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Sequence space properties related to the Wilansky property

Stoudt, Gary Scott, Ph.D.

Lehigh University, 1991

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**SEQUENCE SPACE PROPERTIES
RELATED TO THE WILANSKY PROPERTY**

by

Gary S. Stoudt

A Dissertation

Presented to the Graduate Committee

of Lehigh University

in Candidacy for the Degree of

Doctor of Philosophy

in

Mathematics

Lehigh University

1990

CERTIFICATE OF APPROVAL

Approved and recommended for acceptance as a dissertation in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

December 19, 1990
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December 20, 1990
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DEDICATION

To Zena, Gregg, Jimmy, Gene, and Eric

May all of your dreams come true, as
one of mine has with the completion of this thesis.

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I have come to learn that my Mom and Dad true miracles in the world of parenthood. Your questions about my work always went unanswered; this thesis is my answer to you now. The great pride and joy that I feel because of this work is dwarfed by my pride in simply being your son.

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G.S.S.– December, 1990

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ABSTRACT

A *BK* space is a Banach space of scalar sequences with the property that the coordinate projection maps are continuous. The β dual of a *BK* space X , denoted X^β , is the set of all scalar sequences u such that $\sum_{k=1}^{\infty} u_k x_k$ exists for all x in X . Similarly, the γ dual of a *BK* space X , denoted X^γ , is the set of all scalar sequences u such that $\sup_n |\sum_{k=1}^n u_k x_k| < \infty$ for all $x \in X$.

A *BK* space Z is said to have the Wilansky property (*W3*) if each *BK* space X contained in Z having $X^\beta = Z^\beta$ is closed in Z . This thesis considers the Wilansky property for special *BK* spaces. These special spaces are *AD* spaces, spaces in which $\text{span} \{\delta^n\}$ is dense, and *AK* spaces, spaces in which $\{\delta^n\}$ is a basis. Here δ^n is the sequence whose n th coordinate is 1 and all others are 0.

It is shown that in the Wilansky property it is sufficient to consider *BK*, *AD* subspaces of Z , if Z is itself a *BK*, *AK* space. This is the context of the theorem which states that the new Wilansky property (*W5*) is equivalent to property (*W3*). A *BK* space Z has property (*W5*) if each *BK*, *AD* subspace X of Z with $X^\gamma = Z^\gamma$ has finite codimension in Z . New properties are developed and the equivalence of many of these is shown. The others are ordered by a series of implications.

The most notable of these new properties concerns the presence in Z of sequences that are equivalent to the unit vector basis of ℓ_1 . This occurs if a *BK*, *AK* space Z fails to have one of the new properties. It is interesting to note that the space ℓ_1 , which is known to fail the Wilansky property, is in some way fundamentally linked to the failure of the Wilansky property in general *BK*, *AK* spaces. Previous results concerning the Wilansky property are obtained in this new setting, for instance, a result of Noll and Stadler that states that a *BK*, *AK* space has the Wilansky property if the *AK* part of Z' has finite codimension in Z' .

§1. Introduction

The notion of the β dual of a sequence space X , written X^β , is a special case of multipliers of that space. The β dual of X is the set of all scalar sequences u such that when u is multiplied coordinatewise by any element in X , an element of the sequence space cs is obtained. The space cs is the set of all sequences x such that $\sum x_n$ converges.

It is known [15, 16.3.3] that if D is a dense subset of the bounded sequences ℓ_∞ , then $D^\beta = \ell_1$, the absolutely summable sequences. Note that if D contains c_0 , the bounded sequences that converge to 0, this is clear.

Another notion of duality is the f dual of a sequence space X , denoted X^f . This is the sequence space $\{\{f(\delta^n)\} : f \in X^f\}$, where X^f is the usual dual space of bounded linear functionals on X and δ^n is the sequence that is 1 in the n th coordinate and 0 elsewhere. An FK space is a sequence space which is also a locally convex Fréchet space and has the additional property that the functionals $\{x_n\} \rightarrow x_n$ are continuous. It is also known that if X is an FK space in which the span of $\{\delta^n\}$ (denoted φ) is dense, called an AD space, and if Y is also an FK space then $Y \supseteq X$ if and only if $Y^f \subseteq X^f$. For example, $Y \supseteq c_0$ if and only if $Y^f \subseteq c_0^f = \ell_1$. This last statement is not true if we consider β duals.

A. Wilansky considered these ideas in [15] and they suggested to him the following question: is there an FK space X containing φ with $X \subset c_0$, $X \neq c_0$ and $X^\beta = c_0^\beta = \ell_1$? That is, does the above fact concerning f duals fail for β duals in this case? Another similar question is this: if X is an FK, AD space and $X^\beta = \ell_1$, must $X = c_0$?

G. Bennett considered questions of this type in [2]. Noll and Stadler considered them as well in [5], [7], [8], [9], and [13]. It is known that $X \supseteq Z$ implies $X^\beta \subseteq Z^\beta$

for sequence spaces X and Z . Bennett states that the converse is false (giving an example of Wilansky) and the the converse is still false if one assumes $X \subseteq Z$ and $X^\beta = Z^\beta$ and asks if $X = Z$. A counterexample for this is $X = c_0, Z = \ell_\infty$. Bennett then adds the hypothesis that X be dense in Z and formulates what he calls the Wilansky property (W). An FK space Z has *property (W)* if X an FK space contained in Z with X dense in Z and $X^\beta = Z^\beta$ together imply that $X = Z$.

Bennett shows that c_0 has this property thereby answering Wilansky's original question. Stadler also proves this result in [13]. Motivated by his solution for the space c_0 , Bennett formulates the following Wilansky properties, the first of which is included for completeness only: an FK space Z has *property (W1)* if X a linear subspace of Z and $X^\beta = Z^\beta$ together imply that X is barrelled, that is to say, X is a locally convex space in which every barrel (balanced, convex, absorbing, closed set) is a neighborhood of 0. An FK space Z has *property (W2)* if X an FK space contained in Z and $X^\beta = Z^\beta$ together imply that X is closed in Z .

Bennett goes on to prove that property (W1) is equivalent to property (W2) and that these imply property (W). He also proves that if Z is separable, then these three Wilansky properties are equivalent. In the case that Z is a BK space, property (W2) may be replaced by property (W3): a BK space Z has *property (W3)* if X a BK space contained in Z and $X^\beta = Z^\beta$ together imply that X is closed in Z . It is often said, if the properties are equivalent, that a space "has the Wilansky property".

In the course of his work Bennett cites A. K. Snyder [11], but proceeds in a different direction. This thesis proceeds following some of the ideas from [11]. A relevant result from [11] states that in the following list of statements (i) \Leftrightarrow (ii) \Leftrightarrow (iii) and each is implied by (iv), for a BK space Z , having $\{\delta^n\}$ as a basis. Such

a space is called an *AK* space.

- (i) There does not exist a *BK, AD* space X such that $X \subseteq Z$, $X \neq Z$, and Z is the smallest *AK* space containing X (Z is not “minimal *AK*”).
- (ii) If X is a *BK, AD* space, $X \subseteq Z$, and $X \neq Z$, then $X \widehat{\otimes} bv \neq Z$. ($X \widehat{\otimes} bv$ is the *BK* product of X and bv , which is discussed in section 3.)
- (iii) If X is a *BK, AD* space, $X \subseteq Z$, and $X^\gamma = Z^\gamma$, then $X = Z$.
- (iv) Z_0^γ has finite codimension in Z^γ .

Here the space Z^γ denotes the γ dual of Z . This is defined in a manner similar to Z^β , but the element obtained by coordinatewise multiplication is in the space of bounded series *bs* instead of the space *cs*. Also note here Z_0^γ denotes the closure of φ in Z^γ .

One of the goals of this thesis is to use the ideas formulated in [11], specifically the result above, to develop new Wilansky properties and to establish relationships between these new properties and those of Bennett. It is also hoped that this may shed new light on the Wilansky property. For example, Bennett establishes the Wilansky property for *BK, AK* spaces with β dual also an *AK* space. He also establishes the Wilansky property in other spaces using different techniques. All of the spaces considered in [2] have the property that the closure of φ in the β dual has finite codimension in the β dual. We make use of this fact, thereby unifying the results concerning the spaces in [2]. Using techniques developed in this thesis, one may also prove certain results of Noll and Stadler in [8].

In one of the major results of this thesis (in section IV), considerable effort is used to show that when investigating the Wilansky property, one only needs to consider subspaces X with φ dense. This also will help to establish new equivalences between the various Wilansky properties.

It is also noted that if a space fails to have the new Wilansky property ($W8$), it must contain a nice copy of ℓ_1 . This gives a good idea of what it means for a space to fail the Wilansky property.

Other Wilansky type properties are established later in the thesis, with implications between them. These properties have hypotheses that have the effect of strengthening the conditions on the subspace investigated, for example that the subspace itself is an AK space. This facilitates verification of the Wilansky property for these spaces. Finally, the thesis turns to solid spaces. These are spaces in which an element of the space may be coordinatewise multiplied by a bounded sequence and remain in that space. It is noted that under the condition that the space is a solid BK , AK space, all of these new Wilansky type properties are indeed equivalent.

§2. Preliminaries

This thesis examines properties of sequence spaces. Many different types of sequence spaces with quite varied properties will be encountered. Some of these properties will be discussed here for future reference.

The space of all scalar sequences will be denoted by ω . This is a Fréchet space (locally convex, with topology induced by a complete translation invariant metric) with the topology of pointwise convergence. An *FK space* is a Fréchet space which is made up of sequences and has the property that the coordinate functionals π_n , where $\pi_n(x) = x_n$, are continuous for each n . A *BK space* is the special case of an *FK space* in which the Fréchet space is a Banach space (complete normed space). This thesis deals primarily with *BK spaces*. The following *BK spaces* will be encountered:

$$\begin{aligned} \ell_p &= \left\{ x : \|x\|_p^p = \sum_{k=1}^{\infty} |x_k|^p < \infty \right\} \\ \ell_{\infty} &= \left\{ x : \|x\|_{\infty} = \sup_n |x_n| < \infty \right\} \\ c_0 &= \left\{ x : \lim_n x_n = 0 \right\} \text{ with norm of } \ell_{\infty} \\ bs &= \left\{ x : \|x\|_{bs} = \sup_n \left| \sum_{k=1}^n x_k \right| < \infty \right\} \\ cs &= \left\{ x : \lim_n \sum_{k=1}^n x_k \text{ exists} \right\} \text{ with the norm of } bs \\ bv &= \left\{ x : \|x\|_{bv} = \left| \lim_n x_n \right| + \sum_{k=1}^{\infty} |x_{k+1} - x_k| < \infty \right\} \\ bv_0 &= bv \cap c_0 = \left\{ x \in bv : \lim_n x_n = 0 \right\} \text{ with norm of } bv. \end{aligned}$$

Let φ denote the span of $\{\delta^n\}$, where $\delta_k^n = 0$ for $k \neq n$ and $\delta_n^n = 1$. For an *FK space* X containing φ let X_0 denote the closure of φ in X . If X and Y are

both *BK* spaces, $X + Y$ is a *BK* space under the norm

$$\|z\|_{X+Y} = \inf\{\|x\|_X + \|y\|_Y : z = x + y, x \in X, y \in Y\}.$$

This section discusses facts about *FK* spaces which will be used frequently. They are labelled here for reference. The details for the results in this section may be found in [15].

