{aligned}
a_{11} &= \frac{1}{2} \int_{\lambda}^{\prime} s^2 r dr + 5 \nu \int_{\lambda}^{\prime} s [2f + f'(1+6r^3)r + tr^5] r dr + \\
&+ 25 \int_{\lambda}^{\prime} [(1+\nu)f \{f + f'(1+6r^3)r + tr^5\} + \\
&\quad + f'^2 \{ \frac{1}{2} + (5+\nu)r^3 + (13+5\nu)r^6 \} r^2 + \\
&\quad + f' t \{ 1 + (5+\nu)r^3 \} r^6 + \frac{1}{2} t^2 r^{10}] r dr \\
a_{13} &= \nu \int_{\lambda}^{\prime} s t r^2 dr - 5 \int_{\lambda}^{\prime} t [(1-\nu)f + \\
&\quad + f' \{ 1 + (5-\nu)r^3 \} r + tr^5] r^2 dr \\
a_{14} &= \int_{\lambda}^{\prime} s^2 r dr - 5 \nu \int_{\lambda}^{\prime} s [2f + f'(1+6r^3)r + tr^5] r dr \\
a_{16} &= \nu \int_{\lambda}^{\prime} s (2F' + F''r) r^2 dr + \\
(4.4) \quad &+ \nu^2 \int_{\lambda}^{\prime} s [4F + 2F'(1+3r^3)r + F''r^5] r dr + \\
&- 5 \int_{\lambda}^{\prime} [\{ (1-\nu)f + tr^5 \} (2F' + F''r) + \\
&\quad + f'(2F' + F''r^2) \{ 1 + (5-\nu)r^3 \} r] r^2 dr + \\
&- 5 \nu \int_{\lambda}^{\prime} [(1-\nu)f \{ 4F + 2F'(1+3r^3)r + F''r^5 \} r + \\
&\quad + 2(1-\nu)F \{ f'(1+6r^3) + tr^4 \} r^2 + \\
&\quad + f'F' \{ 2 + 3(5-\nu)r^3 + 2(13-5\nu)r^6 \} r^3 + \\
&\quad + tF' \{ 2 + (5-\nu)r^3 \} r^7 + \\
&\quad + f'F'' \{ 1 + (5-\nu)r^3 \} r^7 + tF''r^{11}] dr
\end{aligned}$$

$$\begin{aligned}
 (4.4) \quad \left\{ \begin{aligned}
 a_{22} &= (1+\nu) \int_{\lambda}^1 t^2 r dr \\
 a_{25} &= 2(1+\nu) \int_{\lambda}^1 t^2 r dr \\
 a_{33} &= \frac{1}{2} \int_{\lambda}^1 t^2 r^3 dr \\
 a_{34} &= \nu \int_{\lambda}^1 t s r^2 dr \\
 a_{36} &= \int_{\lambda}^1 t [2F' + F''r] r^3 dr + \\
 &\quad + \nu \int_{\lambda}^1 [2(1-\nu)F + r\{2 + (5-\nu)r^3\}F' + \\
 &\quad + F''r^5] t r^2 dr \\
 a_{44} &= \frac{1}{2} \int_{\lambda}^1 s^2 r dr \\
 a_{46} &= \nu \int_{\lambda}^1 s [2F' + F''r] r^2 dr + \\
 &\quad + \nu^2 \int_{\lambda}^1 s [4F + 2F'(1+3r^3)r + F''r^5] r dr \\
 a_{55} &= (1+\nu) \int_{\lambda}^1 t^2 r dr \\
 a_{66} &= \nu^2 \int_{\lambda}^1 [2(1-\nu)F\{2F + 2F'(1+3r^3)r + F''r^5\} + \\
 &\quad + F'^2\{2 + 2(5-\nu)r^3 + (13-5\nu)r^6\}r^2 + \\
 &\quad + F'F''\{2 + (5-\nu)r^3\}r^6 + \frac{1}{2}F''^2 r^{10}] r dr + \\
 &\quad + \nu \int_{\lambda}^1 2(1-\nu)F(2F' + F''r) + 2F'^2\{(5-\nu)r^3 + 4\}r + \\
 &\quad + F'F''\{(7-\nu)r^3 + 6\}r^2 + F''^2(1+r^3)r^3] r^2 dr
 \end{aligned} \right.
 \end{aligned}$$

Defining

$$U^* = \frac{E}{2\pi} U$$

equation (4.2) then becomes

$$U^* = \int_0^\infty I(g, g', g'', G, G', G'') dz$$

The necessary condition that the integral U^* be stationary is that its (first) variation vanishes (Ref. 9), so that

$$(4.5) \quad \delta U^* = \delta \int_0^\infty I(g, g', g'', G, G', G'') dz = 0$$

where z is the independent variable.

According to the definition of variation and its commutative property, (4.5) is equivalent to

$$\begin{aligned} (4.6) \quad \delta U^* &= \int_0^\infty \delta I(g, g', g'', G, G', G'') dz \\ &= \int_0^\infty \left(\frac{\partial I}{\partial g} \delta g + \frac{\partial I}{\partial g'} \delta g' + \frac{\partial I}{\partial g''} \delta g'' + \right. \\ &\quad \left. + \frac{\partial I}{\partial G} \delta G + \frac{\partial I}{\partial G'} \delta G' + \frac{\partial I}{\partial G''} \delta G'' \right) dz \\ &= 0 \end{aligned}$$

It is to be noted that g, g', g'', G, G' and G'' are all functions of z , and I does not depend explicitly on z .

Moreover, in the differentiations with respect to g, g', g'' ,

G , G' and G'' , all of six variables comprised in the integrand I have been treated as though they are independent.

Replacing $\delta g'$ by $\frac{d}{dz}(\delta g)$, in accordance with rules of the variational operation, those terms consisting of the variational derivatives can be successively integrated by parts. The general procedure may be illustrated by considering the treatment of two terms in (4.6).

The second term in (4.6) can be written as

$$(4.7) \quad \int_0^\infty \frac{\partial I}{\partial g'} \delta g' dz = \int_0^\infty \left[\frac{\partial I}{\partial g'} \frac{d}{dz} \delta g \right] dz$$

which, on performing integration by parts, yields

$$\begin{aligned} &= \int_0^\infty \frac{d}{dz} \left[\frac{\partial I}{\partial g'} \delta g \right] dz - \int_0^\infty \left[\delta g \frac{d}{dz} \frac{\partial I}{\partial g'} \right] dz \\ &= \left[\frac{\partial I}{\partial g'} \delta g \right]_0^\infty - \int_0^\infty \left[\frac{d}{dz} \frac{\partial I}{\partial g'} \right] \delta g dz \end{aligned}$$

Similarly, the term in (4.6) can be written as

$$(4.8) \quad \int_0^\infty \frac{\partial I}{\partial g''} \delta g'' dz = \int_0^\infty \left[\frac{\partial I}{\partial g''} \frac{d}{dz} \delta g' \right] dz$$

which, on performing integration by parts, yields

$$\begin{aligned} &= \int_0^\infty \frac{d}{dz} \left[\frac{\partial I}{\partial g''} \delta g' \right] dz - \int_0^\infty \left[\frac{d}{dz} \frac{\partial I}{\partial g''} \right] \delta g' dz \\ &= \left[\frac{\partial I}{\partial g''} \delta g' \right]_0^\infty - \int_0^\infty \left[\frac{d}{dz} \frac{\partial I}{\partial g''} \right] \frac{d}{dz} \delta g dz \end{aligned}$$

On performing integration by parts once again, and

considering $\frac{d}{dz} \frac{\partial I}{\partial g''}$ as a variable, one has

$$\begin{aligned} \int_0^\infty \frac{\partial I}{\partial g'} \delta g'' dz &= \left[\frac{\partial I}{\partial g''} \delta g' \right]_0^\infty - \int_0^\infty \frac{d}{dz} \left[\left(\frac{d}{dz} \frac{\partial I}{\partial g''} \right) \delta g \right] dz + \\ &\quad + \int_0^\infty \left[\delta g \frac{d}{dz} \left(\frac{d}{dz} \frac{\partial I}{\partial g''} \right) \right] dz \\ &= \left[\frac{\partial I}{\partial g''} \delta g' - \frac{d}{dz} \frac{\partial I}{\partial g''} \delta g \right]_0^\infty + \\ &\quad + \int_0^\infty \left[\frac{d^2}{dz^2} \frac{\partial I}{\partial g''} \right] \delta g dz \end{aligned}$$

The terms involving the "variables" G , G' and G'' in (4.6) are correspondingly similar to those involving g , g' and g'' . On substituting from (4.7) and (4.8) with the corresponding terms in G 's in (4.6), one obtains

$$\begin{aligned} (4.9) \quad \delta U^* &= \int_0^\infty \left[\left(\frac{\partial I}{\partial g} - \frac{d}{dz} \frac{\partial I}{\partial g'} + \frac{d^2}{dz^2} \frac{\partial I}{\partial g''} \right) \delta g + \right. \\ &\quad \left. + \left(\frac{\partial I}{\partial G} - \frac{d}{dz} \frac{\partial I}{\partial G'} + \frac{d^2}{dz^2} \frac{\partial I}{\partial G''} \right) \delta G \right] dz + \\ &\quad + \left[\left(\frac{\partial I}{\partial g'} - \frac{d}{dz} \frac{\partial I}{\partial g''} \right) \delta g + \frac{\partial I}{\partial g''} \delta g' \right]_0^\infty + \\ &\quad + \left[\left(\frac{\partial I}{\partial G'} - \frac{d}{dz} \frac{\partial I}{\partial G''} \right) \delta G + \frac{\partial I}{\partial G''} \delta G' \right]_0^\infty \\ &= 0 \end{aligned}$$

By direct substitution of I from (4.3) and making some reductions, (4.9) becomes

$$\begin{aligned}
 (4.10) \quad & \int_0^{\infty} \left\{ 2[a_{11}g + (a_{13}-a_{22})g'' + a_{33}g'''] + \right. \\
 & \quad + [a_{14}G + (a_{16}-a_{25}+a_{34})G'' + a_{36}G'''] \} \delta g + \\
 & \quad + \left\{ [a_{14}g + (a_{34}-a_{25}+a_{16})g'' + a_{36}g'''] + \right. \\
 & \quad + 2[a_{44}G + (a_{46}-a_{55})G'' + a_{66}G'''] \} \delta G \Big] dz + \\
 & + \left\{ [(2a_{22}-a_{13})g' - 2a_{33}g''' + (a_{25}-a_{34})G' - a_{36}G'''] \delta g + \right. \\
 & \quad + (a_{13}g + 2a_{33}g'' + a_{34}G + a_{36}G'') \delta g' + \\
 & \quad + [(a_{25}-a_{16})g' - a_{36}g''' + (2a_{55}-a_{46})G' - 2a_{66}G'''] \delta G + \\
 & \quad + (a_{16}g + a_{36}g'' + a_{46}G + 2a_{66}G'') \delta G' \Big\} \Big|_0^{\infty} \\
 & = 0
 \end{aligned}$$

Since the variations δg and δG are independent of each other, i.e., g and G can be varied independently with respect to z , the coefficients of δg and δG in the foregoing expression (4.10) must identically vanish in the prescribed domain, i.e., over the range of z from 0 to ∞ . Provided a vanishing condition is imposed to the second part in (4.10), i.e., the remaining terms outside the integral sign, the first part gives the Euler equations and the second part is known to be the natural (auxiliary) boundary condition.

From (4.10) the following natural boundary condition is obtained.

$$\begin{aligned}
 (4.11) \quad & \left[\left(\frac{\partial I}{\partial g'} - \frac{d}{dz} \frac{\partial I}{\partial g''} \right) \delta g + \frac{\partial I}{\partial g''} \delta g' + \left(\frac{\partial I}{\partial G'} - \frac{d}{dz} \frac{\partial I}{\partial G''} \right) \delta G + \frac{\partial I}{\partial G''} \delta G' \right] \\
 & = \left[(2a_{22} - a_{13})g' - 2a_{33}g''' + (a_{25} - a_{34})G' - a_{36}G''' \right] \delta g + \\
 & + \left[a_{13}g + 2a_{33}g'' + a_{34}G + a_{36}G'' \right] \delta g' + \\
 & + \left[(a_{25} - a_{16})g' - a_{36}g''' + (2a_{55} - a_{46})G' - 2a_{66}G''' \right] \delta G + \\
 & + \left[a_{16}g + a_{36}g'' + a_{46}G + 2a_{66}G'' \right] \delta G' \\
 & = 0
 \end{aligned}$$

and the Euler equations as

$$(4.12) \quad \begin{cases} 2[a_{11}g + (a_{13} - a_{22})g'' + a_{33}g^{IV}] + \\ \quad + [a_{14}G + (a_{16} - a_{25} + a_{34})G'' + a_{36}G^{IV}] = 0 \\ [a_{14}g + (a_{34} - a_{25} + a_{16})g'' + a_{36}g^{IV}] + \\ \quad + 2[a_{44}G + (a_{46} - a_{55})G'' + a_{66}G^{IV}] = 0 \end{cases}$$

The Euler equations are two fourth-order differential equations in $g(z)$ and $G(z)$ with constant coefficients as defined by (4.4). Let us choose the solution for these equations as

$$(4.13) \quad \begin{cases} g(z) = \xi e^{-\gamma z} \\ G(z) = \eta e^{-\gamma z} \end{cases}$$

where γ , in general, is a complex number. Substituting in (4.12) and dividing by $e^{-\gamma z}$, one obtains

$$\begin{aligned} & 2 [a_{11} + (a_{13} - a_{22}) \gamma^2 + a_{33} \gamma^4] \zeta + \\ & + [a_{14} + (a_{16} - a_{25} + a_{34}) \gamma^2 + a_{36} \gamma^4] \eta = 0 \\ & [a_{14} + (a_{16} - a_{25} + a_{34}) \gamma^2 + a_{36} \gamma^4] \zeta + \\ & + 2 [a_{44} + (a_{46} - a_{55}) \gamma^2 + a_{66} \gamma^4] \eta = 0 \end{aligned}$$

