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A view of the elliptical catastrophe machine.

Randy Ellen Davidson

A VIEW OF THE ELLIPTICAL CATASTROPHE MACHINE

by

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ABSTRACT

Among the various applications of catastrophe theory are the catastrophe machines, especially designed for experimental observation. In this paper, the elliptical catastrophe machine is examined as a particular example of one of the seven elementary catastrophes, the cusp. The machine is an ellipse-shaped physical system whose resting positions or stable equilibria at a specific center of gravity can be determined by finding critical points of the gravitational potential energy function. This paper includes direct computation of the potential functions, and/or uses an alternate method, in order to determine the equilibria for the machine lying on both a flat surface and an inclined plane. Analysis leads to confirmation of the Classification Theorem for universal unfoldings (the main theorem of catastrophe theory) which predicts that "catastrophes" or discontinuities in the turning motion of the machine form a cusp.

ABSTRACT

Among the various applications of catastrophe theory are the catastrophe machines, especially designed for experimental observation. In this paper, the elliptical catastrophe machine is examined as a particular example of one of the seven elementary catastrophes, the cusp. The machine is an ellipse-shaped physical system whose resting positions or stable equilibria at a specific center of gravity can be determined by finding critical points of the gravitational potential energy function. This paper includes direct computation of the potential functions, and/or uses an alternate method, in order to determine the equilibria for the machine lying on both a flat surface and an inclined plane. Analysis leads to confirmation of the Classification Theorem for universal unfoldings (the main theorem of catastrophe theory) which predicts that "catastrophes" or discontinuities in the turning motion of the machine form a cusp. Then, the elliptical machine is considered to be lying on an inclined plane and the angles at which the machine rolls down the incline are calculated.

INTRODUCTION

An elliptical catastrophe machine is a physical system providing a clear example of an application of Rene Thom's work in Catastrophe Theory. This, and other catastrophe machines, have been created and viewed to further an understanding of Thom's Classification Theorem [6]. Before actual discussion of the elliptical machine, the theorem and some explanation is presented. The object of the first chapter is to provide a general feeling for the theorem's mathematical assertions and an idea as to how these assertions may be applied to a particular system, such as the elliptical machine.

It is in the second chapter that the elliptical catastrophe machine is critically viewed. Here, the machine is supposed first as resting on a level surface and then on an inclined plane. In each case, there is analysis which includes locating those points at which the system might lie in an equilibrium position. Analysis also includes verification that the system behaves as predicted by catastrophe theory. Development of explicit equations throughout the chapter complements and completes some of the discussion in Poston and Stewart's "Gravitational catastrophe machines", an article which inspired the basic content of this paper, [8]. In particular, sketches in the above-mentioned article indicate that an elliptical machine may or may not roll down the inclined plane, depending on the steepness of the incline. The final section indicates the angle of incline that would first cause an elliptical machine to roll.

CHAPTER 1

1.1 Introduction to Catastrophe Theory

In the physical world, many systems can be easily described by n variables. Thus, let $(x_1,...,x_n) \in \mathbb{R}^n$ describe the state of some system and call $\mathbb{R}^n$ the state space. In addition, there might be r variables whose values directly affect the state. Therefore, call $u \in \mathbb{R}^n$ a control variable and $\mathbb{R}^r$ the control space. For each u, suppose there is a potential function $F : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}$ which governs some aspect of the system in the sense that the system at u will assume a state which is a local minimum of F_u. Think of $F_u : \mathbb{R}^n \to \mathbb{R}$ as a family of potential functions parametrized by $u \in \mathbb{R}^r$.

Specifically, consider the elliptical machine whose center of gravity is altered at will. The state of the system may be described by possible angles θ at which the ellipse may rest; its control variables $(u,v) \in \mathbb{R}^2$ specify the position of center of gravity on the face of the ellipse. Alterations in position of center of gravity do indeed affect at which θ the machine will rest. It is the gravitational potential energy function $F : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$, or think of a family $F_{(u,v)} : \mathbb{R} \to \mathbb{R}$, which describes the gravitational potential for some (θ,u,v). At a particular (u,v), the system assumes a state which is a local minimum of $F_{(u,v)}$. Note that local minima of $F_{(u,v)}$ are points at which the machine rests in stable equilibrium, but local maxima and points of inflection correspond to unstable equilibrium positions so that the ellipse never rests in such positions.

In catastrophe theory, families of functions satisfying certain conditions are classified into one of seven types or catastrophes. Potential functions most often satisfy these conditions so that their classification into one of a finite number of types assists in predicting accurate descriptions of the behavior of a system. Using explicit equations in the second chapter, for gravitational potential functions of the elliptical machine, the main theorem's assertions are found to be true for a particular physical system.

1.2 The Main Theorem of Catastrophe Theory

There are some essential concepts and definitions which must be presented before there can be any understanding of the Classification Theorem, i.e. the Main Theorem. The functions of concern in the following discussion are smooth, i.e. their derivatives of all orders exist and are continuous. These functions are considered as close according to the Whitney C^∞-topology which requires derivatives of all orders as well as functional values to be close.

Definition 1: A <u>germ</u> of a map $f : \mathbb{R}^n \to \mathbb{R}$, at a point $x \in \mathbb{R}^n$, is an equivalence class of functions in which two mappings are equivalent, i.e. have the same germ at x, if they are identical on some neighborhood of x. A germ is similarly defined for a smooth function which maps an open set U, $x \in U$, of a manifold M to a manifold N.

Denote the germ of f as $\tilde{f} : (M,x) \to (N,f(x))$, though simply f is written when it is clearly a germ. In the second definition, note that a diffeomorphism is a smooth homeomorphism with a smooth inverse and a diffeomorphism germ refers to a germ of a diffeomorphism.

Definition 2: Germs $f : (M,x) \to (N,f(x))$ and $g : (M,y) \to (N,g(y))$ are <u>equivalent</u> if there exists diffeomorphism germs $h : (M,x) \to (M,y)$ and $k : (N,f(x)) \to (N,g(y))$ such that the following diagram commutes.

$$
\begin{array}{ccc}
(M,x) & \xrightarrow{\quad f \quad} & (N, f(x)) \\
h \downarrow & & \downarrow k \\
(M,y) & \xrightarrow[\quad g \quad]{} & (N, g(y))
\end{array}
$$

$g \circ h = k \circ f$ in a neighborhood of x. Smooth maps $f, g : M \to N$ are equivalent if there exists diffeomorphisms $h : M \to M$ and $k : N \to N$ such that $g \circ h = k \circ f$.

Next is a definition found essential to the core of catastrophe theory.

Definition 3: $b \in \mathbb{R}^n$ is a <u>singular</u> or <u>critical</u> point (or singularity) of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ if $df(b) = (\frac{\partial f}{\partial x_1}(b), \ldots, \frac{\partial f}{\partial x_n}(b)) = 0$. b is a <u>nondegenerate singularity</u> of f is the Hessian matrix $d^2 f(b) = (\frac{\partial^2 f}{\partial x_i x_j}(b))$, $1 \leq i, j \leq n$, is nonsingular, i.e. $\det(d^2 f(b)) \neq 0$. b is called a <u>degenerate singularity</u> if the determinant of the Hessian matrix is zero.

There is a theorem which states that for a compact manifold M, stable maps $f : M \to \mathbb{R}$ have all critical points as nondegenerate. Stability of a smooth map f requires that small perturbations of f be equivalent to f. Thus, a small perturbation g will have a nondegenerate singular point nearby any nondegenerate singular point of f. Note also that the set of stable maps from $M \to \mathbb{R}$ is found to be an open dense subset of $C^{\infty}(M, \mathbb{R})$; denseness implies that smooth functions can be approximated by stable maps. Thus a function f with a degenerate singular point can be approximated by a function with all its singular points as nondegenerate.

Next are described unfoldings of a germ $\tilde{f}$ and certain kinds of unfoldings, called versal, which are large enough to induce any unfolding of $\tilde{f}$ in the manner described below.

Definition 4: Let $\tilde{f} : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$. An <u>r-parameter unfolding</u> of $\tilde{f}$ is a germ $F : (\mathbb{R}^n \times R^r, 0) \to (\mathbb{R}, 0)$ such that $F(x,0) = f(x)$, $x \in \mathbb{R}^n$.

If $u \in R^r$, denote $F(x,u)$ as $F_u(x)$ and consider F_u to be a parametrized family of germs which take points from $\mathbb{R}^n$ to $\mathbb{R}$. Now an unfolding F of $\tilde{f}$ is itself stable if it is versal; versality of F requires that for any unfolding $G : (\mathbb{R}^n \times \mathbb{R}^s, 0) \to (\mathbb{R}, 0)$ of the same germ $\tilde{f}$ there exists $\tau : \mathbb{R}^s \to \mathbb{R}^r$ such that for all $(x,v) \in \mathbb{R}^n \times \mathbb{R}^s$ in some neighborhood of $(0,0) \in \mathbb{R}^n \times \mathbb{R}^s$, $G_V : \mathbb{R}^n \to \mathbb{R}$ and $F_{\tau(v)} : \mathbb{R}^n \to \mathbb{R}$ are equivalent as germs. F is called universal when r is minimal among versal unfoldings of a germ.

The following Classification of Unfoldings Theorem is useful in understanding one of the assertions of the Main Theorem.

