A study of the creep buckling of shallow spherical shells subjected to external pressure.

Thomas D. Chimner

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A STUDY OF THE
CREEP BUCKLING OF SHALLOW SPHERICAL
SHELLS SUBJECTED TO EXTERNAL PRESSURE

BY

THOMAS D. CHIMNER

A THESIS
PRESENTED TO THE GRADUATE COMMITTEE
OF LEHIGH UNIVERSITY
IN CANDIDACY FOR THE DEGREE OF
MASTER OF SCIENCE
IN
MECHANICAL ENGINEERING

LEHIGH UNIVERSITY
1979
This thesis is accepted and approved in partial fulfillment of the requirements for the degree of Master of Science.

December 11, 1979
Date

Professor in Charge

Chairman of Department
ACKNOWLEDGEMENTS

I would like to express my appreciation and gratitude to Professors Arturs Kalnins and Dean Updike for their guidance. I would also like to thank my wife Jeanine for typing this thesis, and special thanks to Judith Kraycik for her help in editing.
# TABLE OF CONTENTS

## I. INTRODUCTION

<table>
<thead>
<tr>
<th>Subsection</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. Elastic Buckling of Shallow Spherical Shells</td>
<td>2</td>
</tr>
<tr>
<td>B. Creep Buckling of Columns</td>
<td>3</td>
</tr>
<tr>
<td>C. Creep Buckling of Cylindrical Shells</td>
<td>4</td>
</tr>
<tr>
<td>D. Creep Buckling of Shallow Spherical Shells</td>
<td>5</td>
</tr>
<tr>
<td>E. Other Geometries and Simplified Hand Calculation</td>
<td>7</td>
</tr>
</tbody>
</table>

## II. CREEP LAW IDEALIZATIONS

<table>
<thead>
<tr>
<th>Subsection</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. Secondary Creep Laws</td>
<td>8</td>
</tr>
<tr>
<td>B. Primary Creep Laws</td>
<td>9</td>
</tr>
<tr>
<td>C. Extension of Uniaxial Creep Laws to Multiaxial</td>
<td>10</td>
</tr>
<tr>
<td>Stress State</td>
<td></td>
</tr>
</tbody>
</table>

## III. NONLINEAR ANALYSIS OF SHELLS OF REVOLUTION SUBJECTED TO AXISYMMETRIC LOADS USING KSHEL

<table>
<thead>
<tr>
<th>Subsection</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. Implementation of Creep in the Shell's Governing</td>
<td>13</td>
</tr>
<tr>
<td>Equations</td>
<td></td>
</tr>
<tr>
<td>B. Creep Laws Available Using the Current Version of</td>
<td>16</td>
</tr>
<tr>
<td>the KSHEL Program</td>
<td></td>
</tr>
<tr>
<td>C. Test of Creep Laws in KSHEL</td>
<td>17</td>
</tr>
</tbody>
</table>
IV. DISCUSSION OF RESULTS

A. Assumptions in Creep Analysis 22
B. Governing Shallow Shell Parameters 23
C. Creep Buckling Analysis Using a Secondary Creep Law 27
D. Creep Buckling Analysis Using a Primary Creep Law 30
LIST OF TABLES

1. Shallow Spherical Shell Geometries
2. Elastic Buckling Pressures and Applied Loads
3. Creep Law Parameters
4. Comparison of Critical Buckling Times Between KSHEL and Miyazaki (21)
LIST OF FIGURES

1. Creep Law Idealizations
2. Steady State Creep Results for Cylinder With End Caps Under Internal Pressure
3. Primary Creep Law Test of Cylinder With End Caps Under Internal Pressure Using KSHEL
4. Shallow Shell Geometry
5. Elastic Limit Load Calculations for Shallow Spherical Geometry
6. Elastic Buckling Shape of a Spherical Cap Versus λ
7. Effect of n for Shallow Shell λ = 4.45, P/Pcr = .89
8. Creep Deflection Curve for Case 6, λ = 7.0
9. Effect of Pressure on the Creep Buckling Time for Shell Geometry λ = 9
10. Critical Buckling Time as a Function of Pressure λ = 9.0, n=5
11. Effect of Creep Power n for Shallow Shell With λ = 9.0, P/Pcr = .75
12. Deflection Curve for Case 12, λ = 12
13. Creep Deflection Curve for Case 14, α = 5.0
14. Creep Deflection Curve for Case 15, α = 15.0
15. Creep Deflection Curve for Case 13, α = 30.0
16. Eigenvalue Prestress Multiplier as a Function of Time for Case 15, α = 15.0, Wave No. 3
17. Comparison Between Shi (20) and KSHEL for a Simply Supported Cap Subjected to a External Pressure, P=165.5 kPa
18. Hemisphere Loaded Thru a Rigid Boss, P = 1112 N
19. Comparison Between Creep Law Idealizations for Shallow Shell Geometry λ = 7.0, Case 16
20. Comparison Between Creep Law Idealization for Shallow Shell Geometry λ = 4.06, case 20
# LIST OF SYMBOLS

