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Literature survey on restrained columns

M. Ojalvo

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Welded Continuous Frames and their Components

Literature Survey on the Analysis and Design of
Restrained Columns

By
Morris Ojalvo

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1. **INTRODUCTION**

The theory of the behavior of columns in structural frameworks remains at the present time in a rather unsatisfactory state. If one were to consider the problem in its entirety one would take into account the restraints to displacements, longitudinal twist, and rotations at the ends of the column as well as the thrust, bending moments, and twisting moments applied to the column at the ends. The effects of plasticity on the column behavior must also receive special attention.

The present report will concern itself only with references that consider failure by excessive bending about one of the principal axes of the column section. It also will be assumed that the column ends do not translate and that the columns are not loaded or restrained except at their ends. In spite of these restrictions on the scope of the investigation, the type of column considered often occurs in tier building design when the columns are clad in a manner that precludes failure by lateral-torsional buckling or when columns are of a cross section having a high torsional rigidity (box and annular sections). This investigation is concerned with the behavior of columns up to the ultimate
carrying capacity including effects of partial plasticification.

Plastic analysis is concerned with the prediction of the ultimate carrying capacity of a structure and its aim is to base the design upon this capacity. A prediction of the capacity of tension members and laterally restrained beams is relatively simple and reliable as compared with predictions for members in which stability considerations predominate. A beam for example, will develop a plastic hinge at points where the bending moment reaches a prescribed value. A beam-column in contrast develops large areas where the material has plasticized and consequently a simple hinge mechanism is not a sufficiently accurate assumption for the purposes of analysis. It appears therefore, that a study of columns should be undertaken with a view to arriving at a concept of column behavior sufficiently complete to allow its incorporation into the methods of Plastic Analysis of multi-story structures.

This report will take the following form:

a) Comments on the major theoretical and experimental investigations. (They are intended to
describe the present status of knowledge.)

b) A discussion of the essentials of the problem including recommendations on the course that further theoretical investigations should take. A possible design method is also discussed.

c) List of references (with translation of title if in a foreign language).
2. **COMMENTS**

**Axially Loaded Columns**

Axially loaded restrained columns that fail by buckling about a principal axis of the cross section have been extensively treated in references 5, 8, 26, 16, 17 and 18. The problem is basically an Eigenvalue problem in which the framework is loaded only at the joints. There is no flexing of the members until the critical loading is reached. As the loading increases, the bending rigidities of the axially loaded compression members decreases while the rigidities of tension members may either increase or decrease. This is due to the reduction in the modulus of elasticity for members stressed beyond the proportional limit and to the influence of the axial force upon the rotational rigidity at the ends of the column. The basic theory may be found in Ref. 16, while Ref. 5 provides important simplifications that may be used.

**Combined Bending and Compression**

A much more difficult problem is encountered when bending and axial load are applied to the column simultaneously. Deflections begin at the onset of loading and
continue to increase until the ultimate capacity of the column is reached. Fig. 1a illustrates this simultaneous bending and axial loading of a compression member. For small values of W, the bending moments at the ends of the column would cause tensile flexural stresses at the upper side of the top joint. For this condition the column restrains the end rotations of the beams. Further increase of W may reduce the stiffness of the column so that it will tend to rotate excessively at its ends. Since this rotation must equal the rotation of the ends of the beams at the joints, the beams (if strong enough) may start to restrain the column. This will cause the moments at the joints to reduce the zero and then reverse sign (Fig. 1b). Fig. 1c illustrates how the deflection at the center of a column will vary as the load increases. At the beginning, the increase of the deflection is practically linear with W. Further increase of loading causes the deflection to increase more rapidly than the loading until finally the load reaches a maximum value as indicated on Fig. 1c. This is the ultimate load of the column. The non-linearity of the curve of Fig. 1b is due to two separate effects: first, there is the change in the geometry of the structure particularly in the column. Secondly, there is the change in
stiffness of the material where the stresses have exceeded the proportional limit.

**Summary of Currently Used Theoretical Approaches**

Three general theoretical approaches have been used on restrained column problems. In the following, a brief summary of each approach will be given together with some of the references which may be considered representative of that approach.

In the following the nomenclature used in the original reports will be used. In all cases the symbols will be defined where they first appear.