Theorem: If $F : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}$ is a stable unfolding, $r \leq 4$, then F is equivalent to one of the following seven unfoldings:

germ	unfolding	name of catastrophe
x^3	x^3+ux	fold
x^4	x^4+ux^2+vx	cusp
x^5	$x^5+ux^3+vx^2+wx$	swallowtail
x^6	$x^6+ux^4+vx^3+wx^2+cx$	butterfly
x^3+y^3	$x^3+y^3+uxy+vx+wy$	hyperbolic umbilic
x^3-xy^2	$x^3-xy^2+u(x^2+y^2)+vx+wy$	elliptic umbilic
x^2y+y^4	$x^2y+y^4+ux^2+vy^2+wx+cy$	parabolic umbilic

Equivalence of unfoldings of the same germ allows that certain variables, up to equivalence, can be split off in a canonical quadratic form without affecting singularities. The above theorem shows that all but one or two state variables can be split off in this way when the number of control variables r is less than or equal to four. Remarkably, up to equivalence, only seven types of unfoldings arise.

Thom's Classification Theorem for universal unfoldings states:

(i) if A is the set consisting of all smooth functions $F : \mathbb{R}^{n+r} \to \mathbb{R}$ and $r \leq 4$, there is an open dense set $A_0 \subset A$ in the Whitney C^{∞}-topology, such that for all F contained in A_0,

$$M_F = \left\{ (x,u) \in \mathbb{R}^n \times \mathbb{R}^r \mid \frac{\partial F}{\partial x_i}(x,u) = 0,\ i=1,\ldots,n \right\}$$

is an r-manifold.

(ii) if $X_F : M_F \to \mathbb{R}^r$ is the restriction of the projection map $\pi : \mathbb{R}^{n+r} \to \mathbb{R}$, then any singularity of X_F is equivalent to one of seven germs called the elementary catastrophes.

Therefore, if given a family of functions, such as potential functions $F_u : \mathbb{R}^n \to R$, belonging to A_0, a stable unfolding of the germ of F_u will be equivalent to one of the seven unfoldings listed. Since F and X_F are mappings involving different spaces, only after some work can it be shown that for equivalent unfoldings F and G, X_F and X_G are also equivalent; however, once this is shown, it follows that a singularity of X_F is equivalent to one of seven types.

1.3 The Cusp Catastrophe

The cusp catastrophe arises commonly when minimizing potential functions. Quite often these functions are equivalent to the standard unfolding of the germ x^4, i.e. $F : \mathbb{R}^1 \times \mathbb{R}^2 \to \mathbb{R}$ by $F(x,u,v) = x^4 + ux^2 + vx$. The set of critical points of F forms the manifold M_F, which in this case is described as $M_F = \left\{(x,u,v) \in \mathbb{R}^1 \times \mathbb{R}^2 \mid 4x^3 + 2ux + v = 0\right\}$. Let S be the singular set of X_F. Then $S = \left\{(x,u,v) \in M_F \mid 12x^2 + 2u = 0\right\} = \left\{(x,u,v) \in \mathbb{R}^1 \times \mathbb{R}^2 \mid u \in -6x^2,\ v = 8x^3\right\}$. The bifurcation, or catastrophe, set is the image of $X_F : M_F \to \mathbb{R}^r$ and in this case is $\left\{(u,v) \in \mathbb{R}^2 \mid u = -6x^2, v = 8x^3, \text{ for some } x\right\} = \left\{(u,v) \in \mathbb{R}^2 \mid 27v^2 = -8u^3\right\}$. This bifurcation set forms a cusp in $\mathbb{R}^2$ as indicated by Thom's Classification Theorem. Below is a local description of the above sets for the standard unfolding of the germ x^4.

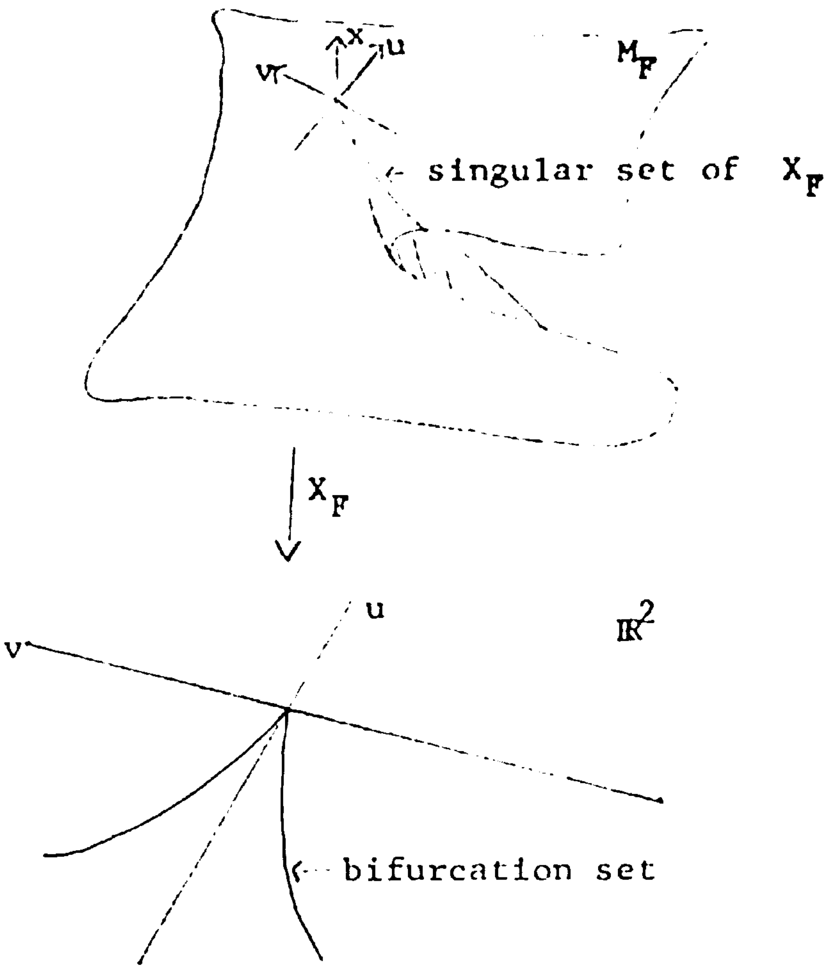


figure 1.3.1

For $(u,v) \in \mathbb{R}^2$, there is an associated potential function $F_{(u,v)}$. These functions vary in the number and location of local extrema so that for each $u \in \mathbb{R}^r$, the local minima, maxima, and inflection points indicate the number and type (stable or unstable) of equilibrium positions the state may assume. As $(u,v) \in \mathbb{R}^2$ varies, potential functions $F_{(u,v)}$ take on minima as described in the following sketch:

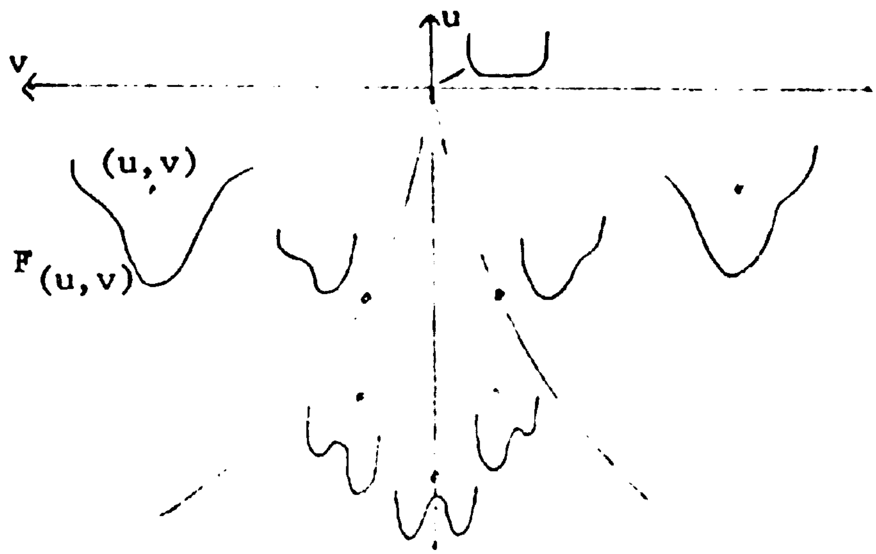


figure 1.3.2

Observe that for points lying outside the cusp, and at $(u,v) = (0,0)$, the system has one stable equilibrium; for each $(u,v) \in \mathbb{R}^2$ within the cusp there are two stable and one unstable; and for points on the cusp line itself, the system has one stable and one unstable equilibrium position.

The concept of a "catastrophe" or discontinuity arises in considering continuously varying control variables and their related effects on the state of the system. Viewing the cusp as an example, choose (u,v) on the outside of the cusp in the lower left quadrant [figure 1.3.2] and let it move toward the right, across the cusp. The state varies continuously until it assumes a new local minimum. According to a convention called Perfect Delay, which applies to the elliptical machine, the system remains at a minimum as long as possible, until that minimum disappears or degenerates. Thus, according to this convention, if (u,v) moves from left to right across the cusp, it will pass the left branch and reach the right branch before the minimum disappears and the system assumes a new local minimum, i.e. the system experiences a catastrophic jump from one state to another. This is easily envisioned in the case of the elliptical machine; as the center of gravity is moved continuously, the system's resting position or stable equilibrium moves continuously, causing the ellipse to turn; but suddenly, upon reaching a certain center of gravity, the machine jumps to a new stable equilibrium, causing the ellipse to turn suddenly and quickly in order to reach its resting position. Note that the catastrophe set consists of those points in $\mathbb{R}^r$, or for the cusp $(u,v) \in \mathbb{R}^2$, where the local minima of the potential functions degenerate and cause jumps in the state of the system.