PROPOSITION 2.1. *Let X and Y be *FK* spaces with $X \subset Y$. Then the topology of X is larger than the topology of Y (on X). They are equal if and only if X is a closed subset of Y . In particular the topology of an *FK* space is unique, that is, there is at most one way to make a vector subspace of ω into an *FK* space.*

Notice that for *BK* spaces this proposition gives useful information about the size of the norms of X and Y . Namely, if $X \subset Y$, then $\|x\|_X \geq C\|x\|_Y$ for some constant $C > 0$. If X is closed in Y , the norms are equivalent, that is, there exist positive constants a and b such that $a\|x\|_X \leq \|x\|_Y \leq b\|x\|_X$. This will be exploited quite often. The following are corollaries to Proposition 2.1.

PROPOSITION 2.2. *Let X, Y , and Z be *FK* spaces with $X \subset Y \subset Z$ and suppose that X is closed in Z . Then X is closed in Y .*

PROPOSITION 2.3. *Let X and Y be *FK* spaces with $X \subset Y$, and let E be a subset of X . Then $\overline{E}^Y = \overline{\overline{E}^X}^Y$, in particular $\overline{E}^X \subset \overline{E}^Y$.*

Note that \overline{E}^X denotes the closure of E in X .

The notion of sections of a sequence is useful in determining the structure of *FK* spaces. For a sequence x and for each positive integer n let $P_n x = \sum_{k=1}^n x_k \delta^k$, the n th section of x . An element x of an *FK* space X is *sectionally bounded* if $\{P_n x\}$ is bounded in X . An *FK* space X is *sectionally bounded* (abbreviated X has *AB*) if

each element of X is sectionally bounded. An FK space X is called an AD space if φ is dense in X , that is, $X_0 = X$. An element x of X is said to have the AK property if $P_n x$ converges to x in X . Finally, X is an AK space if each element of X has the AK property. It is known that an FK space that is both an AD space and has AB is an AK space. For example, $bv_0, c_0, \ell_p (1 \leq p < \infty)$, and cs are all AK spaces.

The notion of multipliers is also important to our study. For sequences x and y , xy denotes the sequence $\{x_n y_n\}$. The multiplier space $M(X, Y)$ is $\{z \in \omega : xz \in Y \text{ for all } x \in X\}$. If X and Y are BK spaces and X contains φ then $M(X, Y)$ is a BK space with norm

$$\|z\| = \sup\{\|xz\|_Y : \|x\| \leq 1\}.$$

The space $M(X, X)$ will be written $M(X)$. A sequence space is called *strongly series summable* if there exists a sequence $\{u^n\}$ in φ such that $\{u^n\}$ is bounded in $M(X)$ and $u^n \rightarrow 1$ in ω , that is, coordinatewise. Here 1 is the constant sequence $\{1, 1, 1, \dots\}$. Special cases of $M(X, Y)$ result in various Köthe-Toeplitz duals which will play an important role in this thesis.

$$M(X, \ell_1) = X^\alpha = \{u \in \omega : \sum_{i=1}^{\infty} |u_i x_i| \text{ converges for all } x \in X\}, \text{ the } \alpha\text{-dual of } X$$

$$M(X, cs) = X^\beta = \{u \in \omega : \sum_{i=1}^{\infty} u_i x_i \text{ converges for all } x \in X\}, \text{ the } \beta\text{-dual of } X$$

$$M(X, bs) = X^\gamma = \{u \in \omega : \sup_n \left| \sum_{i=1}^n u_i x_i \right| < \infty \text{ for all } x \in X\}, \text{ the } \gamma\text{-dual of } X$$

Note that X^γ may also be written $\{u \in \omega : \{\sum_{i=1}^n u_i x_i\} \in \ell_\infty\}$. Recall that if X is a BK space containing φ then X^α, X^β , and X^γ are also BK spaces, because they are multiplier spaces.

It is clear that, for an FK space X containing φ , $\varphi \subseteq X^\alpha \subseteq X^\beta \subseteq X^\gamma$. Furthermore if $X \subseteq Y$ then $Y^t \subseteq X^t$ for $t = \alpha, \beta, \gamma$. If we write $X^{t'}$ for $(X^t)^t$ then it is also clear that $X \subseteq X^{t'}$ for $t = \alpha, \beta, \gamma$. Combining these two facts gives us $X^t = X^{t'}$ for $t = \alpha, \beta, \gamma$. Also useful is the fact that for X and Y BK spaces containing φ , $(X + Y)^t = X^t \cap Y^t$, for $t = \alpha, \beta, \gamma$.

There is another notion of duality which is frequently used. Let X be an FK space containing φ . Define the f -dual of X , written X^f , as $X^f = \{ \{f(\delta^n)\} : f \in X' \}$. Here X' denotes the usual dual space of bounded linear functionals on X . Notice that $u \in X^f$ means that there exists an $f \in X'$ such that $u_n = f(\delta^n)$ for all n . Also note that, by the Hahn-Banach Theorem, $X_0^f = X^f$. If X is a BK space containing φ , then X^f is also a BK space with norm

$$\|u\| = \inf \{ \|f\|_{X'} : u = \{f(\delta^n)\} \}.$$

This follows from the following proposition, given in [15], which will be useful later as well.

PROPOSITION 2.4. *Let X be an FK space containing φ . Then*

- (i) $X^\beta \subseteq X'$, where $u \in X^\beta$ represents a functional $\hat{u} \in X'$ defined by $\hat{u}(x) = \sum_{k=1}^{\infty} u_k x_k$. The map $u \rightarrow \hat{u}$ is an isomorphism into X' . If X is an AK space, the map is onto.
- (ii) X^f is a quotient of X' by the onto map $q : X' \rightarrow X^f$ defined by $q(f) = \{f(\delta^n)\}$.
- (iii) $X^f = X'$, in the sense of (ii) if and only if X is an AD space.

Another useful fact from [15] is the following proposition.

PROPOSITION 2.5. *Matrix maps between FK spaces are continuous.*

The following facts about the relationship between these various duals will be used throughout this thesis. In all of these facts X and Y are FK spaces both containing φ . Once again, the details may be found in [15].

FACT 1. $X^\gamma \subseteq X'$.

FACT 2. *If X is an AD space, $X^\beta = X^\gamma$. If X is an AK space, $X^\beta = X'$.*

FACT 3. *If $X \subseteq Y$, then $Y' \subseteq X'$. If X is an AD space, then $X \subseteq Y \Leftrightarrow Y' \subseteq X'$.*

FACT 4. $X \cap X'^\gamma = \{x \in X : \{P_n x\} \text{ is bounded in } X\}$, that is, the “ AB part” of X .

FACT 5. X has AB if and only if $X' = X^\gamma$.

FACT 6. *Two AD spaces with the same f -dual are equal.*

Note that by Fact 2 this says that two AK spaces with the same β -dual are equal.

The final fact is found in [11].

FACT 7. *If X and Y are BK spaces and Y' is closed in X' , then $Y' = X'$.*

Finally, the notation $Z(\{a^n\})$ will be used to denote the BK space $\{\sum \alpha_n a^n : \alpha \in Z\}$. Here Z is a subspace of ℓ_1 and $\{a^n\}$ is a bounded sequence in some BK space X .

3. BK Products

Before proceeding to the results of this thesis, the reader should be familiar with the notion of the BK product of two BK spaces. BK products are useful in determining the γ dual of a space. The following results are from a seminar given at Lehigh University by A. K. Snyder. More general results concerning FK products may be found in [3].

Let X and Y be BK spaces. Let B denote the set

$$\{xy : \|x\|_X \leq 1, \|y\|_Y \leq 1\}.$$

Define the operator $A : \ell_1(B) \rightarrow \omega$ by $A\lambda = \sum_{i=1}^{\infty} \lambda_i x^i y^i$, where $\lambda \in \ell_1$, $\|x^i\|_X \leq 1$, and $\|y^i\|_Y \leq 1$. Here $\ell_1(B)$ is the ℓ_1 space indexed by the elements of B . In the above sums only countably many λ_i are nonzero. A is certainly a linear map. To show A is continuous consider the maps $\pi_m \circ A$.

$$\begin{aligned} |(\pi_m \circ A)\lambda| &= \left| \sum_{i=1}^{\infty} \lambda_i (x^i y^i)_m \right| \leq \sum_{i=1}^{\infty} |\lambda_i| \|x^i y^i\|_m \\ &\leq \|\pi_m\|_X \|x^i\|_X \|\pi_m\|_Y \|y^i\|_Y \sum_{i=1}^{\infty} |\lambda_i| \end{aligned}$$

In this expression, $\|x^i\|_X \leq 1$, $\|y^i\|_Y \leq 1$, and $\sum_{i=1}^{\infty} |\lambda_i| = \|\lambda\|_{\ell_1(B)}$. Therefore $|(\pi_m \circ A)\lambda| \leq M \|\lambda\|_{\ell_1(B)}$ for $M = \|\pi_m\|_X \|\pi_m\|_Y$, and so $\pi_m \circ A$ is continuous for arbitrary m , hence A is continuous [15, 4.3.2]. Thus the image of this map, $A(\ell_1(B))$, is a BK space with norm

$$\|z\| = \inf \left\{ \sum_{i=1}^{\infty} |\lambda_i| : z = \sum_{i=1}^{\infty} \lambda_i x^i y^i \right\}.$$

Clearly $A(\ell_1(B))$ contains $X \cdot Y$, the set of all sequences xy where $x \in X$ and $y \in Y$. In fact, $A(\ell_1(B))$ is the smallest BK space containing $X \cdot Y$ (note that $X \cdot Y$ need not be a BK space).

PROPOSITION 3.1. $A(\ell_1(B))$ as described above is the smallest BK space containing $X \cdot Y$.

PROOF: The proof will involve the notion of multipliers. Assume that Z is a BK space such that $X \cdot Y \subseteq Z \subseteq A(\ell_1(B))$. It will be shown that $A(\ell_1(B)) \subseteq Z$ yielding $Z = A(\ell_1(B))$. To this end let $u \in A(\ell_1(B))$, say $u = \sum_{i=1}^{\infty} \lambda_i x^i y^i$. It is known that $X \cdot Y \subseteq Z$, so $X \subseteq M(Y, Z)$ and hence $\|xy\|_Z \leq M\|x\|_X \|y\|_Y$ for some positive constant M . Therefore

$$\|u\|_Z = \left\| \sum_{i=1}^{\infty} \lambda_i x^i y^i \right\|_Z \leq M \sum_{i=1}^{\infty} |\lambda_i| \|x^i\|_X \|y^i\|_Y \leq M \|\lambda\|_{\lambda_1(B)} < \infty.$$

Note that $\|x^i\|_X \leq 1$ and $\|y^i\|_Y \leq 1$ here. This tells us that the series converges in Z and therefore $u \in Z$, completing the proof. ■

Henceforth denote $A(\lambda_1(B))$ by $X \hat{\otimes} Y$ and we call $X \hat{\otimes} Y$ the BK product of X and Y . There are three facts about BK products which will prove useful in this thesis.

PROPOSITION 3.2. If X is a BK space containing φ , then $X \hat{\otimes} bv_0$ is an AK space.

Actually, $X \hat{\otimes} Y$ is an AK space if Y is an AK space.

