Certain Decompositions Of Periodic Semigroups And Congruences On Unipotent Semigroups.

Citation

SEDLOCK, JAMES THOMAS. *Certain Decompositions Of Periodic Semigroups And Congruences On Unipotent Semigroups*. 1966, https://preserve.lehigh.edu/lehigh-scholarship/graduate-publications-theses-dissertations/theses-dissertations/certain-1.

66–11,834

SEDLOCK, James Thomas, 1940–
CERTAIN DECOMPOSITIONS OF PERIODIC SEMIGROUPS AND CONGRUENCES ON UNIPOTENT SEMIGROUPS.

Lehigh University, Ph.D., 1966
Mathematics

University Microfilms, Inc., Ann Arbor, Michigan

CERTAIN DECOMPOSITIONS OF PERIODIC SEMIGROUPS AND CONGRUENCES ON UNIPOTENT SEMIGROUPS

by
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A Dissertation

Presented to the Graduate Faculty

of Lehigh University

in Candidacy for the Degree of

Doctor of Philosophy

in

Mathematics

Lehigh University

1966

CERTIFICATE OF APPROVAL

Approved and recommended for acceptance as a dissertation in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

May 11, 1966
(date)

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Accepted, May 11, 1966
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ACKNOWLEDGEMENTS

The author wishes to sincerely thank Professors Mario Petrich and Everett Pitcher for their assistance in the completion of this dissertation.

The availability, useful suggestions, and prompt review of submitted manuscripts on the part of Professor Petrich were of inestimable aid to the author.

Without Professor Pitcher's resolution of the administrative entanglements, this work would have been technically impossible.

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ABSTRACT

The dissertation consists of five chapters, the first four of which deal exclusively with periodic semigroups; the last chapter concerns a certain type of semigroup with one idempotent.

Schwarz [Contribution to the theory of periodic semigroups, Czech. Math. J., 3(1953), 7-21] first introduced the natural equivalence relation $\mathcal{K}$ on a periodic semigroup S. Chapter 1 gives necessary and sufficient conditions that $\mathcal{K}$ coincide with any one of the Green's relations. In particular, $\mathcal{K} = \mathcal{H}$ if and only if S is a semilattice of completely simple semigroups; $\mathcal{K} = \mathcal{L}$ if and only if S is a semilattice of right groups; $\mathcal{K} = \mathcal{R}$ if and only if S is a semilattice of left groups; and $\mathcal{K} = \mathcal{D}$ if and only if S is a semilattice of groups.

Several characterizations of the congruence $\mathcal{K}^*$ generated by $\mathcal{K}$ are found in Chapter 2. From among these it is seen that $\mathcal{K}^*$ is the minimal band congruence on periodic semigroups S, and a $\mathcal{K}^*$-class A is a minimal subsemigroup of S having the property: for $x,y,z \in S$

$$xyz \in A \quad \text{if and only if} \quad xy^2z \in A.$$

Chapter 3 deals with $\mathcal{K}$ and the minimal band, semilattice, and matrix congruences on S, denoted by β, η, Σ respectively. To parallel the known result that $\mathcal{K} = \eta$ if and only if S is weakly commutative, it is proved that $\mathcal{K} = \Sigma$ if and only if S is weakly rectangular. In addition, S is briefly studied in the case when $\mathcal{K}$ is a congruence.

Since a periodic semigroup S is a disjoint union of subsets each containing one idempotent (namely, the $\mathcal{K}$-classes of S), it is natural to ask: when is S isomorphic to the Cartesian product of a rectangular band and a periodic semigroup containing one idempotent. Chapter 4 contains necessary and sufficient conditions that this be the case; one requirement is that the union of the maximal subgroups of S be a rectangular group.

In Chapter 5 we drop the hypothesis of periodicity and consider a semigroup S which is the ideal extension of a group by a z-semigroup (a semigroup containing exactly one idempotent, it being a zero). Such a semigroup can be parametrized by a triple consisting of a z-semigroup, a group, and a partial homomorphism. We give necessary and sufficient conditions on these three components in order that an equivalence relation on S be a congruence. Also, an inductive procedure similar to a technique of Tamura [The theory of construction of finite semigroups III, Osaka Math. J., 10(1958), 191-204] is developed for constructing all congruences of a certain type on S.

INTRODUCTION

A semigroup S is a non-empty set on which there is defined an associative multiplication. S is said to be periodic if each element of S has finite order, the order of an element being the order of the cyclic subsemigroup generated by the element. The class of periodic semigroups has been treated only occasionally in the literature, essential contributions having been made by Schwarz [12] and Yamada [19]. The principal results of the first four chapters of this work involve periodic semigroups. Only in the last chapter is the hypothesis of periodicity dropped.

In Chapter 1 we recall the natural equivalence relation $\mathcal{K}$ on a periodic semigroup S, which was first studied in [12]. The main results of this section are those which give necessary and sufficient conditions that $\mathcal{K}$ coincide with one of the well-known Green's equivalence relations, which are defined for an arbitrary semigroup. Theorems are proved relating $\mathcal{K}$ and each of the Green's equivalences.

The purpose of Chapter 2 is to characterize the congruence $\mathcal{K}^*$ on S generated by the natural equivalence $\mathcal{K}$. From among those descriptions given, we see that $\mathcal{K}^*$ is the minimal band congruence on S and its classes are minimal subsemigroups of S of a certain type.

Just as the first chapter considers $\mathcal{K}$ and the Green's relations, Chapter 3 deals with $\mathcal{K}$ and certain minimal congruences on periodic semigroup S; namely, the minimal band, semilattice, and matrix congruences. Here we find necessary and sufficient conditions in order that $\mathcal{K}$ equal any one of these minimal congruences. In addition, some facts concerning S in which $\mathcal{K}$ is a congruence are proved.

Chapter 4 has but one aim: the derivation of necessary and sufficient conditions in order that periodic semigroup S be a rectangular $\mathcal{K}$-class; that is, conditions that S be isomorphic to the Cartesian product of a rectangular band and a periodic semigroup with one idempotent.

In Chapter 5 we consider a semigroup S, not necessarily periodic, that is the ideal extension of a group by a z-semigroup (a semigroup with one idempotent, it being a zero), of which a periodic semigroup with one idempotent is a particular case. Such a semigroup S can be expressed in terms of a certain triple, and it is found that any congruence on S can be characterized by means of the components of the triple. Finally, an inductive procedure is developed for constructing congruences of a certain type on S.

Any undefined expressions or notation can be found in [2]. The symbol // will be used to denote the end of a proof.

CHAPTER 1

GREEN'S RELATIONS AND THE NATURAL EQUIVALENCE RELATION

Definition 1.1. Given semigroup S the Green's relations $\mathcal{L},\mathcal{R},\mathcal{J},\mathcal{D},\mathcal{H}$ on S are defined as follows: for $a,b \in S$

$$a\mathcal{L}b \quad \text{if and only if} \quad S^1a = S^1b$$

$$a\mathcal{R}b \quad \text{if and only if} \quad aS^1 = bS^1$$

$$a\mathcal{J}b \quad \text{if and only if} \quad S^1aS^1 = S^1bS^1$$

$$\mathcal{D} = \mathcal{L}\circ\mathcal{R} \quad \text{and} \quad \mathcal{H} = \mathcal{L} \cap \mathcal{R},$$

where the symbols $S^1, \circ, \cap$, and all subsequent unexplained notation, are those of [2].

From the definitions it can be proved that $\mathcal{D} = \mathcal{R}\circ\mathcal{L}$ and each relation is an equivalence. These and other fundamental properties of the Green's relations along with an intuitive discussion are given in 2.1 of [2]. It is often useful to observe that the adjunction of an identity to S leaves these relations undisturbed (Theorem 2 of [3]). If $x \in S$, then L_x $[R_x, J_x, D_x, H_x]$ will denote the $\mathcal{L}$ $[\mathcal{R},\mathcal{J},\mathcal{D},\mathcal{H}]$ class of S containing x. In particular, for each idempotent $e \in S$, H_e is the maximal subgroup of S containing e (ex.1, p.61 of [2]).

Definition 1.2. A semigroup S is called periodic if each element of S has a finite order, where the order of $x \in S$ is the order of the cyclic subsemigroup of S generated by x.

The results of this work are concerned primarily with periodic semigroups, and unless specified to the contrary (e.g. "for an arbitrary semigroup,..."), the hypothesis of periodicity will be assumed throughout. Of constant use will be the fact that for each element x of a periodic semigroup, some power of x is idempotent (ex.1, p.20 of [2]).

The following result is Theorem 3 of [3], so the proof will be omitted; it has been proved for a more general case (viz. stable semigroups) in [5]. In light of it we shall henceforth use $\mathcal{D}$ and $\mathcal{J}$ interchangeably.

Theorem 1.3. For a periodic semigroup, $\mathcal{D} = \mathcal{J}$.

The aforementioned fact that some power of each element in a periodic semigroup is idempotent suggests a natural relation on S; namely, the association of those elements whose powers yield the same idempotent.

Definition 1.4. Let S be a periodic semigroup. On S define the relation $\mathcal{K}$ by: for $a,b \in S$
$a\mathcal{K}b$ if and only if there exists an idempotent $e \in S$ and integers m,n such that $a^n = b^m = e$. This relation $\mathcal{K}$, easily verified to be an equivalence relation on S, is called the *natural equivalence relation on a periodic semigroup S*; its classes will be denoted by K^e, e idempotent.

For each $\mathcal{K}$-class K^e, the notions of the maximal group G^e and maximal semigroups contained in K^e were first defined and studied by Schwarz in [12]. Due to the remark immediately preceding Definition 1.2, $H_e = G^e$ for each idempotent e in a periodic semigroup.

Of immediate interest is theorem 1.1.43 of [8], which is stated here without proof.

Theorem 1.5. For each idempotent e in periodic semigroup S, $G^e = eK^e = K^e e$.

Corollary 1.6. If K^e is a subsemigroup of S, then G^e is a two-sided ideal in K^e.

Proof. From the preceding theorem,

$$K^e G^e = K^e e G^e = G^e G^e = G^e$$
$$\text{and } G^e K^e = G^e e K^e = G^e G^e = G^e.$$

The next result is the steppingstone to several later results.

Theorem 1.7. For each idempotent e in a periodic semigroup, $K^e \cap D_e = G^e$.

Proof. Since $G^e \subseteq K^e$ and $G^e = H_e \subseteq D_e$, it is immediate that $G^e \subseteq K^e \cap D_e$.

Conversely, let $z \in K^e \cap D_e$; then $xe \in K^e G^e = G^e \subseteq D_e$. From Corollary 1.1(5) of [1], we have $xe \in R_x \cap L_e$. Hence, $G^e = R_x \cap L_e$ and $G^e \subseteq R_x$. Similarly, $ex \in G^e K^e = G^e \subseteq D_e$ and $ex \in R_e \cap L_x$; so $G^e = R_e \cap L_x$ and $G^e \subseteq L_x$. Thus, $x \in L_x \cap R_x = G^e$.//

Corollary 1.8. For each idempotent e, $K^e \cap L_e = G^e$ and $K^e \cap R_e = G^e$.

Proof. Again $G^e \subseteq K^e$ and $G^e = H_e \subseteq L_e$, so it is immediate that $G^e \subseteq K^e \cap L_e$.

From the preceding lemma, $K^e \cap L_e \subseteq K^e \cap D_e = G^e$. Therefore, $K^e \cap L_e = G^e$.

Similarly, one shows $K^e \cap R_e = G^e$.//

Corollary 1.9. Semigroup S is a union of groups if and only if $\mathcal{K} \subseteq \mathcal{D}$.

Proof. If S is a union of groups, then for each idempotent e, $K^e = G^e = H_e \subseteq D_e$.

Conversely, if $\mathcal{K} \subseteq \mathcal{D}$, then for each idempotent e, $K^e = K^e \cap D_e = G^e$. Hence S is a union of its maximal subgroups.//

Definition 1.10. (6.4 of [9]). Semigroup S is *weakly commutative* if for each $a,b \in S$ there exist $x,y \in S$ and an integer k such that $(ab)^k = xa = by$.

The next corollary generalizes Theorem 14 of [12], which deals with the commutative case.

Corollary 1.11. If S is a weakly commutative periodic semigroup, then $\mathcal{D} \subseteq \mathcal{K} = \eta$ and each maximal subgroup is a $\mathcal{D}$-class of S.

Proof. From the characterization given in [9] of the minimal semilattice congruence η, it follows that $\mathcal{J} \subseteq \eta$ for an arbitrary semigroup. In particular (Theorem 6.7 of [9]), if S is a weakly commutative periodic semigroup, then $\mathcal{D} = \mathcal{J} \subseteq \eta = \mathcal{K}$. Hence, each $\mathcal{K}$-class is a subsemigroup of S and a union of $\mathcal{D}$-classes.

Moreover, combining $\mathcal{D} \subseteq \mathcal{K}$ with Lemma 1.7 above yields $G^e = K^e \cap D_e = D_e$.//

Definition 1.12. Semigroup S is a matrix of semigroups of type α if S is a disjoint union of semigroups of type α $\{S_i \mid i \in I, \text{ I an index set}\}$ such that the induced equivalence relation is a congruence and for each $i,j \in I$, $S_iS_jS_i \subseteq S_i$.

Lemma 1.13. For an arbitrary semigroup S, a $\mathcal{D}$-class D which is a union of groups is a matrix of groups.

Proof. If E is the set of idempotents of D, then $D = \cup\{H_e \mid e \in E\}$ is a disjoint union of groups. Let $e,f \in E$ and $x,z \in H_e$, $y \in H_f$ be arbitrary. Since $R_y \cap L_x$ contains an idempotent, then by Theorem 2.17 of [2] we have $xy \in R_x \cap L_y = H_{xy}$, where $H_{xy} = H_g$ for some $g \in E$. Thus, $H_eH_f \subseteq H_g$ and the induced equivalence relation is a congruence.

Similarly, xy,z,xyz are $\mathcal{D}$-related and $xyz \in R_{xy} \cap L_z = R_x \cap L_x = H_x$; that is, $xyz \in H_{xyz} = H_x = H_e$. Therefore, $H_eH_fH_e \subseteq H_e$.//

Theorem 1.14. On periodic semigroup S the following are equivalent:

(a) S is a matrix of groups

(b) S is simple

(c) S is completely simple

(d) S is bisimple.

Proof. (a) implies (b). If S is a matrix of groups $\{G_i \mid i \in I, \text{ I an index set}\}$, then for each $i,j \in I$ we have

$G_iG_jG_i \subseteq G_i$. From this and the fact that any ideal of S is a union of groups, it follows that S is simple.

(b) implies (c) by Corollary 2.56 of [2].

(c) implies (d) by Theorem 2.51 of [2].

(d) implies (a). Since S is bisimple, then $\mathcal{K} \subseteq \mathcal{D}$; from Corollary 1.9 above S is a union of groups. That is, S is a single $\mathcal{D}$-class which is a union of groups. The result follows from Lemma 1.13.//

The proof of the next proposition follows easily from Corollary 1.6 and the definitions of the terms involved, hence shall be omitted. It is also a consequence of Theorem 7 of [12].

Proposition 1.15. On periodic semigroup S the following are equivalent:

(a) S is a union of groups

(b) each element of S has index one

(c) for each idempotent $e \in S$, $K^e = G^e$

(d) for each idempotent $e \in S$, K^e is a [right, left] simple semigroup.