This set of two homogeneous equations in the arbitrary constants ζ and η possesses non-trivial solutions if, and only if, the determinant of the coefficients vanishes. Thus the characteristic equations of the system, which determine the eigen-values γ_k , become

$$\begin{vmatrix} 2 [a_{11} + (a_{13} - a_{22}) \gamma^2 + a_{33} \gamma^4] & [a_{14} + (a_{16} - a_{25} + a_{34}) \gamma^2 + a_{36} \gamma^4] \\ [a_{14} + (a_{16} - a_{25} + a_{34}) \gamma^2 + a_{36} \gamma^4] & 2 [a_{44} + (a_{46} - a_{55}) \gamma^2 + a_{66} \gamma^4] \end{vmatrix} = 0$$

or by expanding, one finds

$$\begin{aligned} (4.14) \quad & (4a_{33}a_{66} - a_{36}^2) \gamma^8 + \\ & + \left\{ 4[a_{33}(a_{46} - a_{55}) + a_{66}(a_{13} - a_{22})] + \right. \\ & \left. - 2a_{36}(a_{16} - a_{25} + a_{34})^2 \right\} \gamma^6 + \end{aligned}$$

$$\begin{aligned}
& + \left\{ 4[a_{11}a_{66} + (a_{13}-a_{22})(a_{46}-a_{55}) + a_{33}a_{44}] + \right. \\
& \quad \left. - 2a_{36}a_{14} - (a_{16}-a_{25}+a_{34})^2 \right\} \gamma^4 + \\
& + \left\{ 4[a_{11}(a_{46}-a_{55}) + a_{44}(a_{13}-a_{22})] + \right. \\
& \quad \left. - 2a_{14}(a_{16}-a_{25}+a_{34}) \right\} \gamma^2 + \\
& + (4a_{11}a_{44} - a_{14}^2) \\
& = 0
\end{aligned}$$

Equation (4.14) is reducible to a quartic equation if one makes following substitution

$$(4.15) \quad y = \gamma^2$$

and (4.14) can be re-written, on deviding throughout by the factor $(4a_{33}a_{66} - a_{36}^2)$, as

$$(4.16) \quad y^4 + h_3 y^3 + h_2 y^2 + h_1 y + h_0 = 0$$

where

$$(4.17) \quad \left\{ \begin{aligned} h_3 &= \left\{ 4[a_{33}(a_{46}-a_{55}) + a_{66}(a_{13}-a_{22})] + \right. \\ &\quad \left. - 2a_{36}(a_{16}-a_{25}+a_{34}) \right\} / (4a_{33}a_{66} - a_{36}^2) \\ h_2 &= \left\{ 4[a_{11}a_{66} + (a_{13}-a_{22})(a_{46}-a_{55}) + a_{33}a_{44}] + \right. \\ &\quad \left. - 2a_{14}a_{36} - (a_{16}-a_{25}+a_{34})^2 \right\} / (4a_{33}a_{66} - a_{36}^2) \end{aligned} \right.$$

$$\left\{ \begin{aligned} h_1 &= \left\{ 4[a_{11}(a_{46}-a_{55}) + a_{44}(a_{13}-a_{22})] + \right. \\ &\quad \left. - 2a_{14}(a_{16}-a_{25}+a_{34}) \right\} / (4a_{33}a_{66}-a_{36}^2) \\ h_0 &= (4a_{11}a_{44}-a_{14}^2) / (4a_{33}a_{66}-a_{36}^2) \end{aligned} \right.$$

When the stress expressions are found as given by (2.5), in general, the solution of (4.14) is of the following form

$$\gamma = \pm(\alpha_1 \pm i\beta_1) \quad \text{and} \quad \pm(\alpha_2 \pm i\beta_2)$$

Since $g(z)$ and $G(z)$ vanish as $z \rightarrow \infty$, in (4.13) γ can have only positive real parts. Making use of Euler's formula

$$e^{i\theta} = \cos\theta + i \sin\theta$$

one finds that (4.13) can be expressed in the form

$$(4.18) \quad \left\{ \begin{aligned} g(z) &= e^{-\alpha_1 z} (m \cos(\beta_1 z) + n \sin(\beta_1 z)) + \\ &\quad + e^{-\alpha_2 z} (p \cos(\beta_2 z) + q \sin(\beta_2 z)) \\ G(z) &= e^{-\alpha_1 z} (M \cos(\beta_1 z) + N \sin(\beta_1 z)) + \\ &\quad + e^{-\alpha_2 z} (P \cos(\beta_2 z) + Q \sin(\beta_2 z)) \end{aligned} \right.$$

where m, n, p, q, M, N, P and Q are real arbitrary constants yet to be determined.

5. Evaluation of Arbitrary Constants for Axial Functions

We need eight equations in terms of those eight arbitrary constants appearing in the expressions of the axial functions which, as is clear from (4.18), are of the form

$$(5.1) \quad g(z) = e^{-\alpha_1 z} (m \cos(\beta_1 z) + n \sin(\beta_1 z)) + \\ + e^{-\alpha_2 z} (p \cos(\beta_2 z) + q \sin(\beta_2 z))$$

and

$$(5.2) \quad G(z) = e^{-\alpha_1 z} (M \cos(\beta_1 z) + N \sin(\beta_1 z)) + \\ + e^{-\alpha_2 z} (P \cos(\beta_2 z) + Q \sin(\beta_2 z))$$

By using the boundary conditions at $z = 0$, i.e., $\sigma_z(r, 0) = -\sigma_z^\infty(r)$, and $\tau(r, 0) = 0$, and the equations (3.1) and (3.5), we can write

$$(5.3) \quad g(0) + G(0) = 1$$

and

$$(5.4) \quad g'(0) + G'(0) = 0$$

By taking variations of (5.3) and (5.4), we obtain

$$(5.5) \quad \delta g(0) = -\delta G(0)$$

and

$$(5.6) \quad \delta g'(0) = -\delta G'(0)$$

Rearranging the natural boundary conditions given by (4.11) in the following form

$$\left[\left(\frac{\partial I}{\partial g^i} - \frac{d}{dz} \frac{\partial I}{\partial g^i} \right) \delta g + \left(\frac{\partial I}{\partial G^i} - \frac{d}{dz} \frac{\partial I}{\partial G^i} \right) \delta G \right] + \left[\frac{\partial I}{\partial g^i} \delta g^i + \frac{\partial I}{\partial G^i} \delta G^i \right] = 0$$

and using (5.5) and (5.6), we find

$$(5.7) \quad \frac{\partial I}{\partial g^i} - \frac{d}{dz} \frac{\partial I}{\partial g^i} = \frac{\partial I}{\partial G^i} - \frac{d}{dz} \frac{\partial I}{\partial G^i}$$

and

$$(5.8) \quad \frac{\partial I}{\partial g^i} = \frac{\partial I}{\partial G^i}$$

Substituting from (5.1), (5.2) and (4.3) into (5.3), (5.4), (5.7) and (5.8), we obtain four equations relating the integration constants as follows

$$(5.9) \quad m + p + M + P = 1$$

$$(5.10) \quad \alpha_1(m+M) - \beta_1(n+N) + \alpha_2(p+P) - \beta_2(q+Q) = 0$$

$$(5.11) \quad \begin{aligned} & \alpha_1(b_1+b_2b_5)m + \beta_1(-b_1+b_2b_6)n + \\ & + \alpha_2(b_1+b_2b_7)p + \beta_2(-b_1+b_2b_8)q + \\ & + \alpha_1(b_3+b_4b_5)M + \beta_1(-b_3+b_4b_7)N + \\ & + \alpha_2(b_3+b_4b_7)P + \beta_2(-b_3+b_4b_8)Q = 0 \end{aligned}$$

$$(5.12) \quad (b_{11}-b_2b_9)m + 2\alpha_1\beta_1b_2n +$$

$$\begin{aligned}
& + (b_{11} - b_2 b_{10})p + 2\alpha_2 (\beta_2 b_{2q} + \\
& + (b_{12} - b_4 b_9)M + 2\alpha_1 (\beta_1 b_{4N} + \\
& + (b_{12} - b_4 b_{10})P + 2\alpha_2 (\beta_2 b_{4Q} = 0
\end{aligned}$$

where

$$b_1 = (a_{13} - 2a_{22} + a_{25} - a_{16})$$

$$b_2 = (a_{36} - 2a_{33})$$

$$b_3 = (a_{34} - a_{25} + 2a_{55} - a_{46})$$

$$b_4 = (2a_{66} - a_{36})$$

$$b_5 = (3\beta_1^2 - \alpha_1^2)$$

$$b_6 = (3\alpha_1^2 - \beta_1^2)$$

$$b_7 = (3\beta_2^2 - \alpha_2^2)$$

$$b_8 = (3\alpha_2^2 - \beta_2^2)$$

$$b_9 = (\alpha_1^2 - \beta_1^2)$$

$$b_{10} = (\alpha_2^2 - \beta_2^2)$$

$$b_{11} = (a_{13} - a_{16})$$

$$b_{12} = (a_{34} - a_{46})$$

To furnish four more equations, we may insert the axial functions (5.1) and (5.2) in either one of the two equations in (4.12). It is obvious that the solution being obtained from one of the two homogeneous equations satisfies the other identically. Therefore, by grouping coefficients for $\cos \beta_1 z$, $\sin \beta_1 z$, $\cos \beta_2 z$ and $\sin \beta_2 z$, and setting these coefficients of the various cosine and sine terms to be zero, four additional equations expressed in terms of those eight arbitrary constants are obtained as follows:

$$(5.13) \quad c_1 m + c_2 n + c_3 M + c_4 N = 0$$

$$(5.14) \quad c_2 m - c_1 n + c_4 M - c_3 N = 0$$

$$(5.15) \quad c_5 p + c_6 q + c_7 P + c_8 Q = 0$$

$$(5.16) \quad c_6 p - c_5 q + c_8 P - c_7 Q = 0$$

where

$$c_1 = 2[a_{33}(\alpha_1^4 - 6\alpha_1^2\beta_1^2 + \beta_1^4) + (a_{13} - a_{22})(\alpha_1^2 - \beta_1^2) + a_{11}]$$

$$c_2 = 4\alpha_1\beta_1[2a_{33}(\beta_1^2 - \alpha_1^2) - (a_{13} - a_{22})]$$

$$c_3 = [a_{36}(\alpha_1^4 - 6\alpha_1^2\beta_1^2 + \beta_1^4) + (a_{16} - a_{25} + a_{34})(\alpha_1^2 - \beta_1^2) + a_{14}]$$

$$c_4 = 2\alpha_1\beta_1[2a_{36}(\beta_1^2 - \alpha_1^2) - (a_{16} - a_{25} + a_{34})]$$

$$c_5 = 2[a_{33}(\alpha_2^4 - 6\alpha_2^2\beta_2^2 + \beta_2^4) + (a_{13} - a_{22})(\alpha_2^2 - \beta_2^2) + a_{11}]$$

$$c_6 = 4 \alpha_2 \beta_2 [2a_{33}(\beta_2^2 - \alpha_2^2) - (a_{13} - a_{22})]$$

$$c_7 = [a_{36}(\alpha_2^4 - 6\alpha_2^2\beta_2^2 + \beta_2^4) + (a_{34} + a_{16} - a_{25})(\alpha_2^2 - \beta_2^2) + a_{14}]$$

$$c_8 = 2 \alpha_2 \beta_2 [2a_{36}(\beta_2^2 - \alpha_2^2) - (a_{16} - a_{25} + a_{34})]$$

Equations (5.9) to (5.16) together form a set of eight simultaneous linear equations in eight unknowns whose coefficients are given in terms of a_{ij} ($i, j = 1, 2, \dots, 6$), and the eigenvalues α_k, β_k . By inserting these eight constants into (5.1) and (5.2), the axial functions $g(z)$ and $G(z)$ are, therefore, completely determined.

6. Stress Expressions

Having constructed the stress components σ_{ij} ($i, j = r, \theta, z$) in terms of the bi-product variational stress functions and completely determined the basic radial and axial functions, we have the solution of the end effect at once if we substitute all of the functions in (2.8). A superposition on that of the infinitely long concentric hollow cylinder completes the stress formulas of the semi-infinite system which is subjected to a radial temperature distribution.

