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INELASTIC INSTABILITY ANALYSIS OF SPACE FRAMES

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Summary

A method of elastic-plastic analysis for space frames subjected to combined gravity and lateral loads has been developed. It takes into account the secondary bending and twisting moments in the structure due to deformations and the reduction in member stiffnesses caused by axial forces. Inelastic action, which is represented by formation of plastic hinges, is assumed to occur at discrete locations under combined axial force, biaxial bending and twist. The application of the method is illustrated by examples.

Résumé

Une méthode d'analyse élasto-plastique pour cadres tridimensionnels sous charges verticales et latérales a été développée. Les moments secondaires de flexion et de torsion dus aux déformations de la structure et la réduction de la raideur des barres due aux forces axiales sont pris en compte. Il est supposé que l'effet inélastique représenté par la formation de rotules plastiques intervient à des emplacements discrets sous l'action combinée de l'effort axial, de la flexion biaxiale et de la torsion. L'application de la méthode est illustree par des exemples.

1. INTRODUCTION

Much research has been carried out in the recent years to study the strength and behavior of two-dimensional frames and design provisions accounting for the effects of instability are available (1). For three-dimensional, rigidly jointed frames, however, only very limited studies have been reported in the literature, although such frames are used frequently in building construction, especially in the seismic regions. The analysis of a 3-D frame is highly complex and involves a number of important considerations. The first is the appropriate yield surfaces to be used to define the capacity of a cross section subjected to combined axial force, biaxial bending moments and twist. The second consideration is the reduction of member stiffnesses by member forces and the possibility of instability failure of the individual members. The third problem is presence of the secondary PΔ moment due to deflections. The main objective of this paper is to present a method of analysis for 3-D frames, which takes into consideration all these effects.

2. MEMBER STIFFNESS AND OVERALL STIFFNESS

Consider a typical member in a space frame subjected to member forces F_a and F_b with the corresponding displacements u_a and u_b. These forces and displacements are related by the member stiffness matrices:

$$F_a = k_{aa}\, u_a + k_{ab}\, u_b \tag{1}$$

$$F_b = k_{ba}\, u_a + k_{bb}\, u_b \tag{2}$$

where the k's member stiffness matrices. These quantities are associated with the member coordinate system in which the x axis coincides with the longitudinal axis of the member, and y and z axes are major and minor axes of the cross section. Eqs. (1) and (2) can be written with reference to the global coordinate system as follows:

$$\bar{F}_a = \bar{k}_{aa}\,\bar{u}_a + \bar{k}_{ab}\,\bar{u}_b \tag{3}$$

$$\bar{F}_b = \bar{k}_{ba}\,\bar{u}_a + \bar{k}_{bb}\,\bar{u}_b \tag{4}$$

At a joint i as shown in Fig. 1, the force vector is $\bar{p}_i$. The equilibrium condition at this joint requires that

$$\bar{p}_i = \bar{F}_i^{\ell} + \bar{F}_i^{m} + \bar{F}_i^{n} \tag{5}$$

The superscripts in Eq. (5) indicate members. Substitution of Eqs. (3) and (4) with the appropriate subscripts into Eq. (5) results in

$$\bar{p}_i = (\bar{k}_{ii}^{\ell} + \bar{k}_{ii}^{m} + \bar{k}_{ii}^{n})\,\bar{u}_i + \bar{k}_{ij}^{\ell}\,\bar{u}_j + \bar{k}_{ik}^{m}\,\bar{u}_k + \bar{k}_{ih}^{n}\,\bar{u}_h \tag{6}$$