Since points in $\mathbb{R}^r$ are the image of the projection map: $M_F \rightarrow R^r$, it is of interest to notice the positions of the points which map to $u \in \mathbb{R}^r$. M_F for the cusp catastrophe is a smooth manifold with three sheets formed by two fold lines (see fig. 1.3.1); a bottom, middle, and top sheet. For (u,v) in which the state has a stable equilibrium, $(x,u,v) \in M_F$ lies on either the top or bottom sheet, whereas a point on the middle sheet indicates that the system has an unstable equilibrium at the corresponding (u,v). As an example, choose (u,v) inside the cusp where the system assumes two stable and one unstable equilibrium position; three points "above" (u,v) will lie in a vertical line, one in each sheet of M_F. As done in $\mathbb{R}^2$, a path can be traced on the surface of the manifold in order to determine points at which the system jumps. Using the perfect delay convention, a catastrophe occurs when a point on the top sheet reaches the fold line and must jump to the bottom sheet, or a point on the bottom sheet coming from the opposite direction reaches the fold and must jump up in order to continue on the upper sheet. Notice that the jumps occur at different points, depending on the direction points (x,u,v) are moving.

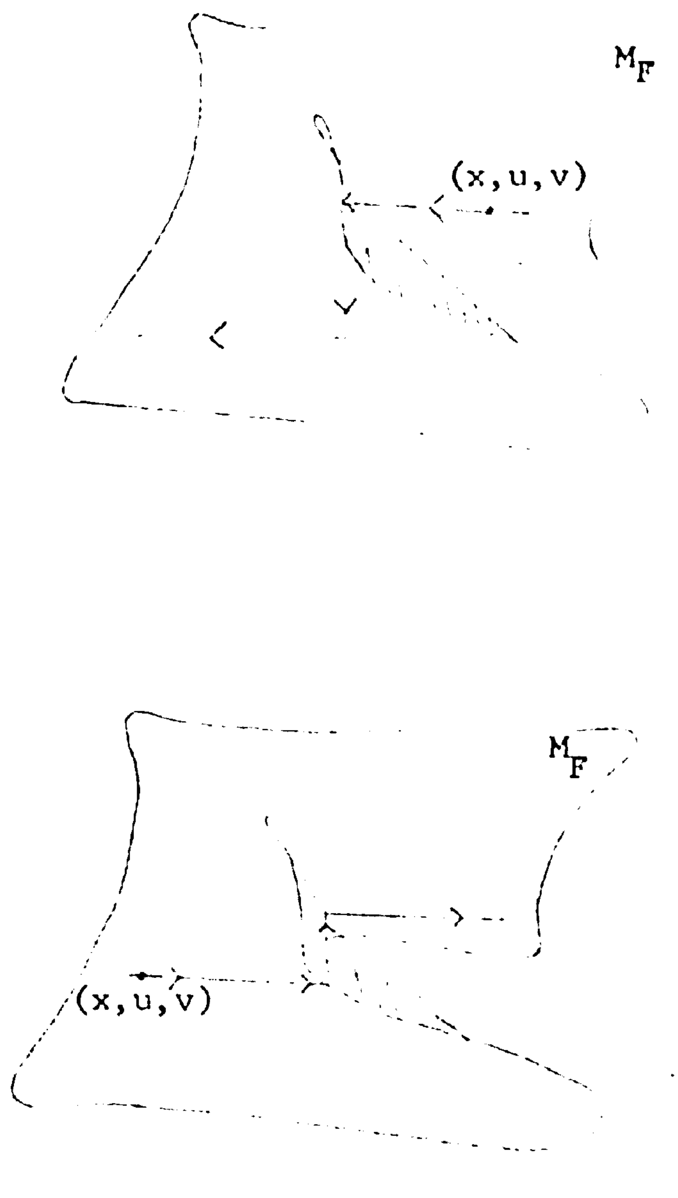
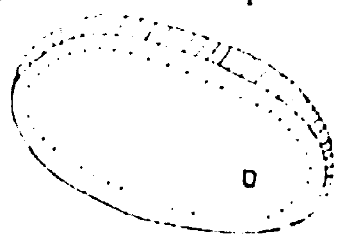


figure 1.3.3

CHAPTER 2

2.1 The Ellipse



The elliptical machine can be simply constructed by connecting two pieces of thin cardboard, each in the shape of an ellipse, using toothpicks. Placing a magnet on either side of one sheet of cardboard determines the center of gravity so that when placed on a flat surface, the ellipse will easily turn as the magnet is moved. An assumption which must be made, though it may not occur this way in practice, is that the weight of the cardboard is neglible compared to the weight of the magnet. Thus the position of the magnet can be considered as the center of gravity. Description of the elliptical machine can be found in Poston and Stewart's "Gravitational catastrophe machines". Consider for the remainder of discussion an ellipse with major axis of length $2a$ and minor axis of length 2, so that its equation is $\frac{x^2}{a^2} + y^2 = 1$, $a > 1$. At various times throughout the chapter, the ellipse will be parametrized by arclength; $s = 0$ occurring at $(1,0)$ and a point (x,y) on the boundary of the ellipse written as $(x(s),y(s))$. At other times, it will be convenient to consider $(x,y) = (r\cos\theta, r\sin\theta)$, where θ is the angle between the major axis and the line R which is formed by (0,0) and a point on the ellipse (x,y). In this last case, $\frac{x^2}{a^2} + y^2 = 1$ implies $\frac{r^2\cos^2\theta}{a^2} + r^2\sin^2\theta = 1$ implies $r = \frac{a}{\sqrt{a^2\sin^2\theta+\cos^2\theta}}$. Therefore $(x,y) = \left(\frac{a\cos\theta}{\sqrt{a^2\sin^2\theta+\cos^2\theta}}, \frac{a\sin\theta}{\sqrt{a^2\sin^2\theta+\cos^2\theta}}\right)$.

2.2 The Ellipse on The Surface

Suppose that the elliptical machine is lying on a flat surface. If the magnet is at a point (u,v) on the inside of the face of the ellipse, the gravitational potential energy of the system is proportional to the perpendicular distance from (u,v) to the tangent line at a point P = (x,y) on which the ellipse is resting. (figure 2.2.1a) Think of the tangent line as the flat surface on which the ellipse lies, and P as the point of contact between the ellipse and that surface (figure 2.2.1b).

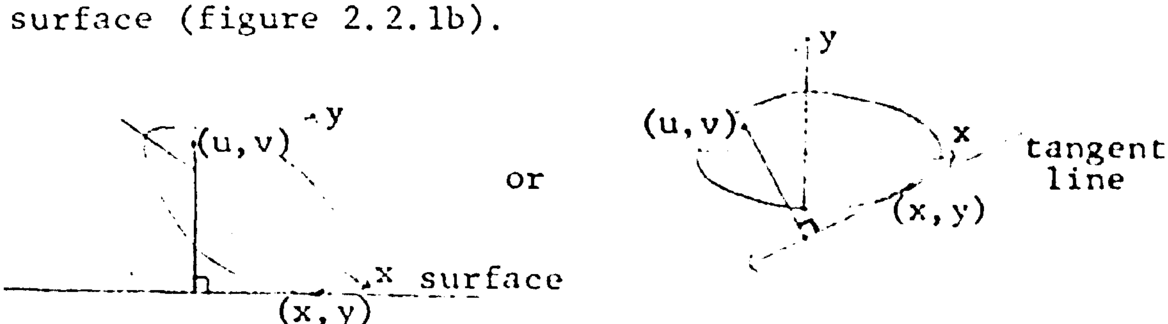


figure 2.2.1 a b

Implicit differentiation of the equation of the ellipse yields $\frac{dy}{dx} = \frac{-x}{a^2 y} = \frac{-r\cos\theta}{a^2 r\sin\theta} = \frac{-\cos\theta}{a^2\sin\theta}$. Thus, the equation of the tangent line at $(x,y) =$

$$\left(\frac{a\cos\theta}{\sqrt{a^2\sin^2\theta+\cos^2\theta}}, \frac{a\sin\theta}{\sqrt{a^2\sin^2\theta+\cos^2\theta}}\right) \text{ is}$$

$$y - \frac{a\sin\theta}{\sqrt{a^2\sin^2\theta+\cos^2\theta}} = \frac{-\cot\theta}{a^2}\left(x - \frac{a\cos\theta}{\sqrt{a^2\sin^2\theta+\cos^2\theta}}\right)$$

$$\text{or } y + \frac{x\cot\theta}{a^2} - \frac{\sqrt{a^2\sin^2\theta+\cos^2\theta}}{a\sin\theta} = 0 .$$

The gravitational potential energy function $F(\theta,u,v)$, where θ is the state variable and u,v are control parameters, is

$$\frac{\left|v + \frac{\cot\theta}{a^2}\cdot u - \frac{\sqrt{a^2\sin\theta+\cos^2\theta}}{a\sin\theta}\right|}{\sqrt{1+\frac{\cot^2\theta}{a^4}}},$$

since the distance from a point (u,v) to a line $Ax + By + C = 0$ is $\frac{|Au+Bv+C|}{\sqrt{A^2+B^2}}$.