Definition 1.16. Semigroup S is a semilattice of semigroups of type α if S is a disjoint union of semigroups of type α $\{S_i \mid i \in I, I \text{ index set}\}$, and for each $i,j \in I$ there exists a $k \in I$ such that $S_iS_j, S_jS_i \subseteq S_k$.

The minimal semilattice congruence η of an arbitrary semigroup has been determined modulo prime ideals in [9]; its congruence classes shall be called $\mathfrak{h}$-classes of the semigroup. As previously mentioned, a periodic semigroup is weakly commutative if and only if the $\mathfrak{h}$ and $\mathcal{H}$ relations coincide (Theorem 6.7 of [9]). This fact shall be used often and without reference in the sequel.

<u>Lemma 1.17</u>. If S is a weakly commutative periodic semigroup and for each idempotent e, $K^e = G^e$, then the set E of idempotents of S is a subsemigroup.

<u>Proof</u>. Since S is weakly commutative, then $\mathfrak{h} = \mathcal{H}$ and each $\mathfrak{h}$-class of S is a maximal group. Clearly, for each $\mathfrak{h}$-class G^e, $E \cap G^e = \{e\}$ is a subsemigroup of S. By Lemma 1 of [10], E is a subsemigroup of S.//

<u>Definition 1.18</u>. (cf. p.1 of [19]). Semigroup S is said to satisfy <u>Condition C</u> if for each $a,b \in S$ and any positive integers m,n there exist positive integers r,s and t such that

$$(ab)^r = (a^m b^n)^s = (b^n a^m)^t.$$

<u>Lemma 1.19</u>. Periodic semigroup S is weakly commutative if and only if it satisfies Condition C.

<u>Proof</u>. If S is weakly commutative, then $\mathfrak{h} = \mathcal{H}$ and S is a semilattice of unipotent homogroups; namely, the $\mathcal{H}$-classes of S. By Theorem 3 of [19], S satisfies Condition C.

Conversely, let S satisfy Condition C. For $a,b \in S$ there exist integers r,t such that $(ab)^r = (ba)^t$. Hence,
$(ab)^r = b[a(ba)^{t-1}] = [(ba)^{t-1}b]a$.//

Definition 1.20. ([18]). Semigroup S is strongly reversible if for each a,b $\in$ S there exist positive integers r,s,t such that $(ab)^r = a^s b^t = b^t a^s$.

Theorem 1.21. The following are equivalent on a periodic semigroup S with set E of idempotents:

(a) S is a union of groups and E is commutative

(b) S is a semilattice of groups

(c) S is weakly commutative and each $\mathcal{K}$-class is a simple semigroup

(d) S is strongly reversible and each $\mathcal{K}$-class is a simple semigroup.

Proof. (a) implies (b). This follows directly from Corollary 2 of [10].

(b) implies (c). Since S is a semilattice of (periodic) groups, then it is trivially a semilattice of unipotent homogroups. By Theorem 3 of [19], S satisfies Condition C, and by Lemma 1.19, it is weakly commutative. That each $\mathcal{K}$-class is a simple semigroup follows from Proposition 1.15.

(c) implies (d). Since each $\mathcal{K}$-class is a simple semigroup, then S is a union of groups. By Lemma 1.17 E is a subsemigroup of S. That S is strongly reversible comes from Corollary 6.8 of [9].

(d) implies (a). Again it is immediate that S is a union of of groups. From the aforementioned Corollary 6.8 of [9], E is commutative.//

Theorem 1.22. The following are equivalent on a periodic semigroup S:

(a) S is a semilattice of groups

(b) Each $\mathcal{D}$-class of S is a group

(c) $\mathcal{K} = \mathcal{D}$

(d) $\mathcal{K} = \eta$, and each left and each right ideal of S is the union of $\mathcal{K}$-classes.

Proof. (a) implies (b). If S is a semilattice of groups, then it is a union of groups and for each idempotent e, $K^e = G^e = H_e$; thus $\mathcal{K} = \mathcal{H}$. It is easily verified that any semilattice congruence on S must be coarser than $\mathcal{K}$, so in this case $\mathcal{K} = \mathcal{H}$ is the minimal semilattice congruence η on S. As noted previously in the proof of Corollary 1.1, for an arbitrary semigroup $\mathcal{J} \subseteq \eta$. Here S is periodic and we have $\mathcal{D} = \mathcal{J} \subseteq \eta = \mathcal{K} = \mathcal{H}$. However, it follows from the definitions that $\mathcal{H} \subseteq \mathcal{L} \subseteq \mathcal{D}$ and $\mathcal{H} \subseteq \mathcal{R} \subseteq \mathcal{D}$; so all of the Green's relations coincide.

We have already seen that each $\mathcal{H}$-class is a maximal subgroup of S.

(b) implies (c). Let D be a $\mathcal{D}$-class of S. By hypothesis it is some maximal subgroup of S; that is, for some idempotent e, $D = G^e \subseteq K^e$. Hence, $\mathcal{D} \subseteq \mathcal{K}$.

On the other hand, if each $\mathcal{D}$-class of S is a group, then S is a union of groups and $\mathcal{K} \subseteq \mathcal{D}$.

(c) implies (d). Since $\mathcal{K} \subseteq \mathcal{D}$, then S is a union of groups; that is, each $\mathcal{K}$-class is a maximal group. From this it follows that each left and each right ideal of S is a union of $\mathcal{K}$-classes. By Theorem 4.5 of [2], S is a semilattice of $\mathcal{D} = \mathcal{K}$-classes; hence, $\mathfrak{h} \subseteq \mathcal{K}$. But $\mathcal{K}$ must be finer than $\mathfrak{h}$, so $\mathcal{K} = \mathfrak{h}$.

(d) implies (a). Again S is weakly commutative, since $\mathfrak{h} = \mathcal{K}$. Recall that in an arbitrary semigroup elements a,b are $\mathcal{J}$-related if and only if for each (two-sided) ideal I either $a,b \in I$ or $a,b \notin I$. Fix idempotent $e \in S$. By hypothesis for any (two-sided) ideal I either $K^e \subseteq I$ or $K^e \cap I = \emptyset$; that is, for any $x,y \in K^e$ either $x,y \in I$ or $x,y \notin I$; that is, for any $x,y \in K^e$, $x\mathcal{J}y$. Therefore, $K^e \subseteq J_e = D_e$. Since e is arbitrary, $\mathcal{K} \subseteq \mathcal{D}$.

This shows that S is a union of groups, or equivalently, that each $\mathcal{K}$-class is a simple semigroup. The result follows from Theorem 1.21.//

Definition 1.23. A subset of a semigroup is called unipotent if it contains one and only one idempotent.

Theorem 1.24. The following are equivalent on a periodic semigroup S:

(a) $\mathcal{K} = \mathcal{D}$

(b) $\mathcal{K} = \mathcal{H}$ and each $\mathcal{L}$-class and each $\mathcal{R}$-class is unipotent

(c) $\mathcal{K} = \mathcal{L}$ and each $\mathcal{R}$-class is unipotent

(d) $\mathcal{K} = \mathcal{R}$ and each $\mathcal{L}$-class is unipotent.

Proof. (a) implies (b). If $\mathcal{K} = \mathcal{D}$, then S is a union of groups and, recalling Lemma 1.7, for each idempotent e

$$H_e = G^e = K^e \cap D_e = D_e.$$

Therefore, $\mathcal{H} = \mathcal{D} = \mathcal{K}$. However, $\mathcal{H} \subseteq \mathcal{L} \subseteq \mathcal{D}$ and $\mathcal{H} \subseteq \mathcal{R} \subseteq \mathcal{D}$, so all Green's relations coincide. Thus, each $\mathcal{L}$-class and each $\mathcal{R}$-class, being a maximal group, is unipotent.

(b) implies (c). Let e be idempotent. By hypothesis $G^e = H_e = K^e$, hence each $\mathcal{H}$-class of S is a group. Therefore L_e, being unipotent and the union of $\mathcal{H}$-classes, is precisely H_e; that is, $L_e = H_e = K^e$. Since e is arbitrary, $\mathcal{L} = \mathcal{K}$.

(c) implies (d). Assuming $\mathcal{L} = \mathcal{K}$, then $\mathcal{K} = \mathcal{L} \subseteq \mathcal{D}$ and S is a union of groups. Let idempotent e be arbitrary. From Corollary 1.8, $H_e = G^e = K^e \cap L_e = L_e \cap L_e = L_e$. Moreover, since R_e is unipotent and a union of $\mathcal{H}$-classes, each $\mathcal{H}$-class being a union of groups, then $R_e = H_e = G^e$. Therefore, $\mathcal{H} = \mathcal{L} = \mathcal{R}$ and the result is immediate.

(d) implies (a). Using the dual proof of that given in (c) implies (d), one arrives at the same conclusion, $\mathcal{H} = \mathcal{L} = \mathcal{R}$. Thus,

$$\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{R} = \mathcal{R} = \mathcal{K}. //$$

Corollary 1.25. In case any of the statements appearing in Theorems 1.21, 1.22, 1.24 are satisfied, then each of the following is true:

(a) every ideal in S is two-sided and a union of groups

(b) for each idempotent e,

$$K^e = G^e = \{x \in S \mid e \in Sx \cap xS,\ e \text{ two-sided identity for } x\}$$

(c) each Green's relation coincides with $\mathcal{K}$, it being a congruence

(d) S is simple if and only if it is a group.

Proof. (a). Let I be an ideal in S. From Theorem 4.6 of [9], I is two-sided; by Theorem 1.22, I is the union of $\mathcal{H}$-classes, each being a maximal subgroup.

(b). This also follows from Theorem 4.6 of [9], and the fact the $Se \cap eS = eSe$ is the set of all elements of S for which e is a two-sided identity.

(c). From Theorem 1.24 it is immediate that the Green's relations coincide with $\mathcal{H}$. Since the product of any $\mathcal{L}$-class and any $\mathcal{R}$-class is contained in a $\mathcal{D}$-class, this equivalence relation is a congruence (Theorem 2.4 of [2]).

(d). We know that S is a union of groups and $\mathcal{H} = \mathcal{D} = \mathcal{J}$.

Moreover, S is simple
if and only if $S^1aS^1 = S$ for each $a \in S$,
if and only if S is a single $\mathcal{H}$-class,
if and only if S is a group.//

The succeeding definition and lemmas are aimed toward determining necessary and sufficient conditions that $\mathcal{L} = \mathcal{H}$. The dual statements relative to $\mathcal{R}$ are omitted.

Definition 1.26. A semigroup is a right group if it is right simple and left cancellative.

Lemma 1.27. For an arbitrary semigroup S, any $\mathcal{R}$-class R which is a union of groups is a right group.

Proof. It is well-known that the set of idempotents of any $\mathcal{R}$-class from a right zero semigroup, and for any $\mathcal{D}$-related idempotents e,f, groups H_e and H_f are isomorphic.

Thus, if E is the set of idempotents of R, and H is a fixed $\mathcal{H}$-class contained in R, it is easily established that $R \cong E \times H$. In light of Theorem 1.27 of [2], R is a right group.//

Lemma 1.28. If an arbitrary semigroup S is a semilattice of right groups, then $\mathfrak{n} = \mathcal{J}$ and each of these right groups is a $\mathcal{J}$-class of S.

Proof. Let $\mathfrak{m}$ be a semilattice congruence on S such that each $\mathfrak{m}$-class is a right group. Now a right group is a completely simple semigroup, thus S is a union of completely simple semigroups. By Theorem 4.5 of [2], S is a semilattice of $\mathcal{J}$-classes; hence $\mathfrak{n} \subseteq \mathcal{J}$. However, we have already noted that $\mathcal{J} \subseteq \mathfrak{n}$ in any semigroup; thus $\mathfrak{n} = \mathcal{J}$.

Also, $\mathfrak{n} \subseteq \mathfrak{m}$, where each $\mathfrak{m}$-class is a right group. Recalling the characterization of a right group given in Lemma 1.27, let $M \cong E \times G$ be an arbitrary $\mathfrak{m}$-class of S. For any $(e,a),(f,b) \in M$ it is easily verified that $S^1(e,a)S^1 = S^1(f,b)S^1$. That is, $x\mathfrak{m}y$ implies $x\mathcal{J}y$; or $\mathfrak{m} \subseteq \mathcal{J} = \mathfrak{n}$. Thus, $\mathfrak{m} = \mathfrak{n} = \mathcal{J}$ and each right group is a $\mathcal{J}$-class of S.//

Theorem 1.29. The following are equivalent on a periodic semigroup S:

(a) $\mathcal{L} = \mathcal{K}$

(b) $\mathcal{D} = \mathcal{R}$ and each $\mathcal{D}$-class is a completely simple semigroup

(c) $\mathcal{H} = \mathcal{K}$ and each $\mathcal{L}$-class is unipotent

(d) S is a semilattice of right groups.

Proof. (a) implies (b). Assuming $\mathcal{L} = \mathcal{K}$, then $\mathcal{K} = \mathcal{L} \subseteq \mathcal{D}$ and S is a union of groups; from Lemma 1.13 each $\mathcal{D}$-class is a completely simple semigroup.

Moreover, for each idempotent e,

$$H_e = G^e = K_e \cap L_e = L_e \cap L_e = L_e .$$

Therefore, $\mathcal{H} = \mathcal{L}$ and $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{H} \circ \mathcal{R} \subseteq \mathcal{R}$; hence $\mathcal{D} = \mathcal{R}$.

(b) implies (c). Since each $\mathcal{D}$-class of S is a completely simple semigroup,, then each $\mathcal{D}$-class is a union of groups, and S is likewise. Thus for each idempotent e, $H_e = G^e = K^e$; that is, $\mathcal{H} = \mathcal{K}$. Moreover, $\mathcal{D} = \mathcal{R}$ so $L_e = D_e \cap L_e = R_e \cap L_e = H_e = G^e$. Hence each $\mathcal{L}$-class is unipotent.

(c) implies (d). By hypothesis $\mathcal{H} = \mathcal{K}$, so $\mathcal{K} = \mathcal{H} \subseteq \mathcal{D}$ and S is a union of groups. But each $\mathcal{L}$-class is unipotent, so for each idempotent e, $L_e = G^e = H_e$. Thus, $\mathcal{L} = \mathcal{H}$,and $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{H} \circ \mathcal{R} \subseteq \mathcal{R}$; hence $\mathcal{D} = \mathcal{R}$.

From Theorem 4.5 of [2] we infer that S is a semilattice of $\mathcal{R}$-classes. In this case each $\mathcal{R}$-class is a union of groups, so the result follows from Lemma 1.27.

(d) implies (a). From Lemma 1.28, $\eta = \mathcal{J} = \mathcal{D}$ and each of these right groups is a $\mathcal{D}$-class of S. Again, a right group is completely simple, so each $\mathcal{D}$-class (and likewise S) is a union of groups.

Let L_e be an $\mathcal{L}$-class of S. Since L_e is a union of $\mathcal{H}$-classes, each a maximal subgroup of S, let H_f be any $\mathcal{H}$-class contained in L_e. The idempotents of any $\mathcal{L}$-class form a left-zero semigroup, so ef = e. However, the idempotents of right group D_e form a right-zero semigroup, so ef = f. That is, e = ef = f; hence, $L_e = H_e = G^e = K^e$. Since $\mathcal{L}$-class L_e is arbitrary, $\mathcal{L} = \mathcal{H}$.//

Definition 1.30. Semigroup S is left weakly commutative if for each $a,b \in S$ there exist $x \in S$ and an integer k such that

$$(ab)^k = bx.$$

Lemma 1.31. Let a periodic semigroup S be a union of groups. Then S is left weakly commutative if and only if $\mathcal{R} = \mathcal{D}$.

