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ANALYTICAL MODELING OF MECHANICAL INTERFACE:
FAILURE PREDICTION

by
E. Thomas Moyer, Jr.

A Thesis
Presented to the Graduate Committee
of Lehigh University
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December 6, 1978
(date)

Professor In Charge

Chairman of Department
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ABSTRACT

A method of analytically modeling the interface between two dissimilar materials is proposed. The influence of these models on the prediction of the location of possible failure sites is investigated. The results of the analysis show that the way in which the modulus of elasticity varies within the interface is as important in modeling as is the average interface modulus. A gradual modulus variation in the interface behaves similar to a stiffer interface than does a steeper modulus variation for a constant average modulus. An interface with a higher average modulus than the average of the bulk materials causes a shift of possible failure locations from those points predicted with a single line interface. The results also show that the thickness of the interface has a large effect on the bulk behavior of the interface. A thicker interface (with a constant modulus being the average of that of the bulk materials) causes a more gradual change in modulus throughout the composite material which behaves similar to an interface with a higher average modulus.

The numerical analysis in this work is carried out using a finite element procedure, which employs fourth order isoparametric elements, to calculate the displacement, stress, and energy fields in the material in question. The location of possible failure sites is predicted by the Strain Energy Density Criterion which assumes failure to occur in a region where the dilatation portion of the strain energy is dominant.
A method of analytically modeling the interface between two dissimilar materials is proposed to account for non-uniformities that could arise due to the method of joining. In most composite material problems, the interface between two materials is often idealized to be a line discontinuity with a sudden change in material properties. The stress and displacement fields are assumed to be continuous across this line as if the bonding were perfect. These assumptions predict reasonably accurate stress fields far from the interface region, but do not always lead to an accurate prediction of the location of failure.

In the proposed model, the interface attains a finite thickness and is modeled by a finite number of material layers whose elastic properties satisfy a line discontinuity. The displacement field is assumed to be continuous across each layer boundary and the stress field is continuous only in an average sense according to the finite element approximations. By varying the interface thickness and layer moduli, it is possible to create models which can alter the failure location within the bonded material. Through a knowledge of the behavior of a given configuration (materials, loading and geometry), the present model allows a more realistic evaluation of the influence of the interface on the composite material behavior. Several examples and their results are discussed in this work. Due to the vast number of interface bonding types and material combinations, it is possible to present models which will cover all possibilities. Nevertheless, some salient features of these models are exhibited.
The stress analysis is based on the two-dimensional finite element procedure. Twelve node isoparametric elements are employed to obtain high resolution. The location of possible failure sites is predicted by the Strain Energy Density Failure Criterion which predicts failure in a region where the dilatation portion of the strain energy is dominant.

The accuracy of the finite element method for use in composite material problems is investigated in the Appendix of this work. By an appropriate choice of grid, extremely accurate results can be obtained using the finite element method. The type of grid features which yield the most accurate results for composite material problems are determined in the Appendix and employed throughout this work.
II. BACKGROUND CONCEPTS

A. The Finite Element Method

The finite element method is a procedure for obtaining approximate solutions to partial differential equations. It is formulated on the basis of variational principles. Elasticity lends itself extremely well to the finite element method as the Potential Energy Theorem [1] provides a strong variational principle upon which the analysis is developed.

The mechanics of the finite element method involve dividing the domain of interest into a number of smaller regions called elements. The variables of interest are represented in functional form by shape functions within the domain of each element and in terms of the value of the variable at specified points on the element boundary (called nodes). These functional relations are then used in the analysis as approximate solutions to the problems in question.

In the case of elasticity, the variables in question are the components of the displacement field, $u_i$. If the shape functions are chosen to identically satisfy displacement boundary conditions, and to insure a continuous displacement field across element boundaries, those components of the displacement field which minimize the potential energy will be the solution to the problem, providing their derivatives satisfy traction boundary conditions in an approximate sense. The potential energy of an elastic system under the influence of surface tractions, $T^m_i$, without body forces or temperature gradients.
can be written as

$$\text{PE} = \sum_{i=1}^{N} \int_{V_i} (\text{SED})_i \, dV - \int_{S} T_i^N u_i \, ds$$  \hspace{1cm} (1)$$

where $(\text{SED})_i$ is the strain energy per unit volume in the $i$th element, $T_i^N u_i \, ds$ is the work done by surface tractions and $N$ is the total number of elements in the volume, $V$, bounded by the surface, $S$. Utilizing Hooke's Law and the strain-displacement relations, the strain energy for an element can be expressed in terms of the nodal displacements as

$$\text{SE} = \frac{1}{2} \sum_{i=1}^{m'} \sum_{j=1}^{m'} k_{ij} u_i u_j$$  \hspace{1cm} (2)$$

where $k_{ij}$ are the components of the element stiffness matrix determined by the shape functions and material constants. $m'$ is the number of degrees of freedom of each element. Substituting (2) into (1), the potential energy of the system can be written as

$$\text{PE} = \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} K_{ij} u_i u_j - \sum_{i=1}^{m} R_i u_i$$  \hspace{1cm} (3)$$

where $K_{ij}$ are the components of the master stiffness matrix, $K_{ij} = \sum_{k=1}^{N} (k_{ij})_k$, for the system and $R_i$ is the resulting vector from the integration of the surface traction terms. $i$ and $j$ refer to the global node numbers and $m$ is the total number of degrees of freedom for the entire system. The minimization of (3) with respect of $u_i$ results in

-5-
\[
\frac{\partial \text{PE}}{\partial u_\ell} = \sum_{i=1}^{m} K_{i\ell} u_i - R_\ell = 0
\]  
\tag{4}

where

\[
\sum_{i=1}^{m} K_{i\ell} u_i = R_\ell
\]  
\tag{5}

The resulting system of linear algebraic equations, (5), are solved for the nodal displacements, \( u_i \).