The axial functions as well as the derivatives which will be utilized to expand the stress expressions (2.8) are now summarized.

$$(6.1) \quad g(z) = e^{-\alpha_1 z} (m \cos \beta_1 z + n \sin \beta_1 z) + e^{-\alpha_2 z} (p \cos \beta_2 z + q \sin \beta_2 z)$$

$$(6.2) \quad g'(z) = e^{-\alpha_1 z} [-\alpha_1 (m \cos \beta_1 z + n \sin \beta_1 z) + \beta_1 (n \cos \beta_1 z - m \sin \beta_1 z)] + e^{-\alpha_2 z} [-\alpha_2 (p \cos \beta_2 z + q \sin \beta_2 z) + \beta_2 (q \cos \beta_2 z - p \sin \beta_2 z)]$$

$$(6.3) \quad g''(z) = e^{-\alpha_1 z} [(\alpha_1^2 - \beta_1^2)(m \cos \beta_1 z + n \sin \beta_1 z) - 2\alpha_1 \beta_1 (n \cos \beta_1 z - m \sin \beta_1 z)] +$$

$$+ e^{-\alpha_2 z} [(\alpha_2^2 - \beta_2^2)(p \cos \beta_2 z + q \sin \beta_2 z) + \\ - 2 \alpha_2 \beta_2 (q \cos \beta_2 z - p \sin \beta_2 z)]$$

A similar set for $G(z)$ can be obtained by substituting the constants M , N , P and Q for m , n , p and q in the expressions for $g(z)$. Thus we find

$$(6.4) \quad G(z) = e^{-\alpha_1 z} (M \cos \beta_1 z + N \sin \beta_1 z) + \\ + e^{-\alpha_2 z} (P \cos \beta_2 z + Q \sin \beta_2 z)$$

$$(6.5) \quad G'(z) = e^{-\alpha_1 z} [-\alpha_1 (M \cos \beta_1 z + N \sin \beta_1 z) + \\ + \beta_1 (N \cos \beta_1 z - M \sin \beta_1 z)] + \\ + e^{-\alpha_2 z} [-\alpha_2 (P \cos \beta_2 z + Q \sin \beta_2 z) + \\ + \beta_2 (Q \cos \beta_2 z - P \sin \beta_2 z)]$$

$$(6.6) \quad G''(z) = e^{-\alpha_1 z} [(\alpha_1^2 - \beta_1^2)(M \cos \beta_1 z + N \sin \beta_1 z) + \\ - 2 \alpha_1 \beta_1 (N \cos \beta_1 z - M \sin \beta_1 z)] + \\ + e^{-\alpha_2 z} [(\alpha_2^2 - \beta_2^2)(P \cos \beta_2 z + Q \sin \beta_2 z) + \\ - 2 \alpha_2 \beta_2 (Q \cos \beta_2 z - P \sin \beta_2 z)]$$

Substituting the foregoing six equations in (2.8) and making some simplification, we have the stress expressions as follows:

$$(6.7) \quad \sigma_r = e^{-\alpha_1 z} [W_r(r) \cos \beta_1 z + X_r(r) \sin \beta_1 z] + \\ + e^{-\alpha_2 z} [Y_r(r) \cos \beta_2 z + Z_r(r) \sin \beta_2 z]$$

where

$$\left\{ \begin{array}{l} W_r(r) = 5(f + r^4 f') m + \\ \quad - \nu(2F + r^4 F') [(\alpha_1^2 - \beta_1^2) M - 2 \alpha_1 \beta_1 N] \\ X_r(r) = 5(f + r^4 f') n + \\ \quad - \nu(2F + r^4 F') [(\alpha_1^2 - \beta_1^2) N + 2 \alpha_1 \beta_1 M] \\ Y_r(r) = 5(f + r^4 f') p + \\ \quad - \nu(2F + r^4 F') [(\alpha_2^2 - \beta_2^2) P - 2 \alpha_2 \beta_2 Q] \\ Z_r(r) = 5(f + r^4 f') q + \\ \quad - \nu(2F + r^4 F') [(\alpha_2^2 - \beta_2^2) Q + 2 \alpha_2 \beta_2 P] \end{array} \right.$$

$$(6.8) \quad \sigma_\theta = e^{-\alpha_1 z} [W_\theta(r) \cos \beta_1 z + X_\theta(r) \sin \beta_1 z] + \\ + e^{-\alpha_2 z} [Y_\theta(r) \cos \beta_2 z + Z_\theta(r) \sin \beta_2 z]$$

where

$$\left\{ \begin{array}{l} W_\theta(r) = 5 [r^5 t + (1 + 5r^3) r f' + f] m + \\ \quad - r t [(\alpha_1^2 - \beta_1^2) m - 2 \alpha_1 \beta_1 n] + \end{array} \right.$$

$$\begin{aligned}
& + \left[(\alpha_1^2 - \beta_1^2) M - 2 \alpha_1 \beta_1 N \right] \left\{ - (rF'' + 2F')r + \right. \\
& \quad \left. - \nu [r^5 F'' + (5r^3 + 2)rF' + 2F] \right\} \\
X_\theta(r) &= 5 \left[r^5 t + (1 + 5r^3)rf' + f \right] n + \\
& \quad - rt \left[(\alpha_1^2 - \beta_1^2)n + 2 \alpha_1 \beta_1 m \right] + \\
& \quad + \left[(\alpha_1^2 - \beta_1^2)N + 2 \alpha_1 \beta_1 M \right] \left\{ - (rF'' + 2F')r + \right. \\
& \quad \left. - \nu [r^5 F'' + (5r^3 + 2)rF' + 2F] \right\} \\
Y_\theta(r) &= 5 \left[r^5 t + (1 + 5r^3)rf' + f \right] p + \\
& \quad - rt \left[(\alpha_2^2 - \beta_2^2)p - 2 \alpha_2 \beta_2 q \right] + \\
& \quad - \left[(\alpha_2^2 - \beta_2^2)P - 2 \alpha_2 \beta_2 Q \right] \left\{ (rF'' + 2F')r + \right. \\
& \quad \left. + \nu [r^5 F'' + (5r^3 + 2)rF' + 2F] \right\} \\
Z_\theta(r) &= 5 \left[r^5 t + (1 + 5r^3)rf' + f \right] q + \\
& \quad - rt \left[(\alpha_2^2 - \beta_2^2)q + 2 \alpha_2 \beta_2 p \right] + \\
& \quad - \left[(\alpha_2^2 - \beta_2^2)Q + 2 \alpha_2 \beta_2 P \right] \left\{ (rF'' + 2F')r + \right. \\
& \quad \left. + \nu [r^5 F'' + (5r^3 + 2)rF' + 2F] \right\}
\end{aligned}$$

$$\begin{aligned}
(6.9) \quad \sigma_z &= e^{-\alpha_1 z} \left[W_z(r) \cos(\beta_1 z) + X_z(r) \sin(\beta_1 z) \right] + \\
&+ e^{-\alpha_2 z} \left[Y_z(r) \cos(\beta_2 z) + Z_z(r) \sin(\beta_2 z) \right]
\end{aligned}$$

where

$$\left\{ \begin{array}{l} W_z(r) = S(m + M) \\ X_z(r) = S(n + N) \\ Y_z(r) = S(p + P) \\ Z_z(r) = S(q + Q) \end{array} \right.$$

$$(6.10) \quad \mathcal{C} = e^{-\alpha_1 z} [W_{\mathcal{C}}(r) \cos(\beta_1 z + X_{\mathcal{C}}(r) \sin(\beta_1 z)] + \\ + e^{-\alpha_2 z} [Y_{\mathcal{C}}(r) \cos(\beta_2 z + Z_{\mathcal{C}}(r) \sin(\beta_2 z)]$$

where

$$\left\{ \begin{array}{l} W_{\mathcal{C}}(r) = t [\alpha_1(m + M) - \beta_1(n + N)] \\ X_{\mathcal{C}}(r) = t [\alpha_1(n + N) + \beta_1(m + M)] \\ Y_{\mathcal{C}}(r) = t [\alpha_2(p + P) - \beta_2(q + Q)] \\ Z_{\mathcal{C}}(r) = t [\alpha_2(q + Q) + \beta_2(p + P)] \end{array} \right.$$

The complete solution of the semi-infinite circular hollow cylinder subjected to a radial temperature distribution is, thus, obtained by substituting (6.7), (6.8), (6.9), (6.10) and (1.1) into (1.2)

III ILLUSTRATIVE EXAMPLE

7. Numerical Results for a Polynomial Temperature Distribution

As an illustration of the use of the developed method, we shall consider an example to obtain the complete solution of the thermal stress problem in a semi-infinite circular hollow cylinder. Since it is attempted to compare the results with experimental data (Ref. 8), the physical dimensions and material properties adopted for this example will be chosen so as to conform to that particular hollow cylinder being investigated experimentally. Hence a concentric circular hollow cylinder of 2.5 in. ID, 3.5 in. OD, 12 in. length with the following properties will be considered:

AISI 1020 hot-rolled mild steel

Modulus of elasticity: $E = 30 \times 10^6$ psi

Poisson's ratio: $\nu = 0.3$

Coefficient of thermal expansion: $\alpha = 6.8 \times 10^{-6} / ^\circ\text{F}$

For solving a particular problem without loss of generality, we shall employ non-dimensional units throughout this example. The stresses will be expressed in $\sigma_{ij} / \alpha E \Delta T$; the strains in $\epsilon_{ij} / \alpha \Delta T$; the temperature in $T / \Delta T$; and the outside radius will be taken as the length unit, as defined previously. Consequently, the most frequently

appearing fractional parameter λ has the value of

$$\lambda = \frac{ID}{OD} = \frac{2.5}{3.5} = 0.714285714$$

We shall use a temperature profile which is a second degree polynomial given by

$$(7.1) \quad \frac{T(r)}{\Delta T} = 12.25 (r^2 - 2r + 1)$$

Substituting (7.1) into (1.1), the stresses for an infinitely long circular hollow cylinder which is equivalent to the portion far away from the end of the semi-infinite hollow cylinder obtained as

$$(7.2) \quad \left\{ \begin{array}{l} \frac{\sigma_r^\infty}{\alpha E \Delta T} = -4.375 r^2 + 11.666667 r - 8.531798 + \\ \quad + 1.240079 r^{-2} \\ \frac{\sigma_\theta^\infty}{\alpha E \Delta T} = -13.125 r^2 + 23.333333 r - 8.531798 + \\ \quad - 1.240079 r^{-2} \\ \frac{\sigma_z^\infty}{\alpha E \Delta T} = -17.5 r^2 + 35 r - 17.063489 \end{array} \right.$$

By substituting (7.2) into (3.2) we obtain,

$$(7.3) \quad \frac{S(r)}{\alpha E \Delta T} = 12.25(1.428571r^2 - 2.875143r + 1.392938)$$

The determination of the radial functions with their derivatives involves considerable calculation and reduc-

tion. By using (3.7), (3.8), (3.9) and (3.15), in conjunction with (3.16) and (3.17), one obtains

$$(7.4) \left\{ \begin{aligned} \frac{t(r)}{\alpha E \Delta T} &= 12.25(0.357143r^3 - 0.952381r^2 + 0.696469r - \\ &\quad - 0.101231r^{-1}) \\ \frac{f'(r)}{\alpha E \Delta T} &= 12.25(0.089284r^4 - 0.317460r^3 + 0.348235r^2 - \\ &\quad - 0.120160 - 0.101231 \ln r) \\ \frac{f(r)}{\alpha E \Delta T} &= 12.25(0.017857r^5 - 0.079365r^4 + 0.116078r^3 - \\ &\quad - 0.018929r - 0.035540 - 0.101231 \ln r) \\ \frac{F(r)}{\alpha E \Delta T} &= 12.25(0.017857r^4 - 0.079365r^3 + 0.116078r^2 - \\ &\quad - 0.018848 - 0.035703r^{-1} - 0.101231 \ln r) \\ \frac{F'(r)}{\alpha E \Delta T} &= 12.25(0.071429r^3 - 0.238095r^2 + 0.23156r - \\ &\quad - 0.101231r^{-1} + 0.03570r^{-2}) \\ \frac{F''(r)}{\alpha E \Delta T} &= 12.25(0.214286r^2 - 0.476190r + 0.232156 - \\ &\quad + 0.101231r^{-2} - 0.071406r^{-3}) \end{aligned} \right.$$

By substituting from (7.4) into (4.4), the coefficients a_{1j} are obtained as follows:

$$\left(\begin{aligned} a_{11} &= 0.027726020 \\ a_{13} &= -0.000050107 \end{aligned} \right.$$

$$(7.5) \quad \left\{ \begin{array}{l} a_{14} = 0.000351910 \\ a_{16} = -0.000010767 \\ a_{22} = 0.000002838 \\ a_{25} = 0.000005676 \\ a_{33} = 0.000000782 \\ a_{34} = -0.000000002 \\ a_{36} = 0.000000111 \\ a_{44} = 0.000140574 \\ a_{46} = 0.000000234 \\ a_{55} = 0.000002838 \\ a_{66} = 0.000001467 \end{array} \right.$$

where the values given above should be multiplied by the factor, $(12.25 \Delta E \Delta T)^2$.

Substituting (7.5) into (4.17) and solving (4.16) by using the Ferrari's method and the Cardan's formulas (Ref. 10) for solving the quartic equation we find the roots as follows

$$(7.6) \quad \left\{ \begin{array}{l} y_I = 0.855155949 \pm i 9.722056235 \\ y_{II} = 33.579024979 \pm i 185.346929450 \end{array} \right.$$

Then (4.15) gives

$$(7.7) \quad \left\{ \begin{array}{l} \gamma_I = \pm (2.303774 \pm i 2.110028) \\ \gamma_{II} = \pm (10.534304 \pm i 8.797303) \end{array} \right.$$

It is seen that there are only two sets of complex roots distinct in their numerical values of the real and imaginary parts, which are,

$$(7.8) \quad \begin{cases} \alpha_1 = 2.303774 & \beta_1 = 2.110028 \\ \alpha_2 = 10.534304 & \beta_2 = 8.797303 \end{cases}$$

To evaluate the integration constants which appear in the expressions of $g(z)$ and $G(z)$, we substitute (7.5) and (7.8) into equations (5.9) through (5.16), which give

$$(7.9) \quad \begin{cases} m = -0.008204 \\ n = -0.001505 \\ p = -0.031353 \\ q = 0.061458 \\ M = 1.036912 \\ N = 0.729091 \\ P = 0.002650 \\ Q = -0.000950 \end{cases}$$

and the axial functions are, therefore, completely determined as

$$(7.10) \quad \begin{cases} g(z) = e^{-2.303774z}(-0.008204\cos 2.110028z + \\ \quad -0.001505\sin 2.110028z) + \\ \quad + e^{-10.534304z}(-0.031353\cos 8.797303z + \\ \quad + 0.061458\sin 8.797303z) \end{cases}$$

$$\begin{aligned}
 G(z) = e^{-2.303774z} & (1.036912 \cos 2.110028z + \\
 & + 0.729091 \sin 2.110028z) + \\
 e^{-10.534304z} & (0.002650 \cos 8.797303z + \\
 & - 0.000950 \sin 8.797303z)
 \end{aligned}$$

Then, equations (6.7) through (6.10) in conjunction with the equations (7.4) and (7.10) give the stress distribution due to the end effect. The computation procedure is shown in Appendix B.

