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Robert T. Reese
Celal N. Kostem
A METHOD TO DETERMINE THE SENSITIVITY OF MATHEMATICAL MODELS IN DETERMINISTIC STRUCTURAL DYNAMICS

by

Robert T. Reese
Celal N. Kostem

Fritz Engineering Laboratory
Department of Civil Engineering
Lehigh University
Bethlehem, Pennsylvania

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ABSTRACT

This report presents a general method for determining the sensitivity of mathematical models describing deterministic structural dynamics. The term sensitivity is defined as the degree to which an individual parameter (mass, stiffness, damping, or forcing function) affects the response behavior of a mathematical model describing a structural system. The proposed method consists of carefully and critically comparing selected responses of multiple configurations representing a particular system. The comparative techniques used include superimposed plots of responses, minimums and maximums of motion and stress resultants and superimposed shock spectra.

The multiple configurations needed can be developed from possible limiting or intermediate values for predetermined parameters. These values may indicate a load-displacement relationship, a geometrical constraint, or possible design alternative. These values are formulated into multiple configurations capable of describing both linear and nonlinear structural behavior.

The emphasis in this study is on the early-time (shock or transient) response of structures. The governing equations of motion are integrated with a Newmark technique with $\gamma$ and $\beta$ equal to zero. Nonlinear structural behavior is often encountered in transient response.
problems as contrasted to usual linear vibrational behavior. Virtually any linear or non-linear load-displacement relationship describing structural behavior can be treated. The non-linear behavior is treated as a pseudo-forcing function in the integration process.

The structures considered herein are modeled as discrete mass systems with one or two degrees of freedom per mass using small-deflection theory. Types of structural systems successfully modeled with this approach include bridges, buildings, automobiles, aircraft, and other flight structures.

The methods currently available to study, analyze, improve, and optimize models are examined in detail for their applicability to the treatment of transient response of structures. None of these methods (perturbation, optimization, identification, boundary value solutions, sensitivity functions, or incomplete models using modal analysis) can be used for or adapted to a general approach for the sensitivity analysis of models. Two basic reasons are involved: (1) the methods do not give satisfactory treatment of non-linear behavior and (2) the computational effort needed for solution and convergence can be very large and would have to be repeated for additional loading environments.

The application of the proposed method of multiple configuration analysis is contained in three detailed example problems.
1. INTRODUCTION

This report presents a general method for determining the sensitivity of a structure subjected to modifications made in the mathematical model of the system. The term sensitivity is defined as the degree to which an individual model parameter (stiffness, damping, forcing function, or inertial property) influences the dynamic response behavior of a structural system. The results of the sensitivity study are directly applicable to the formulation of a model. The sensitivity analysis identifies the parameters having a controlling influence on the dynamic response of a particular system.

If needed, additional analytical and experimental effort can be concentrated on these controlling parameters to obtain as exact a description as possible. A subsequently formulated model will predict dynamic response as accurately as the information available for model development allows. This sensitivity analysis also indicates the range of applicability for a particular model to describe the transient response of a particular system.

The structures considered herein are those capable of being described in the form of mathematical models consisting of discrete-mass systems with one or two degrees of freedom per mass. A resulting model is developed with small-deflection theory and may include material
non-linearities, geometrical constraints and gaps, and rigid body motion. The emphasis in this study is on the shock or transient response of structural systems. A deterministic approach is used in which the material and geometric properties are known or can be approximated explicitly rather than statistically.

1.1 Problem Statement

The need exists in structural engineering to develop a general method to determine the sensitivity of models used to describe dynamical behavior. No general approach is available for the analysis of dynamic response of structural systems with non-linearities and approximate descriptions\(^{(17)}\). Therefore, this dissertation addresses the problem of a general approach to determine the sensitivity of mathematical models describing shock or transient response. In particular, the numerical treatment of the problem will be emphasized because there are only isolated closed-form solutions available that can be applied to the types of problems treated in this study\(^{(4, 27)}\).

Dynamic response is characterized by a multiplicity of effects that is not found in static behavior of structures. This multiplicity of effects is encountered in two ways: (1) the responses are functions of time, and (2) the usual nature of dynamic response is to amplify the effects of an applied forcing function.

In the development of models for structural systems, many assumptions and approximations are needed to define an initial model. The responses obtained from this original model should be viewed as an
indicator of response rather than as an absolute predictor of behavior. There is a distinct tendency on the part of a structural engineer to place excessive trust in a model that is approximately correct.

A general method to determine the sensitivity of structural systems is complicated by the approximations and assumptions needed to formulate a model and the subsequent multiplicity of effects encountered in dynamic response.

1.2 Object and Scope

The objective of this study is the development of a general method to determine the sensitivity of mathematical models describing transient response of structures. Included in this development is the treatment of linear and non-linear structures.

The scope of the study includes a detailed discussion of mathematical modeling techniques as applied to structural dynamics. The goals, uses, and applicability of models, the governing equations, and the problems associated with the formulation and use of models are given. Guidelines for the development of models for transient response are also presented. An evaluation and assessment of methods developed to analyze, improve, and optimize mathematical models is included. The general method based on a multiple configuration analysis\(^{(44)}\) is developed. The method consists of obtaining solutions of different configurations of a model describing a particular system. Selected responses are carefully and critically compared to determine the sensitivity of the models to prescribed sets of modifications.
Detailed example problems are presented demonstrating the applicability of the proposed method.
2. THE ANALYTICAL ART OF MODELING

Methods of structural mechanics have been developed to describe the behavior of structures under the action of applied loadings. The description of the structural behavior is usually expressed in terms of equations or mathematical models. The goal of structural analysis is the development of a model which accurately predicts the response behavior of structural systems.

Models and their subsequent responses can be of use in the following ways:

(1) An initial understanding of the behavior of a particular system can be determined.

(2) The stresses and accelerations given by a model can indicate the extent of instrumentation needed and the corresponding calibration levels for experimental evaluation of a system.

(3) The model behavior can be used in conjunction with test results so that proper interpretation of available data can be made and adjustments in model parameters can be accomplished.

(4) A model based on the experience cited above can be applied to the solution of other similar problems.
This chapter describes the art of modeling from its basic foundations, including the governing equations, the response behavior expected, and the necessary guidelines for the formulation of models which predict the transient response of structures. The problems associated with the use of models, additional needs, and requirements for models of transient response are given.

2.1 The Overall Approach

Structural systems can be represented by the following diagram (35).

\[
\text{INPUT, EXCITATION, FORCING FUNCTION} \rightarrow \text{Defined System with properties and equations is called the "CONSTITUTIVE EQUATION"} \rightarrow \text{OUTPUT, BEHAVIOR, RESPONSE} \quad (2.1)
\]

The goal of mathematical modeling of structures is the accurate description of all parts of Eq. 2.1. With a completely general system definition or constitutive equation, the response of a structural system would be known for any and all forcing functions. However, a completely general constitutive equation would require the adequate identification and description of all conceivable details of all possible inputs. Simplifying assumptions must be made in the formulation of models because of the complexities in developing a general constitutive relationship. Even if a completely general system definition could be obtained through a continuum approach, the resulting equations would be so complicated that their solution would not be tractable (17).
The equations describing structural behavior would be partial differential equations. Since these equations would require a numerical solution for most practical or engineering problems, an alternate formulation of the model is usually made which simplifies the basic relationship, Eq. 2.1, into a solvable form.\(^{(78)}\)

The response of structures to dynamic loadings can be treated deterministically or in a stochastic manner. A stochastic process is controlled by probabilistic laws. The choice of the treatment should be consistent with the overall goal of developing an accurate model for various structural systems. Certainly, there is a degree of randomness in every structural system. It will be shown in Section 3.2 that the probabilistic (stochastic) approach cannot accept significant modifications. For this reason and the fact that most analyses ultimately require known conditions, a deterministic approach is chosen for the sensitivity analysis.

The usual simplifying assumption made in dynamics of continuous systems is the replacement of the continuum by a series of discrete masses connected by springs.\(^{(63,74)}\) The motion of the masses and springs represents the response behavior of the real structure. The masses describe the inertial characteristics (translatory and rotatory), and the springs describe the load-deflection relationships (membrane, bending, and torsion).

The three basic methods of analysis for treating the response of structures due to dynamic loadings are the wave propagation method
(closed-form solution)\(^{(56)}\), the modal superposition method\(^{(84)}\), and
direct numerical integration of the equations of motion\(^{(20,51)}\). The
wave propagation method can handle only a few reflections and
refractions before becoming unwieldy. The modal superposition method
is primarily used for linear elastic structures with known modes of
vibration\(^{(17)}\). The most general approach to the method of analysis for
mathematical models is the direct integration of the equations of
motion\(^{(18,20)}\). A distinct advantage of this approach is that both
geometrical and material non-linearities can be treated\(^{(28)}\).

The information derived from a model should predict the maxi-
mums and minimums of acceleration, velocity, displacement, and stress
resultants for each mass and spring of a given structural system. The
model should also predict the instantaneous shape and amplitude for
the various response-time histories for a given structure. In order to
fulfill these expectations of information desired from a model, it is
necessary to give as much latitude and flexibility as possible in
defining structural systems and forcing functions. This latitude and
flexibility in definition of models may include material non-linearities,
geometric constraints, and various types of forcing functions, such as
impact, blast, and moving loads. A priori knowledge of response
behavior of a structural system is not available nor can it be estimated.
Therefore, a definition of a model which is concise and as complete as
practicable is needed. Then, a model will predict the response behavior
of a given system as accurately as possible based on available infor-
mation.
In summary, a basic goal in mathematical modeling is to investigate the general cause and effect relationship and, specifically the response of structures due to dynamic loadings. Many possible alternatives are needed to formulate a model which accurately predicts the response behavior of a structural system. A deterministic approach has been chosen because significant approximations for structural behavior can be treated. The equations of motion that govern the response of structural systems to dynamic loadings have been discussed with regard to solution techniques. The solution techniques have a considerable influence over the types of models that can be developed. The model describing all parts of a structural system is the basic constitutive relation, Eq. 2.1, in a simplified, solvable form.

2.2 The Governing Equations of Motion

From Newton's second law the governing equations of motion for a structure modeled as a series of lumped (discrete) masses connected by springs are given by (78):

\[ [m][\ddot{x}] + [c^*][\dot{x}] + [k][x] = \{F(t)\} \]  

(2.2)

where

\[ [m] \quad = \text{mass matrix} \]

\[ [c^*] \quad = \text{damping matrix} \]

\[ [k] \quad = \text{stiffness matrix} \]

\[ \{F(t)\} \quad = \text{forcing function} \]

\[ \dot{\cdot} \quad = \text{derivative with respect to time} \]

\[ \{x\} \quad = \text{displacement vector} \]
Equation 2.2 is written for one degree of freedom (translation) per mass definition. Both overall and localized behavior of a structure can be investigated and predicted using a solution technique for Eq. 2.2. Examples of structural systems modeled in this manner include beams (63), buildings (8,48), bridges (45,75), automobiles (43), and missile and flight structures (14,19). To examine models of structural systems cited, it is necessary to include all the appropriate mass and inertial characteristics and stiffnesses.

The solution to Eq. 2.2 depends upon the structural system and forcing function (24). It may be considered to be of the form:

\[ x(t) = \text{homogeneous and particular solutions of} \]
\[ \text{Eq. 2.2} \]
\[ = A e^{-\lambda t}[\cos \omega t + \sin \omega t] \quad (2.3) \]

where

\[ A = \text{a constant} \]
\[ \lambda = \text{a constant depending upon the type of forcing function and damping} \]
\[ \omega = \text{a constant depending on the characteristics of the structural system} \]

There are two parts to the solution: the first part (particular solution) is the transient response, and the second part (homogeneous solution) is the longer time or steady state response (vibration) (21,69). The emphasis in this study is in the transient response domain.
2.3 Vibration and Shock Response in the General Formulation of a Model

The first step in the formulation of a model is the determination of the types of possible response behaviors for a structural system and loading condition\(^{(8,10,13,48,50,59,61,67,81)}\). In dynamics there are two types of response which correspond to the solution represented by Eq. 2.3. These responses are shock and vibration behavior. The shock response eventually decays to a vibratory behavior with sufficient cycles of motion. The vibration response for a real structure will decay with damping unless the motion continues to be forced. It is important to separate shock from vibration because a different solution technique is utilized\(^{(51,78)}\).

Vibrations, even forced response, are essentially linear in behavior. Some non-linearities may be excited, but the usual importance of vibrations is in the linear regime of structural behavior. Thus, the modal superposition method of solution would be applicable\(^{(84)}\). Modal superposition (normal mode method) may not be easily applied to the solution of problems containing complicated forcing functions such as gust, blast, or earthquake behavior. The normal mode approach requires a Fourier series approximation of the excitation. Significant errors can occur in this series approximation because the number of terms of the series is truncated to equal the number of degrees of freedom of the system. Therefore, the response behavior for a structure may be appreciably different depending on the series approximation and model used. In contrast, the integration of the equation of motion supplies all of the excitation to a structure\(^{(19,24,74,78,84)}\).
In forced vibrations with frequencies approaching resonance, the structure tends to exercise any non-linearities that are present. A short duration or transient response usually accompanies an alteration in the structure such as an appearance of a non-linearity. A thorough treatment of the transient response is necessary for proper understanding of the behavior of a structural system because of the variety of possible loading environments. The analyst should be concerned with both types of response, with the importance of vibrations not being minimized and without placing excessive emphasis on shock response. Many methods exist for the solution of vibration problems, and there is much information available to aid the analyst and designer. The analysis of transient response is not as well understood as vibrations, especially in terms of solution techniques. The description of the structure and the forcing function is more difficult to obtain in shock-loaded structures. Presence of geometric or material non-linearities complicate the definition of the structural system. In certain types of structures, such as bridges and buildings, subjected to shock loadings, the analytical solution to the design problem is to perform a static solution and add factors which account for dynamic loadings. These factors, depending upon the type of forcing functions and structural characteristics (including weight and geometry), form the basis for the design of structures for wind, earthquake, and vehicular loadings. The assumptions developed for these factors have a good degree of validity for certain environments in which the
acceleration (inertial or body) forces are not significant. However, the dynamic response due to time dependent loadings is not recognized by these equivalent methods of static analysis. The major interest in structural analysis is the determination of maximum stresses and displacements. Neither of these quantities can be determined in actuality by an equivalent static approach. In the study of other types of structures exhibiting dynamic response (aircraft, missiles, automobiles, reentry bodies, and in earthquake phenomena) the acceleration forces can be significant, and any complexities in structural definition become very important in the accurate determination of structural response. Damping does affect the response, but its contribution in the analysis of transient response is mainly the reduction of the secondary peaks of motion\(^{(49)}\). The effect of damping is minimal when compared with the other approximations which have been made to develop a model\(^{(9,20,35)}\).

There have been many methods of solution developed for the analysis of dynamic response of structural systems\(^{(26,74,78,84)}\). Most problems can be solved with the two methods presented (modal analysis and integration of the equations of motion). It is important for the structural engineer to concentrate on these two methods in order to understand dynamic response. The emphasis in both methods is on the mathematical model. Therefore, the analyst should become well informed of the aspects of modeling.
2.4 Guidelines for the Development of Mathematical Models for Structural Dynamics

Guidelines may now be established for the formulation and development of a model. It must be emphasized that the formulation of a model can be done only after the type of structure is considered, the pertinent characteristics of possible loadings are anticipated, and the subsequent response to the forcing functions is determined (18, 78). The following guidelines have been developed for the analysis of transient response of those structures modeled as lumped mass systems:

1. Determine the possible flow of forces (load paths) that may exist within a structural system.
   a. Note any portions of the structure which may require a complex analysis, such as gaps and non-linearities.
   b. Note any joints or connections.
   c. Note any other parts of the structure which may have an unconventional definition.

2. Determine the known points where information (motion or stress resultants) is desired.

3. Establish the types of forcing functions which will excite the structural system.

4. Determine the mass discretization which is consistent with items 1, 2, and 3.
   a. Masses should be concentrated at known mass points—joints, flanges, panel points in trusses,
and components such as guidance packages and automobile engines.

(b) Any point where information is desired should have a mass definition if motion is of interest.

(c) The various types of forcing functions—blast, impulse, moving loadings—may require a specified discretization so that the forcing function may be properly applied to the structural system.

(d) The mass discretization should be done to minimize the differences between any one mass-spring set because of the influence that sizable differences make on the integration step size (20).

(5) Determine the governing material properties, geometrical configuration, and constraints. A thorough stress and structural analysis must then be done to adequately define the stiffnesses and load-displacement relationships.

(6) Assign damping to each spring to make the model as realistic as possible. An accepted number is three to five percent viscous damping for the majority of structural systems (9, 20, 35, 51).

2.5 Problems Associated with the Use and Formulation of Models

There are three basic problems associated with mathematical models in transient dynamics:
(1) The first problem is the possibility of systematic errors which may occur in any modeling procedure. The only way to overcome any systematic errors is to obtain an independent check on a particular model or to critically examine all assumptions involved in the formulation of a model.

(2) The second problem is the evaluation of the parameters which govern the response of the real (modeled) structure. Regardless of the modeling technique, a real structure will respond according to basic physical laws, and not necessarily to the analyst's interpretation of the relationships describing those laws. The proper determination of the load paths, the stiffnesses, and geometric constraints is not necessarily a simple task but is imperative for the development of an accurate model (22).

(3) The third problem is the determination of the parameters which have a controlling effect on the response behavior of a particular system. This determination is the sensitivity analysis. With the sensitivity determined, analytical and experimental efforts to obtain a more accurate model can be directed. The sensitivity analysis indicates the applicability of a particular model to predict responses of a system.
2.6 Complex Analysis Problems Including Non-linearities

Non-linearities have been mentioned in Chapter 1. The importance of non-linearities and other complexities in analysis is in their controlling influence on the development of a properly formulated model\(^\text{(7,8,71,75)}\). In the formulation of a model, certain structural systems or components are predictable in their behavior to dynamic loadings\(^\text{(14,25,26)}\). Some other parts of structural systems are less predictable analytically\(^\text{(59,67,81)}\). Examples include impact, misfits, joints, and effects of viscoelasticity, plasticity, and hysteresis.

The analyst learns to concentrate his efforts on those less predictable parts of the structure to formulate an accurate model. Even with a detailed analysis technique such as a finite element method or actual testing of parts or all of a structural system, only approximate load-displacement relationships can be determined for complex structures which may contain non-linearities. As often occurs, static tests do not accurately predict the behavior in the transient response regime because of material property differences. In terms of the constitutive equation, Eq. 2.1, the major difficulty in the correct formulation of a model rests with a correct determination of the stiffnesses (springs) involved.

2.7 Parametric Studies

There is a need for parametric studies in the development and formulation of models of structural systems subjected to dynamic loadings. A parametric study may involve the effect that modifications in
parameters have on dynamical behavior \( (23, 25, 53, 75) \). Efforts to correlate test results and model behavior may be viewed as a parametric study because adjustments to model parameters are made so that resulting responses agree with experimental data.

Many possible configurations and alternatives are encountered in the design of a structural system. Many of the responses resulting from dynamic loadings for these alternative design configurations can be assessed using a parametric study. The proposed method of sensitivity analysis may be considered a parametric study with the addition of the needed comparative procedures.

2.8 Summary

The art of modeling is the development of a series of simplifying assumptions and judgments for the formulation of mathematical descriptions of structural systems responding to dynamic loadings. The general approach as presented is the application of the general constitutive equation, 2.1, to the analysis of structural behavior. The problems, uses, and applicability of models have been presented.
3. EXAMINATION OF METHODS DEVELOPED TO STUDY, ANALYZE, AND OPTIMIZE MATHEMATICAL MODELS

The methods contained in this chapter do not apply to the formulation of a general approach to determine sensitivity of structural systems. While certain conclusions, observations and equations presented in this chapter will be used in the development of a general approach, the material is not essential to the method presented in Chapter 4. A chart summary presenting a detailed study and analysis methods is presented in Appendix B.

The purpose of this chapter is to examine the various methods used to formulate more accurate and representative models for structural dynamics. The following are the seven basic approaches available as determined from a literature survey:

(1) Solution of non-linear equations
(2) Perturbation methods
(3) Optimization solution techniques
(4) Identification methods
(5) Boundary value solutions
(6) Sensitivity functions
(7) Theory of incomplete models using modal analysis

There is some overlapping among the approaches; furthermore, a particular
method may have subsets. The basic development of each method will be
delineated. The advantages and disadvantages of each particular method
with respect to its applicability to the improvements of models for
transient dynamics will also be presented.

As an initial prerequisite to the formulation of models for
transient dynamics is the proper treatment of non-linear behavior. The
understanding of non-linearities is an important issue in the analysis
of transient response of structures\(^{(41)}\).

