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the steady state motion of a rigid strip perfectly bonded to an elastic half space.

Robert W. Kolkka

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THE STEADY STATE MOTION
OF A RIGID STRIP PERFECTLY BONDED
TO AN ELASTIC HALF SPACE

by
Robert W. Kolkka

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Professor in Charge

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The problem of steady state motion of a rigid strip perfectly bonded to a linearly elastic half space is analyzed. The problem is reduced to solving a single complex singular integral equation, rather than a system of dual integral equations, for reasons elucidated in the text. The singular integral equation is solved by expansion in terms of weighted Jacobi polynomials. The resulting linear algebraic system of expansion coefficients is solved numerically. Comments on related papers are presented.
INTRODUCTION

The problem to be considered is posed as follows. We have an isotropic linearly elastic half space, upon which a rigid strip is glued with perfect adhesion. The strip may be externally forced or subjected to a steady field of incoming Rayleigh waves (fig 1-0). As is common practice, such a problem is conveniently formulated in terms of the two cartesian displacement components u and v. The governing equation in this case is the Navier equation,

\[(\lambda+\mu)\nabla(\nabla \cdot \vec{u}) + \mu \nabla^2 \vec{u} - \rho \frac{\partial^2 \vec{u}}{\partial t^2} = 0\]  

(1)

where \(\lambda\) and \(\mu\) are the familiar Lame constants, \(\rho\) the mass density, and \(\vec{u} = u\hat{i} + v\hat{j}\). Due to the wave nature of the problem, we do not solve (1) directly for \(u\) and \(v\). It can be shown that the displacement field can be decomposed into the gradient of a scalar potential \(\phi\), and the curl of a vector field \(\vec{\nabla}\) in accordance with the Helmholtz decomposition theorem [1], i.e.
\[ \vec{\nabla} \cdot \vec{\nabla} \phi + \vec{\nabla} \times \vec{\nabla} \psi; \ \nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} \quad \vec{\nabla} = \hat{\mathbf{k}}, \frac{\partial}{\partial z} = 0 \quad (2) \]

Substitution of (2) into (1) can be readily shown to result in the following equation,

\[ \nabla \left[ (\lambda + 2\mu) \nabla^2 \phi - \rho \frac{\partial^2 \phi}{\partial t^2} \right] + \nabla \times \left[ \mu \nabla^2 \psi - \rho \frac{\partial^2 \psi}{\partial t^2} \right] = 0 \quad (3) \]

For the quantities in brackets vanishing separately, we obtain the two uncoupled wave equations

\[ \nabla^2 \phi = \frac{1}{c_1^2} \frac{\partial^2 \phi}{\partial t^2}, \quad c_1 = (\frac{\lambda + 2\mu}{\rho})^{1/2} \]

\[ \nabla^2 \psi = \frac{1}{c_2^2} \frac{\partial^2 \psi}{\partial t^2}, \quad c_2 = (\frac{\mu}{\rho})^{1/2} \quad (4) \]

In this paper, we shall only consider the steady-state solution, in which the time dependence can be eliminated by considering all quantities to have an \( e^{i\omega t} \) time dependence, i.e.

\[ \phi(x,y,t) = \phi(x,y)e^{i\omega t} \quad \psi(x,y,t) = \xi(x,y)e^{i\omega t} \]

\[ u(x,y,t) = U(x,y)e^{i\omega t} \quad v(x,y,t) = V(x,y)e^{i\omega t} \quad (5) \]

Substitutions of (5) into (4) yields the steady-state governing equations
\[ v^2 \varphi + \alpha_1^2 \varphi = 0, \quad \alpha_1 = \frac{\omega}{c_1}; \quad v^2 \xi + \alpha_2^2 \xi = 0, \quad \alpha_2 = \frac{\omega}{c_2} \]  
(6)

Equations (6) together with the appropriate steady-state boundary conditions constitute a mathematically well defined boundary value problem.
FORMULATION

As previously mentioned there are two basic problems which can be considered, the excitation of the strip by a field of incoming Rayleigh waves, or the externally forced motion of the strip generating the steady-state wave field. The two problems are closely related in that they are governed by the same singular integral equation (to be consequently developed). The first problem is of intrinsic geophysical importance, for it represents the interaction of a structure to seismic waves, and therefore warrants a discussion, although it will not be solved in this paper due to the highly complex numerical nature of its solution.

The scheme of the solution is as follows. Proceeding with a generalized method suggested by Thau [12], the otherwise unmanageable total problem is decomposed into the superposition of two simpler and physically fundamental problems, the diffraction of waves by a fixed strip and the radiation of waves caused by the forced motion of an inertialess strip. The solution of the first problem yields the loading on the strip due to the incident waves which tend to force the strip
to move. The solution of the second problem yields the loads acting on the strip due to the resistance of the half space to a forced motion.

We express the decomposition of the displacement field as

\[ \vec{u}(x,y,t) = \vec{u}^f(x,y,t) + \vec{u}_1^s(x,y,t) + \vec{u}_2^s(x,y,t) \]  

(7)

where \( \vec{u}^f \) is the Rayleigh wave field, \( \vec{u}_1^s \) the diffracted field by a fixed strip, and \( \vec{u}_2^s \) the field radiated by the motion of the strip. The mixed boundary value problem for the \( \vec{u}_1^s \) field is defined by

\[ \vec{u}_1^s(x,0,t) = -\vec{u}^f(x,0,t) \quad |x| < a \]

(8)

\[ \sigma_{yy_1}^s(x,0,t) = \sigma_{xy_1}^s(x,0,t) = 0 \quad |x| > a \]

where \( \vec{u}^f(x,0,t) \) is given by

\[ \vec{u}^f(x,0,t) = [-iD(\alpha_R)e^{-i\alpha_R x} \hat{i} + iD(\alpha_R)e^{-i\alpha_R x}]e^{i\omega t} \]

(9)

\[ \hat{\vec{u}} = 1.4667, \quad \alpha_R = \frac{\omega}{c_R} \]

\( D(\alpha_R) \) being the amplitude of the Rayleigh waves, and \( \alpha_R \) the Rayleigh wave no. Solution of (8) yields the loads
\[
P_1(D) = \int_{-a}^{a} \sum_{ij} y(x,0) \, dx \quad Q_1(D) = \int_{-a}^{a} \sum_{xy} y(x,0) \, dx
\]
\[
M_1(D) = \int_{-a}^{a} \sum_{ij} y(x,0) x \, dx
\]

(10)

on a would be immobile strip (\( \sum_{ij} \) being the associated time independent stresses). As a consequence of (8) the mixed boundary value problem for \( \bar{u}_2^s \) is given by

\[
\bar{u}_2^s(x,0,t) = [u_0(\omega) \hat{i} + (v_0(\omega) + \epsilon_0(\omega) x) \hat{j}] e^{i\omega t} \quad |x| < a
\]
\[
\sigma_{yy}^s(x,0,t) = \sigma_{xy}^s(x,0,t) = 0 \quad |x| > a
\]

