

1960

Analysis of frames loaded into the plastic range,
Proc. ASCE, 87 (EM4), (August 1961), Reprint
No. 182 (61-9)

M. Ojalvo

L. W. Lu

Follow this and additional works at: <http://preserve.lehigh.edu/engr-civil-environmental-fritz-lab-reports>

Recommended Citation

Ojalvo, M. and Lu, L. W., "Analysis of frames loaded into the plastic range, Proc. ASCE, 87 (EM4), (August 1961), Reprint No. 182 (61-9)" (1960). *Fritz Laboratory Reports*. Paper 1765.
<http://preserve.lehigh.edu/engr-civil-environmental-fritz-lab-reports/1765>

This Technical Report is brought to you for free and open access by the Civil and Environmental Engineering at Lehigh Preserve. It has been accepted for inclusion in Fritz Laboratory Reports by an authorized administrator of Lehigh Preserve. For more information, please contact preserve@lehigh.edu.

WELDED CONTINUOUS FRAMES AND THEIR COMPONENTS

ANALYSIS OF FRAMES LOADED INTO THE PLASTIC RANGE

by

Morris Ojalvo and Le-Wu Lu

This work has been carried out as a part of an investigation sponsored jointly by the Welding Research Council and the Department of the Navy with funds furnished by the following:

American Institute of Steel Construction
American Iron and Steel Institute
Office of Naval Research (Contract Nonr. 610(03))
Bureau of Ships
Bureau of Yards and Docks

Reproduction of this report in whole or in part is permitted for any purpose of the United States Government.

Fritz Engineering Laboratory
Department of Civil Engineering
Lehigh University
Bethlehem, Pennsylvania

December 1960

Fritz Engineering Laboratory Report No. 276.6

SYNOPSIS

In this paper a precise method of analyzing plane frames stressed into the inelastic range is presented. The method involves the determination of a "compatible" moment and rotation at a joint by the intersection of two moment versus end-rotation curves of the adjoining members. The effects of initial residual stresses and instability of beam-columns are considered in the analysis. It is shown how the behavior of frames may be predicted after the attainment of the ultimate loads. Throughout the paper the frames are assumed to fail by excessive bending in the plane of loading. A complete example is included to illustrate the technique developed.

TABLE OF CONTENTS

	Page
SYNOPSIS	i
TABLE OF CONTENTS	ii
1. INTRODUCTION	1
2. DEVELOPMENT OF THE METHOD	4
3. ILLUSTRATIVE EXAMPLE	11
4. SUMMARY	14
5. ACKNOWLEDGEMENTS	15
6. NOMENCLATURE	16
7. ILLUSTRATIONS	17
8. REFERENCES	26

1. INTRODUCTION

Two assumptions that are commonly made in the analysis of rigid frames, and which reduce the problem to a linear one in structural analysis, are that (a) the material follows Hooke's Law and (b) the deformations of the loaded structure are small in comparison with the dimensions. The latter assumption is cited to justify the formulation of the equilibrium equations on the basis of the undeformed structure.

References 1 and 2 consider the inelastic behavior of the structure but make no provisions for the effects of the deformations or the effects of the axial loads on the moment-curvature relationship of the members. Other studies which consider axial load and deformation effects are limited to cases where the stresses do not exceed the proportional limit in any part of the structure. References 3 and 4 fall into this category of elastic analysis. They are primarily concerned with the determination of sideways buckling load of symmetrical frames when the loads are not applied directly to the joints.

A third and more precise analysis, of which Refs. 5, 6 and 7 are representative, takes into account both inelastic action and deformations but in a somewhat restricted way. It ignores the effect of axial load on the

moment-curvature relationship and although the displacements of the joints and the deflections at points of concentrated load are considered, the influence of the displacements at intermediate points is not fully taken into account.

In spite of the good correlation with experimental results that the methods of Refs. 1,2,3,6 and 7 provide, the interest in the buckling of frames loaded into the inelastic range has made it desirable to develop a more precise method. The present method will consider inelastic action, axial load effects, and the exact shape of the loaded structure. The scope of the method presented here may be seen by referring to Fig. 1. A symmetrical frame having beam and column elements of specified cross sections is subjected to symmetrical loading as shown. For a specified beam length and varying column heights, the curves give the ultimate load according to the different modes of failure assumed and according to the methods used in the computation of the curves.*

At the specified column height on the chart of Fig. 1, line (a) gives the carrying capacity according to simple plastic theory. This theory assumes failure by symmetrical bending without sidesway. It ignores the reduction in the moment capacity of the columns due to

*In Fig. 1 the load P is equal to half of the total applied loads. It is equivalent to the axial force carried by each column.

axial load and due to the secondary moments in the columns resulting from their deformations.

Curve (b) assumes purely elastic behavior, with failure caused by sidesway buckling as shown in the right sketch of Fig. 1. Points on this curve are determined by the methods of Refs. 3 and 4.

Curve (c) gives the ultimate load according to the theory developed in this paper. Failure occurs by excessive symmetrical bending deformations; this mode of failure would occur when a lateral restraint at an upper joint prevents sidesway motion. As will be indicated in an example, the ultimate load is the largest for which an equilibrium configuration of the frame can be found.

Curve (d) gives the true ultimate load of the frame when it is not laterally restrained. In Ref. 8 a procedure is developed for determining a point on this curve. The method may be summarized as follows:

- (1) Assume a value for the load.
- (2) Perform an exact analysis according to the method described in the present paper. The analysis will give the bending stiffnesses of the frame members for the assumed loading.

- (3) Perform a stability analysis to determine whether positive work is required to deform the frame into an adjacent anti-symmetrical shape. This may be performed by the methods outlined in Ref. 9 (where only elastic frames are considered), using the stiffnesses of the partially plastic members (computed in step (2)) in the analysis.
- (4) If step (3) indicates a higher or lower buckling load than the one assumed in step (1), a lower or higher load respectively is assumed until the analysis of step (3) and the assumed load in step (1) are identical.

2. DEVELOPMENT OF THE METHOD

The theoretical analysis is subject to the following limitations:

- (1) The frame is symmetrical.
- (2) The loading is symmetrical.
- (3) The members are prismatic.
- (4) Only deformations due to flexure are considered.
- (5) The members of the structure are made up of columns which have no loads applied to

them except possibly at their ends, and beams which have negligibly small axial loads in them. Although allowance for the axial loads in the beams may be made by a trial procedure, the present analysis ignores these effects because of their small influence on the structural behavior. The effect of axial load in the columns is more critical and it is therefore taken into account.

- (6) The method assumes non-reversability of the stress-strain relationship. It is necessary to specify that during the loading there is no strain reversal of material stressed beyond the elastic limit.