A. The Method of von Karman and Chwalla

The oldest approach to the problem was due to E. Chwalla (Ref. 10). The method is the following: Moment vs. curvature curves are computed at various average axial stresses for the rectangular cross section under consideration. These are based upon an assumed stress-strain curve. By means of the moment-curvature relations it is possible to construct "column deflection curves" using a numerical integration procedure described in Ref. 35. The term
"column deflection curve" has been adopted in this report to refer to the shape that a column will take when held in a bent position by axial forces applied at the ends. Fig. 2 shows a number of such column deflection curves for a given axial load $P = P_1$. For each assumed center deflection a different length and shape of column deflection curve is obtained. However, as long as the material of the column is not stressed beyond the proportional limit, the length of the column deflection curve will remain equal to the length given by the Euler column formula. The numerical integration procedure yields the slope and the deflection of the column deflection curve at intervals along the length corresponding to the increment of length used in the integration.

For a restrained column such as in Fig. 3 the following equation expresses the equality of internal and external moments at the ends of the restrained column:

$$\frac{a}{h} + \frac{p}{h} + \frac{\delta}{h} = 0 \quad (1)$$

where:

- $a$ - transverse deflection of the point of zero moment on the column measured from the lines connecting the column ends.
c - length of elastic restraining beam (see Fig. 3).

e - horizontal distance on the column from the point of zero moment to the support.

h - width of rectangular section.

i - radius of gyration of the rectangular cross section about the axis of bending.

p - eccentricity with which the load is applied to the column.

\( y_0 \) - maximum amplitude of a column deflection curve.

E - modulus of elasticity of the restraining beams.

L - length of restrained column.

\( L_0 \) - half wave length of a column deflection curve.

P - axial load in the column.

\( \sigma_0 \) - average axial stress in the column.

\[
\delta = \frac{3}{2} \cdot \frac{E}{\sigma_0} \cdot \frac{i}{c} \cdot \frac{dy}{dx} \bigg|_{y=a}
\]
The equation relates applied moment, resisting moment, and internal moment at the ends of the beam. By keeping P, c, and the column cross section constant it is possible to determine the relationship between \( y_0 \) and L for several values of L which satisfy Eq. 1. This is accomplished in the following manner.

a) A column deflection curve identified by \( y_0 \) is chosen and its half wave length, \( L_0 \) is determined.

b) By trial a value of \( e \) is determined which gives a slope and a deflection at the support which upon substitution satisfies equation 1.

c) The sum of \( L_0 \) and 2\( e \) gives the length of column L consistent with equilibrium for the given column deflection curve.

d) Steps a to c are repeated for several column deflection curves.

In Fig. 4 the relationship between \( y_0 \) and L is plotted. The horizontal point of tangency gives the longest length consistent with equilibrium.
The solution to the symmetrically loaded and symmetrically restrained column was the only one worked out by Chwalla. His work was confined to rectangular columns of steel having a proportional limit of 27,000 psi.

B. The Method of Baker, Horne, and Roderick

A second approach to the problem is due to J. F. Baker, M. R. Horne, and J. W. Roderick (Ref. 2, 3, 13, 19). The method is based upon a simplified elastic-plastic stress-strain diagram as proposed by Jezek (Ref. 22). Analytical expressions for curvature based on the simplified diagram are integrated to determine the shape of the bent column. Reference 13 of the group is unique because it treats a cross section in which the area is assumed to be concentrated in a thin circular ring. The other references of the group are all concerned with rectangular cross section.

Basically, the method determines the maximum load that a column of given length can carry. That is to say, on a diagram such as in Fig. 1c or Fig. 4, the axial load is taken as the ordinate. The difficulties arise in finding the deflected shape of the column that satisfies the conditions of continuity at the ends for each trial value.
The procedure used to solve the case of a column restrained and loaded as shown in Fig. 5 is as follows:

a) For an assumed value of the total axial load in the column, a trial moment transmitted from the beams to the column is assumed.

b) Since the beams are acting in an elastic manner, it is possible to compute the end rotations of the column.

c) Starting at one end of the column with this rotation, analytical integration is carried out to give the deflected shape of the column.

d) If the trial moment of step (a) was the correct one, then the slope of the column will be parallel to its undeflected direction at the center of the column. If this is not the case the procedure is repeated from step (a).

e) When step (d) gives the proper slope at the center of the column it is known that the boundary conditions are satisfied at the column ends. For this value the center deflection is determined. In
general, up to the maximum axial load, two de-
flected columns may be found which satisfy the
boundary conditions.

f) For several values of the total column compressive
load $P$, the center deflections for the equili-
brium state are plotted as in Fig. 6. The maximum
point on this curve gives the axial load at collapse.

In the computations it was assumed that the beam loads
are applied first and remain constant while the load $F$ is
increased to failure. The axial load referred to as $P$ is
equal to the sum of $F$ and the portion of the transverse
loads $W$ transmitted to the column.