(11)

where \( u_0(\omega), v_0(\omega), \) and \( \epsilon_0(\omega) \) are the displacements of the rigid body motion to be subsequently determined. If \( u_0(\omega), v_0(\omega), \) and \( \epsilon_0(\omega) \) were known, the solution of (11) would yield \( P_2(u_0, v_0, \epsilon_0), Q_2(u_0, v_0, \epsilon_0) \) and \( M_2(u_0, v_0, \epsilon_0) \). Now, the unknowns \( u_0(\omega), v_0(\omega), \) and \( \epsilon_0(\omega) \) can be determined via Newton's Laws, supplemented by continuity of displacement at \( x = -a, x = a \) (in view of the solution to the singular integral equation, the continuity conditions can be seen to be a vital necessity, even though at this stage, in view of
Newton's laws they appear to be extraneous). The equations of rigid body motion are

\[ -mg + P_1(D) + P_2(u_o, v_o, \epsilon_o) = -m\omega^2 v_o \]
\[ Q_1(D) + Q_2(u_o, v_o, \epsilon_o) = -m\omega^2 u_o \]
\[ M_1(D) + M_2(u_o, v_o, \epsilon_o) = -I_0\omega^2 \epsilon_o \]

and the continuity conditions are

\[ x = -a: \quad \bar{u}_1^s(-a,0,t) + \bar{u}_2^s(-a,0,t) = [u_o \hat{i} + (v_o - \epsilon a) \hat{j}] e^{i\omega t} - \bar{u}^f(-a,0,t) \]
\[ x = a: \quad \bar{u}_1^s(a,0,t) + \bar{u}_2^s(a,0,t) = [u_o \hat{i} + (v_o + \epsilon a) \hat{j}] e^{i\omega t} - \bar{u}^f(+a,0,t) \]

The second problem, the externally forced motion of the strip is amenable to a numerical solution, and is the objective of this paper. We consider the rocking problem, that is the strip is steadily rotated back and forth through a prescribed angular tilt \( \epsilon \), and the vertical and horizontal oscillations. The mixed boundary value problem is given as
\[ u(x,0,t) = u_0 e^{i\omega t} \]
\[ v(x,0,t) = \epsilon x e^{i\omega t} + v_0 e^{i\omega t} \quad |x| < a \]
\[ \sigma_{yy}(x,0,t) = \sigma_{xy}(x,0,t) = 0 \quad |x| > a \]

and the dynamic equilibrium condition is simply

\[ M = \int_{-a}^{a} \sum_{yy} (x,0) dx \]
\[ P = \int_{-a}^{a} \sum_{yy} (x,0) dx \quad (15) \]
\[ Q = \int_{-a}^{a} \sum_{xy} (x,0) dx \]

The boundary conditions (14) for \(|x| < a\) can be expressed in terms of \(\varphi\) and \(\xi\) by use of (2), (5), and (7), the result is

\[ \frac{\partial \varphi}{\partial x}(x,0) + \frac{\partial \xi}{\partial y}(x,0) = 0 = U(x,0) \]
\[ |x| < a \quad (16) \]
\[ \frac{\partial \varphi}{\partial y}(x,0) - \frac{\partial \xi}{\partial x}(x,0) = \epsilon x = V(x,0) \]

The remainder of the half plane is stress free, and by use of Hooke's law and (2) and (5), we have that
The problem is now well posed in mathematical language. Eqns. (6) are the governing equations, which are subject to the boundary conditions (16) and (17) and conditions at infinity (i.e. solutions which grow exponentially at infinity are discarded).
DEVELOPMENT OF THE SINGULAR INTEGRAL EQUATION

The complete formulation of the problem commences by applying a Fourier transform on $\phi$ and $\xi$ in the $x$ direction (since $x \in (-\infty, \infty)$). The transforms are defined by

$$\bar{\phi}(k,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x,y)e^{-ikx} \, dx$$

$$\bar{\xi}(k,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \xi(x,y)e^{-ikx} \, dx$$ (18)

Applying the transform to the governing equations (6) results in the two ordinary differential equations

$$\frac{\partial^2 \bar{\phi}}{\partial y^2} - (k^2 - a_1^2)\bar{\phi} = 0, \quad \frac{\partial^2 \bar{\xi}}{\partial y^2} - (k^2 - a_2^2)\bar{\xi} = 0$$ (19)

whose solutions are

$$\bar{\phi}(k,y) = A_1(k)e^{-\frac{1}{2}(k^2 - a_1^2)y}, \quad \bar{\xi}(k,y) = A_2(k)e^{-\frac{1}{2}(k^2 - a_2^2)y}$$ (20)

where $A_1(k)$ are arbitrary functions of $k$, to be determined through use of the boundary conditions (16), (17). Usually $A_1(k)$ are determined by transforming the given boundary conditions, which in most of the
classical problems are not mixed. For the mixed problem, the transforms are inverted by use of Fourier inversion, and substituted into the given boundary conditions, resulting in a system of simultaneous integral equations.

For this problem \( \varphi(x,y) \) and \( \xi(x,y) \) are

\[
\varphi(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A_1(k)e^{-(k^2-\alpha_1^2)\frac{1}{2}y} e^{iky} dk, \quad (21)
\]

\[
\xi(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A_2(k)e^{-(k^2-\alpha_2^2)\frac{1}{2}y} e^{ikx} dk.
\]

Subsequent substitution of (21) into (16), (17) results in the following system of dual integral equations

\[
\sum_{j=1}^{2} \int_{-\infty}^{\infty} R_{1j}^m(k,x)A_j(k)dk = f_1^m(x) \quad m=1: |x|<a, =2: |x|>a \quad i = 1,2 \quad (22)
\]

where it can be easily verified that

\[
R_{11}^1(k,x) = \frac{1}{\sqrt{2\pi}} e^{ikx}, \quad R_{12}^1(k,x) = \frac{-1}{\sqrt{2\pi}} (k^2-\alpha_1^2)^{\frac{1}{2}}, \quad f_1^1(x) = 0 \quad (23)
\]

\[
R_{21}^1(k,x) = \frac{1}{\sqrt{2\pi}} (k^2-\alpha_1^2)^{\frac{1}{2}} e^{ikx}, \quad R_{22}^1(k,x) = \frac{-1}{\sqrt{2\pi}} e^{ikx}, \quad f_2^1(x) = e^x
\]
All that remains is to solve for the unknowns $A_j(k)$ in (22). This can be achieved by a "direct method" as described in [2], [3], and [4]. However, improper choice of expansion functions in the "direct method" will lead to a smoothing of the singularity of the unknown functions of interest, as was shown by Erdogan in [4]. Consequently, to avoid all the controversy and confusion associated with the "direct method", a Green's function approach is used. As a result of this approach, a single complex singular integral equation is generated, the theory and solution of which is relatively straightforward. For a detailed discussion of singular integral equations see [5].

The general Green's function method consists of solving the governing equations for the impulsive inhomogeneities, the resulting solution being referred
to as the Green's functions of the given problem. The actual inhomogeneities are then integrated with the Green's functions kernel over the given region of interest, resulting in the solution of interest.

The governing equations for this problem are homogeneous, but the boundary condition are not. Consequently, equations (6) are solved for the following boundary conditions

\[ \sum_{yy} (x, 0) = - P\delta(x-\eta), \quad \sum_{xy} (x, 0) = Q\delta(x-\eta) \] (24)

i.e. impulsive normal and shear stresses applied at \( x = \eta \). The quantities \( P \) and \( Q \) are extended to be functions of \( \eta \), and represent the normal and shear stresses under the strip. \( P \) and \( Q \) become the unknowns in the ensuing real singular integral equations, which will be integral expressions for the derivatives with respect to \( x \) of the time independent displacements \( U(x, 0) \) and \( V(x, 0) \) under the strip.