A typical structure, shown in Fig. 2, will be used as an illustration of the proposed method. The frame is symmetrical about a vertical axis passing through point C. It is desirable at first to consider the columns and the beam of the frame separately.

In Fig. 3(a) the column element BA is shown with axial load P and moment M_{BA} applied to end B. Figure 3(b) shows how the rotation of end B varies as M_{BA} is slowly increased while the axial load P is maintained constant. A procedure by which the curve in Fig. 3(b) may be obtained follows from

the fact that a column such as BA may be considered as a segment of a column deflection curve. The column deflection curve is defined as the shape that a compressed member (identical to the column member except in length) will take when held in a bent configuration by axial loads applied to the ends. Such a column deflection curve is shown in Fig. 4. It may be identified by the angle θ_0 of the initial tangent. An infinite number of column deflection curves are possible for a single cross section and axial load. For each value of θ_0 the column deflection curve will have a different shape. If θ_0 is small so that the stresses in the column deflection curve do not exceed the proportional limit, the curve is a sine wave and its half wave length is determined by the Euler column formula:

$$L = \pi \sqrt{\frac{EI}{P}} \dots\dots\dots(1)$$

As θ_0 becomes larger, y_m increases. When the proportional limit is exceeded over some part of the column, the length shortens. The bending moment at any section is equal to $P \cdot y$. It will be noticed that the segment AB of the column deflection curve in Fig. 5 resembles the column AB in Fig. 3(a). The only difference between the two curves is that the component of the reaction at A in Fig. 3(a) along the direction AB is P while in Fig. 5 it is $P \cdot \cos \alpha$. This

difference will be examined.

In Fig. 3(a) the angle between the reactive forces at A and the direction AB is:

$$\tan \alpha = \frac{V}{P} = \frac{M_{BA}}{Ph} \dots\dots\dots(2)$$

In Fig. 5 the same angle is given by:

$$\tan \alpha = \frac{y_B}{h} = \frac{M_{BA}}{Ph} \dots\dots\dots(3)$$

It is seen that the angles between the total thrust at A and the direction AB are the same for the segment of the column deflection curve and the actual column.

Next the magnitude of these thrusts will be examined.

In Fig. 3(a) the thrust is:

$$R = \sqrt{P^2 + V^2} \dots\dots\dots(4)$$

When V is equal to 20% of P, R exceeds P by only 2%. With V equal to 10% of P, R exceeds P by 1/2% and for smaller ratios of V to P, R approaches P very rapidly. The above consideration makes it possible to consider the segment of Fig. 5 equivalent to the column of Fig. 3(a) for all practical cases.

There remains the problem of constructing several column deflection curves for the actual column load. A

numerical integration procedure which is based upon the moment-curvature ($M-\phi$) relationship of the column is best employed here because it is easily adaptable to different cross sections and material properties. The moment-curvature relationship is first determined for the actual axial load, material properties, and column cross section. General methods for determining the $M-\phi$ curve and performing the integration for the column deflection curve are outlined in Ref. 9, pgs. 51-55. For rolled steel wide-flange sections with residual stresses due to uneven cooling, Ref. 10 provides information for the determination of the $M-\phi$ curve. The integration gives the slope and deflections at intervals along the column deflection curve corresponding to the interval of length used for the integration. It has been noted in Ref. 11 that with stresses below the proportional limit, the numerical integration gives lengths and maximum ordinates that are in error by less than 0.13%. In the comparisons of Ref. 11 an interval of length equal to four times the radius of gyration of the section was used.

After several column deflection curves have been computed, the moment-rotation curve ($M-\theta$ curve) of Fig. 3(b) may be determined from them. Point B is located on each column deflection curve at a horizontal distance h from one end of the curve (Fig. 5). The moment corresponding

to M_{BA} is $P \cdot y_B$ while the angle θ is the slope of the tangent at B plus y_B/h . Each column deflection curve is thus seen to furnish the coordinates of a point on the $M-\theta$ curve in Fig. 3(b).

One can now consider beam member BD for the purpose of determining an $M_{BD}-\theta$ relation analogous to the $M_{BA}-\theta$ relation for the column. The construction of an $M_{BD}-\theta$ curve for the beam proceeds from the fact that the shear as well as the slope must be zero at the center of the beam*. The deformed shape of the beam in Fig. 6 will be referred to as a beam deflection curve. If one assumes an arbitrary internal bending moment at C (Fig. 6) the bending moment at distance x measured from C is:

$$M_x = -M_C + \frac{wx^2}{2} \dots\dots\dots(5)$$

A positive bending moment is one that causes tensile stresses on the outside of the frame. By using an integration procedure similar to that used for the columns it is possible to construct several beam deflection curves, each for a different value of the center moment M_C . For each beam deflection curve:

$$M_{BD} = -M_C + \frac{w\ell^2}{8} \dots\dots\dots(6)$$

 *When the center section of the beam is fully plastified (formation of a plastic hinge), the $M_{BD}-\theta$ curve becomes simply a straight line parallel to the θ axis; see Fig.9 for example.

The value of θ at x equal to $l/2$ is determined from the numerical integration. Each beam deflection curve therefore provides the coordinates of a point on the beam curve ($M_{BD}-\theta$) in Fig. 7. Since the axial load in the beam is small, a moment-curvature relation for zero axial load may be used for numerical integration without the introduction of an appreciable error. A further refinement to this part of the procedure would require that an estimate be made of V and that a slightly more appropriate $M-\theta$ curve be used for the beam curve integration.

When the two $M-\theta$ curves, one for the column and one for the beam, are plotted as in Fig. 7 the intersection will give the moment and rotation at B (or D) for the equilibrium configuration of the structure. In other words, of all the combinations of moment and rotation at B that are possible in the column and of all the combinations that are possible for the beam, the correct one is the one that is simultaneously compatible for the beam and the column. Once the correct M and θ at B are determined, it is possible to obtain the forces in the rest of the frame by statics. Deflections may be determined by interpolation from column and beam deflection curves yielding values near the correct values of M and θ .

The stiffness of member BA is given by:

$$K_{BA} = \frac{\Delta M_{BA}}{\Delta \theta} \dots\dots\dots(7)$$

On the basis of the Engesser concept of inelastic buckling, in which no unloading is assumed to take place at the instant of buckling, the stiffness of the column is given by the slope of the tangent to the column M- θ curve at its intersection with the beam M- θ curve (Fig. 7). The stiffness K_{BD} (or K_{DB}) for the member BD may be determined in the usual manner by considering the effective flexural rigidity (reduced due to yielding) of all the sections. Detailed procedure for computation can be found in Ref. 8.