Proof. Suppose S is left weakly commutative. Let $a,b \in S$ and $a\mathcal{L}b$. Then $S^1a = S^1b$ and there exist $x,y \in S^1$ such that $a = xb$, $b = ya$. Now the hypothesis that S is left weakly commutative implies there exist $z \in S$ and an integer k such that $(xb)^k = bz$. Also, the index of a is one, for a is in some maximal group, so choose integer $n > k$ such that $a^n = a$. Then $a = a^n = (xb)^n = (xb)^k(xb)^{n-k} = bz(xb)^{n-k} \in bS^1$. Similarly, one shows $b \in aS^1$. Hence, $a\mathcal{R}b$ and $\mathcal{L} \subseteq \mathcal{R}$. Therefore $\mathcal{D} = \mathcal{L}\circ\mathcal{R} \subseteq \mathcal{R}\circ\mathcal{R} = \mathcal{R}$, and $\mathcal{R} = \mathcal{D}$.

Conversely, suppose $\mathcal{D} = \mathcal{R}$. By Theorem 4.5 Of [2], S is a semilattice of $\mathcal{R}$-classes. So for any $a,b \in S$ $ab\mathcal{R}ba$. Hence, $abS^1 = baS^1$ and there exists an $x \in S^1$ such that $ab = b(ax)$.//

Corollary 1.32. Let periodic semigroup S be a union of groups. Then all Green's relations on S coincide if and only if S is weakly commutative.

Proof. If $\mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{D}$, then by the preceding lemma and its dual, S is both right and left weakly commutative. To see that S is weakly commutative, let $a,b \in S$. Then there exist $x,y \in S$ and integers r,s such that $(ab)^r = bx$, $(ab)^s = ya$. Hence,
$(ab)^{rs} = [(ab)^r]^s = (bx)^s = b[x(bx)^{s-1}]$ and
$(ab)^{rs} = [(ab)^s]^r = (ya)^r = [(ya)^{r-1}y]a$.

Conversely, a weakly commutative semigroup is trivially both right and left weakly commutative; so again by the preceding lemma and its dual, $\mathcal{R} = \mathcal{D} = \mathcal{L}$. By definition $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, and we know $\mathcal{D} = \mathcal{J}$.//

Theorem 1.33. The following are equivalent on a periodic semigroup S:

(a) S is left weakly commutative and each $\mathcal{H}$-class is a simple semigroup

(b) $\mathcal{R} = \mathcal{D}$ and each $\mathcal{D}$-class is a completely simple semigroup.

Proof. (a) implies (b). Since each $\mathcal{H}$-class is a simple semigroup, then S is a union of groups. That each $\mathcal{D}$-class is a completely simple semigroup follows from Lemma 1.13. Due to Lemma 1.31, $\mathcal{R} = \mathcal{D}$.

(b) implies (a). Given each $\mathcal{D}$-class is a completely simple semigroup, then S is a union of groups; that is, each $\mathcal{H}$-class is a simple semigroup. Again, S is left weakly commutative by Lemma 1.31.//

Corollary 1.34. In case any of the statements appearing in Theorems 1.29 or 1.33 are fulfilled, then each of the following is true:

(a) $\eta = \mathcal{D}$

(b) The set E of idempotents is a subsemigroup of S

(c) The maximal semilattice homomorphic images of E and of S are isomorphic.

Proof. (a). This is immediate from Lemma 1.28.

(b). In each case S is a union of groups, and the idempotents of each $\mathcal{R} = \mathcal{D} = \eta$-class form a right zero semigroup. The result follows from Lemma 1 of [10].

(c). Let $\eta, \bar{\eta}$ be the minimal semilattice congruences on S and E respectively. We shall write $N_x [\bar{N}_x]$ for the η-class of S [$\bar{\eta}$-class of E] containing the element x. It suffices to show that for e,f ∈ E: $e\bar{\eta}f$ if and only $e\eta f$.

Let $e\bar{\eta}f$. Then efe = e, fef = f (cf. Theorem 1 of [7]) and we have

$$N_e = N_{efe} = N_eN_fN_e = N_eN_eN_f = N_eN_f$$

$$N_f = N_{fef} = N_fN_eN_f = N_eN_fN_f = N_eN_f.$$

Hence, $N_e = N_f$; or $e\eta f$.

Conversely, let $e\eta f$. As noted in (b) above, the idempotents of each $\mathcal{R} = \mathcal{D} = \eta$-class form a right zero semigroup. Therefore, efe = e and fef = f; that is, $e\bar{\eta}f$.//

For the sake of completeness in our discussion of the Green's relations, we conclude this chapter with necessary and sufficient conditions that $\mathcal{K} = \mathcal{H}$. Their proof is immediate from Proposition 1.15 and the oft-mentioned Clifford's theorem (Theorem 4.5 of [2]).

Theorem 1.35. The following are equivalent on a periodic semigroup S:

(a) $\mathcal{K} = \mathcal{H}$

(b) S is a union of groups

(c) S is a semilattice of completely simple semigroups.

CHAPTER 2

CHARACTERIZATIONS OF THE CONGRUENCE GENERATED BY $\mathcal{K}$

The existence of certain maximal decompositions of an arbitrary semigroup, and their corresponding minimal congruences, has been proved in [14]. The minimal semilattice congruence η was mentioned briefly in Chapter 1 and shall be encountered again in a later chapter. Of particular interest in this section is the minimal band congruence β. For on a periodic semigroup S it leads to our first characterization of the congruence $\mathcal{K}^*$ generated by $\mathcal{K}$ — that is, the smallest congruence on S containing the natural equivalence relation $\mathcal{K}$.

Definition 2.1. A congruence σ on a semigroup S is called a band or idempotent congruence if for each $x \in S$, $x\sigma x^2$.

Lemma 2.2. For a congruence σ on a periodic semigroup S, $\mathcal{K} \subseteq \sigma$ if and only if σ is a band congruence.

Proof. From the definition of the $\mathcal{K}$-relation given in 1.4, it follows that for each $x \in S$, $x\mathcal{K}x^2$. If $\mathcal{K} \subseteq \sigma$, then for each $x \in S$, $x\sigma x^2$.

Conversely, let $x\mathcal{K}y$. Again, from the definition of $\mathcal{K}$, there exist integers m,n and an idempotent e such that $x^m = y^n = e$. Since σ is a band congruence, then $x\sigma x^2$, $x^2\sigma x^3$, ..., $x^{m-1}\sigma x^m = e$; thus $x\sigma e$. Similarly, one shows $y\sigma e$. Hence $x\sigma y$.//

Theorem 2.3. For a periodic semigroup S, $\mathcal{K}^*$ is the minimal band congruence on S.

Proof. By Lemma 2.2 $\mathcal{K} \subseteq \beta$, where β is the minimal band congruence on S; therefore $\mathcal{K}^* \subseteq \beta$. However $\mathcal{K} \subseteq \mathcal{K}^*$, so by the same lemma $\mathcal{K}^*$ is a band congruence on S. From the minimality of β, $\beta \subseteq \mathcal{K}^*$.//

<u>Notation 2.4</u>. i) If A,B are subsets of a given set, we shall use the symbolism $A \between B$ to mean $A \cap B \neq \emptyset$, where $\cap$ is the usual set-theoretic intersection and $\emptyset$ is the empty set.

ii) Whenever simplicity of notation warrants it, we shall denote a $\mathcal{K}$-class of a periodic semigroup S by K_i, i some positive integer, rather than the more common K^e, e an idempotent of S.

In light of these remarks and our previously defined notation, the following result is an immediate corollary of Theorem 1.8 of [2].

<u>Theorem 2.5</u>. Let S be a periodic semigroup and $a,b \in S$. Then $a\mathcal{K}^*b$ if and only if there exist $x_i, y_i \in S^1$ and $\mathcal{K}$-classes K_i of S $(i = 1,2,\ldots,n)$ such that

$$\{a\} \between x_1K_1y_1 \between x_2K_2y_2 \between \cdots \between x_nK_ny_n \between \{b\}.$$

<u>Theorem 2.6</u>. (cf. Theorem 4.2 of [4]). On a periodic semigroup S the following are equivalent:

(a) $\mathcal{K} = \mathcal{K}^*$

(b) If $x,y \in S^1$ and $xK^ey \between K^f$, then $xK^ey \subseteq K^f$.

<u>Proof</u>. (a) implies (b). This follows directly from the definition of a congruence.

(b) implies (a). Certainly $\mathcal{K} \subseteq \mathcal{K}^*$; so let $a\mathcal{K}^*b$, where $a \in K^e$ and $b \in K^f$. From the preceding theorem there exist $x_1,y_1,\ldots,x_n,y_n \in S^1$ and $\mathcal{K}$-classes $K_1,\ldots,K_n$ of S such that

$$\{a\} \between K^e \between x_1K_1y_1 \between \cdots \between x_nK_ny_n \between K^f \between \{b\}.$$

By hypothesis, $x_1K_1y_1 \subseteq K^e$, hence

$$\{a\} \between K^e \between x_2K_2y_2 \between \cdots \between x_nK_ny_n \between K^f \between \{b\}.$$

Continuing this procedure yields

$$\{a\} \between K^e \between K^f \between \{b\}.$$

Thus, $e = f$ and $a\mathcal{K}b$.//

Theorem 2.7. Let e be an idempotent in a periodic semigroup S, A the $\mathcal{K}^*$-class containing e, $A_o = K^e$, and for integer $n \geq 1$ $A_n = \bigcup\{xKy \mid x,y \in S^1$, K is a $\mathcal{K}$-class of S, and $xKy \between A_{n-1}\}$. Then $A = \bigcup_{k=1}^{\infty} A_k$.

Proof. If $a \in A$, then $a\mathcal{K}^*e$ and for some $x_1,y_1,\ldots,x_n,y_n \in S^1$ and $\mathcal{K}$-classes $K_1,\ldots,K_n$ we have

$$\{e\} \between x_1K_1y_1 \between \cdots \between x_nK_ny_n \between \{a\}.$$

Hence,

$$x_1K_1y_1 \subseteq A_1,\ x_2K_2y_2 \subseteq A_2, \ldots\ldots,\ a \in x_nK_ny_n \subseteq A_n.$$

Conversely, since $\mathcal{K} \subseteq \mathcal{K}^*$, then $A_o = K^e \subseteq A$. Assume that $A_o, A_1, \ldots, A_{n-1}$ are each contained in A and let $a \in A_n$. For some $x,y \in S^1$ and $\mathcal{K}$-class K, $a \in xKy$ and $xKy \between A_{n-1}$. Therefore, $a \in xKy \between A_{n-1} \subseteq A$. But again $\mathcal{K} \subseteq \mathcal{K}^*$, and A is a $\mathcal{K}^*$-class of S, hence $xKy \subseteq A$; that is, $a \in xKy \subseteq A$. Thus $A_n \subseteq A$.//

Definition 2.8. (cf. [17]). Let K be a fixed $\mathcal{K}$-class of a periodic semigroup S. On S define relation $\mathcal{E}_K$ by: for $a,b \in S$ $a\mathcal{E}_Kb$ if and only if either $a = b$ or there exist $x_1,y_1,\ldots,x_ny_n \in S^1$ such that $\{a\} \between x_1Ky_1 \between x_2Ky_2 \between \cdots\cdots \between x_nKy_n \between \{b\}$.

It is easily verified that $\mathcal{E}_K$ is a congruence on S; in fact it is the minimal congruence σ on S such that K is contained in some congruence class of σ ([17]). Now let $\mathcal{E}_o = \cup\{\mathcal{E}_K \mid K \text{ a } \mathcal{K}\text{-class of } S\}$, and let $\mathcal{E}$ be the congruence on S generated by $\mathcal{E}_o$.

<u>Theorem 2.9</u>. For a periodic semigroup S, $\mathcal{K}^* = \mathcal{E}$.

<u>Proof</u>. From the definition of $\mathcal{E}_K$ it follows that $\mathcal{E}$, the minimal congruence containing $\mathcal{E}_o$, contains the equivalence relation $\mathcal{K}$. By the minimality of $\mathcal{K}^*$, $\mathcal{K}^* \subseteq \mathcal{E}$.

On the other hand, from the characterization of $\mathcal{K}^*$ given in Theorem 2.5 it is immediate that $\mathcal{E}_K \subseteq \mathcal{K}^*$ for each $\mathcal{K}$-class K of S. Hence, $\mathcal{E}_o \subseteq \mathcal{K}^*$ and by the minimality of $\mathcal{E}$, $\mathcal{E} \subseteq \mathcal{K}^*$.//

<u>Definition 2.10</u>. A complex A of semigroup S is called <u>exponential</u> if for any $x,y,z \in S^1$

$$xyz \in A \quad \text{if and only if} \quad xy^2z \in A.$$

<u>Lemma 2.11</u>. Let A be an exponential complex of an arbitrary semigroup S; $x,y \in S$; and m,n positive integers. Then

$$xy \in A \quad \text{if and only if} \quad x^my^n \in A.$$

<u>Proof</u>. If m,n are arbitrary integers, then

$$\begin{aligned} & xy = 1xy \in A \\ \text{if and only if}\quad & 1x^2y = xxy \in A \\ \text{if and only if}\quad & xx^2y = x^2xy \in A \\ & \quad\vdots \qquad\qquad \vdots \\ \text{if and only if}\quad & x^{m-2}x^2y = x^my = x^my1 \in A \end{aligned}$$

$$
\begin{aligned}
&\text{if and only if} && x^m y^2 1 = x^m yy \in A \\
&\text{if and only if} && x^m y^2 y = x^m yy^2 \in A \\
&\qquad\vdots && \qquad\vdots \\
&\text{if and only if} && x^m y^2 y^{n-2} = x^m y^n \in A.//
\end{aligned}
$$

Lemma 2.12. The non-empty intersection of two exponential subsemigroups of an arbitrary semigroup S is an exponential subsemigroup of S.

Proof. Let A,B be exponential subsemigroups of semigroup S. Certainly $A \cap B$, if it is non-empty, is a subsemigroup of S. In this case, for any $x,y,z \in S^1$,

$$
\begin{aligned}
& && xyz \in A \cap B \\
&\text{if and only if} && xyz \in A \text{ and } xyz \in B \\
&\text{if and only if} && xy^2z \in A \text{ and } xy^2z \in B \\
&\text{if and only if} && xy^2z \in A \cap B.//
\end{aligned}
$$

Lemma 2.13. The classes of a band congruence σ on an arbitrary semigroup S are exponential subsemigroups of S.

Proof. It follows directly from the definition of a band congruence that each σ-class is a subsemigroup of S. For any $x \in S$, $x \sigma x^2$; so for any $x,y,z \in S^1$ and σ-class A

$$xyz \in A \quad \text{if and only if} \quad xy^2z \in A.//$$

Lemma 2.14. Let A be an exponential subsemigroup of periodic semigroup S, K a $\mathcal{K}$-class of S, and $x,z \in S^1$. Then

$$xKz \between A \quad \text{if and only if} \quad xKz \subseteq A.$$

Proof. If $K = K^e$ and $y^n = e$, then $xyz \in A$

$$\begin{aligned}
&\text{implies} && (xy)yz = xy^2z \in A \\
&\text{implies} && (xy^2)yz = xy^3z = (xy)y^2z \in A \\
&\text{implies} && (xy^3)yz = xy^4z = (xy^2)y^2z \in A \\
&\quad\vdots && \qquad\vdots \\
&\text{implies} && (xy^{n-1})yz = xez = (xy^{n-2})y^2z \in A.
\end{aligned}$$

Hence, if $a^m = e$, then

$$\begin{aligned}
& && xez = xa^mz = (xa^{m-2})a^2z \in A \\
&\text{implies} && (xa^{m-3})a^2z = (xa^{m-2})az \in A \\
&\text{implies} && (xa^{m-4})a^2z = (xa^{m-3})az \in A \\
&\quad\vdots && \qquad\vdots \\
&\text{implies} && xa^2z = (xa)az \in A \\
&\text{implies} && xaz \in A.
\end{aligned}$$

Therefore, $xKz \subseteq A$.