<table>
<thead>
<tr>
<th>SYMBOL</th>
<th>MEANING</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>Radius of Curvature of Shell</td>
</tr>
<tr>
<td>A</td>
<td>Primary Creep Constant Eq. 2.3</td>
</tr>
<tr>
<td>bi</td>
<td>Thermal Stress</td>
</tr>
<tr>
<td>Bij</td>
<td>Creep Compliance of Material</td>
</tr>
<tr>
<td>C</td>
<td>Material Creep Parameter Eq. 2.5a-c</td>
</tr>
<tr>
<td>C₁, C₂, C₃</td>
<td>Constants Eq. 3.16, 3.20, 3.22</td>
</tr>
<tr>
<td>Ci,j, Kij, Di,j, Ni, Mi</td>
<td>Equations 3.9a-e</td>
</tr>
<tr>
<td>D</td>
<td>Material Creep Compliance</td>
</tr>
<tr>
<td>E</td>
<td>Modulus of Elasticity</td>
</tr>
<tr>
<td>h</td>
<td>Thickness of Shell</td>
</tr>
<tr>
<td>l</td>
<td>Length of Cylinder</td>
</tr>
<tr>
<td>M₀, M₀</td>
<td>Moment Resultants</td>
</tr>
<tr>
<td>N₀, N₀</td>
<td>Membrane Stress Resultants</td>
</tr>
<tr>
<td>n</td>
<td>Stress Power for Variation of Creep Rate With Stress Eq. 2.1</td>
</tr>
<tr>
<td>Po</td>
<td>Reference Strain</td>
</tr>
<tr>
<td>p</td>
<td>Pressure</td>
</tr>
<tr>
<td>P₀cr</td>
<td>Critical Axisymmetric Elastic Buckling Pressure</td>
</tr>
<tr>
<td>n P₀cr</td>
<td>Critical Asymmetric Elastic Buckling Pressure for Wave No. n</td>
</tr>
<tr>
<td>q₀</td>
<td>Elastic Buckling Pressure of a Complete Sphere Eq. 4.2</td>
</tr>
<tr>
<td>Q,N</td>
<td>Effective Shear Resultants</td>
</tr>
<tr>
<td>SYMBOL</td>
<td>MEANING</td>
</tr>
<tr>
<td>------------</td>
<td>-------------------------------------------------------------------------</td>
</tr>
<tr>
<td>$s_1, s_2$</td>
<td>Distance From Axis of Symmetry Along Reference Surface</td>
</tr>
<tr>
<td>$t$</td>
<td>Time</td>
</tr>
<tr>
<td>$t_1, t_2$</td>
<td>Time at Beginning and End of Time Step</td>
</tr>
<tr>
<td>$\Delta t$</td>
<td>Time Step $t_2 - t_1$</td>
</tr>
<tr>
<td>$t^*$</td>
<td>Transition Time Eq. 3.13a</td>
</tr>
<tr>
<td>$U_0$</td>
<td>Reference Strain Rate</td>
</tr>
<tr>
<td>$\bar{W}$</td>
<td>Average Radial Deflection Eq. 4.3</td>
</tr>
<tr>
<td>$W_c$</td>
<td>Center Radial Deflection</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Geometry Parameter Eq. 4.8</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Angle of Shallow Shell See Fig. 4</td>
</tr>
<tr>
<td>$\beta_\phi, \beta_\theta$</td>
<td>Angle of Rotation of Normal</td>
</tr>
<tr>
<td>$\varepsilon, \dot{\varepsilon}$</td>
<td>Total Strain and Strain Rate</td>
</tr>
<tr>
<td>$\varepsilon_c, \dot{\varepsilon}_c$</td>
<td>Creep Strain and Creep Strain Rate</td>
</tr>
<tr>
<td>$\varepsilon_1, \varepsilon_2, \varepsilon_3$</td>
<td>Principal Strains</td>
</tr>
<tr>
<td>$\dot{\varepsilon}_1, \dot{\varepsilon}_2, \dot{\varepsilon}_3$</td>
<td>Principal Strain Rates</td>
</tr>
<tr>
<td>$\varepsilon^<em>, \dot{\varepsilon}^</em>$</td>
<td>Equivalent Strain and Equivalent Strain Rate Eq. 2.7a-b</td>
</tr>
<tr>
<td>$\bar{\varepsilon}_1, \bar{\varepsilon}_2$</td>
<td>Strain at Beginning and End of Time Step</td>
</tr>
<tr>
<td>$\varepsilon_\phi, \varepsilon_\theta$</td>
<td>Strains at Middle Surface</td>
</tr>
<tr>
<td>$\Delta \varepsilon$</td>
<td>Change in Strain Across Time Step $\bar{\varepsilon}_2 - \bar{\varepsilon}_1$</td>
</tr>
<tr>
<td>SYMBOL</td>
<td>MEANING</td>
</tr>
<tr>
<td>--------</td>
<td>---------</td>
</tr>
<tr>
<td>$\xi_\phi$, $\xi_\theta$</td>
<td>Strains Eq. 3.10a-b</td>
</tr>
<tr>
<td>$k_\phi$, $k_\theta$</td>
<td>Curvature Changes at Middle Surface</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Geometry Parameter Eq. 4.1</td>
</tr>
<tr>
<td>$\sigma_1$, $\sigma_2$, $\sigma_3$</td>
<td>Principal Stresses</td>
</tr>
<tr>
<td>$\sigma$, $\dot{\sigma}$</td>
<td>Stress and Stress Rate</td>
</tr>
<tr>
<td>$\sigma^*$</td>
<td>Equivalent Stress Eq. 2.6</td>
</tr>
<tr>
<td>$\overline{\sigma}_1$, $\overline{\sigma}_2$</td>
<td>Stress at Beginning and End of Time Step</td>
</tr>
<tr>
<td>$\Delta \sigma$</td>
<td>Change in Stress Across Time Step $\sigma_2 - \sigma_1$</td>
</tr>
<tr>
<td>$\sigma_n$, $\varepsilon_n$</td>
<td>Nominal Stress and Strain Eq. 4.3</td>
</tr>
<tr>
<td>$\sigma'$, $\dot{\sigma}'$</td>
<td>Cylinder Circumferential Stress and Stress Rate Eq. 3.14a</td>
</tr>
<tr>
<td>$\sigma_\phi$, $\sigma_\theta$</td>
<td>Stresses Eq. 3.1a-b</td>
</tr>
<tr>
<td>$\tau$</td>
<td>Nondimensional Time Eq. 4.6</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Primary Creep Time Power Eq. 2.3</td>
</tr>
<tr>
<td>$\nu$</td>
<td>Poisson's Ratio</td>
</tr>
</tbody>
</table>
ABSTRACT

This paper presents an investigation of the creep buckling of shallow shells subjected to a uniform external pressure. A numerical analysis of various shell geometries and pressures were performed using the computer program KSHEL. A hemisphere loaded thru a rigid boss was analyzed in addition to the shallow shell geometries. Comparison with other authors showed significant differences in the predicted buckling times. A set of nondimensional governing parameters were identified for the shallow shell geometry.

The buckling modes studied were both the axisymmetric snap-thru and bifurcation types. A fourier expansion series is used to express the bifurcation mode. It is assumed that the bifurcation of the shell occurs from an axisymmetric creep state.

The creep material behavior was modeled using both primary and secondary creep laws. The secondary creep behavior was modeled using a power law for the variation of the creep rate with stress. The primary creep analysis compared results from the time hardening and strain hardening creep laws.
I. INTRODUCTION

A. Elastic Buckling of Shallow Spherical Shells

The elastic buckling of shallow spherical shells has been widely researched. Reissner (1) first developed the shallow shell elastic theory. Budiansky (2) solved the axisymmetric elastic buckling problem. Then Huang (3) extended the elastic buckling theory to include asymmetric modes of instability. In the axisymmetric mode, the spherical cap loses stability at a limit point. The equilibrium curve for a shell causes the cap to jump or snap to a new equilibrium state at the limit point. This type of buckling mode is called snap-thru. The shell becomes unstable thru bifurcation in the asymmetric buckling mode. Huang showed that most shallow spherical shells reach a bifurcation point before snap-thru occurs.

The differences between the numerical and experimental elastic results led to the study of geometry imperfections in shallow shells. Various authors (4-10) have studied the imperfection sensitivity problem. Several conclusions can be drawn from this research work. The thinner or flatter the shell becomes the more sensitive it is to imperfections. Also the shells are much more sensitive to changes in curvature than changes in thickness. Imperfections on the order of half the shell thickness can reduce the elastic buckling pressure by 60%. These effects also play an important role in the creep buckling behavior of a shell.
B. Creep Buckling of Columns

The analysis of creep buckling is an important factor in the design of slender components subjected to a high temperature environment. Creep effects will reduce the time independent buckling strength of a component. Thus, a component that is initially stable can pass into an unstable state under creep. In certain cases it is possible for a component to always be stable while deforming under creep. The time at instability is defined as the critical time.