If the analysis is restricted to two dimensions, the displacements at an arbitrary point, \( (x,y) \), can be written as

\[
u_x = \left( \sum_{i=1}^{m^i} N_i(x,y) u_{x1} \right)_E \]  
\tag{6a}

and

\[
u_y = \left( \sum_{i=1}^{m^i} N_i(x,y) u_{y1} \right)_E \]  
\tag{6b}

where \( E \) is the element containing the point \( (x,y) \). Utilizing the strain displacement relations and the generalized Hooke's Law, the strains can be written as

\[
\varepsilon_x = \left( \sum_{i=1}^{m^i} \frac{\partial N_i(x,y)}{\partial x} u_{x1} \right)_E \]  
\tag{7a}

\[
\varepsilon_y = \left( \sum_{i=1}^{m^i} \frac{\partial N_i(x,y)}{\partial y} u_{y1} \right)_E \]  
\tag{7b}
The stress components can be obtained from

$$\sigma_{xy} = \left[ \sum_{i=1}^{m} \left( \frac{\partial N_i(x,y)}{\partial y} u_{x1} + \frac{\partial N_i(x,y)}{\partial x} u_{y1} \right) \right]_2$$

(7c)

where \([D]\) is a matrix of material properties depending on the specified stress condition (i.e., plane stress or plane strain).

For the analysis in this work, a 12 node isoparametric element was used. The element has 4 nodes on each side (including corner nodes) allowing for a cubic variation of the displacement field [2]. The geometry of each element side is allowed to vary according to the same shape functions as the displacement field. Since the geometry and displacement field vary according to the same functional relations, the isoparametric region is easily mapped into a rectangular region with a linear displacement field. A typical element in coordinate and mapped spaces is shown in Figure 1. Stresses and strains are easily calculated in the transformed space and mapped back to the original coordinate space since the mapping is isomorphic.

The present analysis makes use of a two-dimensional and axisymmetric computer program developed for the Naval Ship Research and Devel-

-7-
opment Center by Hilton, Gifford and Lomacky [3]. The program employs
the 12 node isoparametric element discussed above. The integration re-
quired in the analysis is done using a Gauss-Legendre quadrature tech-
nique in the mapped space. Both three and four point approximations
are available as a user option. The linear algebraic equations ob-
tained in the analysis are solved using a Frontal Solution technique
[4]. Stresses and strains are calculated at the quadrature points and
extrapolated to give nodal values.
B. Strain Energy Density Theory - Concept and Criterion

One of the foremost problems in the field of solid mechanics is the prediction of fracture initiation and the onset of crack propagation leading to instability. The classical concepts of yield strength and maximum stress criteria do not adequately predict the onset of fracture, nor do they locate the point of fracture initiation. If the safety of a structure is to be insured, the critical loading and places of fracture initiation must be considered.

A theory of instability of elastic (or inelastic) systems which examines the local strain energy density (SED) field in the domain of interest has been proposed by Sih [5,6]. He has used this theory to predict the onset and direction of crack propagation. The theory itself is not limited to regions containing an existing crack and will therefore be employed to predict instability and to locate possible fracture sites in an uncracked elastic medium.

The fundamental hypotheses used to predict instability by the SED theory are:

Hypothesis 1: Failure will occur in a medium at a point where the strain energy density field has a stationary (minimum) value.

Hypothesis 2: The onset of failure is governed by a material parameter \( (\text{SED})_{cr} \).
The minima of SED occur when the dilatation portion of the strain energy is dominant such as the case of hydrostatic tension. This theory predicts the location of failure and the loading conditions under which this failure will take place. The critical value of the strain energy density for a given material can be determined experimentally. A discussion of this criterion for engineering design purposes is given by Sin and Macdonald in [7].

In this work, the loading conditions are assumed to be below the critical level. The SED theory will be used to locate points of instability in the structures and to identify those places where fracture would initiate if the loads reached the critical level.

Two important aspects of this theory should be pointed out. The SED theory requires no knowledge of the microstructure of the material nor the location of flaws or fissures below the macroscopic level. This aspect allows application of the continuum solutions for stress and energy fields without the inconsistency of scale change. Second, the theory assumes that, until the critical value of SED is reached, the material properties do not change and (providing one stays below the proportional limit) no load dependent corrections need to be made to the linear elasticity solutions. These aspects make application of the theory to structures with complicated loading and geometry relatively straightforward. The elasticity solution can be obtained for very complicated structures by the finite element method. The stress-strain fields obtained in these calculations can then be used to calculate the SED field in the structure. An examination of this field
using the above criterion will predict the points of instability.
C. Analytical Interface Modeling

Two loading cases on a cantilevered rectangular region were investigated in this work. In the first loading case, the specimen was loaded in bending through the layer thicknesses and in tension along the interface boundaries. The effect of varying the interface moduli were investigated with this loading case. In the second loading case, the specimen was loaded in tension and in shear normal to the layer boundaries. The effect of varying the interface thickness was investigated using this loading case by utilizing a single interface layer of constant modulus.