PROOF: Let x and y be elements of X and bv_0 respectively. To show $X \hat{\otimes} bv_0$ is AK , consider $P_n(xy) - xy$. Certainly, $P_n(xy) = xP_n y$, and so

$$\|P_n(xy) - xy\|_{X \hat{\otimes} bv_0} = \|x(P_n y - y)\|_{X \hat{\otimes} bv_0} \leq \|x\|_X \|P_n y - y\|_{bv_0}.$$

Recall that bv_0 is an AK space, yielding $\|P_n y - y\|_{bv_0} \rightarrow 0$, and therefore $\|P_n(xy) - xy\|_{X \hat{\otimes} bv_0} \rightarrow 0$. This says that xy has AK in $X \hat{\otimes} bv_0$. This is true for arbitrary

x and y so $X \cdot Y$ is contained in the “AK part” of $X \widehat{\otimes} bv_0$. It is known [15] that the AK part of a BK space is itself a BK space. Now $X \widehat{\otimes} bv_0$ is the smallest BK space containing $X \cdot Y$, so $X \widehat{\otimes} bv_0$ itself must be the AK part of $X \widehat{\otimes} bv_0$. This is merely the statement that $X \widehat{\otimes} bv_0$ is an AK space. ■

PROPOSITION 3.3. *If X is a BK space, then $X^\gamma = (X \widehat{\otimes} bv_0)^\gamma = (X \widehat{\otimes} bv_0)'$.*

PROOF: Chasing down the definitions,

$$(X \cdot bv_0)^\gamma = M(X \cdot bv_0, bs) = M(X, M(bv_0, bs)) = M(X, bs) = X^\gamma.$$

Note that $M(bv_0, bs) = bv_0^\gamma = bs$. Clearly $X \cdot bv_0 \subseteq X \widehat{\otimes} bv_0$, so $(X \widehat{\otimes} bv_0)^\gamma \subseteq (X \cdot bv_0)^\gamma = X^\gamma$. The proof is completed by showing $X^\gamma \subseteq (X \widehat{\otimes} bv_0)^\gamma$.

Let u be an element of X^γ and let $z = \sum_{i=1}^{\infty} \lambda_i x^i y^i$ be in $X \widehat{\otimes} bv_0$.

$$uz = u \cdot \sum_{i=1}^{\infty} \lambda_i x^i y^i = \sum_{i=1}^{\infty} \lambda_i u(x^i y^i) = \sum_{i=1}^{\infty} \lambda_i [(ux^i) y^i].$$

Now $u \in X^\gamma$, so $\{ux^i\} \in bs$. Therefore uz may be written as

$$\sum_{i=1}^{\infty} \|ux^i\|_{bs} \lambda_i \left(\frac{ux^i}{\|ux^i\|_{bs}} y^i \right),$$

which is an element of $bs \widehat{\otimes} bv_0 = cs \subseteq bs$. Thus $uz \in bs$ for arbitrary $z \in (X \widehat{\otimes} bv_0)^\gamma$.

This is just the statement that $u \in (X \widehat{\otimes} bv_0)^\gamma$. Thus $X^\gamma = (X \widehat{\otimes} bv_0)^\gamma$. The proof of the second equality follows from the preceding proposition and Fact 2. ■

PROPOSITION 3.4. *Let X and Z be BK spaces containing φ . Furthermore let Z be an AK space and $X \subseteq Z$. Then $X^\gamma = Z^\gamma$ if and only if $X \widehat{\otimes} bv_0 = Z$.*

PROOF: Suppose first that $X^\gamma = Z^\gamma$. Then by using the previous proposition $(X \widehat{\otimes} bv_0)^\gamma = Z^\gamma$. By Proposition 3.2, $X \widehat{\otimes} bv_0$ is an AK space. By Fact 2 and Fact

6 we know (two AK spaces with the same γ dual) that the spaces must be equal, that is $X \widehat{\otimes} bv_0 = Z$.

Now suppose that $X \widehat{\otimes} bv_0 = Z$. The preceding proposition yields $X^\gamma = (X \widehat{\otimes} bv_0)^\gamma = Z^\gamma$. ■

4. New Wilansky Properties and Their Relationships

In this section we develop new Wilansky properties. Recall the *BK* space definition of the Wilansky property in [2].

DEFINITION. A *BK* space Z has property (W3) if X a *BK* space, $X \subseteq Z$, and $X^\beta = Z^\beta$ together imply that X is closed in Z .

Following the ideas in [11], define two new Wilansky properties.

DEFINITION. A *BK* space Z has property (W4) if X a *BK* space, $X \subseteq Z$, and $X^\gamma = Z^\gamma$ together imply that X has finite codimension in Z , that is $\dim Z/X < \infty$.

DEFINITION. A *BK* space Z has property (W5) if X a *BK* space with *AD*, $X \subseteq Z$, and $X^\gamma = Z^\gamma$ together imply that $X = Z$.

Note the appearance of the γ dual in these properties, corresponding to Snyder's work. It shall be noted later that the choice of dual is unimportant.

It will be helpful to have an example to help see where these properties may lead. This example investigates a space not having these new Wilansky properties. This failure reveals a copy of ℓ_1 in the space. Recognizing this copy of ℓ_1 requires some results on bases in Banach spaces. This information may be found in [4].

A sequence $\{x_n\}$ in a Banach space X is called a *basis for X* if for each x in X there exists a unique sequence of scalars $\{\alpha_n\}$ such that $x = \lim_n \sum_{k=1}^n \alpha_k x_k$. Note that X is an *AK* space if and only if $\{\delta^n\}$ is a basis for X . A *basic sequence* in a Banach space X is a sequence $\{x_n\}$ that is a basis for its closed linear span $[x_n]$. Also useful is the idea of a block basic sequence. Let $\{x_n\}$ be a basis for a Banach space X , and let $\{p_n\}$ and $\{q_n\}$ be "intertwining" sequences of positive integers (that is, $p_1 < q_1 < p_2 < q_2 < p_3 < q_3 \dots$). Let $y_n = \sum_{i=p_n}^{q_n} a_i x_i$ be a linear combination of the x_i , with a_i nonzero. Then $\{y_n\}$ is called a *block basic*

sequence taken with respect to $\{x_n\}$, or simply a block basic sequence. It will be useful to say that y_n is zero off $[p_n, q_n]$ in this instance, where $[p_n, q_n]$ denotes the integers between and including p_n and q_n . The sequence $\{y_n\}$ is indeed a basic sequence. Two bases $\{x_n\}$ and $\{y_n\}$ for X and Y respectively are *equivalent* if the convergence of $\sum a_n x_n$ is equivalent to that of $\sum a_n y_n$. It is known [4] that $\{x_n\}$ and $\{y_n\}$ are equivalent if and only if there is an isomorphism between X and Y that maps each x_n to y_n . This fact will be of use later in the thesis.

EXAMPLE 4.1.

Consider the space $Z = \ell_2 + bv_0$. This space fails to have the Wilansky property (W4) but *does* have an interesting property concerning the presence of ℓ_1 .

Consider a^k in φ defined by

$$a^k_i = \begin{cases} 1 & \text{for } 2^{2k-1} \leq i \leq 2^{2k} \\ 0 & \text{otherwise.} \end{cases}$$

By using the preceding fact about equivalent bases, it will be shown that $\{a^k\}$ is equivalent to the unit vector basis of ℓ_1 . Note that $\{a^k\}$ is bounded in $Z = \ell_2 + bv_0$.

Consider also u^k defined by

$$u^k_i = \begin{cases} \frac{1}{2^{2k-1}} & \text{for } 2^{2k-1} \leq i < 2^{2k} \\ -\frac{1}{2^{2k-1}} & \text{for } 2^{2k} \leq i < 2^{2k} + 2^{2k-1} \\ 0 & \text{otherwise.} \end{cases}$$

Define the map $P : \ell_2 + bv_0 \rightarrow \ell_2 + bv_0$ by $Pz = \sum_n u^n(z)a^n$. For P to be defined, $\{u^n(z)\}$ must be in ℓ_1 for all z in Z . To show this let $\alpha \in \ell_\infty$. Then

$$\sup_n \left\| \sum_{k=1}^n \alpha_k u^k \right\|_{Z^\gamma} \leq \sup_n \left\{ \left\| \sum_{k=1}^n \alpha_k u^k \right\|_2 + \left\| \sum_{k=1}^n \alpha_k u^k \right\|_{b_*} \right\}$$

yielding that $\{\sum_{k=1}^n \alpha_k u^k\}$ is a bounded sequence in $(\ell_2 + bv_0)^\gamma$ and therefore $\sum_{k=1}^\infty \alpha_k u^k$ converges *weak** in $(\ell_2 + bv_0)^\gamma$, using Proposition 2.4 to identify this

in $(\ell_2 + bv_0)'$. Then for any z in $\ell_2 + bv_0$, the series $\sum_{k=1}^{\infty} \alpha_k u^k(z)$ converges, and hence $u^k(z) \in \ell_{\infty}^{\beta} = \ell_1$ for all z in $\ell_2 + bv_0$.

The map P is bounded by Proposition 2.5 because it may be written as a matrix map. Also, $P(a^k) = \sum_n u^n(a^k) a^n = a^k$ so $R(P) = \ell_1(\{a^n\})$. The map P is also a projection because $P(P(z)) = P(z)$ using $u^n(a^k) = \delta_k^n$. The range $R(P)$ is closed in Z and therefore, since $[a^n]_Z \subseteq \ell_1(\{a^n\})$, $[a^n]_Z = \ell_1(\{a^n\})$. Thus there exists an obvious isomorphism from ℓ_1 to $\ell_1(\{a^n\})$ and therefore $\{a^n\}$ is equivalent to the unit vector basis of ℓ_1 .

This example is completed by showing that $\ell_2 + bv_0$ indeed fails to have property (W4). Let $Y = \{x \in \ell_1 : x_{2n} = x_{2n-1} \text{ for all } n\}$. Then Y is closed in ℓ_1 . Recall the notation $Z(\{a^n\})$ denoting $\{\sum \alpha_n a^n : \alpha \in Z\}$ for a bounded sequence $\{a^n\}$. Let $X = Y(\{a^n\}) \oplus W$. Note here that $\ell_2 + bv_0 = \ell_1(\{a^n\}) \oplus W$ because of the projection defined above. Note also that $\dim(\ell_2 + bv_0)/X = \infty$, because $\dim \ell_1/Y = \infty$, so all that is required to show that $\ell_2 + bv_0$ fails to have property (W4) is to show that $X^\gamma = (\ell_2 + bv_0)^\gamma$.

This may be accomplished by showing instead that $X \widehat{\otimes} bv_0 = \ell_2 + bv_0$ and using Proposition 3.4. Note that $X \subseteq \ell_2 + bv_0$, so $X \widehat{\otimes} bv_0 \subseteq \ell_2 + bv_0$ because $\ell_2 + bv_0$ is an AK space. To show the opposite inclusion it suffices to show that $\{a^n\}$ is bounded in $X \widehat{\otimes} bv_0$. Consider $\{a^{2^{n-1}} + a^{2^n}\}$. For all n , this is bounded in X because $a^{2^{n-1}} + a^{2^n} = \sum_{k=1}^{\infty} (\delta^{2^{n-1}} + \delta^{2^n}) a^k$ and $\delta^{2^{n-1}} + \delta^{2^n}$ is bounded in Y . Notice that $X \widehat{\otimes} bv_0$ is an AB space, so sections are bounded and therefore $P_{4^{n-2}}(a^{2^{n-1}} + a^{2^n}) = a^{2^{n-1}}$ is bounded (recall a^{2^n} is zero off $[2^{4^{n-1}}, 2^{4^n}]$). Using $\|a^{2^n}\| \leq \|a^{2^{n-1}} + a^{2^n}\| + \|a^{2^{n-1}}\|$ it follows that $\{a^n\}$ is bounded in $X \widehat{\otimes} bv_0$ for all n and our example is complete. ■

The techniques used in this example will be employed again in the general

theorems later in this thesis. The failure of a BK, AK space to have property $(W4)$ places a “nice” copy of ℓ_1 in that space. Not surprisingly, ℓ_1 itself fails to have property $(W4)$ as was noted by Bennett [2], and such copies of ℓ_1 certainly exist in ℓ_1 .