The first work in the area of creep buckling was conducted on columns. Early work was limited to experimentation and hand calculation methods. A good review of this early work was given by Hoff (11). To better understand the creep buckling process a dynamic model was developed by Hoff. A dashpot represented the creep material behavior, while the mass represented the load and a spring simulated the stiffness. The dynamic system was always unstable for viscoelastic material. If strain hardening was assumed, the dynamic system would become unstable for only a range of loads.

When analyzing a column, several additional points were brought up. The imperfections in the shape of a column decreased the critical buckling time. Addition of elastic strains also decreased the critical time. Hoff showed large differences in the predicted critical buckling times when comparing various other authors' research.
Several simplified hand methods were developed to predict the critical time for a column. Carlson and Schwope (12) and Shanley (13) used iso-stress-strain curves to predict the column's behavior. The main disadvantage with this method was the plotting of the isochrous stress-strain curves. An additional error was also introduced into the calculations through the conversion of the uniaxial creep data. Also, in an actual column the stresses keep changing during creep, while preparation of the iso-curves assumes a constant time independent stress. Gerard and Papino (14) used a tangent modulus approach with limited success.

C. Creep Buckling of Cylindrical Shells

Development of high speed computers led to the study of more complex components. The first creep buckling work on shells investigated cylindrical geometries. Batterman (15) analyzed the time independent behavior of a cylinder subjected to axial compression. Diamant (16) and Samuelson (17) investigated the creep buckling of a cylindrical shell under axial compression and internal pressure. Only transient creep was assumed in Dimant's paper. The analysis concluded that a strain hardening theory would predict a shorter critical time than the corresponding time hardening theory. Experimental results gave a critical buckling time between the two hardening theories. Samuelson included the effect of time independent plasticity in his analysis. For moderate loadings there was very little effect from plasticity. Inclusion of plasticity is important...
when the plastic strains are of the same order of magnitude as the elastic strains.

Thermal gradients can significantly affect the creep deformation of a shell. Y. S. Pan (18) studied the creep buckling of a cylindrical shell subject to external pressure and thermal gradients. Inclusion of thermal gradients decreased the critical buckling time by 60% in the example presented. This effect could become important for thick shells where significant thermal gradients can occur thru the thickness.

D. Creep Buckling of Shallow Spherical Shells

The creep buckling of shallow spherical shells has received much attention in recent years. Huang (19) studied a viscoelastic clamped spherical shell. A relaxation modulus was used to represent the viscoelastic shell material behavior. Shi et al (20), studied a simple supported spherical shallow shell using a variational theorem for creep. The creep material behavior was idealized using a time hardening model. Shi's study included experimentation of five test shells made of type 6/6 nylon. Comparison of experimentation and numerical results showed smaller strains and longer buckling times occurring for the theoretical work. One explanation of these differences results occurred with a slight variation of the shell's radius.
The first investigation of asymmetric creep buckling of a clamped shallow spherical cap was performed by Miyazaki, et al (21). His results were presented in terms of a nondimensional time hardening creep law. The majority of the work was performed using a power of three stress for the variation of creep strain rate with stress. As in the elastic results, bifurcation of the shell could occur before the snap-thru limit point is reached. The creep bifurcation of the shell can occur in a lower mode than obtained from the time independent elastic case. The buckling mode is dependent on the applied pressure and the creep rate. This is because the creep deformations change the stress distribution within the shell. Miyazaki assumed that the bifurcation of the shell occurs from an axisymmetric stress state. Also, a threshold pressure exists such that below this pressure the shell is always in equilibrium.
E. Other Geometries and Simplified Hand Calculation Methods

Other shell geometries studied include work by Jones (22). He studied a complete sphere subjected to external pressure. Penny and Marroitt (23) investigated a hemisphere loaded thru a rigid boss using a reference stress method.

Simplified hand calculations methods have been developed because of the large amount of computer time needed to solve a creep buckling problem. Mayville and Gordon (24) present a comparative stress method based on elastic results. The main assumption of this simplified approach is that the same primary strain distribution exists in both the creep and elastic buckling problem. Hoff (11) proved that the buckling state for a column is dependent on the slope of the creep curve and not just the strain level. In addition, creep deformations change the stress distribution within the shell subjected to anything other than a pure membrane stress state.

In summary, the creep buckling problem is affected by various conditions. These include the creep rate, loading intensity, thermal gradients, plasticity, and imperfections. In addition, the type of creep behavior idealization also affects the results. A shallow shell can buckle thru either a axisymmetric limit point or asymmetric bifurcation mode. The type of buckling mode depends on the geometry and load intensity.
II. CREEP LAW IDEALIZATIONS

A. Secondary Creep Laws

The creep phenomenon is characterized by three regions of behavior. The initial creep strain rate decelerates with time in the primary phase. This is followed by a region where the creep strain rate remains constant. This second region is referred to as the secondary or steady state creep. The creep behavior ends with a rapid acceleration of the creep strain rate in the tertiary phase. These three regions are shown on figure 1 for an uniaxial specimen.

To describe the three creep regions various creep laws have been proposed. The simplest method is to assume that only secondary creep material behavior occurs. One of the most commonly used laws was suggested by Norton (25) and Bailey (26) as:

\[ \varepsilon_c = U_0 \left( \frac{\sigma}{S_0} \right)^n ; \quad n > 1 \]  \hspace{1cm} (2.1)

For most practical problems the value of the stress exponent ranges from 2.5 to 12.0. A hyperbolic sine law was proposed by Nadai (27) for the secondary creep rate as:

\[ \varepsilon_c = U_0 \times \sinh \left( \frac{\sigma}{S_0} \right) \]  \hspace{1cm} (2.2)
B. Primary Creep Laws

In the primary creep region the material is commonly assumed to be either strain hardening or time hardening. The time hardening law is much simpler to handle in creep calculations. A curve fit of the primary creep data can be made using the following equation:

$$\varepsilon_c = A\sigma^n t^\mu.$$  \hspace{1cm} (2.3)

The constants $A$, $n$, and $\mu$ are dependent on the material, temperature, and stress level of the test.

The creep rate equations differ between the two hardening theories. The time hardening law is derived by differentiating equation 2.3 with respect to time. The time hardening creep rate equation is expressed as:

$$\dot{\varepsilon}_c = \mu A \sigma^n t^{\mu-1}.$$  \hspace{1cm} (2.4a)

The strain hardening creep law is formulated by eliminating the time from equation 2.4a using equation 2.3. The resulting strain hardening law is written as:

$$\dot{\varepsilon}_c = \mu A^{1/\sigma} n/\mu \varepsilon (\mu-1)/\mu.$$  \hspace{1cm} (2.4b)

The two hardening theories predict different results in a time varying stress field. Identical results occur from the two hardening theories for a time independent stress field.