The converse is obvious.//

Corollary 2.15. An exponential subsemigroup of a periodic semigroup is a union of $\mathcal{K}$-classes.

Proof. Taking $x = z = 1$ in the preceding lemma, then for any $\mathcal{K}$-class K and exponential subsemigroup A,

$$K \between A \quad \text{if and only if } K \subseteq A.//$$

Lemma 2.16. Let A be an exponential subsemigroup of a periodic semigroup S and B a $\mathcal{K}^*$-class of S. Then

$$B \between A \quad \text{if and only if} \quad B \subseteq A.$$

Proof. Let $B \between A$ and $b \in B$ be arbitrary. If $z \in B \between A$, then there exist $x_1, y_1, \ldots, x_n, y_n \in S^1$ and $\mathcal{K}$-classes $K_1, \ldots, K_n$ of S

such that $\{z\} \between x_1K_1y_1 \between \ldots\ldots \between x_nK_ny_n \between \{b\}$. However, $z \in A$, so $x_1K_1y_1 \between A$ and from Lemma 2.14 we have $x_1K_1y_1 \subseteq A$. Similarly,

$$x_2K_2y_2 \between A \quad \text{implies} \quad x_2K_2y_2 \subseteq A$$
$$\vdots \qquad\qquad\qquad\qquad \vdots$$

and finally $\quad x_nK_ny_n \between A \quad$ implies $\quad x_nK_ny_n \subseteq A$.

Hence, $b \in A$.

The converse is obvious.//

Remark 2.17. For each element x of an arbitrary semigroup S, there exists at least one exponential subsemigroup of S containing x, namely S itself. Therefore, the intersection of all those exponential subsemigroups of S containing x is contained in each such subsemigroup, and will be called the minimal exponential subsemigroup of S containing x (Lemma 2.12).

Theorem 2.18. For each element x in a periodic semigroup S, the $\mathcal{K}^*$-class containing x is the minimal exponential subsemigroup of S containing x.

Proof. Let B be the $\mathcal{K}^*$-class of x. From Lemma 2.16 it follows that B is contained in every exponential subsemigroup of S containing x.

That B is the minimal exponential subsemigroup of S containing x is immediate from Lemma 2.13.//

Corollary 2.19. Each periodic semigroup is the union of pairwise disjoint minimal exponential subsemigroups.

Proof. This follows directly from the preceding theorem.

<u>Corollary 2.20</u>. The classes of any band congruence σ on a periodic semigroup S are unions of minimal exponential subsemigroups of S.

<u>Proof</u>. Since $\mathcal{H}^*$ is the minimal band congruence on S, then $\mathcal{H}^* \subseteq \sigma$.//

CHAPTER 3

THE NATURAL EQUIVALENCE AND CERTAIN MINIMAL CONGRUENCES

We have already mentioned the existence of certain minimal congruences on an arbitrary semigroup. In the preceding chapter it was shown that the congruence $\mathcal{K}^*$ generated by the natural equivalence $\mathcal{K}$ is the minimal band congruence on a periodic semigroup. In this chapter we investigate the relationships among $\mathcal{K}, \mathcal{K}^*$, the minimal semilattice congruence $\mathfrak{h}$, and the minimal matrix congruence Σ.

As before, a given relation σ on a semigroup S shall be treated as a particular subset of the Cartesian product $S \times S$. Consistent with this notion, when simplicity of expression demands it, we shall write $(x,y) \in \sigma$ to mean $x,y \in S$ and $x\sigma y$.

<u>Definition 3.1</u>. A band congruence σ on a semigroup S is called a <u>semilattice</u> [<u>matrix</u>] congruence if for each $x,y \in S$, $xy\sigma yx$ $[x\sigma xyx]$.

<u>Definition 3.2</u>. Let σ,ρ be congruences on a semigroup S and $\rho \subseteq \sigma$. We define the relation σ/ρ on the factor semigroup S/ρ by: for $x,y \in S$

$$(x\rho, y\rho) \in \sigma/\rho \quad \text{if and only if} \quad (x,y) \in \sigma,$$

where $a\rho$ is the ρ-congruence class of $a \in S$.

It is easily verified that σ/ρ is a congruence on S/ρ. Moreover, from this definition it is immediate that given any semigroup S and congruence ρ on S, then any congruence on S/ρ may be expressed as σ/ρ, where σ is a congruence on S and $\rho \subseteq \sigma$. We use these facts in the statements which follow.

<u>Lemma 3.3</u>. Let σ,ρ be congruences on an arbitrary semigroup S and let $\rho \subseteq \sigma$. Then σ is a band [semilattice, matrix] congruence on S if and only if σ/ρ is a band [semilattice, matrix] congruence on S/ρ.

<u>Proof</u>. σ is a band congruence on S

if and only if for each $x \in S$ $(x,x^2) \in \sigma$

if and only if for each $x \in S$ $(x\rho,x^2\rho) = (x\rho,(x\rho)^2) \in \sigma/\rho$

if and only if σ/ρ is a band congruence on S/ρ.

σ is a semilattice congruence on S

if and only if σ is a band congruence on S and for each $x,y \in S$, $(xy,yx) \in \sigma$

if and only if σ/ρ is a band congruence on S/ρ and for each $x,y \in S$, $((xy)\rho,(yx)\rho) = (x\rho y\rho,y\rho x\rho) \in \sigma/\rho$

if and only if, σ/ρ is a semilattice congruence on S/ρ.

Similarly, one proves the matrix case.//

<u>Theorem 3.4</u>. Let σ be a congruence on an arbitrary semigroup S such that $\beta \subseteq \sigma$, where β is the minimal band congruence on S.

Then σ/β is the minimal band [semilattice, matrix] congruence on S/β if and only if $\sigma = \beta$ [$\sigma = \eta$, $\sigma = \Sigma$].

<u>Proof</u>. Let σ/β be the minimal band [semilattice, matrix] congruence on S/β. Since $\beta \subseteq \sigma$, then from Lemma 3.3 σ is a band [semilattice, matrix] congruence on S. Therefore, $\beta \subseteq \sigma$ [$\eta \subseteq \sigma$, $\Sigma \subseteq \sigma$]. On the other hand, by the same lemma we have that β/β [η/β, Σ/β] is a band [semilattice, matrix] congruence on S/β. From the minimality of σ/β it follows that

$\sigma/\beta \subseteq \beta/\beta$ $[\sigma/\beta \subseteq \eta/\beta, \sigma/\beta \subseteq \Sigma/\beta]$. Thus, as congruences on S, $\sigma \subseteq \beta$ $[\sigma \subseteq \eta, \sigma \subseteq \Sigma]$.

Conversely, let $\sigma = \beta$ $[\sigma = \eta, \sigma = \Sigma]$, and let γ/β be an arbitrary band [semilattice, matrix] congruence on S/β, where γ is a congruence on S and $\beta \subseteq \gamma$. Again, from Lemma 3.3 γ is a band [semilattice, matrix] congruence on S, so $\sigma \subseteq \gamma$. Hence as congruences on S/β, $\sigma/\beta \subseteq \gamma/\beta$. Since σ/β is a band [semilattice, matrix] congruence on S/β, it is the minimal such congruence.//

Corollary 3.5. For an arbitrary semigroup S,

$$\eta = \{(a,b) \in S \times S \mid (a,aba) \in \beta, (b,bab) \in \beta\}.$$

Proof. By the preceding theorem η/β is the minimal semilattice congruence on band S/β. Recalling the results of [7] we have

$$\begin{aligned}\eta &= \{(a,b) \in S \times S \mid (a,b) \in \eta\} = \{(a,b) \in S \times S \mid (a\beta,b\beta) \in \eta/\beta\}\\ &= \{(a,b) \in S \times S \mid a\beta = (a\beta)(b\beta)(a\beta),\ b\beta = (b\beta)(a\beta)(b\beta)\}\\ &= \{(a,b) \in S \times S \mid a\beta = (aba)\beta,\ b\beta = (bab)\beta\}\\ &= \{(a,b) \in S \times S \mid (a,aba) \in \beta,\ (b,bab) \in \beta\}.//\end{aligned}$$

Corollary 3.6. For a periodic semigroup S, $\eta/\mathcal{K}^*[\Sigma/\mathcal{K}^*]$ is the minimal semilattice [matrix] congruence on $S/\mathcal{K}^*$, and on S

$$\eta = \{(a,b) \in S \times S \mid (a,aba) \in \mathcal{K}^*, (b,bab) \in \mathcal{K}^*\}.$$

Proof. This follows directly from Theorem 3.4, Corollary 3.5, and the fact that $\beta = \mathcal{K}^*$ on S.//

Corollary 3.7. Let S be a periodic semigroup with the set E of idempotents of S forming a subsemigroup of S. If $\mathcal{K} = \mathcal{K}^*$, then the maximal semilattice homomorphic images of E and of S are isomorphic; but not conversely.

Proof. Let $a,b \in S$; say $a \in K^e$, $b \in K^f$. Recalling that $\mathcal{K} \subseteq \eta$ and $\mathcal{K}$-classes are unipotent, then we have:

$$(a,b) \in \eta$$

if and only if $(e,f) \in \eta$

if and only if $(e,efe) \in \mathcal{K}$ and $(f,fef) \in \mathcal{K}$

if and only if $e = efe$ and $f = fef$

if and only if $(e,f) \in \bar{\eta}$, where $\bar{\eta}$ is the minimal semilattice congruence on E.

To see that the converse fails, consider the semigroup of p.8, ex.1 of [2]. It is easy to verify that both η and $\bar{\eta}$ are the universal relations on S and E respectively, hence their maximal semilattice homomorphic images are trivially isomorphic. However, $K^e = \{e,a\}$, $K^f = \{f\}$, $K^g = \{g\}$ and $K^eK^g = \{e,f\}$, so that $\mathcal{K}$ is not a congruence.//

Definition 3.8. ([19]). Semigroup S is said to satisfy Condition B if for each $a,b \in S$ and any positive integers m,n there exist positive integers r,s such that $(ab)^r = (a^m b^n)^s$.

Theorem 3.9. On a periodic semigroup S the following are equivalent:

(a) $\mathcal{K} = \mathcal{K}^*$

(b) S satisfies Condition B.

Proof. (a) implies (b). Let $a,b \in S$ and integers m,n be given. Since $\mathcal{K}$ is a band congruence on S (Theorem 2.3), then $(a,a^m) \in \mathcal{K}$, $(b,b^n) \in \mathcal{K}$ and $(ab,a^m b^n) \in \mathcal{K}$. From the definition of the relation $\mathcal{K}$ there exist integers r,s and an idempotent $e \in S$ such that $(ab)^r = (a^m b^n)^s = e$.

(b) implies (a). To see that $\mathcal{K}$ is a congruence, let $a,b,c \in S$ and $(a,b) \in \mathcal{K}$; say $a^n = b^m = e$ and $c^t = f$, where e,f are idempotents. Since S satisfies Condition B there exist integers r,s,u,v such that $(ca)^r = (c^t a^n)^s = (fe)^s$ and $(cb)^u = (c^t b^m)^v = (fe)^v$.

Now it is immediate from the definition of $\mathcal{K}$ that all powers of a single element are $\mathcal{K}$-related. Thus, $((fe)^s,(fe)^v) \in \mathcal{K}$, $((ca)^r,(cb)^u) \in \mathcal{K}$, and $(ca,cb) \in \mathcal{K}$.

Similarly, one can prove that $(ac,bc) \in \mathcal{K}$.//

Definition 3.10. Periodic semigroup S is said to satisfy Condition D if for each $a,b \in S$ there exist $x_i,y_i \in S^1$ and $\mathcal{K}$-classes K_i of S $(i = 1,2,\ldots,n)$ such that

$$\{ab\} \between x_1K_1y_1 \between x_2K_2y_2 \between \cdots \between x_nK_ny_n \between \{ba\}.$$

Theorem 3.11. On a periodic semigroup S the following are equivalent:

(a) $\mathcal{K}^* = \eta$

(b) S satisfies Condition D.

Proof. Recalling the characterization of $\mathcal{K}^*$ given in Theorem 2.5, we have:

$$\mathcal{K}^* = \mathfrak{h}$$

if and only if for each $a,b \in S$, $(ab,ba) \in \mathcal{K}^*$

if and only if for each $a,b \in S$ there exist $x_i,y_i \in S^1$ and $\mathcal{K}$-classes K_i of S $(i = 1,2,\dots,n)$ such that

$$\{ab\} \between x_1K_1y_1 \between \dots \between x_nK_ny_n \between \{ba\}.//$$

For the sake of completeness we include the next result.

Theorem 3.12. On a periodic semigroup S the following are equivalent:

(a) $\mathcal{K} = \mathfrak{h}$

(b) S is weakly commutative

(c) S satisfies Condition C.

Proof. The equivalence of (a) and (b) was proved in Theorem 6.7 of [9]. Statements (b) and (c) were shown to be equivalent in Lemma 1.19.//

Corollary 3.13. In case any of the statements appearing in Theorems 3.11 or 3.12 are satisfied, then Σ is the universal relation on S; but not conversely.

Proof. Let $a,b \in S$ be arbitrary; then $(a,aba) \in \Sigma$ and $(b,bab) \in \Sigma$. In each of the two preceding theorems $\mathcal{K}^* = \mathfrak{h}$, so $(ab,ba) \in \mathcal{K}^* \subseteq \Sigma$. Hence, $(a,ba^2) \in \Sigma$, $(b,b^2a) \in \Sigma$; and $(a,ba) \in \Sigma$, $(b,ba) \in \Sigma$. Therefore, $(a,b) \in \Sigma$.

To see that Σ being the universal relation does not imply $\mathcal{K}^* = \mathfrak{h}$, consider the semigroup S of example 5 of [11]:

$$\begin{array}{c|ccc} & a & b & c \\ \hline a & a & a & a \\ b & a & b & c \\ c & c & c & c \end{array} \quad .$$

Since b is an identity for S, then Σ is the universal relation (Prop. 7 of [11]). Also, S is a band, so $\mathcal{K}$ is the identity relation on S and trivially a congruence. However, aca = a and cac = c; so $(a,c) \in \mathfrak{h}$. Thus, $\mathcal{K} = \mathcal{K}^* \subsetneq \mathfrak{h}$.//

<u>Definition 3.14</u>. Periodic semigroup S is said to satisfy <u>Condition E</u> if for each $a,b \in S$ there exist $x_i, y_i \in S^1$ and $\mathcal{K}$-classes K_i of S $(i = 1,2,\ldots,n)$ such that

$$\{a\} \between x_1K_1y_1 \between x_2K_2y_2 \between \cdots \between x_nK_ny_n \between \{aba\}.$$

<u>Theorem 3.15</u>. On a periodic semigroup S the following are equivalent:

(a) $\mathcal{K}^* = \Sigma$

(b) $\mathfrak{h}$ is the universal relation on S

(c) S satisfies Condition E.