IV DISCUSSION OF RESULTS

Numerical results of the illustrative example obtained with the aid of a LGP-30 digital computer are shown graphically in Figures 3 through 14. Non-dimensional units have been used to scale the coordinates of all graphs. A comparison will be made to the experimental data from an available source (Ref. 8).

Figure 3 shows the thermal stresses in the infinitely long hollow cylinder subjected to a given axisymmetric temperature distribution.

Figure 4 shows the temperature profile used in the numerical example and the resulting axial stress in the infinitely long hollow cylinder. Two radial functions $S(r)$ and $t(r)$ are also included.

With the relative axial distance from the end of the cylinder as a parameter, the radial distribution of the stress components σ_{ij} ($i, j = r, \theta, z$) are plotted individually for the radial, circumferential, axial and shear stresses in Figures 5, 6, 7 and 8.

Figures 9, 10 and 11 show z -dependence of the stress components. Figure 9 shows the axial variations of the total axial and circumferential stresses on the outer curved boundary.

Figure 10 shows the axial variation of the maximum

radial and circumferential stresses due to the end effect alone. The total values of these stresses can be obtained by merely shifting the ordinates to new origins to add the stresses obtained for the infinitely long cylinder.

Figure 11 shows the axial variation of the maximum shear stress due to the end effect alone. These maximum shear stresses occur approximately at the radial location $r/r_0 = 0.83$. In contrast to the general pattern of the axial distributions of radial, circumferential and axial stresses which possess the largest value at the end, decay appreciably at $z = 1.5r_0$, and become virtually negligible at about $z = 2r_0$, the shear stress increases from zero at the end to its peak at approximately $z = 0.24r_0$, and then follows a similar pattern of decay. It is to be noted, however, that the magnitude of the maximum shear stress even at its peak is in the order of one tenth of the other stress components.

The circumferential stresses exerted at the end face, where the magnitudes are the largest in the whole cylinder, are plotted in Figure 12. The resultant curve indicates that the maximum stress is compressive on the inner curved boundary, the peak of the total tensile stress is located approximately at $r/r_0 = 0.9$, and at the outer boundary its magnitude is 32 percent less than the corresponding stress at points far away from the end. It is also seen that the

peak value of the circumferential tensile stress is 45 percent larger than the maximum circumferential tensile stress in the infinitely long hollow cylinder which occur on the outer boundary. On the other hand, the approximate solution based on the thin-walled cylinder assumption gives the maximum stress at the outer boundary with a 25 percent increase over the infinitely long cylinder solution.

The theoretical strains have been calculated from the stresses by using the Hooke's law, and are shown in Figures 13 and 14.

To compare the theoretical results with the available experimental data qualitatively, the experimental and theoretical values of the axial strains on the outer boundary of the hollow cylinder are plotted in Figure 14, where ΔT is the temperature difference between the wall temperatures.

The original experimental work, as given by Reference 8, was conducted with a hollow steel cylinder of 2.5 inches ID, 3.5 inches OD, and 12 inches in length. Well-stirred hot liquid suddenly poured in the cylinder was used as the heat source, and the transient strains and the wall temperatures were recorded.

The experimental results shown in Figure 14. are the axial strains measured at $t = 2$ seconds. In the particular example under consideration, the transient temperature distribution at $t = 2$ seconds may be assumed to have a similar

variation in r as the temperature distribution chosen for the numerical example discussed above. Since the exact experimental temperature distribution is not known, Figure 14 can not be considered a direct verification of the theoretical results. However, considering the equality of the temperature differentials and similarity of the temperature profiles in both cases, at least qualitatively, there is a good agreement between the theoretical and the experimental results.

One of the main advantages of the method developed in this thesis is that separation of the plane strain part will reduce the axisymmetric z -dependent problem to a simpler one handled by the use of variational principles. A further application of the method to the end effect problems in composite cylinders has been outlined by the author (Ref. 12).

As a guide for future investigations, the necessary requirements and the limits of arbitrariness in the construction of stress expressions in the variational stress functions are given fully in the Appendix A.

V CONCLUSIONS

It is apparent that in the analysis of the end effect of a semi-infinite circular hollow cylinder, the utilization of the developed method is advantageous from both conceptual and practical points of view. Several essential conclusions can be drawn as follows:

(1) The difficulties which would be encountered by the classical thermo-elasticity treatment of a semi-infinite hollow cylinder are overcome by this method. The end effect problem can be treated by this method separately with no simplifying assumptions.

(2) This analysis has developed a unique method of formulating the radial functions which are solutions to a set of differential equations obtained from the equilibrium conditions.

(3) The application of the variational approach which eliminates the preliminary restriction that the chosen stress functions satisfy the compatibility equations, reduces the problem partly to an eigenvalue problem. The resulting Euler equations are two fourth-order, linear, constant-coefficient differential equations. The solution of the Euler equations give the axial functions in the form of exponentially decaying expressions and are conceivable on account of the Saint-Venant's principle, i.e., the self-

equilibrating stress distribution applied to the end gives rise only to local effects.

(4) The numerical results show good qualitative agreement with the available experimental data.

(5) In addition to the thermal stress problems in semi-infinite hollow cylinders with any given radial temperature distribution, the method developed above can be applied to other axisymmetric problems (such as shrink-fit and punch problems) with non-uniform end tractions.

VI APPENDICES

A. Construction of Stress Expressions in Variational Stress Functions

Conditions which must be fulfilled by the required stress expressions can be stated as follows:

- (a) To make the Theorem of Minimum Complementary Energy available as a variational approach, the selected stress functions should assure automatic satisfaction of the equilibrium equations (2.3) without satisfying the compatibility equations.
- (b) To prevent the solution from being trivial, it demands results of non-vanishing circumferential and axial stresses on both lateral curved boundaries, where the surface tractions are zero.
- (c) To ensure the axial variation of the stresses being exponential, the Euler equations, for the axial functions, should lead to a reducible (i.e. containing even order derivatives) differential equations with constant coefficients.

For problems of the similar type, there exists several well-known functions but none of them is suitable for the present purpose. The most commonly used Love function (ref. 2) is merely a 'displacement function' satisfying the compatibility equations rather than a 'stress function'

satisfying the equilibrium equations. The well-known Boussinesq functions (Ref. 11) violate the conditions (a) and (c) mentioned above: The requirement that the Boussinesq functions to be harmonic eliminates the possibility of dealing with the bi-product functions as defined by (2.6); and the explicit appearance of z in these stress expressions will cause the Euler equations for the axial functions $g(z)$ and $G(z)$ to have variable coefficients which makes the problem analytically intractable.

By modifying the Sadowsky-Sternberg stress formulas (Ref. 7), Horvay obtained his set of stress expressions for the case of a solid cylinder (Ref. 3); but they are likewise inadequate to apply to the hollow cylinder case because of two additional boundary requirements arising from the inner curved surface. However, some essential points indicated by his results are helpful in the development of the present work. On bearing the aforementioned conditions in mind as a guide, the derivations of (2.5) and (2.8) can now be proceeded as follows:

Let us define a pair of bi-product functions:

$$(A.1) \quad \begin{cases} \varphi(r, z) = f(r) g(z) \\ \phi(r, z) = F(r) G(z) \end{cases}$$

and let,

$$\begin{aligned}
 (A.2) \quad \mathcal{Z}(r, z) &= -\varphi_{rrz} - (r\phi_{rrz} + 2\phi_{rz}) \\
 &= -f''g' - (rF'' + 2F')G'
 \end{aligned}$$

Making use of the second equilibrium equation of (2.3), which reads

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \mathcal{Z}}{\partial r} + \frac{\mathcal{Z}}{r} = 0$$

and integrating with respect to z yields

$$\begin{aligned}
 (A.3) \quad \sigma_z(r, z) &= - \int \left(\frac{\partial \mathcal{Z}}{\partial r} + \frac{\mathcal{Z}}{r} \right) dz \\
 &= \varphi_{rrr} + \frac{1}{r} \varphi_{rr} + r\phi_{rrr} + 4\phi_{rr} + \frac{2}{r} \phi_r \\
 &= (f''' + \frac{1}{r}f'')g + (rF''' + 4F'' + \frac{2}{r}F')G
 \end{aligned}$$

where the integration constant has been taken to be zero.

Similarly, the first equilibrium equation of (2.3) reads

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \mathcal{Z}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

which, in a re-arranged form, is

$$(A.4) \quad \sigma_\theta(r, z) = r \left(\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r}{r} + \frac{\partial \mathcal{Z}}{\partial z} \right)$$

Let

$$\begin{aligned}
 (A.5) \quad \sigma_r(r, z) &= 5(\varphi + r^4\varphi_r) - \nu(2\phi_{zz} + r^4\phi_{rzz}) \\
 &= 5(f + r^4f')g - \nu(2F + r^4F')G''
 \end{aligned}$$

The circumferential stress is obtained by inserting (A.5) in (A.4), which gives

$$\begin{aligned}
 (A.6) \quad \sigma_{\theta}(r, z) &= 5 \left[r^5 \varphi_{rr} + (1+5r^3)r \varphi_r + \varphi \right] + \\
 &\quad - r \varphi_{rrzz} - r(r \phi_{rrzz} + 2 \phi_{rzz}) + \\
 &\quad - \nu \left[r^5 \phi_{rrzz} + (2+5r^3)r \phi_{rzz} + 2 \phi_{zz} \right] \\
 &= 5 \left[r^5 t + (1+5r^3)rf' + f \right] g - rtg'' + \\
 &\quad - r(rF'' + 2F')G'' + \\
 &\quad - \nu \left[r^5 F'' + (2+5r^3)rF' + 2F \right] G''
 \end{aligned}$$

Introducing notation from (2.7) into (A.2), (A.3), (A.5) and (A.6), we have

$$(A.7) \quad \left\{ \begin{aligned}
 \sigma_r &= 5(f + r^4 f')g - \nu(2F + r^4 F')G'' \\
 \sigma_{\theta} &= 5 \left[r^5 t + (1+5r^3)rf' + f \right] g - rtg'' + \\
 &\quad - \nu \left[r^5 F'' + (2+5r^3)rF' + 2F \right] G'' + \\
 &\quad - r(rF'' + 2F')G'' \\
 \sigma_z &= S_f g + S_F G \\
 \tau &= -t_f g' - t_F G'
 \end{aligned} \right.$$

It is to be noted that the axisymmetric nature of the problem, reduces the number of equilibrium equations to

two, as given by (2.3). This fact indicates that there is some freedom in the choice of two stress expressions among those of four participating stress components. However, this arbitrariness is limited to a certain extent. The two expressions of $\mathcal{E}$ and σ_r were so selected that the appropriate combinations of the variational bi-product stress functions with their derivatives not only lead the solution to fulfill the aforementioned fundamental requirements but give the best approximation by the degree of adherence to the compatibility conditions. The latter consideration, though is not a necessary condition while the variational approach is employed, it has been utilized as an assurance of better results in the stage of selecting this very set from a number of trial functions.

B. Flow Diagram Indicating Computation Procedures for Stress Components due to the End-Effect of a Semi-Infinite Circular Hollow Cylinder

The stress expressions due to the end-effect, (6.7) through (6.10) can be generalized as:

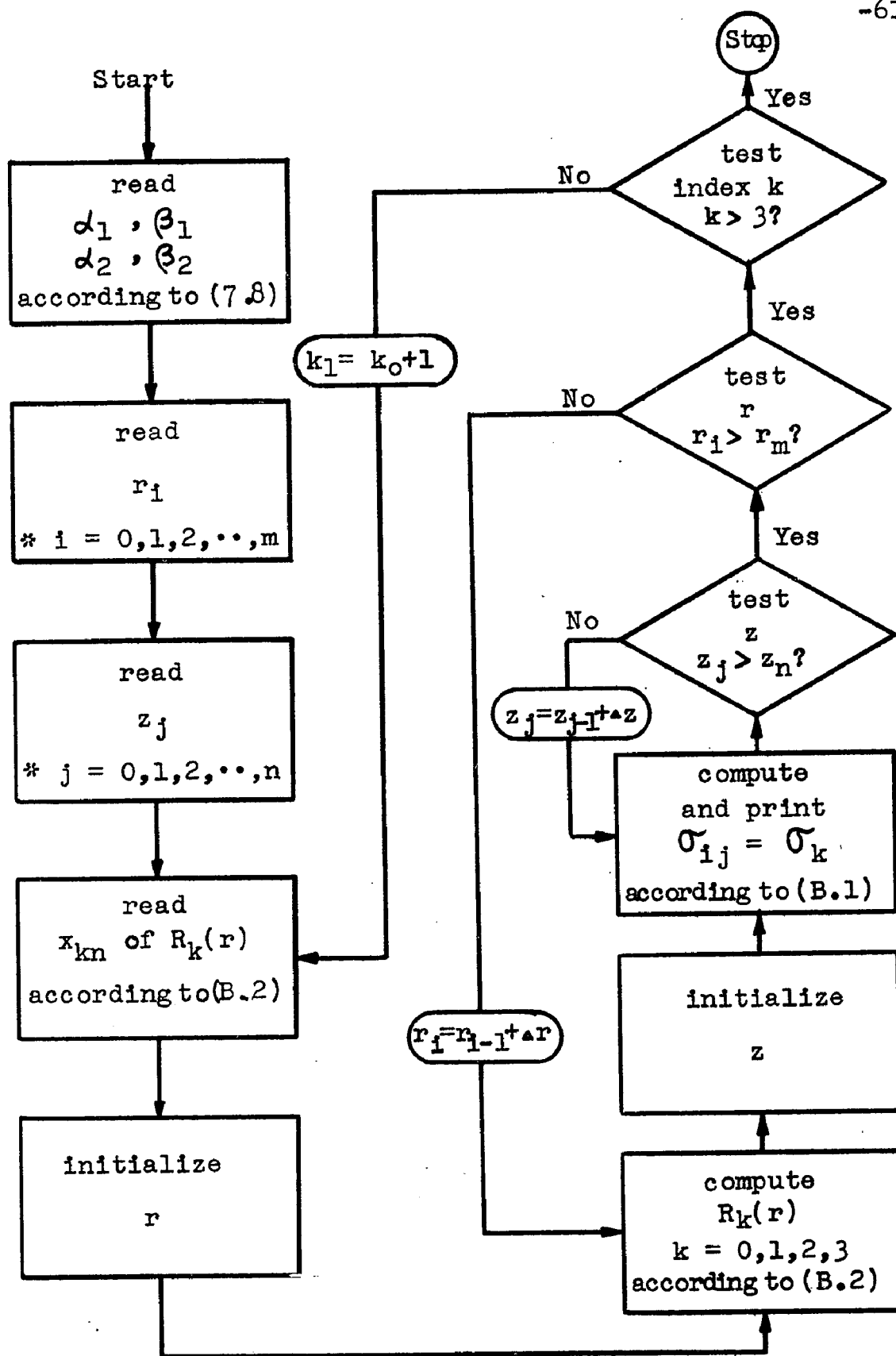
$$(B.1) \quad \begin{aligned} \sigma_k &= \sigma_{1j}(r, z) \\ &= e^{-\alpha_1 z} [W_k(r) \cos \beta_1 z + X_k(r) \sin \beta_1 z] + \\ &\quad + e^{-\alpha_2 z} [Y_k(r) \cos \beta_2 z + Z_k(r) \sin \beta_2 z] \end{aligned}$$

where $(k = 0, 1, 2, 3)$, $\sigma_{rr} = \sigma_r = \sigma_0$, $\sigma_{\theta\theta} = \sigma_\theta = \sigma_1$, $\sigma_{zz} = \sigma_z = \sigma_2$, and $\sigma_{zr} = \tau = \sigma_3$.