The finite element idealization for the first loading case is shown in Figures 2 and 3. The material used for the bottom of the region was bone and the material used for the top of the region was polymethylmethacrylate bone cement. The moduli of elasticity for these materials are

$$E_{\text{bone}} = 2.2753 \times 10^3 \text{ Mega Pascals}$$

and

$$E_{\text{cement}} = 6.8948 \times 10^4 \text{ Mega Pascals}$$

respectively [8]. The normal stress applied to the right end of the region was $1.0 \times 10^3$ Pascals and the shear stress applied to the right end was $1.0 \times 10^3$ Pascals. The total interface ($\delta$) was held constant at .05 times the total thickness, $H$, which was 4 meters. Poisson's
ratio was assumed to be the same for the bone, the cement, and for the interface layers. The value used in the analysis was 0.3. The left end of the specimen was cantilevered (i.e., $u(0,y) = v(0,y) = 0$). The interface was modeled as four layers of equal thickness ($\delta/4$). The moduli of these layers was varied in the analysis. These variations are shown pictorially in Figure 4.

The finite element idealization for the second loading case is shown in Figure 5. Again, the materials used for the bottom and top of the region were bone and cement respectively. The interface was modeled as a single layer with modulus of elasticity

$$E_{\text{interface}} = \frac{(E_{\text{bone}} + E_{\text{cement}})}{2} = 3.561 \times 10^4 \text{ Mega Pascals} \quad (10)$$

Poisson's ratio for the materials was assumed to be 0.3. The total specimen thickness was again 4 meters. The top of the specimen was loaded with uniform tension of $1.0 \times 10^3$ Pascals and shear stress of $-1.0 \times 10^3$ Pascals. The bottom of the region was cantilevered (i.e., $u(x,0) = v(x,0) = 0$). The interface thickness was varied in the analysis.
III. RESULTS FROM THE INTERFACE MODELS

A. Loading Case I

The effect of varied interface moduli was investigated using loading case I. Both the average interface modulus and the variation about that average in the interface was explored. The results of these models were compared to the results from an analysis of the specimen with an idealized line interface.

Figure 6 is a plot of the SED contours for loading case I with a single line interface. The largest relative minimum, which is the one which will reach critical value first, is located on the neutral bending axis in the cement layer (marked with an "X" in the figure). This prediction is consistent with the elementary maximum shear stress criterion for failure. A secondary minimum region occurs in the bone section of the specimen. The relative intensity of this contour is over 100 times less than the minimum contour in the cement layer and would not reach critical intensity until long after the cement had failed.

A linear modulus variation is now introduced to the interface region. The moduli of the interface layers are given in Table I (Model I) and shown pictorially in Figure 4. There are several minimum contours in the specimen. The largest minimum contour lies in the bone layer near the cantilevered end. The SED contours in the bone layer are shown in Figure 7. The predicted failure location is marked with an "X". This contour is 4.8 times more intense than any of the other relative minima; thus, it will reach critical value and cause failure.
<table>
<thead>
<tr>
<th>Layer</th>
<th>Model I</th>
<th>Model II</th>
<th>Model III</th>
<th>Model IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>LAYER I</td>
<td>$1.561\times10^{10}$</td>
<td>$1.000\times10^{09}$</td>
<td>$2.500\times10^{10}$</td>
<td>$1.000\times10^{09}$</td>
</tr>
<tr>
<td>LAYER II</td>
<td>$2.894\times10^{10}$</td>
<td>$1.000\times10^{10}$</td>
<td>$5.000\times10^{10}$</td>
<td>$3.000\times10^{10}$</td>
</tr>
<tr>
<td>LAYER III</td>
<td>$4.228\times10^{10}$</td>
<td>$3.000\times10^{10}$</td>
<td>$7.500\times10^{10}$</td>
<td>$4.714\times10^{10}$</td>
</tr>
<tr>
<td>LAYER IV</td>
<td>$5.561\times10^{10}$</td>
<td>$5.000\times10^{10}$</td>
<td>$9.000\times10^{10}$</td>
<td>$6.428\times10^{10}$</td>
</tr>
<tr>
<td>AVERAGE MODULUS</td>
<td>$3.561\times10^{10}$</td>
<td>$2.704\times10^{10}$</td>
<td>$5.187\times10^{10}$</td>
<td>$3.561\times10^{10}$</td>
</tr>
</tbody>
</table>

The introduction of this interface model has caused the predicted location of failure to be shifted from the cement to the bone layer.