$F(\theta,u,v)$, not defined at $\theta = 0$, involves the absolute value of an expression which can be either positive or negative; however, in finding critical points of F, it is only necessary to take the derivative of the positive expression since a negative sign factors out when taking a derivative and therefore does not affect a set of critical points. Thus, find the derivative with respect to θ of the following expression

$$\frac{v + \frac{u \cot \theta}{a^2} - \frac{\sqrt{a^2\sin^2\theta+\cos^2\theta}}{a \sin \theta}}{\sqrt{1 + \frac{\cot^2\theta}{a^4}}}$$

or

$$\frac{va^2 + u \cot \theta - a \sqrt{a^2+\cot^2\theta}}{\sqrt{a^4+\cot^2\theta}}$$

The derivative $\frac{\partial F}{\partial \theta}$ is

$$\frac{a^2\csc^2\theta\sqrt{a^2+\cot^2\theta}(v\cot\theta-ua^2)+a^2(a^3-a)\cot\theta\csc^2\theta}{\sqrt{a^2+\cot^2\theta}\ (a^4+\cot^2\theta)^{3/2}} = 0$$

when $\frac{1}{\sin^2\theta}\left[v\cot\theta-ua^2 + \frac{(a^3-a)\cos\theta}{\sqrt{a^2\sin^2\theta+\cos^2\theta}}\right] = 0$

Since $\sin^2\theta \neq 0$, $M_F = \left\{(\theta,u,v) \in R^1\times R^2 \mid v - ua^2\tan\theta + \frac{(a^3-a)\sin\theta}{\sqrt{a^2\sin^2\theta+\cos^2\theta}} = 0\right\} = \left\{(\theta,u,v) \mid (v-ua^2\tan\theta)\sqrt{a^2+\cot^2\theta} + (a^3-a) = 0\right\} = \left\{(\theta,u,v) \mid H = 0\right\}$. For a given center of gravity (u,v), the ellipse assumes an equilibrium position at an angle θ, occurring at a local minimum of its gravitational potential energy function $F_{(u,v)}$.

The singular set of $X_F : M_F \to \mathbb{R}^2$ consists of those points $(\theta,u,v) \in M_F$ such that $\frac{\partial H}{\partial \theta} = 0$. Differentiating H,

$$(v-ua^2\tan\theta)\cdot\frac{1}{2}(a^2+\cot^2\theta)^{-1/2}(-2\cot\theta\csc^2\theta)$$

$$-\sqrt{a^2+\cot^2\theta}(ua^2\sec^2\theta)$$

$$= (a^2+\cot^2\theta)^{-1/2}[-\cot\theta\csc^2\theta(v-ua^2\tan\theta)$$

$$-(a^2+\cot^2\theta)(ua^2\sec^2\theta)] = 0$$

when $-v\cot\theta\csc^2\theta - ua^4\sec^2\theta = 0$, i.e. $v = -ua^4\tan^3\theta$. Those points satisfying this equality while also lying on the manifold will satisfy

$$(-ua^4\tan^3\theta-ua^2\tan\theta)\sqrt{a^2+\cot^2\theta} + (a^3-a) = 0 \quad .$$

Therefore, $u = \dfrac{a^3-a}{a^2\tan\theta(a^2\tan^2\theta+1)\sqrt{a^2+\cot^2\theta}} = \dfrac{a^2-1}{a(a^2\tan^2\theta+1)^{3/2}}$

and $v = \dfrac{-(a^2-1)a^4\tan^3\theta}{a(a^2\tan^2\theta+1)^{3/2}} = \dfrac{-a^3(a^2-1)\tan^3\theta}{(a^2\tan^2\theta+1)^{3/2}}$. Taking into account both the positive and negative parts of the square root function, the singular set of X_F is

$$\left\{(\theta,u,v) \in \mathbb{R}^1\times\mathbb{R}^2 \mid u = \frac{\pm(a^2-1)}{a(a^2\tan^2\theta+1)^{3/2}} ,\ v = \frac{\pm a^3(a^2-1)\tan^3\theta}{(a^2\tan^2\theta+1)^{3/2}}\right\} .$$

The bifurcation set is

$$\left\{(u,v) \in \mathbb{R}^2 \mid u = \frac{\pm(a^2-1)}{a(a^2\tan^2\theta+1)^{3/2}},\ v = \frac{\pm a^3(a^2-1)\tan^3\theta}{(a^2\tan^2\theta+1)^{3/2}},\ \text{for some } \theta\right\}.$$

To see that this set forms a cusp, note first that $\tan^2\theta$ can vary from zero to positive infinity. Parametrize the curve by $t = \tan^2\theta$. At $t = 0$, $(u,v) = (\frac{\pm(a^2-1)}{a}, 0)$; this represents two points lying on the u-axis, inside the ellipse. Near $t = 0$, use a first order approximation to derive $(u,v) \approx (\frac{\pm(a^2-1)}{a})(1-\frac{3}{2}a^2 t), \pm \frac{3}{2} a^3(a^2-1)t^{3/2})$. Thus, for a given a, and u near $\frac{\pm(a^2-1)}{a}$, $v = \pm \frac{3}{2} a^3(a^2-1)t^{3/2}$ or $v = \frac{\pm a^3}{\sqrt{\frac{3}{2}(a^2-1)}} [\pm u + \frac{(a^2-1)}{a}]^{3/2}$, i.e. there are cusps on either side of the v-axis. Finally, as t approaches positive infinity, (u,v) approaches $(0,\pm(a^2-1))$. Thus for $1 < a^2 < 2$, the cusps on each side of the v-axis meet at points $(0,\pm(a^2-1))$ and lie inside the ellipse (figure 2.2.2a); whereas, for an ellipse with $a^2 > 2$ (here $a^2-1 > 1$ and $1-a^2 < -1$), neither cusp curve intersects the v-axis within the boundary of the ellipse (figure 2.2.2b).

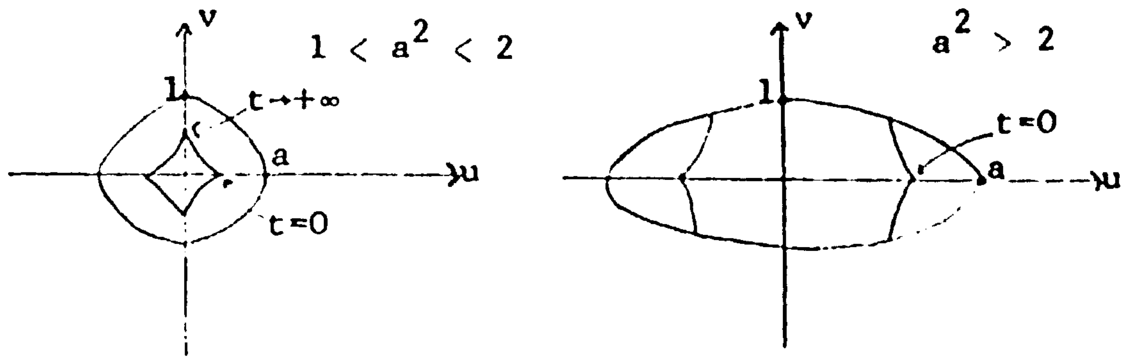


figure 2.2.2 a|b

2.3 An Alternate Method

It is possible to find M_F, and consequently the bifurcation set, without an explicit expression for F. As seen in the last section, finding F for the elliptical machine on the level poses no problem; however, often in nature, and in particular for the elliptical machine on an inclined plane, finding F can be tedious or impossible. Consider the gravitational potential energy function $f(s,u,v)$ for the ellipse on the level as parameterized by arclength. Then it will be shown that $M_f = \left\{(s,u,v) \mid \frac{\partial f}{\partial s} = 0\right\} = \left\{(s,u,v) \mid (u,v) \text{ lies on the normal line to the ellipse at } (x(s),y(s))\right\} = M_F$. Thus, to find M_F, where $F(\theta,u,v)$ is the potential function parameterized by (u,v), it is only necessary to find points (u,v) on the normal line at the point $(x(s),y(s))$ which forms the angle θ.

Recall that $(x(s),y(s))$ is expressible as $(r\cos\theta, r\sin\theta)$, making it easy to find the equation of the normal to the tangent line at a point on the ellipse, in terms of θ.

In the general case, if $f(s)$ is the perpendicular distance from a point P to the tangent to a curve parametrized by arclength s, at a point $(x(s),y(s))$, and if without loss of generality, $P = (0,0)$ such that it does not lie on the curve and R is a line formed by P and $(x(s),y(s))$, the following will hold.

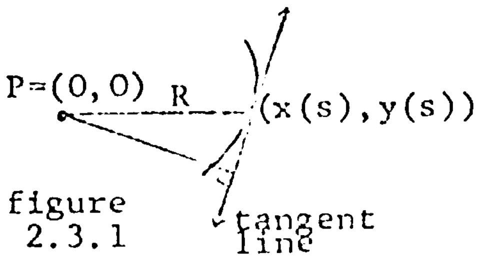


figure 2.3.1

Theorem 1: For points on any curve where $\frac{d}{ds}\left(\frac{dy}{dx}\right) \neq 0$, $\frac{df}{ds} = 0$ if and only if R is perpendicular to the tangent line at $(x(s),y(s))$.