For the illustrated example, the radial functions in (B.1) can be expressed in one general form as:

$$(B.2) \quad \left. \begin{array}{l} W_k(r) \\ X_k(r) \\ Y_k(r) \\ Z_k(r) \end{array} \right\} R_k(r) = x_{k0}r^8 + x_{k1}r^7 + x_{k2}r^6 + x_{k3}r^5 + \\ + x_{k4}r^4 + x_{k5}r^3 + x_{k6}r^2 + x_{k7}r^1 + \\ + x_{k8}r^0 + x_{k9}r^{-1} + x_{k10}r^{-2} + \\ + (x_{k11}r^4 + x_{k12}r^3 + x_{k13}r^2 + x_{k14}r + x_{k15}) \ln r$$

where x_{km} ($n=0,1,\dots,15$) are coefficients in $W(r), X(r), Y(r)$ and $Z(r)$ obtained by using (6.7) through (6.10). The computation procedure for the stresses is given by the following flow diagram.



* pre-assigned values with regular increment Δr or Δz .

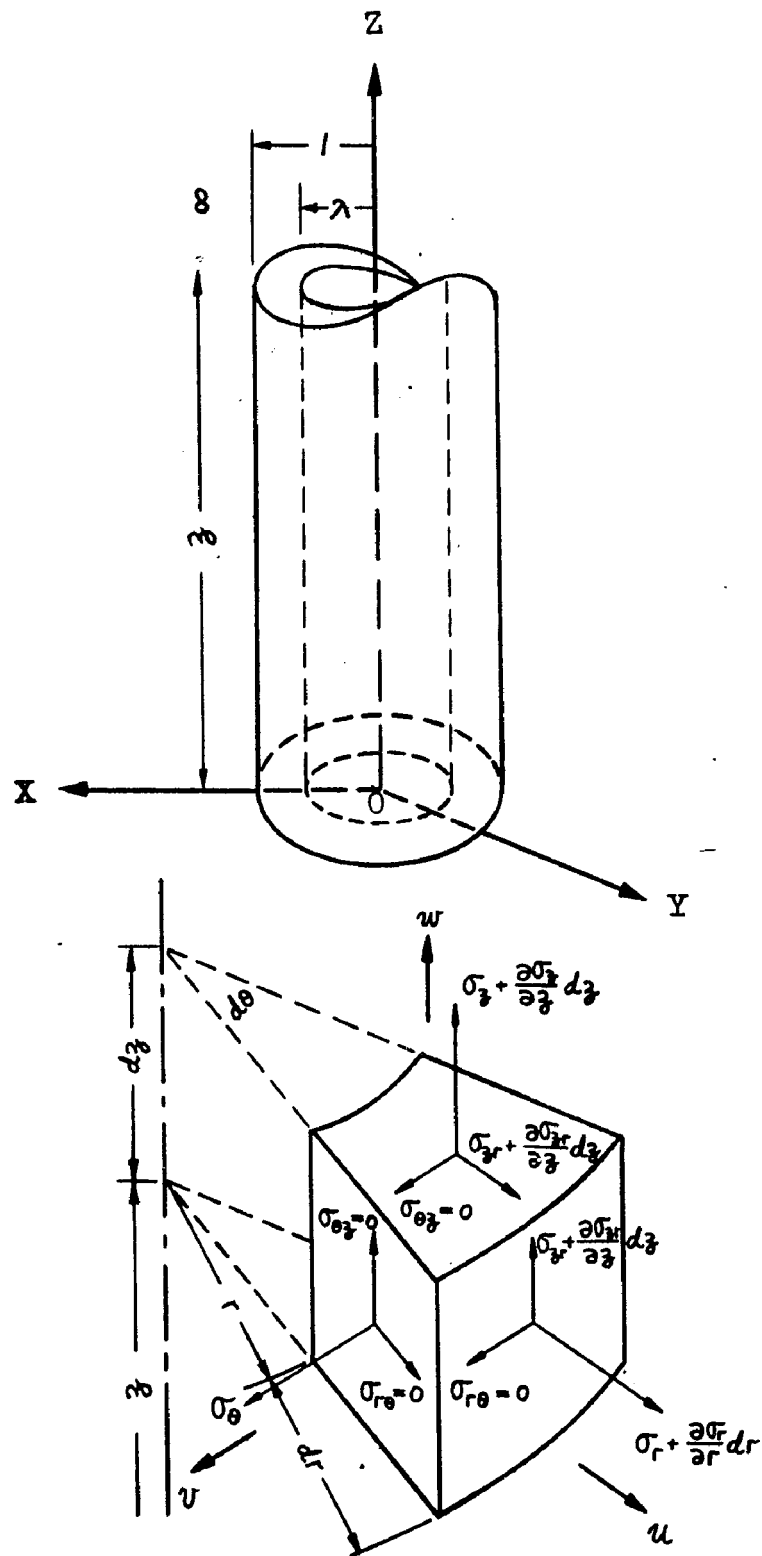


Fig. 1 A Semi-Infinite Circular Hollow Cylinder

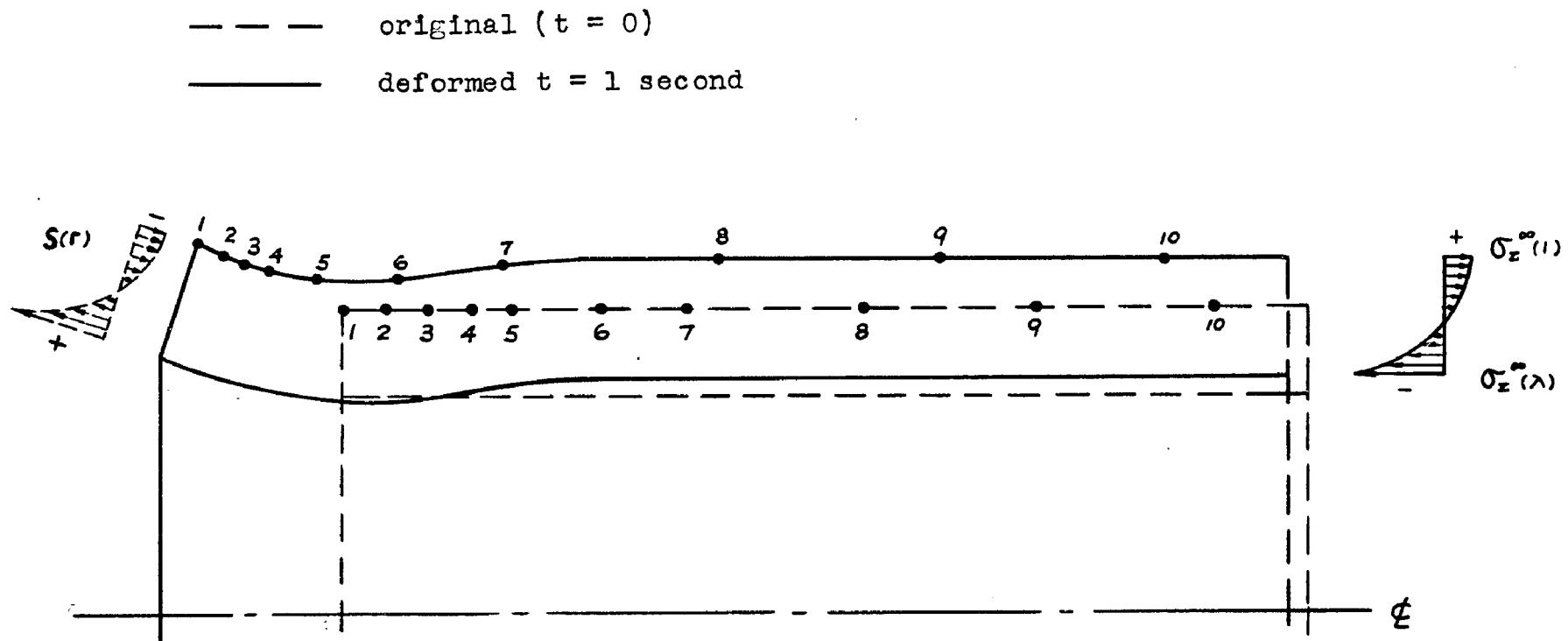


Fig. 2 Approximate Shape of the Cylinder at one second with Magnified Strains (from Ref. 8, Fig. 25)

$$\frac{\sigma_{ij}^{\infty}}{\alpha E \Delta T}$$

-64

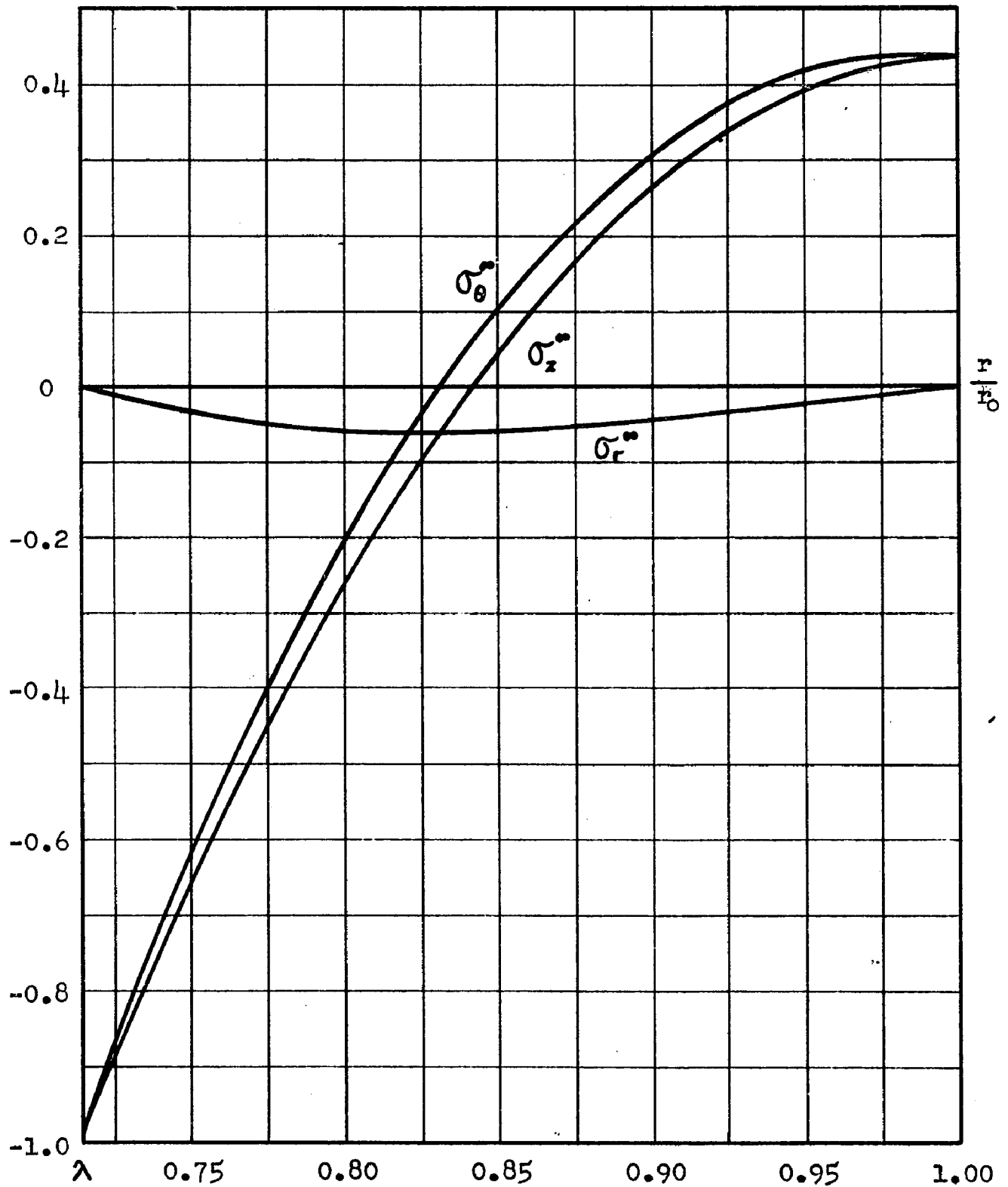


Fig. 3 Distribution of Thermal Stresses in the Semi-Infinite Circular Hollow Cylinder of the Infinite Set