3. ILLUSTRATIVE EXAMPLE

The behavior of the portal frame shown in Fig. 8 will be investigated in this section by using the procedure described above. The dimensions and member sizes of the frame were chosen arbitrarily. The stress-strain diagram for the rolled steel members is assumed to be ideally elastic-plastic with a yield stress level at 33,000 psi. The M- θ curves that were used in the computations were taken from Ref. 10. These curves assumed a well defined residual stress pattern in which a maximum

residual stress of about 10,000 psi occurred at the tips of the flanges. Figure 9 shows $M_{BA}-\theta$ curves obtained from Ref. 11. The $M_{BD}-\theta$ curves for the beam (Fig. 10) were taken from Ref. 8.

The ratio of the force F applied directly to the column top to the distributed load $w\ell/2$ on the beam, γ , is arbitrarily maintained at 25. Solutions for several values of w , corresponding to several stages in the proportional loading of the structure, will be obtained.

The values of w that are considered are $w = 1.22$, 2.03, 3.05 and 4.06 kips per foot. The corresponding average compressive stresses in the columns are $P/A = 3960$, 6600, 9900 and 13,200 psi.* By combining the curves of Figs. 9 and 10, values of θ and M_B are obtained for each value of w (Fig. 11).

It will be noticed from Fig. 11 that intersections of beam and column $M-\theta$ curves occur in pairs until the load reaches a certain maximum value for the frame. At this maximum load the beam and column $M-\theta$ curves become tangent. For a frame which is restrained against a premature sideways buckling failure, a higher load would produce no intersection and thus indicate that the capacity of the frame has been exhausted. For the frame of Fig. 8, the maximum value of w is estimated to

*The axial forces in the columns, corresponding to these values of stresses, are 0.12, 0.2, 0.3 and 0.4 P_y , where P_y is the axial yield load ($P_y = A\sigma_y$) of the column member.

be 4.11 kips per foot. The maximum value of w according to simple plastic theory is 4.94 kips per ft., indicating a reduction of 16.8% of the load-carrying capacity due to beam-column action. Figure 12, which shows the rotation of the joint at B with increasing load, gives a graphic indication of the non-linearity of the structural action.

It is of interest to know whether the ability of the structure to support load diminishes suddenly after the load attains its maximum value or whether the structure's load-carrying capacity is maintained until the deformations become very large. This information is obtainable from the second intersections of the $M-\theta$ curves which trace the load deformation characteristics after the maximum load has been reached. Figure 11 indicates that the unloading will be gradual as long as the $M-\theta$ curve for the column has a long flat portion at or near the maximum moment. It is known from a study of $M-\theta$ curves for rolled steel wide-flange columns that this will generally be true when the slenderness ratio remains below 60 and the average compressive stress is less than 40% of the yield stress of the steel (it is assumed here that the column does not fail except by excessive bending in its plane). (11)

4. SUMMARY

A new method of elasto-plastic analysis for rigid frames has been developed in this paper. It is based on the graphical determination of a moment and rotation at a joint by the intersection of two moment-rotation curves of the adjoining members. In constructing these moment-rotation curves, the effects of initial residual stresses and the instability of beam-columns may be included in the calculations. It is expected that the proposed method would yield results more precise than those obtained by other currently available methods.

The chief importance of the method is that it provides information by which the problem of sidesway buckling of frames in the plastic range may be investigated. It can also be used to determine the carrying capacity of a frame subjected to high axial forces in the columns, but restrained against sidesway movement. It is also of interest when it is desirable to study the behavior of a frame after the attainment of the ultimate load.

The method described herein has been illustrated by reference to a pinned-base symmetrical frame. It is equally valid to the solution of the several frames shown in Fig. 13. In each case appropriate sets of moment-

rotation curves of the beam and column are first constructed, the joint moments are then determined by the intersection of these curves.