<u>Proof</u>. (a) implies (b). For arbitrary $a,b \in S$, $(a,aba) \in \mathcal{K}^*$ and $(b,bab) \in \mathcal{K}^*$. So from Corollary 3.6, $(a,b) \in \mathfrak{h}$.

(b) implies (c). Let a,b be arbitrary in S; hence $(a,b) \in \mathfrak{h}$. Again by Corollary 3.6, $(a,aba) \in \mathcal{K}^*$. In light of Theorem 2.5 there exist $x_i, y_i \in S^1$ and $\mathcal{K}$-classes K_i of S $(i = 1,2,\ldots,n)$ such that

$$\{a\} \between x_1K_1y_1 \between \ldots\ldots \between x_nK_ny_n \between \{aba\}.$$

(c) implies (a). Since S satisfies Condition E, then by Theorem 2.5 for each $a,b \in S$, $(a,aba) \in \mathcal{K}^*$. That is, $\mathcal{K}^*$ is a matrix congruence on S.//

Definition 3.16. Semigroup S is weakly rectangular if for each $a,b \in S$ there exist positive integers r,s such that $a^r = (aba)^s$.

Lemma 3.17. Periodic semigroup S is weakly rectangular if and only if for each $a,b \in S$, $(a,aba) \in \mathcal{K}$.

Proof. Let S be a periodic semigroup. Recalling the definition of the $\mathcal{K}$-equivalence given in 1.4, then:

S is weakly rectangular

if and only if for each $a,b \in S$ there exist integers r,s such that $a^r = (aba)^s$

if and only if for each $a,b \in S$ there exist integers m,n and an idempotent $e \in S$ such that $a^m = (aba)^n = e$

if and only if for each $a,b \in S$, $(a,aba) \in \mathcal{K}$.//

Lemma 3.18. If S is a weakly rectangular periodic semigroup, then $\mathcal{K} = \mathcal{K}^*$.

Remark. In this and any subsequent proof, whenever a zero exponent appears merely delete the term involved; e.g. read a^2b^0c as a^2c.

Proof. Let $a,b,c \in S$ and $(a,b) \in \mathcal{K}$; say $a^n = b^m = e$. Since S is weakly rectangular, there exist integers r,s,t,u such that $a^r = (aca)^s$ and $b^t = (bcb)^u$. For the same reason there exist integers v,w such that

$$
\begin{aligned}
(ca)^v &= [(ca)\{a^{n-1}(aca)^{ns-1}a\}(ca)]^w \\
&= [ca^n(aca)^{ns}]^w = (ca^n a^{nr})^w \\
&= (cee)^w = (ce)^w.
\end{aligned}
$$

Also, there exist integers h,k such that

$$
\begin{aligned}
(cb)^h &= [(cb)\{b^{m-1}(bcb)^{mu-1}b\}(cb)]^k \\
&= [cb^m(bcb)^{mu}]^k = (cb^m b^{mt})^k \\
&= (cee)^k = (ce)^k.
\end{aligned}
$$

Thus, $(ca)^{vk} = (ce)^{wk} = (cb)^{hw}$, and there exist an integer p and an idempotent $f \in S$ such that $(ca)^{vkp} = (ce)^{wkp} = (cb)^{hwp} = f$; that is, $(ca,cb) \in \mathcal{K}$.

This proves that $\mathcal{K}$ is left compatible, similarly one shows it is right compatible.//

Theorem 3.19. On a periodic semigroup S the following are equivalent:

(a) $\mathcal{K} = \Sigma$

(b) S is weakly rectangular.

Proof. (a) implies (b). If $\mathcal{K}$ is a matrix congruence, then for each $a,b \in S$, $(a,aba) \in \mathcal{K}$. That S is weakly rectangular follows directly from Lemma 3.17.

(b) implies (a). From the preceding lemma $\mathcal{K} = \mathcal{K}^*$, and by Lemma 3.17 for each $a,b \in S$ $(a,aba) \in \mathcal{K}$. That is, $\mathcal{K}$ is a matrix congruence on S. Since Σ is a band congruence and hence necessarily coarser than $\mathcal{K}$ (Lemma 2.2), then $\mathcal{K} = \Sigma$.//

Note that $\mathcal{K} = \Sigma$ does not necessarily imply that the set E of idempotents of S form a subsemigroup of S; for example, any completely simple periodic semigroup such that E is not a subsemigroup. Thus, the next definition.

Definition 3.20. Semigroup S is strongly rectangular if for each $a,b \in S$ there exist positive integers r,s,t such that $a^r = a^r b^s a^r = (aba)^t$.

Definition 3.21. ([11]). A band S is called a rectangular band if for each $a,b \in S$, $a = aba$.

Theorem 3.22. On a periodic semigroup S with set E of idempotents the following are equivalent:

(a) S is strongly rectangular

(b) S is weakly rectangular and E is a rectangular band

(c) S is weakly rectangular and E is a semigroup.

Proof. That (a) implies (b) and (b) implies (c) follows directly from the definitions of the terms involved.

(c) implies (a). Let $a,b \in S$; say $a^n = e$ and $b^m = f$ where $e,f \in E$. From Theorem 3.19 $\mathcal{H}$ is a matrix congruence, so $(a,aba) \in \mathcal{H}$ and $(e,efe) \in \mathcal{H}$. Since E is a semigroup and each $\mathcal{H}$-class is unipotent, then $efe \in E$ and $e = efe$. Also, there exist integers r,t such that $a^r = (aba)^t = e$. Hence, $(aba)^{tn} = a^{rn} = e = efe = a^{rn}b^m a^{rn}$.//

Corollary 3.23. If semigroup S is periodic and strongly rectangular, then E is a subsemigroup of S and the maximal matrix homomorphic images of E and of S are isomorphic.

Proof. Since E is a rectangular band, E is its own maximal matrix decomposition. In addition, the $\mathcal{H}$-classes are unipotent and $S/\mathcal{H}$ is the maximal matrix decomposition of S. The result is now immediate.//

Definition 3.24. An element a of semigroup S is called regular if $a = axa$ for some $x \in S$. A subset A of S is called regular if each element of A is regular.

In light of Theorem 2.11 of [2]:

an element a of semigroup S is regular,
if and only if D_a, or R_a, or L_a is regular,
if and only if D_a, or R_a, or L_a contains an idempotent.

These facts shall be used below without reference.

We conclude this chapter with some results concerning a periodic semigroup satisfying Condition B; that is, a periodic semigroup in which $\mathcal{K}$ is a congruence (Theorem 3.9). In each of the remaining statements let S be such a semigroup.

Proposition 3.25. Each regular $\mathcal{R}$-class of S is a right group.

Proof. Because of Lemma 1.27 it suffices to show that each regular $\mathcal{R}$-class of S is a union of groups.

Let R_e be an arbitrary regular $\mathcal{R}$-class of S, where e is an idempotent. Let $a \in R_e$; say $a \in K^f$ and $a^n = f$. Then $aS^1 = eS^1$ so for some $x,y \in S^1$, $a = ex$ and $e = ay$. Since S satisfies Condition B there exist integers r,s such that $(fy)^r = (a^n y)^r = (ay)^s = e$; hence $e \in fS^1$. Also, $f = a^n = (ex)^n \in eS^1$. Thus, $(e,f) \in \mathcal{R}$ and $R_e = R_f$.

However, by Corollary 1.8 $K^f \cap R_f = G^f$, so $a \in K^f \cap R_e = K^f \cap R_f = G^f$. Since a is arbitrary, then R_e is a union of maximal subgroups of S.//

Corollary 3.26. Each regular $\mathcal{D}$-class of S is a matrix of groups.

Proof. Since each regular $\mathcal{D}$-class is a union of regular $\mathcal{R}$-classes (i.e. right groups), and each right group is a completely simple semigroup, then each regular $\mathcal{D}$-class is a union of groups. The result follows directly from Lemma 1.13.//

Certainly a commutative periodic semigroup is weakly commutative, hence $\mathfrak{h} = \mathcal{K}^* = \mathcal{K}$ and such a semigroup satisfies Condition B. The next corollary thus generalizes a result of Schwarz in [12] which deals with the commutative case.

Corollary 3.27. An element x of S is regular if and only if x belongs to some maximal subgroup of S.

Proof. Noting the preceding corollary, then:

$x \in S$ and x is regular,

if and only if x belongs to some regular $\mathcal{D}$-class of S,

if and only if x belongs to some maximal subgroup of S.//

Proposition 3.28. For idempotents $e,f \in S$,

$$(e,f) \in \mathfrak{n} \text{ if and only if } (e,f) \in \mathcal{D}.$$

Proof. In any semigroup, $\mathcal{D} \subseteq \mathcal{J} \subseteq \mathfrak{n}$.

Conversely, let $(e,f) \in \mathfrak{n}$. Since $\mathcal{K} = \mathcal{K}^*$ then from Corollary 3.6, $(e,efe) \in \mathcal{K}$ and $(f,fef) \in \mathcal{K}$. Therefore, for some integers n,m, $e = (efe)^n$ and $f = (fef)^m$. Hence, $e \in S^1fS^1$, $f \in S^1eS^1$ and $(e,f) \in \mathcal{J}$. But in a periodic semigroup $\mathcal{D} = \mathcal{J}$.//

Theorem 3.29. Each $\mathfrak{n}$-class N of S contains a $\mathcal{D}$-class D which is the union of all maximal subgroups of N, a completely simple semigroup, and the kernel of N.

Proof. From Corollary 3.26 and the preceding proposition it follows that the $\mathcal{D}$-class D containing the idempotents of N is the union of all maximal subgroups of N and a completely simple semigroup (Theorem 1.14). Moreover, since D is simple, to show that D is the kernel of N it suffices to prove that D is an ideal in N.

So, let $x \in K^f \subseteq N$ and $a \in G^e \subseteq D$. Then $xa \in K^fG^e \subseteq K^fK^e \subseteq K^g$ for some idempotent g since $\mathcal{K}$ is a congruence; say $(xa)^n = g = (fe)^m$

for integers m,n. Also, D is a matrix of groups hence efe $\in G^e$; say r is an integer such that $(efe)^r = e$. Then we have

$xa = x(ae) = xa(efe)^{mr} = xae(fe)^{mr} = xa(fe)^{mr} = xa(xa)^{nr} = (xa)^{nr+1}$.

Hence xa, having index one, lies in a maximal subgroup (Theorem 7 of [12]); that is, $xa \in G^g \subseteq D$.

Similarly one proves that $ax \in D$.//

Remarks 3.30. i) These last two propositions indicate that the $\mathfrak{h}$-classes of S can be determined modulo the regular $\mathcal{D}$-classes of S (recall $\mathcal{K} \subseteq \mathfrak{h}$), and conversely the regular $\mathcal{D}$-classes by the $\mathfrak{h}$-classes.

ii) The idempotents of each $\mathfrak{h}$-class of S need not form a subsemigroup of S. For, consider a completely simple periodic semigroup S. It satisfies Condition B (by Theorem 1.14 $\mathcal{K} = \Sigma$ is a congruence) and since it is bisimple the set E of idempotents of S is contained in a single $\mathcal{D}$-class of S. Hence, E is contained in a single $\mathfrak{h}$-class of S, and $\mathfrak{h}$ is the universal relation on S. However, E need not be a subsemigroup of an arbitrary completely simple periodic semigroup.

In the notation of Proposition 3.29, we may say that the idempotents of N form a subsemigroup of N if and only if the idempotents of completely simple semigroup D form a subsemigroup of D.

Necessary and sufficient conditions that the idempotents of a completely 0-simple semigroup form a subsemigroup were studied in [15]; The case without zero can be easily deduced. Three of these are given in Proposition 3.32, which is stated and proved for an arbitrary semigroup.

Definition 3.31. ([10]). A semigroup is a rectangular group if it is isomorphic to the Cartesian product of a rectangular band and a group.

Theorem 3.32. On a completely simple semigroup S with set E of idempotents the following are equivalent:

(a) E is a subsemigroup of S

(b) E is a rectangular band

(c) S is a rectangular group

(d) Each inverse of an idempotent is idempotent.

Proof. (a) implies (b). Let $e,f \in E$. Since S is bisimple and a union of groups, then from Lemma 1.13 S is a matrix of groups and $H_eH_fH_e \subseteq H_e$. Hence, $efe \in H_e$, and $efe = e$.

(b) implies (c). Again, since S is completely simple, the maximal subgroups of S are isomorphic to each other. Fix $e \in E$. For any $f \in E$ the mapping $x \longrightarrow exe$, $x \in H_f$, is an isomorphism of H_f onto H_e (Theorem 2.20 of [2]).

Thus, define mapping $S \longrightarrow E \times H_e$ by $x \longrightarrow (f,exe)$, where $x \in H_f$, $f \in E$. Clearly the mapping is well-defined, one-to-one, and onto. Moreover, for any $x,y \in S$ (say $x \in H_f, y \in H_g$ where $f,g \in E$) we have $exye = exfgye = exfefgegye = exefgeye = exeye = (exe)(eye)$.

Therefore, $xy \longrightarrow (fg, exye) = (fg, exeeye) = (f, exe)(g, eye)$. Hence S, being isomorphic to $E \times H_e$, is a rectangular group.

(c) implies (d). Let $e \in E$ and $a \in S$ be an inverse of e; say $a \in H_f$ where $f \in E$. It is easily verified that the idempotents of any rectangular group form a rectangular band. Therefore, $e = efe$, $f = fef$ and f is an inverse of e. But no $\mathcal{H}$-class of S may contain more than one inverse of e. (Theorem 2.18 of [2]). Hence, $a = f$.

(d) implies (a). Let $e, f \in E$. Since S is completely simple, then by Theorem 2.17 of [2], $ef \in R_e \cap L_f$. Let $H_g = R_f \cap L_e$, where $g \in E$.

Recalling that the idempotents of each $\mathcal{R}[\mathcal{L}]$-class form a right [left]-zero semigroup, then $(ef)g(ef) = e(fg)ef = (eg)ef = eef = ef$ and $g(ef)g = (ge)fg = g(fg) = gg = g$. That is, ef is an inverse of idempotent g; thus ef is idempotent.//

CHAPTER 4

NECESSARY AND SUFFICIENT CONDITIONS THAT A PERIODIC SEMIGROUP BE A RECTANGULAR $\mathcal{K}$-CLASS

Definition 4.1. ([15]). A z-semigroup is a semigroup containing exactly one idempotent, it being a zero.

Notation 4.2. i) If semigroup S contains a zero, we shall write S^* for $S\setminus\{0\}$.

ii) Whenever clarity demands that the operation $\bullet$ in semigroup S be specified, we shall say: let $(S,\bullet)$ be a semigroup.

Theorem 4.3. For a periodic semigroup S, $\mathcal{K} = \Sigma$ if and only if $\mathcal{K}$ is a congruence and S is an ideal extension of a completely simple periodic semigroup H by a periodic z-semigroup.

Proof. If $\mathcal{K} = \Sigma$, then trivially $\mathcal{K}$ is a congruence. From preceding results we have that S satisfies Condition B (Theorem 3.9) and $\mathfrak{h}$ is the universal relation on S (Theorem 3.15). Hence, recalling that each regular $\mathcal{D}$-class of S is a union of groups (Corollary 3.26), and for idempotents e,f, $e\mathfrak{h}f$ if and only if $e\mathcal{D}f$ (Proposition 3.28); it follows that S has precisely one regular $\mathcal{D}$-class, namely $H = \bigcup\{G^e \mid e \in E\}$, where E is the set of idempotents of S. By Theorem 3.29, H is a completely simple semigroup and the kernel of S.