In spite of the involved mathematics, the method has
been made to yield solutions for rectangular columns with
equal elastic end restraints when the applied end moments
are equal and also when they are equal but opposite in sense.
(This latter solution has been strongly questioned in the
closure to Ref. 40. It appears instead that the solution
is applicable with modification to the case of one end
pinned and the other end elastically restrained with the
applied moment existing only at the restrained end.) The method can take into account an upper and a lower yield stress in the material and also the effects of strain reversal when some of the material in the column exceeds the yield stress and subsequently unloads during the application of the column loads. It must be emphasized that the latter results can only be achieved with the expenditure of considerably more effort than is required when unloading is ignored.

C. Bijlaards Method

A third approach to the problem is due to P. P. Bijlaard (Ref. 4, 5, 7). This method must be considered an approximate one as it assumes that the shape of the deflected column may be taken as a branch of a sine wave for the purpose of determining the slope at the end of the column and the deflection at the point in the column where the bending moment is zero. This assumption has its origins in Ref. 28. The following equation for the column of Fig. 7 relates the average stress in the column, the length between restrained ends, the area of the column, the rotational stiffness at the ends, and the maximum deflection from a chord joining the points of inflection.
\[ y_m = \frac{A \sigma_a e}{A \sigma_a \cos \frac{\pi L}{2\lambda} + \frac{\pi}{\lambda} \beta \sin \frac{\pi L}{2\lambda}} \] (2)

The terms in Equation 2 are defined below:

- \( y_m \) - Maximum deflection from the chord as indicated in Fig. 7.
- \( A \) - Area of the rectangular cross section.
- \( \sigma_a \) - Average stress.
- \( e \) - Eccentricity of the load.
- \( L \) - Length of the column between restrained ends.
- \( \lambda \) - Length between points of contraflexure (half wave length of a column deflection curve).
- \( \beta \) - Rotational stiffness coefficients at the column ends.

Equation 2 is used together with a set of curves computed by Chwalla which gives the half wave length of a column deflection curve for a given \( y_m \) and average stress. To determine the greatest average stress that a restrained column will support, it is necessary to assume several values of \( \lambda \) and to determine \( y_m \) and \( \sigma_a \) from Equation 2 and Chwalla's curves. The largest \( \sigma_a \) gives the greatest carrying capacity of the column.
More complicated expressions similar to equation 2 are derived to cover the general case when the end restraints are unequal and the eccentricities at the ends differ. (7)

Tests

Tests of restrained columns that have been carried out at Cambridge University are reported in References 2 and 3. The tests were carried out on small scale models of either rectangular or I cross sections. The columns of I cross sections were tested so that bending occurred about the minor axis. The restraints were exceptionally stiff and were such that they remained elastic during the entire loading. Single and double-curvature bending of the columns were achieved by loading as illustrated in Fig. 8 and Fig. 5. The loads were first applied to the restraining beams; the axial load F was then increased until the columns collapsed. Good agreement was obtained between the observed collapse load and the theoretical computations performed according to the method of Baker, Horne, and Roderick. Figure 9 is a reproduction from Ref. 2 which indicates the degree of agreement between theory and test results. The test results are for the
loading case indicated in Fig. 5 of this report.

Tests on larger specimens were carried out at Cornell and are reported in References 4 and 15. The columns were of rectangular and I cross section. The I section columns were bent about the minor axis. Rotational restraint was provided by coil springs at each end of the column, while the compressive load was applied with equal eccentricity at each end. The ultimate loads were compared with computed values by Bijlaard's method and found to be in close agreement. The following table from Ref. 4 gives a comparison of the test results with the results computed according to Bijlaard's method.

<table>
<thead>
<tr>
<th>Specimen No.</th>
<th>Value of e/r</th>
<th>Ultimate Mean Stress in kips per sq. in.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Experimental</td>
</tr>
<tr>
<td>8 AS</td>
<td>0</td>
<td>32.20</td>
</tr>
<tr>
<td>11 AS</td>
<td>0.5</td>
<td>27.00</td>
</tr>
<tr>
<td>10 AS</td>
<td>1.0</td>
<td>24.20</td>
</tr>
<tr>
<td>9 AS</td>
<td>2.0</td>
<td>22.60</td>
</tr>
<tr>
<td>17 US</td>
<td>0</td>
<td>37.78</td>
</tr>
<tr>
<td>20 US</td>
<td>0.5</td>
<td>32.60</td>
</tr>
<tr>
<td>21 US</td>
<td>1.0</td>
<td>30.20</td>
</tr>
<tr>
<td>19 US</td>
<td>2.0</td>
<td>26.00</td>
</tr>
</tbody>
</table>