From (17) the expressions for the time independent stresses in terms of \( \phi \) and \( \xi \) are
The transformed stresses can be readily verified to be

$$\sum_{yy} (x,y) = \mu[\alpha^2 \varphi - 2 \frac{\partial^2 \varphi}{\partial x^2} + 2 \frac{\partial^2 \psi}{\partial x^2} - 2 \frac{\partial^2 \xi}{\partial x^2 \partial y}]$$  \hspace{1cm} (25)$$

$$\sum_{xy} (x,y) = \mu[2 \frac{\partial^2 \varphi}{\partial x \partial y} - \alpha^2 \xi - 2 \frac{\partial^2 \xi}{\partial x^2}]$$

Substituting (20) into (26) (and setting y = 0), transforming (24), and then equating the two results yields two equations to determine the two unknowns $A_j(k)$ in terms of the two other unknowns $P$ and $Q$ (it just amounts to switching unknowns from $A_j(k)$ to $P$ and $Q$). The equations are

$$\begin{bmatrix} 2k^2 - \alpha_2^2 & 2ik(k^2 - \alpha_2^2)^{1/2} \\ -2ik(k^2 - \alpha_1^2)^{1/2} & 2k^2 - \alpha_1^2 \end{bmatrix} \begin{bmatrix} A_1(k) \\ A_2(k) \end{bmatrix} = \frac{-e^{-ik\eta}}{\mu/2\pi} \begin{bmatrix} P \\ Q \end{bmatrix}$$ \hspace{1cm} (27)$$

Simple application of Cramer's rule to (27) results in
\[ A_1(k) = \frac{\beta(k; \eta) e^{-ik\eta}}{\mu/2\pi F(k)} \]
\[ A_2(k) = \frac{\alpha(k; \eta) e^{-ik\eta}}{\mu/2\pi F(k)} \]
\[ \beta(k; \eta) = [-P(\eta) (2k^2 - \alpha_1^2) + 2iQ(\eta)k(k^2 - \alpha_1^2)^{1/2}] \]
\[ \alpha(k; \eta) = [-Q(\eta) (2k^2 - \alpha_2^2) - 2iP(\eta)k(k^2 - \alpha_2^2)^{1/2}] \]
\[ F(k) = (2k^2 - \alpha_2^2)^2 - 4k^2(k^2 - \alpha_1^2)^{1/2}(k^2 - \alpha_1^2)^{1/2}, \]

where \( F(k) \) is the well known Rayleigh function.

Substitution of (28) into (21) yields the expressions for \( \varphi(x, y; \eta) \) and \( \varphi(x, y; \eta) \),
\[
\varphi(x, y; \eta) = \frac{1}{2\pi \mu} \int_{-\infty}^{\infty} \frac{\beta(k; \eta)}{F(k)} e^{-\left((k^2 - \alpha_1^2)^{1/2}y + ik(x-\eta)\right)} dk \tag{29}
\]
\[
\xi(x, y; \eta) = \frac{1}{2\pi \mu} \int_{-\infty}^{\infty} \frac{\alpha(k; \eta)}{F(k)} e^{-\left((k^2 - \alpha_2^2)^{1/2}y + ik(x-\eta)\right)} dk
\]

Integral expressions for \( \frac{\partial U}{\partial x} (x, y) \) and \( \frac{\partial V}{\partial x} (x, y) \) can be obtained from (29) by use of (2), the result is
\[
\frac{\partial U}{\partial x}(x, y) = \frac{1}{2\pi \mu} \int_{-a}^{a} \left\{ \int_{-\infty}^{\infty} \left[ \frac{-k^2 \beta(k; \eta)}{F(k)} \right] e^{i k (x-\eta)} d\eta \right\} d\eta
\]

\[
\frac{\partial V}{\partial x}(x, y) = \frac{1}{2\pi \mu} \int_{-a}^{a} \left\{ \int_{-\infty}^{\infty} \left[ \frac{-i k (k^2 - \alpha_2^2)^{\frac{1}{2}} (k^2 - \alpha_1^2)^{\frac{1}{2}}}{F(k)} \right] e^{i k (x-\eta)} d\eta \right\} d\eta
\]

(30)

Substitution of (28) into (30) enables (30) to be expressed in a more convenient form as

\[
\frac{\partial U}{\partial x}(x, y) = \frac{1}{2\pi \mu} \int_{-a}^{a} \left[ \int_{-\infty}^{\infty} \frac{P(\eta) H_{11}(x, \eta, y) d\eta}{P(\eta) H_{21}(x, \eta, y) d\eta} \right]
\]

(31)

\[
\frac{\partial V}{\partial x}(x, y) = -\frac{1}{2\pi \mu} \int_{-a}^{a} Q(\eta) H_{12}(x, \eta, y) d\eta
\]

\[
\frac{\partial V}{\partial x}(x, y) = -\frac{1}{2\pi \mu} \int_{-a}^{a} Q(\eta) H_{22}(x, \eta, y) d\eta
\]

where
\[ H_{ij}(x, \eta, y) = \int_{-\infty}^{\infty} \left[ D_{ij}(k) e^{-\frac{1}{2} (k^2 - \alpha_1^2)y} + E_{ij}(k) e^{-\frac{1}{2} (k^2 - \alpha_2^2)y} \right] e^{ik(x-\eta)} \, dk \]  

(32)

and \( D_{ij}(k) \) and \( E_{ij}(k) \) are given in Appendix A. To obtain expressions for \( \frac{\partial U}{\partial x} (x, 0) \) and \( \frac{\partial V}{\partial x} (x, 0) \), the limit as \( y \to 0 \) cannot be taken directly inside the integral (32) because the integrals are not uniformly convergent without the exponential damping. However, the limit may be taken inside if the divergent contribution of the integrand can be separated in a manner such that the singular nature of that contribution is well known. This may be accomplished as follows.

Letting
\[ D_{ij}(k)e^{-\frac{1}{2} (k^2 - \alpha_1^2)y} = [D_{ij}(k)e^{-\frac{1}{2} (k^2 - \alpha_1^2)y} - D_{ij}^\infty(k)e^{-|k|y}] + D_{ij}^\infty(k)e^{-|k|y} \]  

(33)

\[ E_{ij}(k)e^{-\frac{1}{2} (k^2 - \alpha_2^2)y} = [E_{ij}(k)e^{-\frac{1}{2} (k^2 - \alpha_2^2)y} - E_{ij}^\infty(k)e^{-|k|y}] + E_{ij}^\infty(k)e^{-|k|y} \]

where
\[ D_{ij}^\infty(k) = \lim_{k \to \infty} D_{ij}(k), \quad E_{ij}^\infty(k) = \lim_{k \to \infty} E_{ij}(k) \] (34)

It can be readily shown that \( E_{ij}^\infty(k) + D_{ij}^\infty(k) \) are constants (in the case of \( \Phi_{12} \) and \( \Phi_{21} \), they are constants multiplied by \( \text{sgn}(k) \)), the resulting limit singularity of whose integral is well known.

The limit as \( y \to 0 \) can now be taken inside the remaining integrals, and \( H_{ij}(x, \eta) \) becomes

\[
H_{ij}(x, \eta) = \int_{-\infty}^{\infty} [(D_{ij}(k) - D_{ij}^\infty(k)) + (E_{ij}(k) - E_{ij}^\infty(k))] e^{ik(x-\eta)} dk
\]

\[
+ \lim_{y \to 0} 2 \int_{0}^{\infty} L_{ij}^\infty(k) e^{-|k|y \cos k(x-\eta)} dk
\]

\[
+ \lim_{y \to 0} 2i \int_{0}^{\infty} M_{ij}^\infty(k) e^{-|k|y \sin k(x-\eta)} dk
\]

where

\[
D_{ij}^\infty(k) + E_{ij}^\infty(k) = L_{ij}^\infty(k) \bigg|_{\text{even}} + M_{ij}^\infty(k) \bigg|_{\text{odd}} \] (36)