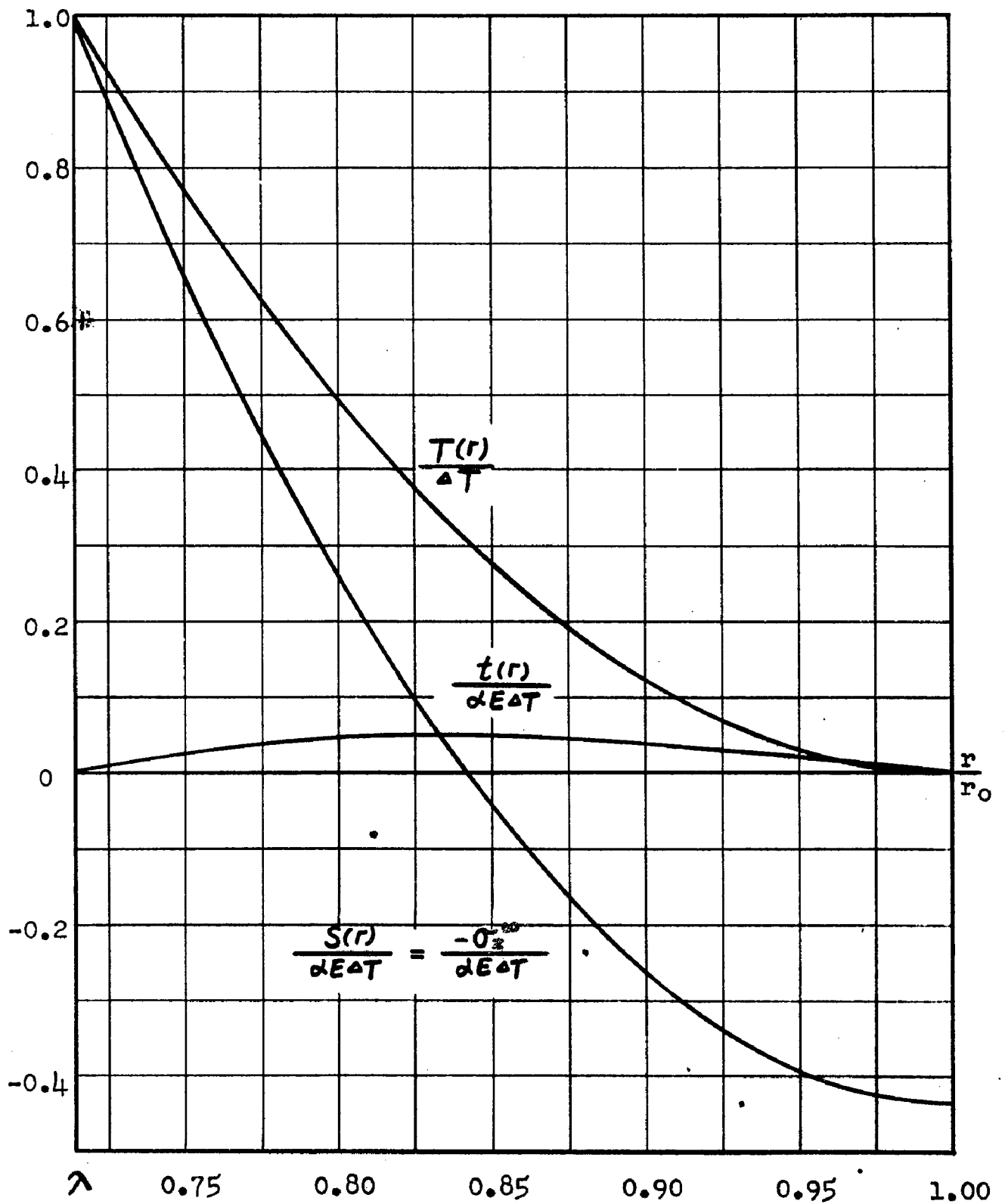


Fig. 4 Radial Distribution of Temperature, Axial Stress, and Radial Functions $S(r)$ and $t(r)$