3.1 Solution of Non-linear Equations

In the determination of the response of structures due to
dynamic loadings, the methods currently used are primarily limited to
linear elastic structures\(^{(6,15,26,61)}\). One of the most important
aspects of the dynamic response of structures is the determination of
structural behavior for a complete set of loadings, including failure
conditions\(^{(12,81)}\). Most structures behave non-linearly to failure,
particularly those in the dynamic response domain\(^{(39,45)}\). Inclusion of
both types of response (linear and non-linear) facilitates the assessment
of the limits of usefulness of a given system. Non-linearities should
be treated because then models are capable of describing the widest
variety of possible structural responses. Therefore, any method used
to determine the dynamic response of structures should include both
linear and non-linear behavior, particularly to determine the transient
response of structures.
Non-linear behavior may be thought of in a physical sense in the behavior of structures or in a mathematical sense in terms of definition of the governing equations. In the physical behavior of structural systems, examples of non-linearities were given in Sections 2.3 and 2.6. In the equations of motion, the non-linear behavior is given by a stiffness matrix which is a function of displacement $[k(x)]$. This is expressed as follows:

$$[m][x] + [c^*][x] + [k(x)][x] = [F(t)] \quad (3.1)$$

A new solution to the equations of motion must be obtained with a change in $[k(x)]$. (A new solution would be required for any other coefficient of Eq. 3.1 which is subject to modification.)

The observation is made that the general expression for structural systems is a non-linear rather than linear system of equations (17). Equation 3.1 is a non-linear equation by definition because the stiffness matrix $[k(x)]$ is a function of the independent variable $x$.

Bauer (5) states that there exists no satisfactory analytical solution to non-linear equations. However, Gabrielson (18), Kavanaugh (28,29), and Stricklin, et al. (70,71) have developed an efficient solution procedure for non-linear equations of motion. This method of solution can be expressed by rewriting the equations of motion in the following manner:

$$[m][x] + [c^*][x] + [k][x] = [F(t)] + [k_n][x] \quad (3.2)$$
The only new term in Eq. 3.2 is the \([k_n^J]\) which is the non-linear part of the stiffness matrix. \([k_n^J]\) is placed on the right hand side of the equation and is considered to be an additional forcing function. To determine the dynamic response, the equations are solved numerically. It is also possible to analyze static non-linear problems by critically damping the normal mode solution. The treatment of non-linearities described by Eq. 3.2 has been successfully implemented to the solution of both static and dynamic problems\(^{(18,29)}\).

Other solutions have also been obtained for the solution to non-linear equations. These methods usually involve different forms of matrix operations. One method is to obtain an additional inverse for each change in the stiffness matrix. Another approach is to use an iterative technique with the original solution to the equations\(^{(28,70)}\). Either of these two basic approaches requires significant storage requirements for a computer; whereas, with the use of Eq. 3.2, the problem of storage requirements is greatly reduced.

In summary, the solution of non-linear equations is necessary to determine the behavior of a structure over its range of intended use and service. Furthermore, the evaluation of solution techniques as reported in the literature shows that the treatment of a non-linearity as an additional forcing function is accurate and the most efficient method yet developed\(^{(29,70,71)}\). This method is used for the solution of non-linear equations described herein.
3.2 Perturbation Methods

Methods of improvement of mathematical models had much of their original formulation in terms of perturbation methods (44, 68). An improved model may be viewed as the originally developed model with sufficient perturbations and alterations. Modifications to structural behavior can be easily viewed as perturbed solutions to original description (80).

Many methods exist for the mathematical formulation of the perturbation method. The formulation most useful for dynamic response will be developed herein. The approach is to examine the characteristic equation of the stiffness matrix describing the structural system. The characteristic roots (eigenvalues) $\lambda_1$ of the original model can be determined from Eq. 2.1 by the following relationship:

$$[k]\{x\} = \lambda^2 [m]\{x\} \quad (3.3)$$

$\{\ddot{x}\}$ has been transformed because of the harmonic nature of the expected response, i.e., $\{\ddot{x}\} = \lambda^2 \{x\} (16)$. The perturbed characteristic roots may be written as

$$\lambda_1 = \lambda_0 + \epsilon \lambda_1 + \epsilon^2 \lambda_2 + \epsilon^3 \lambda_3 + \ldots \quad (3.4)$$

The above expression is a power series expansion for a root of Eq. 3.3 modified to include the effects of perturbations to the stiffness matrix.

Once the characteristic values and vectors (frequencies and mode shapes) have been determined, the dynamic response can be investigated (26).
The displacements, stress resultants, and stresses can be evaluated for the original and perturbed models.

The perturbation method can also be applied to a probabilistic approach as in the analysis of tolerance differences associated with mass-produced items, such as automobiles. Whereas, the other methods presented in this chapter are primarily applicable to deterministic problems. There is a degree of randomness involved in the assembly of each automobile and this randomness can have significant effect on the response behavior in dynamic response situations (44).

The randomness associated with a given structural system can be expressed as a modification to the stiffness matrix of Eq. 2.2. This modification can be considered to be of the form

\[ [k_{\text{ran}}] = [k + \epsilon] \]  

(3.5)

where

\[ \epsilon = \text{random addition to the stiffness matrix for a given system} \]

The solution to Eq. 3.5 is obtained in a similar manner to Eq. 3.4 and the subsequent manipulations involved to obtain useable results for acceleration and stress resultants. Inherent in Eqs. 3.4 and 3.5 is the requirement that any modifications be small percentage kinds of changes (34,54). This approach needs the evaluation of an excessive number of terms of the series expansion, Eq. 3.4, to converge to a solution for a given structural system in which significant modifications have been made.

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3.3 Optimization Approach

The optimization approach parallels the perturbation method. The basic difference between the two techniques is the interpretation and treatment of the modifications. The modifications may include any of the parameters used to describe the structural system. In both methods, the effects of changes are examined through similar solution techniques for characteristic values \(^{(16)}\). The optimization procedure can react to modifications and develop alternate structural configurations \(^{(85)}\). The reaction of this approach is based on constraints and error criterion. The perturbation method cannot react as such.

The optimization approach consists of the formulation of a set of constraints and a function to be optimized. This function is a mathematical description of a particular structural system such that weights, stresses, frequencies, or other parameters may be formulated in an optimization procedure. In the optimization process a developed function is "driven" toward a most favorable state based on the constraints imposed.

Various methods (steepst descent, conjugate gradient) have been developed to process the function to the optimum state \(^{(38,55,62,66)}\). The function may consist of more than one parameter and the constraints may also be formulated with multiple conditions.

The optimization method has not been applied to transient analysis with non-linear differential equations \(^{(30)}\). The original intent of this study was to apply some type of optimization technique to the
development of accurate models describing transient response. The optimization approach will not be used in this study because of the following reasons:

1. The optimization procedure requires a function and constraints. It is not always possible to supply these constraints and to determine what parameters in a model to optimize.

2. The computational effort will be large. The necessity of performing many iterations per time step to have responses conform to constraints requires considerable computational effort even for small systems.

3. There is no guarantee that a model optimized for one set of forcing functions will be applicable to other loading environments. Therefore, the optimization would need to be repeated as a general procedure for each forcing function.

3.4 Identification Methods

The method of identification is primarily used in conjunction with test results. This method is similar to the optimization method in Section 3.3. One of the purposes of testing is to verify the methods of analysis used to predict structural behavior. With adequate test data, it is possible to develop a mathematical model describing the test configuration.
Although this procedure appears simple enough, there are three basic problems that make this method difficult to apply: (1) sufficient and usable test data are not always available, (2) small modifications in a model may make drastic changes in the response behaviors (this change phenomenon is difficult to treat analytically), and (3) acceleration-time histories cannot be predicted accurately because it is the lower modes of a structure that are examined (23).

This method of identification is developed with the use of an error criterion. The criterion is used in conjunction with an expression which describes the combined behavior of the model and test results. This can be expressed as follows:

\[
E = \sum_{i}^{N_i} [\omega_{ie} - \omega_{it}]^2
\]

where
\[
E = \text{error criterion} \\
N_i = \text{number of modal shapes considered} \\
\omega_{ie} = \text{ith experimental eigenvalue} \\
\omega_{it} = \text{ith theoretical eigenvalue}
\]

Implied in Eq. 3.6 is the fact that the frequencies of the model are adjusted to minimize the error criterion. When the frequencies are modified, the springs and/or masses are required to be changed, i.e.,
\[
\omega = \sqrt{k/m}.
\]

The error criterion, Eq. 3.6, is minimized to produce a model which is correctly formulated for frequency. To this point, this method does not appear different from the optimization approach.
However, there are basic differences which are inherent in the treatment of Eq. 3.6. These differences will be explained below.

With additions to Eq. 3.6, Hall, et. al.\(^{(23)}\) use a weighting function and a polynomial expansion of the error criterion to improve the Eq. 3.6 for a minimization technique.

Lion\(^{(40)}\) and Mahalingham\(^{(41)}\) develop a different formulation. The model parameters are adjusted according to the steepest descent law which is a functional derivative.

\[
\alpha_i = -k \left( \frac{\partial F}{\partial \alpha_i} \right)
\]  

(3.7)

where

\(k = \) a small constant

\(F = \) performance (error) criterion which is a functional of \(\alpha_i\)

\(\alpha_i, \alpha'_i = \) model parameters

The approach by Lion has been extended to treat a certain class of non-linearities. However, non-linearities must be treated with a described set of predetermined coefficients. This means that structural non-linearities could not be treated without an a priori knowledge of their behavior. A separate programming effort would be required for each non-linearity and for each forcing function used.

Marchesini and Picci\(^{(42)}\) use an approach similar to Eq. 3.6. With the addition of series expansion to account for as many modes of vibration as possible, their approach is like that of Aleksandroskii\(^{(2)}\),
but it develops response kernels \( k_i \) which are computable functions from series manipulations, i.e., \( k_i(t, \omega_1, \omega_2, \omega_3, \ldots \omega_i) \). These kernels can describe both linear and non-linear behavior through series expansions.

3.5 Boundary Value Solutions

Optimal control theory is concerned with the development and design of systems to bounds and constraints (11). In an effort to develop an adequate method for solutions in optimal control theory, the two-point boundary value problem technique was developed (3). Basically, a two-point boundary value solution method consists of a set of initial and final (terminal) conditions specified for a system of equations. These equations are integrated over the time interval and checked against the values specified at the terminal point. The process is repeated until the integrated equations match the initially specified terminal conditions within an error bound (77).

A two-point boundary value problem for a time interval \((t_0, t_f)\) can be stated as

\[
y_i = f_i(y_1, y_2, \ldots, y_n, t)
\]

where

- \( n \) = number of first order differential equations to be solved over the time interval \((t_0, t_f)\)
- \( t_0 \) = initial time
- \( t_f \) = final time
The initial conditions are
\[ y_i(t_0) = c_i \quad i = 1, 2, 3, \ldots, r \quad (3.9) \]

The terminal conditions are
\[ y_{im}(t_f) = c_{im} \quad i = 1, 2, 3, \ldots, n - r \quad (3.10) \]

In Eqs. 3.9 and 3.10 the following quantities are defined:

- \( r \) = boundary conditions specified at the initial value of the independent variable \( t \)
- \( n - r \) = boundary conditions specified at the final value of variable \( t \) at the end of the interval

This method can be applied to any order differential equation. Therefore, the two-point boundary value approach can be applied to the methods that are developed herein.

Any nth-order differential equation can be rewritten in a series of first order equations as follows:

\[
y^{(n)}(t) = g(y, y^{(1)}, y^{(2)}, \ldots, y^{(n-1)}, t) \quad (3.11)
\]

where
\[ y^{(k)} = \frac{d^k y}{dt^k} \]

and the first order set of equations is defined as

\[
y_1 = y, \quad y_2 = y_1 = y^{(1)} \\
y_3 = y_2 = y_1 = y^{(2)} \\
\vdots \\
\cdots \\
\vdots \\
y_n = y_{n-1} = \ldots = y^{(n-1)} \\
y_{n+1} = y_n - 1 = \ldots = y^{(n-1)}
\]

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where \[ y_k = \frac{dy_k}{dt} \]

Then the first order equations can be written as

\[ \dot{y}_n = g(y_1, y_2, \ldots, y_n, t) \]

\[ \dot{y}_n - 1 = y_n \]

\[ \dot{y}_n - 2 = y_n - 1 \]

\[ \vdots \]

\[ \dot{y}_1 = y_2 \]

Therefore, the second order equations of motion, Eq. 2.2, can be written as two first order equations per mass per degree of freedom. This is expressed as

\[ \frac{\dot{y}_n}{dt} = f(y, \dot{y}, t) \]  

(3.14)

\[ \frac{dy_n}{dt} = \dot{y}_n \]

The two-point boundary value problem can be reformulated as an initial value problem (32). This manipulation greatly enhances the solution technique. In terms of numerical integration, the treatment of an initial value problem is efficient and accurate (20).

Shooting methods can be thought of as a generalized Newton-Raphson method for the solution of equations (58). The term shooting comes from the similarity to projectile motion. By adjusting the slope of an initial value at \( t_o \), the slope is sought that will result in the
of an initial value at \( t_0 \), the slope is sought that will result in the final solution at \( t_f \) being satisfied or "hitting" the target \((76)\).

The method cannot be applied if the boundary values are not known. For example, the behavior of structural systems with rigid body motion is not capable of being described at time \( t_f \). That is, the true or actual displacements, velocities, and stress resultants are not necessarily known at the terminal time after sufficient cycles of motion have been examined.

3.6 Sensitivity Functions

With respect to differential equations, the term sensitivity describes the dependence of the solution of the equations to the coefficients describing them. In a physical sense, the term sensitivity also has a similar meaning of behavioral dependence on certain parameters describing a structural system. In both cases, the sensitivity functions can describe the influence one parameter has on the response of the remainder of the system. At this point, it should be noted that this method is a departure from the other methods because a much broader viewpoint is provided. The viewpoint is that the entire structure can be studied for the effect that any and all parameters have on the localized as well as overall behavior of a given system \((33)\).

Consider an oscillating system that is described by a structure of non-linear differential equations of second order.

\[
f_i (\dot{q}, \ddot{q}, q, t, p) = 0
\]

\((3.15)\)
where
\[ p = \text{denotes a set of parameters that describe possible alterations to a physical system} \]
\[ q = \text{displacement vector with derivatives} \]
\[ t = \text{independent variable time} \]

The system of first order sensitivity functions with respect to an individual parameter \( p \) is expressed in definition form as the following and summer over all parameters (79, 83).

\[
\frac{\partial f}{\partial q}[\dot{u}] + \frac{\partial \dot{f}}{\partial q} \dot{u} = \frac{\partial f}{\partial q}[u] = - \frac{\partial F}{\partial p} \tag{3.16}
\]

where
\[ u = \partial q/\partial p = \text{sensitivity function of the first order} \]
\[ \frac{\partial f}{\partial q} = \text{matrix of generalized masses} \]
\[ \frac{\partial \dot{f}}{\partial q} = \text{damping matrix} \]
\[ \frac{\partial f}{\partial q} = \text{rigidity (stiffness) matrix} \]
\[ \frac{\partial F}{\partial p} = \text{matrix of generalized forces} \]

Equation 3.16 is the equations of motion Eq. 2.2 rewritten with partial derivatives of response (1).

This subsequent set of equations, Eq. 3.16, is integrated and solved. The solution obtained is in plot form of response as a function of time. These plots are the sensitivity functions or measure of influence that the parameters have on the response of a system (82). These plots are the essence of this method. With the use of the plots,
a model can be examined to determine the effects of all possible modifications of all alterable components within a structure.

Suzuki\(^{(72)}\) proved that all first order sensitivity functions can be obtained from a single model, Eq. 3.16. He further states that the first order sensitivity functions are of sufficient usefulness that any system can be studied effectively with them. With second order sensitivity functions used, the size of the matrices, Eq. 3.16, depend on the number of parameters \(p\) and the other level of the sensitivity function. The matrices could be of order \(n \times n\) where \(n\) is the number of degrees of freedom of the original system. Suzuki's contribution reduces the size of the matrices so that much smaller systems can describe the sensitivity functions\(^{(1)}\). The size of the matrices still depend on the number of parameters \(p\) which may be altered and the possible forcing functions considered\(^{(83)}\).

Thompson and Kohr\(^{(73)}\) extend the approach to include a choice of parameters to facilitate non-linear analysis. Their method is to compensate various parts of a system which improves or reduces the sensitivity function. In essence, it is a "weighting" function used to improve the correction given to parts of a model. The treatment of non-linearities is done with a quasi-linearization procedure and this limits the application of the sensitivity functions. It is necessary to resolve Eq. 3.16 for each step in the linearization process.

Unruh\(^{(77)}\) used a method of minimization to predict optimal characteristics describing a dynamical system. The procedure is to
identify the appropriate variables involved and expand the derivatives in terms of a Taylor series. The series representation is truncated, and response behavior is constrained (minimized) for prescribed behavior.

### 3.7 Methods Using Incomplete Models with Modal Analysis

This approach is primarily concerned with the inadequate or approximate description of a model. In Section 2.6, the concept was developed of the approximate nature of a model that describes a real structural system. That is, a model is incomplete in the sense that it is not fully accurate in describing structural behavior (6,64). In terms of testing, sufficient tests cannot be done to fully verify a model. As often occurs, a model is used to predict response of a structure subjected to other loading environments in which partial or no testing has been done. This additional environment could emphasize some other approximations used to develop a model. These other approximations may not have been exercised in initial testing of a system. This discussion is hypothetical in nature, but in practice this shift of emphasis in structural response is a usual occurrence with different loading environments (19,27).

Berman and Flannelly (6) have developed an approach for linear elastic structures. The modal analysis technique forms the basis for this method as given in Eq. 3.3 which is the eigenvalue equation for Eq. 2.2. Equation 3.3 can be rewritten as
\[ [m]^{-1} [k][x] = \lambda_i^2 \{x\} \]  
\[ [c] [m][x] = \frac{1}{\lambda_i^2} \{x\} \]

where
\[ [c] = [k]^{-1} \]

An orthogonality relationship can be stated as
\[ \{x_i\}^t [m]\{x_j\} = \begin{cases} 0 & \text{if } i \neq j \\ m_i & \text{if } i = j \end{cases} \]  
\[\text{(3.18)}\]

Equations 3.17 and 3.18 can be rewritten with some matrix multiplications, and the following results:
\[ [k] = [m][x] \lambda_i^2 \{x\}^{-1} \]
\[ [c] = \{x\} \frac{1}{\lambda_i^2} \{x\}^{-1} [m]^{-1} \]  
\[\text{(3.19)}\]
\[ \{x\}^{-1} = \frac{1}{m_i} \{x\}^T [m] \]

Equation 3.19 can be modified by using the expression for \( \{x_i\}^{-1} \) and substituting into the first two relationships. The result is
\[ [k] = [m][x] \frac{\lambda_i^2}{m_i} \{x\}[m] \]  
\[\text{(3.20)}\]
\[ [c] = \{x\} \frac{1}{\lambda_i^2 m_i} \{x\}^T \]  
\[\text{(3.21)}\]
Equations 3.20 and 3.21 can be written in summation form as

\[
[k] = \sum_{i=1}^{P} \frac{\lambda_i^2}{m_i} [m][x_i][x_i]^T
\]  

(3.22)

\[
[c] = \sum_{i=1}^{P} \frac{1}{\lambda_i^2 m_i} [x_i][x_i]^T
\]  

(3.23)

where

\[P = \text{number of natural frequencies, masses,}
\]

and mode shapes of the discretized structural system.

Two additional relationships are necessary for the approach developed by Berman and Flanelly. The impedance matrix of a system for a sinusoidal forcing function is defined as

\[
[z] = \frac{1}{\omega} \sum_{i=1}^{P} \frac{\Omega_i^2}{m_i} \left(g + i[\frac{\omega}{\Omega_i} - 1]\right)[m][x][x]^T
\]  

(3.24)

where

\[\omega = \text{forcing frequency}
\]

\[g = \text{structural damping}
\]

The mobility matrix \([y]\) is defined as

\[
[y] = \frac{1}{\omega} \sum_{i=1}^{P} \frac{1}{\Omega_i m_i} \frac{g - i \left[ \left( \frac{\omega}{\Omega_i} \right)^2 - 1 \right]}{\left[ \left( \frac{\omega}{\Omega_i} \right)^2 - 1 \right] + g^2} [x_i][x_i]^T
\]  

(3.25)

The impedance \([z]\) is a velocity matrix. The mobility matrix \([y]\) is \([x]^{-1}\). The quantity measured in testing is \([y]\), and \([z]\) is obtained through analysis.
For an incomplete or approximately correct model, the following expressions are given

\[
[k_{\text{inc}}] = \sum_{i=1}^{N} \frac{\lambda_i^2}{m_i} [m] [x_i] [x_i]^T [m] \quad (3.26)
\]

\[
[c_{\text{inc}}] = \sum_{i=1}^{N} \frac{1}{\lambda_i^2 m_i} [x_i] [x_i]^T \quad (3.27)
\]

where

\text{inc} = \text{incomplete model}

\[N = \text{number of modes}\]

and

\[
[z_{\text{inc}}] = -\sum_{i=1}^{N} \frac{\Omega_i^2 m_i}{m_i} \left[ g + i \left( \frac{\omega}{\Omega_i} \right)^2 - 1 \right][m] [x] [x]^T [m] \quad (3.28)
\]

\[
[y_{\text{inc}}] = \omega^2 \sum_{i=1}^{N} \frac{1}{\Omega_i^2 m_i} \left[ \omega^2 \left( \frac{\Omega_i}{\omega} \right)^2 - 1 \right] [x_i] [x_i]^T \quad (3.29)
\]

Revised mass and stiffness matrices can be calculated using Eqs. 3.26, 3.27, 3.28, and 3.29. These recalculated quantities form the improved model.