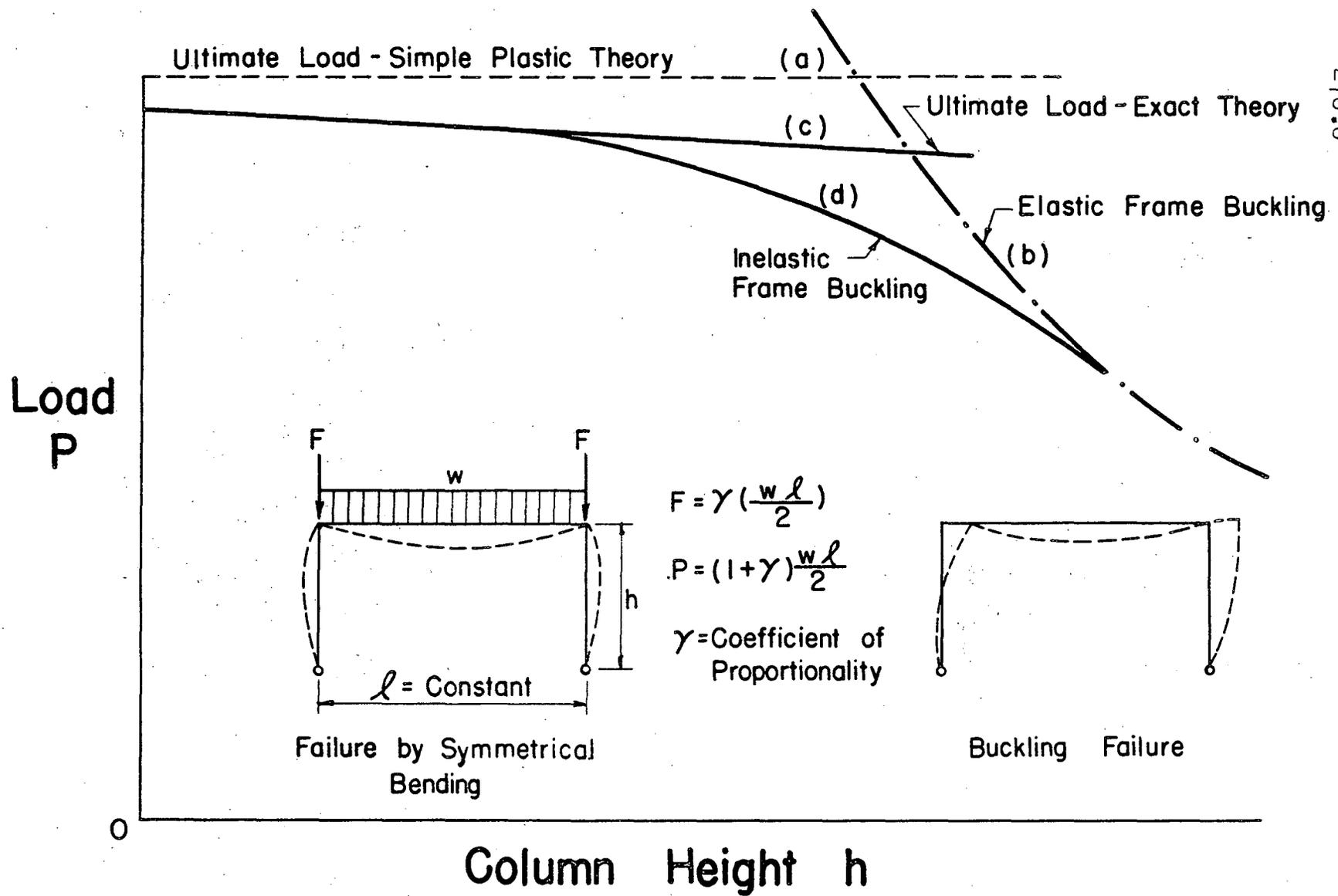
5. ACKNOWLEDGEMENTS

This study is part of the general investigations "Welded Continuous Frames and Their Components" currently being carried out at the Fritz Engineering Laboratory of the Civil Engineering Department of Lehigh University under the general direction of Lynn S. Beedle. The investigation is sponsored jointly by the Welding Research Council and the Department of the Navy, with funds furnished by the American Institute of Steel Construction, American Iron and Steel Institute, Office of Naval Research, Bureau of Ships, and Bureau of Yards and Docks.

The authors wish to express special thanks to Theodore V. Galambos and George C. Driscoll, Jr. for their constructive suggestions which have been incorporated in this paper.

6. NOMENCLATURE

A	= Area of a column cross section (in^2)
E	= Modulus of elasticity (ksi)
F	= Concentrated load applied to the top of a column (kips)
I	= Moment of inertia about axis of bending (in^4)
K_{BA}, K_{BD}	= Stiffness of frame member ($\frac{\text{ft-kips}}{\text{radian}}$)
L	= Half wave length of a column deflection curve (in)
M, M_C, M_x	= Internal bending moments (ft-kips)
M_{BA}, M_{BD}	= Joint moments (ft-kips)
P	= Axial load in column (kips)
R	= Resultant thrust in column (kips)
V	= Shear in column, horizontal reactive force at frame support (kips)
h	= Column length (ft)
ℓ	= Span length (ft)
w	= Distributed load on the beam ($\frac{\text{kips}}{\text{ft}}$)
x, y	= Coordinates
y_m	= Maximum deflection of a column deflection curve (in)
y_B	= Deflection at B on a column deflection curve (in)
$\theta, \theta_B, \theta_C$	= Rotation of a joint (radians)
θ_0	= Initial slope on a column deflection curve (radians)
ϕ	= Curvature of a member ($\frac{\text{radians}}{\text{in.}}$)
Δ	= Symbol indicating a small increment
α	= Arctan y_B/h or Arctan V/P (radians)
γ	= Coefficient of proportionality



276.6

FIG.1 LOAD-CARRYING CAPACITY OF FRAMES

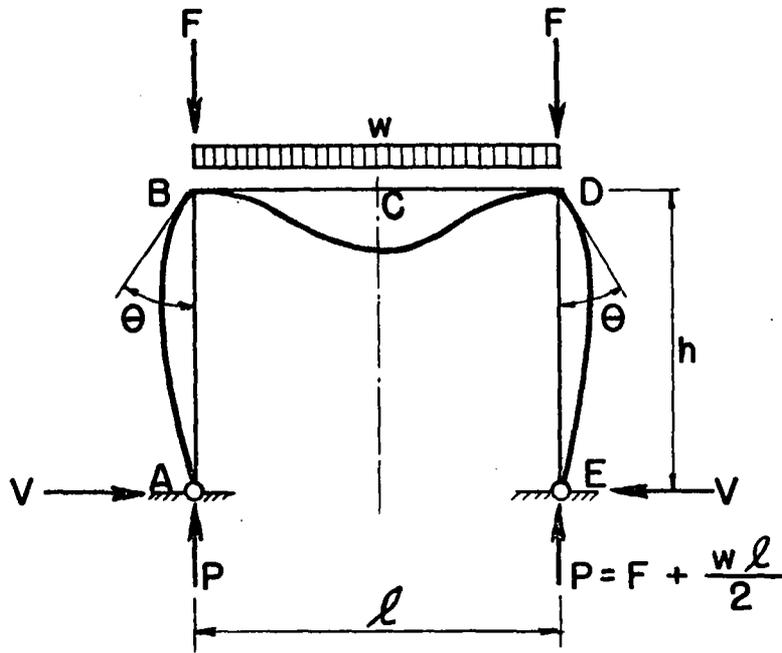


FIG.2

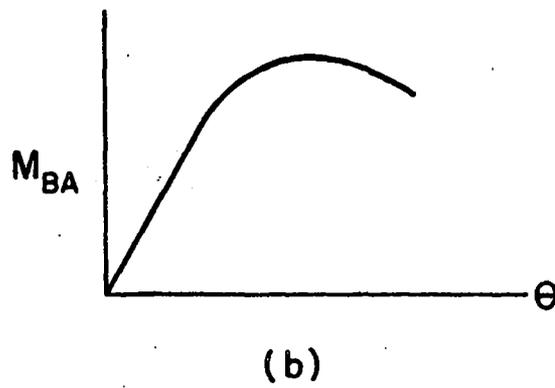
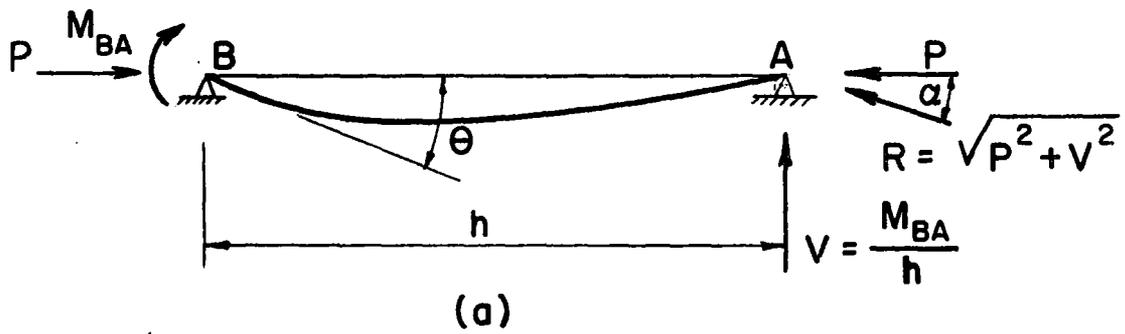