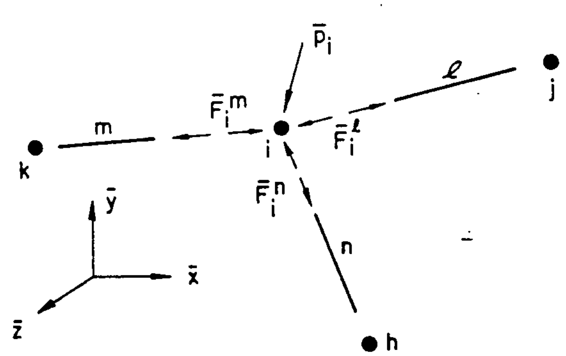

Fig. 1 Equilibrium at Joint i in Global Coordinate System

Finally, by assembling the equilibrium conditions at all the joints, the following simultaneous equations can be obtained.

$$P = K\,U \tag{7}$$

where

$$P = \begin{pmatrix} \bar{p}_1 \\ \bar{p}_2 \\ \cdot \\ \cdot \\ \cdot \\ \bar{p}_n \end{pmatrix} \qquad U = \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \cdot \\ \cdot \\ \cdot \\ \bar{u}_n \end{pmatrix} \quad \text{for n joints} \tag{8}$$

and K is referred to as the "overall stiffness matrix" of the frame.

3. YIELD SURFACE

As the supplied loads increase, plastic hinges will form one by one at the critical sections in the structure. The formation of the plastic hinge is governed by the applicable yield surfaces for the cross section. Several studies have been made to determine the yield surfaces for the case of combined axial force, biaxial bending and twist (2). For wide-flange sections, which are frequently encountered in practice, the yield surfaces are very complex and usually cannot be written in a single mathematical expression. In this study the spherical yield surface defined by the following equation is used.

$$\left(\frac{P_x}{P_{xo}}\right)^2 + \left(\frac{M_x}{M_{xo}}\right)^2 + \left(\frac{M_y}{M_{yo}}\right)^2 + \left(\frac{M_z}{M_{zo}}\right)^2 = 1.0 \qquad (9)$$

in which P_x is the axial force, P_{xo} the yield axial force, M_x the twisting moment, M_{xo} the full plastic twisting moment, M_y and M_z the bending moments about the y and z axes, and M_{yo} and M_{zo} the full plastic bending moments about the same axes.

4. INCREMENTAL LOAD FACTOR

Suppose that the response of the structure has been determined for the n-th incremental loading. The load vector is p^n, and the member end force vector at end i is $(F_i)^n$. Also, the displacement of the joint is $(u_i)^n$. At this time, n plastic hinges have formed in the structure and the modified overall stiffness matrix K^n has been obtained. For any additional load, K^n can be used in the analysis until the next plastic hinge forms.

Consider now the (n+1)th loading. If a unit set of incremental load, dP_o, is applied to the structure, the corresponding incremental joint displacements can be obtained from

$$dP_o = K^n \cdot dU_o^{(n+1)} \qquad (10)$$

The incremental member end force vectors are then obtained using the modified member stiffnesses. For example, for a typical member,

$$(dF_i)_o^{(n+1)} = (k_{ij})^n \; (du_i)_o^{(n+1)} + (k_{ji})^n (du_j)_o^{(n+1)} \qquad (11)$$

which is similar to Eqs. (1) or (2). $(k_{ij})^n$ and $(k_{ji})^n$ are the modified member stiffness matrices, and $(dF_i)_o^{(n+1)}$ and $(du)_o^{(n+1)}$ are respectively, the incremental member end force and the incremental joint displacement.

If the next plastic hinge forms at an incremental load dP, the incremental load factor is given by

$$d\lambda = \frac{dP}{dP_o} \qquad (12)$$

Then, the accumulated member end forces, joint displacements and loads are expressed as

$$(F_i)^{(n+1)} = (F_i)^n + d\lambda (dF_i)_o^{(n+1)} \qquad (13)$$

$$(u_i)^{(n+1)} = (u_i)^n + d\lambda(du_i)_o^{(n+1)} \quad (14)$$

$$P^{(n+1)} = P^{(n)} + d\lambda(dP_o) \quad (15)$$

In order to obtain the incremental load factor, $d\lambda$, the yield surface

$$G(F) = 1.0 \quad (16)$$

must be used. Substitution of Eq. (13) into Eq. (16) results in

$$G\left((F_i)^{(n+1)}\right) = G\left((F_i)^n + d\lambda(dF_i)_o^{(n+1)}\right) = 1.0 \quad (17)$$

Applying Eq. (17) to each of the possible plastic hinge locations, a set of $d\lambda$ values can be obtained. The smallest value of the set is the true incremental load factor.

