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Stability of Steel Structures

ANALYSIS OF INITIALLY CROOKED, END RESTRAINED STEEL COLUMNS

by

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INTRODUCTION

The extensive research carried out during the last three decades on steel columns in the U.S. and abroad has resulted in a much better understanding of their behavior and failure mode. Long columns generally fail by elastic instability, whereas intermediate columns are likely to fail by inelastic instability which occurs after partial yielding has taken place at the critical sections. For short columns, instability failure may occur when parts of the critical sections have been stressed beyond the yield and into the strain-hardening range. Since the intermediate columns are used most frequently in engineering structures, much of the research has been focused on the problem of inelastic instability.

Among the factors that affect the strength of a column in a structural framework, the following have been found to be important: (1) residual stresses resulting from the manufacturing or fabrication process, (2) initial crookedness, (3) end restraints provided by the adjacent members or supports, and (4) eccentricities of the applied load.

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The presence of residual stresses causes earlier yielding at the critical sections, which results in a reduction of the overall stiffness of the column. Bending moment develops in an initially crooked column as soon as the axial load is applied, and the combined action of the bending moment and axial load again causes the critical sections to yield earlier than in an initially straight column. Residual stress and initial crookedness, therefore, tend to weaken the column and reduce its load-carrying capacity. On the other hand, the effect of end restraints is to increase the column's overall stiffness and to reduce its lateral deflection with a resulting increase in strength. When the load is applied eccentrically, an additional bending moment is generated in the column, which also tends to reduce the axial capacity of the column. In some situations, however, the load eccentricity may cause an increase in the strength of a crooked column. (This point will be discussed later in the paper.) The relative importance of these effects depends on the length of the column, the mechanical properties of steel, and the axis of bending.

The past research has paid much attention to the effects of residual stress and initial crookedness. The effect of load eccentricity has been studied by considering the column as a beam-column and treating the bending moment produced by the eccentricity as an externally applied moment. A full study of the influence of all the important factors has not yet been attempted.

In this paper, a general method of column analysis which can simultaneously take into account all the aforementioned factors is presented. It is applicable to long, intermediate, as well as short
columns. The method is first applied to develop theoretical predictions of the load-deflection relationships of some previously tested columns. A detailed parametric study of the effects of initial crookedness and end restraint for a given column is then made. Also included in a separate study examining the effects of varying residual stress distribution and axis of bending on the strength of columns with a constant crookedness.

**BEHAVIOR OF COLUMNS FALING BY INELASTIC INSTABILITY**

Before presenting the method of analysis, a general description of the load-deflection behavior of columns with and without end restraint or load eccentricity is given. The description will facilitate the understanding of some of the results obtained from the analysis. It is assumed that columns always fail by inelastic instability and that for the case of a straight column buckling or bifurcation of equilibrium position occurs after the column has been partially yielded.

An initially crooked and end restrained column of length is L shown in Fig. 1. The crookedness \( v \) varies with the distance \( z \) from the left end. The modulus of elasticity of the material is \( E \), and the moment of inertia of the cross section is \( I \). An eccentric load \( P \) is applied with eccentricities \( e_a \) and \( e_b \), and produces deflection \( w \). The stiffness of the end restraints, which are represented by springs, are \( R_a \) and \( R_b \), and the restraining moments acting at the two ends are \( R_a \theta_a \) and \( R_b \theta_b \), where \( \theta_a \) and \( \theta_b \) are the end rotations produced by the load \( P \) (or the slopes of the deflection curve \( w \)). For the numerical studies to be described later, it has been found convenient to specify \( R_a \) and \( R_b \) in a non-dimensional manner in terms of \( EI/L \) of the column.
Consider first the case where the axial load is concentrically applied, that is, $e_a = e_b = 0$. Several possible load-deflection curves of the column are shown in Fig. 2, where $v_m$ and $w_m$ represent, respectively, the initial crookedness and deflection at the mid-height. If the column is perfectly straight ($v_m = 0$) and without end restraint ($R = 0$), inelastic buckling will take place when the applied load reaches the tangent modulus load of a pinned end column. This load is referred to as the critical load, or $P_{cr}$; beyond this load, the behavior of the column is represented by curve (a) in Fig. 2. The load eventually reaches a maximum value at which the column fails by inelastic instability. If the column has an initial crookedness $v_m$, deflection $w_m$ will take place as soon as the first load is applied and will increase continuously as the load is increased. This behavior is shown as curve (b). Failure of the column is again due to inelastic instability, but occurring at a reduced maximum load. This reduction is dependent on the magnitude of $v_m$. When the column is partially restrained at the ends ($R = 0$), its load-deflection relationship is represented by curve (c) or (d). For sufficiently large values of $R$, the ultimate strength of the crooked column can exceed that of the straight column.