Berman and Flanelly make these observations:

(1) The dominant terms of \([k]\) and \([z]\) will be missing, and therefore, \([k_{\text{inc}}]\) and \([z_{\text{inc}}]\) will not be similar to the matrices describing the real system.

(2) The dominant terms of \([c]\) and \([y]\) are retained in \([c_{\text{inc}}]\) and \([y_{\text{inc}}]\). These matrices describe the response due to applied forces and will approach to the true values for the complete system.
Therefore, the incomplete model contains the first N modes of the real structure.

Ip, et. al.\(^{(27)}\) use a similar approach to the proposed incomplete model. In place of the incomplete model, Ip uses test results and employs Laplace transformations and a minimization technique to calculate the eigenvalues for a model. The eigenvalues calculated are for an improved model which correlates with test results. (This approach is quite similar to the Identification method as presented in Section 3.4.)

Sevin and Pilkey\(^{(64,65)}\) have developed a method to determine the minimums and maximums of response to forcing functions acting on a single degree of freedom system. An incomplete description is given for the system, but the min-max values are obtained within bounds. This method could be applied to large degree of freedom systems, but the computational effort is great. Moreover, the approach is limited to a narrow range of possible inputs. The method could not be used even with very basic modifications.

3.8 Discussion

The methods presented in this chapter are summarized and compared in tabular form in Appendix B. In addition, the proposed method contained in Chapter 4 is also included in Appendix B. The purpose of the tabular summary is to give a concise understanding of the applicability of a particular method as a general technique to
analyze transient response. A general discussion is given here to draw observations about the various methods.

Many methods have been formulated for the study and development of mathematical models for transient response of structures. The treatment of non-linear structures was presented as an initial prerequisite for the formulation of a model. Non-linear behavior is often encountered in transient dynamics because of the large-amplitude short-duration loading. The method chosen for the solution of non-linear equations as developed by Gabrielson, et. al. \(^{(18,28,70)}\), is accurate and efficient. This method is applicable as a general approach to the analysis of non-linear structures. However, the treatment of non-linearities as an additional (pseudo) forcing function cannot be analyzed with rigorous mathematical scrutiny.

The use of modal superposition can treat only a limited class of problems. Therefore, the normal mode method cannot be used as a general approach for analysis of transient response. There are two basic reasons for this: (1) The inability to readily analyze non-linear structures, and (2) the lack of adequate criterion to determine the number of frequencies used to describe transient phenomenon.

Those methods which specifically allow non-linearities, that is, perturbation, identification, and boundary value solution require a significant computational effort for convergence. The optimization and identification methods are dependent upon experimental data or other useable constraints for application purposes.
The boundary value problem approach and the method of sensitivity functions are the two most promising in terms to determine a general approach for the sensitivity analysis of structural systems. However, both methods encounter extensive problems conceptually and computationally with non-linear transient response problems. The incomplete model approach is more fundamental from a modeling viewpoint because it is concerned with a proper assessment of and the effects that approximations and assumptions have in the formulation of a model.

Because of the following reasons, the need exists for an alternate approach to determine the sensitivity of models describing the shock response of structural systems.

(1) The treatment of non-linearities in terms of a general solution approach is unsatisfactory.

(2) The dependence on experimental data or specified sets of constraints for some methods voids their application to the study of systems which do not have available test data.

(3) The methods do not place adequate emphasis on the approximate nature of the description of a real structural system.

(4) The basic assumptions involved in the development of each of these methods as presented restrict their application to a limited class of models.
Therefore, no satisfactory analytical technique is available as a general approach to determine the sensitivity of non-linear mathematical models for transient dynamics. Furthermore, the prospects of the development of a general analytical approach are not promising, the reason being that it is not presently conceivable that any mathematical method can be applied to a solution technique for non-linear equations. The method of solution used herein for non-linear equations cannot be examined mathematically with respect to uniqueness of solution and any errors in the solution process.

The only possible alternative is to use the multiple configuration analysis of structures as proposed by Melosh and Luik(44). A multiple configuration analysis is the solution of mathematical models of similar configurations describing a given structural system. The solutions obtained are compared and conclusions drawn with respect to the response behavior of the system. While the multiple configuration analysis is the only method available which investigates response behavior over the full range of intended usefulness including failure conditions, this approach has not been applied to the analysis of shock response of structures.

The intent of this study is to present the multiple solution approach to the sensitivity analysis of mathematical models in the analysis of transient dynamics.
4. A GENERAL METHOD FOR THE SENSITIVITY ANALYSIS
OF STRUCTURAL SYSTEMS

This study presents a general approach to determine the sensi-
tivity of mathematical models describing structural systems subjected
to dynamic loadings. The proposed method consists of careful and
critical comparative analyses of selected responses obtained from the
solution of multiple configurations of a model for a particular
structure. The sensitivity analysis (the degree to which an individual
parameter affects the response behavior of a system) is determined from
the comparative procedures.

The models developed using this method may have linear and non-
linear behavior, geometric constraints including gaps, and various types
of forcing functions. The analysis of virtually any structural system
that can be modeled using the guidelines developed in Chapter 2 can
benefit from the use of this approach.

This chapter includes the development of the governing equations
and the solution process for this approach. The comparative techniques
used for the sensitivity analysis are examined. The developmental
procedures for the formulation of multiple configurations are presented.
Further, possible applications for the proposed approach are given.
Guidelines for the use of the method are established, and the advantages
and disadvantages are discussed.
4.1 Governing Equations

The governing equations of motion, Eq. 2.2, for a single degree of freedom per mass are rewritten as follows for multiple configuration analysis (m, c, k, x, and t were previously defined):

\[
\begin{bmatrix}
m 
\end{bmatrix}_{n,n,p} \begin{bmatrix}
\ddot{x} 
\end{bmatrix}_{n,p} + \begin{bmatrix}
c 
\end{bmatrix}_{n,n,p} \begin{bmatrix}
\dot{x} 
\end{bmatrix}_{n,p} + \begin{bmatrix}
k 
\end{bmatrix}_{n,n,p} \begin{bmatrix}
x 
\end{bmatrix}_{n,p} = \begin{bmatrix}
F(t) 
\end{bmatrix}_{n,p}
\]

(4.1)

where

\[
\begin{align*}
n &= \text{number of degrees of freedom in a particular system} \\
p &= \text{number of multiple configurations}
\end{align*}
\]

The equations are considered initially as three-dimensional matrix equations. Each configuration is developed as one of pth elements of Eq. 4.1 and then the equations of motion are solved simultaneously.

Equation 4.1 can be expanded for additional degrees of freedom per mass, that is, to analyze shear and bending motion.

\[
\begin{align*}
[m][\dddot{x}] + [c][\ddot{x}] + [d'][\theta] + [k][x] + [\ell'][\theta] &= \begin{bmatrix} F(t) \end{bmatrix} \\
[j][\ddot{\theta}] + [c'][\dot{x}] + [d][\dot{\theta}] + [k'][x] + [\ell][\theta] &= \begin{bmatrix} M(t) \end{bmatrix}
\end{align*}
\]

(4.2) (4.3)

where

\[
\begin{align*}
n,n,p &= \text{dimensions of matrices} \\
n,p &= \text{dimensions of vectors} \\
c,d &= \text{damping matrices} \\
c',d' &= \text{coupled damping matrices} \\
j &= \text{inertial matrix}
\end{align*}
\]
\[ k, \ell = \text{stiffness matrices} \]
\[ k', \ell' = \text{coupled stiffness matrices} \]
\[ M(t) = \text{rotational (moment) forcing function} \]

(The coupled stiffnesses are needed to relate motion in one degree of freedom to motion in another degree of freedom.) Systems with additional degrees of freedom per mass can be described by expanding Eqs. 4.2 and 4.3.

The solution of Eqs. 4.1, 4.2, and 4.3 is accomplished through the technique used to formulate a series of first order equations as given by Eq. 3.14. For the single degree of freedom per mass system described by Eq. 4.1, two first order differential equations can be developed. Each additional degree of freedom (Eqs. 4.2 and 4.3) requires two first order equations. To simplify the development and solution of the first order equations, only the equations for the single degree of freedom per mass will be presented. A similar treatment would be given the equations for additional degrees of freedom.

Equation 4.1 can now be rewritten to facilitate the solution process and to accommodate a multiple configuration analysis. The following set of equations is written for an individual mass \( i \):

\[
\begin{align*}
\sum_{i=1}^{p} \ddot{x}_i &= \sum_{i=1}^{p} \frac{d x_i}{d t} = \sum_{i=1}^{n} m_i^{-1} \left[ c \ddot{x}_i + k x_i + \right. \\
&\left. + \sum_{r=1}^{n} \sum_{s=1}^{f} f_{r,s}(\dot{x}_i, t, x_i) \right] \\
\sum_{i=1}^{p} \frac{d x_i}{d t} &= \sum_{i=1}^{x_i} x_i
\end{align*}
\]

\[ (4.4) \]

\[ (4.5) \]
where

\[ p = \text{number of multiple configurations} \]
\[ x_i = \text{displacement of mass } i \]
\[ \dot{x}_i = \text{velocity of mass } i \]
\[ \ddot{x}_i = \text{acceleration of mass } i \]
\[ m_i = \text{mass } i \quad (\neq 0) \]
\[ c = \text{damping coefficient} \]
\[ k = \text{stiffness coefficient} \]

\[ \sum_{s=1}^{nZ} \sum_{r=1}^{ff} f_{r,s}(\dot{x}_i, t, x_i) = \text{contribution of all forcing functions and forces corresponding to deflection of non-linear load displacement relationships acting on mass } i \text{ at time } t \]

\[ ff = \text{number of forcing functions (may be constant, time varying, or a series summation)} \]
\[ nZ = \text{number of non-linear stiffnesses (virtually any load-displacement relationship)} \]

Equation 4.4 is the solution of Eq. 4.1 for acceleration (\( \ddot{x} \)).

Any term of Eq. 4.4 can be modified as \( i \) varies from one configuration to another. It is possible to modify the stiffness (linear and non-linear), damping, masses (and inertias), and forcing functions from one configuration to another. The modifications may be made in any manner and there is no restriction on the number of modifications made to a particular configuration except that the number of modifications cannot exceed the total number of parameters per configuration.
Equations 4.4 and 4.5 can be treated as an initial value problem for solution\(^{(19,21,52)}\). Only the initial conditions \((x, t,\) and \(h)\) must be specified at each time step \(h\) of the integration. All other quantities can be calculated from the relationships among the derivatives. With Eqs. 4.4 and 4.5 in the form most suitable for numerical integration, the solution process will be developed in the subsequent section.

### 4.1.1 Solution of Equations

The integration of the first order equations of motion (Eqs. 4.4 and 4.5) is treated using the Newmark \(\gamma - \beta\) technique\(^{(51)}\). With the motion (displacements, velocities, and accelerations) history determined, the stress resultants can be determined from the displacements and stiffness properties.

The governing equations of Newmark's integration technique are\(^{(20)}\):

\[
\begin{align*}
  x_{t+1} &= x_t + \dot{x}_t \, h + (0.5 - \beta) \ddot{x}_t \, h^2 + \beta \dddot{x}_{t+1} \, h^2 \quad (4.6) \\
  \dot{x}_{t+1} &= \dot{x}_t + (1.0 - \gamma) \ddot{x}_t \, h + \gamma \dddot{x}_{t+1} \, h \quad (4.7)
\end{align*}
\]

where

\(x_t, \dot{x}_t, \ddot{x}_t\) = displacement, velocity, and acceleration of a particular mass 

at time \(t\)

\(\gamma, \beta\) = scalar constants

\(h\) = time interval or step size
\[ t = \text{initial time of interval} \]
\[ t+1 = \text{final time of interval } h \]

With \( \gamma \) and \( \beta \) nonzero, the integration process is an iterative procedure because \( \ddot{x}_{t+1} \) is unknown when \( x_{t+1} \) and \( \dot{x}_{t+1} \) are calculated. Thus, a value of \( \ddot{x}_{t+1} \) is assumed and \( x_{t+1}, \dot{x}_{t+1}, \) and \( \ddot{x}_{t+1} \) are determined for a particular cycle of iteration. With the calculated value of \( \ddot{x}_{t+1} \) used in Eqs. 4.6 and 4.7, the iterative procedure is repeated until the difference between any two values of \( \ddot{x}_{t+1} \) is less than some predetermined error criterion \( \varepsilon \). Then the values of \( x_{q+1}, \dot{x}_{q+1}, \) and \( \ddot{x}_{q+1} \) are assumed to be correct and the procedure moves to the next point \((t+2)\).

\( \gamma \) and \( \beta \) are constants in Eqs. 4.6 and 4.7 and can be made equal to zero. With this choice, Eqs. 4.6 and 4.7 are reduced to Taylor's series approximations by neglecting third and higher order terms \((56)\). These rewritten equations are given by:

\[ x_{t+1} = x_t + \dot{x}_t \, h + 0.5 \ddot{x}_t \, h^2 \quad (4.8) \]

\[ \dot{x}_{t+1} = \dot{x}_t + \dddot{x}_t \, h \quad (4.9) \]

There are two observations made regarding Eqs. 4.8 and 4.9: (1) no iteration is required because no information is needed about point \((t+1)\) on the right hand sides of the equations, and (2) considerable computational time can be saved.

With the use of \( \gamma \) and \( \beta \) equal to zero, Eqs. 4.4 and 4.5 are rewritten for an individual mass for a single degree of freedom per
mass for the Newmark integration technique and multiple configuration analysis\(^{(57)}\). (Subscripts \(i\) and \(t\) denote the mass and internal time-point respectively.)

\[
\begin{align*}
\sum_{t=t_0}^{t_f} \sum_{i=1}^{p} x_{ti} &= \sum_{t=t_0}^{t_f} \sum_{i=1}^{p} [x_{ti} + x_{ti} h + h^2] \\
(\frac{1}{m_i})(c x_{ti} + k x_{ti} + \sum_{s=1}^{nI} f_{r,s}(x_{r,s}, t, x_{ti}))
\end{align*}
\]

\[(4.10)\]

\[
\begin{align*}
\sum_{t=t_0}^{t_f} \sum_{i=1}^{p} \dot{x}_{ti} &= \sum_{t=t_0}^{t_f} \sum_{i=1}^{p} [\dot{x}_{ti} + (h/m_i)(c x_{ti} + k x_{ti})] \\
\sum_{s=1}^{nI} f_{r,s}(x_{r,s}, t, x_{ti}))
\end{align*}
\]

\[(4.11)\]

The appropriate substitutions for \(x_t\) have been made in Eqs. 4.10 and 4.11. The integration process is summed over all masses so that the total response behavior of a particular system can be determined.

With the integration process completed for an interval \(h\), the stress resultants can be evaluated from the deformed shape of the structure and its accompanying stiffness properties. The parameters and stiffness matrix used herein are developed in Appendix C. The definitions used in the stiffness matrix allow for nonprismatic members and shear deformation. The stiffness matrix (Eq. C-17 from Appendix C) for a structure connecting any two discrete-mass points \((i\) and \(j)\) is as follows (see Figs. 1 and 2):
\[
\begin{bmatrix}
V_i \\
M_i \\
V_j \\
M_j
\end{bmatrix} =
\begin{bmatrix}
-a_{ij} & -b_{ij} & a_{ij} & c_{ij} \\
-b_{ij} & -(d_{ij} + b_{ij} L_{ij}) & b_{ij} & d_{ij} \\
a_{ij} & b_{ij} & -a_{ij} & -c_{ij} \\
c_{ij} & d_{ij} & -c_{ij} & (-d_{ij} + c_{ij} L_{ij})
\end{bmatrix}
\begin{bmatrix}
y_i \\
\theta_i \\
y_j \\
\theta_j
\end{bmatrix}
\]

(4.12)

where

\begin{align*}
V_i, V_j & = \text{shear at mass i, j} \\
M_i, M_j & = \text{moment at mass i, j} \\
y_i, y_j & = \text{deflection at mass i, j} \\
\theta_i, \theta_j & = \text{rotation at mass i, j}
\end{align*}

\begin{align*}
a_{ij} & = \text{shear at i caused by a unit displacement} \\
& \quad \text{with no rotation at j} \\
b_{ij} & = \text{moment at i caused by a unit displacement} \\
& \quad \text{with no rotation at j} \\
c_{ij} & = \text{shear at i caused by a unit rotation with} \\
& \quad \text{no displacement at j} \\
d_{ij} & = \text{moment at i caused by a unit rotation with} \\
& \quad \text{no displacement at j} \\
L_{ij} & = \text{distance between masses i and j}
\end{align*}

For \( \theta_i = \theta_j = 0 \), Eq. 4.12 can be modified to a stiffness matrix for a one degree of freedom per mass system involving \( a_{ij} \) only by replacing \( V_i \) with \( P_i \) and \( V_j \) with \( P_j \) shown in Fig. 1.
The internal forces in the spring $i - j$ can be calculated by the appropriate matrix manipulations of Eq. 4.12 and by equilibrium conditions for the spring-mass system in which the stress resultants in the springs are equal and opposite to forces acting on the masses. The following equations result from such a procedure.

\[
V_i = a_{ij} (y_i - y_j) + b_{ij} \theta_i - c_{ij} \theta_j
\]

\[
V_j = a_{ij} (y_j - y_i) - b_{ij} \theta_i + c_{ij} \theta_j
\]

\[
M_i = b_{ij} (y_i - y_j) + (d_{ij} + b_{ij} L_{ij}) \theta_j - d_{ij} \theta_j
\]

\[
M_j = c_{ij} (y_i - y_j) - d_{ij} \theta_j + (d_{ij} - c_{ij} L_{ij}) \theta_j
\]

(4.13)

When the displacements (relative $(y_i - y_j)$ and actual $\theta_i$ and $\theta_j$ shown in Fig. 2) have been calculated, then the stress resultants can be calculated for each step of the integration process. The stress resultants must be summed over each particular configuration and for the model as a whole to determine the responses of the system.

The generation of the motion and stress resultant histories is completed. The following section is presented to explain the details for input descriptions and computer programming necessary to formulate the additional configurations and solve the equations of motion.
4.1.2 Input Details and Programming Aspects

The details regarding the input descriptions for structural systems and multiple configuration analysis are described in this section. Specific details involving the programming effort needed to solve the systems of equations is described.

The computer program is written so that each input description is treated as an independent entity\(^{(19)}\). Each description has its own individual name and associated arguments (variables on a particular card). Therefore, each description may be checked against a predefined set of vocabulary and possible error conditions.

Because a diagonal mass matrix is used, \(\{x\}\) and \(\{\dot{x}\}\) from Eqs. 4.10 and 4.11 respectively can be processed as a single vector for all configurations. With an appropriate incrementation of the number of degrees of freedom \(n\), it is possible to determine the responses for the same mass or spring simultaneously. This relationship among the responses is given in equation form by

\[
R_i = f(R(i + cn* inc))
\] (4.14)

where

- \(R_i\) = motion or stress resultant history of mass or spring \(i\)
- \(cn\) = configuration number such that \(1 < cn \leq p\) where \(p\) is defined as the total number of configurations
- \(inc\) = incremental integer such that \(n < inc < 100\)
n = number of masses in an original or bench mark model

100 = arbitrary upper limit on the total number of masses summed overall configurations

In the overall description certain groupings of information (problem type and conditions, inertial properties, boundary conditions, springs, plotting requests) are usually developed for any system analyzed. The additional configurations needed to determine the sensitivity of a particular system can be generated by supplying only the changes to an original system. The groupings cited above can be separated into the descriptions which remain constant during the integration phase of solution process and those descriptions that are subject to modification. A single input description (flag word) with two arguments delineates the formation of multiple models. The delineation is such that all descriptions prior to the flag word are constant and all descriptions subsequently are subject to modification. The two arguments are cn and inc used in Eq. 4.14. All other input descriptions are given the additional argument NX such that

\[ \text{NX} = 0 \quad \text{no changes in description} \quad (4.15) \]

\[ \text{NX} = \text{cn arguments contained in this description are used in this configuration and replace those arguments used on a previous configuration} \]

After the arrays have been initialized, the data read in, and possible error conditions checked, the models are formed from the input
descriptions as shown in the flow chart in Appendix C. The models for the multiple configurations of a particular system are formed and the first order equations of motion, Eqs. 4.10 and 4.11, are developed.