A model with softer interface was investigated next. The interface moduli used were those given for Model II in Table I. The average modulus is now 1.3 times less than for Model I. The only relative minimum SED contour in the region occurs in the cement layer on the neutral bending axis near the cantilevered end. The location is almost exactly the same as predicted by the rigid bone interface. The SED contours in the cement layer are shown in Figure 8. The predicted failure location is marked with an "X".
An interface with a stiffer average modulus was explored in Model III. The average modulus is now 1.45 times larger than in Model I. Many relative minima were predicted with this model, all of whose intensities were fairly close. The SED contours for the entire specimen are shown in Figure 9. A finer analysis of the regions near these minima reveal that the largest minima occur in the bone layer. They are about 1.5 times larger than those in the cement. The contour marked with an "X" on the figure is the largest relative minimum contour in the bone layer. It is approximately 1.05 times more intense than the other minimum contours in the bone layer. The significance of this factor is subject to question due to the inaccuracies of the finite element results. Considering the results from Model I, the prediction of failure at the contour near the cantilevered end of the bone layer is a reasonable and consistent prediction. For absolute affirmation of this result, a much more detailed grid requiring far more computer memory would be needed.

The effect of modulus distribution holding the average overall modulus constant was investigated in Model IV. The overall average modulus was the same as in Model I, but the variation in the interface was changed. The first layer was assigned a modulus much lower than the bone. The next three layers were assigned a linear variation in moduli with a much greater slope than in Model I. The modulus of the top interface layer was almost as stiff as the cement. The interface moduli for Model IV are shown in Table I. Figure 10 is a plot of the SED contours in the entire specimen for Model IV. A finer analysis in the bone layer revealed no relative minima in that layer. Thus, the only
relative minimum contour is in the cement layer marked on the figure with an "X". Failure is thus predicted at about the same location with Model IV as with a single line interface.

A comparison of the models studied in this section reveal that for the geometry and loading in loading case I, failure can occur in either the bone or cement layer depending on the stiffness of the interface. If the average modulus is larger than the average of the bone and cement moduli, failure is predicted in the bone layer, (i.e., Model III). Thus, the stiffer interface causes a shift of the failure point from the cement layer to the bone. It is believed that this phenomenon is due to the fact that the stiffer modulus causes a stress state of pure dilatation to occur in the bone layer. Thus, the SED in this region is due mainly to dilatation. The stress contours for the dominant stress component (σx) are shown in Figure 11.

A comparison of the results from Models I and IV reveal that the way in which the interface modulus varies can have a large effect on the prediction of failure. In Model IV, the modulus variation is less gradual than in Model I. It thus, more closely, approaches the single line interface model. The failure prediction from Model IV is hence similar to that for the model with a single line interface. The gradual change in moduli in Model I caused a shift in failure location from the cement to the bone. Hence, a more gradual change of moduli within the interface implies that the effect of this variation on the specimen is the same as with the larger average modulus. The more gradual variation causes a larger effective interface stiffness with respect to its effect on the entire specimen.
B. Loading Case II

The effect of varied interface thickness was investigated using loading case II. The interface was modeled as a single layer of material with modulus given in (10). The thickness was started at .05 times the total thickness and varied both higher and lower. The location of failure predicted from each of these models was compared.

Figure 12 is a plot of the SED contours for loading case II with an interface thickness 5% of the total specimen thickness. Two relative minima are predicted with this interface model (marked with an "X" in the figure). Using a more detailed analysis, the minimum contour located in the bone layer (lower layer) is found to be about 2.8 times more intense than the minimum contour in the cement layer. Thus, failure should take place in the bone layer first as the minimum contour on that layer will reach a critical value first.

The next interface thickness investigated was 2.5% of the total specimen thickness. The SED contours for the specimen with this interface thickness is shown in Figure 13. Two relative minima are predicted at approximately the same location as with the model whose interface thickness was 5% of the total specimen thickness. A more detailed analysis revealed that the minimum contour located in the bone layer was only 1.5 times as intense as the minimum contour in the cement layer. Failure is still predicted in the bone layer first, but now the stability of the cement layer is not much greater than that of the bone.
The interface thickness was next increased. The SED contours for the specimen with interface thickness 7.5% of the total specimen thickness is shown in Figure 14. Again, two relative minima are observed at about the same location as in the two previous models (marked with an "X" in the figure). The major difference in the SED contours for this model compared with those from the other models is that the minimum in the bone layer is 8.1 times more intense than the minimum in the cement layer. Failure is predicted in the bone layer far before the stability of the cement layer will be in question. Thus, the thicker interface preserves the stability of the stiffer material for this loading and geometry.

The final model investigated had an interface thickness of .5% of the total specimen thickness. The SED contours for this model are shown in Figure 15. Two relative minima are observed at the same locations as before and marked in the figure with an "X". The intensity of the minimum in the bone layer is now 1.5 times greater than the intensity of the minimum located in the cement layer. This model yields almost identical results to those obtained with the model whose interface thickness was 2.5% of the total specimen thickness.

A comparison of the results for the four models discussed above reveals that for the loading and geometry of loading case II, the thicker interface (with a modulus being the average of the moduli of the bulk materials) causes the SED minimum in the bone layer to be much more intense than the SED minimum in the cement layer. Since the intensity in the bone layer is approximately the same for each of the
models, the thicker interface has the effect of increasing the stability of the cement layer. This is due to the fact that the thicker interface causes a more gradual change in modulus through the specimen and, while the average modulus is unchanged, the effective stiffness of the specimen is increased with respect to the stability of the cement layer. The stability of the bone layer is fairly insensitive to interface thickness and all models predict the onset of instability at about the same loading.
C. Discussion and Conclusion

The modeling of composite interfaces is of primary importance to analysts who deal with material bonds. The soldering, welding, cementing and epoxying of materials has a large effect on the material properties at the joint and often has a large effect on the stress and energy fields in the entire structure. The proposed technique suggests that these interface regions be modeled as a system of laminate composite layers with variable modulus and thickness.