Now recall some results concerning the embedding property $Y < Z$ for BK spaces Y and Z , as found in [11] and [12]. These are required to prove equivalences between these new Wilansky properties.

DEFINITION. Let Y and Z be BK spaces with $\varphi \subseteq Y \subseteq Z$. $Y < Z$ means that $X = Z$ whenever X is a $BK, \varphi \subseteq X$, is such that $X + Y = Z$.

PROPOSITION 4.2 [LEMMA 3.1 IN [12]]. Let Y and Z be BK spaces with $Y \subseteq Z$ and φ dense in Y (Y an AD space). Then the following are equivalent:

- (i) $X = Z$ whenever X is a BK space containing φ such that $X + Y = Z$
- (ii) $\dim Z/X < \infty$ whenever X is a BK space such that $X + Y = Z$.

Note that this says (ii) may be taken in place of (i) in the definition of the embedding property $Y < Z$. Also useful is the following:

PROPOSITION 4.3 [LEMMA 6(I)[11]]. Let Y and Z be BK spaces with $Y \subseteq Z$ and φ dense in Y . If Y is compactly embedded in Z , then $Y < Z$.

This embedding property yields a relatively easy proof of an equivalence between property $(W4)$ and Bennett’s property $(W3)$. Some lemmas are required; the second is mentioned by Bennett in [2].

LEMMA. Let X be a sequence space and let u be an element of X^γ . Furthermore let α be an element of bv_0 . Then αu is in X^β .

PROOF: Since $u \in X^\gamma, ux \in bs$ for all $x \in X$. Since $\alpha \in bv_0 = bs^\gamma, \alpha v \in cs$ for all $v \in cs$.

Combining these, ux is in bs for all x in X , so $\alpha(ux)$ is in cs for all x in X . This says that αu is in X^β . ■

LEMMA. For sequence spaces X and Y , $X^\beta \subseteq Y^\beta$ implies $X^\gamma \subseteq Y^\gamma$, hence $X^\beta = Y^\beta$ implies $X^\gamma = Y^\gamma$.

PROOF: Let u be an element of X^γ . Then by the preceding lemma, αu is in $X^\beta \subseteq Y^\beta$ for all α in bv_0 . Because αu is in Y^β , $\alpha u y$ is in cs for all y in Y and all α in bv_0 . Looking at this another way, $u y$ is in $(bv_0)^\beta$ for all y in Y . It is known that $(bv_0)^\beta = bs$ so $u y$ is in bs for all y in Y , which says that u is in Y^γ . ■

It should be noted here that the converse of the previous lemma is true as well. This was proven by Noll in [6, Proposition 2]. Note also that this says that the Wilansky property may be stated in terms of either γ duals or β duals. Applying these lemmas yields:

PROPOSITION 4.4. For Z a BK, AK space, Z has property (W3) if and only if Z has property (W4).

PROOF: Suppose Z has property (W4). Let X be a BK space with $X \subseteq Z$ and $X^\beta = Z^\beta$. By the preceding lemma $X^\gamma = Z^\gamma$ and because Z has property (W4) we have $\dim Z/X < \infty$. Thus X is closed in Z and therefore Z has property (W3).

Now suppose Z has property (W3). Let X be a BK space with $X \subseteq Z$ and $X^\gamma = Z^\gamma$. Then $X^\beta \subseteq X^\gamma = Z^\gamma = Z^\beta \subseteq X^\beta$, where the second equality follows from Fact 2 because Z is an AK space. Thus $X^\beta = Z^\beta$. Let $\mathcal{X} = X + Z(\{\frac{\delta^n}{n}\})$. Then $\varphi \subseteq \mathcal{X} \subseteq Z$, \mathcal{X} is a BK space, and $\mathcal{X}^\beta = Z^\beta$. Furthermore, \mathcal{X} is dense in Z since $\varphi \subseteq \mathcal{X} \subseteq Z$ and Z is an AK space. The space Z has property (W3), so \mathcal{X} is closed in Z and hence $\mathcal{X} = X + Z(\{\frac{\delta^n}{n}\}) = Z$ because \mathcal{X} is dense in Z . The

space $Z(\{\frac{\epsilon^n}{n}\})$ is compactly embedded in Z , so $Z(\{\frac{\epsilon^n}{n}\}) < Z$ by Proposition 4.3. By the definition of the embedding $Z(\{\frac{\epsilon^n}{n}\}) < Z$, since $X + Z(\{\frac{\epsilon^n}{n}\}) = Z$, $\dim Z/X < \infty$, by Proposition 4.2. Thus, Z has property (W4). ■

There is also an equivalence between property (W4) and property (W5). This equivalence provides a link between the ideas of Snyder and the results of Bennett, and the results of Noll and Stadler. This link is discussed in the next section. The proof of the equivalence will require substantially more work and some technical lemmas.

LEMMA. *Let Z be a BK, AK space and let $z \in Z$. If $\{u^n\}$ is bounded in Z^γ and $u^n \rightarrow 0$ in ω (coordinatewise), then $\|u^n z\|_{b_s} \rightarrow 0$.*

PROOF: Begin by assuming $\|u^n\|_{Z^\gamma} \leq M$ for some positive constant M . Let $\epsilon > 0$ be given. Because Z is an AK space, fix a positive integer k so that $\|z - P_k z\|_Z < \frac{\epsilon}{2M}$. Because $u^m \rightarrow 0$ in ω and k is fixed, choose a positive integer M so that $\|u^m(P_k z)\|_{b_s} < \frac{\epsilon}{2}$, for $m > M$. Thus,

$$\begin{aligned} \|u^m z\|_{b_s} &= \|u^m(z - P_k z) + u^m(P_k z)\|_{b_s} \\ &\leq \|u^m(z - P_k z)\|_{b_s} + \|u^m(P_k z)\|_{b_s} \\ &\leq \|u^m\|_{Z^\gamma} \|z - P_k z\|_Z + \|u^m(P_k z)\|_{b_s} \\ &< M \frac{\epsilon}{2M} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Therefore $\|u^n z\|_{b_s} \rightarrow 0$. ■

Recall that $\ell_1(\{y^n\})$ denotes the BK space $\{\sum \lambda_n y^n : \lambda \in \ell_1\}$ for some bounded sequence $\{y^n\}$ in a BK space Y .

LEMMA. $[\ell_1(\{y^n\})]^\gamma = \{u \in \omega : \{u y^n\} \text{ is bounded in } b_s\}$

PROOF: Let $\{uy^n\}$ be bounded in bs and let z be an element of $\ell_1(\{y^n\})$, say, $z = \sum_{n=1}^{\infty} \lambda_n y^n$. Then

$$\begin{aligned} \|uz\|_{bs} &= \left\| u \sum_{n=1}^{\infty} \lambda_n y^n \right\|_{bs} \\ &= \left\| \sum_{n=1}^{\infty} \lambda_n (uy^n) \right\|_{bs} \leq \sup_n \|uy^n\|_{bs} \|\lambda\|_1 < \infty. \end{aligned}$$

This shows that u is an element of $[\ell_1(\{y^n\})]^\gamma$. Conversely, let $u \in [\ell_1(\{y^n\})]^\gamma$. This implies that $u : \ell_1(\{y^n\}) \rightarrow bs$ defined by $u(y) = uy$ is a bounded operator. The sequence $\{y^n\}$ is bounded in $\ell_1(\{y^n\})$ and so $\{uy^n\}$ is bounded in bs , by the boundedness of the operator. ■

LEMMA. Let Z be a BK, AK space, and let X be a BK space containing φ compactly embedded in Z . Then $Z^\gamma \rightarrow X^\gamma$ is a compact map.

Note that X containing φ is required to make X^γ a BK space.

PROOF: Let $\{u^n\}$ be bounded in Z^γ , say $\|u^n\|_{Z^\gamma} \leq 1$ for all n . By Proposition 2.4, $\{u^n\}$ may be considered a bounded sequence in Z' , so we may assume $u^n \rightarrow u$ in ω for some $u \in Z'$ (by going to a subsequence if necessary). Thus $u^n - u \rightarrow 0$ in ω and $\{u^n - u\}$ is bounded in Z^γ . The proof is completed by showing $\{u^n - u\}$ has a subsequence that converges to 0 in X^γ .

Recall that the norm on X^γ is given by

$$\|u\|_{X^\gamma} = \sup\{\|ux\|_{bs} : \|x\|_X \leq 1\}.$$

With this in mind, for each n , choose x^n with $\|x^n\| \leq 1$ and $\frac{1}{2}\|u^n - u\|_{X^\gamma} \leq \|(u^n - u)x^n\|_{bs}$. The sequence $\{x^n\}$ is bounded in X so, by going to a subsequence if necessary, assume $x^n \rightarrow z$ in Z . This follows because X is compactly embedded

in Z . Note that

$$\begin{aligned} \|(\mathbf{u}^n - \mathbf{u})x^n\|_{b_s} &= \|(\mathbf{u}^n - \mathbf{u})x^n - (\mathbf{u}^n - \mathbf{u})z + (\mathbf{u}^n - \mathbf{u})z\|_{b_s} \\ &\leq \|(\mathbf{u}^n - \mathbf{u})(x^n - z)\|_{b_s} + \|(\mathbf{u}^n - \mathbf{u})z\|_{b_s} \\ &\leq \|\mathbf{u}^n - \mathbf{u}\|_{Z^\gamma} \|x^n - z\|_Z + \|(\mathbf{u}^n - \mathbf{u})z\|_{b_s}. \end{aligned}$$

Now, by the first lemma, $\|(\mathbf{u}^n - \mathbf{u})z\|_{b_s} \rightarrow 0$, and $\|\mathbf{u}^n - \mathbf{u}\|_{Z^\gamma} \|x^n - z\|_Z \rightarrow 0$ because $x^n \rightarrow z$ in Z , so $\|(\mathbf{u}^n - \mathbf{u})x^n\|_{b_s} \rightarrow 0$. Therefore, $\|\mathbf{u}^n - \mathbf{u}\|_{X^\gamma} \rightarrow 0$ by the first inequality above and the proof is completed. ■

Some results concerning the Meyer-König Wilansky Zeller property of embeddings, as discussed in [11], are now required. These results facilitate the proof that property (W4) implies property (W5).

DEFINITION. *If U is an FK space contained in a sequence space V , the embedding of U into V has the Meyer-König Wilansky Zeller property, abbreviated $U \rightarrow V$ has MKWZ, if $X \cap V \subset U$ implies $X \cap U$ is closed in X for each FK space X .*

Alternately, it is known $U \rightarrow V$ has MKWZ if and only if for each FK space X , the equality $X \cap V = U$ implies that U is closed in X . This will be the characterization used here.

LEMMA. *Let U and V be BK spaces. If $U \rightarrow V$ is compact, then $U \rightarrow V$ has MKWZ.*

PROOF: Let X be an FK space. Suppose $X \cap V = U$. By the above characterization, showing U is closed in X completes the proof. To this end consider a sequence $\{\mathbf{u}^n\}$ in U such that $\|\mathbf{u}^n\|_U \leq 1$ for all n and $\mathbf{u}^n \rightarrow 0$ in X . Since $U \rightarrow V$ is compact, $\mathbf{u}^n \rightarrow v$ for some $v \in V$, by going to a subsequence if necessary.