The second and third integral in (35) may be readily evaluated, and as a result (35) takes on the form
\[ H_{11}(x, \eta) = K_{11}(x, \eta) - \frac{(\lambda-1)}{2} \pi \delta(x-\eta) \]

\[ H_{12}(x, \eta) = K_{12}(x, \eta) + \frac{i(\lambda+1)}{2} \frac{1}{\eta-x} \]

\[ H_{21}(x, \eta) = K_{21}(x, \eta) + i\frac{(\lambda+1)}{\eta-x} \]

\[ H_{22}(x, \eta) = K_{22}(x, \eta) - \frac{(\lambda-1)}{2} \pi \delta(x-\eta) \] (37)

where

\[ K_{ij}(x, \eta) = \int_{-\infty}^{\infty} \left[ (D_{ij}(k)-D_{ij}^{\infty}(k)) + (E_{ij}(k)-E_{ij}^{\infty}(k)) \right] e^{ik(x-\eta)} dk \] (38)

and \( \lambda = \frac{\lambda+3\mu}{\lambda+\mu} \). Finally the expression for \( \frac{\partial U}{\partial x} (x,0) \) and \( \frac{\partial V}{\partial x} (x,0) \) are

\[ \frac{\partial U}{\partial x} (x,0) = -\frac{(\lambda-1)}{4\mu} P(x) + \frac{(1+\lambda)}{4\pi\mu} \int_{-a}^{a} Q(\eta) \frac{d\eta}{\eta-x} \]

\[ + \frac{1}{2\pi\mu} \int_{-a}^{a} P(\eta)K_{11}(x,\eta)d\eta \frac{i}{2\pi\mu} \int_{-a}^{a} Q(\eta)K_{12}(x,\eta)d\eta \]

\[ \frac{\partial V}{\partial x} (x,0) = \frac{(\lambda-1)}{4\mu} Q(x) + \frac{(1+\lambda)}{4\pi\mu} \int_{-a}^{a} \frac{P(\eta)}{\eta-x} d\eta \]

\[ - \frac{1}{2\pi\mu} \int_{-a}^{a} P(\eta)K_{21}(x,\eta)d\eta - \frac{1}{2\pi\mu} \int_{-a}^{a} Q(\eta)K_{22}(x,\eta)d\eta \] (39)
Instead of solving (39) as a coupled system of singular integral equations, the system can be combined into one complex singular integral equation for a single unknown complex valued function \( \varphi(x) \).

Defining the unknown complex function \( \varphi(x) \), and the known complex input function \( f(x) \) as

\[
\varphi(x) = P(x) + iQ(x), \quad f(x) = \frac{\partial U}{\partial x}(x,0) - i \frac{\partial V}{\partial x}(x,0)
\]

(39) becomes

\[
\begin{split}
\frac{4\mu}{1+\Lambda} f(x) &= \frac{1}{4\pi} \int_{-a}^{a} \frac{\varphi(\eta)}{\eta-x} \, d\eta - \gamma \varphi(x) \\
&\quad + \frac{2}{\pi(1+\Lambda)} \int_{-a}^{a} [\varphi(\eta)K_1(x,\eta) + \varphi^*(\eta)K_2(x,\eta)] \, d\eta
\end{split}
\]  

(41)

where

\[
K_1(x,\eta) = \frac{K_{11}(x,\eta) + K_{22}(x,\eta) - [K_{12}(x,\eta) + K_{21}(x,\eta)]}{2}
\]

\[
\gamma = \frac{\Lambda-1}{\Lambda+1}
\]

\[
K_2(x,\eta) = \frac{K_{11}(x,\eta) - K_{22}(x,\eta) + [K_{12}(x,\eta) - K_{21}(x,\eta)]}{2}
\]

Before proceeding to solve (41), it must be noted that the kernels \( K_{ij}(x,\eta) \) as given by (38) cannot be evaluated in closed form analytically, nor can they be
evaluated numerically as given. To evaluate the kernels, the transform variable \( k \) must be extended to complex arguments, and then and only then can the integrals be evaluated, via the appropriate contour integration (which is the subject of the next section).
EVALUATION OF THE FREDHOLM
KERNELS VIA CONTOUR INTEGRATION

The explicit form of the kernels \( K_{ij}(x, \eta) \) in (38) are

\[
K_{11}(x, \eta) = \int_{-\infty}^{\infty} \left[ \frac{k^2 G(k)}{F(k)} - \frac{a_1^2}{2(a_1^2 - a_2^2)} \right] e^{ik(x-\eta)} dk
\]

\[
K_{12}(x, \eta) = \int_{-\infty}^{\infty} \left[ \frac{a_2 k(k^2 - a_2^2)^{1/2}}{F(k)} - \frac{|k| a_2^2}{2k(a_1^2 - a_2^2)} \right] e^{ik(x-\eta)} dk \tag{43}
\]

\[
K_{21}(x, \eta) = \int_{-\infty}^{\infty} \left[ \frac{a_2 k(k^2 - a_1^2)^{1/2}}{F(k)} - \frac{|k| a_2^2}{2k(a_1^2 - a_2^2)} \right] e^{ik(x-\eta)} dk
\]

\[
K_{22}(x, \eta) = K_{11}(x, \eta) \quad G(k) = (2k^2 - a_2^2) - 2(k^2 - a_2^2)\frac{1}{2}(k^2 - a_1^2)\frac{1}{2}.
\]

So it is readily observed that the kernels \( K_{ij}(x, \eta) \) are of the general form

\[
K_{ij}(x, \eta) = \int_{-\infty}^{\infty} \Omega_{ij}(k)e^{ik(x-\eta)} dk
\tag{44}
\]

We want to evaluate the right hand side of (44) by closing the appropriate contour, and choosing the appropriate branches of \((\xi^2 - a_1^2)^{1/2}\) and \((\xi^2 - a_2^2)^{1/2}\), where \(\xi\) is the complex variable \(k + is\) (see fig 2-0).
fig 2-0

BRANCH CUTS

Cu

L₁

L₂

C₁

α₁

α₂

αᵣ
From fig 2-0, we observe that (Cauchy's theorem)

$$\int_C \Omega_{ij}(\xi)e^{i\xi(x-\eta)}d\xi = 0 \quad (45)$$

where $C$ is the entire contour closed in the upper half plane (for $x-\eta > 0$) or closed in the lower half plane ($x-\eta < 0$). For $x-\eta > 0$, we have that

$$\int_{-\infty}^{\infty} \Omega_{ij}(k)e^{ik(x-\eta)}dk = -\int_{L_1} \Omega_{ij}(\xi)e^{i\xi(x-\eta)}d\xi \quad (46)$$

$$+ i\pi \sum \text{Res}(\Omega_{ij}(\xi)e^{i\xi(x-\eta)};\alpha_R, -\alpha_R)$$

where $\pm\alpha_R$ are the roots of the Rayleigh function. The contributions from $C_u$ and $C_{\ell}$ go to zero as $C_u$ and $C_{\ell}$ go to infinity. The loops $L_1$, $L_2$ around the branch cuts assume the following form (the superscript $+$ refers to the limiting value from the left and the superscript $-$ refers to the limiting value from the right).
\[
\int_{L_1} \Omega_{ij}(\xi)e^{i\xi(x-\eta)}d\xi = 1 \int_0^\infty [\Omega_{ij}^+(-\lambda)-\Omega_{ij}^-(-\lambda)]e^{-i\lambda(x-\eta)}d\lambda \\
+ \int_0^{\alpha_1} [\Omega_{ij}^+(-\lambda)-\Omega_{ij}^-(-\lambda)]e^{-i\lambda(x-\eta)}d\lambda \\
+ \int_{\alpha_1}^{\alpha_2} [\Omega_{ij}^+(-\lambda)-\Omega_{ij}^-(-\lambda)]e^{-i\lambda(x-\eta)}d\lambda \\
+ \int_{\alpha_2}^{\alpha_1} [\Omega_{ij}^+(-\lambda)-\Omega_{ij}^-(-\lambda)]e^{i\lambda(x-\eta)}d\lambda
\]

\[
\int_{L_2} \Omega_{ij}(\xi)e^{i\xi(x-\eta)}d\xi = -1 \int_0^\infty [\Omega_{ij}^+(-is)-\Omega_{ij}^-(-is)]e^{s(x-\eta)}ds \\
+ \int_0^{\alpha_1} [\Omega_{ij}^+(-\lambda)-\Omega_{ij}^-(-\lambda)]e^{i\lambda(x-\eta)}d\lambda \\
+ \int_{\alpha_1}^{\alpha_2} [\Omega_{ij}^+(-\lambda)-\Omega_{ij}^-(-\lambda)]e^{i\lambda(x-\eta)}d\lambda \\
+ \int_{\alpha_2}^{\alpha_1} [\Omega_{ij}^+(-\lambda)-\Omega_{ij}^-(-\lambda)]e^{i\lambda(x-\eta)}d\lambda
\]