Since $E \subseteq H$, it is now immediate that S is an extension of H by periodic z-semigroup S/H, the Rees factor semigroup of S modulo H.

Conversely, let $a,b \in S$; say $a \in K^e$ and $b \in K^f$ where $e,f \in E$. Then $e,f \in H$, which is a matrix of groups (Theorem 1.14), so $G^eG^fG^e = G^e$. Since $\mathcal{K}$ is a congruence, $aba \in K^eK^fK^e \subseteq K^e$.

Thus, $a\mathcal{K}aba$ and S is weakly rectangular (Lemma 3.17). In light of Theorem 3.19, $\mathcal{K} = \Sigma$.//

Theorem 4.4. Let $(Z,\bullet)$ be a periodic z-semigroup and $(G,\wedge)$ a periodic group.

If ϕ is a partial homomorphism from the partial groupoid Z^* into G, then the semigroup $(Z^* \cup G,\circ)$ is an ideal extension of G by Z, where operation $\circ$ is defined by

$$x \circ y = \begin{cases} x \bullet y & \text{if} \quad x,y \in Z,\ x \bullet y \neq 0 \\ x\phi \wedge y\phi & \text{if} \quad x,y \in Z,\ x \bullet y = 0 \\ x\phi \wedge y & \text{if} \quad x \in Z,\ y \in G \\ x \wedge y\phi & \text{if} \quad x \in G,\ y \in Z \\ x \wedge y & \text{if} \quad x,y \in G. \end{cases}$$

In fact, the semigroup $(Z^* \cup G,\circ)$ is a unipotent periodic semigroup with maximal group G.

Conversely, each unipotent periodic semigroup $(K,\bullet)$ with maximal subgroup G and idempotent e is the ideal extension of G by K/G determined by the partial homomorphism

$$x \longrightarrow x \bullet e \ (= e \bullet x),\ x \in (K/G)^*,$$

in the above fashion.

Proof. The first sentence is a corollary of Theorem 4.19 of [2]. Clearly, the extension is periodic and unipotent (viz. the identity e of G), and G is contained in the maximal group $\bar{G}$ of the extension.

To see that $G = \bar{G}$, suppose $a \in \bar{G} \cap Z^*$. Then $a = a \circ e = a\phi \wedge e = a\phi$, which is impossible since by definition $\phi: Z^* \longrightarrow G$. Hence, $\bar{G} \cap Z^* = \emptyset$ and $G = \bar{G}$.

Conversely, it is trivial that $(K/G, \bullet)$ is a periodic z-semigroup and $(G, \bullet)$ a periodic group. Noting that for any $x, y \in K$

$$(x \bullet y) \bullet e = (x \bullet e) \bullet (y \bullet e) = (x \bullet e) \bullet y = x \bullet (y \bullet e),$$

then the operation $\bullet$ of the extension $(K = (K/G)^* \cup G, \bullet)$ derived from the given partial homomorphism reduces to:

$$x \bullet y = \begin{cases} x \bullet y & \text{if } \; x, y \in K, \; x \bullet y \notin G \\ (x \bullet y) \bullet e & \text{otherwise.} \end{cases}$$

That is, the operation of the extension coincides with that of $(K, \bullet)$.//

Remark 4.5. Henceforth we shall denote each unipotent periodic semigroup by a triple (Z, G, ϕ), where Z is a periodic z-semigroup, G is a periodic group, and ϕ is a partial homomorphism from Z^* into G.

Theorem 4.6. Unipotent periodic semigroups (Z_1, G_1, ϕ_1) and (Z_2, G_2, ϕ_2) are isomorphic if and only if there exist isomorphisms $\gamma: Z_1 \longrightarrow Z_2$ and $\xi: G_1 \longrightarrow G_2$ such that for each $x \in Z_1^*$

$$x\phi_1\xi = x\gamma\phi_2.$$

Proof. Let $\beta: (Z_1, G_1, \phi_1) \longrightarrow (Z_2, G_2, \phi_2)$ be an isomorphism. It is easily verified that $\gamma: Z_1 \longrightarrow Z_2$ given by

$$x\gamma = \begin{cases} x\beta & \text{if } x \in Z_1^* \\ 0 & \text{if } x = 0 \end{cases}$$

and $\xi = \beta|_{G_1}$ are each isomorphisms.

Now let $x \in Z_1^*$ be arbitrary. Then $x\phi_1$, $(x\phi_1)x \in G_1$ and $[(x\phi_1)x]\beta = (x\phi_1\beta)(x\beta)$. So

$$[(x\phi_1)x]\xi = (x\phi_1\xi)(x\gamma). \qquad \text{(A)}$$

Also, since $x\phi_1\xi \in G_2$, $x\gamma \in Z_2$, then $(x\phi_1\xi)(x\gamma) \in G_2$ and

$$(x\phi_1\xi)(x\gamma) = (x\phi_1\xi)(x\gamma\phi_2). \qquad \text{(B)}$$

Combining equations (A) and (B) yields

$$[(x\phi_1)x]\xi = (x\phi_1\xi)(x\gamma\phi_2). \qquad \text{(C)}$$

From equation (C) and the fact that $(x\phi_1)x = (x\phi_1)(x\phi_1)$ we have $(x\phi_1\xi)(x\phi_1\xi) = [(x\phi_1)(x\phi_1)]\xi = [(x\phi_1)x]\xi = (x\phi_1\xi)(x\gamma\phi_2)$. Using cancellation on the first and last terms of this equation yields $x\phi_1\xi = x\gamma\phi_2$.

Conversely, define $\beta\colon (Z_1^* \cup G_1) \longrightarrow (Z_2^* \cup G_2)$ by

$$x\beta = \begin{cases} x\xi & \text{if } x \in G_1 \\ x\gamma & \text{if } x \in Z_1^*. \end{cases}$$

It is clear that β is well-defined, one-to-one, and onto.

We need check five cases to show that it is a homomorphism.

1) $x,y \in G_1$: Then $xy \in G_1$ and $(xy)\beta = (xy)\xi = (x\xi)(y\xi) = (x\beta)(y\beta)$.

2) $x,y \in Z_1^*$ and $xy \in Z_1^*$: $(xy)\beta = (xy)\gamma = (x\gamma)(y\gamma) = (x\beta)(y\beta)$.

3) $x,y \in Z_1^*$ and $xy \in G_1$: Then $(x\gamma)(y\gamma) \in G_2$ and $(xy)\beta = (xy)\xi = [(x\phi_1)(y\phi_1)]\xi = (x\phi_1\xi)(y\phi_1\xi) = (x\gamma\phi_2)(y\gamma\phi_2)$ $= (x\gamma)(y\gamma) = (x\beta)(y\beta)$.

4) $x \in G_1$, $y \in Z_1^*$: Then $xy \in G_1$ and

$(xy)\beta = (xy)\xi = [x(y\phi_1)]\xi = (x\xi)(y\phi_1\xi) = (x\xi)(y\gamma\phi_2)$

$= (x\xi)(y\gamma) = (x\beta)(y\beta)$.

5) $x \in Z_1^*$, $y \in G_1$: Then $xy \in G_1$ and the computation is symmetric to that of 4).//

Corollary 4.7. Let $\mathcal{H}$-classes K^e, K^f of periodic semigroup S be subsemigroups. Then K^e and K^f are isomorphic if and only if there exist isomorphisms $\gamma\colon K^e/G^e \longrightarrow K^f/G^f$ and $\xi\colon G^e \longrightarrow G^f$ such that for each $x \in (K^e/G^e)^*$

$$(xe)\xi = (x\gamma)f.$$

Proof. The $\mathcal{H}$-class K^e is given the triple $(K^e/G^e, G^e, \phi_e)$, where $\phi_e\colon (K^e/G^e)^* \longrightarrow G^e$ is the natural partial homomorphism $x \longrightarrow xe$. Similarly for K^f.

The desired result follows from the preceding theorem.//

The next theorem can be deduced from the proof of Theorem 1 in [13].

Theorem 4.8. Let S be a periodic semigroup with subsemigroup E of idempotents and K a unipotent periodic semigroup. In order that S and $E \times K$ be isomorphic it is necessary and sufficient that the $\mathcal{H}$-classes of S be mutually isomorphic subsemigroups, and there exist a family of isomorphisms

$$P = \{\rho_{e,f}\colon K^e \longrightarrow K^f \mid e,f \in E\}$$

with the properties:

(A) for each $e,f,g \in E$, $\rho_{e,f} = \rho_{e,g} \circ \rho_{g,f}$

(B) for $a \in K^e$, $b \in K^f$, $ab = (a\rho_{e,ef})(b\rho_{f,ef})$.

Proof. Necessity. If $S \cong E \times K$, then it follows that $\{\{e\} \times K \mid e \in E\}$ is the set of $\mathcal{K}$-classes of S, each being a subsemigroup. Also, for $e,f \in E$, the correspondence $\rho_{e,f}$: $\{e\} \times K \longrightarrow \{f\} \times K$ given by $(e,x) \longrightarrow (f,x)$ is an isomorphism.

It remains only to verify that the family of isomorphisms $P = \{\rho_{e,f} \mid e,f \in E\}$ has the desired properties.

(A) for any $e,f,g \in E$ and $(e,x) \in \{e\} \times K$, then $(e,x)\rho_{e,f} = (f,x)$ and $(e,x)\rho_{e,g} \circ \rho_{g,f} = (g,x)\rho_{g,f} = (f,x)$.

(B) For $(e,x) \in \{e\} \times K$ and $(f,y) \in \{f\} \times K$, then $[(e,x)\rho_{e,ef}][(f,y)\rho_{f,ef}] = (ef,x)(ef,y) = (efef,xy) = (ef,xy) = (e,x)(f,y)$.

Sufficiency. Fix an arbitrarily chosen idempotent 1 of S. (Note that 1 need not be an identity for S.) Define correspondence ϕ: $S \longrightarrow E \times K^1$ by $a \longrightarrow (e,a\rho_{e,1})$ where $a \in K^e \subseteq S$.

1) ϕ is one-to-one: Let $a \in K^e$, $b \in K^f$; then

$a\phi = b\phi$ implies $(e,a\rho_{e,1}) = (f,b\rho_{f,1})$

whence $e = f$ and $a\rho_{e,1} = b\rho_{f,1}$

thus $a\rho_{e,1} = b\rho_{e,1}$

therefore $a = b$.

2) ϕ is onto: If $(e,x) \in E \times K^1$, then since $\rho_{e,1}$ is onto there exists some $a \in K^e$ such that $a\rho_{e,1} = x$. Hence, $a\phi = (e,a\rho_{e,1}) = (e,x)$.

3) ϕ is a homomorphism: Let $a \in K^e$, $b \in K^f$. Due to hypothesis (B) and the fact that each $\mathcal{K}$-class is a semigroup, it follows that $ab \in K^{ef}$. Hence,

$$\begin{aligned}(ab)\rho_{ef,1} &= [(a\rho_{e,ef})(b\rho_{f,ef})]\rho_{ef,1} \\ &= (a\rho_{e,ef}\rho_{ef,1})(b\rho_{f,ef}\rho_{ef,1}) = (a\rho_{e,1})(b\rho_{f,1}).\end{aligned}$$

Therefore,

$$\begin{aligned}(ab)\phi &= (ef,(ab)\rho_{ef,1}) = (ef,(a\rho_{e,1})(b\rho_{f,1})) \\ &= (e,a\rho_{e,1})(f,b\rho_{f,1}) = (a\phi)(b\phi). //\end{aligned}$$

Definition 4.9. If E is a rectangular band (Definition 3.21), then in the case of Theorem 4.8 we shall say S is a rectangular $\mathcal{K}$-class with family P of isomorphisms; or, when family P is not specified, simply a rectangular $\mathcal{K}$-class.

In particular, if K is a periodic z-semigroup, then we call S a rectangular z-semigroup with family P of isomorphisms, or more simply a rectangular z-semigroup.

Theorem 4.10. Let S be a periodic semigroup with subsemigroup E of idempotents. In order that S be a rectangular $\mathcal{K}$-class it is necessary and sufficient that $\mathcal{K} = \Sigma$ and there exist a family of isomorphisms

$$\Gamma = \{\gamma_{e,f}\colon K^e/G^e \longrightarrow K^f/G^f \mid e,f \in E\}$$

with the properties:

(A') for each $e,f,g \in E$, $\gamma_{e,f} = \gamma_{e,g} \bullet \gamma_{g,f}$

(B') for $a \in K^e/G^e$ and $b \in K^f/G^f$, $ab = (a\gamma_{e,ef})(b\gamma_{f,ef})$

(C') for $a \in (K^e/G^e)^*$ and $f \in E$, $faef = (a\gamma_{e,f})f$.

Proof. We first note that for each $\mathcal{H}$-class K^e of a periodic semigroup, e commutes with each element of K^e. Hence the side of multiplication by idempotent f in the right half of condition (c') is irrelevant, for $a\gamma_{e,f} \in K^f$.

Necessity. If E is a rectangular band and $S \cong E \times K$ for some unipotent periodic semigroup K, then again $\{\{e\} \times K \mid e \in E\}$ is the set of $\mathcal{H}$-classes of S and $\{\{e\} \times G \mid e \in E\}$ the set of maximal subgroups of S, where G is the maximal subgroup of K. Clearly, any pair of $\mathcal{H}$-classes $\{e\} \times K$, $\{f\} \times K$ are isomorphic under the correspondence $(e,x) \longrightarrow (f,x)$. For simplicity of notation in the following calculations, we shall write K^e for $\{e\} \times K$, G^e for $\{e\} \times G$, E for the set $E \times \{1\}$ of idempotents of S, and e for idempotent (e,1) — where 1 is the identity of maximal subgroup G.

For any $(e,a) \in K^e$ and $(f,b) \in K^f$, $(e,a)(f,b)(e,a) = (e,aba) \in K^e$; that is, $(e,a)\mathcal{H}(e,a)(f,b)(e,a)$. So by Lemma 3.17 and Theorem 3.19, $\mathcal{H} = \Sigma$.

Also, if $e,f \in E$, define $\gamma_{e,f}: K^e/G^e \longrightarrow K^f/G^f$ by $(e,x) \longrightarrow (f,x)$. [Note: This definition is meant to include the case when $(e,x) = \{e\} \times G \in K^e/G^e$.] Then $\Gamma = \{\gamma_{e,f} \mid e,f \in E\}$ is a family of isomorphisms with the properties:

(A') For each $e,f,g \in E$ and any $(e,a) \in K^e/G^e$,

$$(e,a)\gamma_{e,g}\gamma_{g,f} = (g,a)\gamma_{g,f} = (f,a) = (e,a)\gamma_{e,f}.$$

(B') For $(e,a) \in K^e/G^e$ and $(f,b) \in K^f/G^f$,

$$\begin{aligned}[(e,a)\gamma_{e,ef}][(f,b)\gamma_{f,ef}] &= (ef,a)(ef,b) = (efef,ab) \\ &= (ef,ab) = (e,a)(f,b).\end{aligned}$$

(C') For $(e,a) \in (K^e/G^e)^* = K^e \setminus G^e$ and $(f,1) \in E \times \{1\}$,

$(f,1)(e,a)(e,1)(f,1) = (fef,a1) = (f,a1)$ and

$[(e,a)\gamma_{e,f}](f,1) = (f,a)(f,1) = (f,a1)$.