This, however, ensures that u^n converges to 0 in $X \cap V$. Thus U is closed in X by Proposition 2.1, because $X \cap V = U$. ■

It should now be pointed out that for a BK, AK space Z and $y^n \rightarrow 0$ in Z that $Z^\gamma \rightarrow [\ell_1(\{y^n\}) + Z(\{\frac{\epsilon^n}{n}\})]^\gamma$ has MKWZ. This is a consequence of the previous lemmas because $y^n \rightarrow 0$ in Z yields that $\ell_1(\{y^n\})$ is compactly embedded in Z and therefore $\ell_1(\{y^n\}) + Z(\{\frac{\epsilon^n}{n}\})$ is compactly embedded in Z and contains φ .

With these results the equivalence of the Wilansky properties (W4) and (W5) under certain hypotheses may be deduced.

THEOREM 4.5. *Let Z be a BK, AK space. Then Z has property (W4) if and only if Z has property (W5).*

PROOF: Half of this proof is easy. Let Z have property (W4), and let X be a BK, AD space with $X \subseteq Z$ and $X^\gamma = Z^\gamma$. By (W4), $\dim Z/X < \infty$. However, X is an AD space, so $X = Z$.

The proof of the converse is quite difficult. Let X be a BK space with $X \subseteq Z$ and $X^\gamma = Z^\gamma$. A few simplifying assumptions may be made. In fact, these assumptions, while tedious to prove, yield the result instantly.

- (1) X contains φ . If it does not, let $\mathcal{X} = X + Z(\{\frac{\epsilon^n}{n}\})$. Then \mathcal{X} is a BK space with $\varphi \subseteq \mathcal{X} \subseteq Z$ and $\mathcal{X}^\gamma = Z^\gamma$. This is because $X \subseteq \mathcal{X} \subseteq Z$ and $X^\gamma = Z^\gamma$. If we can prove the result for \mathcal{X} , namely that $\dim Z/\mathcal{X} < \infty$, then $\dim Z/X < \infty$ as well. This follows because $Z(\{\frac{\epsilon^n}{n}\}) < Z$, and since $\dim Z/\mathcal{X} < \infty$, $Z = X + Z(\{\frac{\epsilon^n}{n}\}) + F$, where F is a finite dimensional space. These two facts yield $\dim Z/X + F < \infty$, and therefore $\dim Z/X < \infty$.
- (2) X is separable. This requires constructing a new space Y and showing that if the result holds for Y then it also holds for X . The construction of Y is as follows. The fact that Z is an AK space (and hence separable) yields a

sequence $\{z^n\}$ that is dense in the unit ball of Z . The fact that $X^\gamma = Z^\gamma$ yields $X\widehat{\otimes}bv_0 = Z$ by Proposition 3.4. Thus $z^n = \sum_k \lambda_k^n x^{nk} y^{nk}$, where λ^n is bounded in ℓ_1 , $\|x^{nk}\|_X \leq 1$, and $\|y^{nk}\|_Y \leq 1$ for all n, k (recall the definition of BK products in Section 3). Let Y be $\ell_1(\{x^{nk}\})$.

It is clear that $Y \subseteq X$, Y is BK , and Y is separable. All that is lacking is $Y^\gamma = Z^\gamma$. This is shown by proving $Y\widehat{\otimes}bv_0 = Z$ and again citing Proposition 3.4. Clearly $Y\widehat{\otimes}bv_0 \subseteq X\widehat{\otimes}bv_0 = Z$, so to show $Y\widehat{\otimes}bv_0 = Z$ it suffices to show that $\{z^n\}$ is bounded in $Y\widehat{\otimes}bv_0$. The norm of z^n in $Y\widehat{\otimes}bv_0$ is given by

$$\|z^n\| = \inf\left\{\sum_{k=1}^{\infty} |\lambda_k| : z^n = \sum_{k=1}^{\infty} \lambda_k x^k y^k, \|x^k\|_Y \leq 1, \|y^k\|_{bv_0} \leq 1\right\}.$$

Certainly $\|x^{nk}\|_Y \leq 1$ because $Y = \ell_1(\{x^{nk}\})$. Because of this fact, $z^n = \sum_k \lambda_k^n x^{nk} y^{nk}$ is an element of $Y\widehat{\otimes}bv_0$ as well as an element of $X\widehat{\otimes}bv_0$. The supremum of the norms of z^n in $Y\widehat{\otimes}bv_0$ is therefore less than or equal to $\sup_n \sum_k |\lambda_k^n|$. By virtue of the presence of z^n in the unit ball of $Z = X\widehat{\otimes}bv_0$, this last supremum is bounded. Therefore $\{z^n\}$ is bounded in $Y\widehat{\otimes}bv_0$, yielding $Y\widehat{\otimes}bv_0 = Z$ and, by Proposition 3.4, $Y^\gamma = Z^\gamma$.

The construction of a separable BK space Y with $Y \subseteq X \subseteq Z$ and $Y^\gamma = Z^\gamma$ is complete. Now if the result holds for Y , that is, $\dim Z/Y < \infty$, then $\dim Z/X < \infty$ because $Y \subseteq X$.

- (3) X is an AD space. This will allow the use of property (W5). Because X is separable, let $\{x^n\}$ be dense in the unit ball of X . Because $X \subseteq Z$ and Z is an AK space, we may choose an increasing sequence of positive integers $\{r_n\}$ such that $\|x^n - P_{r_n} x^n\|_Z \rightarrow 0$. Let $y^n = x^n - P_{r_n} x^n$. Consider the following spaces: $\ell_1(\{x^n\})$, $\ell_1(\{P_{r_n} x^n\})$, and $\ell_1(\{y^n\})$. Note that

$\ell_1(\{x^n\}) \subseteq \ell_1(\{P_{r_n} x^n\}) + \ell_1(\{y^n\})$. Note also that $\ell_1(\{x^n\}) = X$ and $\ell_1(\{y^n\})$ is compactly embedded in Z .

Consider the space $W = \ell_1(\{P_{r_n} x^n\}) + Z(\{\frac{\delta^n}{n}\})$. This is a *BK, AD* space and is contained in Z . It is an *AD* space because $\{P_{r_n} x^n\} \subseteq \varphi$, and therefore $\ell_1(\{P_{r_n} x^n\}) \cap \varphi$ is dense in $\ell_1(\{P_{r_n} x^n\})$, and because φ is dense in $Z(\{\frac{\delta^n}{n}\})$.

Recall that $X = \ell_1(\{x^n\}) \subseteq \ell_1(\{P_{r_n} x^n\}) + \ell_1(\{y^n\})$. Thus

$$X \subseteq \ell_1(\{P_{r_n} x^n\}) + Z(\{\frac{\delta^n}{n}\}) + \ell_1(\{y^n\}) \subseteq Z.$$

By taking γ -duals and noting that $X^\gamma = Z^\gamma$ we see that

$$[\ell_1(\{P_{r_n} x^n\}) + Z(\{\frac{\delta^n}{n}\}) + \ell_1(\{y^n\})]^\gamma = Z^\gamma.$$

This may be written $W^\gamma \cap [\ell_1(\{y^n\}) + Z(\{\frac{\delta^n}{n}\})]^\gamma = Z^\gamma$. By virtue of the lemma preceding this theorem, $Z^\gamma \rightarrow [\ell_1(\{y^n\}) + Z(\{\frac{\delta^n}{n}\})]^\gamma$ has MKWZ and therefore Z^γ is closed in W^γ . This fact, however, implies that $Z^\gamma = W^\gamma$, by the following argument: $W^\gamma = (W \widehat{\otimes} bv_0)^\gamma = (W \widehat{\otimes} bv_0)^\prime$, by Proposition 3.3. Thus $Z^\gamma = Z^\prime$ is closed in $(W \widehat{\otimes} bv_0)^\prime$ and therefore by Fact 7, these f -duals are equal and hence the stated γ -duals are equal.

The construction of a *BK, AD* space W with $W^\gamma = Z^\gamma$ is now complete. If the result holds for W , namely that $\dim Z/W < \infty$, the result holds for X as well. This follows by yet another application of the embedding property $Y < Z$. Because $\dim Z/W < \infty$, there is a finite dimensional space F such that

$$\begin{aligned} Z &= W + F = \ell_1(\{P_{r_n} x^n\}) + Z(\{\frac{\delta^n}{n}\}) + F \\ &\subseteq \ell_1(\{x^n\}) + \ell_1(\{y^n\}) + Z(\{\frac{\delta^n}{n}\}) + F \subseteq Z. \end{aligned}$$

Recall that $\ell_1(\{y^n\}) + Z(\{\frac{\epsilon^n}{n}\})$ is compactly embedded in Z , yielding $\{\ell_1(\{y^n\}) + Z(\{\frac{\epsilon^n}{n}\}) < Z$ by Proposition 4.3. But then $\dim Z / \{\ell_1(\{x^n\}) + F\} < \infty$. Recall also that $\ell_1(\{x^n\}) = X$, yielding $\dim Z/X < \infty$.

The proof of the converse of the theorem, namely that Z has property (W4) if it has property (W5), follows easily due to (1), (2), and (3).

Let Z be a BK, AK space with property (W5). Let X be a BK space with $X \subseteq Z$ and $X^\gamma = Z^\gamma$. By (1), (2), and (3) it may be assumed that X is also an AD space. Using property (W5), $X = Z$. Now certainly $\dim Z/X < \infty$. ■

Up to this point in the thesis, it is known that for a BK, AK space, all five stated Wilansky properties are equivalent. Returning to the beginning of this investigation, the result of Snyder in [11] given in the Introduction may be restated. Recall that a BK, AK space Z is called a *minimal AK space* if there exists a BK space X with AD such that $X \subset Z, X \neq Z$, and Z is the smallest AK space containing X .

PROPOSITION 4.6. *Let Z be a BK, AK space. The following conditions (i) through (iii) are equivalent and each is implied by (iv).*

- (i) Z is not a minimal AK space.
- (ii) If X is a BK space with $AD, X \subset Z$, and $X \neq Z$, then $X \hat{\otimes} bv \neq Z$.
- (iii) Z has property (W5).
- (iv) Z'_0 has finite codimension in Z^γ .

It may be worth noting part of the proof of this result. A BK, AD space X with $\dim X^\gamma/X'_0 < \infty$ actually has AB and hence is an AK space. [Corollary 2,11]. Therefore if X satisfies these hypotheses and also satisfies $X^\gamma = Z^\gamma$ for a BK, AK space Z , then, since X must also be an AK space, $X = Z$ by Fact

6. This is a quite straightforward proof. The hypothesis $\dim X^\gamma / X_0^\gamma$ strengthens the conditions on the subspace X , facilitating the proof that Z has the Wilansky property.

This result along with Theorem 4.4, Theorem 4.5, and Proposition 4.6 yields a new proof of a result of Noll and Stadler in [8].

COROLLARY. (Noll, Stadler) Let Z be a BK, AK space. If the AK part of Z' has finite codimension in Z' , then Z has the Wilansky property (W) .

PROOF: All that is needed is to note that because Z is an AK space, Z' may be identified with Z^f as in Proposition 2.4. By Fact 5, $Z^f = Z^\gamma$, so the AK part of Z' is nothing but Z_0^γ . Combine this with (iv) implies (ii) in the above proposition, and the fact that $(W5)$ implies (W) for BK, AK spaces, to complete the proof. ■

5. New Wilansky Type Properties and Copies of ℓ_1

This section considers further implications of the Wilansky property. Along the way, more Wilansky type properties are introduced, and relationships between them are given. One of these properties was encountered in the example in Section 4. It is worthwhile to note that in all of the properties in this section, the subspaces considered are AD spaces. This is a consequence of the assumptions made in the proof of Theorem 4.5.