The forcing functions (applied-time dependent forces, constant forces, and pseudo-non-linear stiffnesses) are added together. The integration process described by Eqs. 4.10 and 4.11 is evaluated. The vector sets of motion ([x], [ẋ], [y], [ẏ], [θ], or [θ̇]) and time ([t]) are stored for all models. The stress resultants are evaluated according to Eq. 4.13 from the displacements. In addition, in each time step a search is performed for the minimum and maximum values for acceleration, velocity and displacement of each mass and the stress resultant in each spring.

Any and all of the response histories for a particular system can be plotted with a similar incrementation for comparison purposes as given by Eq. 4.14. The plotting is usually a selective procedure because it is not normally feasible to plot the responses of each mass and spring of an entire system. Details regarding the actual procedures used for plotting and incrementation of appropriate indexes is found in Ref. 57.

4.1.3 Critique on Multiple Configuration Analysis

One of the reasons that a multiple configuration analysis with independent simultaneous solutions is used for transient response is to overcome the difficulties encountered in treating the approximations needed to define a structural system. The original derivation of this
approach by Melosh and Luik (44) used subsequent solutions based on series expansions of an original solution. There is a tacit assumption made that only "small" changes are needed to modify an original model to accurately predict structural behavior. As nearly as "small" can be defined implies that the parameters need to be modified by only 10 to 20 percent before an accurate model results (54, 56, 66). Appendix E contains a discussion of the effects of expanding subsequent solutions in terms of an original solution and the treatments of modifications made to a particular system.

There is no guarantee that the modifications to model parameters needed to define a particular system are bounded by 10 to 20 percent types of change in an original model. Structural systems are not that "well-defined".

There are two important observations made in Appendix E:

(1) Simultaneous solution of independent configurations is needed because of the excessive computational time needed to properly evaluate series representations of solutions.

(2) The error build-up associated with the remainder terms in the series expansions could lead to serious convergence problems in the integration of the equations of motion.

A detailed discussion for the reasons that simultaneous solutions solutions are needed as opposed to independent solutions is given in
Appendix F. From that discussion and Section 4.1.2, two important facts are noted:

(1) Multiple simultaneous solutions are computational time savers when compared with multiple independent solutions.

(2) Considerable convenience is offered to the analyst by formulation of additional models by using only modifications to an original system.

Therefore, the multiple configuration analysis with simultaneous independent solution procedures is the most appropriate method yet developed as a general approach to determine the sensitivity of structural systems to dynamic loadings.

There is a considerable developmental effort (often man-years) needed for large and sophisticated computer programs. These programs are basically mathematical modeling procedures. The multiple configuration analysis approach offers a method to obtain as much information as possible from these modeling procedures.

4.2 Comparison Procedures for Sensitivity Analysis

With the development and solution of equations describing multiple configurations of a particular system, the responses needed for the sensitivity analysis are available. Without suitable comparative procedures, the determination of the sensitivity of a structure could not be determined.
Since comparative procedures are needed, the question can be raised as to what is needed, what is available, and what can be developed. Two basic needs should be met: (1) determination of the sensitivity of localized portions of a structure and (2) an understanding of the sensitivity of an entire structure.

Methods to analyze shock response data have developed three basic approaches for the comparison of data: (1) superimposed plots of responses for localized behavior, (2) a min-max screening of responses to develop an overall understanding of the structural behavior of a system, and (3) frequency decomposition techniques such as shock and energy spectra\(^{(24)}\). Variations of the above methods exist but do not add sufficient information to a sensitivity analysis to warrant their use.

Statistical methods cannot be used because a sufficiently large data base has not been developed by the solution of a few models. A sensitivity study can be accomplished in significantly smaller computational effort than required for a statistical approach\(^{(44)}\).

With the application of the three approaches for analysis of shock data, the sensitivity of structural systems can be determined for modifications made to a mathematical model describing a particular system. The development of other nonstatistical methods and techniques has not resulted in approaches that are applicable or a general method for sensitivity analysis\(^{(31)}\).
4.2.1 Superimposed Plots

The plots of response histories obtained from the multiple configuration analysis approach can be superimposed. From this superposition technique, a simple comparative procedure is developed for the sensitivity analysis of models subjected to modifications. The basic aim of this comparative procedure is an examination of the response histories for similarities and dissimilarities in behavior. The degree of difference $\Delta R_i(t)$ in a response behavior for a mass or spring may be represented by:

$$\sum_{i=2}^{P} \Delta R_i(t) = \sum_{i=2}^{P} (R_i(t) - R_{bm}(t))$$

where

- $R_{bm}(t)$ = response of mass or spring for benchmark (usually the original) model
- $R_i(t)$ = response of mass or spring to be compared from the multiple configurations
- $p$ = number of multiple configurations

(Eq. 4.4) (4.16)

In cases where phase differences in a set of response histories are not appreciable, the superimposed plots can be used directly to determine the sensitivity of the various models.

As $\Delta R_i(t)$ in Eq. 4.16 approaches zero for a set of compared responses, then the histories are essentially the same for all time.
examines. The magnitude of $\Delta R_i(t)$ is problem dependent and, in general, is nonzero or at least different from zero. The magnitude of this difference depends upon: (1) the motion or stress resultant history being compared, that is, accelerations fluctuate more rapidly than do displacements), (2) the type of system being analyzed, and (3) the excitation given the structure. As a result, it is not feasible to establish criteria governing the use of these differences. However, once the differences in Eq. 4.16 have been determined for a set of responses, then the sensitivity of a model to modifications is established.

In model response histories exhibiting phase differences, the superimposed plots can be of use in a sensitivity analysis by:

(1) The minimums and maximums of responses for each configuration can be noted on a set of plots.

(2) Some of the behavior of the lower modes can be examined (that is, lowest bending or membrane frequency).

(3) Correlation of model responses and experimental data can be accomplished. The usual form of dynamic test results is a strain or acceleration-time history. These test histories can be compared directly with model responses. (Strain-time relationships need to be converted to stress resultants for comparison purposes.)
The subsequent sections use additional data analysis techniques with superimposed plots to give additional methods to determine the sensitivity of mathematical models.

4.2.2 Min-Max Screening (Responses)

A min-max screening is a superimposed plot or possible table summary of a particular set of peak responses for an entire structural system. The superimposed plots compare behavior at specific points but do not indicate overall system behavior. With the usual phase differences encountered in the responses multiple configurations, a min-max screening is essentially independent of time. (The screening depends only on the interval of time examined \((t_0 \text{ to } t_f)\) as in Eqs. 4.10 and 4.11.)

The relationship for the min-max screening process is as follows using Eqs. 4.10, 4.11, and 4.13.

\[
\sum_{i=1}^{p} \left[ \sum_{t=t_0}^{t_f} \max_{\max_{i=1}^{p}} f(V_i, M_i, x_i, \ddot{x}_i, x_i, t) \right] = \sum_{i=1}^{p} \left[ \sum_{t=t_0}^{t_f} \max_{\max_{i=1}^{p}} f(V_i, M_i, x_i, \ddot{x}_i, x_i, t) \right] 
\] (4.17)

\[
\sum_{i=1}^{p} \left[ \sum_{t=t_0}^{t_f} \min_{\min_{i=1}^{p}} f(V_i, M_i, x_i, \ddot{x}_i, x_i, t) \right] = \sum_{i=1}^{p} \left[ \sum_{t=t_0}^{t_f} \min_{\min_{i=1}^{p}} f(V_i, M_i, x_i, \ddot{x}_i, x_i, t) \right] 
\] (4.18)

Similar min-max responses can be evaluated for springs for relative displacements and stress resultants. The min-max screening is shown to be independent of time.

One particular application of the min-max screening process is to determine the effect of modifications in a model have on the
stresses in a structure. Maximum stresses are often used as a design criteria. These stresses are determined for all models analyzed.

Superimposed plots of min-max responses demonstrate the sensitivity of an entire system to modifications made to a model.

4.2.3 Shock and Energy Spectra

A shock spectrum is defined as a plot of an individual response behavior (acceleration) of a multitude of single degree of freedom spring-mass systems subjected to a particular shock input. An energy spectrum is a finite, Fourier transform of a time history. The use of both techniques for sensitivity analysis is not needed. Although the techniques are completely different in concept, a relationship does exist between certain forms of these two spectral decomposition techniques. The relationship between the two techniques is demonstrated in Appendix G.

The shock spectrum will be used herein because it is more familiar to structural engineers. The shock spectrum can be primary or residual in nature. A primary spectrum indicates behavior that has maximum responses occurring during the application of the excitation. A residual spectrum exhibits behavior that has maximum responses which occur after the application of the transient forcing function.

The usual definition for a shock spectrum consists of the motion of a single-degree of freedom oscillator attached to a base fixed in space. With \( x \) as the displacement of the mass, the governing equation of motion, Eq. 2.2, for an undamped system is as follows:

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\[ \ddot{x} + \omega^2 x = \frac{F(t)}{m} \quad (4.19) \]

where

\[ \omega = \text{natural frequency of the oscillator} \]

\[ F(t) = \text{forcing function applied to the mass} \]

For a system initially at rest, so that \( x = \dot{x} = 0 \), the maximum value of \( x \) for \( t > 0 \) is

\[ D = \max \left[ x(t, \omega) \right] \quad (4.20) \]

where \( D = \text{shock spectrum of the response behavior and specifically, the displacement spectrum} \)

Shock spectra can also be calculated digitally from response histories. This digital calculation can be done with some modifications to Eqs. 4.19 and 4.20. However, once the response history has been generated using Eqs. 4.10 and 4.11, then the following alternate formulation to the shock spectra will be used herein. The undamped shock spectra for accelerations can be calculated from a time history as

\[ \zeta(t) = \int_{t_0}^{t_f} \ddot{x}(t - \tau)[2 \sin \omega_n(t - \tau)]d \tau \quad (4.21) \]

where \[ \zeta(t) = \text{relative acceleration time history} \]

\( t_0, t_f = \text{initial and final times of the response to be decomposed} \)
\[ \ddot{x}(t - \tau) = \text{convolution the } \dot{x}(t) \text{ history} \]
\[ \omega_n = \text{undamped natural frequency} \]

Equation 4.21 can be rewritten for numerical integration in the following manner because \( \ddot{x}(t) \) is really not continuous.

\[
\sum_{i=1}^{P} \xi_i = \sum_{i=1}^{P} \frac{h}{\omega_n} \sum_{t=t_0}^{t_f} \ddot{x}_i \sin [\omega_n (t - t_0)]
\] (4.22)

Equation 4.22 also has the multiple configuration analysis procedure included in the summation.

With the superposition of these plots obtained from 4.22, the following guidelines are formulated for the use of superimposed spectra:

1. The correlation of multiple models for acceleration response is determined by evaluation of the plots. Similar shock spectra indicate similar responses.

2. If the high frequency content of the shock spectra differs, then acceleration responses are different. If the low frequency modes are similar, then the stress resultants are not significantly different.

3. Drastic differences in accelerations can be vividly demonstrated.

Thus, the application of superimposed spectra techniques to the sensitivity analysis of structural models is focused on accelerations and stresses. In addition, the spectra techniques overcome the problem of phasing dissimilarities in response histories.
4.3 Development Procedures for Multiple Configurations

Modifications and adjustments to a model are needed for a multiple configuration analysis. Each structural system is unique and has its own characteristics. Therefore, general procedures for the development of multiple configurations will be presented. In addition, one specific example contained in Appendix H is given. The development of multiple configurations is based on the following considerations:

(1) A variety of limiting conditions usually exist for various model parameters. There are many assumptions needed to define the material and geometrical properties so that model parameters can be evaluated. These limiting conditions can arise from analytical attempts to define parameters. Examples of limiting conditions include geometrical constraints (such as gaps) and various yield strengths as they affect stiffness characteristics.

(2) The limiting conditions often give opportunities for wide ranges of possible responses. Therefore, intermediate points lying among the limiting values can be used to describe the parameters subject to question. Often the limiting values are idealized conditions and do not reflect possible behavior.
(3) The alternatives offered in a design process often result in multiple configurations to be analyzed.

The results of the example problem presented in Appendix H showed the following conclusions:

(1) Multiple configurations can be developed from various support conditions (pinned, fixed, and elastic) used for analysis of beams.

(2) The modifications made to model parameters may not be "small" in order to describe possible structural behavior.

4.4 Possible Applications of Proposed Method

With the general approach for the sensitivity analysis of structural systems developed, the following possible applications are formulated:

(1) The integrity of a particular structural system can be examined throughout its intended service life. The combinations of loadings and configurations that are critical in the determination of this integrity can be studied and compared. The sensitivity analysis is particularly useful in determining the governing combination of loading and configuration.
The assumptions and approximations used to develop models can be examined with a sensitivity analysis to determine the controlling parameters for response behavior.

Parametric studies involving design alternatives can be investigated.

Min-max responses for the configurations analyzed can be examined to determine overall system response with particular emphasis on design aspects.

4.5 Guidelines for the Use of Multiple Configuration Analysis

The following guidelines are proposed for the use of the method of multiple configuration analysis as applied to the analysis of transient response of structural systems:

1. Develop an original model of a given structural system. In this development, special emphasis should be given to any and all parts of the given system that need assumptions and approximations for their definition.

2. Develop multiple configurations. These additional models should reflect the limiting and/or intermediate conditions of the assumptions and approximations used in the formulation of the original model.

3. Using a set of appropriate forcing functions, the multiple configurations are excited and solutions
result. These results are in superimposed plot form and min-max sets of responses.

(4) The sensitivity analysis of a model is determined from a careful and critical analysis of the results obtained from the multiple configurations.

(5) If necessary, additional cycles of steps (2), (3), and (4) may be performed.

4.6 Advantages and Disadvantages

Included in the advantages of the proposed method are these:

(1) The sensitivity of models representing dynamic response of structural systems can be determined.

(2) A wide variety of structural configurations can be modeled, solved, and compared. The method is not limited to the analysis of linear elastic structures.

(3) The min-max responses of a system are calculated for a given excitation.

(4) Test results can be used in conjunction with the model responses.

(5) The method is a computational time saver over independent solutions and comparison procedures.

(6) The development of additional configurations is accomplished by using only the modifications to an original model description.
(7) Multiple modifications to a particular model can be treated.

(8) A direct means is given of assessing the assumptions and approximations used in model development.

The main disadvantages of this approach are the following:

(1) It is not a rigorous mathematical approach.

(2) There is no guarantee that the most accurate model can be developed. Only the sensitivity of a system can be determined.
5. APPLICATION OF MULTIPLE CONFIGURATION ANALYSIS IN TRANSIENT DYNAMICS

The purpose of this chapter is to present detailed example problems which were used to verify the proposed method to determine the sensitivity of models describing transient dynamics. With an examination of the results contained herein, an overview of this study can be gained, and an understanding of the approach can be facilitated.

Because of the general nature of approach, it is not feasible to provide examples which illustrate every conceivable structural system that can be treated with the analysis of multiple configurations. For this reason, three examples are presented which demonstrate typical kinds of problems encountered in the analysis of structures subjected to dynamic loadings. These problems consist of the following: (1) the verification of a model which is based on a correlation of model responses and experimental data, (2) a parametric study assessing the effects of design alternatives, and (3) the analysis of a non-linear structural system.

A brief description of each of the problems is given initially. Subsequent sections of this chapter are concerned with a detailed presentation of each example problem. Included in each section are the purposes and goals of the problem, the development of the models, the responses obtained, a discussion of the responses, and the sensitivity
analysis for the structural system. Detailed drawings of each example cannot be supplied, but sufficient data are given to show how the models were developed.

The first example is the analysis of the Summit Bridge which is located approximately 8 miles south of Newark, Delaware on Route 896(45). The portion of the bridge analyzed were the main (1200 ft.) spans over the Chesapeake and Delaware Canal. This example demonstrates the use of the multiple configuration analysis in two ways: (1) The assessment of the effects that approximations needed to formulate mathematical models have on response behavior. The limiting conditions that bound possible structural behavior are incorporated into the models for the superstructure and one supporting bent. (2) The comparison of model behavior and experimental data demonstrates that the sensitivity of a mathematical model can be determined. The parameters controlling dynamic response can be identified.

The second example consists of a parametric study which demonstrates the effect design alternatives have on the response behavior of a missile system subjected to a shock loading. The analysis of multiple configurations treats significantly different models of the missile system. In addition, the proposed method establishes the maximum forces and accelerations needed for design purposes.

The third example considers a nose cone excited by a sweeping blast wave. The nose cone contains an internal cantilever beam that is free to displace until it impacts a coil spring support system. The emphasis in this problem is on the complex nature of an impact.
problem and the use of the multiple configuration analysis to determine the governing parameters for dynamic response.

In each problem, the sensitivity analysis is accomplished by the examination of selected responses using the comparative procedures developed in Section 4.2.

5.1 Summit Bridge Superstructure and North Anchor Span Bent

This example problem consists of the analysis of the dynamic response of a portion of the superstructure and the north supporting bent of the Summit Bridge shown in Fig. 3. The bridge was excited by two trucks side by side traversing the bridge from right to left at 30 miles per hour. The passage of these vehicles induced a perceptible vibrational response in the superstructure and anchor span bent. This vibrational behavior was considered a possible detriment to the integrity of the bridge because of fatigue considerations. Therefore, the bridge and supporting bent were analyzed to determine the significance of these vibrations.

5.1.1 Goals of Analytical Study

Among the goals of the analytical study are the following uses of the proposed method of sensitivity analysis:

(1) The development of models that predict the dynamical behavior of the superstructure and bent.

(2) To assess the approximations and assumptions needed to define the mathematical models.
(3) To determine the stress levels at the base of the north anchor span bent.

(4) To determine the cause of the noticeable vibrational behavior. The assumption was made that the change of lengths in the bottom chords of the twin trusses because of vehicular loading caused a displacement history of the top of the anchor span bent. This assumption needed verification.

Because of the presence of the vibrational behavior, possible corrective actions were formulated and the effect of these actions needed to be assessed. One corrective action gives two possible limiting conditions for the formulation of multiple configurations. The superstructure was design with pins positioned at the ends of the suspended span as shown in Fig. 3. The effect of pins was to produce the behavior of a hinge in the superstructure. The superstructure can then be considered as a statically determinate structure.

An inspection of these pins showed that they were frozen and inoperable. The corrective action of restoring full action of the pins is one limiting condition for structural behavior. Another limiting condition is to assume that the superstructure acts as a truss continuous over four supports. Models formulated with these conditions needed a multiple configuration analysis approach for solution.
5.1.2 Formulation of Models for Superstructure and Anchor Span Bent

The lumped-mass model for the superstructure is shown in Fig. 3 and the model parameters are summarized in Table 1. The guidelines developed in Chapter 2 were used to formulate the model. The model was formulated by assuming the twin trusses could be replaced by an equivalent beam. The beam has varying EI and AG and a section of the superstructure as modeled is given in Fig. 4. The information needed to develop the model is summarized in Table 2.

The mass discretization is shown in Fig. 4. The number of masses used to describe the structure followed these considerations:

(1) Masses were needed to describe the support conditions (boundary conditions can be specified for masses-not springs).

(2) Masses were located at panel and hinge points because part of the weight of the superstructure was concentrated there.

(3) Enough masses were needed between the supports to adequately describe the dynamic response of the equivalent beam.

(4) The choice of a symmetrical arrangement of masses for the cantilever and suspended spans simplified the calculations.

(5) More detailed information was desired in the left anchor span adjacent to the bent so more masses were used.
Therefore, fifteen masses were used to describe the structure as shown in Fig. 4. This arrangement of masses was adequate for the application of the vehicular loading.

The weight and inertia for each mass is concentrated at the mass center for the cross-section and length \( L_i \) of each mass (Fig. 4). The weight \( W_i \) of mass \( i \) is given by:

\[
W_i = w_i L_i \quad \text{(lbs.)} \tag{5.1}
\]

where

\[ w_i = \text{calculated weight per foot of the superstructure} \]

(Table 2 and Ref. 45)

\[ L_i = \text{Length of mass } i \]

The inertia \( J_i \) of each mass \( i \) is calculated by the following relationship in which \( n.m \) is the number of members.

\[
J_i = \sum_{i=1}^{n.m} \left( J_{o_i} + W_{o_i} d_i^2 \right) \quad \text{(lb-in.}^2) \tag{5.2}
\]

where

\[ J_{o_i} = \text{inertia of member } i \text{ about its own mass center} \]

\[ W_{o_i} = \text{weight of member } i \]

\[ d_i = \text{distance from member mass center to overall mass center} \]

With the use of \( EI \) and \( AG \) properties given in Table 2 and \( L_{ij} \) obtained from the mass discretization, the stiffness properties can -76-
be calculated for use in Eqs. 4.10, 4.11, and 4.12. (The equations used for the stiffness properties are C-14, C-15, and C-16 of Appendix C.) Five percent viscous damping was assigned to each spring in the system.