Through a judicious choice of interface model, the stress and energy fields in a composite material can be more accurately predicted than with the normal single line assumption classically used for composite interfaces. This choice can be made through a knowledge of the overall behavior of the structure in question (i.e., location of points of failure, overall deformation and stress concentration near material boundaries). A comparison of models to find those features which most closely predict the overall characteristics of the problem in question should yield an accurate model with which to predict the details of the stress and energy fields in a structure.

The models discussed in this work demonstrate that the effective stiffness of an interface can be varied both by varying the average modulus, the layer thicknesses and the abruptness of modulus discontinuity within the interface. The location of predicted points of instability is very sensitive to changes in these features. Thus, a careful modeling should yield an accurate prediction of the location of points of instability in a structure.
The Appendix of this work examines the obtainable accuracy for the stress analysis of composite materials using the finite element method. With a careful choice of grid features, extremely accurate results can be obtained with the finite element method. Thus, analysis of composite structures with complicated loading and geometries can be carried out with relatively little difficulty. After obtaining the stress and energy fields for a given problem, the SED instability criterion can be employed to predict points of instability and the onset of fracture.

The type of modeling described in this work is limited to those interface regions which are approximated by isotropic layers. This assumption may not be realistic for the interfaces between materials which are highly fibrous or extremely crystalline. Materials which possess these characteristics could be modeled either with an anisotropic interface or matrix material interfaces. Sih et al have investigated these situations for composite materials with cracks [9]. Their models account not only for the non-homogeneity of the interface but also for the lack of isotropy which may exist in certain materials. A universal model which can accurately characterize all materials is, thus, not possible. The choice of model must depend on the problem at hand.
Figure 1: Isoparametric Element in Coordinate and Mapped Spaces
Figure 2: Idealization for the First Loading Case
Figure 3: Idealization of Interface Region for the First Loading Case
Figure 4: Interface Moduli for First Loading Case (x 10^10 Pascals)
Figure 5: Idealization for the Second Loading Case
Contour Values
1 - 1.27E-04
2 - 1.34E-02
3 - 2.68E-02
4 - 5.34E-02
5 - 1.60E-01

Figure 6: SED Contours for Loading Case I with a Single Line Interface (Joules/Meter^2)
Figure 7: SED Contours in the Bone Layer for Model I
(Joules/Meter²)

Contour Values
1 - 1.36E - 04
2 - 6.54E - 04
3 - 1.17E - 03
4 - 1.69E - 03
5 - 4.28E - 03
Figure 8: SED Contours in the Cement Layer for Model II (Joules/Meter²)

Contour Values

1 - 9.59E - 05
2 - 1.42E - 02
3 - 4.25E - 02
4 - 8.48E - 02
5 - 1.70E - 01
Figure 10: SED Contours for Model IV (Joules/Meter^2)

Contour Values:
1 - 5.39E - 05
2 - 1.36E - 02
3 - 4.08E - 02
4 - 9.51E - 02
5 - 1.63E - 01
Figure 11: X Stress Contours for Model III (Pascals)
Figure 12: SED Contours (Joules/Meter^2) for Second Loading Case with Interface Thickness .05 x H

Contour Values
1 - 7.49E-05
2 - 5.18E-04
3 - 9.61E-04
4 - 1.40E-03
5 - 1.85E-03
6 - 2.29E-03
7 - 5.39E-03
Figure 15: SED Contours (Joules/Meter²) for Second Loading Case with Interface Thickness .005 x H
REFERENCES


APPENDIX: THE TWO MATERIAL PROBLEM

A. Analytic Solution

Consider the plane strain problem for the two material medium shown in Figure A-1. Assuming each layer to be a homogeneous and isotropic strip, the governing differential equations for the $i^{th}$ strip can be written as [1]

$$\mu_i \gamma^2 u_i + (\lambda_i + \mu_i) \frac{\partial}{\partial x} \left( \frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y} \right) = 0 \quad (A-1-a)$$

$$\mu_i \gamma^2 v_i + (\lambda_i + \mu_i) \frac{\partial}{\partial y} \left( \frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y} \right) = 0 \quad (A-1-b)$$

where $u_i$ and $v_i$ are the $x$ and $y$ components of the displacement field in the $i^{th}$ material and $\mu_i$ and $\lambda_i$ are the Lamé constants for the $i^{th}$ material.

Assuming symmetric loading, $u_i$ will be an odd function in $x$ and $v_i$ will be an even function in $x$. Restricting the analysis to finite loading so that the displacements as well as their derivatives damp out rapidly enough as $|x| \to \infty$ for the functions to be absolutely integrable [10], the displacements can be written in terms of Fourier integrals as

$$u_i(x,y) = \frac{2}{\pi} \int_0^\infty \phi_i(\alpha,y) \sin(\alpha x) d\alpha \quad (A-2-a)$$

$$v_i(x,y) = \frac{2}{\pi} \int_0^\infty \psi_i(\alpha,y) \cos(\alpha x) d\alpha \quad (A-2-b)$$
where \( \phi_i(\alpha,y) \) and \( \psi_i(\alpha,y) \) are unknown functions to be determined from the field equations (A-1). Inserting the assumed solutions (A-2) into the field equations and solving the resulting ordinary differential equations, the displacement components can be written as