In the case where Z is a BK, AK space, the Wilansky property ($W5$) implies other properties of Z which are elaborated upon here.

PROPOSITION 5.1. *Let Z be a BK, AK space with property ($W5$). Then Z has the property that if X is a BK space with AD and contained in Z , and if Z_0^γ is closed in X^j , then $X = Z$.*

PROOF: This proposition clearly follows from the fact that Z_0^γ closed in X^j implies that $X^\gamma = Z^\gamma$. This fact is shown, using the properties of the various dual spaces set forth in the preliminaries section and also using properties found in [15]. Note that Z_0^γ closed in X^j implies that $(Z_0^\gamma)^j = X^{jj}$. But $(Z_0^\gamma)^j = Z^{\gamma j}$ so $Z^{\gamma j} = X^{jj}$. Now Z^γ , having a monotone norm, has AB so $Z^{\gamma\gamma} = Z^{\gamma j}$ by Fact 5. Hence $Z^{\gamma\gamma} = X^{jj}$ and taking γ -duals gives us $Z^\gamma = Z^{\gamma\gamma\gamma} = X^{jj\gamma}$. Since $X \subseteq Z, Z^\gamma \subseteq X^\gamma \subseteq X^j$, so $Z^\gamma \cap X^j = Z^\gamma$ and thus $Z^\gamma = Z^\gamma \cap X^j = X^j \cap X^{jj\gamma}$. Now by Fact 4, Z^γ is the “ AB part” of X^j . However, the “ AB part” of X^j is, in fact, X^γ . Thus $X^\gamma = Z^\gamma$. It remains to verify that the “ AB part” of X^j is X^γ .

First, the space X^γ , having monotone norm, has AB . It follows from $X^\gamma \subseteq X^j$ and Proposition 2.1 that X^γ is contained in $X^j \cap X^{jj\gamma}$, the “ AB part” of X^j .

Conversely, because X is an AD space, $X \subseteq X^{jj}$ [15, 7.2.15]. Therefore $X^{jj\gamma} \subseteq$

X^γ and $X' \cap X'^{\gamma\gamma} \subseteq X' \cap X^\gamma = X^\gamma$. Thus the "AB part" of X' is contained in X^γ and the proof is completed. ■

For reference, make the following definition: Z has property (W6) if Z has the property stated in the above proposition. Thus for BK, AK spaces property (W5) implies property (W6). Property (W6) in turn implies another property. This is precisely the property encountered in the example in Section 4. Note that the techniques used in the proof will mirror those of the example. The facts concerning bases in Banach spaces given earlier are used again here. Also required is a known renorming lemma.

LEMMA. If Z is a Banach space with $Z = X \oplus Y$, one may define a new norm $\|f\|$ on Z' by $\|f\| = \|f|_X\|_{X'} + \|f|_Y\|_{Y'}$. This new norm is equivalent to the standard norm on Z' .

Note that $f|_X$ denotes the functional f restricted to X . An additional lemma is required.

LEMMA. If Y is a BK space contained in ℓ_1 , $f \in Y'$, and $\{z^n\}$ is a bounded sequence in a BK space Z , then there exists an $\hat{f} \in Y(\{z^n\})'$ such that $\|f\|_{Y'} = \|\hat{f}\|_{Y(\{z^n\})'}$ and $f(y) = \hat{f}(Ty)$ where $y \in Y$ and T is the obvious map $T : Y \rightarrow Y(\{z^n\})$.

The proof follows because T is a quotient map. The details may be found in [14] or any other such text.

THEOREM 5.2. Let Z be a BK, AK space with property (W6). Then there does not exist a pair of block basic sequences $\{z^n\}$ in Z and $\{u^n\}$ in Z^γ with the following properties: $u^n(z^k) = \delta_k^n$, u^n is zero off the same interval $[p_n, q_n]$ as z^n , $\{z^n\}$ complemented in Z by a bounded linear projection $P(z) = \sum_n u^n(z)z^n$, and

$\{z^n\}$ equivalent to the unit vector basis of ℓ_1 .

Note that ℓ_1 does not have the Wilansky property, as was shown in [2]. The space ℓ_1 contains such a block basic sequence, namely $\{\delta^n\}$.

PROOF: Let $\{z^n\}$ and $\{u^n\}$ be block basic sequences as described in the statement of the proposition. Suppose z^n and u^n are both zero off $[p_n, q_n]$. There exists a *BK, AD* space $Y \subseteq \ell_1$ with infinite codimension in ℓ_1 such that c_0 is closed in Y' [see Appendix]. This space Y will be used to construct a *BK, AD* space W with $W \subseteq Z$, Z'_0 closed in W' , but $W \neq Z$. This will show that Z does not have property (W6).

Define the space W by $W = \{z \in Z : Pz = \sum_n y_n z^n \text{ for some } y \in Y\}$. Note that $P(W) = \{\sum_n y_n z^n : y \in Y\}$. The space W is a *BK* space and also $W = P(W) \oplus N(P)$ because $N(P|_W) = N(P) \cap W = N(P)$. Clearly $W \subseteq Z$ and $W \neq Z$.

To show that W is an *AD* space let $z \in W$ be such that $z = x + \sum_n y_n z^n$ with $x \in N(P)$ and $y \in Y$. Because Y itself is an *AD* space, choose $\{s^m\}$ in φ such that $s^m \rightarrow y$ in Y . Thus $\sum_n s_n^m z^n$ is in $P(W) \cap \varphi$ and $\sum_n s_n^m z^n \rightarrow \sum_n y_n z^n$ in $P(W)$. Recall that Z is an *AK* space so $P_{q_n} x \rightarrow x$ in Z (hence in $N(P)$). Note that $P_{q_n} x$ is in $N(P) \cap \varphi$. Combining these two results yields $P_{q_n} x + \sum_n s_n^m z^n \rightarrow x + \sum_n y_n z^n$ in W and therefore W is an *AD* space.

The fact that W has infinite codimension in Z follows because Y has infinite codimension in ℓ_1 . It remains to show that Z'_0 is closed in W' . To show Z'_0 is closed in W' it suffices to show that the norms on these spaces are equivalent by Proposition 2.1.

Let u be an element of φ with $u_n = f(\delta^n)$, where $f \in Z'$. Recall that by hypothesis we have a projection P on Z so that $Z = P(Z) \oplus N(P)$. Using the lemmas

preceding this proposition,

$$\begin{aligned}
\|u\|_{Z^\gamma} &\sim \|u\|_{Z^f} = \|f\|_{Z'} \\
&\sim \|f\|_{N(P)'} + \|f\|_{P(Z)'} \\
&\sim \|f\|_{N(P)'} + \|f\|_{\ell_1(\{z^n\})'} \\
&\sim \|f\|_{N(P)'} + \|\hat{f}\|_\infty \\
&\sim \|f\|_{N(P)'} + \|v\|_\infty
\end{aligned}$$

The first line is due to the fact that Z is an AK space. The second line is by the first lemma, the fourth line is by the second lemma, and the fifth line is the fact that the dual of ℓ_1 may be identified with ℓ_∞ . Note also that v is actually in c_0 because $u \in \varphi$ implies $f(z^n) = 0$ for large n . Continue the equivalences as follows.

$$\begin{aligned}
&\sim \|f\|_{N(P)'} + \|v\|_\infty \\
&\sim \|f\|_{N(P)'} + \|v\|_{Y^f} \\
&\sim \|f\|_{N(P)'} + \|\hat{f}|_{Y'}\|_{Y'} \\
&\sim \|f\|_{N(P)'} + \|f\|_{Y(\{z^n\})'} \\
&\sim \|f\|_{N(P)'} + \|f\|_{P(W)'} \sim \|f\|_{W'} = \|u\|_{W^f}
\end{aligned}$$

Here the second line is due to the fact that c_0 is closed in Y^f , and the other equivalences follow exactly as the first set of equivalences.

This list of equivalences yields that Z_0^γ is closed in W^f , again using Proposition 2.1. ■

Once again, for reference, make the following definition: A BK, AK space Z has *property (W7)* if Z contains no block basic sequence $\{z^n\}$, equivalent to the unit vector basis of ℓ_1 , and with $\{z^n\}$ complemented in Z by a bounded linear

projection P defined by $P(z) = \sum_n u^n(z)z^n$. Here $\{u^n\}$ is a bounded sequence in Z^γ with $u^n(z^k) = \delta_k^n$ and also u^n is zero off the same interval $[p_n, q_n]$ that z^n is. This gives an idea of the structure of a space with the Wilansky property.

So for a BK, AK space, property (W6) implies property (W7). This is the content of the preceding theorem. In this situation property (W7) in turn implies another property. A BK space Z containing φ has property (W8) if it has the following property: If X is a BK, AD space contained in Z and $X^\gamma = Z^\gamma$, then Z_0^γ is closed in X' .

THEOREM 5.3. *Let Z be a BK, AK space with property (W7). Then Z also has property (W8).*

PROOF: Prove the contrapositive; so suppose there exists a BK, AD space X with $X \subseteq Z, X^\gamma = Z^\gamma$, but Z_0^γ not closed in X' . Then, using Proposition 2.1, there exists an increasing sequence of positive integers $\{r_n\}$ and a sequence $\{a^n\} \subseteq \varphi$ with $\|a^n\|_{Z^\gamma} = 1, \|a^n\|_{X'} < \frac{1}{2^n}$, and $a^n_k = 0$ for $k \notin [r_n, r_{n+1})$.

Because $\|a^n\|_{Z^\gamma} = 1$, for each n , and given $\epsilon > 0$, there exists $z^n \in Z$ with $\{z^n\}$ bounded in $Z, z^n_k = 0$ for $k \notin [r_n, r_{n+1})$, and $a^n(z^n) > 1 - \epsilon$. Thus $\{z^n\}$ is a block sequence. By using the sequence $\{\frac{z^n}{a^n(z^n)}\}$, assume further that $a^n(z^n) = 1$. No longer does $\|z^n\|_Z = 1$ but the $\{z^n\}$ are still bounded.

In order to find the necessary projection, show that for any sequence w in $\ell_\infty, \sum_n w_n a^n$ is in $X^\gamma = Z^\gamma$. This will yield an isomorphism between ℓ_1 and $\ell_1(\{z^n\})$. Let $w \in \ell_\infty$. Then $\sum_n w_n a^n \in X'$ because $\|a^n\|_{X'} < \frac{1}{2^n}$. We know that X^γ is the AB part of X' , so to show $\sum_n w_n a^n \in X^\gamma$ it suffices to show that $\sum_n w_n a^n$ has AB in X' . To this end, for any positive integer m with $r_{n-1} \leq m < r_n$,

$$\|P_m(\sum_k w_k a^k)\|_{X'} = \|\sum_{k=1}^{n-1} w_k a^k + P_m(w_n a^n)\|_{X'}$$

because $a_k^n = 0$ off $[r_n, r_n + 1)$. Furthermore,

$$\begin{aligned} \left\| \sum_{k=1}^{n-1} w_k a^k + P_m(w_n a^n) \right\|_{X'} &\leq \sum_{k=1}^{n-1} |w_k| \frac{1}{2^k} + \|P_m(w_n a^n)\|_{X'} \\ &\leq \sum_{k=1}^{n-1} |w_k| \frac{1}{2^k} + C \|P_m(w_n a^n)\|_{X^\gamma}, \end{aligned}$$

using $X^\gamma \subseteq X'$ and Proposition 2.1. Using the fact that the P_m 's are bounded on X^γ yields

$$\sum_{k=1}^{n-1} |w_k| \frac{1}{2^k} + C \|P_m(w_n a^n)\|_{X^\gamma} \leq \|w\|_\infty + CK \|a^n\|_{X^\gamma} \leq 2\|w\|_\infty + CK.$$

Thus for all $w \in \ell_\infty$, $\|P_m(\sum w_n a^n)\|_{X'}$ is bounded and therefore $\sum_n w_n a^n \in X^\gamma = Z^\gamma$.