The original model for the superstructure consists of the equivalent beam continuous (no hinges) over four simple supports. The twin trusses were designed to supply the bending stiffness for the superstructure. However, the bridge deck and diagonal members of the truss had a significant shear stiffness. The area of the diagonal members and 20 percent of the reinforced concrete deck area (\(n = 15\) for transformed area) were used to calculate the shear stiffnesses. Among the other assumptions used to develop the model were the following:

1. Each connection in the trusses behaved as a rigid connection.
2. Roughness of the bridge surface and dynamical behavior of the vehicles were neglected.
3. The bridge deck did not offer bending resistance.
4. The wind and other bracing did not offer bending or shear stiffness.
5. There was no torsional deformation of the twin trusses. Only in-plane deformations were allowed.
6. \(EI\) and \(AG\) were assumed constant for each spring. The variations in \(EI\) and \(AG\) along
the superstructure were localized into each spring.

The first alternate configuration analyzed consisted of the original model with the left support replaced by an elastic foundation. This foundation is equivalent elastic spring of the anchor span bent.

The second configuration analyzed consisted of the original model with elastic support with full action of the hinges. The model of a hinge is shown in Fig. 4. An additional mass \( j \) (with properties \( W_j/2 \) and \( J_j/2 \) from Eqs. 5.1 and 5.2) is connected to mass \( i \) on the other side of the hinge. (Mass \( i \) has properties \( W_i/2 \) and \( J_i/2 \).) The continuity of the structure through the hinge (between \( i \) and \( j \)) involves only a shear stiffness \( (b_{ij} = c_{ij} = d_{ij} = l_{ij} = 0) \). The additional mass \( j \) is connected to mass \( j + 1 \) with the original stiffness for spring \( i - j + 1 \).

The three model configurations used to analyze the behavior of the superstructure are shown in Fig. 5.

The elevation views and the model for the north anchor span bent are shown in Fig. 6. A single model of the superstructure and bent could not be developed. It should be noted that the developed procedure based on Eqs. 4.2 and 4.3 was for two degrees of freedom per mass. The condition of a maximum of two degrees per mass does not allow the same deformations to occur in a model as in the real structure.
The model and model data are summarized in Table 3. The mass discretization was based on the following information:

(1) The bent was analyzed as an equivalent beam. Higher order oscillations (such as torsional vibration of the bent) did not appear to be possible. Test results confirmed this behavior.

(2) A mass point was needed at the upper third point so that model displacements could be compared with a measured displacement history.

(3) A mass was needed at the top of the bent so that the assumed length change history of the bottom chord of the superstructure could be applied to the bent.

(4) Experimental strain data were measured adjacent to the base of the bent. A spring connecting two masses was needed to match this behavior.

(5) Since the connections at the columns at the bents were major concentrations of mass, masses were located at these connections.

(6) One mass was used between each connection to describe the behavior of the bent between the connections.

The weights and inertias were calculated for the various masses by using Eqs. 5.1 and 5.2 and the portion of the members indicated on Fig. 6.
The stiffness calculation were simplified by the mass discretization because $EI$, $AG$, and $L_{ij}$ were the same for each mass. An original model was developed for the bent by considering the base of the bent as fixed.

One alternate configuration was developed for the bent. The base of the bent (Fig. 3) was designated fixed on the plans. However, there was no guarantee that this condition is true for the vehicular loading. Therefore, the stiffness of the bent foundation spring needed to be determined. As often occurs, only a minimal amount of information was available to calculate this spring. Among the unknowns regarding the bent to foundation spring are the following (Fig. 7):

1. The number, size, spacing, and location of the bolts connecting the bent to the foundation. (An inspection determined that these bolts had rusted.)

2. The dimensions of the foundation and footing.

3. The location, size, number, and type of piles supporting the foundation.

4. The soil conditions including the location of the water table.

The configuration used for the analysis of the dynamic response of the anchor span bent consisted of the original model with a bent foundation stiffness given in Table 3. The values chosen reflected that the stiffness parameters for this spring should be larger than the values for the bent itself.
5.1.3 Responses Obtained

The forcing functions used to excite the superstructure are given in Fig. 8. (For those masses fixed against translation, the \( F(t) \) was not included.) The forcing functions are varied depending on the length \( L_i \) of each of the masses \( m_i \).

The plot of the vertical displacement of the centerline of the suspended span is given in Fig. 9. This plot shows that the displacement of the superstructure with full action of the hinges (1.5 in.) is greater than either of the other models (1.0 in.). The curves for the first two models are identical until the vehicle passes the centerline. There is some deviation in this displacement behavior after the vehicles pass the centerline.

The behavior of the superstructure with the hinges shows that no displacement of the centerline occurs until vehicles are on the suspended span. Also noted is the much larger residual displacement behavior of the superstructure with the hinges than without.

The reaction time history at Pier 5 is given in Fig. 10. This plot is based on 70 percent of the support being supplied by one of the columns as shown in Fig. 6. Included in Fig. 10 are the plots obtained from the test data. The experimental force-time history was obtained by converting the digitized strain record into an axial stress resultant for the bent.

The first plot (upper) shows the test record as it was converted. The high frequency content of this plot results from the use
of a crude hand digitization process in which only the peaks of a record were recorded. In an effort to obtain a more realistic record, the reaction-time history, \( R(t) \), was smoothed according to the following five-point scheme:

\[
R(t) = 0.2\phi AE \sum_{i=1}^{n} (e_{i-2} + e_{i-1} + e + e_{i+1} + e_{i+2})
\]

where

- \( A \) = cross-sectional area of one column
- \( E \) = modulus of elasticity
- \( e_i \) = value of experimental strain at \( t_i \)
- \( n \) = number of points in history
- \( \phi \) = factor to account for 70 percent of load transmitted to one column (\( \phi = 1.4 \))

The same maximum axial force (100 kips) is found in all three models. The first two configurations exhibit similar reaction histories with the original model exhibiting the least deviation from the test results. The model with the hinges exhibited a larger residual oscillatory axial force in the bent.

The excitation given the model of the anchor span bent is a displacement-time history applied to the top of bent. This history was obtained from the properties of the structure as it deflected under the vehicular loading as shown in Fig. 11. The change in length of the bottom chords of the trusses is related to the rotation of the superstructure over Piers 4 and 5 as given by:
\[ \delta_{\text{BENT TOP}}(t) = \theta_1(t)(L_1) + \theta_6(t)(L_6) \quad (5.4) \]

where:
- \( \delta_{\text{BENT TOP}}(t) \) = change in length of bottom chord
- \( \theta_1(t) \) = rotation history of mass 1
- \( \theta_6(t) \) = rotation history of mass 6 of superstructure (Pier 4)
- \( L_1 \) = length from mass center (Fig. 4) to bottom chord = 40 ft.
- \( L_6 \) = length from mass center to bottom chord = 72 ft.

The assumption is also made that the superstructure is fixed against longitudinal translation at Pier 4.

The rotation histories for the superstructure for mass 1 and mass 6 are shown in Figs. 12 and 13 respectively. The rotation histories obtained from the original model were used in Eq. 5.4.

The displacement history for the upper third point of the north anchor span bent is given in Fig. 14. The data is bracketed by the model responses.

The moment-time history adjacent to the base is given in Fig. 15. The experimental strain data was converted to moment time history. The responses for the two configurations analyzed bracket the rest results.
5.1.4 Discussion of Results and Sensitivity of Models

The results obtained from the multiple configuration analysis and the experimental data suggest the following observations regarding the dynamical behavior of the superstructure:

(1) The effect of the hinges is to magnify the behavior of the superstructure over the existing continuous condition with the frozen pins. Both greater and larger residual displacements occurred in the hinged structure (Fig. 9). Although it is known that a statically determinate structure would not be as stiff as an indeterminate structure, the relative differences in dynamic responses would not be known without this analysis.

(2) The experimental determination of the effect of the hinges on the behavior of the superstructure would be a difficult task. To free the pins would require supporting the suspended span, removing the old pins, and inserting new pins. However, there is no guarantee that with new pins the superstructure will behave as a hinged structure. Therefore, an analytical study is the only feasible way of assessing the effect of the hinges.

(3) The model for the hinge (Fig. 4d) is valid because of the response history for this configuration as
shown in Figs. 9 and 10. The displacement of the centerline in the hinged structure would not occur until the vehicles have passed the hinges located on the south end of the suspended span. Further evidence of the validity of the hinge model is offered by the fact that the maximum reactions at Pier 5 are the same (100 kips).

(4) The elastic support offered by the axial stiffness of the anchor span bent does not appreciably affect the reactions at Pier 5 nor the deflection history of the centerline. The response and deflection histories for the first two configurations (original and elastic support models) are basically similar. Only with the vehicles in the suspended span of the bridge do differences in deflection and reaction occur.

(5) The response behavior of the original model for the superstructure exhibits the closest agreement with experimental data of the three configurations analyzed.

(6) The roughness of the deck surface and the dynamical characteristics of the test vehicles did not have particular significance in the analysis of dynamic response. The agreement between test data and model response showed that the assumptions used
to develop the model for the superstructure were valid.

The sensitivity analysis of the superstructure consists of the following:

(1) The superstructure is sensitive to the action of the hinges. The effect of the hinges is to produce larger live loads and residual deflections. The multiple configuration analysis demonstrates this sensitivity in Fig. 9.

(2) As a result of the reaction time history (Fig. 10) the superstructure is not significantly affected by elastic support of the anchor span bent (Fig. 10).

(3) The original model for the superstructure accurately predicts the dynamic response behavior. The sensitivity analysis demonstrates that the assumptions and approximations used to formulate the mathematical model were valid. Vehicle characteristics, surface roughness, and any other affects not considered were insignificant in the behavior of the superstructure.

From the results obtained for the behavior of the anchor span bent and the experimental behavior, the following observations can be made:
(1) The assumption is valid that the change of length of the bottom chords of the trusses is responsible for the vibration of the anchor span bent, Eq. 5.4. The validity of this assumption is verified by the agreement shown between model behavior and experimental data for the displacement of the upper third point (Fig. 14). The responses obtained from the two configurations essentially bracket the test results.

(2) The assumptions made for the stiffness of the base-foundation spring are of use in determining the behavior of the bent (Fig. 15). The moment histories obtained from the two configurations bound the experimental data.

(3) The moment histories for the experimental data and the second configuration exhibit close agreement for the vehicles in the first two spans (right anchor and suspended). These two histories do exhibit differences when the vehicles are in the left anchor span. This behavior suggests that the true bent-foundation is a function of vehicle position. However, bounds have been established.²

The sensitivity of the anchor span bent consists of the following:
(1) The displacement history of the bent is not appreciably affected by the choice of a realistic base-foundation stiffness.

(2) The moment history at the base of the bent is sensitive to the choice of base-foundation stiffness. The two configurations differed by a factor of two for most of the histories.

(3) The stress resultants at the base of the bent were determined and the maximum stresses showed close agreement with experimental behavior. The sensitivity analysis showed that the models (superstructure and bent) were applicable to the analysis of the dynamic behavior of the bridge.

5.2 Shock-Driven Missile System

This example problem demonstrates the capability of the multiple configuration analysis procedure to do a parametric study. The purpose of this problem is to determine the effects that design alternatives have on the behavior of different configurations of the missile system including the instrumentation package. The sensitivity analysis assesses the similarities and differences in responses.

In a system containing various components (instrumentation, radar, telemetry and engines) considerable modifications are made to a particular component before a final design results. Because of the
interactive nature and multiplicity of effects encountered in dynamics, it is necessary to assess the effects of modifications as a design process evolves. In particular the maximum responses and response histories need to be determined for each design examined.

5.2.1 Parametric Study

The sketch and discrete-mass model of the shock-driven missile system are shown in Fig. 16. The model description is summarized in Table 4. The structure is excited by the acceleration-time history applied to the driver mass. This excitation is applied to the missile system through an attachment fixture. The missile system consists of an axisymmetric cone containing an instrumentation package. The excitation applied to the driver mass (D.M.) is a high amplitude (50 g rigid body acceleration) shock for a short duration (0.00025 sec.). This half-sine forcing function induces dynamical behavior in the missile system. This behavior is similar to a flight separation shock environment in which one stage of a structure is separated from another (13,59).

The mounting for and design of the instrumentation package consisted of a range of possible configurations which could be bound by two limiting conditions and an intermediate value. The multiple configuration analysis approach assesses the responses of the system and determines the information needed for design purposes. The following items are of particular interest:

(1) The determination of the acceleration and stress resultant histories and shock spectra needed for design purposes.
(2) The limiting values for stress resultants (stresses) in the missile shell (case) need to be evaluated.

(3) The assessment of the effects that the alternate configurations for the mounting system/instrumentation package have on the behavior of the entire system.

5.2.2 Formulation of Models

In accordance with the guidelines developed in Chapter 2, an axial (one translational degree of freedom per mass) model was developed as shown in Fig. 16. Mass 1 is the driver mass, masses 2 through 14 are the missile shell structure with layered ablative covering, and mass 15 is the instrumentation package. Spring 1-2 is for the attachment fixture, springs 2-3 through 13-14 are for the missile shell, and spring 11-15 is the mounting for the instrumentation package.

The original model was developed with a value for spring 11-15 of $10^6$ lbs./in. The limiting conditions for the alternate design possibilities resulted in values of $10^5$ lbs./in. and $10^7$ lbs./in., respectively. The development of these limiting conditions is given in Appendix I.

The weights of the layered masses $W_i$ were obtained from the following relationship

$$W_i = \sum_{m=1}^{z} \pi d_m t_m L_i \rho_m$$  \hspace{1cm} (5.5)
where

\[ d_m = \text{diameter of shell/layer at mass point} \]
\[ t_m = \text{thickness of layer } j \]
\[ L_i = \text{length of mass } i \]
\[ \rho_m = \text{weight density of layer (lbs./in.}^3) \]
\[ z = \text{number of layers of material} \]

The axial stiffness \( k_{ij} \) for an individual spring was calculated by

\[ k_{ij} = \pi d_{ij} t_{ij} E/L_{ij} \quad (5.6) \]

where

\[ E = \text{modulus of elasticity of aluminum (10}^7 \text{ lbs./in.}) \]
\[ L_{ij} = \text{length between masses } i \text{ and } j \]

The layers of ablative covering were neglected in Eq. 5.6 because they did not contribute to the stiffness. The data used to formulate the model are summarized in Table 5 and the sketches for Eqs. 5.5 and 5.6 are given in Fig. 17.

5.2.3 Behavioral Responses

Only those plots of responses for the individual springs and masses which indicate the desired effects are included in the presentation of this example. Other plots were available, but they are not included because they do not add significantly to the analysis of the dynamic response.
In Figs. 18 through 23 the acceleration-time histories and corresponding shock spectra for the missile shell (masses 10 through 15) and the instrumentation package are shown. Figures 24, 25, and 26 are the force-time plots for springs connecting masses 10 through 15. Figures 27 through 30 present the maximum tension and compression forces and maximum positive and negative accelerations for the entire system.

The acceleration-time plots and corresponding shock spectra for the missile case (Figs. 18 through 23) exhibit the following characteristics:

1. The maximum accelerations for masses 10 through 14 are similar for all configurations. The largest difference found is for mass 12 (18 to 23 g's) as shown in Fig. 20. These maximums all occur in the first cycle of dynamic response and at similar times (2 to 3 milliseconds-msec.).

2. The shapes of the acceleration histories do exhibit phase differences. The shock spectra demonstrates that the frequencies above 600 Hz are considerably different in each superimposed spectra. The shock spectra are similar below 600 Hz) is apparent in all three curves as shown by the shock spectra.
The force-time histories for the missile shell indicate the following effects:

(1) The peak forces all occur in the first cycle of response as shown in Figs. 24 and 25. The values do vary with the largest difference found in spring 10-11 of 6.0 to 7.9 kips for the softer \((10^5\) lbs./in.) to the stiffer \((10^7\) lbs./in.) systems respectively.

(2) The basic shapes of these curves are different for each spring. The basic lower modes are apparent, but phase differences mask the similarities of these curves. The phase differences are most pronounced in the plots of forces in springs 10-11 and 12-13.

The curve of force-time for the spring describing the mounting/instrumentation package demonstrates these characteristics:

(1) The maximum force transmitted to the package is 3.2 kips in the stiffer system as shown in Fig. 26. The other maximums are 2.9 and 2.1 kips for the two softer systems.

(2) The shapes of the responses are considerably different.

(3) The basic lower mode of vibration \((100\) Hz) is present in the plot.
The min-max screening for internal forces gives the following information.

(1) The maximum tension forces in the missile system vary in the different configurations as shown in Fig. 27. The stiffer \(10^7\) system has the most maximum forces from the attachment fixture (base) to the mounting point. The maximum forces in the missile shell from the mounting to the nose are found in the softer \(10^5\) system.

(2) The maximum compressive forces in the missile system do vary appreciably except at spring 10-11 (adjacent to the mounting point) and at spring 11-15 (the mounting) as shown in Fig. 28. The magnitude of the compressive forces for each spring is greater than the tension force.

The min-max screening for acceleration shows the following characteristics:

(1) Masses 2 through 8 of the missile case experience the same positive accelerations for all configurations as shown in Fig. 29. The softer system shows the greatest maximum accelerations for masses 9 through 14 of the missile shell. The largest difference in maximum positive acceleration is shown in mass 14 (38 g's to 30 g's) for the softer to the stiffer systems.
(2) None of the maximum negative accelerations are above 15.5 g's for masses 2 through 12 as shown in Fig. 30. Only masses 13 and 14 have accelerations above 15.5 g's. The two largest differences occur at the nose of the missile case which shows 28 g's for the softer system and 21 g's for the stiffer system, and at mass 5 which shows 15.5 g's for the stiffer system and 8.5 g's for the softer system.

(3) The softer system has the most negative maximums of acceleration.

(4) The peak positive accelerations are larger than the absolute values for the maximum negative accelerations.

5.2.4 Discussion of Results and Sensitivity Analysis

The following observations can be made regarding the responses obtained from the multiple configuration analysis for the shock-driven missile system:

(1) The modifications to the stiffness for the mounting/instrumentation package do not significantly affect the maximum stresses in the shell. Only at spring 9-10 (adjacent to the mounting) do appreciable differences occur (Fig. 22).

(2) The stiffer model \((10^7)\) transmits the maximum force to the shell between the attachment and
the mounting. The softer system \(10^5\) transmits the maximum force to the shell between the mounting and the nose. Since the behavior of the shell is linear elastic and the shell does not experience buckling problems, the maximum forces for design purposes are determined.

(3) The acceleration-time curves exhibit significant differences as shown in Figs. 18 through 23. From the shock spectra and the force time histories, the following characteristics are noted: (a) the mounting does affect the accelerations, but (b) the stresses (min-max) are not significantly affected.

(4) The different mounting/instrumentation package affects only the higher frequency (above 600 Hz) behavior of the shell.

The responses for the instrumentation package suggest the following observations:

(1) The maximum force transmitted to the package is 3.2 kips. Even though order of magnitude differences exist in the springs, the range for maximum forces given the package is 3.2 to 2.0 kips. This is not a particularly significant different design condition for components.
(2) The maximum accelerations are not greatly different (13 to 21 g's) but they do occur at different times.

(3) The response histories for force and acceleration exhibit considerable phasing differences.

The sensitivity analysis of the parametric study of the modifications to the stiffness for the mounting/instrumentation package demonstrate the following observations:

(1) The modifications affect only the higher frequencies (above 600 Hz) of the missile shell structure. The acceleration and force histories are significantly affected, but the maximum values are not appreciably affected. Only at spring 9-10 do sizable differences occur. Therefore, the model for the missile shell is an adequate representation of the case structure for this excitation.

(2) The modifications do affect the response histories of force and acceleration for the mounting/instrumentation package. Even though order of magnitude differences exist in the range of stiffnesses, the peak forces (2.1 to 3.2 kips) and the maximum accelerations (13 to 21.5 g's) are not greatly different.
5.3 Blast Loading of Nose Cone Containing a Cantilever Beam

The purpose of this example is to demonstrate the use of the multiple configuration approach to analyze an impact problem. The impact occurs in the nose cone system when the free end of the cantilever beam impacts the coil spring support system shown in Fig. 31. The structural system was excited by a 50 g rigid body blast wave that contacted the nose first and swept over the surface of the cone at 5000 ft./sec.