Next, consider the case when the load is applied with an eccentricity $e$, same at both ends, that is, $e_a = e_b = e$ (Fig. 3). The column is initially crooked and unrestrained. Three possible situations exist.
If the eccentricity $e$ and the crookedness $v_m$ are on the opposite sides of the straight line connecting the column ends ($e = -e_1$), there will be a reduction in the column strength due to the added moment $P_{e_1}$. On the other hand, if $e$ and $v_m$ are on the same side ($e = +e_1$), the strength of the column may be enhanced by the moment $P_{e_1}$ which acts in the direction opposite to that of the $P_v$ moment. At a larger eccentricity, for example $e = +e_2 > e_1$, the counteracting moment $P_{e_2}$ may be sufficient to force the column to deflect and eventually fail in the direction opposite to the initial crookedness. This phenomenon has been observed in previous column tests.

**METHOD OF ANALYSIS**

A general method, which can take into account almost all the known factors affecting column behavior, has been developed to perform precise load-deflection analyses of columns. The specific factors that are included in the development are:

1. Residual stresses
2. Initial crookedness
3. End restraints
4. Load eccentricities
5. Variation in mechanical properties of material over cross section
6. Basic stress-strain characteristics of material
7. Loading, unloading and reloading of yielded fibers.

Any pattern of residual stress distribution can be included, and the stresses may vary through the thickness of the component plate. Any variation of initial crookedness along the length of column can be
incorporated. The restraints and load eccentricities may be equal or unequal at the two ends. By allowing variation in the mechanical properties of material over the cross section, it is possible to include hybrid columns or columns with non-uniform strength properties. Any type of stress-strain relationship, including bilinear, trilinear and nonlinear can be incorporated in the analysis for application to columns having different material characteristics. The bilinear and trilinear relationships are commonly used to analyze steel columns with the option of including the effect of strain hardening. A major difference between the method presented in this paper and those employed previously in analyzing initially crooked columns and beam-columns is that it does not use any pre-determined moment-thrust-curvature (M-P-\(\phi\)) curves in the integration process. The basic input is the stress-strain relationship and the necessary M-P-\(\phi\) relationships are generated internally as needed. The stress history of all the elements in a column cross section is carefully followed in the analysis, and any occurrence of unloading or reloading of the yielded elements can be detected and its effect is included in the generation of the M-P-\(\phi\) relationships. The method makes no assumption with regard to the shape of the initial crookedness or of the deflected column under load. The analysis provides the complete load-deflection curve of a column including both the ascending and descending branches.

The ascending branch of the curve is obtained by calculating the deflections for a series of successively increasing loads. An increment of 0.1 \(P_y\), where \(P_y\) is the axial yield load of the column, is used in the initial calculations. At each load, the deflected shape of the column is
determined by an iterative numerical integration procedure, the details of which will be explained later. After a number of load increments, the trial load will eventually exceed the maximum load that the column can sustain. This is indicated by failure of the iterative procedure to produce a deflected configuration which is compatible with the prescribed boundary conditions of the column. When this happens, the calculation is returned to the last trial load for which a compatible deflection has been found. A new set of calculation, starting with this load and using an increment of \(0.01 P_y\), is then carried out. Once again, the successive calculations eventually will show a stable load (below the maximum) and an unstable load (above the maximum), with a difference of \(0.01 P_y\). This means that the stable load is now within \(0.01 P_y\) of the maximum load. The entire process is repeated again with the size of the load increment reduced to \(0.001 P_y\) and eventually to \(0.0001 P_y\). The maximum load can thus be determined to an accuracy of \(0.01\%\) of \(P_y\). However, because of the discretization of the column (to be explained later) and the convergence criteria that have been adopted in the numerical integration, the real accuracy achieved is likely to be somewhat less.

To obtain the load-deflection curve of the column beyond the maximum load, the same method can still be applied except that the analysis must now be performed by increasing the deflection of the column. In actual calculations, however, it has been found to be more convenient to use increments of end rotation. For each selected end rotation, an equilibrium load is found using the same iterative procedure. If it is desired to determine only the maximum load, not the entire load-deflection curve, this part of the analysis can be omitted.
Referring to the column shown in Fig. 1, the equations of equilibrium for any section located at a distance \( z \) from the left end are:

\[
M = P \left( e_a + v + w \right) - R_a \theta_a - V_a z \quad (1)
\]

and

\[
N = P \quad (2)
\]

in which \( M \) and \( N \) are, respectively, the bending moment and axial force acting on the section, and \( V_a \) the reaction at end A. This reaction is given by

\[
V_a = \frac{[P e_a (1-\beta) - (R_a \theta_a - R_b \theta_b)]}{L} \quad (3)
\]

in which \( \beta \) is the eccentricity ratio, \( e_b/e_a \). Equilibrium requires that \( M \) and \( N \) be equal to the internal resisting moment and axial force of the section which can be calculated by integrating the normal stress \( \sigma \) over the cross section (positive for compression). Thus,

\[
M = \int_A \sigma y dA \quad (4)
\]

and

\[
N = \int_A \sigma dA \quad (5)
\]

in which \( y \) is the distance from \( dA \) to the centroidal axis of the cross section.