5. MEMBER STIFFNESS MODIFICATION DUE TO PLASTIC HINGE FORMATION

Figure 2 shows a member subjected to incremental member end forces and incremental displacements at its two ends. Plastic hinges have already formed at both ends. In order to obtain modified stiffness for this member, the following conditions must be considered. First, the incremental displacement at plastic hinge consists of two parts: elastic and plastic parts. Therefore,

$$du_a = du_a^e + du_a^p \quad (18)$$

$$du_b = du_b^e + du_b^p \quad (19)$$

The incremental member end forces are related to the incremental elastic displacements by elastic member stiffness matrices. The relationships are

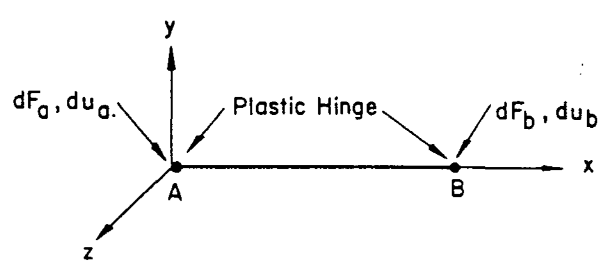

Fig. 2 Member with Plastic Hinges

$$dF_a = k_{aa}\, du_a^e + k_{ab}\, du_b^e \quad (20)$$

$$dF_b = k_{ba}\, du_a^e + k_{bb}\, du_b^e \quad (21)$$

From normality condition of the incremental plastic deformation, each of the du_a^p and du_b^p must be in a direction normal to the yield surface. The normality conditions can be expressed as

$$du_a^p = \mu_a N_a \quad (22)$$

$$du_b^p = \mu_b N_b \quad (23)$$

where μ_a and μ_b are unknown scalers, and N_a and N_b are the unit vectors normal to the yield surface.

The member end force at the plastic hinge must stay on the yield surface. Thus, the incremental member force must be on the plane to the yield surface. From this condition,

$$N_a^t \, dF_a = 0 \tag{24}$$

$$N_b^t \, dF_b = 0 \tag{25}$$

can be obtained. Eliminating μ_a, μ_b, du_a^e, du_a^p, du_b^e and du_b^p from the above equations, the modified member stiffness matrices after plastic hinge formation can be obtained

$$\begin{aligned}
k_{aa}^M &= k_{aa} - k_{aa}N_a(e_3N_b^tk_{ba} - e_4N_a^tk_{aa}) - k_{ab}N_b(e_1N_b^tk_{ba} - e_2N_a^tk_{aa}) \\
k_{ab}^M &= k_{ab} - k_{ab}N_b(e_1N_b^tk_{bb} - e_2N_a^tk_{ab}) - k_{aa}N_a(e_3N_b^tk_{bb} - e_4N_a^tk_{ab}) \\
k_{ba}^M &= k_{ba} - k_{ba}N_a(e_3N_b^tk_{ba} - e_4N_a^tk_{aa}) - k_{bb}N_b(e_1N_b^tk_{ba} - e_2N_a^tk_{aa}) \\
k_{bb}^M &= k_{bb} - k_{bb}N_b(e_1N_b^tk_{bb} - e_2N_a^tk_{ab}) - k_{ba}N_a(e_3N_b^tk_{bb} - e_4N_ak_{ab})
\end{aligned} \tag{26}$$

in which

$$\begin{aligned}
e_1 &= \frac{d_4}{d_1d_4 - d_2d_3} & e_2 &= \frac{d_2}{d_1d_4 - d_2d_3} \\
e_3 &= \frac{-d_3}{d_1d_4 - d_2d_3} & e_4 &= \frac{-d_1}{d_1d_4 - d_2d_3} \\
d_1 &= N_b^tk_{bb}N_b & d_2 &= N_b^tk_{ba}N_a \\
d_3 &= N_a^tk_{ab}N_b & d_4 &= N_a^tk_{aa}N_a
\end{aligned} \tag{27}$$