FIG.3

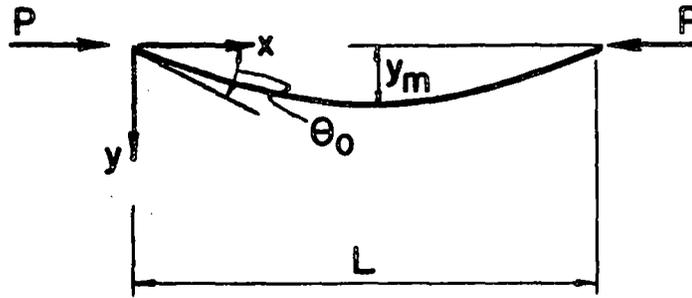


FIG.4 COLUMN DEFLECTION CURVE

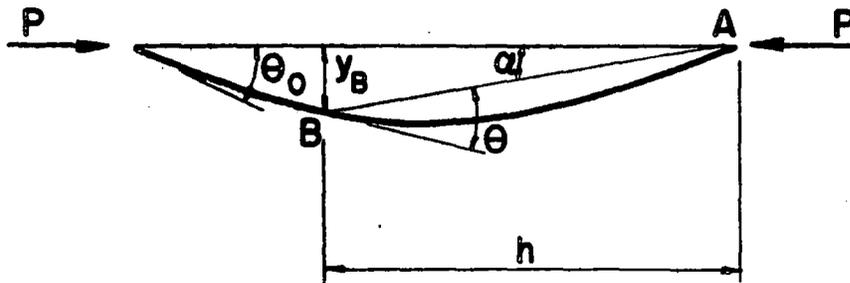


FIG.5

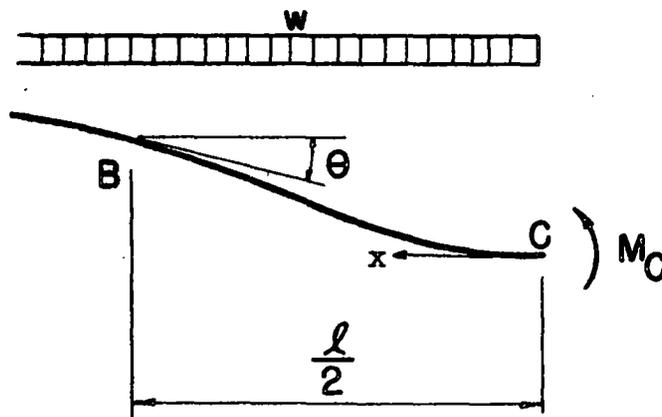


FIG.6 BEAM DEFLECTION CURVE