The stress \( \sigma \) acting on the element \( dA \) is a function of the strain \( \varepsilon \)

\[
\sigma = f(\varepsilon) \quad (6)
\]

\( \varepsilon \) consists of three parts

\[
\varepsilon = \varepsilon^r + \varepsilon^p + \phi y \quad (7)
\]

in which \( \varepsilon^r \) is the residual strain, \( \varepsilon^p \) the strain at the centroid or the axial strain (in the elastic range \( \varepsilon^p = P/\text{AE} \)), and \( \phi \) the curvature.

Equations (1) through (7) are the fundamental equations of the problem, and numerical procedures are usually employed to obtain their solutions.
For a given load $P$, the deflected shape of the column, as defined by the end rotations $\theta_a$ and $\theta_b$, is sought. A value of $\theta_a$ is first assumed (for columns with unsymmetrical restraints both $\theta_a$ and $\theta_a$ must be assumed) and numerical integration is then carried out to determine the deflected shape of the column for the assumed $\theta_a$ (Fig. 4). The procedure adopted is very similar to the one developed previously for analyzing laterally beam-columns (5). The column is divided into many short segments, and the deflection and the slope at the end (nodal point) of each segment are calculated by a numerical integration scheme. Suppose that the calculation has reached nodal point $n-1$, and the deflection $w_{n-1}$, the slope $\theta_{n-1}$, the bending moment $M_{n-1}$, the curvature $\phi_{n-1}$, and the axial strain $\varepsilon^n_{n-1}$ have all been calculated. For the next segment whose length is $\Delta L_n$, the following approximate formulas can be used to calculate the deflection and the slope at point $n$

$$w_n = w_{n-1} + \theta_{n-1} \Delta L_n - \frac{1}{2} \phi_{n-1} \Delta L_n^2$$

and

$$\theta_n = \theta_{n-1} - \phi_{n-1} \Delta L_n$$

in which $\phi_{n-1}$ is the curvature at the mid-point $n-\frac{1}{2}$ of the segment. This curvature, yet to be determined, is a function of the bending moment and the axial force acting at $n-\frac{1}{2}$, which according to Eqs. (1) and (2), are given by

$$M_{n-\frac{1}{2}} = P(e_a + v_{n-\frac{1}{2}} + w_{n-\frac{1}{2}}) - R \theta_a - V Z_{n-\frac{1}{2}}$$

and

$$N_{n-\frac{1}{2}} = P$$

The deflection at $n-\frac{1}{2}$ is
Substituting Eq. (12) into Eq. (10) results in the following expression for $\phi_{n-\frac{1}{2}}$

$$\phi_{n-\frac{1}{2}} = \frac{2(\varepsilon + \nu_{n-\frac{1}{2}} + \varepsilon_{n-1} + \theta_{n-1} - \frac{\Delta L}{2} - \frac{1}{2} \phi_{n-\frac{1}{2}} (\frac{\Delta L}{2})^2)}{(\Delta L)^2}$$

It is apparent from Eqs. (10) through (13) that a direct solution of $\phi_{n-\frac{1}{2}}$ is not possible, and an iterative procedure must therefore be devised. Trial values of $\phi_{n-\frac{1}{2}}$ and $\varepsilon_{n-\frac{1}{2}}$ are first assumed (convenient trial values would be the known $\phi_{n-1}$ and $\varepsilon_{n-1}$ from the already-completed calculations) and the total strain $\varepsilon$ at any point in the cross section is calculated from Eq. (7) and the corresponding stress $\sigma$ from Eq. (6). With $\sigma$ known throughout the section, Eq. (4) can then be used to calculate the bending moment $M_{n-\frac{1}{2}}$. Because of the complex patterns of residual strain distribution present in most of the structural shapes, the required integration of Eq. (4) is best performed numerically by subdividing the cross section into a large number of small elements, and each element is assumed to have a uniform residual strain $\varepsilon^r$ and total strain $\varepsilon$. Figure 5 shows a wide-flange section subdivided into many small elements, each with an area of $\Delta A_j$. The stress $\sigma_j$ acting on each element is again determined from Eq. (6) for the total strain $\varepsilon_j$. Equation (4) now assumes the following form:

$$M = \sum_{j} \sigma_j Y_j \Delta A_j$$

which can be easily applied to calculate the desired bending moment at $M_{n-\frac{1}{2}}$. Substitution of $M_{n-\frac{1}{2}}$ into Eq. (13) gives a new value of $\phi_{n-\frac{1}{2}}$ which is to be compared with the assumed $\phi_{n-\frac{1}{2}}$. If the two values do not agree,
the above process of calculation must be repeated. Satisfactory agreement is obtained if the assumed and the calculated values differ by less than 0.5%. The $\phi_{n-\frac{1}{2}}$ value thus determined satisfies Eq.(10) and (13). Recall that the axial strain $\varepsilon_{n-\frac{1}{2}}^p$ is also assumed at the beginning of the iterative calculation. This strain is related to the axial force $N_{n-\frac{1}{2}}$ which must satisfy Eq.(11). It is, therefore, necessary to check if the stresses $\sigma_j$ associated with the $\phi_{n-\frac{1}{2}}$ just obtained would satisfy Eq.(11). This can be done by substituting the $\sigma_j$ values into Eq.(15) which numerical form is:

$$N = \sum_j \sigma_j \Delta A_j$$  \hspace{2cm} (15)

If the $N_{n-\frac{1}{2}}$ found is not equal to the axial force $P$ as required by Eq.(11), a new $\varepsilon_{n-\frac{1}{2}}^p$ must be selected and the process of calculation is repeated (including the iterative calculation performed previously to obtain $\phi_{n-\frac{1}{2}}$). Satisfactory convergence is reached if the calculated $N_{n-\frac{1}{2}}$ is within 0.00001 $P$ of the axial load. When this occurs, the search for the correct values of $\phi_{n-\frac{1}{2}}$ and $\varepsilon_{n-\frac{1}{2}}^p$ is completed, and the corresponding $\phi_{n-\frac{1}{2}}$ can be substituted into Eqs.(8) and (9) to determine $w_n$ and $\theta_n$ at nodal point $n$. This completes all the required calculation for the segment $\Delta L_n$.

The same calculation is repeated for all the remaining segments.

When the calculation for the last segment is completed, the resulting deflection, if not equal to zero, may show a vertical displacement $w_b$ at end $B$. A non-zero $w_b$ indicates that the $\theta_a$ should be tried. It is convenient to select the new $\theta_a$ to be equal to the initial $\theta_a$ minus $w_b/L$ if $w_b$ is a downward displacement, or $\theta_a$ plus $w_b/L$ if $w_b$ is an upward displacement.

The entire segment-by-segment integration is then repeated for the new $\theta_a$, and, at the end of the calculations, another $w_b$ is found. Using the two $w_b$ values and the corresponding $\theta_a$ values, a third $\theta_a$ can be selected by
linear interpolation. The integration is again repeated. Further repetitions of the process may be required, each time resulting in an improved $\theta_a$. The correct $\theta_a$, therefore, the correct deflected shape of the column, is found if the $w_b/L$ at the end of the calculations is less than $1/1000$ of the assumed $\theta_a$.

It has been mentioned that the portion of the load-deflection curve beyond the maximum load is obtained by using deflection or end rotation increments. In this case, to obtain each point on the curve, an end rotation $\theta_a$ is selected and the interative process is used to find the corresponding equilibrium load. The same numerical integration procedure can be employed to calculate the deflected shape of the column after a trial value of $P$ is assumed. At the end of the integration, the final deflection $w_b$ again may not be equal to zero. An adjustment is now made on $P$ (not on $\theta_a$) and repeated calculations are carried out for a series of successively improved $P$ values. Once again, the correct $P$ is found if the ratio $w_b/L$ is less than $1/1000$ of the selected $\theta_a$.

A comprehensive computer program which can perform all the numerical calculations with the various previously-stated convergence criteria has been prepared. It can be used to analyze columns of any cross sectional shape and with any type of residual stress distribution. The column may be divided into any number of segments, but experience has shown that accurate results can be obtained using as few as four to six segments. In all the calculations performed for this study, the columns are divided into seven equal segments with eight nodal points. The mid-point of the fourth segment coincides with mid-height of the column.

**ANALYTICAL PREDICTION OF TEST COLUMN BEHAVIOR**

The method is first applied to generate the load-deflection curves
of some pinned-end columns which were tested in previous studies conducted at Lehigh University. The purpose of the present work is twofold: (1) to obtain experimental verification of the method of analysis, and (2) to develop analytical predictions for selected columns whose behavior has not heretofore been substantiated by theory. The columns selected had varying amounts of initial crookedness which were carefully measured before testing. Included in the selection are: (1) two concentrically loaded columns, (2) one eccentrically loaded column with small positive eccentricity, and (3) one eccentrically loaded column with large positive eccentricity. The load-deflection behavior of these columns has already been described and is illustrated by curves (b), (c) and (d) in Fig. 3.