The specific integral expressions resulting from explicit calculation of (47) and (48) are all real integrals which do not present any great difficulty in numerical calculation. The particular choice of the bent branch cuts was in accordance with the physical exponential decay at infinity (in the elastic half space, that is). An entire discussion of this type of contour integration is presented in [6].
The explicit expressions for $\Omega_{ij}(\xi)$ are

$$
\Omega_{11}(\xi) = \frac{\xi^2 [2\xi^2 - a_2^2 - 2(\xi^2 - a_2^2)^{1/2}(\xi^2 - a_1^2)^{1/2}] - a_1^2}{2(\xi^2 - a_2^2)^{1/2} - 4\xi^2(\xi^2 - a_2^2)^{1/2}(\xi^2 - a_1^2)^{1/2} - 2(\xi^2 - a_2^2)}
$$

$$
\Omega_{12}(\xi) = \frac{a_2^2 \xi (\xi^2 - a_2^2)^{1/2}}{2(\xi^2 - a_2^2)^{1/2} - 4\xi^2(\xi^2 - a_2^2)^{1/2}(\xi^2 - a_1^2)^{1/2} - 2(\xi^2 - a_2^2)} - \frac{a_2^2 |\xi|}{2(\xi^2 - a_2^2)}
$$

$$
\Omega_{21}(\xi) = \frac{a_2^2 \xi (\xi^2 - a_1^2)^{1/2}}{2(\xi^2 - a_2^2)^{1/2} - 4\xi^2(\xi^2 - a_2^2)^{1/2}(\xi^2 - a_1^2)^{1/2} - 2(\xi^2 - a_2^2)} - \frac{a_2^2 |\xi|}{2(\xi^2 - a_2^2)}
$$

(49)

$$
\Omega_{22}(\xi) = \Omega_{11}(\xi)
$$

The results of the contour integrations are

$$
K_{11}(x, \eta) = \int_{-\infty}^{\infty} \Omega_{11}(k)e^{ik(x-\eta)}dk
$$

$$
= \int_{-\infty}^{\infty} \frac{a_2^2}{a_1} \frac{4k^2 - a_2^2}{(2k^2 - a_2^2)^4 - 16k^4 (k^2 - a_1^2)^2} \frac{e^{-ik(x-\eta)}}{e^{ik(x-\eta)}dk}
$$

$$
+ i\pi [\text{Res}_{11}(a_R) + \text{Res}_{11}(-a_R)]
$$
\[ K_{12}(x, \eta) = \int_{-\infty}^{\infty} \Omega_{12}(k) e^{ik(x-\eta)} dk \]

\[ = -i \text{sgn}(x-\eta) \int_{0}^{\infty} \frac{2a_{1}^{2}s(s^{2}+a_{2}^{2})^{1/2}}{(2s^{2}+a_{2}^{2})^{2}-4s^{2}(s^{2}+a_{1}^{2})^{1/2}(s^{2}+a_{2}^{2})^{1/2}} \]

\[ + \frac{a_{2}^{2}}{(a_{1}^{2}-a_{2}^{2})} e^{-s|x-\eta|} ds \]

\[ - \int_{0}^{a_{1}} \left[ \frac{2a_{2}^{2}k(k^{2}-a_{2}^{2})^{1/2}}{(2k^{2}-a_{2}^{2})^{2}-4k^{2}(k^{2}-a_{1}^{2})^{1/2}(k^{2}-a_{2}^{2})^{1/2}} \right] \]

\[ - \frac{a_{2}^{2}}{(a_{1}^{2}-a_{2}^{2})} e^{-ik|x-\eta|} dk \]

\[ - \frac{a_{2}^{2}}{(a_{1}^{2}-a_{2}^{2})} \int_{a_{1}}^{a_{2}} \left[ \frac{2a_{2}^{2}k(k^{2}-a_{2}^{2})^{1/2}(2k^{2}-a_{2}^{2})^{2}}{(2k^{2}-a_{2}^{2})^{4}-16k^{4}(k^{2}-a_{1}^{2})(k^{2}-a_{2}^{2})} \right] \]

\[ - \frac{a_{2}^{2}}{(a_{1}^{2}-a_{2}^{2})} e^{-ik|x-\eta|} dk + i\pi[\text{Res}_{12}(a_{R}) + \text{Res}_{12}(-a_{R})] \]

\[ K_{21}(x, \eta) = \int_{-\infty}^{\infty} \Omega_{21}(k) e^{ik(x-\eta)} dk \]

\[ = -i \text{sgn}(x-\eta) \int_{0}^{\infty} \frac{2a_{1}^{2}s(s^{2}+a_{2}^{2})^{1/2}}{(2s^{2}+a_{2}^{2})^{2}-4s^{2}(s^{2}+a_{1}^{2})^{1/2}(s^{2}+a_{2}^{2})^{1/2}} \]

\[ + \frac{a_{2}^{2}}{(a_{1}^{2}-a_{2}^{2})} e^{-s|x-\eta|} ds \]
The residues are calculated in the following manner. We want to calculate

\[ \text{Res}(\omega_{ij}(k) e^{ik(x-\eta)}; a_R) + \text{Res}(\omega_{ij}(k) e^{ik(x-\eta)}; -a_R) \]  

(51)

\( \omega_{ij}(k) \) can be re-expressed as

\[ \omega_{ij}(k) = \frac{R_{ij}(k) + d_{ij}(k) F(k)}{F(k)} \]  

(52)

where \( R_{ij}(k) \) and \( d_{ij}(k) \) are given by
\( R_{11}(k) = k^2 G(k) \)
\( d_{11}(k) = \frac{-a_1^2}{2(a_1^2 - a_2^2)} \)
\( R_{12}(k) = a_2^2 k (k^2 - a_2^2)^{1/2} \)
\( d_{12}(k) = \frac{|k|a_2^2}{2k(a_1^2 + a_2^2)} \) \( (53) \)
\( R_{21}(k) = a_2^2 k (k^2 - a_1^2)^{1/2} \)
\( d_{21}(k) = d_{12}(k) \)
\( R_{22}(k) = R_{11}(k) \)
\( d_{22}(k) = d_{11}(k) \)
\( G(k) = (2k^2 - a_2^2) - 2(k^2 - a_2^2)^{1/2}(k^2 - a_1^2)^{1/2} \)
\( F(k) = (2k^2 - a_2^2)^2 - 4k^2(k^2 - a_2^2)^{1/2}(k^2 - a_1^2)^{1/2} \)

So the residues are given by

\[
\text{Res}(\Omega_{ij}(k)e^{ik(x-\eta)};\alpha_R) + \text{Res}(\Omega_{ij}(k)e^{ik(x-\eta)};\alpha_R) = \frac{R_{11}(\alpha_R) i\alpha_R(x-\eta)}{F'(\alpha_R)e} + \frac{R_{11}(-\alpha_R) - i\alpha_R(x-\eta)}{F'(-\alpha_R)e} \]