Before proving this, note that the slope of the tangent line is $\frac{dy/ds}{dx/ds} = \frac{dy}{dx}$ and the equation of the tangent line is $y - y(s) = \frac{dy}{dx}(x - x(s))$. The distance from $(0,0)$ to the tangent line is $\frac{\left|-y(s) + \frac{dy}{dx}x(s)\right|}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} = f(s)$.

For brevity in the following proof, coordinates of points on the curve, $x(s)$ and $y(s)$, are written as x and y.

Proof: $f(s) = (-y + \frac{dy}{dx} x)[1+(\frac{dy}{dx})^2]^{-1/2}$

$$\frac{df}{ds} = (-y + \frac{dy}{dx}x)(\frac{-1}{2})[1+(\frac{dy}{dx})^2]^{-3/2} \cdot 2 \frac{dy}{dx} \frac{d}{ds}(\frac{dy}{dx})$$

$$+ [1+(\frac{dy}{dx})^2]^{-1/2}[\frac{-dy}{ds} + \frac{d}{ds}(\frac{dy}{dx})]$$

$$= [1+(\frac{dy}{dx})^2]^{-3/2}[(y - \frac{dy}{dx} x)\frac{dy}{dx} \frac{d}{ds}(\frac{dy}{dx})$$

$$+ (1+(\frac{dy}{dx})^2)(\frac{-dy}{ds} + \frac{dy}{dx} \frac{dx}{ds} + x\frac{d}{ds}(\frac{dy}{dx}))] = 0$$

if and only if

$$y\frac{dy}{dx} \frac{d}{ds}(\frac{dy}{dx}) - x(\frac{dy}{dx})^2\frac{d}{ds}(\frac{dy}{dx}) + x \frac{d}{ds}(\frac{dy}{dx})$$

$$+ x(\frac{dy}{dx})^2 \frac{d}{ds}(\frac{dy}{dx}) = \frac{d}{ds}(\frac{dy}{dx})[y\frac{dy}{dx} + x] = 0$$

Thus, if $\frac{d}{ds}(\frac{dy}{dx}) \neq 0$, $\frac{df}{ds} = 0$ if and only if $y\frac{dy}{dx} + x = 0$.

This implies $\frac{dy}{dx} = \frac{-x}{y}$, i.e. the tangent line is normal to R, as desired.

Proposition 1: For an ellipse, $\frac{d}{ds}(\frac{dy}{dx}) \neq 0$.

Proof: Without loss of generality, the equation of ellipse remains $\frac{x^2}{a^2} + y^2 = 1$. Since $\frac{dy}{dx} = \frac{-x}{a^2 y}$,

$\frac{d}{ds}(\frac{dy}{dx}) = \frac{x\frac{dy}{ds} - y\frac{dx}{ds}}{a^2y^2}$. Thus, $\frac{d}{ds}(\frac{dy}{dx}) = 0$ if and only if $x\frac{dy}{ds} - y\frac{dx}{ds} = 0$ if and only if $x\frac{dy}{ds} = y\frac{dx}{ds}$, i.e. $\frac{dy}{dx} = \frac{y}{x}$. Note though, that $\frac{dy}{dx}$ represents the slope of the tangent line while $\frac{y}{x}$ is the slope of R, with both lines passing through the same point $(x(s),y(s))$. $\frac{dy}{dx} = \frac{y}{x}$ implies that R is actually the tangent line, contradicting an assumption that (0,0) does not lie on the curve.

Generalizing for $f(s,u,v)$, $\frac{\partial f}{\partial s} = 0$ if and only if (u,v) lies on the normal to the ellipse at $(x(s),y(s))$. Points $(\theta,u,v) \in M_F$ are those which satisfy the equation of the normal to the tangent line at $(x(s),y(s))$; the normal line at $(x(s),y(s))$ is

$$y - \frac{a \sin \theta}{\sqrt{a^2\sin^2\theta+\cos^2\theta}} = \frac{a^2}{\cot\theta}(x - \frac{a \cos \theta}{\sqrt{a^2\sin^2\theta+\cos^2\theta}}) \quad \text{or}$$

$$y - xa^2\tan\theta + \frac{(a^3-a)}{\sqrt{a^2+\cot^2\theta}} = 0.$$

Direct computation, in the previous section of this chapter, gave precisely the same set of points for M_F.

At this point, computation of the bifurcation set is an easy task, as already done. It is interesting that this curve is simply the envelope of the normals to the ellipse, the envelope being that curve which is tangent to each member of a family of curves at some point. Also, the envelope of a family of parametrized curves $H(\theta,u,v)$, such as the normals to the ellipse, is described by $\left\{(u,v) \mid \text{there is some } \theta \text{ satisfying } H = 0 \text{ and } \frac{\partial H}{\partial \theta} = 0\right\}$. This set is at once recognizable as the catastrophe set for a family of potential functions $F(\theta,u,v)$ when considering $\frac{\partial F}{\partial \theta} = H$.

The above asserts that given a center of gravity (u,v), local minima and maxima of the associated potential function are boundary points (x,y) for which (u,v) lies on the normal to the ellipse at (x,y). Each (u,v) lies on either two or four normal lines; points lying inside the cusp lie on four and have two local minima, the system jumping from one to the other according to the convention of perfect delay. Outside the cusp, each (u,v) lies on two normals and has one local minimum. [diagrams in 8]

2.4 The Ellipse on an Incline

An elliptical catastrophe machine might also be placed on a sloped surface of α degrees, $0 < \alpha < 90$. In doing so, suppose that frictional forces play no role in determining the gravitational potential energy function $F_{(u,v)}$ which governs the system. Since slightly more complicated than in the case of a flat surface, a subset of both M_F and the catastrophe set are found using the alternative method presented in the previous section.

In order to find the gravitational potential for the machine on an incline, consider a fixed position of the ellipse, labeling its point of contact with the sloped plane P_0. As the ellipse is rolled, a new point of contact P_1 is made, with s measuring the arclength from P_0 to P_1.

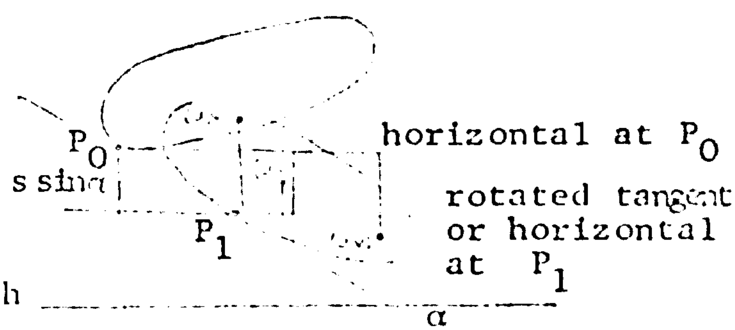


figure 2.4.1

Gravitational potential is proportional to the height from (u,v) to the horizontal line passing through P_0; this is equal to $d - s \sin\alpha$, where d is the signed distance from (u,v) to the horizontal line passing through P_1. This horizontal line may also be viewed as the tangent line rotated through α.

In an attempt to express gravitational potential explicitly in terms of θ, a term including arclength s arises. Using an elliptic integral to evaluate this distance would be impractical when finding the partial derivative of F, with respect to θ, for M_F. To avoid such difficulties, an attempt is made, as in the case of the ellipse on the level, to show the potential function $G(s)$ satisfies, in this case, $\frac{dG}{ds} = 0$ if and only if R forms an angle of $(90+\alpha)^o$ with the tangent line at $(x(s),y(s))$ if and only if R is perpendicular to the tangent line rotated through α^o. Again, consider a general curve and without loss of generality, let P be $(0,0)$ and R represent the line formed by P and $(x(s),y(s))$.

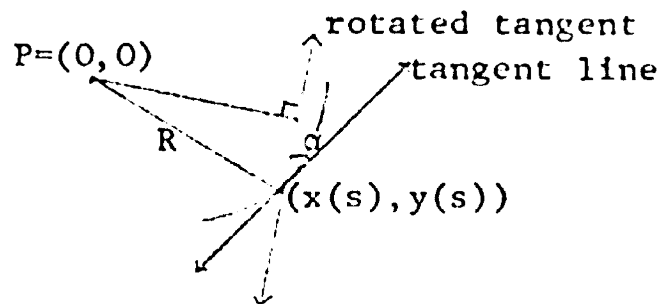


figure 2.4.2

To find $G(s)$, find the distance from a point $P = (0,0)$ to the rotated tangent at $(x(s),y(s))$ on a parametrized curve. For finding the equation of the rotated tangent, note that the slope of the tangent is $\frac{dy}{dx}$, the slope of the rotated tangent with respect to the tangent is $\tan \alpha$, and therefore the slope of the rotated tangent is $\dfrac{\frac{dy}{dx} + \tan \alpha}{1 - \frac{dy}{dx}\tan \alpha}$. The equation is

$$(y-y(s))(1 - \frac{dy}{dx}\tan \alpha) - (x-x(s))(\frac{dy}{dx} + \tan \alpha) = 0$$

and the signed distance

$$d = \frac{\pm\left|y(s)\left(\frac{dy}{dx}\tan\alpha - 1\right) + x(s)\left(\frac{dy}{dx} + \tan\alpha\right)\right|}{\sqrt{\left(1 - \frac{dy}{dx}\tan\alpha\right)^2 + \left(\frac{dy}{dx} + \tan\alpha\right)^2}}.$$

Call the expression within the absolute value sign E. Then, considering $+d$ and $E > 0$, or $-d$ and $E < 0$, in $G(s)$, let

$$f(s) = \left[y\frac{dy}{dx}\tan\alpha - y + x\frac{dy}{dx} + x\tan\alpha\right]\left[\left(1 - \frac{dy}{dx}\tan\alpha\right)^2 + \left(\frac{dy}{dx} + \tan\alpha\right)^2\right]^{-1/2} - s\sin\alpha.$$

As before, coordinates $x(s)$ and $y(s)$ are written as x and y.