Now define the projection $P : Z \rightarrow Z$ by $Pz = \sum_n a^n(z) z^n$. Note that $\sum_n |a^n(z)| < \infty$ for all z because $\sum_n \operatorname{sgn} a^n(z) a^n$ is in Z^γ (hence Z^β) from above. The map P is bounded as it is a matrix map between BK spaces. Note also that $Pz^k = \sum_n a^n(z^k) z^n = z^k$.

The range $R(P)$ is $\ell_1(\{z^n\})$, so $\ell_1(\{z^n\})$ is closed in Z implying that $[a^n]_Z = \ell_1(\{z^n\})$. There exists an obvious isomorphism from ℓ_1 to $\ell_1(\{z^n\})$ implying that $\{z^n\}$ is equivalent to the unit vector basis of ℓ_1 . Note that the condition on the projection is satisfied as well. ■

The fact that Z fails to have property (W8) places a copy of ℓ_1 inside of Z . Again it is interesting to note that ℓ_1 fails to have the Wilansky property. This is an idea running through all of the new Wilansky properties introduced in this section.

For the next Wilansky property, recall the notion of the characteristic $\tau(V)$ of a linear subspace V of a dual space X' . This will bring the Wilansky property in

touch with another set of ideas. The number $r = r(V)$ is the greatest number such that $B(V, 1) = \{f \in V : \|f\| \leq 1\}$ (the unit ball of V) is $\sigma(X', X)$ dense (that is, *weak** dense) in the r ball of X' , $B(X', r) = \{f \in X' : \|f\| \leq r\}$. Properties relating to $r(V)$ may be found in [10]. A few of these facts may be worth noting here. Let X be a Banach space with a basis $\{x_n\}$. Let $\{f_n\}$ be the associated sequence of coordinate functionals, that is, $f_n(x^k) = \delta_k^n$. Then $V = [f_n]$ is a subspace of X' that is $\sigma(X', X)$ dense in X' . Also note that certainly $0 \leq r(V) \leq 1$. It may happen that $r(V) = 0$ for a $\sigma(X', X)$ dense linear subspace V of X' . However, if $V = [f_n]$, where $\{f_n\}$ is an associated sequence of coordinate functionals to a basis $\{x_n\}$, then $r(V) > 0$. An equivalent way to express $r(V)$ is given by $r(V) = \max\{r : B(X', r) \subseteq \overline{B(V, 1)}^{weak^*}\}$. Here $r(X'_0)$ will be considered. Note that because X is an *AD* space, X^f is identified with X' by Proposition 2.4. Thus the notion of $r(X^f_0)$ makes sense.

DEFINITION. A *BK* space Z has property (W9) if X *BK*, *AD*, $X \subseteq Z$, $X^\gamma = Z^\gamma$, and $r(X^f_0) > 0$ together imply that $X = Z$.

PROPOSITION 5.4. If a *BK*, *AK* space has property (W8) then it has property (W9).

PROOF: Suppose Z has property (W8) and let X be a *BK*, *AD* space contained in Z with $X^\gamma = Z^\gamma$ and $r(X^f_0) > 0$. By (W8), Z^γ_0 is closed in X^f . Let $u \in B(X^f, r) \cap Z^\gamma$. From $r(X^f_0) > 0$ there exists a sequence $\{u^n\}$ in the unit ball of X^f_0 such that $u^n \rightarrow u$ in the *weak** topology. The space $Z^\gamma_0 = X^f_0$, and therefore the sequence $\{u^n\}$ chosen above is in Z^γ_0 as well as in X^f_0 . Again, because Z^γ_0 is closed in X^f , $\|u^n\|_{Z^\gamma} \sim \|u^n\|_{X^f} \leq 1$ and therefore $\|u^n\|_{Z^\gamma} \leq B$ for some positive constant B . Since $u^n \rightarrow u$ *weak**, $\|u\|_{Z^\gamma} \leq B$ as well, being the *weak** limit point of $\{u^n\}$ in X^f . By virtue of the fact that $Z^\gamma \subseteq X^f$, $C\|u\|_{X^f} \leq \|u\|_{Z^\gamma}$, for some constant

$C > 0$. Therefore, combining inequalities, $C\|u\|_{X^f} \leq \|u\|_{Z^\gamma} \leq \frac{C}{r}\|u\|_{X^f}$, where the second inequality follows because $u \in B(X^f, r)$. Thus the norms on X^f and Z^γ are equivalent and therefore Z^γ is closed in X^f by Proposition 2.1 (recall that u was an arbitrary choice). Thus $Z^\gamma = X^f$, by Fact 7. Since $Z^\gamma = X^\gamma$, Fact 5 implies that X has AB . Thus X is also an AK space and Fact 6 yields $X = Z$ and thus Z has property (W9). ■

DEFINITION. A BK space Z is said to have property (W10) if there are no non-trivial BK, AD subspaces X of Z that are strongly series summable with $X^\gamma = Z^\gamma$.

Recall that a space X is strongly series summable if there exists a sequence $\{u^n\}$ in φ such that $\{u^n\}$ is bounded in $M(X)$ and $u^n \rightarrow 1$ in ω .

A BK, AK space Z has property (W10) if it has property (W9). The proof of this will require a known fact about multipliers.

LEMMA. For a BK space X containing φ , $M(X) \subseteq M(X^f)$.

Note that since $M(X) \subseteq M(X^f)$, $\|u\|_{M(X^f)} \leq C\|u\|_{M(X)}$ for some positive constant C . Here it is assumed that $M(X)$ and $M(X^f)$ are both BK spaces (see Proposition 2.1). Thus if a sequence is bounded in $M(X)$ then it is also bounded in $M(X^f)$. This is a useful fact as well.

PROPOSITION 5.5. If a BK, AK space Z has property (W9) then it has property (W10).

PROOF: Let X be a BK, AD space contained in Z and strongly series summable. Furthermore, let $X^\gamma = Z^\gamma$. The proof will be completed by showing, using (W9), that $X = Z$ and thus Z has property (W10). Let u be in the unit ball of X^f . The space X is strongly series summable, so there exists a sequence $\{u^n\}$ in φ such that $\{u^n\}$ is bounded in $M(X)$ (hence in $M(X^f)$) and $u^n \rightarrow 1$ in ω . Because $u^n \rightarrow 1$

in ω , $u^n u \rightarrow u$ coordinatewise. The sequence $\{u^n u\}$ is uniformly bounded, hence bounded, so there exists a *weak** cluster point of $\{u^n u\}$. This must be u . Thus $u^n u \rightarrow u$ *weak**. The sequence $\{u^n u\}$ is in φ because $\{u^n\} \subseteq \varphi$. All together this is just saying that $r(\varphi) > 0$. Therefore $r(X'_0) > 0$ because φ is in X'_0 . Now by using property (W9), $X = Z$. ■

It is noted here that there is an interesting fact concerning strongly series summable spaces that provides us with a nice direct connection between between properties (W8) and (W10).

PROPOSITION 5.6. *Let X be a BK, AD space that is also a strongly series summable space. Then if X'_0 is closed in X' , X is an AB space (hence an AK space).*

PROOF: It suffices to prove that $X^\gamma = X'$. This implies, by Fact 5, that X is an AB space.

Because X is strongly series summable, there exists a sequence $\{u^n\} \subseteq \varphi$ bounded in $M(X)$ with $u^n \rightarrow 1$ in ω . Note that $\{u^n\}$ is bounded in $M(X')$ as well. Let $u \in X'$. The sequence $\{u^n u\}$ is bounded in X'_0 , hence in X'_0 . Then $u^n u$ must converge to u *weak** in X^γ , as in the proof of Proposition 5.4. Note that Proposition 2.4 is used implicitly here. Therefore $X' \subseteq X^\gamma$ which implies $X' = X^\gamma$. ■

This proposition yields an alternate proof of the fact that for a BK, AK space, having Property (W8) implies having Property (W10).

COROLLARY. *A BK, AK space Z has Property (W10) if it has Property (W8).*

PROOF: Let Z have Property (W8) and let $X \subseteq Z$ be a BK, AD space that is strongly series summable with $X^\gamma = Z^\gamma$. Showing $X = Z$ completes the proof.

By virtue of Property (W8), Z^γ_0 is closed in X' which implies X^γ_0 is closed in X' since $X^\gamma = Z^\gamma$. But now the previous proposition implies that X is an AK space. Thus X and Z are two AK spaces with the same γ -dual and therefore Fact 6 ensures that $X = Z$. ■

Currently these Wilansky properties are “ordered” in the following manner for a BK, AK space: $(W5) \Rightarrow (W6) \Rightarrow (W7) \Rightarrow (W8) \Rightarrow (W9) \Rightarrow (W10)$. With further hypotheses on Z , these implications may be reversed, which is shown later. However, with no further hypotheses we have a partial list of converses.

PROPOSITION 5.7. *If a BK, AK space Z has property (W10), then it has property (W7). Thus properties (W7) through (W10) are equivalent in this case.*

PROOF: It is shown that if Z fails to have property (W7) then it fails to have property (W10).

There exists a projection $P : Z \rightarrow Z$, defined by $Pz = \sum_n a^n(z)z^n$ where $\{z^n\}$ is a block basic sequence, say with $z^n_k = 0$ off $[r_n, s_n]$, equivalent to ℓ_1 's unit vector basis. Also a^n_k is zero off $[r_n, s_n]$. This is used to construct a space in Z that will cause Z to fail to have property (W10). Let $Y = \{\lambda \in \ell_1 : \sum_n n|\lambda_{2n} - \lambda_{2n-1}| < \infty\}$. This space may be considered as the intersection of ℓ_1 with $(\ell_1)_A$, where A is a suitable matrix, and therefore it is a BK, AD space. See [15] for details. Note that this space is used in [2] to show that ℓ_1 does not have the Wilansky property.

Define the space $U = \{u \in Z : Pu = \sum y_n z^n \text{ where } y \in Y\}$. If $T : Z \rightarrow \omega$ is given by $Tz = \{n(a^{2^n}(z) - a^{2^{n-1}}(z))\}$, then $U = \{u \in Z : Tu \in \ell_1\} = (\ell_1)_T \cap Z$ and $\|z\|_U = \|z\|_Z + \sum n|a^{2^n}(z) - a^{2^{n-1}}(z)|$. Again, see [15] for details. Thus U is certainly a BK space.

This space U is also an AD space. This is shown in a manner identical to that of the space W in Theorem 5.2. Here $U = P(U) \oplus N(P)$. The details are therefore

omitted.