Figure 3 shows an example of the creep behavior predicted by the two primary theories. This example is a cylinder subjected to an instan-
taneous step load at point C. A membrane stress field exists in the cylinder. Curves O-A and O-B are for the cylinder subjected to the two time independent loads. When assuming a time hardening material, the shift to the creep curve follows a constant time path C-D. Thus, the total creep strain follows the path OCG. The strain hardening law assumes that the strain stays constant when shifting to the new creep curve. Thus, the material would take path C-E when using a strain hardening law and would follow the path OCF. The strain hardening law predicts larger strains when subjected to a stress field which increases with time.

Creep laws can also be developed by combining a secondary and primary creep law. The two regions are separated at a specified transition time. The creep strain rate for the two different laws are equated at this transition time. An example of this procedure will be given latter in the text.

C. Extension of the Uniaxial Creep Laws to a Multiaxial Stress State

So far the various creep laws have been limited to only uniaxial stress states. To extend these laws to a multiaxial stress state, a criteria must be developed. The following discussion is taken from Hult (28). It is assumed that the creep strain is incompressible and that the creep strain rates are proportional to the principal shear stresses. Based on the above two assumptions, it can be shown that the creep strain rates are given by the equation:
\[ \varepsilon_{c1} = 2/3 \ C (\sigma_1 - \frac{1}{2} (\sigma_2 + \sigma_3 )) \]  

\[ \varepsilon_{c2} = 2/3 \ C (\sigma_2 - \frac{1}{2} (\sigma_3 + \sigma_1 )) \]  

\[ \varepsilon_{c3} = 2/3 \ C (\sigma_3 - \frac{1}{2} (\sigma_2 + \sigma_1 )) \]

The problem then becomes one of describing the value \( C \) in the above equations. A number of methods can be developed to describe the value of \( C \) for the material being used. One method is to follow the procedures developed to describe the the plastic deformation of ductile metals without creep. This is done by using the definition of equivalent stress which is given as:

\[ \sigma^* = (1/\sqrt{2}) \left( (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right)^{\frac{1}{2}} \]  

The three principal stresses are given by \( \sigma_1, \sigma_2, \) and \( \sigma_3 \) in the above equation. Similar equations can be developed for the equivalent strain and equivalent strain rate. These equations have the form:

\[ \dot{\varepsilon}^* = \sqrt{2}/3 \left( (\dot{\varepsilon}_1 - \dot{\varepsilon}_2)^2 + (\dot{\varepsilon}_2 - \dot{\varepsilon}_3)^2 + (\dot{\varepsilon}_3 - \dot{\varepsilon}_1)^2 \right)^{\frac{1}{2}} \]  

\[ \varepsilon^* = \sqrt{2}/3 \left( (\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2 \right)^{\frac{1}{2}} \]

The value of \( C \) can now be determined by using an uniaxial creep law and the equivalent stress and strain equations. For a steady state multiaxial stress state the creep strain rate is defined as:

\[ \dot{\varepsilon}_c = f(\sigma^*) \]  

For a constant stress tension test the numerical factors can be chosen such that \( \varepsilon^* = \varepsilon \), and \( \varepsilon^* = \varepsilon \). Thus using equations 2.5a-b we can calculate \( C=3\hat{\varepsilon}^*/2\alpha^* \). This is written as:

\[ \varepsilon_{c1} = \frac{\dot{\varepsilon}^*/\sigma^*}{(\sigma_1 - \frac{1}{2} (\sigma_2 + \sigma_3 ))} \]  

\[ \varepsilon_{c2} = \frac{\dot{\varepsilon}^*/\sigma^*}{(\sigma_2 - \frac{1}{2} (\sigma_3 + \sigma_1 ))} \]  

\[ \varepsilon_{c3} = \frac{\dot{\varepsilon}^*/\sigma^*}{(\sigma_3 - \frac{1}{2} (\sigma_2 + \sigma_1 ))} \]
To use equations 2.9a-c a particular secondary creep law must be substituted for the strain rate $\dot{\epsilon}^*$. The extension of the uniaxial creep law from equation 2.1 would have the form:

$$\dot{\epsilon}_c^* = U_0 \left(\sigma^*/S_0\right)^n$$  \hspace{1cm} (2.10)

The creep strain rate can be calculated by using equation 2.10 as:

$$\dot{\epsilon}_{c1} = \left(U_0/S_0^n\right) (\sigma^*)^{n-1} \left(\sigma_1 - \frac{1}{3} (\sigma_2 + \sigma_3) \right)$$ \hspace{1cm} (2.11)

Similar equations for $\epsilon_{c2}$ and $\epsilon_{c3}$ can also be developed.

The same procedure can be used to extend the uniaxial primary creep laws to describe a multiaxial stress state. If the time hardening law given in equation 2.4a is utilized, the creep strain rate is defined as:

$$\dot{\epsilon}_{c1} = A \sigma^{n-1} t^\mu \left(\sigma_1 - \frac{1}{3} (\sigma_2 + \sigma_3) \right)$$ \hspace{1cm} (2.12)

For the strain hardening law the creep strain rates are defined as:

$$\dot{\epsilon}_{c1} = \mu A^{1/\mu} \sigma^ {n/\mu} \epsilon^{1-1/\mu} \left(\sigma_1 - \frac{1}{3} (\sigma_2 + \sigma_3) \right)$$ \hspace{1cm} (2.13)

Here again similar expressions can be written for $\epsilon_{c2}$ and $\epsilon_{c3}$.
III. NONLINEAR ANALYSIS OF SHELLS OF REVOLUTION SUBJECTED TO AXISYMMETRIC LOADS USING KSHEL

A. Implementation of Creep in the Shell's Governing Equations

The solution to the governing shell equations are solved both incrementally and iteratively. The notation used and the numerical method is presented in two papers written by A. Kalnins (29, 30). The shell stresses are expressed by assuming a plane stress mechanical behavior of the form:

\[
\sigma_\phi = B_{11} \varepsilon_\phi + B_{12} \varepsilon_\theta + b_1 \\
\sigma_\theta = B_{12} \varepsilon_\phi + B_{22} \varepsilon_\theta + b_2
\]

The coefficients \(B_{ij}\) and \(b_i\) are calculated using the stresses and total strains. These coefficients are calculated at the beginning of the time step \((t-\Delta t)\). An example will be used to demonstrate how the above compliance coefficients are calculated. The example is an uniaxial specimen subjected to creep. The creep behavior will be described using a steady state power law. The total strain rate is written as follows:

\[
\varepsilon = \dot{\sigma}/E + \sigma^R/D
\]

The dot in the above equation represents differentiation with respect to time. The governing equations are solved by incrementing the time, and then iterating to find the stress. A linear stress variation is assumed across the time step and expressed as:

\[
\sigma = \bar{\sigma}_1 + (\bar{\sigma}_2 - \bar{\sigma}_1) (t - t_1) / (t_2 - t_1)
\]
The subscripts 1 and 2 denote the beginning and the end of the time step, respectively. The nonlinear creep law is also linearized by expanding it using a binomial expansion series and neglecting higher order terms of $\Delta \sigma$. If we integrate equation 3.2 with respect to time, the total change in the strain becomes:

$$\Delta \epsilon = \Delta \sigma / E + 1/D \int_{t_1}^{t_2} \sigma^n + n \bar{\sigma}^{n-1} \frac{\Delta \sigma}{\Delta t} \tau + \ldots \, dt$$