Theorem 2: For points on any curve, where $\frac{d}{ds}\left(\frac{dy}{dx}\right) \neq 0$ and $\frac{ds}{dx} > 0$, $\frac{df}{ds} = 0$ if and only if R is perpendicular to the tangent line rotated through α^{o} if and only if R lies on the rotated normal line at $(x(s),y(s))$.

Proof: $$\frac{df}{ds} = \left[y\frac{dy}{dx}\tan\alpha - y + x\frac{dy}{dx} + x\tan\alpha\right]\left(-\frac{1}{2}\right)\left[\left(1 - \frac{dy}{dx}\tan\alpha\right)^2 + \left(\frac{dy}{dx} + \tan\alpha\right)^2\right]^{-3/2} 2\left[\left(1 - \frac{dy}{dx}\tan\alpha\right)\left(-\tan\alpha\frac{d}{ds}\left(\frac{dy}{dx}\right)\right) + \left(\frac{dy}{dx} + \tan\alpha\right)\frac{d}{ds}\left(\frac{dy}{dx}\right)\right]$$

$$+ \left[\left(1 - \frac{dy}{dx}\tan\alpha\right)^2 + \left(\frac{dy}{dx} + \tan\alpha\right)^2\right]^{-1/2} \left[y\frac{d}{ds}\left(\frac{dy}{dx}\right)\tan\alpha\right.$$

$$\left. + \frac{dy}{dx}\frac{dy}{ds}\tan\alpha - \frac{dy}{ds} + x\frac{d}{ds}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\frac{dx}{ds} + (\tan\alpha)\frac{dx}{ds}\right] - \sin\alpha$$

$$= \left[\left(1 - \frac{dy}{dx}\tan\alpha\right)^2 + \left(\frac{dy}{dx} + \tan\alpha\right)^2\right]^{-3/2} \left\{\left[-y\frac{dy}{dx}\tan\alpha + y - x\frac{dy}{dx}\right.\right.$$

$$\left. - x\tan\alpha\right]\left[\left(\frac{dy}{dx}\tan\alpha - 1\right)\frac{d}{ds}\left(\frac{dy}{dx}\right)\tan\alpha + \left(\frac{dy}{dx} + \tan\alpha\right)\frac{d}{ds}\left(\frac{dy}{dx}\right)\right]$$

$$+ \left[\left(1 - \frac{dy}{dx}\tan\alpha\right)^2 + \left(\frac{dy}{dx} + \tan\alpha\right)^2\right]\left[y\frac{d}{ds}\left(\frac{dy}{dx}\right)\tan\alpha\right.$$

$$\left. + \frac{dy}{dx}\frac{dy}{ds}\tan\alpha + x\frac{d}{ds}\left(\frac{dy}{dx}\right) + \frac{dx}{ds}\tan\alpha\right] - \sin\alpha\left[\left(1 - \frac{dy}{dx}\tan\alpha\right)^2\right.$$

$$\left.\left. + \left(\frac{dy}{dx} + \tan\alpha\right)^2\right]^{3/2}\right\}$$

$$= \left[\left(1 - \frac{dy}{dx}\tan\alpha\right)^2 + \left(\frac{dy}{dx} + \tan\alpha\right)^2\right]^{-3/2} \left\{\left[-y\frac{dy}{dx}\tan\alpha\right.\right.$$

$$\left. + y - x\frac{dy}{dx} - x\tan\alpha\right]\left[\frac{dy}{dx}\frac{d}{ds}\left(\frac{dy}{dx}\right)\tan^2\alpha + \frac{dy}{dx}\frac{d}{ds}\left(\frac{dy}{dx}\right)\right]$$

$$+ \left[1 + \left(\frac{dy}{dx}\right)^2\tan^2\alpha + \left(\frac{dy}{dx}\right)^2 + \tan^2\alpha\right]\left[y\frac{d}{ds}\left(\frac{dy}{dx}\right)\tan\alpha\right.$$

$$\left. + \frac{dy}{dx}\frac{dy}{ds}\tan\alpha + x\frac{d}{ds}\left(\frac{dy}{dx}\right) + \frac{dx}{ds}\tan\alpha\right] - \sin\alpha\left[\left(1 + \left(\frac{dy}{dx}\right)^2\right)\right.$$

$$\left.\left.\left(1 + \tan^2\alpha\right)\right]^{3/2}\right\}.$$

$\frac{df}{ds} = 0$ when

$$0 = [-y\frac{dy}{dx}\tan\alpha + y - x\frac{dy}{dx} - x\tan\alpha][\frac{dy}{dx}\frac{d}{ds}(\frac{dy}{dx})(1+\tan^2\alpha)]$$

$$+ [(1+(\frac{dy}{dx})^2)(1+\tan^2\alpha)][y\frac{d}{ds}(\frac{dy}{dx})\tan\alpha + \frac{dy}{dx}\frac{dy}{ds}\tan\alpha$$

$$+ x\frac{d}{ds}(\frac{dy}{dx}) + \frac{dx}{ds}\tan\alpha] - \sec^2\alpha\tan\alpha(1 + (\frac{dy}{dx})^2)^{3/2}$$

$$= \sec^2\alpha\Big\{-y(\frac{dy}{dx})^2\frac{d}{ds}(\frac{dy}{dx})\tan\alpha + y\frac{dy}{dx}\frac{d}{ds}(\frac{dy}{dx}) - x(\frac{dy}{dx})^2\frac{d}{ds}(\frac{dy}{dx})$$

$$- x\frac{dy}{dx}\frac{d}{ds}(\frac{dy}{dx})\tan\alpha + y\frac{d}{ds}(\frac{dy}{dx})\tan\alpha + \frac{dy}{dx}\frac{dy}{ds}\tan\alpha + x\frac{d}{ds}(\frac{dy}{dx})$$

$$+ \frac{dx}{ds}\tan\alpha + y(\frac{dy}{dx})^2\frac{d}{ds}(\frac{dy}{dx})\tan\alpha + (\frac{dy}{dx})^3\frac{dy}{ds}\tan\alpha$$

$$+ x(\frac{dy}{dx})^2\frac{d}{ds}(\frac{dy}{dx}) + \frac{dy}{dx}\frac{dy}{ds}\tan\alpha - \tan\alpha(1+(\frac{dy}{dx})^2)^{3/2}\Big\}$$

$$= \sec^2\alpha\Big\{(y\frac{dy}{dx}\frac{d}{ds}(\frac{dy}{dx}) - x\frac{dy}{dx}\frac{d}{ds}(\frac{dy}{dx})\tan\alpha + x\frac{d}{ds}(\frac{dy}{dx})$$

$$+ y\frac{d}{ds}(\frac{dy}{dx})\tan\alpha) + (2\frac{dy}{dx}\frac{dy}{ds}\tan\alpha + \frac{dx}{ds}\tan\alpha + (\frac{dy}{dx})^3\frac{dy}{ds}\tan\alpha)$$

$$- \tan\alpha(1+(\frac{dy}{dx})^2)^{3/2}\Big\}$$

$$= \sec^2\alpha\Big\{\frac{d}{ds}(\frac{dy}{dx})[y\frac{dy}{dx} - x\frac{dy}{dx}\tan\alpha + x + y\tan\alpha]$$

$$+ \frac{dx}{ds}\tan\alpha[1+2(\frac{dy}{dx})^2 + (\frac{dy}{dx})^4] - \tan\alpha(1+(\frac{dy}{dx})^2)^{3/2}\Big\} .$$

This is 0 when

$$\frac{d}{ds}(\frac{dy}{dx})[x(1 - \frac{dy}{dx}\tan\alpha) + y(\frac{dy}{dx} + \tan\alpha)]$$

$$+ \frac{dx}{ds}\tan\alpha(1+(\frac{dy}{dx})^2)^2 - \tan\alpha(1+(\frac{dy}{dx})^2)^{3/2} = 0 .$$

Note that $(ds)^2 = (dx)^2 + (dy)^2$ implies that

$1 + (\frac{dy}{dx})^2 = (\frac{ds}{dx})^2$ so that $\frac{dx}{ds}\tan\alpha[1+(\frac{dy}{dx})^2]^2$ becomes $(\frac{ds}{dx})^3\tan\alpha > 0$ and $\tan\alpha[1+(\frac{dy}{dx})^2]^{3/2}$ also becomes $(\frac{ds}{dx})^3\tan\alpha > 0$, meaning that the last two terms in $\frac{df}{ds}$ cancel. By assumption, $\frac{d}{ds}(\frac{dy}{dx}) \neq 0$. Thus, $\frac{df}{ds} = 0$ if and only if $x(1 - \frac{dy}{dx}\tan\alpha) + y(\frac{dy}{dx} + \tan\alpha) = 0$ if and only if $-\frac{x}{y} = \dfrac{\frac{dy}{dx} + \tan\alpha}{1 - \frac{dy}{dx}\tan\alpha}$ if and only if R lies on the rotated normal to the tangent line at $(x(s),y(s))$, as desired.