\( (54) \)

and

\[
F'(k) = 8k[(2k^2-a_2^2)-(k^2-a_2^2)^{1/2}(k-a_2^2)^{1/2} - 4k^3(\frac{k^2-a_2^2}{k^2-a_1^2})^{1/2} + \left(\frac{k^2-a_1^2}{k^2-a_2^2}\right)^{1/2}]
\]
\[
F'(\alpha_R) = -F'(-\alpha_R) \]

\( (55) \)
The results of the calculations are

\[ \sum \text{Res}_{11}(\alpha_R, -\alpha_R) = \frac{2R_{11}(\alpha_R)}{F'(\alpha_R)} \sin \alpha_R(x-\eta) \]

\[ \sum \text{Res}_{12}(\alpha_R, -\alpha_R) = \frac{2R_{12}(\alpha_R)}{F'(\alpha_R)} \]

\[ \sum \text{Res}_{21}(\alpha_R, -\alpha_R) = \frac{2R_{21}(\alpha_R)}{F'(\alpha_R)} \cos \alpha_R(x-\eta) \]

\[ \sum \text{Res}_{22}(\alpha_R, -\alpha_R) = \sum \text{Res}_{11}(\alpha_R, -\alpha_R) \]

The complete expressions for the kernels \( K_{ij}(x, \eta) \), which can be used in computation are given in appendix B.
METHOD OF SOLUTION

In view of the forthcoming calculations we will solve the complex singular integral equation on \([-1,1]\) instead of \([-a,a]\). Introducing the dimensionless variables

\[
x' = \frac{x}{a} \quad t = \frac{\eta}{a} \quad k' = ak \quad a_i' = \frac{a\omega_i}{c_i} \quad y' = \frac{y}{a}
\]

\[p(t) = P(\eta) \quad q(t) = Q(\eta)\]

and then deleting the primes (letting it be understood that all variables now appearing are dimensionless) for simplicity of appearance. Consequently (41) becomes

\[
\frac{4\mu}{1+\lambda} f(x) = \frac{1}{4\pi} \int_{-1}^{1} \frac{\varphi(t)}{t-x} \, dt - \gamma\varphi(x) - \int_{-1}^{1} [\varphi(t)L_1(x,t) + \varphi^*(t)L_2(x,t)] \, dt
\]

where

\[
L_1(x,t) = \frac{2}{\pi(1+\lambda)} K_1(x,t)
\]

The Fredholm kernels \(L_1(x,t)\) and \(L_2(x,t)\) are bounded. Hence aside from a multiplicative constant, the singular behavior of the function \(\varphi(t)\) at \(\pm 1\)
is determined by the dominant part of the singular integral equation. The integral equation (58) will be solved under the assumption that \( \varphi(t) \) satisfies a Hölder condition on every closed part of the interval \([-1,1]\) not containing the endpoints, and its behavior near the endpoints is such that it may be represented by

\[
\varphi(x) = w(x) \phi(x) \quad w(x) = (1-x)^{\alpha}(1+x)^{\beta}, \quad |x| < 1
\]

(60)

where the function \( \phi(x) \) is Hölder-continuous in the closed interval \([-1,1]\), and

\[-1 < \text{Re}(\alpha) < 0, \quad -1 < \text{Re}(\beta) < 0\]

(61)

physically this means that the stresses \( \sum_{yy} (x,0) = p(x) \) and \( \sum_{xy} (x,0) = q(x) \) are continuous in the open interval \((-1,1)\) and have integrable singularities at \( x = \pm 1 \).

The "fundamental function", \( w(x) \), of the integral equation may be obtained, within a multiplicative constant, from the homogeneous dominant part given by

\[
\frac{1}{1!} \int_{-1}^{1} \frac{w(t)}{t-x} \, dt - \gamma w(x) = 0
\]

(62)
Ignoring the constant, the solution of (62) satisfying (61) may be written as, [5].

\[ w(x) = (1-x)^\alpha (1+x)^\beta, \quad \alpha = -\frac{1}{2} - i\theta, \quad \beta = -\frac{1}{2} + i\theta \]

\[ \theta = \frac{1}{2\pi} \ln \Lambda \]  \hspace{1cm} (63)

To solve the singular integral equation (58), rather than following the regularization methods described in [5], which in this case become extremely cumbersome, we will follow the technique described in [7]. Noting that the fundamental functions \( w(x) \) of (58) is the weight of the Jacobi polynomials \( p_n^{(\alpha, \beta)}(x) \), we will express the solution in the following form

\[ \varphi(x) = \sum_{n=1}^{\infty} c_n w(x) p_n^{(\alpha, \beta)}(x) \]  \hspace{1cm} (64)

Substitution of (64) into (58), using the identity [8]

\[ \frac{1}{\pi} \int_{-1}^{1} \frac{w(t)p_n^{(\alpha, \beta)}(t)}{t-x} \, dt = \gamma w(x) p_n^{(\alpha, \beta)}(x) \]

\[ = \frac{(1-\gamma^2)^{1/2}}{2} p_{n-1}^{-\alpha, -\beta}(x) \quad |x| \leq 1 \]  \hspace{1cm} (65)

and defining

35
\[ R_n(x) = \int_{-1}^{1} w(t) P_{n-1}^{(\alpha, \beta)}(t) L_1(x, t) dt, \]  
\[ S_n(x) = \int_{-1}^{1} w(t) * P_{n-1}^{*(\alpha, \beta)}(t) L_2(x, t) dt \]

results in

\[ \frac{\partial}{\partial x} f(x) = \sum_{n=2}^{\infty} c_n \frac{(1-\gamma^2)^{1/2}}{\gamma} P_{n-2}^{(-\alpha, -\beta)} \]

\[ - \sum_{n=1}^{\infty} c_n R_n(x) - \sum_{n=1}^{\infty} c_n S_n(x) \]

Assuming (66) can be expressed in terms of Jacobi polynomial expansions \( p_n^{(-\alpha, -\beta)}(x) \) (which is absolutely reasonable, since they are integral expressions of bounded functions), we have

\[ R_n(x) = \sum_{m=1}^{\infty} a_{nm} p_{m-1}^{(-\alpha, -\beta)}(x) \]
\[ S_n(x) = \sum_{m=1}^{\infty} b_{nm} p_{m-1}^{(-\alpha, -\beta)}(x) \]

\[ a_{nm} = \frac{1}{n_{m-1}} \int_{-1}^{1} \frac{p_{m-1}^{(-\alpha, -\beta)}(x)}{w(x)} R_n(x) dx \]
\[ b_{nm} = \frac{1}{n_{m-1}} \int_{-1}^{1} \frac{p_{m-1}^{(-\alpha, -\beta)}(x)}{w(x)} S_n(x) dx \]
where the expressions for $a_{nm}$ and $b_{nm}$ are determined via orthogonality [9]

$$
\int_{-1}^{1} \frac{p_{i}^{(-\alpha,-\beta)}(x)p_{m}^{(-\alpha,-\beta)}(x)}{w(x)} \, dx
$$

$$
= \begin{cases} 
0 & j \neq m \\
(2) & j = m 
\end{cases} 
$$

(69)

If we then multiply (67) by $\frac{p_{m}^{(-\alpha,-\beta)}(x)}{w(x)}$ and integrate from $-1$ to $1$, we arrive at a simple infinite system of algebraic equations to determine $c_n$. They are

$$
\frac{4\mu \varepsilon i \gamma_{m-1}}{(1+\gamma)} = \frac{c_{m+1}(1-\gamma^2)^{1/2}}{2i} - \sum_{n=1}^{\infty} c_n a_{nm}
$$

$$
- \sum_{n=1}^{\infty} c_n^* b_{nm} \quad f(x) = i\varepsilon .
$$

(70)