\[
\begin{align*}
\phi_i(x,y) &= \frac{2}{\pi} \int_0^\infty \left[ (A_{11} + A_{21}y)e^{\alpha y} + (A_{31} + A_{41}y)e^{-\alpha y} \right] \times \sin(\alpha x) \, d\alpha \\
\psi_i(x,y) &= \frac{2}{\pi} \int_0^\infty \left[ (-A_{11} + A_{21}(\kappa_1/\alpha-y))e^{\alpha y} + (A_{31} + A_{41}(\kappa_1/\alpha+y))e^{-\alpha y} \right] \\
&\quad \times \cos(\alpha x) \, d\alpha
\end{align*}
\]

where \( \kappa_1 = 3-4\nu_1 \) and \( A_{ij} \) are functions of the transform variable, \( \alpha \), to be determined from the boundary and continuity conditions. The material constants \( v_1 \) and \( \nu_1 \) are the shear modulus and Poisson's ratio for the \( i \)th material respectively. The stresses at any point in the \( i \)th material can be obtained from Hooke's Law and the strain-displacement relations and written as

\[
\begin{align*}
\frac{\sigma_{xx1}}{2\mu_1} &= \frac{2}{\pi} \int_0^\infty \left[ \alpha(A_{11} + A_{21}y) + 2\nu_1A_{21} \right] e^{\alpha y} + [\alpha(A_{31} + A_{41}y) \\
&\quad - 2\nu_1A_{41}]e^{-\alpha y}) \cos(\alpha x) \, d\alpha \quad \text{(A-4-a)}
\end{align*}
\]

\[
\frac{\sigma_{yy1}}{2\mu_1} = \frac{2}{\pi} \int_0^\infty \left[ -\alpha(A_{11} + A_{21}y) + 2(1-\nu_1)A_{21} \right] e^{\alpha y} - [\alpha(A_{31} + A_{41}y) \\
&\quad + 2(1-\nu_1)A_{41}]e^{-\alpha y}) \cos(\alpha x) \, d\alpha \quad \text{(A-4-b)}
\]
\( \frac{\sigma_{xy1}}{2\mu_1} = \frac{2}{\pi} \int_0^\infty \{[a(A_{11} + A_{21})] - (1-2\nu_1)A_{21}\} e^{\alpha y} - [a(A_{31} + A_{41})] + (1-2\nu_1)A_{41} \} e^{-\alpha y} \sin(\alpha x) d\alpha \) 

\( (A-4-c) \)

Stress and displacement continuity at the interface between the two materials requires that

\[ u_1(x,\delta) = u_2(x,\delta); \quad v_1(x,\delta) = v_2(x,\delta) \] 

\( \sigma_{yy1}(x,\delta) = \sigma_{yy2}(x,\delta); \quad \sigma_{xy1}(x,\delta) = \sigma_{xy2}(x,\delta) \) 

\( (A-5-a) \)

\( (A-5-b) \)

From the equations for the displacements and stresses, the continuity conditions can be written in terms of the \( A_{ij} \) as

\[ e^{\alpha \delta} A_{11} + \delta e^{\alpha \delta} A_{21} + e^{-\alpha \delta} A_{31} + \delta e^{-\alpha \delta} A_{41} = e^{\alpha \delta} A_{12} + \delta e^{\alpha \delta} A_{22} \]

\[ \quad + e^{-\alpha \delta} A_{32} + \delta e^{-\alpha \delta} A_{42} \] 

\( (A-6-a) \)

\[ - e^{\alpha \delta} A_{11} + (\kappa_1/\alpha-\delta) e^{\alpha \delta} A_{21} + e^{-\alpha \delta} A_{31} + (\kappa_1/\alpha+\delta) e^{-\alpha \delta} A_{41} \]

\[ = - e^{\alpha \delta} A_{12} + (\kappa_2/\alpha+\delta) e^{\alpha \delta} A_{22} + e^{-\alpha \delta} A_{32} \]

\[ + (\kappa_2/\alpha+\delta) e^{-\alpha \delta} A_{42} \] 

\( (A-6-b) \)

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\[ \mu_1 a e^{\alpha \delta} A_{11} + \mu_1 e^{\alpha \delta}(\alpha \delta - 1 + 2\nu_1) A_{21} - \alpha \mu_1 e^{-\alpha \delta} A_{31} \]
\[ - \mu_1 e^{-\alpha \delta}(\alpha \delta + 1 - 2\nu_1) A_{41} = \mu_2 a e^{\alpha \delta} A_{12} \]
\[ + \mu_2 e^{\alpha \delta}(\alpha \delta - 1 + 2\nu_2) A_{22} - \alpha \mu_2 e^{-\alpha \delta} A_{32} \]
\[ - \mu_2 e^{-\alpha \delta}(\alpha \delta + 1 - 2\nu_2) A_{42} \]

On the boundaries, \( y=0, H; 0<x<\infty \), stress will be specified in the form

\[ \sigma_{yy}(x,0) = q(x) \quad \sigma_{yy}(x,H) = R(x) \quad (A-7-a) \]
\[ \sigma_{xy}(x,0) = s(x) \quad \sigma_{xy}(x,H) = T(x) \quad (A-7-b) \]