Proposition 2: For an ellipse, the assumptions in the above theorem are satisfied.

Proof: Again, without loss of generality, consider the ellipse $\frac{x^2}{a^2} + y^2 = 1$.

(i) Proposition 1 in section 2.3 showed $\frac{d}{ds}(\frac{dy}{dx}) \neq 0$.

(ii) Since the ellipse is symmetrical in shape, confine the remainder of this section's discussion to those points which lie on the bottom of the ellipse, including the point $(-a,0)$. As the ellipse rolls down the inclined plane from its initial point of contact P_0, arclength s increases; also view x as increasing as points move from left to right along the bottom of the ellipse. From this, observe that $\frac{ds}{dx} = \pm\sqrt{1+(\frac{dy}{dx})^2}$ is positive for points under consideration, i.e. $\frac{ds}{dx} > 0$.

The following will show that for points lying on a rotated normal line at a point $(x(s),y(s))$ on the bottom of the ellipse, the gravitational potential function $G(s)$ is precisely the function $f(s)$ used in Theorem 2. This occurs because for such points, E is found to be positive and $+d$ must be chosen. The slope of a rotated normal line at $(x(s),y(s))$ is

$$\frac{y(s)}{x(s)} = \frac{-(1 - \frac{dy}{dx}\tan\alpha)}{\frac{dy}{dx} + \tan\alpha}. \quad \text{Therefore} \quad \tan\alpha = \frac{x(s)+y(s)\frac{dy}{dx}}{x(s)\frac{dy}{dx} - y(s)}.$$

Substituting in E, $\dfrac{[y(s)-x(s)\frac{dy}{dx}]^2+[x(s)+y(s)\frac{dy}{dx}]^2}{x(s)\frac{dy}{dx} - y(s)}$ is positive if the denominator is positive. For the ellipse $\frac{x^2}{a^2} + y^2 = 1$, $x(s)\frac{dy}{dx} - y(s) > 0$. To see this, first notice that for $x(s) > 0$ and $y(s) < 0$, $\frac{dy}{dx} = \frac{-x(s)}{a^2 y(s)} > 0$ so that the above expression is positive for such points on the boundary of the ellipse. Since $x(s)\frac{dy}{dx} - y(s)$ is a continuous function on the bottom of the ellipse and since $\frac{dy}{dx} \neq \frac{y(s)}{x(s)}$, this function never passes through zero and therefore does not become negative. Observe that at $(0,-1)$ the value of the function is one; that for $x(s) < 0$ and $y(s) < 0$, $\frac{dy}{dx} < 0$ and the function is still positive; and as $x(s)$ approaches $-a$, the function approaches positive infinity. Thus for points on the rotated normal to a point on the ellipse, $E > 0$. Now, in viewing figure 2.4.1, it becomes apparent that if (u,v) lies below the rotated tangent line, $-d$ must be chosen in $G(s)$; however, the rotated tangent is actually a horizontal line whose normal has points which certainly lie above the rotated tangent line. Therefore $+d$ is chosen for those points on some rotated normal.

Generalizing, Theorem 2 showed that for $f(s,u,v)$, $\frac{df}{ds} = 0$ if and only if (u,v) lies on a rotated normal line at $(x(s),y(s))$. Thus, for points that do lie on a rotated normal to $(x(s),y(s))$, $G(s) = f(s)$ and $\frac{dG(s)}{ds} = 0$.

Though it seems true that Theorem 2 holds for the potential function $G(s)$, it is a complicated task to show. Discussion in section 2.5 requires that the theorem holds for $G(s)$ when center of gravity is (0,0).

Claim: For $(u,v) = (0,0)$, $\frac{dG(s)}{ds} = 0$ if and only if (0,0) lies on the rotated normal at $(x(s),y(s))$.

Proof: ($\Leftarrow$) Previous work showed that for points on a rotated normal $\frac{dG(s)}{ds} = 0$.

($\Rightarrow$) Recall that $E = y(s)(\frac{dy}{dx}\tan\alpha - 1) + x(s)(\frac{dy}{dx} + \tan\alpha)$ is the expression inside the absolute value sign in $G(s)$. If (0,0) lies above the horizontal line at P_1, it will be shown that $E > 0$ and therefore

$$G(s) = \frac{E}{\sqrt{(1 - \frac{dy}{dx}\tan\alpha)^2 + (\frac{dy}{dx} + \tan\alpha)^2}} - s\sin\alpha = f(s).$$

Also, for (0,0) below the horizontal line at P_1, $E < 0$ so that

$$G(s) = \frac{-|E|}{\sqrt{(1 - \frac{dy}{dx}\tan\alpha)^2 + (\frac{dy}{dx} + \tan\alpha)^2}} - s\sin\alpha = f(s).$$

Since the theorem showed that the claim is true for any (u,v) when $G(s) = f(s)$, this will prove the claim.

Continue to consider points on the bottom of the ellipse $\frac{x^2(s)}{a^2} + y^2(s) = 1$. For such points, $\frac{dy}{dx} = \frac{-x(s)}{a^2 y(s)}$ and $E = y(s)(\frac{dy}{dx} \tan \alpha - 1) + x(s)(\frac{dy}{dx} + \tan \alpha) = x(s) \tan \alpha (1 - \frac{1}{a^2}) - \frac{1}{y(s)}$. Since $y(s) \leq 0$ for

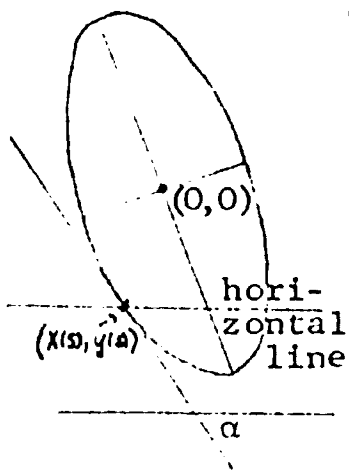


figure 2.4.3

any point on the bottom of the ellipse, for points with $x(s) \geq 0$, $E > 0$. It can be seen geometrically that $(0,0)$ lies above the horizontal line at points where $x(s) > 0$. (figure 2.4.3) Now, if $x(s) < 0$ then $(0,0)$ lies above the horizontal line at $(x(s),y(s))$ if and only if the y-intercept of the rotated tangent at $(x(s),y(s))$ is negative. The y-intercept is $y = y(s)(1 - \frac{dy}{dx} \tan \alpha) - x(s)(\frac{dy}{dx} + \tan \alpha) = -x(s) \tan \alpha (1 - \frac{1}{a^2}) + \frac{1}{y(s)} = -E$. $y < 0$ if and only if $E > 0$. Therefore, for $x(s) < 0$, $(0,0)$ lies above the horizontal line if and only if $E > 0$. (figure 2.4.4) Thus, if $(0,0)$ lies above the horizontal line at $(x(s),y(s))$ on the bottom of the ellipse, $E > 0$ and if $(0,0)$ lies below the horizontal, $E < 0$.

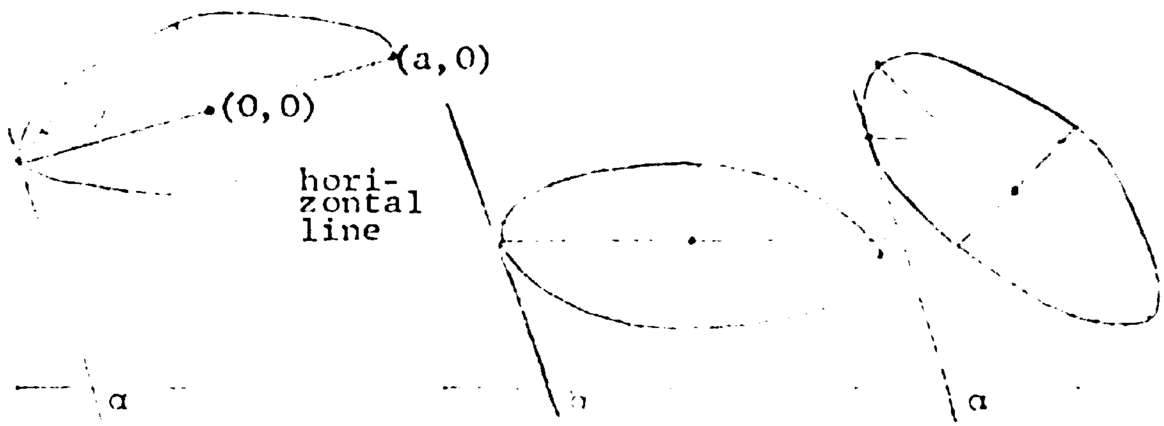


figure 2.4.3

For $G(s,u,v)$, it seems probable that $\frac{\partial G}{\partial s} = 0$ if and only if (u,v) lies on a rotated normal of α^0 at $(x(s),y(s))$, though this was not proved completely. Without finding F in terms of θ, it can be said that M_F at least contains (and probably equals) the set of points (θ,u,v) such that (u,v) lies on the normal to a rotated tangent line at a point $(x(s),y(s))$, corresponding to the angle θ. The envelope of the rotated normals to the ellipse at least forms a subset of the bifurcation set and is found in the expected manner, as described previously. Remember that $\frac{\partial}{\partial \theta}$ of the family of rotated normal lines must be found; thus, consider the equation of the rotated normal lines.