The integral expressions for $a_{nm}$ and $b_{nm}$ can be numerically approximated via quadratures in which the related orthogonal polynomials are the Chebyshev polynomials of the first and second kind [9]. The approximations are
\[ a_{nm} = \frac{1}{h_{m-1}} \sum_{k=1}^{\hat{N}} \frac{\nu}{N+1} \sin^2 \left( \frac{k\nu}{N+1} \right) p^{(-\alpha, -\beta)}(x_k) \left( \frac{1-x_k}{1+x_k} \right)^{1+i\vartheta} \]

\[
\left[ \sum_{j=1}^{N} \frac{1+t_j}{N(1-t_j)} \right] \frac{i\vartheta}{p^{(\alpha, \beta)}}(t_j)L_1(x_k, t_j) \right] \]

\[ b_{nm} = \frac{1}{h_{m-1}} \sum_{k=1}^{\hat{N}} \frac{\nu}{N+1} \sin^2 \left( \frac{k\nu}{N+1} \right) p^{(-\alpha, -\beta)}(x_k) \left( \frac{1-x_k}{1+x_k} \right)^{1+i\vartheta} \]

\[
\left[ \sum_{j=1}^{N} \frac{1+t_j}{N(1-t_j)} \right] \frac{-i\vartheta}{p^{(\alpha, \beta)}}(t_j)L_2(x_k, t_j) \right] \]

\[ t_j = \cos \left( \frac{(2j-1)}{2N\pi} \right) \quad x_k = \cos \left( \frac{k\nu}{N+1} \right) \]

With \( a_{nm} \) and \( b_{nm} \) known (in the numerical sense), (70) can be solved approximately by truncating the infinite series with an appropriate number of terms, and thus we have the solution for the bonding stresses \( p(x) \) and \( q(x) \).

Numerical results are presented in the next section.
### Table 1: $c_i$'s

#### Vertical Oscillations

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### ROCKING OSCILLATIONS

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DISCUSSION OF RESULTS

The solution presented here is a sixteen term expansion, i.e.

$$\varphi(x) = \sum_{n=1}^{16} c_n w(x) P_{n-1}^{(\alpha, \beta)}(x).$$  \(72\)

The expansion coefficients $C_n$ are given in Table 1.

In calculating the expansion coefficients, kernel integrals of the type

$$\int_0^\infty \frac{2a_2 s (s^2 + a^2)^{1/2}}{(2s^2 + a^2)^2 - 4s^2 (s^2 + a_1^2)^{1/2}(s^2 + a_2^2)^{1/2}}$$

$$+ \frac{a_2^2}{(a_2^2 - a_1^2)} e^{-s|x-\eta|} ds$$

were evaluated by the use of a Gauss-Laguerre integral quadrature. It was observed that accuracy of approximately one percent was achieved with twenty quadrature points. For integrals of the type

$$\int_0^{a_2} \frac{2a_2^2 k (k^2 - a_2^2)^{1/2}(2k^2 - a_2^2)^2}{(2k^2 - a_2^2)^4 - 16k^4 (k^2 - a_1^2) (k^2 - a_2^2)^2}$$

$$- \frac{a_2^2}{(a_2^2 - a_1^2)} e^{-1k|x-\eta|} dk$$

48
a twenty point Gauss-Legendre quadrature was sufficient
to guarantee accuracy to within one percent. Twenty-
five quadrature points were necessary for the Chebyshev
polynomial quadratures of the first and second kind, to
obtain accuracy to within five percent.

A decision of vital importance which one is faced
with deliberating is "How many coefficients do I take
in the series expansion?" The answer of course depends
on the accuracy you desire; if on one extreme you wish
to obtain results to within a tenth of a percent, then
you take as many terms as is economically feasible.
Economic feasibility of course brings us to the "law
of diminishing returns to costs." In fig (3-0), a plot
of \[ K = \text{Real}\left( \sum_{n=1}^{N} C_n \beta^\alpha (1) \right) \] (analogus to a stress
intensity factor in fracture mechanics) versus \( \frac{1}{N} \),
where \( N \) is the number of coefficients taken, is given.
It should be observed that the limiting value of \( K \)
is \(-.37980544\) (i.e. as \( N \to \infty \)). A curve fit is given
by
\[
K = A + \frac{B}{N^\alpha} \quad (75)
\]
\[
A = -.37980544 \quad B = -.51268067 \quad \alpha = 2.19.
\]
Thus the use of sixteen terms of the expansion gives results accurate to approximately one percent, the measure being with respect to $K$.

The reliable dimensionless frequency range $\alpha_2 = \frac{\omega}{c_2}$ was determined to be between .01 and 1.0 (results are tabulated for .01 → .10, and 1.0). Even with sixteen terms reliable results could not be obtained for the very high and low frequency limits (i.e. $\alpha_2 = 10.0$, $\alpha_2 = .0001$). It is the author's estimation that limit frequency results could very possibly be obtained by using more terms in the expansion, although this would not improve results for quantities derived from the expansion. It should be pointed out that limit frequency results aren't of crucial physical importance regardless, as will be discussed later.

The results tabulated have been for $\nu = 1/4$ (Poisson's ratio) and $\lambda = 3-4\nu$ for plane strain, so

$$\nu = 1/4, \lambda = 2, \theta = \frac{1}{2\pi} \ln \lambda = .11032$$

(76)

For $\omega = 0$, the closed form static solution is

$$\varphi(x) = -\frac{4\mu\epsilon}{\sqrt{\lambda}} p_{1}^{(\alpha,\beta)}(x)$$

(77)

and for the normalization
we have that
\[
\frac{4\mu e h_{\alpha-\beta}}{1+W} = 1 \quad h_{\alpha-\beta} = 1.553054.
\] (78)

\[
\frac{4\mu e}{\sqrt{\lambda}} = 1.365972188
\] (79)

and the numerical solution for \(\alpha_2 = .01\) is
\[
c_2 = (-1.3659643+i.0000843)
\] (80)

which is in good agreement with (79).

The resultant moment for the rocking problem is given by
\[
M = \text{Real}\left(\int_{-1}^{1} \sum_{n=1}^{N} c_n w(x) x P_{n-1}^{(\alpha,\beta)}(x) dx\right)
\] (81)

which can be evaluated in closed form by use of
\[
x = 2P_1 + 2i\theta P_0
\] (82)

and results in
\[ \sum_{n=1}^{N} \int_{-1}^{1} c_n \mathcal{P}_{n-1}(\alpha, \beta)(x)xw(x)dx = 2c_2 h_1^{(\alpha, \beta)} + 2i\theta c_1 h_0^{(\alpha, \beta)} \]

\[ M = 2c_2 h_1^{(\alpha, \beta)} - 2\theta c_1 h_0^{(\alpha, \beta)} \]

\[ h_1^{(\alpha, \beta)} = \frac{1}{2\pi} (3/2-i\theta)\Gamma(3/2+i\theta) \quad h_0^{(\alpha, \beta)} = -\frac{2\pi^{3/4}}{\Gamma(3/4)} \]

\[ = \frac{1}{2} (1/4+\theta^2)^{1/2}/\cosh \pi \theta. \]

The application of the results pertain to the areas of solid state electronics, machine design, and seismic interaction.

Frequently in solid state electronics we encounter situations in which a very small metallic chip is glued to a quartz substrate, typically \( a = .001 \) meters and \( c_2 = 3400 \) meters/second, so

\[ \frac{c_2}{a} = 3400000 \quad \omega = \frac{c_2}{a} \alpha_2 \]

\[ \alpha_2 = .01 \rightarrow 1.0 \Rightarrow \omega = 34,000 \rightarrow 3,400,000 \text{ cycles/second}, \] the upper frequency limit being much in excess of what is of interest. The lower frequencies are typical of those which occur in solid state devices.