(3.4)

The above equation reduces to:

$$\Delta \epsilon = \Delta \sigma \left[ 1/E + n \bar{\sigma}^{n-1} \Delta t/2D + \bar{\sigma}^{n} \Delta t/D \right]$$

(3.5)

The above equation is solved for $\sigma_2$ in terms of the initial stress and strain. Equation 3.5 is rewritten as:

$$\bar{\sigma}_2 = B \bar{\epsilon}_2 + b$$

(3.6)

The creep compliance is written as:

$$B = 1/ (1/E + n \bar{\sigma}^{n-1} \Delta t/2D)$$

(3.7)

The stresses are assumed to remain constant across the time step in most numerical analysis. The disadvantage of this assumption is that many more time steps must be used in the analysis. Also, the numerical solution can diverge from the actual results if too large of a time step is taken.

The shell stress resultants, and stress couples are related to the extensional strains and curvature changes by means of:
\[ N_\phi = C_{11} \varepsilon_\phi + C_{12} \varepsilon_\theta + K_{11} k_\phi + K_{12} k_\theta + \bar{N}_1 \]  
\[ N_\theta = C_{12} \varepsilon_\phi + C_{22} \varepsilon_\theta + K_{12} k_\phi + K_{22} k_\theta + \bar{N}_2 \]  
\[ M_\phi = K_{11} \varepsilon_\phi + K_{12} \varepsilon_\theta + D_{11} k_\phi + D_{12} k_\theta + \bar{M}_1 \]  
\[ M_\theta = K_{12} \varepsilon_\phi + K_{22} \varepsilon_\theta + D_{12} k_\phi + D_{22} k_\theta + \bar{M}_2 \]

The coefficients \( C_{ij}, K_{ij}, D_{ij}, N_i, M_i \) are calculated with the following integrals through the thickness:

\[ C_{ij} = \int B_{ij} \, dz \]  
\[ K_{ij} = \int B_{ij} \, z \, dz \]  
\[ D_{ij} = \int B_{ij} \, z^2 \, dz \]  
\[ N_i = \int b_i \, dz \]  
\[ M_i = \int b_i \, z \, dz \]

The strain distribution through the thickness is given by the Kirchoff-Love hypothesis. This is:

\[ \xi_\phi = \varepsilon_\phi + k_\phi z \]  
\[ \xi_\theta = \varepsilon_\theta + k_\theta z \]

The strains \( \varepsilon_\phi \) and \( \varepsilon_\theta \), are the extensional strains at the reference surface, and \( k_\phi \) and \( k_\theta \) are the curvature changes.

The coefficients \( B_{ij} \) and \( b_i \) are calculated at discrete values along the reference surface and through the thickness in the computer program KSHEL. These values are saved for later use. To evaluate the integrals 3.9a-e numerically, Simpon's rule is used with nine equally spaced points through the thickness. The values of \( C_{ij}, K_{ij}, D_{ij}, N_i \) and \( M_i \) are then stored at discrete points along the reference surface.
Once the coefficients of equations 3.8a-e are calculated and stored, the shell's governing equations reduce to a nonlinear boundary value problem. The governing equations are solved for the fundamental variables $U_\phi$, $W$, $\beta_\phi$, $N_\phi$, $Q$, and $M_\phi$. In addition, the generalized strain components $\varepsilon_\phi$, $\varepsilon_\phi$, $k_\phi$, and $k_\phi$ are solved. Now the strain components $\varepsilon_\phi$ and $\varepsilon_\phi$ may be found from 3.10a-b, and the stress components $\sigma_\phi$ and $\sigma_\phi$ from equation 3.1a-b. A new time step is taken and the entire process is repeated.

B. Creep Laws Available Using the Current Version of the KSHEL Program

The computer program KSHEL (31) has the capability to model either a primary or secondary creep material behavior. For secondary creep, both the power and hyperbolic sine laws for variation of the creep rate with stress, are available. The general form of the steady state creep strain rate is:

$$\dot{\varepsilon}_c = U_0 G(\sigma/So)$$  \hspace{1cm} (3.11)

Where $U_0$ is the reference strain rate and $So$ is the reference stress. The function of $G(\sigma/So)$ becomes either a power law or a hyperbolic sine law. This is written as:

$$G (\sigma/So) = (\sigma/So)^n \quad \text{Power Law} \hspace{1cm} (3.12a)$$

$$G (\sigma/So) = \text{Sinh} (\sigma/So) \quad \text{Hyperbolic Sine Law} \hspace{1cm} (3.12b)$$
The primary creep laws available are either time hardening or strain hardening. The hardening laws are formed by dividing the creep curve into a secondary and primary region. The time at the separation point between the two regions is called $t^*$, the creep strain rate is equated from the primary and secondary creep laws. The transition time $t^*$ becomes:

$$t^* = \left[ \frac{\mu}{1-\mu} \right] \left[ \frac{P_o G(\sigma/\sigma_o)}{U_o F(\sigma/\sigma_o)} \right]$$ \hspace{1cm} (3.13a)

The values of $\sigma_o$, $\mu$, $P_o$, and $U_o$, are constants. The function $F(\sigma/\sigma_o)$ is expressed as in equations 3.12a-d, but describes the primary creep region. A definition of the constants and how to determine their value is given on page 44-2 of the KSHEL users manual (31). The primary creep strain becomes:

$$\varepsilon_c = \left[ \frac{P_o}{1-\mu} \right] F(\sigma/\sigma_o) (t/t^*)^\mu$$ \hspace{1cm} (3.13b)

C. Test of the Creep Laws in KSHEL

To test the implementation of the secondary and primary creep laws within KSHEL, a test problem was developed. The test consisted of analyzing a step load change on a cylinder with end caps under internal pressure. The boundary conditions at the starting edge of the cylinder are $u_\phi$, $Q$, $\beta_\phi$, and $N_\phi$ equal to zero. At the end of the cylinder, $u_\phi$, $\beta_\phi$, and $Q$ are set to zero. In addition, $N_\phi$ is set to the axial pressure load. These boundary conditions and loading produce a membrane state of stress in the cylinder. The longitudinal and circumferential stresses become the principal stresses in the problem.
The stresses are defined as:
\[ \sigma_0 = \frac{pa}{2h} = \sigma' \]  
\[ \sigma_\phi = \frac{pa}{h} = 2\sigma' \]  
(3.14a)  
(3.14b)

The equivalent stress is calculated using equation 2.6 as \( \sqrt{3} \sigma' \).