If only end a is yielded, the e values are

$$e_1 = e_2 = e_3 = 0, \text{ and } e_4 = -\frac{1}{d_4} \tag{28}$$

Similarily, if only end b is yielded, the e values are

$$e_1 = \frac{1}{d_1} \text{ and } e_2 = e_3 = e_4 = 0 \tag{29}$$

6. MEMBER STIFFNESS REDUCTION DUE TO MEMBER FORCES

The instability effects can be included by developing a set of algebraic equations governing the behavior of individual members. These relationships are analogous to the slope-deflection equations commonly used in the analysis of planar structures. Figure 3 shows a member subjected to axial force P_x, shear forces P_y and P_z, twisting moment M_x, and bending moments M_y and M_z.

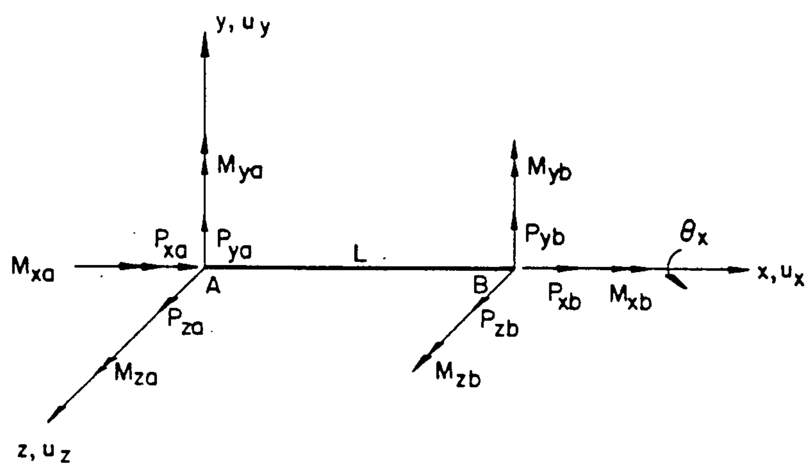

Fig. 3 Member Subjected to Axial Force, Biaxial Bending and Twist

The following equations govern the relationships between the applied forces and the angle of twist θ_x and the displacements u_y and u_z (3).

$$EI_y u_z^{iv} + P_{xa} u_z'' - M_{xb} u_y''' + \frac{2}{L}(M_{za} + M_{zb})\theta_x' + \{-M_{za} + \frac{x}{L}(M_{za} + M_{zb})\}\theta_x'' = 0 \qquad (30)$$

$$EI_z u_y^{iv} + P_{xa} u_y'' + M_{xb} u_z''' + \frac{2}{L}(M_{ya} + M_{yb})\theta_x' + \{-M_{ya} + \frac{x}{L}(M_{ya} + M_{yb})\}\theta_x'' = 0 \qquad (31)$$

$$EI_w \theta_x^{iv} - (GJ - P_{xa} r_o^2)\,\theta_x'' + \{-M_{ya} + \frac{x}{L}(M_{ya} + M_{yb})\}u_y'' + \{-M_{za} + \frac{x}{L}(M_{za} + M_{zb})\}\,u_z'' = 0 \qquad (32)$$