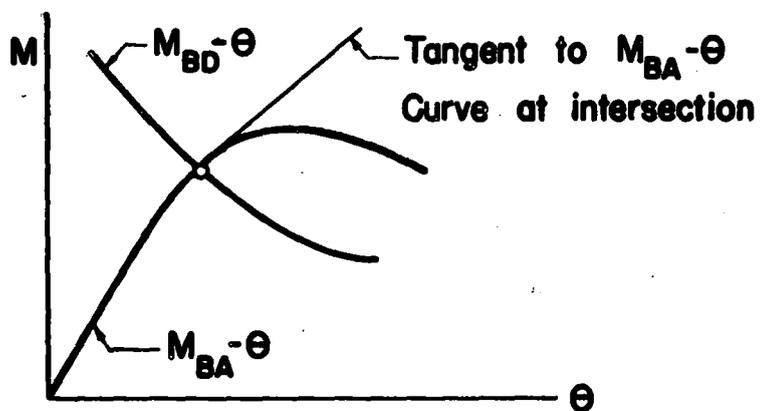


FIG. 7

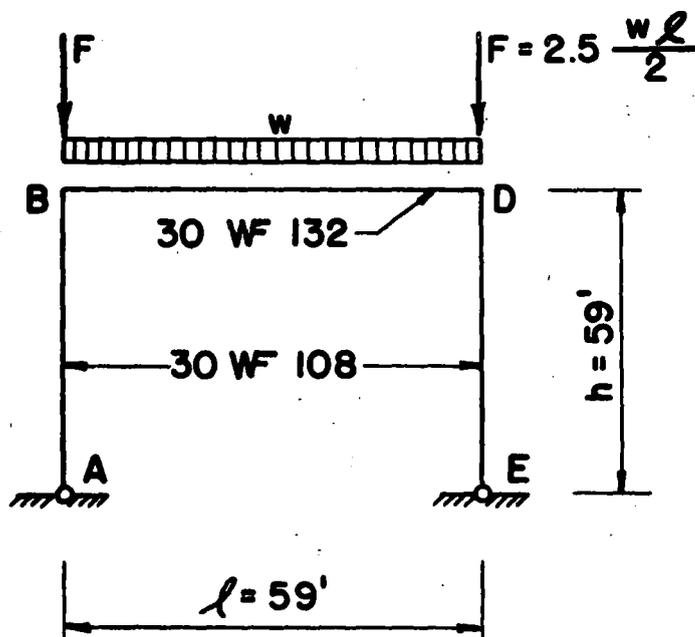


FIG. 8 ILLUSTRATIVE EXAMPLE

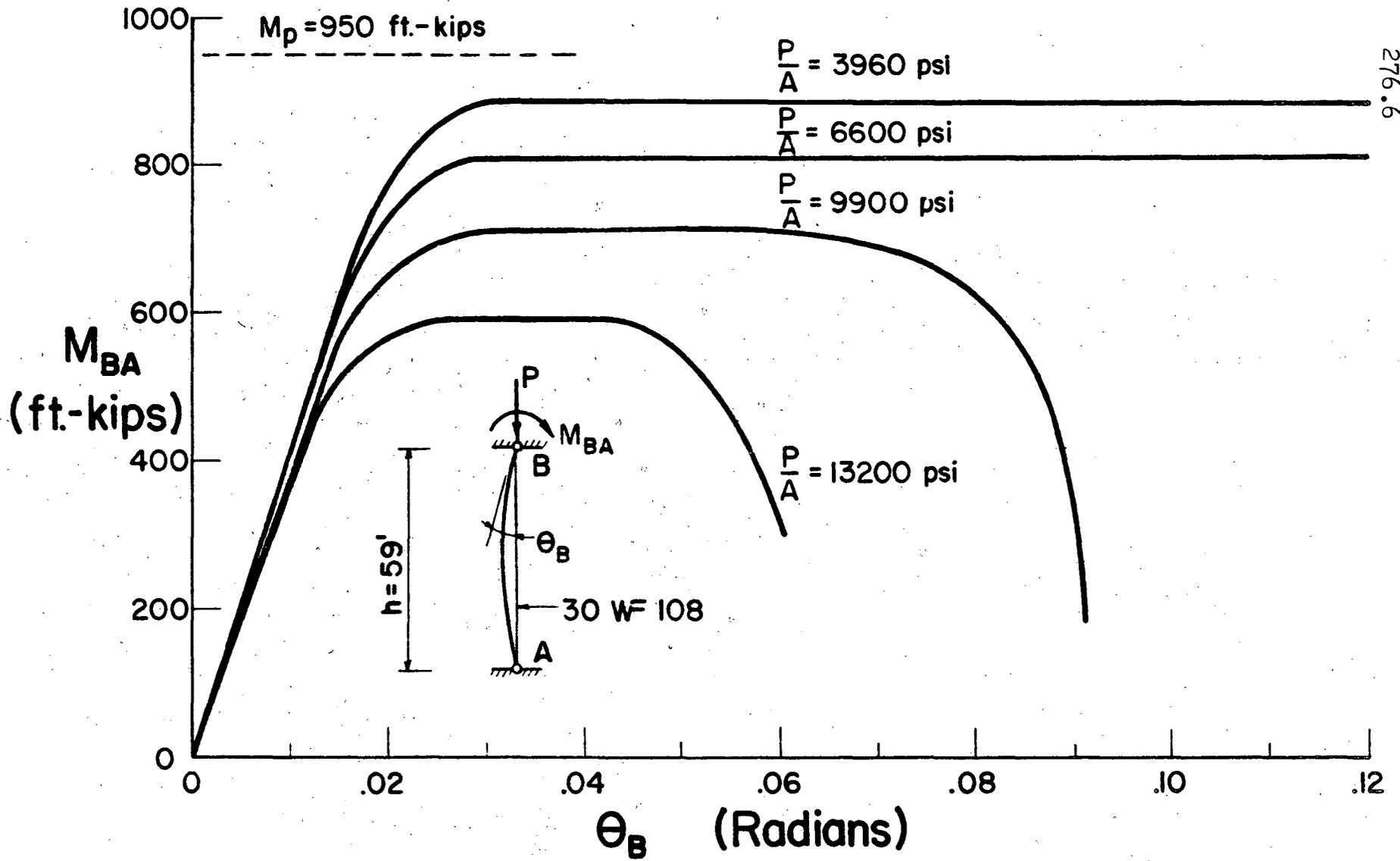