In the first test, a secondary power law was used to describe the creep material behavior. The creep strain rates are calculated using equations 2.10 as:
\[ \dot{\varepsilon}_0 = 0.0 \]  
\[ \dot{\varepsilon}_\phi = C_1 (\sigma')^n \]  
(3.15a)  
(3.15b)

The constant \( C_1 \) in the above equation is defined as:
\[ C_1 = U_0 (1/S_0)^n \sqrt{3}^{n-1} (3/2) \]  
(3.16)

As seen from the above equation, the axial creep strain is zero for this problem. The total strain rates are calculated by adding the elastic and creep strain rates.
\[ \dot{\varepsilon}_\phi = ((1-2\nu)/E) \dot{\sigma}' \]  
\[ \dot{\varepsilon}_\theta = ((2-\nu)/E) \dot{\sigma}' + C_1 (\sigma')^n \]  
(3.17a)  
(3.17b)

To define the total strains, the above equations are integrated with respect to time.
\[ \varepsilon_\phi = ((1-2\nu)/E)\sigma' \]  
\[ \varepsilon_\theta = ((2-\nu)/E)\sigma' + C_1 (\sigma')^n t \]  
(3.18a)  
(3.18b)

A similar expression is developed for the circumferential strain rate when a time hardening law is used.
\[ \dot{\varepsilon}_\theta = ((2-\nu)/E)\dot{\sigma}' + C_2 (\sigma')^n t^{n-1} \]  
(3.19)

The constant \( C_2 \) is a function of the creep law constant given in equation (2.3).
\[ C_2 = (3/2) A (3)^{(n-1)/2} \]  
(3.20)
Again the axial strain is only a function of the elastic strain. If a strain hardening law is assumed, the circumferential strain rate becomes:

\[ \dot{\varepsilon}_\phi = \frac{(2-v)}{E} \dot{\sigma} + C_3 (\sigma^*)^{n/\mu + 1} \varepsilon^* \left( 1 - \frac{1}{\mu} \right) \]  

(3.21)

In this case, the constant \( C_3 \) is defined as:

\[ C_3 = \lambda^1 \sqrt{3}^{n/\mu} (3/2) \]  

(3.22)

\( \varepsilon^* \) is the equivalent strain as defined in equation 2.7b.

The test cylinder was assumed to have the following properties:

- \( a = 500 \text{ mm} \) (radius)
- \( l = 10 \text{ mm} \) (length)
- \( h = 10 \text{ mm} \) (thickness)
- \( E = 30 \text{ MPa} \) (modulus of elasticity)
- \( v = .30 \) (Poisson's ratio)

The loading condition at the beginning of the analysis was:

- \( p = 1000 \text{ kPa} \)
- \( N_\phi = 250 \text{ N/mm} \)

At a time of 0.2 seconds, the load is stepped instantaneously to the following values:
For the steady state creep tests, the power law for the variation of the creep strain rate with stress was used with the following parameters:

\[ U_0 = 0.001 \text{ mm/mm/sec} \]
\[ S_0 = 10 \text{ MPa} \]

The creep strain results, from the computer program KSHEL, are shown on figure 2. The computer results agreed exactly with the closed form solution.

Both the strain hardening and the time hardening primary creep was tested. The following is a list of the creep parameters used in the analysis:

\[ P_0 = 1.707 \times 10^3 \text{ mm/mm} \]
\[ \mu = 0.5 \]
\[ n = 3 \]
\[ t^* = 0.40 \text{ sec} \]

The general form of the hardening creep law is given by equation 3.13. A plot of the circumferential creep strain for both primary laws are shown on figure 3 using the step load change previously.
described. The creep curves obtained for the two time independent stress levels were also plotted for comparison. Again, these results agreed exactly with the theoretical solutions given in the beginning of this section.

In conclusion, a cylinder under a membrane stress state was utilized to test the various creep laws. The secondary and primary power laws, for the variation of creep rate with stress were used in the test. In all three cases, the numerical results agreed exactly with the theoretical solutions.
IV. DISCUSSION OF RESULTS

A. Assumptions in the Analysis

The main assumptions of the analysis are as follows:

1. Shell theory is used to develop the governing equations.

2. No presence of imperfections, thermal gradients or strains in the shell.

3. Only axisymmetric creep is considered. Asymmetric bifurcation of the shell occurs from an axisymmetric stress state.

4. The external pressure load is applied instantaneously and held constant for the duration of the analysis.

5. The uniaxial creep curve is expressed with a power law. The extension of the uniaxial creep law to a multiaxial stress state is performed using a creep potential law of the Von Mises type.

6. Isotropic linear behavior in the elastic range.

7. Only one material is allowed.
8. The shell is assumed to be clamped on the outside edge. Exceptions to this are case numbers 17 and 18, which have a simply supported edge.

9. The nonlinear rotational terms were considered in the governing equations. The nonlinear radial terms are not included in the analysis.

B. Governing Shallow Shell Parameters

The shallow spherical shell is characterized by several governing parameters. Figure 4 shows the definition of the shallow shell geometry. Huang (3) demonstrated that the elastic buckling pressure is a function of \( \lambda \) and \( \frac{P_{cr}}{q_0} \). The parameter \( \lambda \), defines the shell geometry and is written as:

\[
\lambda = 2 \left( 3(1-\nu^2) \right)^{\frac{3}{4}} \left( \frac{H}{h} \right)^{\frac{1}{2}}
\]  

The critical elastic buckling pressure of the shallow spherical shell is defined as \( P_{cr} \). Huang expressed the critical pressure as a function of the elastic buckling pressure of a complete sphere. The equation for the buckling pressure of a complete sphere is defined as:

\[
q_o = 2E \left( \frac{h}{a} \right) \left( \frac{1}{3(1-\nu^2)} \right)^{\frac{1}{2}}
\]  

The critical pressure ratio ranges between .5 and 1.2 for the axisymmetric mode of failure. The critical elastic buckling mode is asymmetric for shallow spherical geometries having \( \lambda \) greater than 5.5. The shell buckles with a higher number of waves around the circumference as \( \lambda \) increases above 5.5.
Table 1 lists the different geometries studied. The critical elastic buckling pressures are listed on Table 2. The lowest time independent critical pressure was also listed in those cases where \( \lambda \) was greater than 5.5. In addition, the number of waves around the circumference are given for the asymmetric buckling modes. For the various geometries studied, a check of the time independent limit load pressures were made using the computer program KSHEL. The limit analysis is preformed by incrementing the pressure until the solution convergence is not obtained. The analysis is repeated using a smaller pressure range to refine the critical pressure calculation. Then the critical pressure value is checked by performing an axisymmetric eigenvalue analysis of the shell. A plot of the load limit and load calculations are shown on figure 5. The values obtained from KSHEL were in good agreement with the results presented by Huang. For various shallow shell geometries, the axisymmetric buckling mode shapes are shown on figure 6.