Inserting the stress components from (A-4) into (A-7), the boundary conditions can be written as
\[- \alpha A_{11} + 2(1-\nu)A_{21} - \alpha A_{31} - 2(1-\nu)A_{41} \]

\[= \int_{0}^{\infty} \frac{q(x)\cos(\alpha x)dx}{2\mu_1} \]  \hspace{1cm} (A-8-a)

\[e^{\alpha H}[2(1-\nu)A_{22} - \alpha(A_{12}+A_{22}H)] - e^{-\alpha H}[\alpha(A_{32}+HA_{42}) + 2(1-\nu)A_{42}] = \int_{0}^{\infty} \frac{R(x)\cos(\alpha x)dx}{2\mu_2} \]  \hspace{1cm} (A-8-b)

\[\alpha A_{11} - (1-2\nu)A_{21} - \alpha A_{31} - (1-2\nu)A_{41} \]

\[= \int_{0}^{\infty} \frac{s(x)\sin(\alpha x)dx}{2\mu_1} \]  \hspace{1cm} (A-8-c)

\[[\alpha(A_{12}+A_{22}H) - (1-2\nu)A_{22}]e^{\alpha H} - [\alpha(A_{32}+HA_{42}) + (1-2\nu)A_{42}]e^{-\alpha H} = \int_{0}^{\infty} \frac{T(x)\sin(\alpha x)dx}{2\mu_2} \]  \hspace{1cm} (A-8-d)

Equations (A-6) and (A-8) form a system of eight linear algebraic equations in the eight unknowns \( A_j \). The system can be solved for the \( A_j \) as functions of \( \alpha \) and the integrals of (A-3) and (A-4) can be evaluated to determine the stress and displacement fields.

If three point bend loading conditions are assumed as illustrated in Figure A-2, the boundary conditions become

\[\sigma_{yy}(x,0) = -P\delta(x); \sigma_{yy}(x,H) = -P\delta(x-x_0)\]  \hspace{1cm} (A-9-a)
\[ \frac{\partial}{\partial x} \sigma_{xy}(x,y) = 0; \quad \sigma_{xy}(x,H) = 0 \quad (A-9-b) \]

Inserting these conditions into (A-8), the boundary conditions can be written in terms of the \( A_j \) as

\[ \begin{align*}
- \alpha A_{11} + 2(1-\nu_1)A_{21} - \alpha A_{31} - 2(1-\nu_1)A_{41} &= - \frac{p}{2\mu_1} \\
 e^{\alpha H} [2(1-\nu_2)A_{22} - \alpha (A_{12} + A_{22}H)] - e^{-\alpha H} [\alpha (A_{32} + HA_{42})] &= - \frac{p}{2\mu_2} \cos(\alpha x_0) \\
\alpha A_{41} - (1-2\nu_1)A_{21} - \alpha A_{31} - (1-2\nu_1)A_{41} &= 0 \\
\alpha (A_{12} + A_{22}H) - (1-2\nu_2)A_{22} \] \\
t + (1-2\nu_2)A_{42} &= 0 \\
\end{align*} \quad (A-10-a) \]

Equations (A-6) and (A-10) form the system of equations which will be solved for the \( A_j \).

The integrals of (A-3) and (A-4) will be evaluated by Gauss-Legendre quadrature techniques. Equations (A-6) and (A-12) will be solved numerically using Gauss-Seidell iteration at each quadrature point and used in evaluation of the integrals.
B. The Finite Element Solution and Comparison of the Results

The relative accuracy of the finite element method for use in composite material problems was examined through a comparison of the results for the three point bend problem (discussed in the preceding section) from the Fourier integral and finite element solutions. The moduli of elasticity used in the analysis were

\[ E_1 = 2.275E + 03 \text{ Mega Pascals} \]

and

\[ E_2 = 6.8948E + 04 \text{ Mega Pascals} \]

which are the moduli for bone and polymethylmethacrylate bone cement respectively. The ratio of the load, \( P \), to the vertical height, \( H \), was taken to be

\[ P/H = 3.81E + 03 \text{ Pascals} \]

Poisson's ratio was assumed to be the same in each material and assigned the value of .3. Small variations from this value had negligible effect on the results and would only be Poisson's ratios for the two materials differed greatly. \( \delta \) was taken to be .5\(^*H\).

The finite element method demands continuity of the displacement field across an element boundary but does not require a continuous stress field. The only requirements on the stress field are that each
element must satisfy local equilibrium, and that the entire structure must be in global equilibrium. Thus, the finite element analysis allows for the possibility of a discontinuous stress field across an element boundary provided the average stress at the boundary is continuous.

The lack of necessary stress continuity at the boundary led to experimentation with different grid patterns to determine what features a grid which would accurately analyze composite materials should possess. For a laminate composite, a minimal amount of stress discontinuity was predicted when using a grid which had only one element through the material thickness and a very fine mesh along the material interface. This insured stress continuity within each material and caused a minimum amount of normal and shear stress discontinuity at the material interface. The grid used for the two material problem is shown in Figure A-3.