Sufficiency. Recalling the proof of Theorem 4.3, we know that $\mathcal{K} = \Sigma$ implies $H = \cup\{G^e \mid e \in E\}$ is a completely simple semigroup and the kernel of S. Since H is a matrix of groups (Theorem 1.14) and E is a semigroup, it is immediate that E is a rectangular band.

Moreover, any pair of maximal subgroups G^e, G^f are isomorphic under the correspondence $\xi_{e,f}: G^e \longrightarrow G^f$ given by $x \longrightarrow fxf$ (Theorem 2.20 of [2]). Therefore, for $e,f \in E$ define $\rho_{e,f}: K^e \longrightarrow K^f$ by

$$x\rho_{e,f} = \begin{cases} x\gamma_{e,f} & \text{if } x \in (K^e/G^e)^* = K^e \setminus G^e \\ fxf & \text{if } x \in G^e. \end{cases}$$

By condition (C') of the hypothesis, for any $x \in (K^e/G^e)^*$ and $f \in E$, $(xe)\xi_{e,f} = fxef = (x\gamma_{e,f})f$. Thus, from Corollary 4.7, each $\rho_{e,f}$ is an isomorphism.

To complete the proof we verify that family of isomorphisms $P = \{\rho_{e,f} \mid e,f \in E\}$ has properties (A) and (B) of Theorem 4.8.

(A) Let $e,f,g \in E$. If $x \in K^e \setminus G^e$, then

$x\rho_{e,f} = x\gamma_{e,f} = x\gamma_{e,g}\gamma_{g,f} = x\rho_{e,g}\rho_{g,f}$. If $x \in G^e$, then

$x\rho_{e,f} = fxf = fexef = fgexegf = fgxgf = x\rho_{e,g}\rho_{g,f}$.

(B) Let $a \in K^e$, $b \in K^f$. We need consider four cases.

1) $a \in K^e \setminus G^e$, $b \in K^f \setminus G^f$:

$$(a\rho_{e,ef})(b\rho_{f,ef}) = (a\gamma_{e,ef})(b\gamma_{f,ef}) = ab.$$

2) $a \in G^e$, $b \in G^f$:

$(a\rho_{e,ef})(b\rho_{f,ef}) = (efaef)(efbef)$

$= (efe)a(fef)b(fef) = eafbf = ab.$

3) $a \in G^e$, $b \in K^f \setminus G^f$: Noting hypothesis (C'),

$(a\rho_{e,ef})(b\rho_{f,ef}) = (efaef)(b\gamma_{f,ef}) = (efa)[(ef)(b\gamma_{f,ef})]$

$= (efa)(efbfef) = (efea)(efbfef) = (ea)(efbf) = abf = ab,$

where the last equality follows from the facts that $ab, abf \in G^{ef}$ and the mapping $G^{ef} \longrightarrow G^f$ given by $x \longrightarrow fxf$ is an isomorphism.

4) $a \in K^e \setminus G^e$, $b \in G^f$: The calculation here is symmetric to that of case 3).//

Lemma 4.11. Let S be a periodic semigroup with subsemigroup E of idempotents. If $\mathcal{K}$ is a congruence and $H = \cup\{G^e \mid e \in E\}$ is an ideal in S, then the system $(\cup\{K^e/G^e \mid e \in E\}, \circ)$ is a semigroup, where operation $\circ$ is defined by:

$$a \circ b = \begin{cases} ab & \text{if } a \in K^e \setminus G^e,\ b \in K^f \setminus G^f \text{ and } ab \in K^{ef} \setminus G^{ef} \\ G^{ef} & \text{if } a \in K^e \setminus G^e,\ b \in K^f \setminus G^f \text{ and } ab \in G^{ef} \\ G^{ef} & \text{if either } a = G^e \text{ of } b = G^f. \end{cases}$$

Proof. Closure is evident, so it remains to verify associativity.

Let $a \in K^e/G^e$, $b \in K^f/G^f$ and $c \in K^g/G^g$. If either $a = G^e$ or $b = G^f$ or $c = G^g$ or $abc \in G^{efg}$, then $(a \circ b) \circ c = G^{efg} = a \circ (b \circ c)$. If $a \in K^e \setminus G^e$, $b \in K^f \setminus G^f$, $c \in K^g \setminus G^g$ and $abc \notin G^{efg}$, then it is impossible that either $ab \in G^{ef}$ or $bc \in G^{fg}$, because H is an ideal in S and $G^{ef} \subseteq H$, $G^{fg} \subseteq H$. In such a case it must be that

$ab \notin G^{ef}$ and $bc \notin G^{fg}$; thus

$(a \bullet b) \circ c = (ab) \circ c = (ab)c = a(bc) = a \bullet (bc) = a \circ (b \circ c)$.//

Definition 4.12. In the case of the preceding lemma we call $H = \bigcup\{G^e \mid e \in E\}$ the heart of S, and the defined semigroup $(\bigcup\{K^e/G^e \mid e \in E\}, \circ)$ the shell of S.

Since $\mathcal{K} = \Sigma$ implies the hypothesis of Lemma 4.11 (Theorem 4.3), then the shell of S has meaning in Theorem 4.10, which we restate as:

Theorem 4.13. Let S be a periodic semigroup with subsemigroup E of idempotents. In order that S be a rectangular $\mathcal{K}$-class it is necessary and sufficient that:

I) $\mathcal{K} = \Sigma$

II) Shell of S is a rectangular z-semigroup with family Γ of isomorphisms such that for $a \in (K^e/G^e)^*$ and $f \in E$, then $faef = (a\gamma_{e,f})f$, where $\gamma_{e,f} \in \Gamma$.

Theorem 4.14. Let S be a periodic semigroup with subsemigroup E of idempotents. Then S is a rectangular $\mathcal{K}$-class if and only if

I) $\mathcal{K}$ is a congruence

II) Heart of S is a rectangular group

III) Shell of S is a rectangular z-semigroup with family Γ of isomorphisms such that for $a \in (K^e/G^e)^*$ and $f \in E$, then $faef = (a\gamma_{e,f})f$, where $\gamma_{e,f} \in \Gamma$.

Proof. Assume S is a rectangular $\mathcal{K}$-class. By the preceding theorem we know $\mathcal{K} = \Sigma$, so K is a congruence. Moreover, from Theorem 4.3 we have seen that the heart H of S is a matrix of groups, and each pair G^e, G^f is isomorphic under the correspondence $x \longrightarrow fxf$, $x \in G^e$. Fixing $G^1 \subseteq H$ it is easily verified that $H \longrightarrow E \times G^1$ given by $x \longrightarrow (e, 1x1)$, where $x \in G^e$, is an isomorphism. Hence, H is a rectangular group. Part III) follows directly from Theorem 4.13.

Conversely, in order that the shell of S be meaningful, we must show that the heart H of S is an ideal in S. Since E is a semigroup and H a rectangular group, it is immediate that E is a rectangular band.

Let $a \in H$, $b \in S$; say $a \in G^e$, $b \in K^f$. Then $ab \in G^eK^f \subseteq K^{ef}$; say $(ab)^n = ef$. Hence, $ab = eab = (efe)ab = efab = (ab)^n ab = (ab)^{n+1}$; so ab, having index one, must lie in G^{ef}. Similarly, one can prove $ba \in G^{fe}$. Thus, H is an ideal in S.

It remains only to show that $\mathcal{K} = \Sigma$ (Theorem 4.13). Let $a \in K^e$ and $b \in K^f$. Since $\mathcal{K}$ is a congruence and E a rectangular band, then $aba \in K^eK^fK^e \subseteq K^{efe} = K^e$; that is, $a\mathcal{K}aba$.//

CHAPTER 5

CONGRUENCES ON A UNIPOTENT SEMIGROUP

As established in the preceding chapter, a unipotent periodic semigroup can be expressed as a triple (Z,G,ϕ), where Z is a periodic z-semigroup, G a periodic group, and $\phi: Z^* \longrightarrow G$ a partial homomorphism (Remark 4.5). More generally, a semigroup S which is an ideal extension of a group G by a z-semigroup Z can be represented by the triple (Z,G,ϕ), where again $\phi: Z^* \longrightarrow G$ is a partial homomorphism (Theorem 4.19 of [2]).

Hereafter, we shall take the symbol $S = (Z,G,\phi)$ to mean that S is a semigroup of the latter, more general class, without explicitly mentioning the fact. Observe that periodicity of S is assumed nowhere in this chapter.

The purpose of this section is to characterize any congruence on $S = (Z,G,\phi)$ in terms of its three components.

<u>Notation 5.1</u>. i) For semigroup S, let $\mathcal{C}(S)$ be the set of all congruences on S.

ii) If ρ is an equivalence relation on $S = (Z,G,\phi)$, let $A = \{x \in S \mid x\rho y \text{ for some } y \in G\}$; trivially, A is a union of ρ-classes of S. If, in addition, A is an ideal in S, define the equivalence relations τ,σ on S/A and A respectively by:

$$\begin{cases} a\tau b & \text{if and only if} \quad a\rho b \text{ or } a = b = 0, \\ a\sigma b & \text{if and only if} \quad a\rho b. \end{cases}$$

In all the results of this chapter, whenever ρ is an equivalence relation on $S = (Z,G,\phi)$, A shall have this meaning; and if A is an ideal in S, the symbols τ, σ shall be reserved for these particular equivalences.

Lemma 5.2. Let ρ be an equivalence relation on $S = (Z, G, \phi)$, A an ideal in S, and $\sigma \in \mathcal{C}(A)$.

(1) If $x \in A \setminus G$, then $x\sigma(x\phi)$.

(2) If $x \in S \setminus A$ and $a \in A$, then $(ax)\sigma[a(x\phi)]$ and $(xa)\sigma[(x\phi)a]$.

Proof. (1) Since $x \in A$ there exists $y \in G$ such that $x\sigma y$. Hence, $(xe)\sigma(ye)$, where e is the identity of G. But $ye = y$ and $xe = (x\phi)e = x\phi$. Therefore, $(x\phi)\sigma y$ and $x\sigma(x\phi)$.

(2) If $ax \in G$, then $ax = a(x\phi)$ and trivially $(ax)\sigma[a(x\phi)]$. If $ax \in A \setminus G$, then $a(x\phi) = (a\phi)(x\phi) = (ax)\phi$. From part (1) above, $(ax)\sigma[(ax)\phi]$; thus, $(ax)\sigma[a(x\phi)]$.

Similarly one proves that $(xa)\sigma[(x\phi)a]$.//

Definition 5.3. Let ρ be an equivalence relation on a semigroup S.

i) The equivalence ρ is called *0-restricted* if $\{0\}$ is a ρ-class of S.

ii) A subset T of S is said to be *saturated by ρ* if T is a union of ρ-classes of S.

Theorem 5.4. Let ρ be an equivalence relation on $S = (Z, G, \phi)$. Then $\rho \in \mathcal{C}(S)$ if and only if

(1) A is an ideal in S

(2) $\tau \in \mathcal{C}(S/A)$ and τ is 0-restricted

(3) $\sigma \in \mathcal{C}(A)$

(4) If $x,y \in S\setminus A$ and $x\tau y$, then $(x\phi)\sigma(y\phi)$.

Proof. Assume that $\rho \in \mathcal{C}(S)$.

(1) Let $x \in S$ and $a \in A$. By definition of set A, there exists an element $b \in G$ such that $a\rho b$. Thus, $(ax)\rho(bx)$ and $(xa)\rho(xb)$, where bx and $xb \in G$ because G is an ideal in S. Therefore $ax, xa \in A$, and A is an ideal in S.

(2) and (3). These statements follow directly from the definitions of τ and σ.

(4) Let $x,y \in S\setminus A$ and $x\tau y$. From the definition of τ we have $x\rho y$, and since ρ is a congruence $(xe)\rho(ye)$, where e is the identity of G, $xe = (x\phi)e = x\phi \in G$, and $ye = (y\phi)e = y\phi \in G$. That is, $(x\phi)\sigma(y\phi)$.

Conversely, assume that conditions (1) through (4) hold in S. In addition, the conditions in Lemma 5.2 will be used often in the computation which follows.

Let $x\rho y$ and $c \in S$. To prove that ρ is a congruence we consider four cases.

I) $x,y \in S\setminus A$ and $c \in S\setminus A$.

i) $cx \in S\setminus A$: By hypothesis $x\tau y$ and $\tau \in \mathcal{C}(S/A)$, so $(cx)\tau(cy)$. Since τ is 0-restricted, then $cy \in S\setminus A$. Hence, $(cx)\rho(cy)$.

ii) $cx \in A$: As above $(cx)\tau(cy)$, and since τ is 0-restricted, $cy \in A$. Now $x\tau y$ implies $(x\phi)\sigma(y\phi)$; and $\sigma \in \mathcal{C}(A)$ so

$$(c\phi)(x\phi)\sigma(c\phi)(y\phi). \qquad (*)$$

If $cx \in G$, then $(c\phi)(x\phi) = cx$. If $cx \in A\setminus G$, then $(c\phi)(x\phi) = (cx)\phi$, where by Lemma 5.2 $[(cx)\phi]\sigma(cx)$. Similarly, if $cy \in G$, then $(c\phi)(y\phi) = cy$; and if $cy \in A\setminus G$, then $(c\phi)(y\phi) = (cy)\phi$, where $[(cy)\phi]\sigma(cy)$. In any case, (*) above becomes $(cx)\sigma(cy)$, or $(cx)\rho(cy)$.

II) $x,y \in S\setminus A$ and $c \in A$. Then $cx,cy \in A$. By hypothesis $x\tau y$, which implies $(x\phi)\sigma(y\phi)$; and $\sigma \in \mathcal{C}(A)$ so $[c(x\phi)]\sigma[c(y\phi)]$. Due to Lemma 5.2, $(cx)\sigma(cy)$, or $(cx)\rho(cy)$.

III) $x,y \in A$ and $c \in S\setminus A$. Again $cx,cy \in A$. Now $x\rho y$ implies $x\sigma y$, and $\sigma \in \mathcal{C}(A)$, so $[(c\phi)x]\sigma[(c\phi)y]$. From the afore-mentioned lemma, $(cx)\sigma(cy)$, or $(cx)\rho(cy)$.

IV) $x,y \in A$ and $c \in A$. Once more $cx,cy \in A$. Since $x\rho y$, then $x\sigma y$; and $(cx)\sigma(cy)$ because $\sigma \in \mathcal{C}(A)$. That is, $(cx)\rho(cy)$.

Therefore, ρ is left compatible; dually one can prove it is right compatible.//

<u>Corollary 5.5</u>. Let ρ be an equivalence relation on $S = (Z,G,\phi)$ such that G is saturated by ρ. Then $\rho \in \mathcal{C}(S)$ if and only if

(1) $\tau \in \mathcal{C}(Z)$ and τ is 0-restricted

(2) $\sigma \in \mathcal{C}(G)$

(3) If $x,y \in Z^*$ and $x\tau y$, then $(x\phi)\sigma(y\phi)$.

<u>Proof</u>. Since G is saturated by ρ, then $A = G$, which is a (two-sided) ideal in S. In this case, $S/G \cong Z$ and $S\setminus G = Z^*$.

The result is now immediate from the preceding theorem.//

Theorem 5.6. Let $S = (Z,G,\phi)$ and

$$P = \{\rho \in \mathcal{C}(S) \mid \text{for each } \rho\text{-class } B,\ B \between G\}.$$

Then there exists a one-to-one correspondence between P and the set $\mathcal{C}(G)$ of all congruences on G.