In Eq. (32), EI_w represents the warping rigidity of the cross section, GJ the St. Venant's torsional rigidity, and r_o is the polar radius of gyration. In this study, approximate solutions of these equations are obtained by assuming that θ_x, u_y and u_z may be expressed as fourth order polynomials with unknown coefficients. These coefficients can be determined by applying the boundary conditions at the two ends. The solutions for θ_x, u_y and u_z are then substituted into the governing equations, and each member can be expressed by a linear combination of the boundary displacements. Since the final expressions of member end forces are very lengthy, they are omitted here. Relationships between the member end forces and the joint displacements, including the secondary effects, can be written as

$$\begin{aligned} F_a &= \phi_1\,(F_a, F_b, Q)\,u_a + \phi_2\,(F_a, F_b, Q)\,u_b \\ F_b &= \phi_3\,(F_a, F_b, Q)\,u_a + \phi_4\,(F_a, F_b, Q)\,u_b \end{aligned} \qquad (33)$$

where Q is a cross sectional property, and ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are functions of the member end forces and the cross sectional properties. The ϕ's may therefore be considered as the reduced member stiffness matrices. The basic equation to be solved becomes

$$P = K(P) \cdot U \tag{34}$$

in which the overall stiffness is a function of the applied load.

The reduced member stiffness matrices obtained by the above method have been used to calculate the displacements of some biaxially loaded wide-flange columns which have been solved previously by a more exact procedure (4). The displacements calculated by the present method show good agreement with the "exact" solutions for short and medium length columns (weak-axis slenderness ratios between 40 and 80).

7. SECOND-ORDER ANALYSIS

A procedure which incorporates all the theoretical developments presented herein has been developed for performing second-order analysis for space frames. Since the relationships between member end forces and joint displacements defined by Eq. (33) are not linear, it is necessary to use an iterative process to obtain solutions for the displacements at a given level of load. It is also very difficult to obtain a value of the applied load at which one set of the member forces exactly satisfies the yield condition according to Eq. (16). In the development of the computational procedure, the following bound on the yield condition has been adopted

$$0.95 < G(F) < 1.05 \tag{35}$$

The analysis is carried out either with increasing load or with increasing deflection. The ascending portion of the load-deflection is usually obtained with load increments as follows:

1. Compute the reduced member stiffnesses using the member forces determined at the end of the previous step.

2. Determine the displacements by Eq. (34) for a pre-selected load increment, and obtain the member forces from Eq. (33). If the difference between these displacements and those of previous step is less than 0.1%, go to Step 3. If not, repeat this step for a larger load increment.

3. Check whether or not one set of the member forces would satisfy the yield condition. If so, modify the member forces according to Eq. (26).

4. Increase the applied load and repeat the process from Step 1.

When the load reaches the maximum, the solution for displacements in Step 2 usually does not converge. The load must therefore be decreased in Step 4.

8. EXAMPLES

Example 1 The pentagonal frame shown in Fig. 4 is subjected to five vertical loads, one at each column top, with a combined total of 25P. In addition, a horizontal load P is applied to one of the columns as shown. All members are made of W8 x 24. The length of the girders is equal to 30 r_y and the height of the columns is 60 r_y, where r_y is the radius of gyration about the y-axis of the W8 x 24 section.