The concentrically loaded columns are selected from a group of heavy European columns which were tested as part of the cooperative study with the European Convention for Constructional Steelwork (ECCS). The procedure adopted for these tests followed the ECCS recommendations which require the test column to be "geometrically aligned" with respect to the centerline of the testing machine. The purpose of the alignment is to achieve a concentric loading condition. The test load was applied continuously to the column at a prescribed rate, and the "dynamic" load-deflection curve was recorded automatically as the test progressed. The results of the tests have already been published (10), but no attempt has yet been made to provide theoretical predictions for these test columns. Figure 6 shows comparisons of the analytical and experimental load-deflection curves of the two HEM 340 columns manufactured in Italy. All the analyses are performed using the dynamic stress-strain characteristics determined from the tension coupon tests and the measured
residual stresses and initial crookedness. For each test column, two analyses are made: one includes the effect of strain hardening (dashed line) and the other neglects it (dot-dashed one). Both analyses take into account the effect of elastic unloading and reloading of the yielded fibers. When the two analyses gave essentially the same results, the one that includes the effect of strain hardening is shown. For both columns, the analytical predictions show remarkably good agreement with the test results. For the column with \( L/r = 50 \), the effect of strain hardening becomes quite pronounced after the attainment of the maximum load, and a close prediction of the unloading response can be obtained by including this effect in the analysis. On the other hand, strain hardening appears to have very little effect on the behavior of the column with \( L/r = 95 \) because the two analyses give almost the same load-deflection curve. This study also shows that it is possible to develop the dynamic load-deflection relationship of a test column by using the dynamic mechanical properties if the strain rate in the column test is not too different from the strain rate specified in the coupon test. Further study of this observation is needed.

An example of eccentrically loaded column with small positive eccentricity is shown in Fig. 7. The column is a welded H column with A514 steel flanges and A36 steel web and was included in a pilot program carried out to study the strength of hybrid columns (7). Before testing the column was aligned under load by the so-called "old Lehigh method". The alignment was based on readings from the strain gages which were mounted at the mid-height and at each end of the column. The goal of the alignment is to achieve a reasonably uniform strain distribution
in the column during the early stages of testing. If the column is initially crooked, in order to achieve uniform distribution, the alignment load as well as the test load must be applied with an eccentricity. As illustrated in Fig. 3, this eccentricity would cause an apparent increase of the capacity of the test column. For the selected column, the results given in Fig. 7 show that this increase is from 0.652 $P_y$ for $e/L = 0$ to 0.745 $P_y$ for $e/L = 0.000469$ which is the value of eccentricity adopted in the calculation. This value of $e$ is determined in such a way that in the elastic range the deflection at the mid-height produced by the bending moment $P_e$ along the column offsets completely the deflection produced by the $P_v$ moment. In this calculation, the variation of $v$ is assumed to be sinusoidal although the actual measured variation can also be used. The load-deflection curve calculated with this value of $e$ shows that the column remains essentially straight until the applied load exceeds about 50% of the maximum value. This behavior was also observed during the test. As shown in Fig. 7, the calculated maximum load agrees very closely with experimental load. All the analyses were performed without considering the effect of strain hardening because data on strain hardening characteristics were not available from the original study.

A column, which was among the seven heavy rolled columns that were tested to study the different methods of column testing (11), is selected to examine the behavior of eccentrically load columns with large positive eccentricity. Some basic information about the column is given in Fig. 8. The test piece was not cold straightened after rolling and the column therefore has a larger-than-acceptable initial
crookedness. The same load alignment procedure was also used to align the column. Because of the large crookedness, a correspondingly large load eccentricity was likely involved in the testing. The large eccentricity led to a rapid increase of the bending moment which eventually caused the column to bend in the direction opposite to the crookedness. As shown in Fig. 8, this behavior and the entire load-deflection behavior can be successfully predicted by the analytical procedure. The $e$ value used is determined in the same manner as described above.

The close correlation between the analytical and the experimental results for all the columns selected indicates that the behavior of steel columns can be accurately predicted if data on mechanical property, residual stress and initial crookedness are available. Good prediction for the behavior of columns tested according to the ECCS procedure can be obtained by treating them as concentrically loaded columns. On the other hand, columns tested by the old Lehigh method would behave very much like an eccentrically loaded column with positive eccentricity and should be analyzed as such. The method of analysis developed in this paper can be effectively used for both cases.

**PARAMETRIC STUDY 1**

After the computed program has been thoroughly verified, extensive parametric studies are carried out to investigate the influence of the various factors mentioned in the Introduction on the strength of columns. In these studies, the columns are assumed to be concentrically loaded and the effect of strain hardening is neglected. Two separate studies have been performed. Study 1 examines the influences of initial crookedness and end restraint on the strength of the W8 x 31 column made of A36
steel. Bending is about the minor axis, Study 2 investigates the effects of varying residual stress pattern and magnitude, yield stress of steel, axis of bending and end restraint.