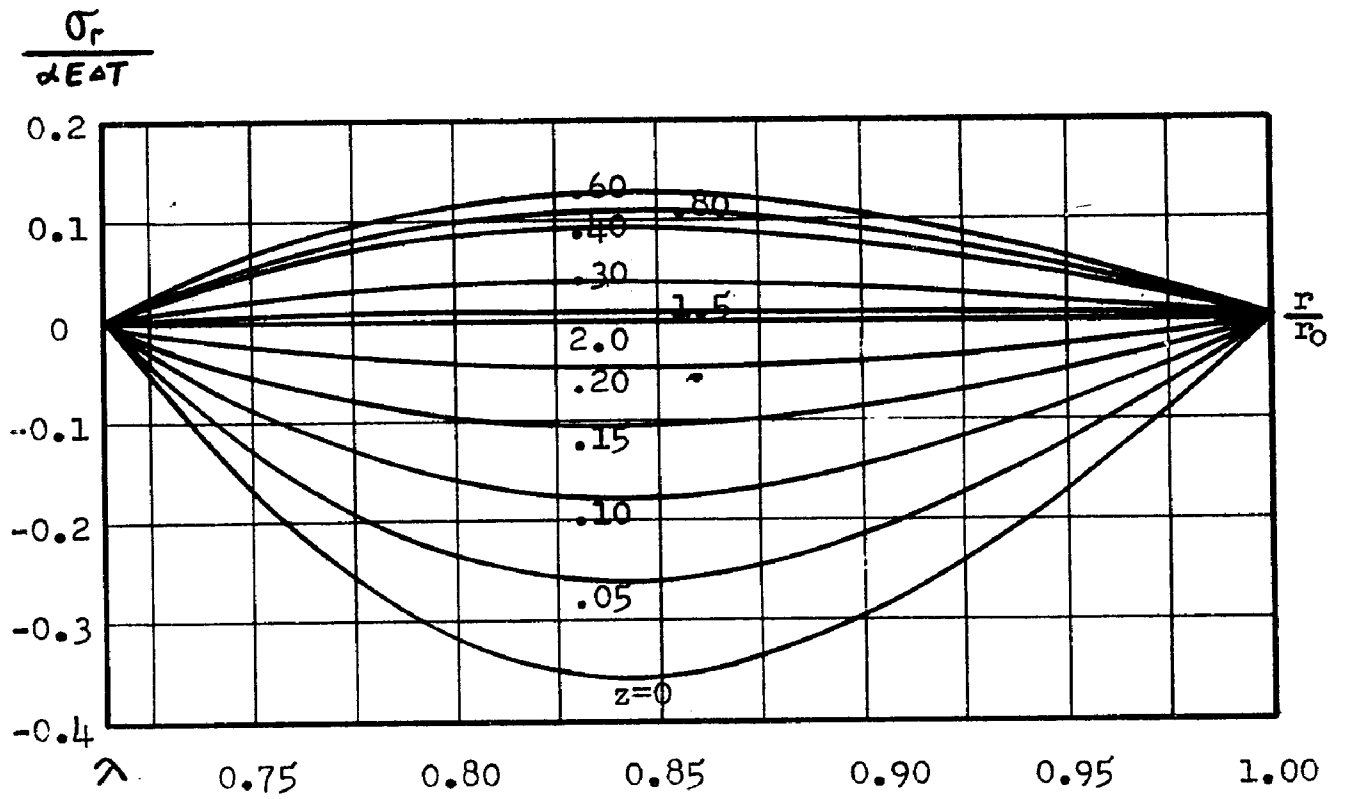


Fig. 5 Radial Distribution of Radial Stress
due to the End Effect

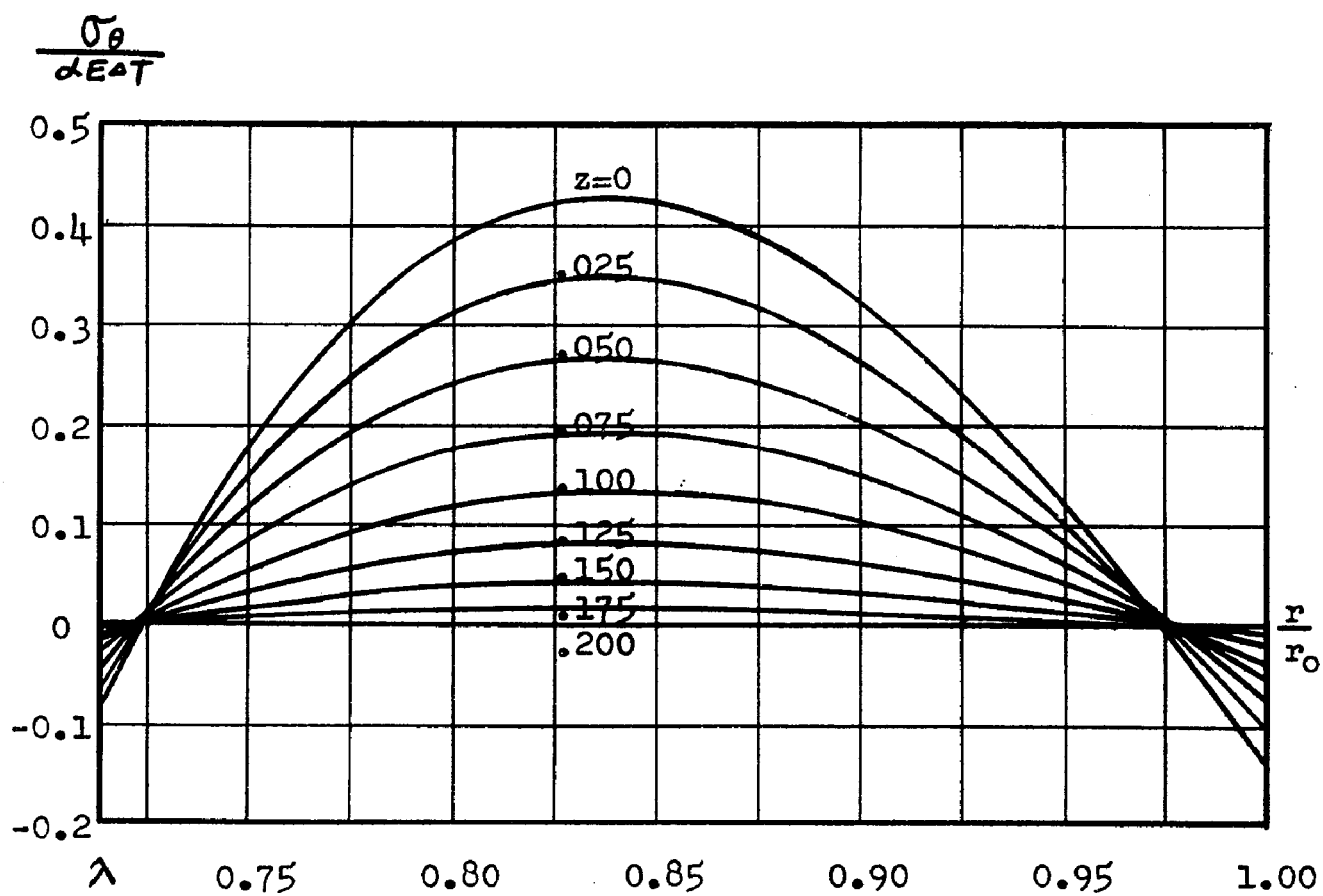


Fig. 6 Radial Distribution of Circumferential Stress due to the End Effect

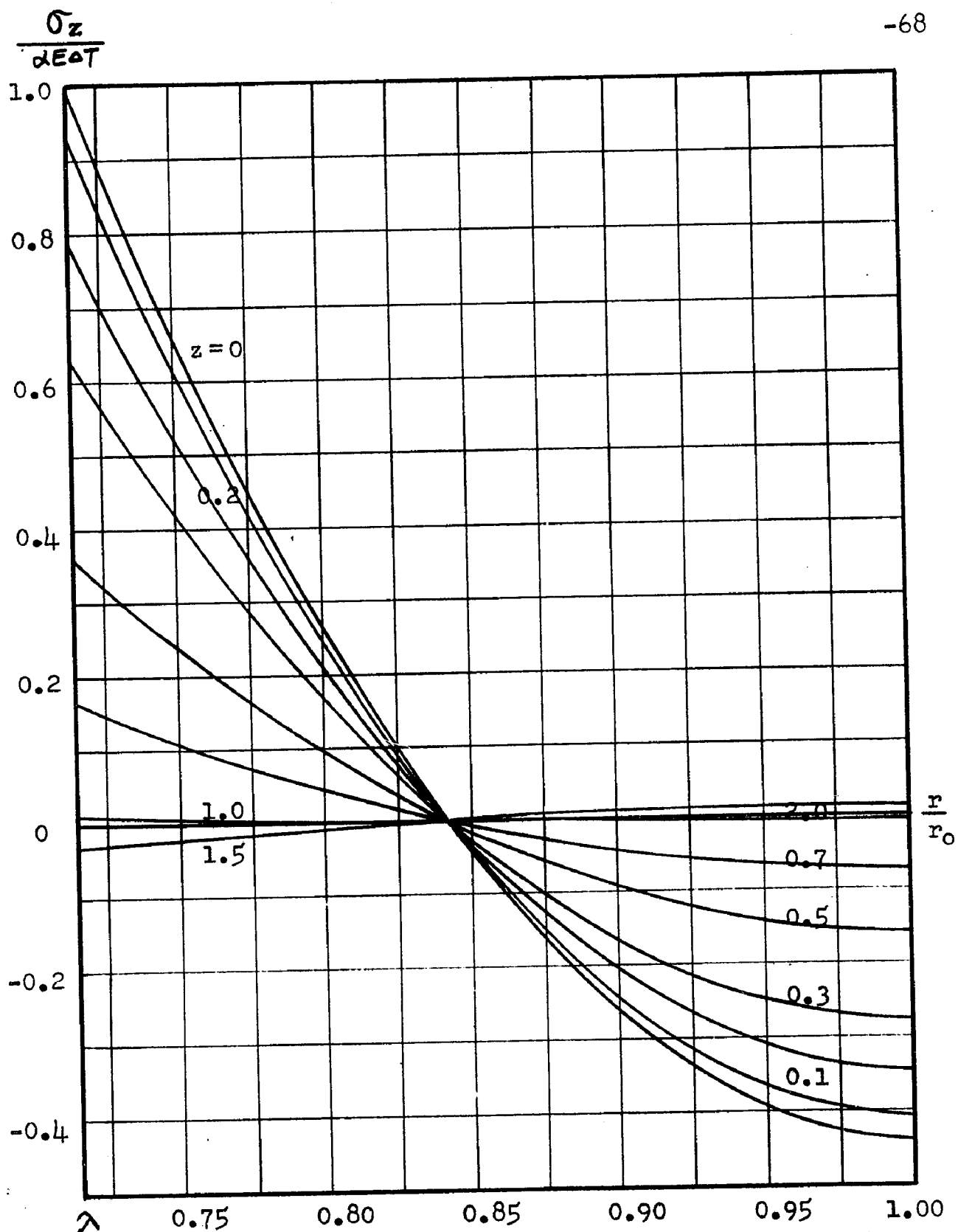


Fig. 7 Radial Distribution of Axial Stress due to the End Effect

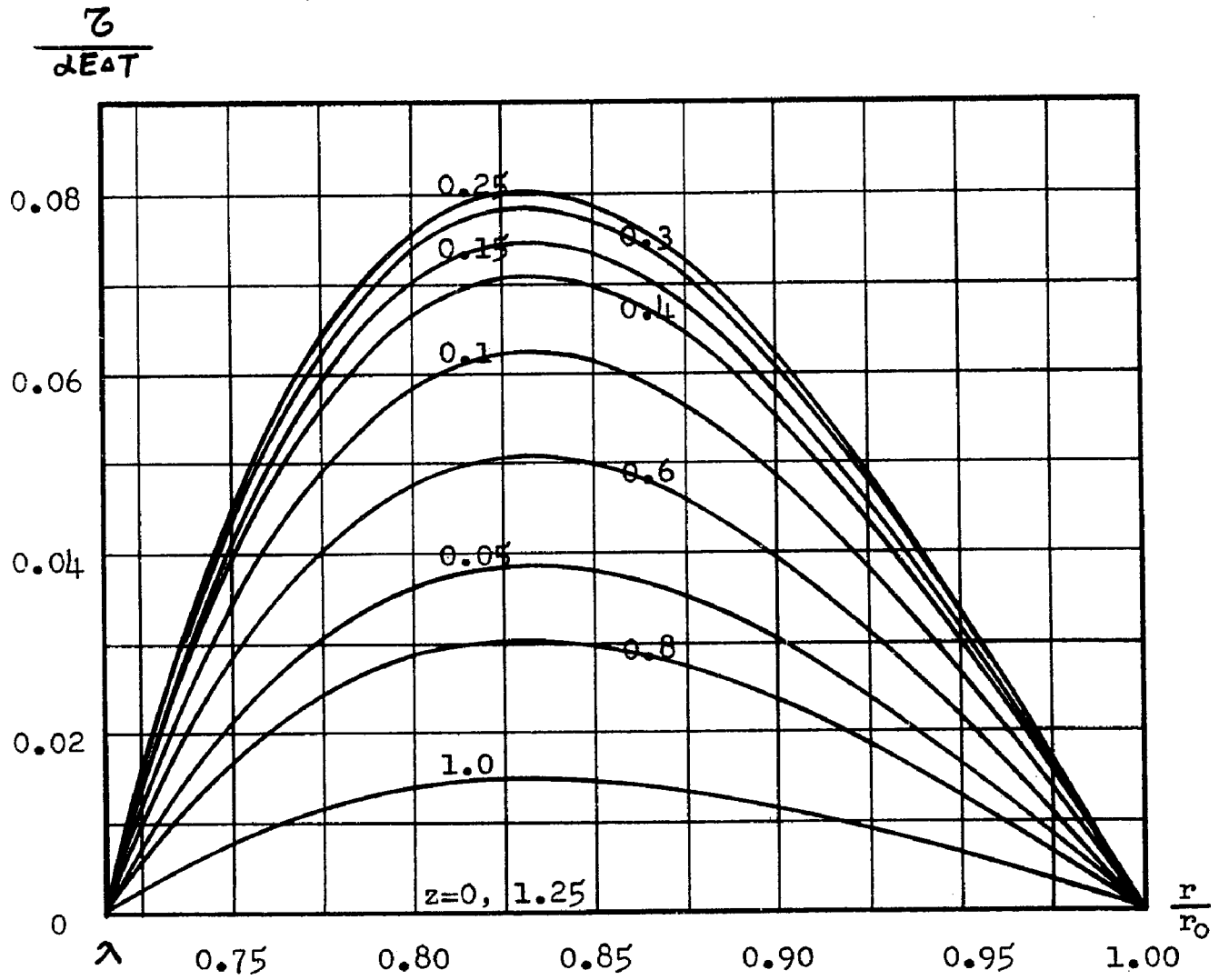


Fig. 8 Radial Distribution of Shear Stress due to the End Effect

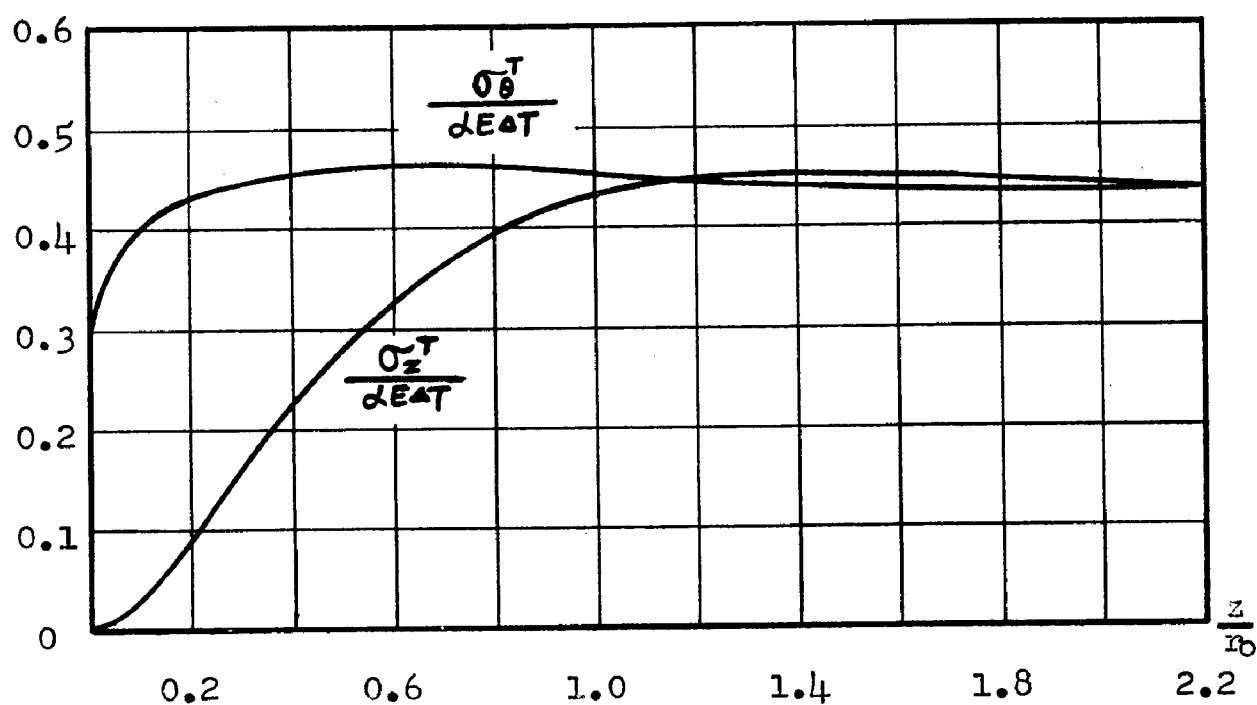


Fig. 9 Axial Distribution of Circumferential and Axial Stresses on the Outer Boundary of the Semi-Infinite Circular Hollow Cylinder

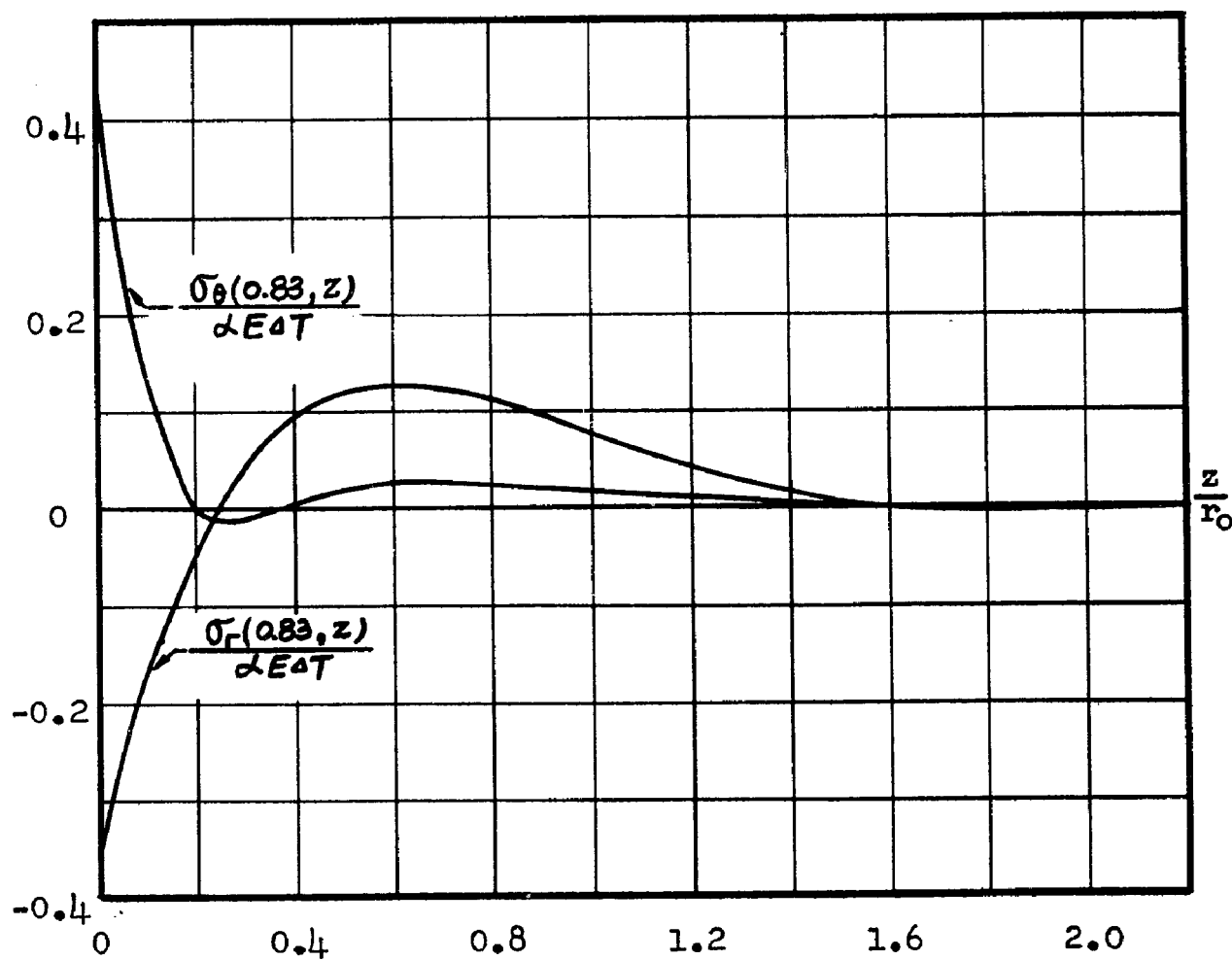


Fig. 10 Axial Distribution of Circumferential and Radial Stresses at $r=0.83r_0$ (Radial Maximum Values)