Proof. First, note that each $\rho \in P$ has the property:

$$a\rho b \quad \text{if and only if} \quad (ae)\rho(be), \qquad (1)$$

where e is the identity of G. The implication to the right is obvious. To see the converse, let $x \in S$. Then there exists $y \in G$ such that $x\rho y$, hence $(xe)\rho y$ and $x\rho(xe)$. Since $a\rho(ae)$ and $b\rho(be)$, then $(ae)\rho(be)$ implies $a\rho b$.

Now, given $\alpha \in \mathcal{C}(G)$ define relation $\bar{\alpha}$ on S by:

$$a\bar{\alpha}b \quad \text{if and only if} \quad (ae)\alpha(be). \qquad (2)$$

It is easily verified that $\bar{\alpha} \in P$ and $\bar{\alpha}|_G = \alpha$.

Given $\rho \in P$, then certainly $\rho|_G \in \mathcal{C}(G)$; denote it by ρ^*. Using the statements (1) and (2) above we have: $a\rho b$

if and only if $(ae)\rho(be)$

if and only if $(ae)\rho^*(be)$

if and only if $a\overline{\rho^*}b$;

that is, $\rho = \overline{\rho^*} = \overline{\rho|_G}$.

Therefore, the correspondences

$$f: P \longrightarrow \mathcal{C}(G) \text{ and } g: \mathcal{C}(G) \longrightarrow P$$

given by $\rho \longrightarrow \rho|_G$ and $\alpha \longrightarrow \bar{\alpha}$

respectively, are such that

$\rho(f \circ g) = (\rho|_G)g = \overline{\rho|_G} = \rho$

and $\alpha(g \circ f) = (\bar{\alpha})f = \bar{\alpha}|_G = \alpha$.//

Remarks 5.7. Let ρ be an equivalence relation on $S = (Z,G,\phi)$ such that A is an ideal in S.

i) The preceding theorem affords a means of determining whether or not $\sigma \in \mathcal{C}(A)$. Namely, $\sigma \in \mathcal{C}(A)$ if and only if $\rho|_G \in \mathcal{C}(G)$. In this sense the congruences $\mathcal{C}(A)$ of Theorem 5.4 are known modulo the normal subgroups of G.

ii) We proceed next to develop a method of constructing all congruences $\mathcal{C}(S/A)$ on a z-semigroup S/A of a certain type. The technique used is similar to that of Tamura in [16], in which he constructs all finite z-semigroups.

Notation 5.8. i) (cf. [16]). Let S be a z-semigroup and $I_1 = \{0\}$. For integer $n \geq 2$ form I_n inductively as:

$$I_n = \{x \in S \mid x \in S \setminus (\bigcup_{k=1}^{n-1} I_k)\, ;\ xy, yx \in \bigcup_{k=1}^{n-1} I_k \text{ for each } y \in S\}.$$

Clearly, for each positive integer n, $\bigcup_{k=1}^{n} I_k$ is a two-sided ideal in S. From the given characterization of I_n it follows that if $x \in I_n$, then there exists $y \in S$ such that either xy or yx belongs to I_{n-1}.

ii) To avoid repeated use of cumbersome notation, we shall hereafter write J_n for $\bigcup_{k=1}^{n} I_k$.

Definition 5.9. i) A z-semigroup S will be called I-saturated if $S = \bigcup_{k=1}^{\infty} I_k$.

ii) (cf. [16]). Let S be a z-semigroup and $x,y \in S$, $y \neq 0$. We say that x is a multiple of y, denoted $y > x$, if $x \in Sy \cup yS \cup SyS$. If $x \in S$ and there exist elements $x_2, \ldots, x_n \in S$

such that $x = x_1 > x_2 > \ldots\ldots > x_n = 0$, then we shall say that x has a chain of length n. Trivially, 0 has exactly one chain, it of length one. Also, each non-zero element x of S has a chain of length 2; namely, $x = x_1 > x_2 = 0$. For $x \in S$ the number $D[x] = \max\{n \mid x \text{ has a chain of length } n\}$ is called the depth of x, if such a maximum exists.

Lemma 5.10. For a z-semigroup S, $x \in I_n$ if and only if $D[x] = n$.

Proof. In case $n = 1$, then $x \in I_1 = \{0\}$ if and only if $D[x] = 1$. Assume the result valid for integers $1,2,\ldots,n-1$.

Let $x \in I_n$. An arbitrary chain connecting x and 0 is of the form

$$x = x_1 > x_2 > \ldots. > x_k = 0,$$

where x_2, being a multiple of x_1, belongs to J_{n-1}. By the induction hypothesis, $D[x_2] \leq n-1$. Hence,

$$x_2 > x_3 > \ldots. > x_k = 0,$$

being a chain connecting x_2 and 0, is of length $\leq n-1$. Thus, $k-1 \leq n-1$ and $k \leq n$.

This proves that x has no chain of length greater than n, and $D[x] \leq n$. From the induction hypothesis, $D[x] = n$.

Conversely, let $D[x] = n$; due to the induction hypothesis $x \in S \setminus J_{n-1}$. Moreover, for each $y \in S$ all chains of the form

$$x > xy > \ldots\ldots > 0 \text{ and } x > yx > \ldots\ldots > 0$$

are of length $\leq n$. Therefore, for each $y \in S$ all chains of the form

$$xy > \ldots\ldots > 0 \text{ and } yx > \ldots\ldots > 0$$

are of length $\leq n-1$. That is to say, for each $y \in S$, $D[xy] \leq n-1$ and $D[yx] \leq n-1$. Again, from the induction hypothesis, $xy, yx \in J_{n-1}$ for each $y \in S$.

By the definition of I_n in 5.8, $x \in I_n$.//

The first of the next two corollaries follows trivially from the lemma.

Corollary 5.11. In order that a z-semigroup S be I-saturated it is necessary and sufficient that each element of S has finite depth.

Corollary 5.12. If z-semigroup S is I-saturated, then S satisfies the D.C.C on principal ideals.

Proof. If not, let $S^1xS^1 \supset S^1yS^1 \supset \ldots$ be a non-terminating strictly descending chain of principal ideals in S. Then $x > y > \ldots$ is a non-terminating chain in S, and the depth of x would not be finite.//

Remark 5.13. In view of Corollary 5.11, one might describe an I-saturated z-semigroup S by saying that for each element $x \in S$, the lengths of all those chains connecting x and 0 are bounded. If there is a common bound h on the lengths of all chains in S, then $S = \bigcup_{k=1}^{h} I_k$; otherwise, $S = \bigcup_{k=1}^{\infty} I_k$.

Lemma 5.14. Let z-semigroup S be I-saturated. If n is an integer such that $S \setminus J_n \neq \emptyset$, then $I_{n+1} \neq \emptyset$.

Proof. Let $x \in S \setminus J_n = (\bigcup_{k=1}^{\infty} I_k) \setminus J_n = \bigcup_{k=n+1}^{\infty} I_k$; say $x \in I_m$ where $m > n$. From the definition of I_m there exists $y \in S$ such that xy or $yx \in I_{m-1}$ (let it be xy). Similarly, there exists $z \in S$ such that either xyz or $zxy \in I_{m-2}$ (say zxy). Repeating this a total of $(m-1)$ times one gets a chain

$$x = x_m > xy = x_{m-1} > zxy = x_{m-2} > \dots > x_1 = 0$$

of length m such that $x_i \in I_i$, $i = 1,2,\dots,m$. Since $1 < n+1 \leq m$, then $x_{n+1} \in I_{n+1}$.//

As for examples of I-saturated z-semigroups, certainly finite z-semigroups are such. And if a finite z-semigroup S is inflated (see ex. 10, p. 99 of [2]) by sets containing infinitely many elements, the resulting inflation of S is an I-saturated z-semigroup of infinite order. (Just observe that each element of the inflation has finite depth.)

Now assume that S is an I-saturated z-semigroup, so that the results of Lemmas 5.10, 5.14, Corollaries 5.11,5.12 apply. We intend to derive a 0-restricted congruence on S and claim that every 0-restricted congruence on S can be so constructed.

Let $I_1 = \{0\}$ and define I_n as in 5.8. Choose an arbitrary equivalence relation π_2 on I_2, and let $\overset{2}{-}$ be the induced 0-restricted equivalence on J_2. Since J_2 is a null semigroup, trivially $\overset{2}{-}$ is a congruence on J_2.

We proceed to define equivalence relation $\overset{n}{-}$ on J_n $(n \geq 3)$ inductively as follows. First, choose an equivalence π_n on I_n having the property:

(a) if $x,y \in I_n$ and $x\pi_n y$, then for each $z \in S$

$$zx \overset{n-1}{—} zy \text{ and } xz \overset{n-1}{—} yz.$$

Then define $\overset{n}{-}$ on J_n by:

(b) $x \overset{n}{-} y$ if and only if $\begin{cases} \text{either } x \overset{n-1}{—} y & \text{if } x,y \in J_{n-1} \\ \text{or } \quad x\pi_n y & \text{if } x,y \in I_n. \end{cases}$

By induction one can easily show that each $\overset{n}{-}$ is a congruence on J_n.

In countably many steps, this method yields a congruence on S. It is completely characterized by the sequence $\{\pi_2, \pi_3, \pi_4, \ldots\}$ where π_2 is an arbitrary equivalence on I_2, and each π_n $(n \geq 3)$ is an equivalence relation on I_n having property (a).

Conversely, let ρ be a 0-restricted congruence on S; again let $I_1 = \{0\}$ and define I_n as in 5.8.

Trivially, I_1 is saturated by ρ; so suppose $I_2, \ldots, I_{n-1}$ are likewise saturated by ρ. If $x \in I_n$ and $x\rho y$, by the induction hypothesis $y \in S \setminus J_{n-1}$. In addition, for each $z \in S$, $(xz)\rho(yz)$ and $(zx)\rho(zy)$. Consequently, $y \in S \setminus J_{n-1}$ and $yz, zy \in J_{n-1}$ for each $z \in S$; that is, $y \in I_n$. This proves that each I_k is saturated by ρ.

Let $\pi_2 = \rho|_{I_2}$, and $\overset{2}{-}$ be the 0-restricted equivalence on J_2 induced by π_2. Certainly $\overset{2}{-}$, which is the restriction of ρ to J_2, is a congruence on J_2.

Again proceeding inductively for $n \geq 3$, let $\pi_n = \rho|_{I_n}$, which has property (a) above, and define $\overset{n}{-}$ on J_n as in (b). Since each

I_k is saturated by ρ, it is immediate that each $\overset{n}{-}$ coincides with the restriction of ρ to J_n.

Therefore, the derived congruence on S agrees with ρ. In terms of the aforementioned sequence, we might say

$\rho = \{\rho|_{I_2}, \rho|_{I_3}, \rho|_{I_4}, \ldots\}$.

We conclude this chapter by combining these last results with Theorem 5.6 to derive all congruences of a certain type on $S = (Z, G, \phi)$.

Let A be an ideal in $S = (Z, G, \phi)$ such that z-semigroup S/A is I-saturated. Note that A necessarily contains G, the kernel of S. Arbitrarily choose a congruence π_1 on G and define equivalence relation σ on A by:

(c) $a\sigma b$ if and only if $(ae)\pi_1(be)$, where e is the identity of G.

Due to the proof of Theorem 5.6 and Remark 5.7, σ is a congruence on A.

For z-semigroup S/A define each set I_n as previously in 5.8. Now choose an equivalence π_2 on I_2 having the property:

(d) if $x, y \in I_2$ and $x\pi_2 y$, then $(x\phi)\sigma(y\phi)$. Let $\overset{2}{-}$ be the 0-restricted congruence on J_2 induced by π_2, and continue as before in (a) and (b) to inductively define congruence $\overset{n}{-}$ on J_n.

<u>Lemma 5.15</u>. If equivalence π_2 on I_2 has property (d), then each congruence $\overset{n}{-}$ on J_n $(n \geq 2)$ has the property:

(e) if $x \overset{n}{-} y$, then $(x\phi)\sigma(y\phi)$.

Proof. Since $\overset{2}{-}$ is the 0-restricted congruence on J_2 induced by π_2, then it is clear that (e) holds for $n = 2$. Assume that it is valid for integers $2,3,\ldots,n-1$.

Let $x \overset{n}{-} y$; from the definition of $\overset{n}{-}$ in (b) either $x \overset{n-1}{-\!-} y$ or $x\pi_n y$. In the former case the result follows from the induction hypothesis. If $x,y \in I_n$ and $x\pi_n y$, due to the definition of I_n there exists $z \in S$ such that either zx or $xz \in I_{n-1}$; say it is xz. Since π_n has property (a), then $xz \overset{n-1}{-\!-} yz$; so by the induction hypothesis, $[(xz)\phi]\sigma[(yz)\phi]$. But $xz,yz \in I_{n-1} \subseteq S\setminus A \subseteq S\setminus G$ and ϕ is a partial homomorphism on $S\setminus G$, hence $(xz)\phi = (x\phi)(z\phi)$ and $(yz)\phi = (y\phi)(z\phi)$; that is, $(x\phi)(z\phi)\sigma(y\phi)(z\phi)$. Due to cancellation in group A/σ, $(x\phi)\sigma(y\phi)$.//

If τ is the congruence induced on S/A by the congruences $\overset{n}{-}$, $n = 1,2,3,\ldots$, then the congruences σ on A and τ on S/A satisfy the conditions of Theorem 5.4. Thus, the induced equivalence relation on $S = (S/A)^* \cup A$ is a congruence. Note that it is characterized by the sequence

$$\{A,\pi_1,\pi_2,\pi_3,\ldots\},$$

where A is an ideal in S such that S/A is I-saturated, π_1 is a congruence on G, π_2 is an equivalence on I_2 having property (d), and each π_n $(n \geq 3)$ is an equivalence on I_n having property (a).

The following is a summary of the construction of congruence $\{A,\pi_1,\pi_2,\pi_3,\ldots\}$ on $S = (Z,G,\phi)$:

(1) Choose ideal A of S such that S/A is an I-saturated z-semigroup, and define I_n for S/A as in 5.8.

(2) Arbitrarily choose a congruence π_1 on G and define congruence σ on A as in (c).

(3) Choose an equivalence relation π_2 on I_2 having property (d) and let $\overset{2}{-}$ be the 0-restricted equivalence on J_2 induced by π_2.

(4) For $n \geq 3$, choose equivalence π_n on I_n having property (a) and inductively define $\overset{n}{-}$ on J_n as in (b).

If one wishes to express the second parameter of the congruence by a normal subgroup N of G rather than by a congruence π_1 on G, then step (2) becomes:

(2') Arbitrarily choose a normal subgroup N of G and define congruence σ on A by:

(c') $a\sigma b$ if and only if $(ae)(be)^{-1} \in N$.

Thus, the constructed congruence is given by

$$\{A, N, \pi_2, \pi_3, \dots\}.$$

Conversely, if ρ is a congruence on $S = (Z, G, \phi)$ such that z-semigroup S/A (see Notation 5.1) is I-saturated, then

$$\rho = \{A, \rho|_G, \rho|_{I_2}, \rho|_{I_3}, \dots\}.$$

Or, alternately, if N is the kernel of $\rho|_G$,

$$\rho = \{A, N, \rho|_{I_2}, \rho|_{I_3}, \dots\}.$$

Corollary 5.16. If $S = (Z, G, \phi)$ is such that z-semigroup S/G is I-saturated, then the above procedure yields all congruences on S.

Proof. Since each ideal A in S contains group G, then S/G I-saturated implies S/A likewise.//

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