**Influence of Initial Crookedness** Figure 9 shows the non-dimensional maximum strength vs. slenderness ratio curves of the W8 x 31 column without end restraint. The non-dimensional slenderness ratio \( \lambda \) is defined by

\[
\lambda = \frac{1}{\pi} \sqrt{\frac{\sigma_y}{E}} \frac{L}{r}
\]

in which \( \sigma_y \) is the yield stress and \( r \) the radius of gyration about the axis of bending. The usual "Lehigh Pattern" of residual stress distribution is assumed with a maximum compressive stress of 0.3\( \sigma_y \) occurring at the flange tips. The \( v_m/L \) values are 0, 1/1500, 1/1000 and 1/500.

For the case \( v_m/L = 0 \) (perfectly straight column), inelastic instability failure governs the strength of the columns for \( P/P_y \) between 0.7 and 1.0. The initial crookedness has the greatest effect at a \( \lambda \) of 1.2 (see also Fig.14). This has also been observed in the study by Batterman and Johnston (2) in which the contribution of the column web was neglected.

**Influence of End Restraint** The increase in strength due to end restraint has been studied for a series of selected \( R \) values and the results are shown in Fig.10. The initial crookedness is maintained at 0.001. For \( \lambda \) values greater than about 0.75, all the curves are nearly parallel to each other, indicating that the strength increase is approximately constant for a given increase of \( R \). It should, however, be pointed out that \( R \) is defined in terms of \( EI/L \) of the column and represents a relative measure of the stiffness of the restraint. On each curve, the
actual end restraints are different for different values of $\lambda$. The amount of restraint at the ends of a long column is less than that of a short column. To produce the same increase of column strength, the required end restraint is therefore less for the long column.

The results presented in Fig. 10 indicate that the increase in strength due to end restraint becomes smaller as the $R$ values becomes larger. This can also be seen from Fig. 11 in which the strength increases are plotted as a function of $R$ for six selected values of $\lambda$. The increases are expressed as percentages of the capacities of the respective columns with zero end restraint.

Fig. 10 can be used directly to determine the strength of a column with a known end restraint. The procedure which is currently used to analyze restrained columns is an indirect one and makes use of the effective length factor $K$. This factor is usually determined from buckling analysis of an initially straight column and not from maximum strength consideration. The adequacy of this procedure can now be examined, using the curve for $R = 0$ as the reference. For a column with $\lambda = 1.1$, Fig. 10 shows that its strength is increased from $0.542 P_y$ to $0.740 P_y$, when $R$ is increased from zero to $2EI/L$. For the same end restraint, an elastic buckling analysis gives a $K$ factor of 0.776 and the capacity of the column is found to be $0.673 P_y$. Thus, the increase in strength determined by the $K$-factor approach is about 34% less than the exact value. Similar examinations show that the $K$-factor gives lower estimates of column strength for the entire range of $\lambda$ shown in Fig. 10.

**Combined Influence of Crookedness and Restraint** Further calculations have been performed for a variety of crookedness and end restraint combinations to study their influence on column.
Several small values of end restraint have been selected for the initial study. They intend to simulate the restraints that exist at the end of a column which is connected to its adjoining members through flexible connections. Figure 12 shows the selected results of this study for three columns with λ equal to 0.5, 0.9 and 1.5. The R values are 0, 0.1, 0.2, 0.3 and 0.4 EI/L and the v_m/L varies from 0 to 0.002. The results for R = 0 are taken from Fig. 9 and those for R = 0.4 and v_m/L = 0.001 are from Fig. 10. For the column with λ = 0.5, the reduction in strength due to a crookedness of 0.001 can be offset completely by the increase produced by an end restraint equal to 0.38 EI/L. For λ = 0.9, however, the same crookedness causes a much larger reduction which cannot be fully compensated by the effect of a small end restraint. (R equal to or less than 0.4 EI/L).

The results of calculations for a large number of R and v_m/L combinations are presented in two different ways in Figs. 13 and 14. In Fig. 13 the maximum strengths of three columns with λ = 0.5, 0.9 and 1.5 are plotted against the restraining factor R for four different values of v_m/L. In Fig. 14 the reductions in strength due to initial crookedness and given for various columns with λ values between 0.5 and 2.0 and v_m/L values of 1/1500, 1/1000 and 1/500. Each plot is for a constant R. For convenience of comparison, the reduction is expressed as a percentage of the capacity of the respective column if v_m = 0. Several interesting observations may be made from Fig. 14:

(1) For each combination of R and v_m/L, there exists a value of λ at which the strength reduction is maximum.
(2) The $\lambda_p$ value becomes larger as $R$ increases. For the case $v_m/L = 1/1000$, $\lambda_p$ is equal to 1.20 for $R = 0$ and 1.55 for $R = 2.0$ EI/L.

(3) For a given $v_m/L$, the peak reduction becomes smaller as $R$ becomes larger. For $v_m/L = 1/1000$, the peak reduction decreases from 31% to 26% for an increase of $R$ from zero to 2.0 EI/L.