Machine design lengths are typically of the order
\[ a = .1 \rightarrow 1 \text{ meters and } c_2 = 3100 \text{ meters/second, so} \]

\[
\frac{c_2}{a} = 3100 \rightarrow 31000
\]

\[ a_2 = .01 \rightarrow 1.0 \Rightarrow \omega = 31 \rightarrow 3100 \text{ cycles/second } a = .01m \]

\[ \omega = 3100 \rightarrow 31000 \text{ cycles/second } a = 1.0m \]

The frequencies for \( a = 1 \) meter are at the upper limit of the physically interesting spectrum. Finally seismic interactions are characterized by \( a = 100 \text{ ft. and } c_2 \approx 10,000 \text{ ft./second}, \) this gives frequencies \( \omega = 1 \rightarrow 10 \text{ cycles/second}, \) which is exactly the range for Rayleigh waves generated by earthquakes.
APPENDIX A

\[ D_{11}(k) = \frac{k^2(2k^2-a_2^2)}{F(k)} \quad E_{11}(k) = \frac{-2k^2(2k^2-a_2^2)^{1/2}(k^2-a_1^2)^{1/2}}{F(k)} \]

\[ D_{12}(k) = \frac{2k^3(2k^2-a_2^2)^{1/2}}{F(k)} \quad E_{12}(k) = \frac{-k(k^2-a_2^2)^{1/2}(2k^2-a_2^2)}{F(k)} \]

\[ D_{21}(k) = \frac{-k(k^2-a_1^2)^{1/2}(2k^2-a_2^2)}{F(k)} \quad E_{21}(k) = \frac{2k^3(k^2-a_1^2)^{1/2}}{F(k)} \]

\[ D_{22}(k) = \frac{-2k^2(k^2-a_1^2)^{1/2}(k^2-a_2^2)^{1/2}}{F(k)} \quad E_{22}(k) = \frac{k^2(2k^2-a_2^2)}{F(k)} \]

\[ D_{11}^\infty(k) + E_{11}^\infty(k) = \frac{\alpha_1^2}{2(\alpha_1^2-\alpha_2^2)} \quad D_{12}^\infty(k) + E_{12}^\infty(k) = \frac{|k|\alpha_2^2}{2k(\alpha_1^2-\alpha_2^2)} \]

\[ D_{21}^\infty(k) + E_{21}^\infty(k) = \frac{|k|\alpha_2^2}{2k(\alpha_1^2-\alpha_2^2)} \quad D_{22}^\infty(k) + E_{22}^\infty(k) = \frac{\alpha_1^2}{2(\alpha_1^2-\alpha_2^2)} \]

\[ F(k) = (2k^2-a_2^2)^2 \quad F^\infty(k) = \lim_{k \to \infty} F(k) \]

\[ -4k^2(k^2-a_1^2)^{1/2}(k^2-a_2^2)^{1/2} = 2k^2(\alpha_1^2-a_2^2) \]

\[ G(k) = (2k^2-a_2^2)-2(k^2-a_2^2)^{1/2}(k^2-a_1^2)^{1/2} \quad G^\infty(k) = \lim_{k \to \infty} G(k) = \alpha_1^2 \]
APPENDIX B

\[ K_{11}(x, \eta) = \int_{0}^{a_2} \frac{4k^2 \alpha_2^2 (2k^2 - \alpha_2^2)(k^2 - \alpha_2^2)^{1/2}(k^2 - \alpha_1^2)^{1/2}}{(2k^2 - \alpha_1^2)^2 - 4k^2 (k^2 - \alpha_2^2)^2 (k^2 - \alpha_1^2)^2} e^{-ik(x-\eta)} \, dk \]

\[ - \frac{2\pi R_{11}(\alpha_R)}{F'(\alpha_R)} \sin \alpha_R(x-\eta) \]

\[ K_{12}(x, \eta) = -i \, \text{sgn}(x-\eta) \int_{0}^{\infty} \left[ \frac{2a_2^2 s(s^2 + \alpha_2^2)^{1/2}}{2s^2 + \alpha_2^2 - 4s^2 (s^2 + \alpha_1^2)^{1/2}(s^2 + \alpha_2^2)^{1/2}} + \frac{a_2^2}{(\alpha_1^2 - \alpha_2^2)} \right] e^{-s|x-\eta|} \, ds \]

\[ - \int_{0}^{a_1} \left[ \frac{2a_2^2 k(k^2 - \alpha_2^2)^{1/2}}{(2k^2 - \alpha_1^2)^2 - 4k^2 (k^2 - \alpha_2^2)^2 (2k^2 - \alpha_1^2)^2} - \frac{a_2^2}{(\alpha_1^2 - \alpha_2^2)} \right] e^{-ik(x-\eta)} \, dk \]

\[ - \int_{a_1}^{a_2} \left[ \frac{2a_2^2 k(k^2 - \alpha_2^2)^{1/2}(2k^2 - \alpha_2^2)^2}{(2k^2 - \alpha_1^2)^2 - 16k^4 (k^2 - \alpha_1^2)(k^2 - \alpha_2^2)} - \frac{a_2^2}{(\alpha_1^2 - \alpha_2^2)} \right] e^{-ik(x-\eta)} \, dk \]

\[ + \frac{2\pi R_{12}(\alpha_R)}{F'(\alpha_R)} \cos \alpha_R(x-\eta) \]
\[
K_{21}(x, \eta) = -i \; \text{sgn}(x-\eta) \int_0^\infty \left[ \frac{2\alpha_2^2 s^2 (s^2 + \alpha_1^2)^{1/2}}{(2s^2 + \alpha_2^2)^2 - 4s^2 (s^2 + \alpha_1^2)^{1/2}(s^2 + \alpha_2^2)^{1/2}} \right. \\
+ \frac{\alpha_2^2}{(\alpha_1^2 - \alpha_2^2)} e^{-s|x-\eta|} ds \\
\left. - \int_0^{\alpha_1} \frac{2\alpha_2^2 k (k^2 - \alpha_1^2)^{1/2}}{(2k^2 - \alpha_2^2)^2 - 4k^2 (k^2 - \alpha_1^2)^{1/2}(k^2 - \alpha_2^2)^{1/2}} - \frac{\alpha_2^2}{(\alpha_1^2 - \alpha_2^2)} e^{-ik|x-\eta|} dk \right] \\
- \int_{\alpha_1}^{\alpha_2} \frac{8\alpha_2^2 k^3 (k^2 - \alpha_1^2)(k^2 - \alpha_2^2)^{1/2}}{(2k^2 - \alpha_2^2)^4 - 16k^4 (k^2 - \alpha_1^2)(k^2 - \alpha_2^2)} - \frac{\alpha_2^2}{(\alpha_1^2 - \alpha_2^2)} e^{-ik|x-\eta|} dk \\
+ \frac{2\pi i R_{21}(\alpha_R)}{F'(\alpha_R)} \cos \alpha_R(x-\eta) \\
\]

\[
K_{22}(x, \eta) = K_{11}(x, \eta) 
\]
REFERENCES


VITA

The author was born on August 27, 1954 at Westover A.F.B. in Chicopee Falls, Mass.; the first son of Richard P. and Mary M. Kolkka.

After graduation in June 1972 from Valley Stream South High School in Valley Stream, N.Y., he entered Rutgers University in New Brunswick, New Jersey. He was awarded the degree of Bachelor of Arts in Applied Mathematics from Rutgers College in May of 1976.

In May 1976, the author accepted a research position in wave propagation at Lehigh University, and is now currently teaching at Lehigh University. He expects to be awarded the Degree of Master of Science in Applied Mechanics in June 1979.