Fig. 9 $M_{BA}-\theta_B$ CURVES OF THE COLUMN

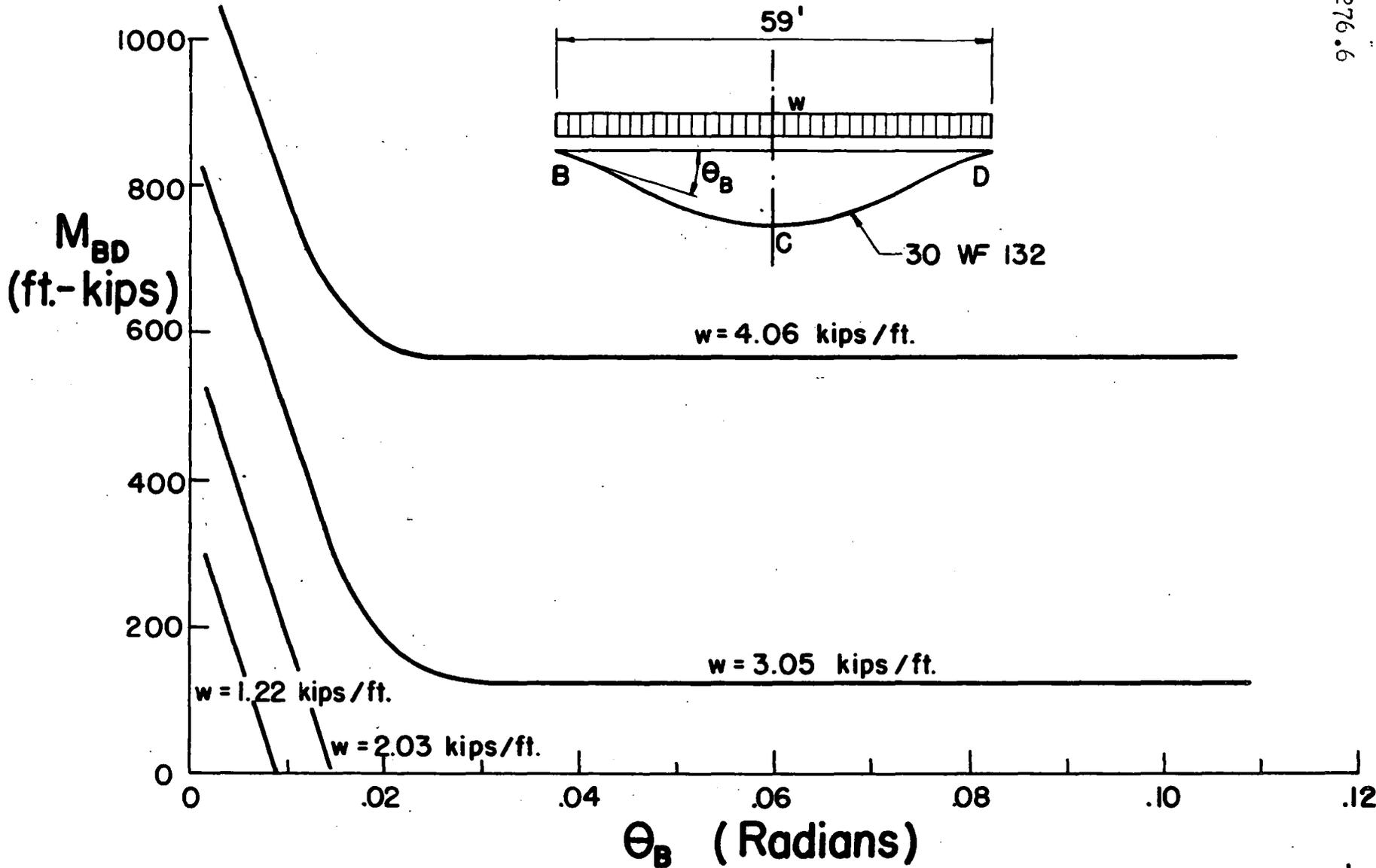


FIG.10 $M_{BC}-\theta_B$ CURVES OF THE BEAM

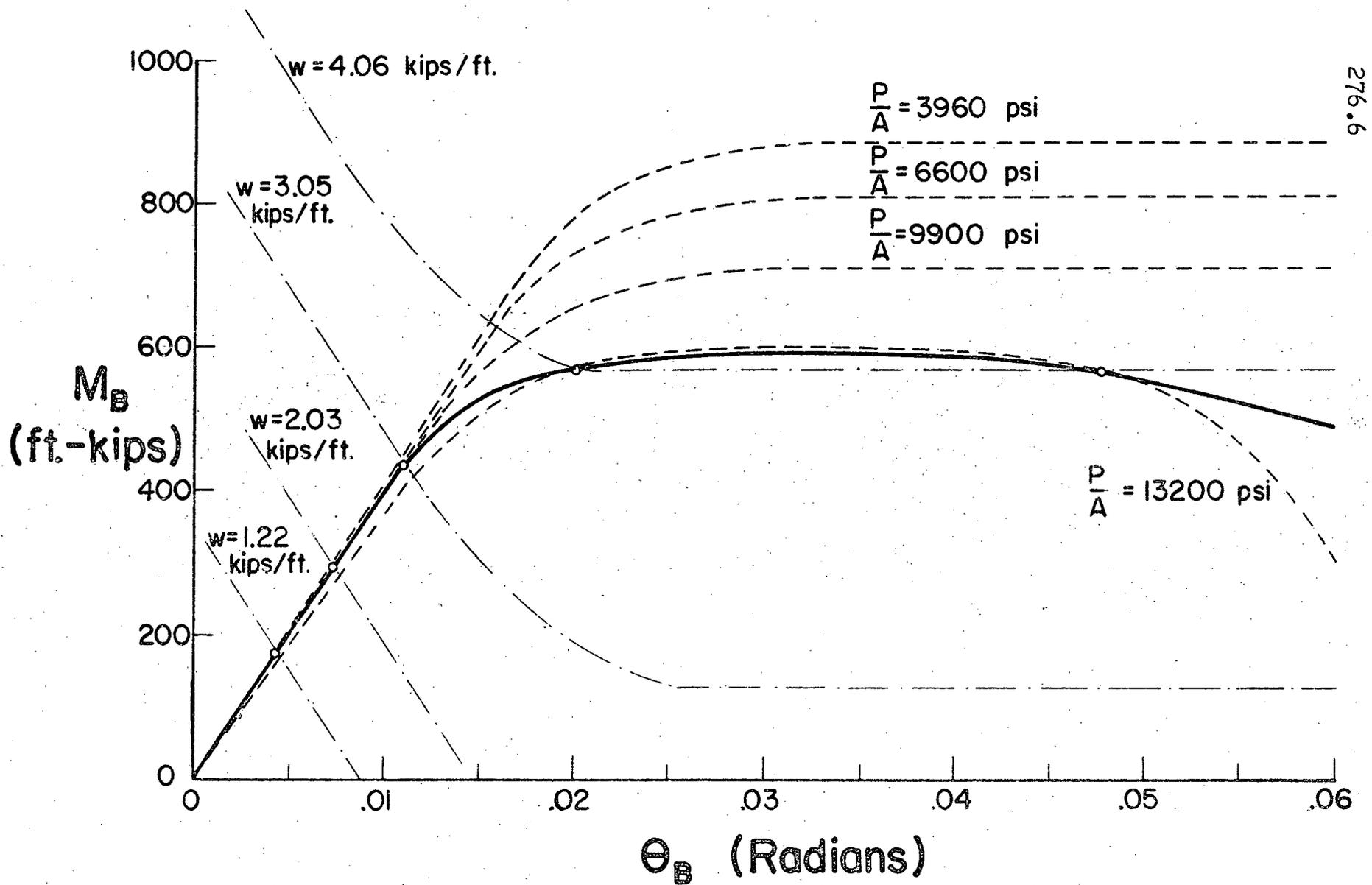
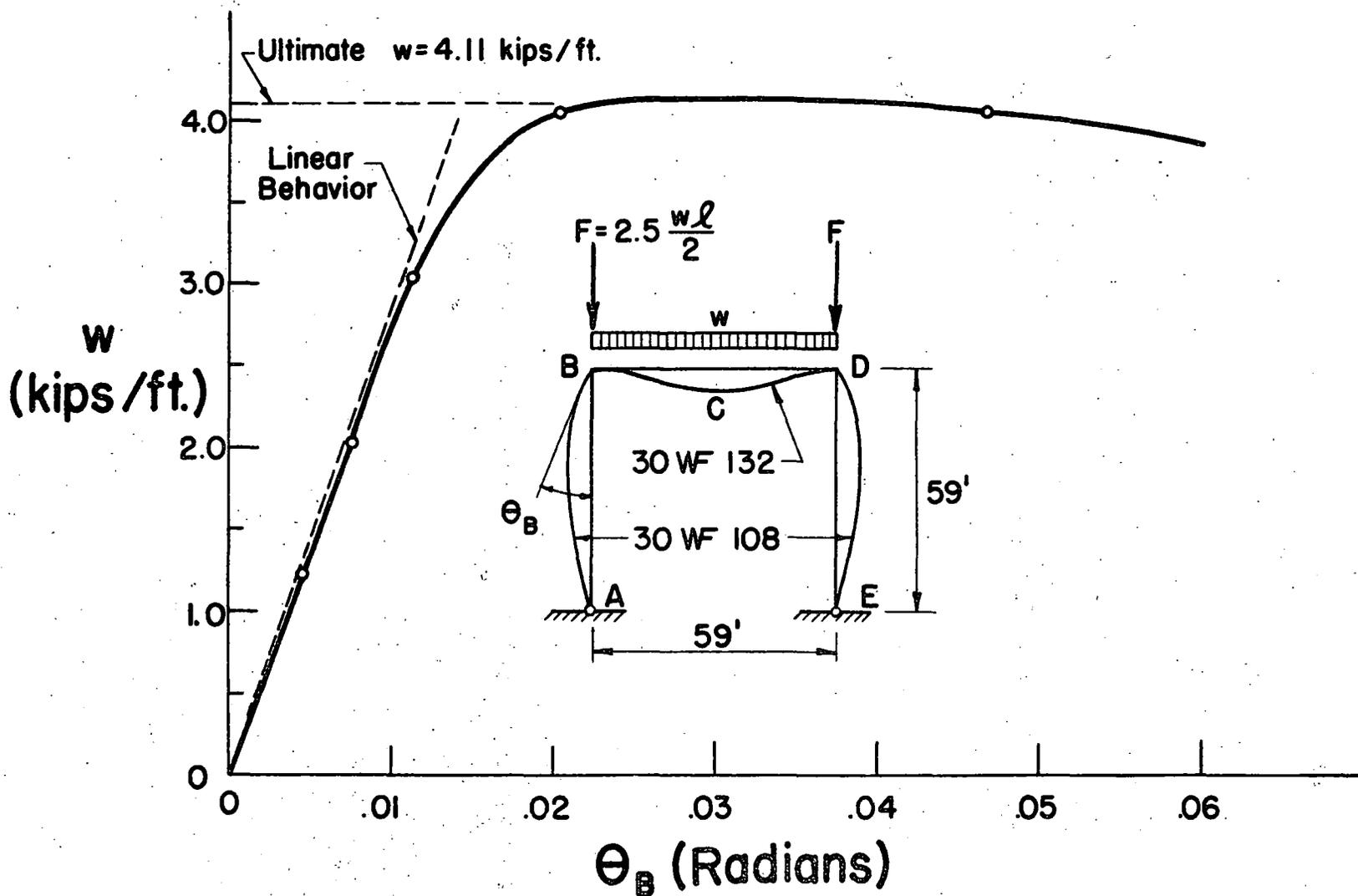