The results shown in Fig. 13 can be used to determine the amount of end restraint required to produce an increase in strength which is exactly equal to the reduction caused by a given initial crookedness. The procedure is illustrated for the case $\lambda = 1.5$ and $v_m/L = 1/1000$. The required end restraint is found to be 0.8 EI/L. If the procedure is repeated for different value of $\lambda$, a curve relating the required $R$ to $\lambda$, as shown in Fig. 15, can be constructed. For $v_m/L = 1/1000$, the maximum required $R$ is 2.45 EI/L occurring at $\lambda = 1.2$.

**PARAMETRIC STUDY 2**

In the second study, the emphasis is on residual stress variation and axis of bending (x and y axes). Research carried out at Lehigh University and elsewhere (9) has shown that the magnitude and distribution of residual stresses in a structural member depend on many factors, the most important of which are manufacturing method (rolled and welded), grade of steel, and size (light and heavy). These factors have been carefully considered in the selection of the three rolled sections and three welded sections included in this study. The residual stress distribution in these sections has been studied in detail in the previous investigation (1,3,6,7,8). The rolled sections are W12 x 50 and W14 x 426,
both made of the old A7 steel, and W8 x 31 of A514 steel. Among the welded sections, one is a H12 x 16 section fabricated from flame-cut (FC) plates of A36 steel, another is a H15 x 290 section fabricated from universal mill (UM) plates of the same material, and the third is a hybrid section, H7 x 21, with A514 flanges and A36 web, all flame cut. The residual stress measurements made previously indicate that among the six selected sections the W8 x 31 has the smallest compressive residual stress and the H15 x 290 has the largest compressive residual stress. Also, in the H15 x 290 section, the pattern of the residual stress distribution is the least favourable as far as column strength is concerned. All the calculations made for this study are based on the measured material properties and residual stresses and a $v_m/L = 0.001$.

The results of the calculations made for the six columns are shown in Fig. 16 for $R = 0$ and in Fig. 17 for $R = 0.2 \, EI/L$. The W8 x 31 column bent about the x axis (curve 1) is the strongest (in relation to its axial yield load, $P_y$) and the H15 x 290 column bent about the y axis (curve 12) is the weakest. For a given $\lambda$, the difference between curves 1 and 12 defines a band width, which represents the range of variation of the strength of all the columns with the same $\lambda$. The largest difference between the curves occurs at a $\lambda$ around 0.9.

A comparison of Figs. 16 and 17 shows that the presence of end restraint tends to reduce the strength difference among the various curves. Between curves 1 and 12, the maximum difference is $0.36 \, P_y$ in Fig. 16 and $0.34 \, P_y$ in Fig. 17. The plot in the lower part of Fig. 17 shows the increase in strength due to end restraint for four selected cases (curves 1, 2, 4 and 12). The increase is the largest for the weakest column, H15 x 290.
Another way of studying the reduction in column strength variation due to end restraint is illustrated in Fig. 18, where the ultimate strength curves of the W8 x 31 column bent about the x axis (curves 1) and the H15 x 290 column bent about the y axis (curves 12) are given. The curves are for three R values: 0, 0.2, and 0.6 EI/L. At a given λ, a band width is found as the difference between curves 1 and 12. For the same λ an average strength, \( P_{ave} \), can be determined as the middle point of the band. The ratio between half of the band width and \( P_{ave} \) represents the maximum deviation in the strength of all the columns from \( P_{ave} \). This ratio, expressed as percent of \( P_{ave} \), is plotted against λ for the three selected R values in the lower part of Fig. 18. For λ values less than 1.3 this ratio has a significant dependence on R. At λ = 0.9, for instance, it is equal to 0.27 (or 27% deviation from \( P_{ave} \)) for R = 0 and equal to 0.19 for R = 0.6 EI/L. Hence, an increase in end restraint produces a reduction in the spread of column strength curves.

**SUMMARY AND CONCLUSIONS**

A general method for analyzing long, intermediate and short steel columns, which fail by in-plane instability, has been presented. The method takes into account the effects of residual stress, initial crookedness, end restraint, and eccentricity of applied load and provides the complete load-deflection curve of a column including both the ascending and descending portions. All the necessary integrations are carried out numerically, and it is possible to include any pattern of residual stress distribution, any variation in crookedness, and any non-decreasing stress-strain relationships. Also, any combination of end restraints and eccentricities may be specified at the ends of the column. Unlike most
of the previously developed methods, the basic input in the present method is the stress-strain relationship, instead of some pre-determined moment-thrust-curvature relationships. The strain history of the individual elements in a cross section is carefully followed, and any unloading or reloading of the yielded elements can be conveniently incorporated into the analysis.