$$y - y(s) = \left(\frac{\frac{dy}{dx}\tan\theta - 1}{\frac{dy}{dx} + \tan\theta}\right)(x - x(s)),$$

where $(x(s),y(s))$ represents a point on the ellipse and (x,y) is any point on the coordinate plane. Recalling that $\frac{dy}{dx} = \frac{-\cot\theta}{a^2}$ and $r = \frac{a}{\sqrt{a^2\sin^2\theta+\cos^2\theta}}$ where $(x(s),y(s)) = (r\cos\theta, r\sin\theta)$, the above equation becomes

$$(*)\quad y\left(\frac{-\cot\theta}{a^2}+\tan\alpha\right)+x\left(\frac{\cot\theta}{a^2}\tan\alpha+1\right) - \frac{a\sin\theta}{\sqrt{a^2\sin^2\theta+\cos^2\theta}}\left(\frac{-\cot\theta}{a^2}+\tan\alpha\right) - \frac{a\cos\theta}{\sqrt{a^2\sin^2\theta+\cos^2\theta}}\left(\frac{\cot\theta}{a^2}\tan\alpha+1\right) = 0\ .$$

Next, find $\frac{\partial}{\partial\theta}$ of the above expression, i.e.

$$\frac{\partial}{\partial\theta}\left\{x+y\tan\alpha+\cot\theta\left[\frac{x\tan\alpha-y}{a^2}\right] + \frac{1}{\sqrt{a^2\sin^2\theta+\cos^2\theta}}\left[\frac{\cos\theta}{a} - a\sin\theta\tan\alpha - \frac{\cos\theta\cot\theta\tan\alpha}{a} - a\cos\theta\right]\right\}$$

$$= -\csc^2\theta\left[\frac{x\tan\alpha-y}{a^2}\right] + \frac{1}{\sqrt{a^2\sin^2\theta+\cos^2\theta}}\left[\frac{-\sin\theta}{a} - a\tan\alpha\cos\theta + \tan\alpha(\cos\theta\csc^2\theta+\cot\theta\sin\theta)+a\sin\theta\right]$$
$$+\left[\frac{\cos\theta-a^2\tan\alpha\sin\theta-\tan\alpha\cos\theta\cot\theta-a^2\cos\theta}{a}\right]\left[-(a^2\sin^2\theta+\cos^2\theta)^{-3/2}(a^2-1)(\cos\theta\sin\theta)\right] = 0$$

when

$$\frac{-\csc^2\theta(x\tan\alpha-y)}{a^2}(a^2\sin^2\theta+\cos^2\theta)^{3/2}$$

$$+ (a\sin\theta + \frac{\cos^2\theta}{a})[(a^2-1)(\sin\theta-\tan\alpha\cos\theta)+\tan\alpha\cot\theta\csc\theta]$$

$$+ \frac{(a^2-1)\cos\theta\sin\theta}{a}[(a^2-1)\cos\theta+a^2\tan\alpha\sin\theta+\tan\alpha\cos\theta\cot\theta]$$

$$= \frac{-\csc^2\theta(x\tan\alpha-y)}{a^2}(a^2\sin^2\theta+\cos^2\theta)^{3/2}+a(a^2-1)\sin^3\theta$$

$$+ a\tan\alpha\cos\theta + \frac{(a^2-1)\cos^2\theta\sin\theta}{a} - \frac{(a^2-1)\tan\alpha\cos^3\theta}{a}$$

$$+ \frac{\tan\alpha\cos\theta\cot^2\theta}{a} + \frac{(a^2-1)^2\cos^2\theta\sin\theta}{a} + \frac{(a^2-1)\tan\alpha\cos^3\theta}{a} = 0.$$

The catastrophe set is contained in (or equal to)

$$\Big\{(u,v) \mid \text{for some } \theta,\ (\theta,u,v) \text{ satisfies } (*) \text{ and}$$

$$\frac{-\csc^2\theta}{a^2}(u\tan\alpha-v)(a^2\sin^2\theta+\cos^2\theta)^{3/2} + a(a^2-1)\sin^3\theta$$

$$+ a\tan\alpha\cos\theta + a(a^{2-1})\cos^2\theta\sin\theta + \frac{\tan\alpha}{a}\cos\theta\cot^2\theta = 0\Big\}.$$

It can be verified that when $\alpha = 0$, the above set reduces to the bifurcation set for the ellipse on the level.

2.5 The Tendency of the Ellipse to Roll

Similar to the case of the ellipse on the level, local minima and maxima of $F_{(u,v)}$ are likely to be boundary points (x,y) for which (u,v) lies on the rotated normal to (x,y); however, in this case of the ellipse on an incline, there might be (u,v) which lie on no rotated normal. At such points, the system may not have, and at $(u,v) = (0,0)$ does not have, any stable equilibrium positions and the ellipse rolls off the sloped plane. Consider a fixed a in the equation of the ellipse. When the elliptical machine is placed on a slope, with center of gravity at the center of the ellipse, the question of what angles the machine will automatically roll off the slope arises.

Let the center of gravity be the center of the ellipse $(0,0)$. The equation of the ellipse is $\frac{x^2}{a^2} + y^2 = 1$ or $y = \frac{\sqrt{a^2-x^2}}{a}$ and $\frac{dy}{dx} = \frac{-x}{a^2 y}$. Recall that

$$y(\frac{dy}{dx} + \tan \alpha) - y(s)(\frac{dy}{dx} + \tan \alpha) - x(\frac{dy}{dx}\tan \alpha - 1)$$
$$+ x(s)(\frac{dy}{dx}\tan \alpha - 1) = 0$$

is an equation for the rotated normal to the ellipse at $(x(s),y(s))$. When $(0,0)$ lies on <u>some</u> rotated normal line,

$$x(s)(\frac{dy}{dx}\tan \alpha - 1) - y(s)(\frac{dy}{dx} + \tan \alpha) = 0 \quad \text{or}$$

$$x(\frac{-x}{a^2 y} \tan \alpha - 1) - y(\frac{-x}{a^2 y} + \tan \alpha) = 0 ,$$

substituting for $\frac{dy}{dx}$ and writing $x(s)$ and $y(s)$ as x and y. Then,

$$\frac{-x^2 \tan\alpha}{a^2 y} - x + \frac{x}{a^2} - y\tan\alpha = 0$$

$$\text{or} \quad \tan\alpha = \frac{(x-a^2x)y}{x^2+a^2y} \quad .$$

Substituting $y = \sqrt{\frac{a^2-x^2}{a}}$,

$$\tan\alpha = \frac{-x(a^2-1)\sqrt{a^2-x^2}}{a^3}$$

$$\text{or} \quad (x^2)^2 - a^2(x^2) + \frac{a^6\tan^2\alpha}{(a^2-1)^2} = 0 \ .$$

When $(0,0)$ lies on the normal to some rotated tangent line at $(x(s),y(s))$, the above quadratic in x^2 must hold. Determine for what α, x^2 has real solutions where $0 \leq x^2 \leq a^2$. To do so, find those α where the discriminant of the quadratic is greater than or equal to zero.

$$a^4 - \frac{4a^6\tan^2\alpha}{(a^2-1)^2} \geq 0 \quad .$$

$$\text{i.e.} \quad \tan^2\alpha \leq \frac{a^4(a^2-1)^2}{4a^6} = \frac{(a^2-1)^2}{4a^2}$$

$$\text{i.e.} \quad \frac{1-a^2}{2a} \leq \tan\alpha \leq \frac{a^2-1}{2a} \quad .$$

Choosing only positive values for α,

$$\tan \alpha \leq \frac{a^2-1}{2a} \quad \text{or} \quad \alpha \leq \arctan\left(\frac{a^2-1}{2a}\right).$$

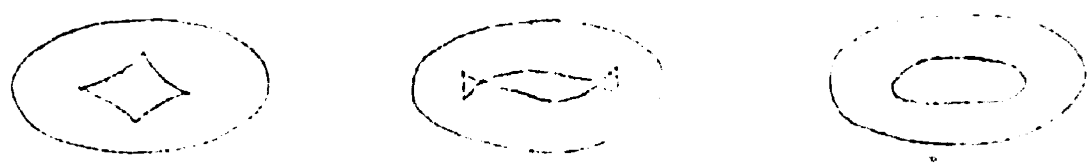
The elliptical machine will roll off an incline of $\alpha > \arctan\left(\frac{a^2-1}{2a}\right)$ when center of gravity is $(0,0)$.

The following chart gives an idea as to how steep the incline can reach before a fixed ellipse will automatically roll down (arctans are approximations).

$a (a > 1)$	$\alpha \leq$ (in degrees)
1.01	½
1.1	5
1.6	26
2	37
3	53
4	61

The closer the ellipse is to the shpae of a circle, the smaller the slope need be to cause the ellipse to roll.

Lastly, as α increases, more and more points (u,v) will cause the elliptical machine to roll down the inclined plane. The appearance of such (u,v) breaks the cusp shaped bifurcation set into two shapes that look like swallowtails. As α increases, fewer (u,v) lie on four rotated normal lines; and finally, all points lies on at most two normal lines and the swallowtails disappear completely. Thus, cusp singularities disappear and only fold singularities remain so that as α increases, the catastrophe manifolds change from manifolds consisting of both cusp points and folds to those consisting only of folds. Below are sketches of the bifurcation sets as α increases. For more complete sketches, see Poston and Stewart's article [8].



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