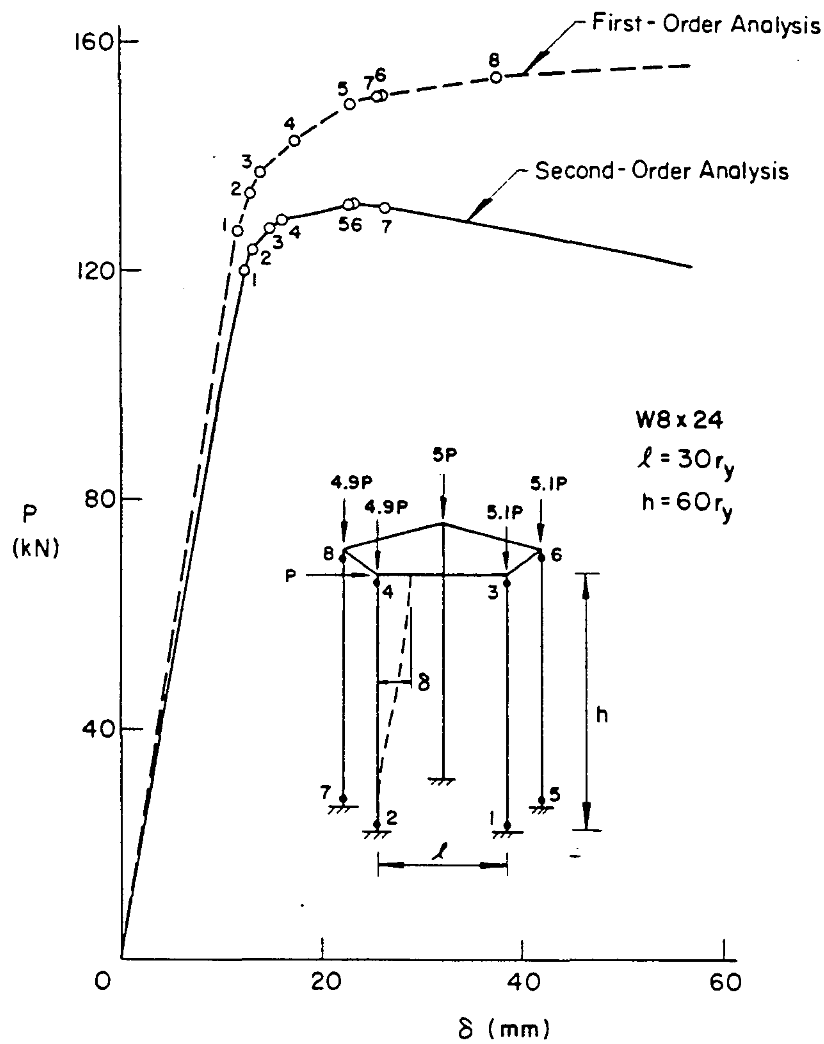

Fig. 4 Load-Deflection Curves of Frame of Example 1

The results of the first- and second-order analyses are given in Fig. 4. The combined effects of member and frame instability reduce the ultimate strength of the structure by about 16%. The load-deflection curve of the first-order analysis shows that the formation of eight plastic hinges in the four columns does not result in failure because the frame is still capable of sustaining additional loads by utilizing the torsional resistance of the center column. The second-order analysis indicates that the maximum load is reached after the formation of the sixth hinge and that the seventh one is on the descending portion of the load-deflection curve.

Example 2 As a second example, the two-story triangular frame given in Fig. 5 is analyzed. Each column in the upper story is subjected to a vertical load of 4P. An additional load of 2P is applied to each of the lower columns. Two horizontal loads are applied as shown; they cause twist of the entire structure.

In the first-order analysis, five plastic hinges form at the ultimate load in the columns of the lower story. The structure can then rotate freely about