The radial deflection was normalized using the following scheme. The shell's radial deflection was normalized by dividing the shell thickness. The center radial deflection was used in predicting the critical buckling time in the early work of this thesis. Budiansky (2) showed that, in general, the center deflection is not always the point of maximum radial deflection. The question becomes whether to plot the maximum deflection, or just the center value. Budiansky concluded that the shell's behavior is better characterized by using an average radial deflection. Thus, in my later work, the average deflection of the shell was used based on the following:

\[
W = \left( \frac{1}{S_2 - S_1} \right) \int_{s_1}^{s_2} \left( w^2 + u_\phi^2 \right) ds^{\frac{3}{2}} \quad (4.3)
\]
This average deflection worked very well in all the analysis work performed.

The time was normalized by dividing the nominal creep strain by the classical buckling elastic strain for a sphere. The classical buckling strain is expressed as:

\[ \sigma_n = \frac{q_k a}{2h} \]  \hspace{1cm} (4.4a)

\[ \varepsilon_n = \frac{\sigma_n (1-\nu)}{E} \]  \hspace{1cm} (4.4b)

Assuming the creep law has the general form as listed in equation 2.3, then the nondimensional time is written as:

\[ \tau = \frac{\varepsilon_c}{\varepsilon_n} = \left( A(\sigma_n)^{n-1} \right)^{1/(1-\nu)} \]  \hspace{1cm} (4.5)

The above equation can be simplified as:

\[ \tau = \frac{(AE/(1-\nu))\sigma_n^{n-1}}{1} \]  \hspace{1cm} (4.6)

The time power constant \( \mu \) is equal to one for the case of a secondary creep behavior. This normalized time allows secondary creep results to be immediately transformed to the case of time hardening creep. The secondary creep time case becomes either contracted or expanded for the time hardening case.

As previously stated, the governing equations are solved by incrementing the time step. The size of the time step is estimated from the creep strain rate. The computer program KSHEL predicts the time step increment based on the following equation:

\[ \Delta t = \Omega \times \text{EEQ}/\text{STRT} \]  \hspace{1cm} (4.7)
Omega is a constant that is entered by the user, which usually ranges between one to twenty. The predicted maximum equivalent strain is EEQ and the maximum strain rate is STRT. The time step was assumed to remain constant throughout the creep buckling analysis. Equation 4.7 was used to estimate the time step at the start of the creep analysis. The use of equation 4.7 to estimate the time at only the start of each step causes too large of a time step near the critical time.

A scheme based on the number of iterations to a satisfactory accuracy was developed in an effort to adjust the time step size. The results at each time step were saved for possible later use. A solution taking less than three iterations caused the next time step to be increased. The time step remained constant if the solution was solved in three iterations. The time step was decreased if more than three iterations were needed to solve the governing equations. The calculations were restarted at the beginning of the time step if the solution did not converge. The next time step was set to half the proceeding value in this case. This restarting procedure was done a maximum of three times. This new time step scheme resulted in a much better definition of the shell's behavior near the critical buckling time.

A new time step estimation scheme was developed for use with a primary creep law. Equation 4.7 cannot be used for a primary creep law because the creep strain rate is infinite when the time is zero. To
avoid this problem, an option to input the time step was incorporated into KSHEL. The time step size was estimated using a comparable secondary creep analysis. Equation 4.6 is used to convert the secondary creep time steps to an equivalent time hardening analysis. This option also benefited the secondary creep analysis by allowing better control of the time step near the critical time.

C. Creep Buckling Analysis Using a Secondary Creep Law

Various shell geometries were analyzed using a secondary creep power law. Table 3 lists the creep law parameters and the type of creep law used. The creep law constant $A$ was arbitrarily set to obtain a critical buckling time of approximately 24 hours. The constant $A$ in case numbers 17, 18, and 19 were set to the specific values listed in their respective references.

A comparison of the radial deflection verses time curves for various powers of $n$ at $\lambda=4.45$ are shown on figure 7. As expected, the actual critical buckling time decreases as the power $n$ is increased. These curves show the few number of points needed to define the curves. The critical time occurs at the point where the slope of the deflection curves goes to infinity. A similar plot for $\lambda=7$ is given on figure 8.

The effect of pressure and power $n$ were analyzed for a shell geometry having $\lambda=9$. A deflection plot is given on figure 9 for the various
loads analyzed. A plot of the critical time, as a function of the pressure, is shown on figure 10. Two values of the power \( n \) were analyzed for a pressure ratio of .750. The results of the radial deflection are plotted on figure 11. The nondimensional time shows little change in the results.

A deflection plot for \( \lambda=12 \) is given on figure 12. As seen from all the plots the shell buckles at approximately the same deflection to thickness ratio.

Three comparison cases from Miyazaki (21) were analyzed using the computer program KSHEL. A steady state power law for the variation of stress with creep rate was used in the analysis. The power \( n \) was set to 3 in all three test cases. Miyazaki presented his results in terms of a nondimensional time hardening law. The results presented in reference (21) were converted to the nondimensional time given by equation 4.6. When the constant \( \mu \) is equal to one, the time hardening law reduces to the secondary creep law. Figures 13, 14, and 15 present the radial deflection as a function of time for the three different geometries analyzed. The external pressure load was set to 80\% of the critical time independent buckling pressure which corresponded to the lowest mode of failure. The geometry parameter used by Miyazaki was given as:

\[
\alpha = a \beta^2 / h \tag{4.8}
\]

Miyazaki investigated both the axisymmetric snap-thru and the asymmetric bifurcation creep buckling modes. The critical buckling time
was determined by the mode which resulted in the lowest buckling time. Miyazaki demonstrated that a shell can buckle under creep in a lower mode than obtained from the time independent case. The creep buckling modes shape depends on the creep rate and pressure load ratio. An example of this is given in case 13. This shell has a geometry parameter $\alpha$ equal to 30. The lowest time independent buckling pressure is obtained for the asymmetric mode of 5. The lowest creep buckling time is found for the axisymmetric snap-thru mode.

An example of creep buckling of a shell thru bifurcation is given in case 15. The critical creep buckling time occurs for the asymmetric mode of 3. This is the same mode predicted for the elastic buckling case. Buckling the shell into an asymmetric mode was determined by using an eigenvalue buckling analysis. The axisymmetric creep state was used to prestress the shell for the eigenvalue buckling analysis. A plot of the prestress multiplier from the eigenvalue buckling analysis as a function of time is given on figure 16. This case corresponds to $\alpha = 15$ in Miyazaki's paper. A comparison of the predicted creep buckling time between KSHEL and Miyazaki is listed in Table 4. As seen in the table the predicted critical time obtained from KSHEL is much shorter than given in reference (21). One reason for this difference could be the fact that Miyazaki uses a finite element approach. The use of finite elements in a shell analysis tends to produce a stiffer shell. A comparison of the strain displacement and curvature equations between the two methods was also made. The governing axisymmetric equations presented by Miyazaki
were converted to the same notation used in the computer program KSHEL. The governing axisymmetric equations given by Miyaziki are as follows:

\[ \varepsilon_\phi = u_{\phi,s} + \frac{1}{2} (w,s)^2 \] (4.9a)
\[ \varepsilon_\theta = \frac{(u_\phi \cos \phi + w \sin \phi)}{r} \] (4.9b)
\[ k_\phi = -w,ss \] (4.9c)
\[ k_\theta = -((\cos \phi)/r)(w,s) \] (4.9d)