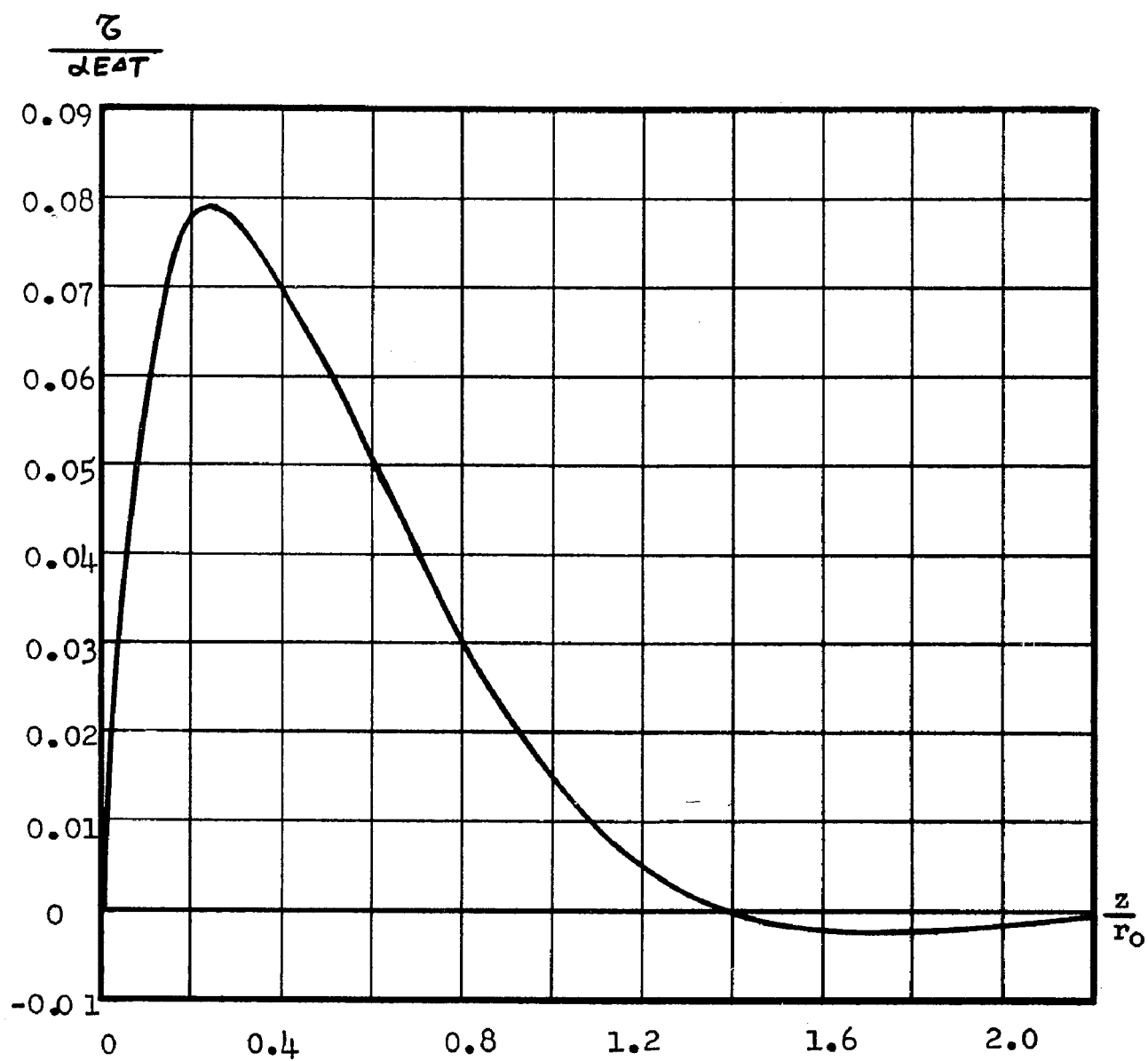


Fig. 11 Axial Distribution of Shear Stress at $r = 0.83r_0$ (Radial Maximum Values)

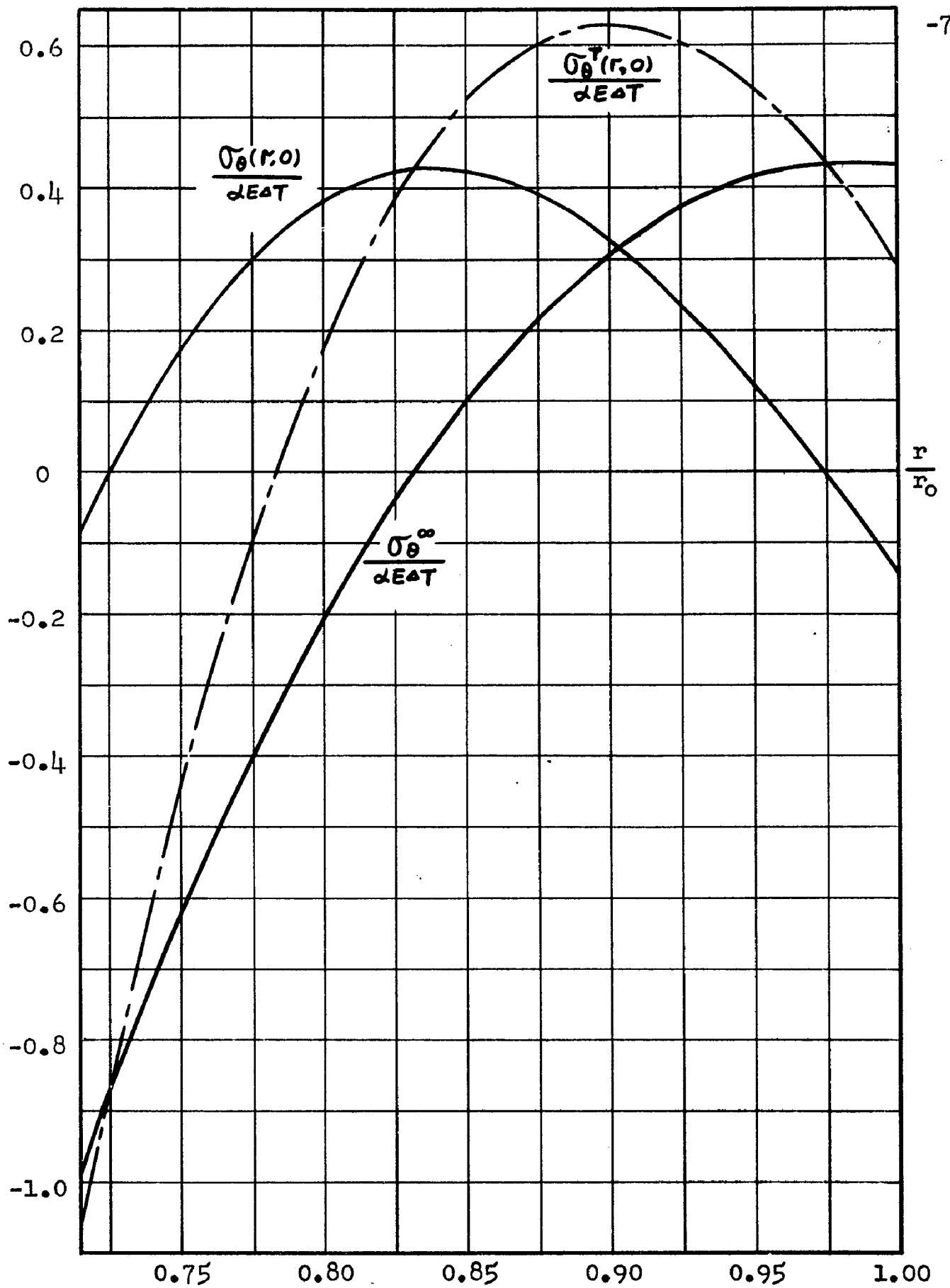


Fig. 12 Radial Distribution of Circumferential Stress at the End Section of the Semi-Infinite Circular Hollow Cylinder

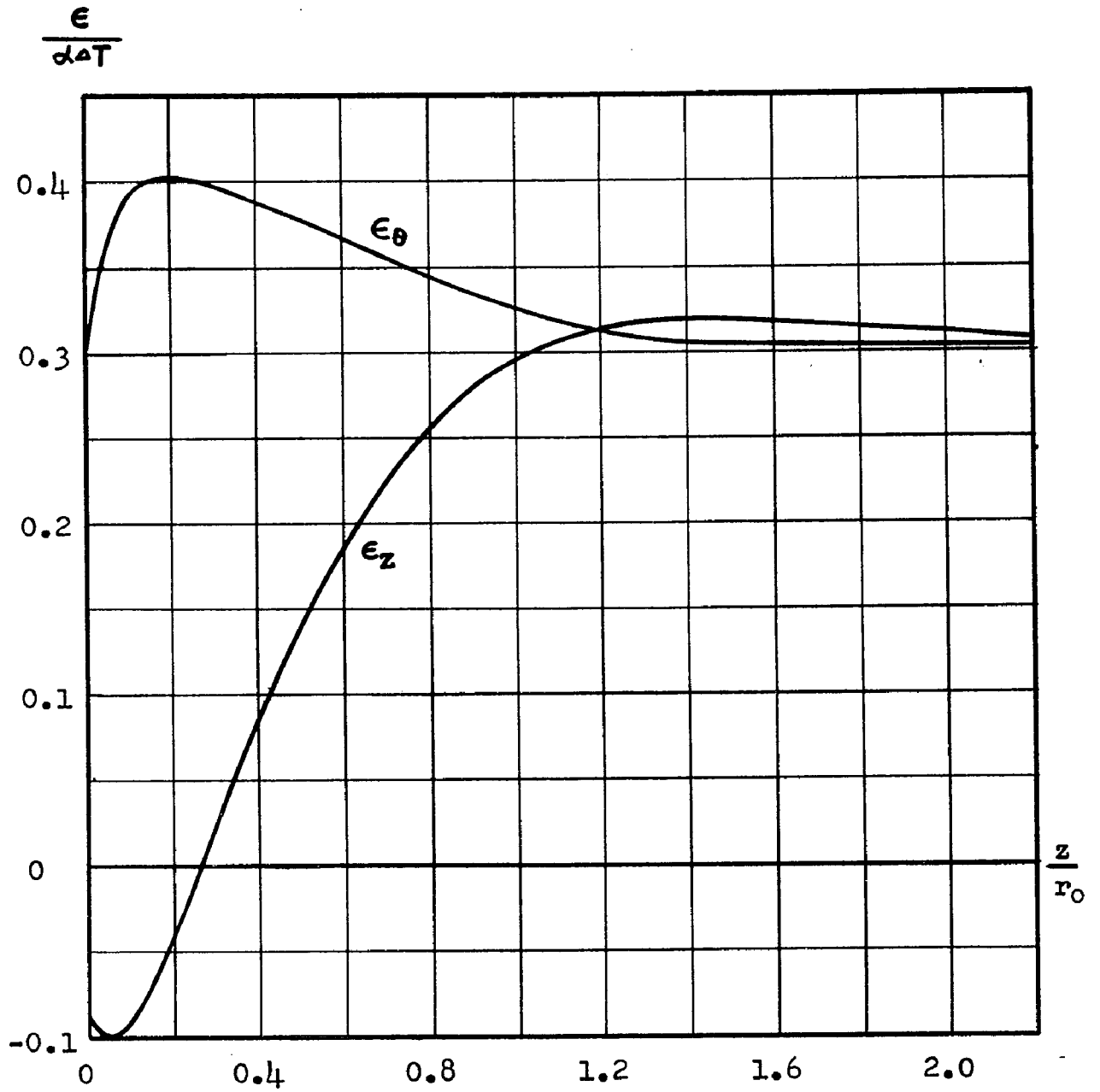


Fig. 13 Axial Distribution of Circumferential and Axial Strains on the Outer Boundary of the Semi-Infinite Circular Hollow Cylinder

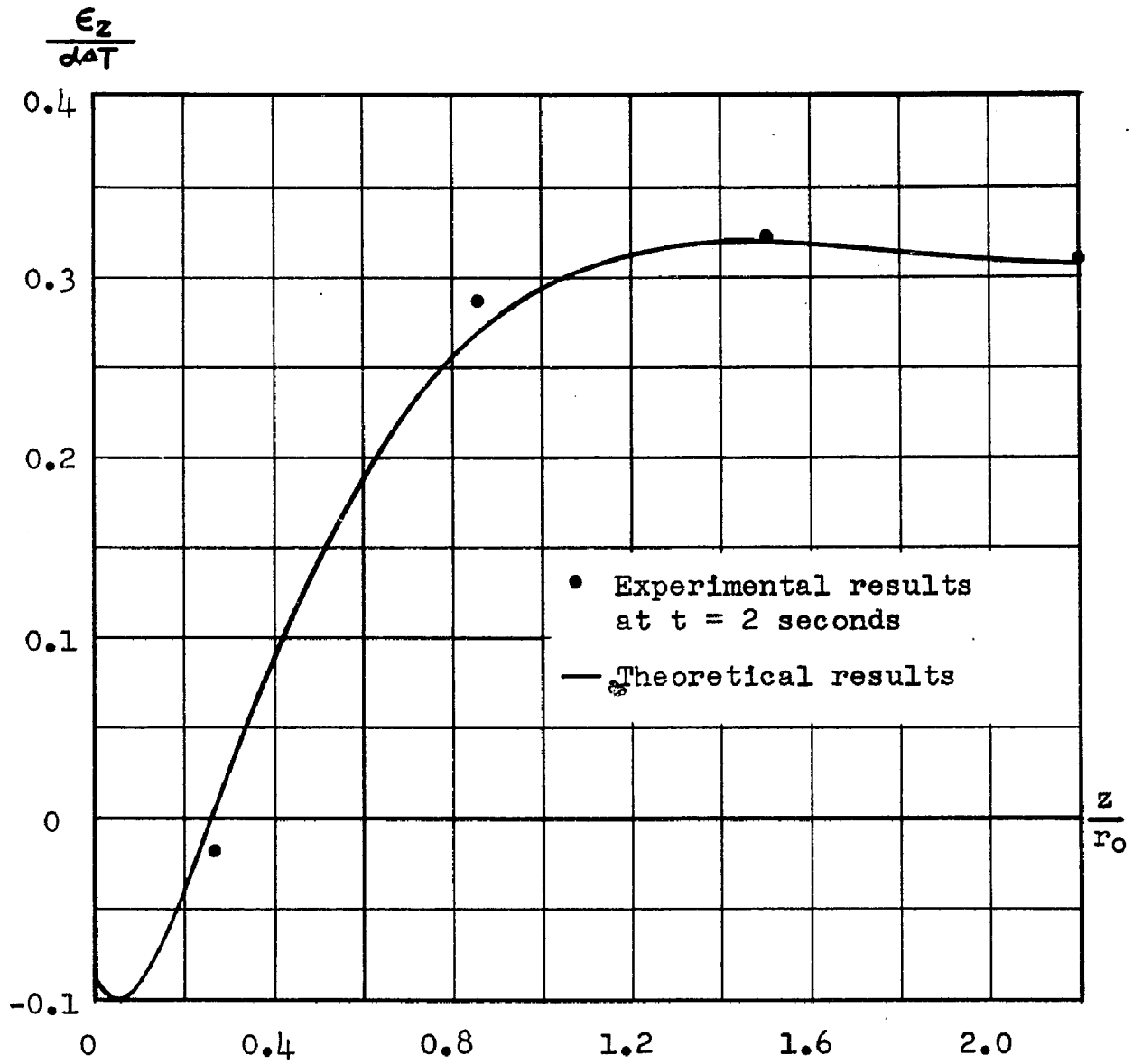


Fig. 14 Comparison of Theoretical and Experimental Results of Axial Strains on the Outer Boundary of the Semi-Infinite Circular Hollow Cylinder Near the End

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