It is also true that U is strongly series summable. To prove this, let $u^n = P_{s_n}(1)$, the s_n th section of the constant sequence of 1's. For each n , $u^n \in M(U)$ since $u^n \in \varphi$. Certainly $u^n \rightarrow 1$ in ω , so it remains to show that $\{u^n\}$ is bounded in $M(U)$. Let z be an element of the unit ball of U . Then for each m ,

$$\begin{aligned} \|u^m z\|_U &= \|u^m z\|_Z + \sum_{n=1}^{\infty} n |a^{2^n}(u^m z) - a^{2^{n-1}}(u^m z)| \\ &\leq M \|z\|_Z + \sum_{n=1}^{\infty} n |a^{2^n}(P_{s_m} z) - a^{2^{n-1}}(P_{s_m} z)| \\ &\leq M \|z\|_Z + \sum_{n=1}^m n |a^{2^n}(z) - a^{2^{n-1}}(z)| \\ &\leq M \|z\|_Z + \sum_{n=1}^{\infty} n |a^{2^n}(z) - a^{2^{n-1}}(z)| \leq M. \end{aligned}$$

Necessary in the above list of inequalities is the fact that $\|u^m z\|_Z = \|P_{s_m} z\|_Z \leq M \|z\|_Z$ for some positive constant M . This is true because Z is an AK space (hence has AB) and thus uniform boundedness may be applied to the maps $z \rightarrow P_{s_m} z$. The fact that a^{2^n} is zero off $[r_{2^n}, s_{2^n}]$ is used as well. Hence $\{u^n\}$ is bounded in $M(U)$.

Finally, $U^\gamma = Z^\gamma$. To show this consider the space

$$X = \left\{ \sum_n t_n (z^{2^n-1} + z^{2^n}) : t \in \ell_1 \right\} + N(P).$$

Clearly $X \subseteq U \subseteq Z$ and $X \widehat{\otimes} bv_0 \subseteq X \widehat{\otimes} bv_0 = Z$. It suffices to show that $\{z^n\}$ is bounded in $X \widehat{\otimes} bv_0$ because then $X \widehat{\otimes} bv_0 = Z$ and hence $X^\gamma = U^\gamma = Z^\gamma$.

The sequence $\{z^{2^n-1} + z^{2^n}\}$ is certainly bounded in $X \widehat{\otimes} bv_0$ and $X \widehat{\otimes} bv_0$ is an AB space so $P_{s_{2^n-1}}(z^{2^n-1} + z^{2^n}) = z^{2^n-1}$ is bounded in $X \widehat{\otimes} bv_0$ for all n . Now, since $\|z^{2^n}\| \leq \|z^{2^n-1} + z^{2^n}\| + \|z^{2^n-1}\|$, $\{z^{2^n}\}$ is bounded in $X \widehat{\otimes} bv_0$ as well. Together

these imply that $\{z^n\}$ is bounded in $X \widehat{\otimes} bv_0$. So we have a *BK, AD* space U which is strongly series summable, and strictly contained in Z with $U^\gamma = Z^\gamma$. This shows that Z cannot have property (W10). ■

Now consider the case that Z is a *BK, AK* space and in addition is *solid*, that is, $\ell_\infty \cdot Z = Z$. If Z has property (W7) then it also has property (W4). Note in this situation that all of the stated Wilansky properties are equivalent.

THEOREM 5.8. *If Z is a *BK, AK* space and in addition is solid, then if Z has property (W7) it also has property (W4). Thus all Wilansky properties are equivalent in this case.*

PROOF: Assume that Z fails to have property (W4). Note that this guarantees that Z^γ is not an *AD* space or else Proposition 4.6 would be violated. Because Z^γ is not an *AD* space there exists $u \in Z^\gamma / Z_0^\gamma$ such that $\|\sum_{k=r_n}^{s_n} u_k \delta^k\|_{Z^\gamma} \geq a > 0$ for some constant $a > 0$ and some intertwining sequences of positive integers $\{r_n\}$ and $\{s_n\}$. Let $u^n \in \varphi$ (and also in Z^γ) be defined by $u^n = \sum_{k=r_n}^{s_n} u_k \delta^k$. Thus $\|u^n\|_{Z^\gamma} \geq a > 0$. Now, as in the proof of Theorem 5.3, there exists $\{z^n\}$ bounded in Z with $u^n(z^n) = 1$ and z^n_k zero off $[r_n, s_n]$.

Define $P : Z \rightarrow Z$ by $P(z) = \sum_n u^n(z) z^n$. Note that $\{u^n(z)\}$ is in ℓ_1 . This follows from

$$\begin{aligned} \sum_{n=1}^{\infty} |u^n(z)| &= \sum_{n=1}^{\infty} \left| \sum_{k=r_n}^{s_n} u_k z_k \right| \leq \sum_{n=1}^{\infty} \sum_{k=r_n}^{s_n} |u_k z_k| \\ &\leq \sum_{k=1}^{\infty} |u_k z_k| < \infty \end{aligned}$$

because u is in Z^γ which is equal to Z^α by the solidness of Z . See [10] for this result. P is a bounded operator because it may be written as a matrix map between *BK* spaces. The map P is also a projection because $P(P(z)) = P(z)$

using $u^n(z^k) = \delta_k^n$. The range $R(P) = \ell_1(\{z^n\})$ is closed in Z and therefore, since $[z^n]_Z \subseteq \ell_1(\{z^n\})$, $[z^n]_Z = \ell_1(\{z^n\})$. There exists an obvious isomorphism from ℓ_1 to $\ell_1(\{z^n\})$ and therefore $\{z^n\}$ is equivalent to the unit vector basis of ℓ_1 . The projection also clearly has the other required properties so Z fails to have property (W7) and the proof is completed. ■

Appendix

The following is an example given by A. K. Snyder of a BK, AD space Y with Y_0^γ closed in Y' that is not an AB space. Recall that in these circumstances Y^γ closed in Y' *does* imply that Y is an AB space.

Begin by constructing a space Y strictly contained in ℓ_1 with $Y \widehat{\otimes} bv = \ell_1$. Note that this implies $Y^\gamma = \ell_\infty$ by Proposition 3.3. Choose Y by the following construction. Let $\{J_n\}$ be a partition of N into infinite subsets. Write each J_n as an increasing sequence $\{r_k^n\}$. Define ℓ_1^n to be $\{x \in \ell_1 : x_i = 0 \text{ for } i \notin J_n\}$ which is a closed subspace of ℓ_1 . Then ℓ_1 may be considered as $\bigoplus_n \ell_1^n$ where the norm is the sum of the norms on ℓ_1^n .

Fix n . Let $\{a^{n,k}\} \subseteq \varphi$ be dense in the unit ball of ℓ_1^n . This is possible because ℓ_1 is a separable space. Note that $a^{n,k}$ is zero off J_n .

Fix n and k . Because $a^{n,k}$ is in φ , choose m such that $a_{r_i^n}^{n,k} = 0$ for $i \geq m$ (recall $J_n = \{r_k^n\}$). Define $b^{n,k}$ by $b_{r_{i+m}^n}^{n,k} = -a_{r_i^n}^{n,k}$ for $1 \leq i \leq m$ and $b_i^{n,k} = 0$ for all other i . Note $b^{n,k}$ is $a^{n,k}$ shifted out past m . Note also that $b^{n,k} \in \ell_1^n$ and $P_{r_m^n}(a^{n,k} + b^{n,k}) = a^{n,k}$.

Define the following BK space X :

$$X = \left\{ \sum_{n,k} \lambda_{n,k} (a^{n,k} + b^{n,k}) : \sum_{n,k} |\lambda_{n,k}| \leq \infty \right\}.$$

Think of ℓ_1 as being doubly indexed; then X is the image of ℓ_1 under the obvious map.

(1) Note that $\dim \ell_1/X = \infty$. This is because the annihilator of X is infinite dimensional, as it contains χ_{J_m} for all $m \neq n$, where χ_{J_m} is the sequence that is 1 on J_m and 0 elsewhere.

(2) Note also that $X \widehat{\otimes} bv = \ell_1$. The proof is as follows. Note $\|a^{n^k} + b^{n^k}\|_X \leq 1$ because $a^{n^k} + b^{n^k}$ is the image of δ^{n^k} in ℓ_1 . Thus $\|a^{n^k}\|_{X \widehat{\otimes} bv} \leq 1$ also because $a^{n^k} = (a^{n^k} + b^{n^k} \cdot 1) \cdot r_m^{n^k}$ in $X \widehat{\otimes} bv$ where $r_m^{n^k}$ is the sequence with 1 in each coordinate up to $r_m^{n^k}$ and 0 afterward. Now note that $\{a^{n^k}\}$ is dense in the unit ball of ℓ_1 to achieve the result.

Because φ is not necessarily in X , consider instead the space $Y = X + \{\frac{1}{n}\}\ell_1$.

(3) $\dim \ell_1/Y = \infty$. Suppose not. That is, suppose there exists a finite dimensional space F with $X + \{\frac{1}{n}\}\ell_1 + F = \ell_1$. It is known that $\{\frac{1}{n}\}\ell_1 \leq \ell_1$ (Proposition 4.3), so $\dim \ell_1/(X + F) \leq \infty$ which would imply that $\dim \ell_1/X \leq \infty$ which contradicts (1) above.

(4) The space Y is an AD space. By a result of Bennett [1], $\overline{\varphi^Y} = \overline{\varphi}^{\{\frac{1}{n}\}\ell_1} + \overline{\varphi \cap X^X} = \{\frac{1}{n}\}\ell_1 + \overline{\varphi \cap X^X} = \{\frac{1}{n}\}\ell_1 + X = Y$. Also used is the fact that $X \cap \varphi$ is dense in X . This is true by the definition of X ; recall, a^{n^k}, b^{n^k} are elements of φ .

(5) Note $Y \widehat{\otimes} bv = \ell_1$. This is true by (2) because $X \subseteq Y$. Therefore $Y^\gamma \subset \ell_1$. This is true by (2) because $X \subseteq Y$. Therefore $Y^\gamma = \ell_\infty$ and $Y_0^\gamma = c_0$.

(6) The space c_0 is closed in Y' . Prove this by showing that for $u \in \varphi$ and $\|u\| = 1, \|u\|_{Y'} \geq \frac{2}{5}$ and using Proposition 2.1. Consider only $u \in \varphi$ because φ is dense on c_0 . Let $u \in \varphi$ with $\|u\|_\infty = 1$. Choose m such that $|u_m| = 1$. Choose s such that $m \in J_s$. Choose $n \in J_s$ such that $u_i = 0$ for $i \leq n$ (recall $u \notin \varphi$ and J_s is an infinite set). Note that $\frac{\delta^m + \delta^n}{2} \in \ell_1^s$. The sequence $\{a^{n^k}\}$ is dense in the unit ball of ℓ_1^n so choose k such that $\|a^{n^k} - (\frac{\delta^m + \delta^n}{2})\|_1 < 1/10$. Because of this inequality it is certain that $a^{n^k} \neq 0$. This ensures that b^{n^k} is nonzero beyond where u is nonzero so $|u(a^{n^k} + b^{n^k})| = |u(a^{n^k})|$ (recall the definition of b^{n^k}). Note

that $\|a^{*k} + b^{*k}\|_X \leq 1$. Combining this information yields

$$\begin{aligned}
 |u(a^{*k} + b^{*k})| &= |u(a^{*k})| = \left| u\left(\frac{\delta^m + \delta^n}{2}\right) - u\left(\frac{\delta^m + \delta^n}{2}\right) + u(a^{*k}) \right| \\
 &\geq \left| u\left(\frac{\delta^m + \delta^n}{2}\right) \right| - \left| u\left(\frac{\delta^m + \delta^n}{2}\right) - u(a^{*k}) \right| \\
 &\geq \frac{1}{2} - \|u\|_\infty \left\| \frac{\delta^m + \delta^n}{2} - a^{*k} \right\| \\
 &\geq \frac{1}{2} - 1\left(\frac{1}{10}\right) = \frac{2}{5}.
 \end{aligned}$$

Thus Y is the space with the stated properties.

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Vita

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