The comma in the above equation refers to differentiation with respect to the variables that are listed after the comma. The corresponding equations for a axisymmetric spherical shell from KSHEL are as follows:

\[ \varepsilon_\phi = u_{\phi,s} + \frac{1}{2} \beta_\phi \left( \frac{w}{R_\phi} \right)^2 \] (4.10a)
\[ \varepsilon_\theta = \frac{(u_\phi \cos \phi + w \sin \phi)}{r} \] (4.10b)
\[ k_\phi = -w,ss + u_{\phi,s}/R_\phi \] (4.10c)
\[ k_\theta = -((\cos \phi)/r)(w,s) \] (4.10d)

As seen in the comparison of the governing equations, the rotational term \( \beta_\phi \) is approximated by \( dw/ds \) in reference (21). Also, several other terms in the governing equations were not included by Miyaziki.

D. Creep Buckling Analysis Using a Primary Creep Law

The creep buckling of a simply supported shallow shell was studied by Shi (20). The creep behavior was modeled using a time hardening creep law. The shell and creep parameters are given in Tables 1, 2, and 3 under case number 17 and 18. Case 17 was first run on KSHEL using a secondary creep law to estimate the time step for in the
primary creep analysis. Shi's work included both a numerical and experimental investigation. A comparison of Shi's work and KSHEL is shown on figure 17. The results presented by Shi were converted to agree with the nondimensionalizing scheme presented in this paper. The secondary creep results from KSHEL were also plotted for comparison. The differences between the secondary and time hardening results from KSHEL were due to the assumption that the stress is linear across the time step. A smaller time step would decrease the deviation between the secondary and time hardening cases. In addition, the shell was also analyzed using a strain hardening creep law. The time and strain hardening theories predict about the same results because of the small variation of the stress field with respect to time. The results from KSHEL predict slightly higher strains than reported by Shi for his numerical work. The numerical method from both the computer program KSHEL and Shi predict much smaller strains than experimental values given by Shi. One reason for this could be the effect of imperfections in the test shell. Also, the creep properties could be different from the uniaxial tension test for a bending stress field.

The creep buckling of a hemisphere loaded through a rigid boss was studied by Marriott and Penny (23). A uniaxial tension test was made to determine the material's creep properties. The tension test creep data was fitted using a time hardening creep law. The geometry, load, and creep parameters are listed on Tables 1, 2, and 3 under case number 19. Marriott presented both a reference stress calcula-
tion method and experimental results. Figure 18 shows the predicted deflection of the boss under creep. The same model was analyzed in KSHEL using a strain hardening creep law as a comparison. The results showed the strain hardening theory to be in closer agreement with the experimental values reported by Shi. The differences between the numerical and experimental results could be due to imperfections in the test shells. Also, The difference between the material's tension and bending creep properties could account for the differences.

Two additional shallow shell geometries were studied using a primary creep law. A comparison was made between the results obtained from a secondary, time hardening, and strain hardening creep law idealization. The first geometry investigated is listed under case number 16. This shell has a geometry parameter \( \lambda \) equal to 7.0. The same model was analyzed in case 6 using a secondary creep law. A plot of the radial deflection results is shown on figure 19. The large differences between the strain and time hardening results were due to the variation of the stress field with respect to time. The greater this stress variation is, the larger the difference becomes between the two hardening theories. The last geometry to be investigated is listed under case 14. Figure 20 shows the average radial deflection results as a function of time for the three different creep laws used. Here, again, large differences are predicted in the results between the time hardening and strain hardening theories.
In conclusion, a study of the creep buckling behavior of shallow shells has been presented. Several geometries were analyzed using the computer program KSHEL. The program KSHEL has the capability to analyze any axisymmetric shell subjected to creep. The shallow shell geometry was chosen because of its simplicity and reference in other literature. The various creep laws studied include a secondary power law and the strain and time hardening primary creep laws. The buckling modes considered are both the axisymmetric snap-thru and asymmetric bifurcation types.
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### TABLE 2

Elastic Buckling Pressures and Applied Loads

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<th>Load</th>
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TABLE 3

Creep Law Parameters

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TABLE 4

Comparison of Critical Buckling Times
Between KSHEL and Miyazaki (21)

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<th>Case No.</th>
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<td>1.66</td>
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Figure 2: Steady State Creep Results for Cylinder With End Caps Under Internal Pressure

-39-
Figure 3: Primary Creep Law Test of Cylinder With End Caps Under Internal Pressure Using KSHEL

- Strain Hardening
- Time Hardening

- $p = 1200$ kPa
- $p = 1000$ kPa
Figure 4: Shallow Shell Geometry
Figure 5: Elastic Limit Load Calculations for Shallow Spherical Geometry
Figure 6: Elastic Buckling Shape of a Spherical Cap Versus $\lambda$

RADIAL DIMENSION $r/r_o$
Figure 7. Effect of \( n \) for Shallow Shell

\[ \lambda = 4.45, \ P_{cr} = 0.89 \]
Figure 8: Creep Deflection Curve for Case 6, λ = 7.0
Figure 9: Effect of Pressure on the Shell Buckling Time for Shell Geometry $\lambda = 9$.
Figure 10. Critical Buckling Time as a Function of Pressure
\[ \lambda = 9.0, n = 5 \]
Figure 11: Effect of Creep Power n for Shallow Shell With $\lambda = 9.0$, $P / P_{cr} = 0.75$
Figure 12: Deflection Curve for Case 12, $\lambda=12$
Figure 14: Creep Deflection Curve for Case 15, $\alpha=15.0$

Figure 15: Creep Deflection Curve for Case 13, $\alpha=30.0$
Figure 16: Eigenvalue Prestress Multiplier as a Function of Time for Case 15, $\alpha=15.0$, Wave No. 3

$\tau_{cr}=2.55$
Figure 17. Comparison between Shi (20) and KSHEL for a Simply Supported Cap Subjected to a External Pressure, P=165.5 kPa
Figure 18. Hemisphere Loaded Thru a Rigid Boss, $P = 1112 \text{ N}$
Figure 19: Comparison Between Creep Law Idealizations for Shallow Shell Geometry $\lambda = 7.0$, Case 16
REFERENCES


BIOGRAPHY

I was born July 28, 1953 in Royal Oak, Michigan to the parents of Donald and Therese Chimner. I did my undergraduate work at Lawrence Institute of Technology in Southfield, Michigan. In June, 1975 I received a Bachelor of Science Degree, with honors, in Mechanical Engineering. After graduating from Lawrence Institute of Technology I started working at Air Products and Chemicals Inc., where I am presently employed as a Senior Design Engineer.