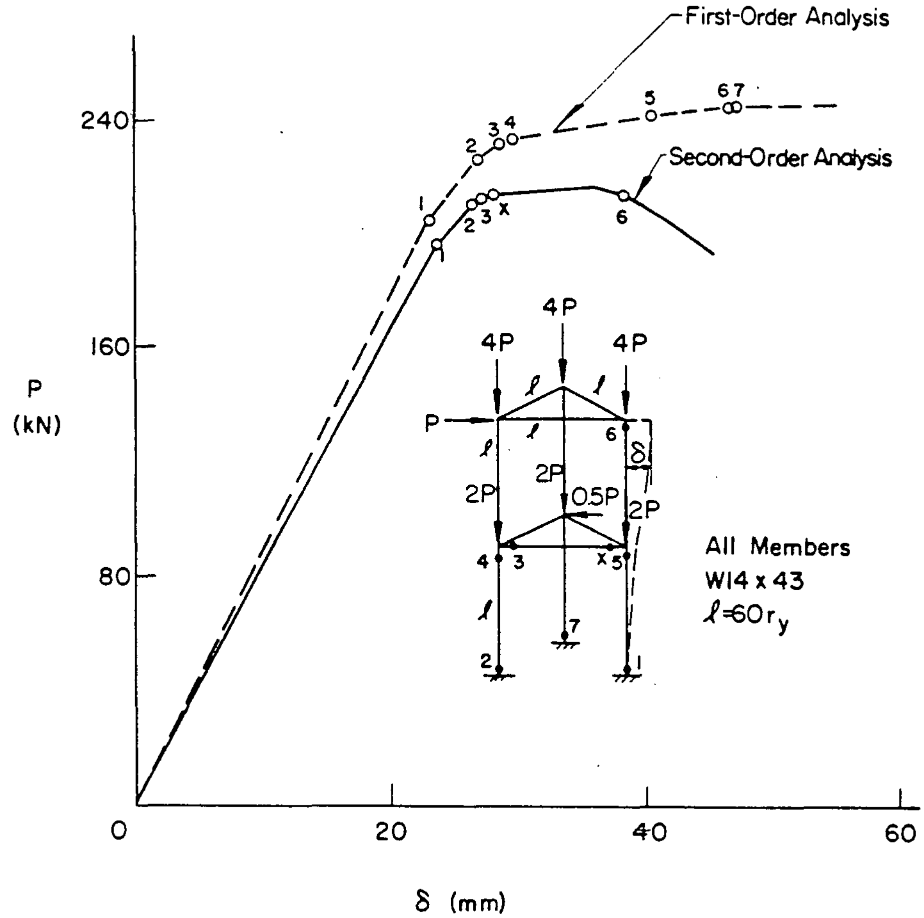

Fig. 5 Load-Deflection Curves of Frame of Example 2

the center column. The results of the second-order analysis show significant influence of frame instability. Even in the elastic range, the high vertical loads reduce substantially the overall lateral stiffness of the structure. Also, the slope of the unloading portion of the load-deflection curve is considerably steeper than that observed in Example 1.

9. SUMMARY AND CONCLUSION

Presented in this paper is a method for elastic-plastic, second-order analysis of three-dimensional frames. In its development both the member and frame instability effects have been included. The method is quite general and can be applied to different types of two-and three-dimensional frames, with and without the inclusion of these effects.

Since the members in a space frame are usually subjected simultaneously to axial force, biaxial bending and twist, an appropriate yield surface, which defines the combinations of these forces that cause plastic hinges to form, must be used. Two such surfaces, one defined by a four-dimensional sphere and the other developed by fitting the numerically calculated yield surfaces by polynomial functions (5), have been examined in developing the method. The spherical surface has been used in the examples presented because of its simplicity. After the yield surface has been selected, the normality condition is then used to determine the plastic deformations at the hinges, which are necessary in computing the spatial deflections of the overall structure. This has been described in general terms in the paper.

The member instability effect has been included by obtaining an approximate solution to the basic differential equations for a biaxially loaded column. The solution is in algebraic form and can be conveniently incorporated into the computation.

The load-deflection curve obtained from the second-order analysis usually consists of an ascending or loading portion and a descending or unloading portion. When the vertical load is low, the instability effects would not significantly alter the location and the order of plastic hinge formation. On the other hand, when large vertical load is applied, the instability effects would reduce the overall stiffness and the strength of the structure. Also, in this case, the descending portion of the load-deflection curve tends to have a steeper slope.

A number of important problems are recommended for further study in order to understand more fully the behavior of frames. A more detailed study of the appropriate yield surfaces to be used for space frames is necessary and additional analysis should be performed with different surfaces. Another problem is related to the assumption that plastic hinges can form only at the ends of members. Although some evidence is available to support this assumption, a more complete study is warranted. Finally, an experimental program for space frames with substantial instability effects is suggested.

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