The finite element results were in good agreement with those obtained from the analytic solution for the dominant stress components, $\sigma_x$. Figure A-4 shows the plot of both solutions as a function of the vertical position, $Y/H$, at $X=0$. The numerical results deviate slightly from the analytic solution as the point load is approached ($Y/H = 0$). Figure A-5 is a plot of both solutions as a function of the vertical position, $Y/H$, at $X/H = 6$ (below the far point load). While qualitatively both solutions predict the same behavior of the stress field, the analytic solution exhibits a leveling off of the stress component between $Y/H = .6$ and $Y/H = .8$ which the finite element results fail to predict. Both solutions predict that the stress component will tend
to $\infty$ as the point load is approached (the finite element analysis will predict a finite value for the stress at the point of loading due to the nature of the shape functions, but as the grid is taken finer and finer in the region of loading, this value becomes larger indicating the validity of extrapolation to infinity). Figure A-6 is a plot of both solutions as a function of the vertical position, $Y/H$ at $X/H = 2$. Far from the point loads, the finite element results match the analytic results almost exactly.

The finite element calculations predicted the shear stress component, $\sigma_{xy}$, very accurately. Figure A-7 shows the plot of both analytic and finite element solutions as a function of the vertical position, $Y/H$, at $X/H = 2$. The two solutions match almost identically. Notice that the shear stress at $X/H = 2$ is 20 times smaller than the dominant stress, (i.e., $|\sigma_{xy}^{max}(Y,2.)| = |20. * \sigma_{xy}^{max}(Y,2.)|$, thus, a slight error will have negligible effect on the total solution at that point. Figure A-8 shows the plot of both solutions as a function of the vertical position, $Y/H$, at $X/H = 6$. (directly below the far point load). The shear stress predictions from the finite element results agree very well with the analytic solution even very near the point loads. This demonstrates that if the grid is carefully chosen, the effect of the finite element allowance for a discontinuous stress field has a negligible effect on the shear stress solution.

The third in-plane stress component, $\sigma_y$, is the least important (far from the region of loading) and the least accurate. Except near the far point load (and very near the center point load), $\sigma_y$ is at
least three orders of magnitude smaller than $\sigma_x$. The finite element errors on the stress calculations are on the same order of magnitude as $\sigma_y$ and thus have a very large effect on these results. Near the point loads, as $\sigma_y$ gets larger in magnitude, the results get better. Figure A-9 shows a plot of both solutions as a function of the vertical position, $Y/H$, at $X/H = 6.$, (i.e., just below the far point load). Both solutions predict the same behavior as $Y/H \rightarrow 1$. The interface discontinuity in $\sigma_y$ distorts the finite element solution somewhat and predicts small positive stresses in the softer material where the analytic solution predicts small negative stresses. The inaccuracies that occur are due to the failure of the finite element method to account for the stress singularity at the point load and the allowance for interface stress discontinuity. These inaccuracies have a negligible effect on the total solution except near the point load where $\sigma_y$ becomes dominant.

The strain energy density criterion was used to predict failure due to fracture in the material. An inspection of the SED plots versus vertical distance demonstrates that there exist local minima along the neutral axis ($Y/H = .71$) for all $X$ (i.e., $\partial\text{SED}/\partial y = 0.$ at $Y/H = .71$). The SED along the neutral axis is an even and increasing function of $X$ and thus the stationary value is located at $X=0.$ and $Y/H = .71$ (i.e., $\partial\text{SED}/\partial x = \partial\text{SED}/\partial y = 0.$ at $X=0.$ and $Y/H = .71$). A plot of SED versus vertical distance, $Y/H$, at $X=0.$ is shown in Figure A-10. The errors in the finite element results for $Y/H < .5$ are due to the inability of the finite element calculations to predict zero shear stress at $X=0.$ Figure A-11 shows a plot of SED versus vertical distance, $Y/H$, at $X/H = 49.$
5. The two solutions agree very well due to the accurate stress predictions in this region.

Very accurate results can be obtained for composite material problems using the finite element method if a proper grid is chosen. Finite elements can adequately predict the qualitative behavior of the stress and energy fields for a composite material even with a grid which is slightly less accurate than the optimum would be for a given problem. Since the analytic solution is usually not known, optimum grid choice is a difficult procedure. In general, experience tells us that finite element results using 12-node isoparametric elements will be accurate within 5-10% if a reasonable grid which is sensitive to regions with high displacement gradients is used [11].
Figure A-3: Finite Element Idealization of Three Point Bend Problem
Figure A-4: \( x \) - Stresses versus \( y/H \) at \( x = 0 \).
Figure A-5: X - Stresses versus Y/H at X/H = 6.
Figure A-6: X - Stresses versus Y/H at X/H = 2.
Figure A-8: Shear Stress Versus Y/H at X/H = 6.

- Finite Element Solution
- Analytic Solution
Figure A.9: Y-Stresses versus Y/H at X/H = 6.
Figure A-10: SED versus Y/H at X = 0

0 - Analytic Solution
A - Finite Element Solution

SED (Joules/Meter²)
VITA

The author was born in Philadelphia, Pennsylvania on July 8, 1955 to E. Thomas and Carolyn M. Moyer.

After graduation in June 1973 from Cheltenham High School in Wyn- cote, Pennsylvania, he entered Moravian College in Bethlehem, Pennsylvania. He was awarded the Degree of Bachelor of Science with Honors in Physics from Moravian College in May of 1977.

In September 1977, the author accepted a position as a University Fellow at Lehigh University which position he still holds. He expects to be awarded the Degree of Master of Science in Applied Mechanics in January 1979.