FIG.11 MOMENT VS. JOINT ROTATION RELATIONSHIP



276.6

FIG.12 LOAD VS. JOINT ROTATION RELATIONSHIP

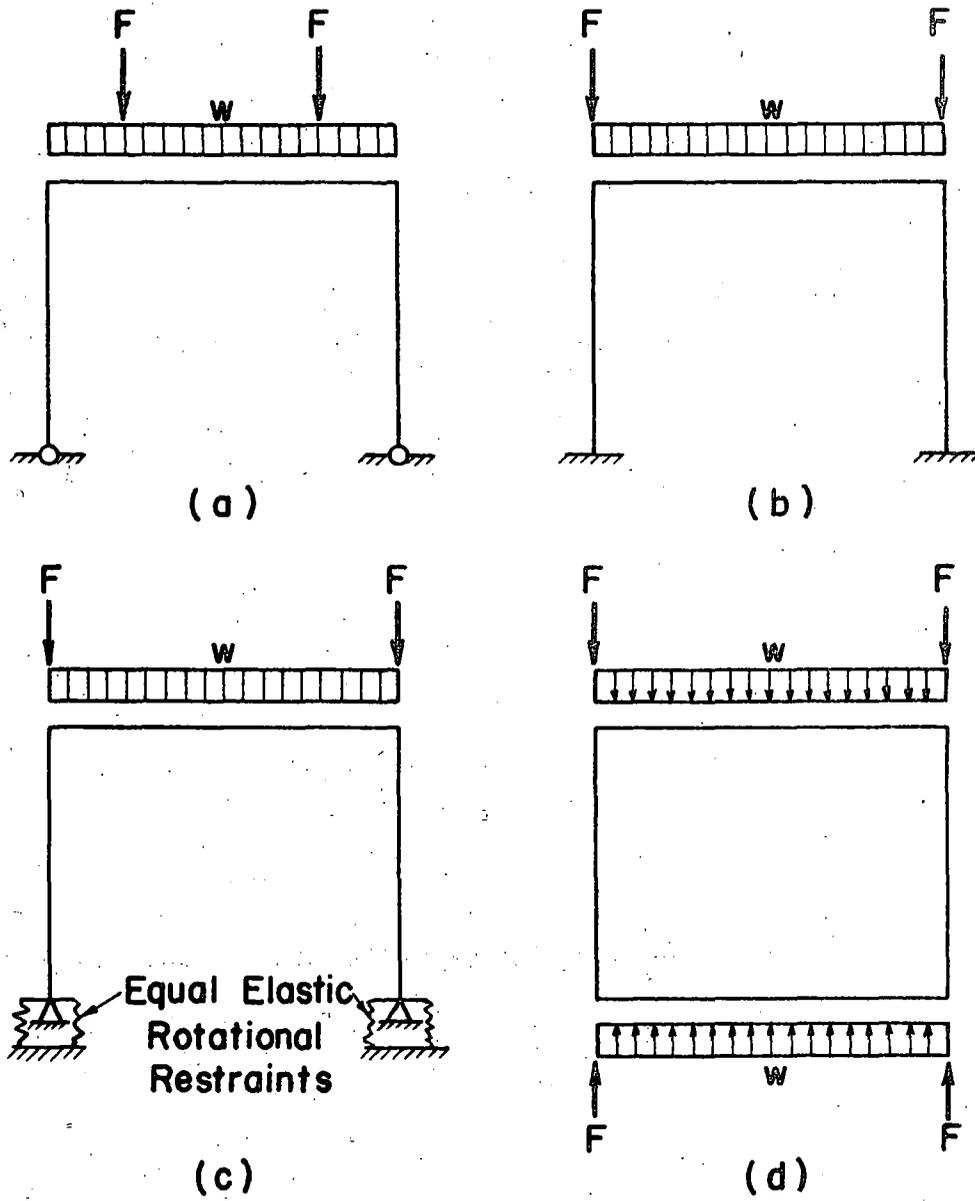


FIG.13

8. REFERENCES

1. Hrennikoff, A.
THEORY OF INELASTIC BENDING WITH REFERENCE TO
LIMIT DESIGN, Transactions of the ASCE, 113,
p.213 (1948)
2. Sawyer, H. A. Jr.
ELASTIC-PLASTIC DESIGN OF SINGLE-SPAN BEAMS AND
FRAMES, Proceedings of the ASCE, 81, (Proc. Sep.
No. 851) (December 1955)
3. Chwalla, E.
DIE STABILITAT LOTRECHT BELASTETER RECHTECKRAHMEN,
Der Bauingenieur, 19, p.69 (1938)
4. Masur, E. F.; Donnell, L. H.; and Chang I. C.
STABILITY OF FRAMES IN THE PRESENCE OF PRIMARY
BENDING MOMENTS, Paper presented at the Conference
on Structural Mechanics, Purdue University (1960)
5. Ang, A. H.
ANALYSIS OF FRAMES WITH NON-LINEAR BEHAVIOR, Pro-
ceedings of the ASCE, 86 (EM3) p. 1 (June 1960)
6. Rawlings, B.
THE ANALYSIS OF PARTIALLY PLASTIC REDUNDANT STEEL
FRAMES, Australian Journal of Applied Science,
7(1) p. 10 (1956)
7. Roderick, J. W.
THE ELASTO-PLASTIC ANALYSIS OF TWO EXPERIMENTAL
PORTAL FRAMES, The Structural Engineer, 38 (8)
p. 245 (August 1960)
8. Lu, L. W.
STABILITY OF ELASTIC AND PARTIALLY PLASTIC FRAMES,
Ph.D. Dissertation, Lehigh University, 1960
9. Timoshenko, S.
THEORY OF ELASTIC STABILITY, McGraw-Hill Book Co.,
New York, 1936
10. Ketter, R. L.; Kaminsky, E. L.; and Beedle, L. S.
PLASTIC DEFORMATION OF WIDE-FLANGE BEAM COLUMNS,
Transactions of the ASCE, 120, p.1028 (1955)
11. Ojalvo, M.
RESTRAINED COLUMNS, Ph.D. Dissertation, Lehigh
University, 1960