The major problem associated with the treatment of an impact problem is the description of the load-displacement relationships for the parts of the structure experiencing contact. Peculiar to this type of problem is a "seating" or "meshing" of the parts in contact. The analysis of this seating phenomenon is an example of a complex analysis problem described in Section 2.6.

This structural system was constructed to evaluate various types of accelerometers. The accelerometers were expected to ascertain both low and high frequency responses so that overall and localized structural behavior could be determined. The mounting for the accelerometers is shown as part of the cantilever beam in Fig. 31. Although test results are not available for this system as analyzed with the impact phenomenon, this example demonstrates that the sensitivity analysis determine the pertinent information needed for the assessment of the models and the impact behavior.
5.3.1 Study of a Non-linear Structure and Formulation of Models

This section is organized so that the description of the non-linear spring describing the impact phenomenon is investigated as the first part of the formulation of the model. Then, the nose cone and cantilever beam will be modeled.

This impact problem is considered a non-linear problem because the load-displacement relationship for beam-coil spring interaction is a non-linear relationship. The non-linearity is composed of two phases: (1) the nominal gap of 0.01 inches, and (2) the variable slopes of the coil spring and support system as shown in Fig. 32 for the original model.

A detailed study of contact stresses shows that the force transmitted between the parts in contact is proportional to the area in contact. However, full area of contact is not made initially and the "seating" or "meshing" of the contacted parts is apparent. After the seating has occurred, a usually stiffening load-displacement relationship occurs. The analysis of these two phases of the load-displacement relationship is critical for determining structural behavior. Without a sensitivity analysis, it would not be known to what degree each phase of load-displacement curve influences the responses of the beam and the system.

The usual procedure used to describe a non-linear load-displacement curve is to use some form of quasi-linearization\(^{(3,5)}\). Consistent with approach, four load-displacement curves were developed as shown.
in Fig. 32. Care was taken to isolate the effects of the two phases of the curve as described previously.

The first curve was used for the original model. This curve is composed of a 0.01 in. gap, an initial spring of 50 lbs./in. (coil spring), and stiffer second slope of $10^4$ lbs./in. reflecting the support system. The second curve (model 1) has an initial gap of 0.005 in., a seating of 0.015 in., an initial slope of 50 lbs./in., and the final slope of $10^4$ lbs./in. The third relationship (model 2) consisted of a seating of 0.01 in., a stiffer initial slope of 75 lbs./in., a second slope of 150 lbs./in., and a final slope of $10^4$ lbs./in. The fourth curve (model 3) had no initial gap, an initial slope of 33 lbs./in., a second slope of 100 lbs./in., a third slope of 200 lbs./in., and a final slope of $10^4$ lbs./in.

An original model was developed for the nose cone and the cantilever beam. The model parameters are summarized in Table 6. The data needed to develop the models are summarized in Table 7.

The mass discretization was chosen for the nose cone so that the sweeping blast could be applied realistically. The weights of the shell masses were calculated according to Eq. 5.5. The mass discretization for the beam was chosen so that sufficient frequencies could be modeled to agree with possible test results. The weights of the beam masses were calculated by:

$$w_i = \pi d_i^2 L_i \rho_i / 4$$  \hspace{1cm} (5.7)
Masses 1 through 15 represented the nose cone, mass 16 is the back cover, mass 17 through 20 are the cantilever beam with mass 19 representing the accelerometer mounting as shown in Fig. 31.

The stiffnesses for the beam were calculated using the equations given in Appendix C. The stiffness coefficients for the nose cone were also calculated using the equations of Appendix C modified for conical elements (57). These equations for a constant thickness cone are given below and the modeling of a conical shell structure is shown in Fig. 33:

\[
\begin{align*}
    a_{ij} &= -D/(AD - BC) \\
    b_{ij} &= (C - DL_{ij})/(AD - BC) \\
    c_{ij} &= B/(AD - BC) \\
    d_{ij} &= (BL_{ij} - A)/(AD - BC)
\end{align*}
\]

\[\text{*(C-5 through C-8)}\]

where

\[
\begin{align*}
    A &= \left[ \ln \frac{L}{d} - \frac{1}{2} \left( \frac{d}{L} \right)^2 + 2 \left( \frac{d}{L} \right) - \frac{3}{2} \\
    &\quad + \sin^2 \theta (1 + \nu)(1 - \left( \frac{d}{L} \right)^2) \right] / \pi E t \sin^3 \theta
\end{align*}
\]

\[
\begin{align*}
    B &= C = \left[ (1 - \left( \frac{d}{L} \right)^2) - 2 \sin^2 \theta (1 + \nu)(1 - \left( \frac{d}{L} \right)^2) \right] / \pi E t d \sin^3 \theta
\end{align*}
\]

\[
\begin{align*}
    D &= \left[ (1 - \left( \frac{d}{L} \right)^2)(1 + 2(1 + \nu) \sin^2 \theta) \right] / \pi E t d^2 \sin^3 \theta
\end{align*}
\]
and

\[
t = \text{thickness of shell}
\]

\[
\theta = \text{cone half angle}
\]

\[
\nu = \text{Poisson's ratio}
\]

\[
d = \text{distance from cone apex to smaller diameter of cone segment}
\]

\[
L = \text{distance from cone apex to larger diameter of cone segment}
\]

\[
L_{ij} = \text{distance between mass centers}
\]

\[
E = \text{modulus of elasticity}
\]

The four load-displacement relationships for the cantilever beam support system (spring 14-20) were incorporated into four configurations of the model for the nose cone-beam system.

Only those responses of the beam and nose cone that are useful in the sensitivity analysis of the impact phenomenon and structural behavior of the system are included.

5.3.2 Response Behavior

The plots of maximum and minimum moments in the nose cone and cantilever beam demonstrate the following characteristics:

1. The maximum negative moments for the nose cone are significantly greater than the maximum positive moments as shown in Figs. 34 and 35.

The negative moments are essentially constant along the length of the nose cone.
(2) The maximum positive moments for the cantilever beam occur at the base. The maximum negative moments occur at the accelerometer mounting (see insets in Figs. 34 and 35). The peak magnitudes of the min-max moments are similar although model 1 (smoothed curve and 50 lbs./in. initial slope) exhibits the largest number of maximums for the beam.

The plots of min-max accelerations for the nose cone and beam demonstrate the following behavior:

(1) All four configurations have the same maximum positive accelerations as shown in Fig. 36.

(2) The four configurations also show that the maximum negative accelerations are the same as shown in Fig. 37.

(3) The summary for the beam exhibits similar peak positive and negative accelerations as shown in the insets in Figs. 36 and 37.

As shown in the plots of acceleration-time, three collisions occur between the beam and coil spring support system. The time span covered also shows that the maximums of accelerations associated with each mass have been reached as shown in Figs. 38 through 41. The maximum moments have also been reached as shown in Figs. 42 and 43.
The response histories for linear acceleration and corresponding shock spectra suggest the following observations:

1. Phase differences are present in all plots of acceleration. The shock spectra demonstrate that similar frequencies are present in all configurations.

2. Model 1 does exhibit a lower dominant frequency (1100 Hz) in contrast to approximately 1200 Hz for other configurations (Figs. 38 through 41).

3. The maximum accelerations vary from mass to mass with the original model having the majority (six of eight) of the maximum responses for the beam.

The responses for moment in the cantilever beam demonstrate the following behavior:

1. The curves are very similar as shown in Figs. 42 and 43. Some phase differences are apparent, particularly after the second and third impacts.

2. The maximum moment occurs at the base of the beam.

3. The two dominant frequencies of 100 and 1200 Hz are visible in the moment history of the base of the beam (spring 11-17).
(4) The greatest difference shown in the histories is for model 1 (smoothed curve and softer initial slope).

The force-time history for the beam coil spring support system interaction exhibits the following characteristics:

(1) The effects of the softer (50, 75, 100 lbs./in.) and stiffer \( \left(10^6 \text{ lbs./in.}\right) \) parts of the load-displacement curves are vividly demonstrated (Fig. 44). The nearly vertical portion of the force-time curve is a function of the stiffer part of the load-displacement relationship. The rounded parts of the responses are functions of the softer initial slopes.

(2) The beam-spring support system interacts in only one direction.

(3) The maximum responses for all four configurations are very similar.

(4) The response histories are very nearly alike for the time span shown.

5.3.3 Discussion of Results and Sensitivity Analysis

From the responses obtained from the multiple configuration analysis for the nose cone and cantilever beam system, the following observations can be made and the sensitivity analysis explained:

(1) The effect of different initial loading slopes does not significantly change the response
behavior of the beam-spring support interaction. The sensitivity analysis demonstrates that the responses are similar. The lack of difference in behavior for the configurations is problem dependent and should not be construed as indicative of the responses of all impact problems.

(2) The sensitivity analysis shows that the more significant behavioral differences among the configurations is for the smoothing effect. This smoothing effect is most noticeable in the responses for the original model and model 1. The differences in responses are noted for accelerations of and moments in the beam. The shock spectra shows the difference in the second dominant frequency.

(3) This structural system provides the necessary low (100 Hz) to high frequency (1900 Hz) structural behavior needed to evaluate accelerometers. The sensitivity analysis demonstrates that the behavior of the beam is isolated and relatively insensitive to the impact phenomenon.

(4) The impact of the beam does not affect the min-max responses of accelerations and moments in the nose cone (Figs. 34 - 37). Further, the response histories are not significantly affected (Figs. 38 - 44).
6. SUMMARY, CONCLUSIONS, AND FUTURE RESEARCH

This chapter presents a summary, specific conclusions for each of the example problems given in Chapter 5, general conclusions for this study, and recommendations for future research.

6.1 Summary

The principal objective of this dissertation is the development of a general method to determine the sensitivity of mathematical models describing deterministic transient dynamics.

The need for this sensitivity analysis is described in Chapter 1. This need is emphasized because of the approximations usually needed to define the models of structural systems to analyze dynamic response.

The area of modeling as presented in Chapter 2 is represented as an analytical procedure requiring assumptions and engineering judgment. Specific guidelines for the development of and problems associated with the use of models are also given.

The various methods for the study, analysis, improvement, and optimization of mathematical models are examined in Chapter 3. A table summary of the advantages and disadvantages of the various methods and the proposed method are given in Appendix B.
In Chapter 4 the proposed method of multiple configuration analysis to determine the sensitivity of mathematical models subjected to modifications is presented. The mathematical formulation of the equations of motion and their solution and development of multiple configurations of an individual model are given. The details of computer usage and programming are given. The comparative procedures used to determine the sensitivity analysis are the superimposed plot, the min-max screening of responses, and the shock spectra of response behavior.

Detailed applications of this approach are presented as example problems contained in Chapter 5. The use of the three techniques for sensitivity analysis are demonstrated. In each example, the sensitivity analysis of the particular system is determined.

6.2 Specific Conclusions

Based on the detailed example problems presented in Chapter 5, specific conclusions for the sensitivity analysis of each problem will be established. The following conclusions are reached for the Summit Bridge:

(1) The behavior of the superstructure was demonstrated to be basically independent of the effects of the elastic support at pier 5. Only with the vehicles in the suspended span did responses differ for displacements and internal forces.
(2) The behavior of the superstructure was greatly influenced by the presence of hinges at the ends of the suspended span. The sensitivity analysis showed the degree to which these hinges affected the responses.

(3) The hinge was correctly modeled.

(4) The correlation with test results showed that the responses of the original model exhibited the closest agreement with the experimental data.

(5) The assumption that the vibration of the anchor span bent was caused by change in lengths of the bottom chords of the trusses was verified.

(6) The experimental moment history at the nose of the bent was bounded by the responses obtained from the multiple configuration analysis.

(7) The displacement history of the upper part of the bent was not appreciably affected by the choice of a realistic base-foundation stiffness.

(8) The maximum stresses were determined at the base of the bent as based on the use of the limiting conditions used in the formulation of the multiple configurations.
(9) The sensitivity analysis demonstrated that the integrity of the bridge was not impaired.

Stated below are the conclusions reached for the sensitivity analysis of the shock-driven missile system:

(1) The maximum stresses in the missile shell were not appreciably affected.

(2) The effect of modifications in the mounting/instrumentation were essentially changes to the high frequency behavior of the system.

(3) The maximum accelerations of and forces transmitted to the instrumentation package were determined. Any subsequent modifications to the mounting/package system within the limiting conditions should have responses within the values determined.

The following conclusions are reached for the behavior of the nose cone and beam system:

(1) The impact of the beam did not appreciably affect the behavior of the nose cone.

(2) The smoothing effect on the load-displacement relationships is more critical than the initial slopes.

(3) The beam was shown to be relatively insensitive to various configurations. Therefore, the
responses of the beam would give a good
basis of comparison for the evaluation
of accelerometers.

6.3 General Conclusions

The following general conclusions can be established as based on this study and the example problems:

(1) The use of sensitivity analysis is an effective tool to determine the applicability of a model to treat transient dynamics.

(2) The multiple configuration analysis is needed to investigate typical models of structural systems needing significant modifications for the formulation of a model.

(3) The application of a general method for the analysis of models for structural dynamics has been made.

(4) The structural engineer should be concerned with modal analysis methods for linear structural behavior and integration of the equations of motion for the treatment of transient analysis.

(5) The methods available are not applicable as a general method to determine sensitivity of models.
(6) The comparative techniques used are all that are necessary to determine sensitivity of models that may include non-linear behavior.

(7) The application of shock spectra to multiple degree of freedom systems has been made.

6.4 Future Research

The following areas of recommended future research are based on the specific and general conclusions established and observations made concerning this study:

(1) The study of this type of approach in the context of a stochastic treatment could be done.

(2) The application of the proposed method to analyze parts of a structure rather than the entire system. The development of data analysis techniques to facilitate this substructure analysis could be done.

(3) Mathematical methods are needed to establish uniqueness of solution and error bounds for the analysis of non-linear systems. The work of Kavanaugh, et al (19,28,71) needs to be developed in a mathematically tractable form.

(4) The application of multiple configuration analysis and appropriate comparative procedures
to other methods describing structural behavior, such as the finite element approach.

(5) While the proposed method can give the maximum and minimum responses for each configuration analyzed, maximum responses for a particular system may not be known. The need exists to determine maximum responses for a multiple degree of freedom system based on the analysis of a few configurations. The development of this technique would be an extension of the work of Sevin and Pilkey (65).

(6) The application of the proposed method to combine various types of responses for a particular system such as axial and bending, torsional and bending, and axial and torsional. Also, limits of such application should be investigated, particularly in the non-linear range.
# TABLE 1

## SUMMARY OF MODELS FOR SUMMIT BRIDGE SUPERSTRUCTURE

<table>
<thead>
<tr>
<th>MASS NO.</th>
<th>WEIGHT</th>
<th>INERTIA</th>
<th>SPRING</th>
<th>STIFFNESS (see Appendix C)</th>
<th>LENGTH</th>
</tr>
</thead>
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<td></td>
<td></td>
<td>lb-in²</td>
<td>i - j</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Kips</td>
<td>x 10⁻¹¹</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>in</td>
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<td>1.3</td>
<td>0+1 - 16</td>
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</table>

* = original model (see Fig. 5)

□ = original plus elastic support (bent spring)

+ = original with bent spring and hinges (see Fig. 4)
### TABLE 2

**SUMMARY OF MODEL DATA FOR SUPERSTRUCTURE OF SUMMIT BRIDGE**

<table>
<thead>
<tr>
<th>MASS NO.</th>
<th>LENGTH OF MASS $L_i$ (ft)</th>
<th>WEIGHT PER FOOT (kips/ft)</th>
<th>MASS CENTER FROM BOTTOM CHORD (in)</th>
<th>WEIGHT INERTIA $(\text{lb} \cdot \text{in}^2) \times 10^{-11}$</th>
<th>EQUIVALENT SECTION PROPERTIES AT MASS $i-j$</th>
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</thead>
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<td></td>
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<td>$i-j$</td>
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<td>60.</td>
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<td>615.</td>
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<tr>
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</tr>
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<td>60.</td>
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<td>840.</td>
<td>.8414</td>
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<tr>
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<td>30.</td>
<td>16.0</td>
<td>470.</td>
<td>.50</td>
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</tr>
<tr>
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<td></td>
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</tr>
<tr>
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<td>120.</td>
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<td>2.60</td>
<td>11-12</td>
</tr>
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<td>30.</td>
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<td>470.</td>
<td>.50</td>
<td>12-13</td>
</tr>
<tr>
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<td>60.</td>
<td>16.0</td>
<td>840.</td>
<td>.8414</td>
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<td>16.0</td>
<td>615.</td>
<td>4.0</td>
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<td>2.60</td>
<td>15</td>
</tr>
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<td>30.</td>
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<td>498.</td>
<td>.1349</td>
<td>16</td>
</tr>
<tr>
<td>16</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td></td>
<td>one-half of mass 8</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td></td>
<td>one-half of mass 10</td>
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</table>

-115-
TABLE 3
SUMMARY OF CONFIGURATIONS FOR
ANCHOR SPAN BENT OF SUMMIT BRIDGE

<table>
<thead>
<tr>
<th>MASS NO.</th>
<th>WEIGHT</th>
<th>INERTIA</th>
<th>SPRING</th>
<th>STIFFNESS (see Appendix C)</th>
<th>LENGTH</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>lbs.</td>
<td>lb-in²</td>
<td>i - j</td>
<td>aᵢjbᵢj cᵢjcᵢj dᵢjdᵢj</td>
<td>in.</td>
</tr>
<tr>
<td>1</td>
<td>2090.</td>
<td>.464</td>
<td>1 - 2</td>
<td>.196 .155 .155 .778</td>
<td>158.</td>
</tr>
<tr>
<td>2</td>
<td>1600.</td>
<td>.359</td>
<td>2 - 3</td>
<td>.196 .155 .155 .778</td>
<td>158.</td>
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<tr>
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<td>4525.</td>
<td>2.635</td>
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<td>158.</td>
</tr>
<tr>
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<td>1600.</td>
<td>.326</td>
<td>4 - 5</td>
<td>.196 .155 .155 .778</td>
<td>158.</td>
</tr>
<tr>
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<td>158.</td>
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<tr>
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<td>.236</td>
<td>6 - 7</td>
<td>.196 .155 .155 .778</td>
<td>158.</td>
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<tr>
<td>7</td>
<td>2720.</td>
<td>.464</td>
<td>7 - 8</td>
<td>1.0 .5 .5 1.0</td>
<td>100.</td>
</tr>
<tr>
<td>8</td>
<td>20000.</td>
<td>1.0</td>
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MODEL DATA FOR
ANCHOR SPAN BENT OF SUMMIT BRIDGE

<table>
<thead>
<tr>
<th>MASS NO.</th>
<th>LENGTH OF MEMBERS</th>
<th>WEIGHT OF MASS i</th>
<th>EQUIVALENT SECTION PROPERTIES AT MASS i - j</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>in.</td>
<td>lbs.</td>
<td>i - j</td>
</tr>
<tr>
<td>1</td>
<td>6.5' Column</td>
<td>800</td>
<td>1 - 2</td>
</tr>
<tr>
<td>13.0' Horiz.</td>
<td>1290</td>
<td>2 - 3</td>
<td>6.91</td>
</tr>
<tr>
<td>2</td>
<td>13.0' Column</td>
<td>1600</td>
<td>3 - 4</td>
</tr>
<tr>
<td>3</td>
<td>13.0' Column</td>
<td>1600</td>
<td>4 - 5</td>
</tr>
<tr>
<td>20.0' Diagonal</td>
<td>1635</td>
<td>5 - 6</td>
<td>6.91</td>
</tr>
<tr>
<td>13.0' Horiz.</td>
<td>1290</td>
<td>6 - 7</td>
<td>6.91</td>
</tr>
<tr>
<td>4</td>
<td>13.0' Column</td>
<td>1600</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>13.0' Column</td>
<td>1600</td>
<td></td>
</tr>
<tr>
<td>20.0' Diagonal</td>
<td>1635</td>
<td></td>
<td></td>
</tr>
<tr>
<td>13.0' Horiz.</td>
<td>1290</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>13.0' Column</td>
<td>1600</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>6.5' Column</td>
<td>800</td>
<td></td>
</tr>
<tr>
<td>13.0' Horiz.</td>
<td>1290</td>
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<td></td>
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</tbody>
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TABLE 4