Remarks

A, annealed; U, unannealed; All cross sections 1-1/2 square All slenderness ratios = 81 Stiffness of end restraints = 1850 kip inches per radian
Load applied with equal eccentricity at each end
\( e = \text{eccentricity} \)
\( r = \text{radius of gyration} \)
3. DISCUSSION AND RECOMMENDATIONS

The theoretical treatment of restrained columns for the cases where the members are subjected to a purely axial load up to the instant of buckling may be considered to be in a relatively satisfactory state as a result of the investigations reported in Ref. 5, 8, 26, 16, 17 and 18. The information supplied by these references is applicable to the buckling of members in a plane framework. The theoretical treatment of structures in which the columns are normally called upon to support bending as well as axial loads is not nearly as complete.

For the latter case extensive work has been carried out with the purpose of determining the collapse load of isolated columns with end restraints. The theoretical solutions for isolated restrained columns that have been developed so far are only applicable to columns of rectangular or annular cross sections and to an I cross section for bending about the minor axis. With the exception of Bijlaard's solution the method are also inadequate to cover the general case of unequal end restraints and unequal applied end moments.
A limitation of Bijlaard's approach is that it is an approximate one and also that it can be applied with confidence only to rectangular cross sections of steel. According to Ref. 11, 23, and 14, the shape of the cross section has a strong influence on the stability of unrestrained eccentrically loaded columns. In Ref. 8, Bleich suggests that this effect may be accounted for by a suitable shape factor and goes on to recommend shape factors for various sections ranging from 0.7 to 1.3. Bleich further states that his suggested shape factors must be considered as crude approximations and that the effect of the cross sectional form upon the buckling strength of eccentrically loaded columns is by no means sufficiently cleared up.

It became apparent from the foregoing remarks on isolated restrained columns that attention should be focused on the development of a general method for solving isolated restrained columns of any cross section. Advances in the techniques for the determination of moment-curvature relationships as outlined in Ref. 27 and 35 make an approach based upon a numerical integration procedure (as Chwalla adopted) highly desirable. To meet this need a general approach was developed in Ref. 29.
Only Ellis \(^{(13)}\) makes an attempt at extending the theory so that it will indicate whether a continuous column in a tier type structure will be safe. The proposed analysis of Ref. 13 considers that the beams of the structure have developed sufficient plastic hinges so that a column extending from the footing to the roof line may be considered as a continuous compression member with known moments and axial loads supplied at the supports. The spacings of the supports to this compression member correspond to the story heights of the structure. The analysis amounts to a complete elastic-plastic investigation of the continuous compression member and as such it is doubtful that it will find favor in design offices because of its complexity. Any further extension of this to include the restraining influence of the beams seems even more unlikely as far as a practical design procedure is concerned.

The extension of plastic analysis and design to non-sway multi-story frames may very well result in the adoption of a method of analysis which allows simple plastic theory to be used in the design of the beams together with a simplified analysis of the columns. Ref. 20 proposes such an analysis. It does not permit the stresses in the column to
The extension of this method at least for single axis bending so that some plasticity in the column will be permissible is desirable and possible by the use of nomographs similar to those introduced in Ref. 29. For example, the beams may first be designed according to the plastic method and the unbalanced beam moments at a joint may be distributed to the columns according to some approximation such as the one suggested in Ref. 20. The columns may then be selected so that when each segment between story elevations is considered, the column with the known end moments will be safe from collapse regardless of whether the yield stress is exceeded. This method would be suitable only when exceed the yield point and in addition it proposes a separate check of the column for lateral-torsional buckling in the elastic range. This method treats each column between story elevations as a separate entity and does not consider the continuity requirements at the column ends. In this respect the procedure represents the same philosophy of column design as is presently found in American building specifications. Several examples of columns designed according to the concepts of Ref. 20 are to be found in Ref. 1.
lateral-torsional buckling is not a factor. It will be appreciated that in this case the columns would be restraining the end rotations of the beams.

It is possible that design office methods in the foreseeable future will not consider column continuity since such procedures must of necessity amount to a complex elastic-plastic analysis of the entire structure. The further development of methods which discount part of the beneficial effects of continuity but which aim at utilizing the entire plastic strength appears likely.
ACKNOWLEDGEMENTS

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Fig. 1a

Fig. 1b

Fig. 1c

Fig. 2
Fig. 6

Fig. 7

Fig. 8
Comparison of observed and theoretical relationships between total axial load and central deflexion for a stanchion bent in single curvature.

Fig. 9
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