The method has been applied to develop theoretical predictions of the load-deflection relationships of some test columns. It is shown that the columns, which were geometrically aligned and tested according to the ECCS procedure, behave like a concentrically loaded column and the response can be closely predicted using the dynamic stress-strain properties of the material. On the other hand, the best way to predict the behavior of those columns, which were aligned to achieve uniform strain distribution under load, is to treat them as eccentrically loaded columns. The study show that the eccentricity of the test load can cause an apparent increase of the capacity of the columns (Figs. 7 and 8).

Two parametric studies on concentrically loaded columns, which included such variables as magnitude of initial crookedness, amount of end restraint, pattern of residual stress distribution, and axis of bending, have been described. The results of these studies show that

1. For the W8 x 31 columns bent about the y-axis, the strength increase due to a given increase of R is approximately constant for \( \lambda \) greater than about 0.75 (Fig. 10). Also, this increase becomes smaller as R becomes larger.

2. The usual approach of using effective length factor K to account for end restraint gives a lower estimate of the strength of a restrained column with initial crookedness (Fig. 10).
3. For each combination of \( \frac{v_m}{L} \) and \( R \), there exists a \( \lambda \) value at which the strength reduction due to initial crookedness is maximum. This \( \lambda \), referred to as \( \lambda_p \) in Fig. 14, becomes larger as \( R \) increases.

4. The end restraint required to produce an increase in column strength which will completely offset the reduction due to initial crookedness varies considerably with \( \lambda \) and reaches its maximum at \( \lambda_p \) (Fig. 15).

5. The range of variation in column strength due to variation in residual stress distribution is less in restrained columns than in pinned end columns (Fig. 18).

The method presented in this paper has already been extended to beam-columns with initial crookedness. This work will be described in a future report.
APPENDIX I - REFERENCES


APPENDIX II - NOTATION

The following symbols are used in this paper:

- $A$ = cross-sectional area;
- $E$ = modulus of elasticity;
- $e$ = eccentricity;
- $I$ = moment of inertia;
- $K$ = effective length factor;
- $L$ = length of column;
- $M$ = bending moment;
- $N$ = normal force at section;
- $P$ = axial load;
- $R$ = rotational stiffness of end restraint;
- $r$ = radius of gyration;
- $V$ = end reaction;
- $v$ = initial crookedness of column;
- $w$ = deflection of column;
- $y$ = distance from centroidal axis;
- $z$ = distance from end A;
- $\beta$ = ratio of eccentricity at end A to that at end B;
- $\varepsilon$ = strain;
- $\theta$ = slope of deflected column;
- $\lambda$ = non-dimensional slenderness ratio
- $\sigma$ = stress; and
- $\phi$ = curvature
Subscripts

\begin{align*}
a &= \text{end A}; \\
b &= \text{end B}; \\
cr &= \text{critical}; \\
m &= \text{mid-height}; \\
y &= \text{yield}; \\
j &= j\text{th element}; \text{ and} \\
n &= n\text{th segment}.
\end{align*}

Superscripts

\begin{align*}
r &= \text{residual}; \text{ and} \\
p &= \text{peak of curve or quantity related to P}.
\end{align*}
Fig. 1 Initially Crooked, End Restrained Column
Initially Straight Column

End Restraint = $R_2$

No End Restraint

$R_2 > R_1$

Fig. 2 Load-Deflection Behavior of Concentrically Loaded Column
Fig. 3 Load-Deflection Behavior of Eccentrically Loaded Column
Fig. 4 Segment of a Deflected Column

Fig. 5 Subdivision of Cross Section
Fig. 6 Analytical and Experimental Load-Deflection Curves of Concentrically Loaded Columns (1 kip = 4.45 kN, 1 in. = 25.4 mm)
Fig. 7 Analytical and Experimental Load-Deflection Curves of Eccentrically Loaded Hybrid Column with Small Eccentricity
W 12 x 161 y axis
A 36
L/r = 50
$\nu_m/L = -0.00151$
Bending Opposite to Direction of Initial Crookedness

Fig. 8 Analytical and Experimental Load-Deflection Curves of Eccentrically Loaded Column with Large Eccentricity
Fig. 9 Influence of Initial Crookedness on Maximum Strength of Columns
Fig. 10 Influence of End Restraint on Maximum Strength of Column
Fig. 11 Increase in Column Strength Due to End Restraint

\[ \frac{P_R - P_{R=0}}{P_{R=0}} \% \]

W8 x 31 y Axis
A36
\( v_m/L = 0.001 \)
Fig. 12 Combined Influence of Initial Crookedness and End Restraint (Small End Restraint Cases)
Fig. 13 Combined Influence of Initial Crookedness and End Restraint
Fig. 14 Reduction in Column Strength Due to Initial Crookedness

Fig. 15 End Restraint Required to Offset Effect of Initial Crookedness
Fig. 16 Maximum Strength of Six Selected Columns, No End Restraint
Fig. 17 Maximum Strength of Six Selected Columns with End Restraint
Fig. 18 End Restraint and Reduction of Spread of Column Curves