SUMMARY OF CONFIGURATIONS FOR
SHOCK-DRIVEN MISSILE SYSTEM

<table>
<thead>
<tr>
<th>MASS NO.</th>
<th>WEIGHT lbs.</th>
<th>SPRING i - j</th>
<th>AXIAL SPRING $a_{ij}$ = lbs/in $\times 10^{-6}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10000.</td>
<td>1 - 2</td>
<td>10.0</td>
</tr>
<tr>
<td>2</td>
<td>200.</td>
<td>2 - 3</td>
<td>9.5</td>
</tr>
<tr>
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<td>200.</td>
<td>3 - 4</td>
<td>9.0</td>
</tr>
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<td>200.</td>
<td>4 - 5</td>
<td>8.5</td>
</tr>
<tr>
<td>5</td>
<td>200.</td>
<td>5 - 6</td>
<td>8.0</td>
</tr>
<tr>
<td>6</td>
<td>200.</td>
<td>6 - 7</td>
<td>7.5</td>
</tr>
<tr>
<td>7</td>
<td>200.</td>
<td>7 - 8</td>
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</tr>
<tr>
<td>8</td>
<td>160.</td>
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<td>6.5</td>
</tr>
<tr>
<td>9</td>
<td>230.</td>
<td>9 - 10</td>
<td>6.0</td>
</tr>
<tr>
<td>10</td>
<td>200.</td>
<td>10 - 11</td>
<td>6.0</td>
</tr>
<tr>
<td>11</td>
<td>120.</td>
<td>11 - 12</td>
<td>6.0</td>
</tr>
<tr>
<td>12</td>
<td>60.</td>
<td>12 - 13</td>
<td>5.0</td>
</tr>
<tr>
<td>13</td>
<td>50.</td>
<td>13 - 14</td>
<td>5.0</td>
</tr>
<tr>
<td>14</td>
<td>50.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>150.</td>
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</tr>
<tr>
<td></td>
<td></td>
<td>□11 - 15</td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>+11 - 15</td>
<td>10.0</td>
</tr>
</tbody>
</table>

* = original model

□ = modified mounting/instrumentation

package $k_{11-15} = 0.1 \times 10^6 = 10+5$

+ = modified mounting/instrumentation

package $k_{11-15} = 10.0 \times 10^6 = 10+7$
### TABLE 5

**SUMMARY OF MODEL DATA FOR SHOCK-DRIVEN MISSILE SYSTEM**

<table>
<thead>
<tr>
<th>MASS NO.</th>
<th>LENGTH in.</th>
<th>DIAMETER</th>
<th>( t_{\text{SHELL}} ) ( \rho = 0.1 )</th>
<th>( t_{\text{LAYER}} ) ( \rho = 0.2 )</th>
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<tbody>
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<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
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<td>3</td>
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<td>26.</td>
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<td>0.22</td>
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<td>24.</td>
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<td>0.24</td>
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<td>0.26</td>
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<td>20.</td>
<td>0.1</td>
<td>0.28</td>
</tr>
<tr>
<td>7</td>
<td>10.</td>
<td>18.</td>
<td>0.1</td>
<td>0.30</td>
</tr>
<tr>
<td>8</td>
<td>9.</td>
<td>16.</td>
<td>0.12</td>
<td>0.20</td>
</tr>
<tr>
<td>9</td>
<td>11.</td>
<td>14.</td>
<td>0.20</td>
<td>0.30</td>
</tr>
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<td>9.</td>
<td>12.</td>
<td>0.15</td>
<td>0.20</td>
</tr>
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<td>10.</td>
<td>10.</td>
<td>0.10</td>
<td>0.20</td>
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<td>10.</td>
<td>8.</td>
<td>0.10</td>
<td>0.10</td>
</tr>
<tr>
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<td>10.</td>
<td>6.</td>
<td>0.10</td>
<td>1.5</td>
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<td>14</td>
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<td>4.</td>
<td>0.10</td>
<td>0.25</td>
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<tr>
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<td>-</td>
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</table>
**TABLE 6**

SUMMARY OF CONFIGURATIONS FOR
NOSE CONE WITH INTERNAL CANTILEVER BEAM

<table>
<thead>
<tr>
<th>MASS NO.</th>
<th>WEIGHT (lbs.)</th>
<th>INERTIA (lb.-in²)</th>
<th>SPRING STIFFNESS (lbs/ in, in-lbs/radian x 10⁻⁶)</th>
<th>LENGTH (in)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>i - j</td>
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<td>0.443</td>
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<td>2 - 3</td>
<td>.1758</td>
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<td>.5342</td>
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<td>8.855</td>
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<td>.7187</td>
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<td>1.669</td>
<td>27.90</td>
<td>12-13</td>
<td>2.194</td>
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<td>2.718</td>
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<td>0.019</td>
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<td>0.020</td>
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</table>

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### TABLE 7
MODEL DATA FOR NOSE CONE AND CANTILEVER BEAM

<table>
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<tr>
<th>MASS NO.</th>
<th>FROM STATION</th>
<th>TO STATION</th>
<th>MASS CENTER OF GRAVITY</th>
<th>RADIUS FROM CENTERLINE in</th>
<th>THICKNESS in</th>
</tr>
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<td>0.72</td>
<td>0.72</td>
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<td>10.282</td>
<td>1.55</td>
<td>0.52</td>
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<tr>
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<td>22.542</td>
<td>2.85</td>
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<td>29.145</td>
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<td>46.938</td>
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</tr>
<tr>
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<td>42.461</td>
<td>44.711</td>
<td>43.586</td>
<td>0.25</td>
<td>0.25</td>
</tr>
<tr>
<td>18</td>
<td>44.711</td>
<td>46.961</td>
<td>45.836</td>
<td>0.25</td>
<td>0.25</td>
</tr>
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<td>48.961</td>
<td>47.961</td>
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<td>0.95</td>
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<tr>
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<td>48.961</td>
<td>50.961</td>
<td>49.961</td>
<td>0.25</td>
<td>0.25</td>
</tr>
</tbody>
</table>

aluminum: $E = 10^7$ lbs/in

$\rho = 0.10$ lbs/in$^3$
Masses i and j connected with a massless spring showing positive stress resultants

Definition of terms used in stiffness matrix

\[ a_{ij} = \text{shear at } i \text{ caused by a unit displacement at } j \text{ with no rotation at } j \]

\[ b_{ij} = \text{moment at } i \text{ caused by a unit displacement at } j \text{ with no rotation at } j \]

\[ c_{ij} = \text{shear at } i \text{ caused by a unit rotation at } j \text{ with no displacement at } j \]

\[ d_{ij} = \text{moment at } i \text{ caused by a unit rotation at } j \text{ with no displacement at } j \]

Fig. 1 BEAM ELEMENT, SIGN CONVENTION, AND TERMS NEEDED FOR STIFFNESS MATRIX

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Fig. 2 DEFORMED BEAM ELEMENT
LUMPED-MASS MODEL OF SUMMIT BRIDGE
ANCHOR, CANTILEVER, SUSPENDED, CANTILEVER, AND ANCHOR SPANS

![Diagram of the lumped-mass model of Summit Bridge with labeled spans and a test vehicle.]

**ELEVATION VIEW SUMMIT BRIDGE**

**TEST VEHICLE**

Fig. 3 SUMMIT BRIDGE: ELEVATION VIEW AND LUMPED-MASS MODEL AND TEST VEHICLE
(a) Typical Mass Section of Superstructure

(b) Mass Center

(c) Superstructure and Model

(d) Model of Hinge

Fig. 4 MODELING OF SUMMIT BRIDGE SUPERSTRUCTURE
Fig. 5 MULTIPLE CONFIGURATIONS FOR SUPERSTRUCTURE
Fig. 6 SUMMIT BRIDGE: TWIN BENTS AND ANCHOR SPAN BENT MODELS

LUMPED-MASS MATHEMATICAL MODELS FOR BOTH CONFIGURATION (models for bending in direction of longitudinal axis of bridge)
Fig. 7 DETAIL OF FOUNDATION FOR ANCHOR SPAN BENT
Fig. 8 SUMMIT BRIDGE FORCING FUNCTIONS FOR EACH MASS AS CIRCLED
LUMPED-MASS MODEL OF SUMMIT BRIDGE
ANCHOR, CANTILEVER, SUSPENDED, CANTILEVER, AND ANCHOR SPANS

in.

\[ t = 0.0 \text{ sec.}, \text{ front axle enters right anchor span at pier 2} \]
\[ t = 28.3 \text{ sec.}, \text{ rear axle passes from left anchor span at pier 5} \]

Fig. 9 PLOT OF VERTICAL DISPLACEMENT OF M (MASS 9) VS. TIME

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LUMPED-MASS MODEL OF SUMMIT BRIDGE
ANCHOR, CANTILEVER, SUSPENDED, CANTILEVER, AND ANCHOR SPANS

Fig. 10 REACTION AT PIER 5 (AXIAL FORCE IN BENT) WITH ORIGINAL AND AVERAGED TEST DATA
Fig. 11 DISPLACEMENT OF BENT TOP DUE TO ANGULAR ROTATIONS OF SUPERSTRUCTURE AT PIERS 4 AND 5

\[ \delta_{\text{BENT TOP}}(t) = \theta_1(t) L_1 + \theta_6(t) L_6 \]
LUMPED MASS MODEL OF SUMMIT BRIDGE
ANCHOR, CANTILEVER, SUSPENDED, CANTILEVER, AND ANCHOR SPANS

Fig. 12 ROTATION OF SUPERSTRUCTURE (MASS 1) AT PIER 5
(TWO TRUCKS RIGHT TO LEFT AT 30 MPH)

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LUMPED MASS MODEL OF SUMMIT BRIDGE
ANCHOR, CANTILEVER, SUSPENDED, CANTILEVER, AND ANCHOR SPANS

![Diagram of the bridge model]

Fig. 13  ROTATION OF SUPERSTRUCTURE (MASS 6) AT PIER 4
(TWO TRUCKS RIGHT TO LEFT AT 30 MPH)
Fig. 14  DISPLACEMENT VS. TIME FOR UPPER THIRD POINT (MASS 3) FOR ANCHOR SPAN BENT
MOMENT

ELEVATION VIEW
ANCHOR SPAN BENT

LUMPED-MASS MODEL

(two trucks
right to left at
30 mph on main bridge

Fig. 15 MOMENT VS. TIME FOR BASE OF ANCHOR SPAN BENT (SPRING 6-7)

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LUMPED-MASS STRUCTURAL MODEL

\[ k_{ij} = A_{ij} E_{ij} / L_{ij} \]

CROSS-SECTIONAL VIEW
SHOCK-DRIVEN MISSILE SYSTEM

Fig. 16 SHOCK-DRIVEN MISSILE SYSTEM
Weight

\[ W_i = \sum_{j=1}^{z} \pi d_m t_m L_i \rho_m \]  \hspace{1cm} (5.5)

Stiffness \hspace{1cm} (d_{ij} = (d_i + d_j)/2, \ E = E_s, \ E_s \gg E_l)

\[ a_{ij} = k_{ij} = \pi d_{ij} t_{shell} E / L_{ij} \]  \hspace{1cm} (5.6)

\[ k_{ij} = 2 \pi E t_{shell} \sin \theta \cos \theta / \ln (L/d) \]

Fig. 17 MODELING OF MISSILE SHELL FOR SHOCK-DRIVEN MISSILE SYSTEM
**Fig. 18** COMPARED ACCELERATIONS AND SHOCK SPECTRA FOR MASS 10 (MISSILE CASE)

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Fig. 19 COMPARED ACCELERATIONS AND SHOCK SPECTRA FOR MASS 11 (MISSILE CASE AND INSTRUMENTATION MOUNT)
MatheMatical model

Fig. 20 Compared accelerations and shock spectra for mass 12 (missile case)
Fig. 21 COMPARED ACCELERATIONS AND SHOCK SPECTRA FOR MASS 13 (MISSILE CASE)
Fig. 22 COMPARED ACCELERATIONS AND SHOCK SPECTRA FOR MASS 14 (NOSE OF MISSILE)
Fig. 23 COMPARSED ACCELERATIONS AND SHOCK SPECTRA FOR MASS 15 (INSTRUMENTATION PACKAGE)
Fig. 24 COMPARSED INTERNAL FORCES IN MISSILE CASE AT SPRINGS 10-11 AND 11-12
Fig. 25 COMPARRED INTERNAL FORCES IN MISSILE CASE AT SPRINGS 12-13 AND 13-14
Fig. 26  COMPARED INTERNAL FORCES IN MOUNTING FOR INSTRUMENTATION PACKAGE AT SPRING 15-11

F(t)  FORCES VS. TIME SPRING 15-11
Time in milliseconds - Response in Kips

KIPS

-4 -2 0 2 4
0 10 20 30

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Fig. 27 MAXIMUM TENSION FORCES IN MISSILE CASE AND MOUNTING
Fig. 28 MAXIMUM COMPRESSIVE FORCES IN MISSILE CASE AND MOUNTING

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Fig. 29 MAXIMUM POSITIVE ACCELERATIONS IN MISSILE CASE AND INSTRUMENTATION PACKAGE
Fig. 30  MAXIMUM NEGATIVE ACCELERATIONS IN MISSILE CASE AND INSTRUMENTATION PACKAGE
Lumped-Mass Model

Typical $F(t)$ for each mass of Nose Cone (longitudinal velocity of sweeping blast wave is 5000 ft./sec.)

Cross-Sectional View of Nose Cone With Cantilever Beam

Fig. 31 NOSE CONE WITH INTERNAL CANTILEVER BEAM
Fig. 32 LOAD-DEFLECTION CURVES (P (lbs) vs. δ (in)) FOR COIL SPRING/CANTILEVER BEAM FOR MULTIPLE CONFIGURATIONS OF SPRING 14-20

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Cross-Section of Nose Cone and Beam

Fig. 33 MODELING OF NOSE CONE AND CANTILEVER BEAM
Fig. 34 MAXIMUM POSITIVE MOMENTS IN NOSE CONE AND CANTILEVER BEAM

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Fig. 35 MAXIMUM NEGATIVE MOMENTS IN NOSE CONE AND CANTILEVER BEAM
Fig. 36  MAXIMUM POSITIVE ACCELERATIONS FOR NOSE CONE WITH CANTILEVER BEAM

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Fig. 37  MAXIMUM NEGATIVE ACCELERATIONS FOR NOSE CONE WITH CANTILEVER BEAM
Fig. 38 COMPARED LINEAR ACCELERATIONS AND SHOCK SPECTRA AT BASE OF CANTILEVER BEAM (MASS 17)
Fig. 39 COMPARED LINEAR ACCELERATIONS AND SHOCK SPECTRA OF CANTILEVER BEAM (MASS 18)
Fig. 40 COMPARED LINEAR ACCELERATIONS AND SHOCK SPECTRA OF ACCELEROMETER MOUNTING (MASS 19)
Fig. 41 COMPARED LINEAR ACCELERATIONS AND SHOCK SPECTRA OF FREE END OF CANTILEVER BEAM (MASS 20)
Fig. 42 COMPARED MOMENTS ALONG CANTILEVER BEAM: SPRINGS 11-17 (BASE) AND 17-18
Fig. 43 COMPARED MOMENTS ALONG CANTILEVER BEAM: SPRINGS 18-19 AND 19-20
Nose Cone with Cantilever Beam
Lumped-Mass Model

FORCE

FORCE VS. TIME IN NONLINEAR SPRING 14-20

Fig. 44 COMPARED FORCES IN NONLINEAR SPRING 14-20
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APPENDIX A

GLOSSARY

Analytical (Closed-Form) Solution: A mathematical solution that does not require a numerical or iterative approach.

Approximations: The engineering judgment and assumptions needed to define material properties, geometric constraints, forcing functions, and damping so that a model can be formulated.

Behavior: The observable and computable action of a structural system which has been subjected to various loading conditions.

Configuration: The description of a structural system in terms of a model with a particular set of parameters describing the system.

Constituitive Equation: The complete mathematical relationship describing the behavior of a structural system. The relationship consists of the material and geometric properties needed to define the structure.
Damping: The physical phenomenon associated with the dissipation of energy in cyclic motion. The usual forms are structural or hereditary (material and geometric), viscous (proportional to velocity), and Coulomb friction (proportional to displacement).

Degrees of Freedom: Independent deflection configurations in a generalized coordinate system.

Deterministic: The description of a structural system and forcing function with known parameters.

Discrete-Mass: The concentration at a point in space of the mass and inertial characteristics of a part of a structure.

Domain: The general sphere of interest and influence of the dynamic response of structures.

Dynamic Load: An applied forcing function having a frequency content which is near or above the natural frequencies of a structural system.

Excitation: An applied set of conditions given a structural system sufficient to produce dynamic response. Examples include impulse, blast, initial displacements and velocities, and impact.
Forcing Function: The application of a time-dependent loading system to a structure. The loading system may produce static as well as dynamic responses, although forcing functions are usually considered dynamic in nature. (see excitation, load/loading).

\[ g: \] The acceleration of gravity (386.4 in. sec.\(^2\)).

Geometric Properties: The given sizes, shapes, thicknesses, and configuration of a structural system.

History: A record (time versus magnitude) of occurrence of the dynamic behavior of a part of a structural system.

Homogeneous Solution: The solution to a differential equation in which the right hand side of the equation is equal to zero. In dynamics, the solution is composed of terms which are harmonic in nature.

Limiting Conditions: Conditions which place constraints of possible structural behavior, such as, a fixed or pinned support condition for a beam or for maximum displacements.

Linear Elastic: In the response of structures to a loading system, the adherence of the behavior to straight line load-displacement (stress-strain) relationships.
Load/Loading: The application of a set of applied forces to a structural configuration.

Load Path (Flow of Forces): The manner in which the applied forces are distributed throughout a given system. The distribution depends on the type of loading and the relative stiffnesses and springs of a given structure.

Lumped-Mass: (see Discrete-Mass)

Material Properties: The needed mechanical properties of Young's modulus, Poisson's ratio, and yield strength required to define the stress-strain behavior of a structural system.

Mathematical Model: A series of equations which describe a structural system and applied forces. The solution of the set of equations is the response behavior of a given system.

Min-Max Screening (Response): A summary of a particular set of peak responses for the entire structural system. The sets of responses may include minimum and maximum accelerations, velocities, displacements, forces, moments, and shears. The major use of these summaries is to show maximum behaviors which occur and make these peaks independent of time.
Modal Superposition (Normal Mode Method): The differential equations of motion are decoupled when the displacements are expressed in terms of the normal modes of the structure. With the superposition of the normal modes the displacement-time histories can be calculated. Then the corresponding response behaviors for the structure can be determined.

Model Parameter: A quantity describing a specified coefficient of the governing equations of motion.

Modeling: The analytical art of describing structural systems in terms of equations. The responses of structures are studied through the solutions of the equations. Therefore the meanings for models, modeling, and solutions are interchangeable.

Modifications/Modified Models: The alteration of the model parameters from one configuration to another. The modifications result in different models describing a particular structural system.

Multiple Configuration Analysis: The solution of different models of a particular structure. The different models are developed through efforts to study, analyze, and predict possible responses of a particular system.
Multiplicity of Effects: In dynamical behavior, the responses are functions of time and the effect of the applied forcing function produces amplification within a structure.

Non-linear Behavior/Non-linearities: The material and geometric conditions other than linear which describe a structural system. Examples include gaps and any other arbitrary load-deflection relationship.

Optimization: An organized procedure using constraints and a function used to produce the most favorable structural configuration or model.

Parametric Study: The analysis of various structural systems to determine the effects of alteration in selected parameters describing a model.

Particular Solution (Integral): The solution of a differential equation which is an addition to the homogeneous solution. The particular solution is for a nonzero right hand side of the equation.

Phase Differences: The change in shape of one response history compared to another. The peaks and valleys occur at different time for the compared histories.
Probabilistic: A general formulation and treatment of the analysis of structures in which the random aspects of structures and forcing functions are emphasized and analyzed.

Regime: The area of interest and concern as controlled by a particular set of governing equations or conditions.

Responses: The time-histories of behavior of motion (acceleration, velocity, and displacement) and stress resultants (forces, shears, and moments).

Selected Comparisons: The response histories chosen to be examined in detail. It is not usually necessary to examine the behavior of each mass and spring of a system to determine the sensitivity of a structure.

Sensitivity (Analysis): The determination of the degree to which modifications in model parameters affect the response behavior of a particular system.

Series Approximation (Representation): The evaluation of a function (or sets of functions) with a numerical procedure.

Shock Response: The application of an excitation to a structural system producing short-time behavior. This behavior is transient in nature and usually decays rapidly with repeated cycles of motion.
Small Deflection Theory: The use of small angle theory (\( \tan \theta = \sin \theta = \theta \)) in the development of load-displacement relationships for structures.

Springs: The load-deflection relationships between any two mass points in a lumped-mass model. The springs may be linear or non-linear.

Stiffness: The matrix describing the linear load-deflection relationship between any two mass points in a model (see spring).

Stochastic: That process which proceeds with time and is governed by probabilistic laws. The stochastic approach is a basic method used to formulate mathematical models.

Structural Dynamics: That branch of structural mechanics which deals with the response behavior of structural systems due to time-dependent loadings.

Structural System: A physical object composed of parts, shapes, and sizes assembled for a specific purpose. The behavior of the total integrated structure is usually more complicated than any of the individual parts.
Transient Response: (see Shock Response)

Verified Model: A model whose responses agree with known or accepted behavior. The usual source of authentication of a model is in correlation with measured responses.

Vibration Behavior: The physical phenomenon of repeated motion about an equilibrium position as demonstrated by many structural systems.

Wave Propagation Method: The general solution to the wave equation

\[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \]

as applied to various structural configurations. The partial differential equation does have a closed form solution for a structural membrane.

\[ u(x,t) = f(x + ct) + g(x - ct) \]
**APPENDIX B**

**CHART SUMMARY AND EXAMINATION OF THE EFFECTIVENESS AND APPLICABILITY OF THE METHODS USED TO STUDY AND DEVELOP ACCURATE MATHEMATICAL MODELS**

<table>
<thead>
<tr>
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<th>DEPENDENT ON CONSTRAINTS TEST RESULTS, OR BOUNDARY CONDITIONS ((H))</th>
<th>METHOD OF SOLUTION CHANGE WITH PROBLEM ((I))</th>
<th>USE OF ERROR CRITERION ((J))</th>
<th>EMPHASIS ON STRESSES OR ACCELERATIONS ((K))</th>
<th>MATHEMATICALLY TRACTABLE SOLUTION ((L))</th>
<th>CONVERGENCE PROBLEMS IN SOLUTION TECHNIQUES ((M))</th>
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<td>YES (F, G, K, N)</td>
<td>This method needs the addition of comparative techniques to be applicable to the analysis of a limited class of problems.</td>
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<tr>
<td>OPTIMIZATION</td>
<td>NO (Ref. 55,62)</td>
<td>This method is not applicable as a general approach. The method needs a &quot;well-posed&quot; problem and &quot;custom tailoring&quot; to each particular problem.</td>
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<td>IDENTIFICATION</td>
<td>NO</td>
<td>This approach needs test results or predetermined coefficients to be applicable. It also has considerable difficulty with non-linear problems.</td>
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<td>This method breaks down when the terminal conditions cannot be specified as in rigid body motion.</td>
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<td>SENSITIVITY FUNCTIONS</td>
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<td>This approach is particularly adapted to linear systems but cannot be generally applied to non-linear systems.</td>
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<td>INCOMPLETE MODELS</td>
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<td>This approach is applicable to linear systems where stresses are the important consideration.</td>
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<td>YES</td>
<td>This approach is applicable as a general technique with the comparative procedures used. This approach is shown to be a time-saver over independent solutions.</td>
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APPENDIX C

DERIVATION OF PARAMETERS AND STIFFNESS MATRIX

From equilibrium conditions for the beam-mass system shown in Fig. 1, the following equations can be developed for a cantilever beam fixed as mass 1

\[ \delta = A P + B M \]  \hspace{1cm} (C-1)

\[ \theta = C P + D M \]  \hspace{1cm} (C-2)

where

- \( \delta \) = deflection
- \( \theta \) = rotation
- \( A, B, C, D \) = constants
- \( P \) = applied shear at right or free end
- \( M \) = applied moment at right or free end

For the condition where \( \theta \) is zero and \( \delta = 1 \), Eqs. C-1 and C-2 can be written as

\[ 1 = A P + D M \]

\[ 0 = C P + D M \]

With appropriate manipulations,

\[ P = D/(A D - B C) \]  \hspace{1cm} (C-3)

\[ M = -C/(A D - B C) \]  \hspace{1cm} (C-4)
For the condition where $\delta$ is zero and $\theta = 1$, Eqs. C-1 and C-2 can be written as

\[
0 = A P + B M \\
1 = C P + D M
\]

With the appropriate substitutions made in Eqs. C-3 and C-4, the following relationships are found:

\[
P = -B/(A D - B C) \\
M = A/(A D - B C)
\]

With the use of the definitions for $a_{ij}$, $b_{ij}$, $c_{ij}$, $d_{ij}$ and $L_{ij}$ as shown in Fig. 1, equilibrium considerations lead to the following relationships:

for $\delta = 1$ and $\theta = 0$

\[
\Sigma F_{\text{vertical}} = 0 \quad a_{ij} = -P \\
\Sigma M_i = 0 \quad b_{ij} = -P L_{ij} - M
\]

Therefore,

\[a_{ij} = -D/(A D - B C) \quad (C-5)\]

and

\[b_{ij} = (C - DL_{ij})/(A D - B C) \quad (C-6)\]

for $\delta = 0$ and $\theta = 1$

\[
\Sigma F_{\text{vertical}} = 0 \quad c_{ij} = -P \\
\Sigma M_i = 0 \quad d_{ij} = -P L_{ij} - M
\]

Therefore,

\[c_{ij} = B/(A D - B C) \quad (C-7)\]

\[d_{ij} = (B L_{ij} - A)/(A D - B C) \quad (C-8)\]
The constants A, B, C and D can be evaluated for a prismatic cantilever beam for bending and shear deformation from the load deflection relationships for $\delta$ and $\theta$.

$$
\delta = P \frac{L^3}{3} E I + M \frac{L^2}{2} E I + (\alpha)P \frac{L}{A_{ij}} G
$$

$$
\theta = P \frac{L^2}{2} E I + M \frac{L}{E I}
$$

where

- $P \frac{L^3}{3} E I$ = deflection due to load
- $M \frac{L^2}{2} E I$ = deflection due to moment
- $P \frac{L^2}{2} E I$ = rotation due to load
- $M \frac{L}{E I}$ = rotation due to moment
- $(\alpha)P \frac{L}{A_{ij}} G$ = shear deflection
- $A_{ij}$ = cross-sectional area
- $E$ = modulus of elasticity
- $I$ = moment of inertia
- $L = L_{ij}$ = length between i and j
- $(\alpha)$ = coefficient based on the form of the cross-section (see Roark, R. J., Formulas for Stress and Strain, McGraw-Hill, 4th Ed. p. 129, 1965)

Then

$$
A = \frac{L^3}{3} E I + (\alpha) \frac{L}{A_{ij}} G
$$

$$
B = C = \frac{L^2}{2} E I
$$

$$
D = \frac{L}{E I}
$$

A, B, C and D are the flexibility coefficients for the beam elements. $a_{ij}$, $b_{ij}$, $c_{ij}$, and $d_{ij}$ are the elements needed for the stiffness matrix.
Substituting Eqs. C-11, C-12, and C-13 into Eqs. C-5, C-6, C-7, and C-8, the following relationships result:

\[ a_{ij} = -12 \frac{E I A_{ij} G}{(L_{ij})^3 A_{ij} G + 12(\alpha) E I L_{ij}} \]  
(C-14)

\[ b_{ij} = -c_{ij} = -6 \frac{E I A_{ij} G}{(L_{ij})^2 A_{ij} G + 12(\alpha) E I} \]  
(C-15)

\[ d_{ij} = 2 \frac{E I [L_{ij}^2 A_{ij} G - 6(\alpha) E I] / (L_{ij})^3 A_{ij} G + 12(\alpha) E I L_{ij}} \]  
(C-16)

With the development of the appropriate parameters, equilibrium conditions, and deformed beam element shown in Fig. 2, the following stiffness matrix is formulated.

\[
\begin{bmatrix}
V_i \\
M_i \\
V_j \\
M_j
\end{bmatrix} =
\begin{bmatrix}
-a_{ij} & -b_{ij} & a_{ij} & c_{ij} \\
-(d_{ij} + b_{ij} L_{ij}) & b_{ij} & d_{ij} & 0 \\
\text{symmetric} & -a_{ij} & -c_{ij} & 0 \\
(-d_{ij} + c_{ij} L_{ij}) & 0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
Y_i \\
\Theta_i \\
Y_j \\
\Theta_j
\end{bmatrix}
\]  
(C-17)

where

- \( V_{i,j} \) = shear at masses i and j
- \( M_{i,j} \) = moments at mass i and j
- \( Y_{i,j} \) = deflection at masses i and j
- \( \Theta_{i,j} \) = rotation at masses i and j
It should be noted that only \( a_{ij}, b_{ij}, c_{ij}, d_{ij}, \) and \( L_{ij} \) are needed to define the load-displacement relationship between any two mass points. Therefore, only these five arguments are needed for a unique input description and the resulting stiffness matrix can be formulated within the computer program.
APPENDIX D

LOGICAL FLOW CHART AND PHASE SUMMARY OF COMPUTER PROGRAM

PHASE 1
Initialization of Arrays, Restarting, End of File

PHASE 2
Read/Write Data and Perform Error Checks

PHASE 3
Processing Input Data to Develop Stiffness Matrices, Forcing Functions, and Formulation of First Order Equations, Develop Multiple Configurations

PHASE 4
Integration of the Equations of Motion for all Configurations

PHASE 5
Determine the Min-Max of Response Histories, Preparation of Restart Data, Transfer Control to Phase 1

READ MODIFICATIONS TO ORIGINAL MODEL

FORMULATE MODIFIED PARAMETERS

INITIALIZE VARIABLES AND ARRAYS IN MAIN PROGRAM AND SUBROUTINES, DO END OF FILE CHECK

READ/WRITE INPUT DATA

ANY ERROR CONDITIONS IN DATA?

YES

TERMINATE PROBLEM

NO

MULTIPLE SOLUTIONS DESIRED?

YES

DEFINE INDEXES, CHECK INPUT, AND RESTART OPTION

CONSTRUCT MASS, DAMPING, AND STIFFNESS MATRICES

PERFORM ADDITIONAL CHECKS ON DATA

DEVELOP THE NECESSARY COEFFICIENTS FOR THE FIRST ORDER EQUATIONS

α
ANY ERROR CONDITIONS FOR COEFFICIENTS?

BEGIN INTEGRATION OF EQUATIONS OF MOTION

OBTAIN ANY NONLINEAR SPRING DEFINITIONS AND COMBINE WITH F(t)

EVALUATE DERIVATIVES

INTEGRATE VELOCITY VECTOR

STORE VELOCITIES AND DISPLACEMENTS OF MULTIPLE MODELS

EVALUATE RELATIVE DISPLACEMENTS

DETERMINE STRESS RESULTANTS

WITH MULTIPLE CONFIGURATIONS?

DETERMINE STRESS RESULTANTS FOR ALL CONFIGURATIONS

PRINT SOLUTION DATA WITH MIN-MAX SUMMARY

PLOT DATA

PRINT SOLUTION DATA WITH MIN-MAX SUMMARY

SUPERIMPOSED PLOTS OF RESPONSES

SUPERIMPOSED SHOCK SPECTRAS

SENSITIVITY ANALYSIS OF STRUCTURAL SYSTEM

TERMINATE PROBLEM

NO

YES

NO

YES

Ph.3

Ph.4

Ph.5

α

β

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APPENDIX E

PROBLEMS ASSOCIATED WITH SERIES REPRESENTATION OF MODIFIED SOLUTIONS

In the original development of the multiple configuration analysis (re-analysis) of structural systems, solutions to modified models of a particular system were expanded in terms of an original system \((30, 44)\). This expansion process uses the implied assumption that the changes are small. The problems associated with series representations of modified solution are: (1) the evaluation of the terms needed for the series approximation can be a considerable computational effort, and (2) the remainder terms of the series can lead to an error build-up and propagation in the integration procedure.

The first step in a multiple configuration analysis is to obtain a modified model. The usual modification is to the stiffness matrix of Eq. 2.2. A modified model can be formulated through a power (Taylor's) series representation about the original stiffness matrix by the following:

\[
[K_n] = [K + \delta] \quad (E-1)
\]

where

\[
[K_n] = \text{modified stiffness matrix}
\]

\[
[K + \delta] = \text{original stiffness matrix with modified terms}
\]

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models. 

where 

\[ R_n \] = remainder after n terms 

The evaluation of Eq. E-2 is needed only to formulate multiple models. Even with a significant number of terms to be evaluated, the modified models can be formulated.

However, the solution of the original model and the series expansion of the original solution into subsequent solutions becomes a difficult computational task for situations when \( \delta \) is greater than 20 percent. The subsequent solutions for displacement \( x_n(t) \) can be represented in terms of the original solution by

\[ x_n(t) = x(t) + \alpha(t) \]  

(E-3)

where 

\[ \alpha(t) \] = terms from subsequent solutions of 

Eq. E-2 substituted into Eq. 2.2 

and 

\[ x_n(t) - x(t) \] can be expanded as:

\[ f[x_n(t) - x(t)] = \epsilon f'[x(t)] + \frac{\epsilon^2}{2!} f''[x(t)] + \frac{\epsilon^3}{3!} f'''[x(t)] + R_n \]  

(E-4)
For solution purposes, let n be the number of calculations per time step of the Newmark integration method. For the analysis of \( p \) multiple configurations, the number of calculations per time step is \( p \times n \).

With the expansion of Eqs. E-2 and E-4, the following relationships result:

(1) The number of calculations for a three term Taylor's series expansion is approximately \( 3n \) depending on how the derivatives are evaluated.

(2) A three term Taylor's expansion is limited to 
\[ \delta < (0.10 \text{ to } 0.20) \]

Therefore, the practical aspects for solution of linear systems, the series expansion of subsequent solutions requires fewer calculations per time step than the multiple configuration analysis.

However, for \( \delta > 0.20 \), the multiple configuration analysis procedure needs fewer calculations. The usual definition of a system has some possible non-linear conditions. The inclusion of non-linear behavior would require calculation of Eq. E-2 at each time step in addition to solution of Eq. E-4 and the number of calculations per time step could go as high as \( 9 \times n \times n \).

As a result, the series expansion of subsequent solutions in terms of an original solution is limited to \( \delta < 0.20 \) for transient dynamics.
APPENDIX F

COMPUTATIONAL TIME-SAVING WITH MULTIPLE CONFIGURATION ANALYSIS

The following are three basic unknowns in the solution of transient response problems:

(1) The real-time at and cycle in which the peak responses of motion and stress resultants occur. Although these maximums usually occur in the first cycle of motion, there is no guarantee the peaks will occur in the first cycle.

(2) The amount of real-time needed to give sufficient information so that a thorough understanding of structural response is obtained.

(3) The amount of computational (computer) time needed to solve a system.

The equations of motion, Eq. 2.2, are treated herein as an initial value problem (Section 4.1). As a result, the solution process may be restarted.

A problem can be solved with sufficient restarts so that item (3) is not a problem.

The choice now is whether to perform a set of independent solutions or do multiple independent solutions. (It was shown in -197-
Appendix E that independent solutions are needed.) With the use of independent solutions, a specific model configuration would be solved for a time interval \((t_0 - t_f)\). However, there is no guarantee that this time interval will satisfy items (1) and (2) for the other configurations. It is not necessarily known which configuration will require the largest \(t_f\) to satisfy (1) and (2). Therefore, \(t_f\) must be chosen arbitrarily large for a particular independent solution.

The particular advantage of the simultaneous multiple configuration analysis is that it is a computational time-saver when compared with independent solutions. This is because the solutions are examined con-currently. If items (1) and (2) are not satisfied, the problem can be restarted until (1) and (2) are satisfied. The choice for \(t_f\) is only as large as it needs to be.

If it is possible to view the compared responses as they are generated (that is, a display console on line with a computer), then the multiple configuration analysis procedure is a significant time saver because of the approach described to satisfy (1) and (2). In addition, the proposed method is a great convenience to the analyst.
APPENDIX G

RELATIONSHIP BETWEEN SHOCK AND ENERGY SPECTRA

The relationship between shock and energy spectra can be shown using Eq. 4.21 and three definitions. The Fourier transform of a time history is given by

\[ x(f) = \int_{t_0}^{t_f} x(t) e^{-2 \pi j \omega t} dt \]  

where

- \( x(t) \) = time history
- \( j \) = imaginary number \((\sqrt{-1})\)
- \( \omega \) = frequency in Hz
- \( x(f) \) = modulus of Fourier transform
- \( e \) = base of natural logarithms

The Fourier spectrum is defined as

\[ x(f) = x(f) e^{-j\theta(f)} \]

where

- \( x(f) = R_e^2 \text{Re}[x(f)] + Im^2 \text{Im}[x(f)] \)
- \( \theta(f) = \text{phase angle of Fourier transform} \)
  \[ = \tan^{-1} \left[ \frac{\text{Im} x(f)}{R_e x(f)} \right] \]
- \( R_e = \text{real part of a complex quantity} \)
- \( \text{Im} = \text{imaginary part of a complex quantity} \)
By expanding \( \sin w_n (t - \tau) \) and \( \cos w_n (t - \tau) \) and expressing \( \ddot{x}(t) \) from Eq. 4.21 in terms of its Fourier spectrum, the following results:

\[
\omega_n \zeta(t) = [Re x (w_n)] \sin \omega_n t + [Im x (w_n)] \cos \omega_n t \quad (G-3)
\]

With an undamped system, the response will be sinusoidal with \( \zeta(t) \) and \( \zeta(t) \) as initial conditions. The maximum value of this residual response is given by

\[
(Cr)_{\text{max}} = \zeta(t) \sqrt{\left(\frac{\omega_n^2 + 1}{\omega_n^2}\right)} \quad (G-4)
\]

Substituting Eq. G-3 and its derivative for \( \zeta(t) \) into Eq. G-4, the following results

\[
\omega_n (Cr)_{\text{max}} = \sqrt{\left[Re x (w_n)\right]^2 + \left[Im x (w_n)\right]^2} \quad (G-5)
\]

The right side of Eq. G-5 is equivalent to the right side of Eq. G-2. The relationship between the shock and energy spectra is given by \( \omega_n^{-1} \) for the left sides of Eqs. G-2 and G-5.
APPENDIX H

BEAM SUPPORT CONDITIONS USED IN THE FORMULATION OF
MULTIPLE CONFIGURATIONS

The purpose of this appendix is to demonstrate a problem with a set of limiting and possible intermediate conditions that can be used in the formulation of multiple configurations. This example consists of two beams A and B with identical material properties and geometric cross-section shown in Fig. H-1. The only difference between the two beams is the support condition for the right end of the beams. Beam A is simply supported while Beam B is fixed. The two support conditions are idealized boundary conditions which may be approached in the behavior of real structures. The true support condition lies somewhere between pinned and fixed.

For a uniform static loading as shown the comparisons for moment and deflection at centerline show that Beam A is 2.0 times and 2.5 times greater than Beam B. The true answers for this beam lies somewhere in between.

In the lumped-mass models shown, the only part of the system that is less predictable analytically is the spring between the beam and support.

Therefore, the limiting conditions for the right support can be used for the original and one additional configuration. Other
possibilities include intermediate values of stiffness. As an initial estimate, the value of the stiffness would probably be greater than the stiffness of the adjacent part of the beam. (The mass of the support is fixed and would not enter into the dynamic response calculations.)

Depending on the support conditions, a wide variety of responses can be obtained for a particular excitation. It should be emphasized that the modifications from one model to another are not small.
Comparisons for uniform static loading

<table>
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<th>BEAM B</th>
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<td>Deflection at Midspan</td>
<td>$2.5 \cdot (5wL^4/384EI)$</td>
<td>$1.0 \cdot (wL^4/192EI)$</td>
</tr>
<tr>
<td>Moment at Midspan</td>
<td>$2.0 \cdot (wL^2/8)$</td>
<td>$1.0 \cdot (wL^2/16)$</td>
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</table>

Fig. H-1 COMPARISONS OF EFFECTS OF SUPPORT CONDITIONS FOR A BEAM: POSSIBLE MULTIPLE CONFIGURATIONS

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APPENDIX I

DEVELOPMENT OF MULTIPLE CONFIGURATIONS FOR SHOCK-DRIVEN MISSILE SYSTEM

The development of alternate configurations is based on the changes in the stiffnesses of the individual load paths composing the spring between the mounting mass center and the center of gravity of the instrumentation package. $k_1$ as shown in Fig. I-1 represents the axial stiffness of the flange of the shell. $k_2$ shows the stiffness of the flange of the instrumentation package. (To obtain $k_1$ and $k_2$, both stiffnesses would be analyzed as the load-deflection relationship for a ring element.)

$k_3$ reflects the stiffness of the case of the instrumentation package. $k_4$ is the stiffness of the bottom plate of the package. $k_5$ is the spring between the bottom plate and the mass center of the package.

The equivalent spring $k_e$ is defined by:

$$k_e = \sum_{i=1}^{5} \frac{1}{k_i}$$

The range of $k_e$ for this problem was $10^5$ lbs./in. ≤ $k_e$ ≤ $10^7$ lbs./in. depending on the values chosen for an individual part i. The relatively softer stiffness dominate the terms of Eq. E-1.
Cross-Section of Missile System

Fig. I-1 LOAD PATHS AND STIFFNESSES FOR MULTIPLE CONFIGURATIONS OF MOUNTING AND INSTRUMENTATION PACKAGE FOR SHOCK-DRIVEN MISSILE SYSTEM
ACKNOWLEDGMENTS

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