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Buckling Strength of Elastic Frames
By Le-Wu Lu

1. Theoretical Analysis

The modified three-moment equation method is used to solve the problem.

\[
\frac{L_i}{EI_i} S_i M_i^l + \left( \frac{L_i}{EI_i} C_i + \frac{L_{i+1}}{EI_{i+1}} C_{i+1} \right) M
\]

\[
+ \frac{L_{i+1}}{EI_{i+1}} S_{i+1} M_{i+1}^r + (\delta p_{i+1} - \delta p_i) + (F_{i+1}^r - F_i^l) = 0 \quad \cdots (a)
\]

When a structure passes from one deformation configuration into another, there will be changes of reactions and deformations, these changes will in turn cause variations of moments \( \Delta M \), axial forces \( \Delta P \), and the member rotations \( \Delta \theta \). Consequently, there will be variations of \( s \), \( c \), and \( F \), of each member due to the change of axial force.

\[
\frac{L_i}{EI_i} \left( S_i \Delta M_i^l + \Delta S_i M_i^l + C_i \Delta M_i^r + \Delta C_i M_i^r \right)
\]

\[
+ \frac{L_{i+1}}{EI_{i+1}} \left( C_{i+1} \Delta M_{i+1}^l + \Delta C_{i+1} M_{i+1}^l + S_{i+1} \Delta M_{i+1}^r + \Delta S_{i+1} M_{i+1}^r \right)
\]

\[
+ (\Delta p_{i+1} - \Delta p_i) + (\Delta F_{i+1}^r - \Delta F_i^l) = 0 \quad \cdots (b)
\]

Instability will occur when the applied load has reached such a value that both of these equations are satisfied.
Before buckling

\[ P = a \left( \frac{WL}{2} \right) \]
\[ \bar{P} = (1 + a) \frac{WL}{2} \]

Immediately after buckling

From the symmetry

\[ V_L = V_R = \bar{P} \]
\[ M_L = M_R = -H \bar{P} \]  \hspace{1cm} (1)

From (a)

\[ \left( \frac{h}{EI_c} C_1 + \frac{L}{EI_b} C_2 \right) M_L + \frac{L}{EI_b} S_2 M_R + F_2 = 0 \] \hspace{1cm} (2)

where the subscripts 1 and 2 refer to members AB and BD.

\[ C_1 = \frac{1}{(k_1 h)^2} \left( (-k_1 h \cot k_1 h) \right) \]
\[ C_2 = \frac{1}{(k_2 L)^2} \left( 1 - k_2 L \cot k_2 L \right) \]
\[ S_2 = \frac{1}{(k_2 L)^2} \left( \frac{k_1 L}{\sin k_1 L} - 1 \right) \]
\[ F_2 = \frac{W L}{H} \left( \frac{1}{k_2 L} \left( 1 - \cos k_2 L \sin k_2 L - \frac{1}{2} \right) \right) \]
Then equation (2) takes the form

\[
\frac{H}{P} \left( 1 - k_1 \cot k_1 h \right) + k_2 h \tan \frac{k_2 L}{2} - \frac{W L}{H} \left( \frac{1}{k_2 L} \tan \frac{k_2 L}{2} - \frac{1}{2} \right) = 0 \quad \cdots (2)
\]

For the case with roof load \( w \) only, then \( P = \frac{w L}{2} \), and eq. (4) takes the form

\[
\frac{2H}{wL} \left( 1 - k_1 \cot k_1 h \right) + k_2 h \tan \frac{k_2 L}{2} - \frac{W L}{H} \left( \frac{1}{k_2 L} \tan \frac{k_2 L}{2} - \frac{1}{2} \right) = 0 \quad \cdots (5)
\]

Next consider the equilibrium of members when the frame is in the state of unstable equilibrium, the following equations can be readily obtained.

\[
\begin{align*}
V_L + \Delta V_L &= \bar{P} - 2\bar{P} \frac{\Delta R}{L} \\
M_L + \Delta M_L &= -(H + \delta H) h + V_L \Delta R \\
M_R + \Delta M_R &= -(H + \delta H) h + V_R \Delta R
\end{align*}
\]

Combining eq. (6) with eq. (4) leads to

\[
\begin{align*}
\Delta V_L &= -2\bar{P} \frac{\Delta R}{L} \\
\Delta V_R &= 2\bar{P} \frac{\Delta R}{L} \\
\Delta M_L &= -\delta H h + \bar{P} \Delta R \\
\Delta M_R &= -\delta H h - \bar{P} \Delta R
\end{align*}
\]

or

\[
\begin{align*}
\frac{\partial V_L}{\partial R} &= -2 \frac{\bar{P}}{L} & \frac{\partial V_L}{\partial H} &= 0 \\
\frac{\partial V_R}{\partial R} &= 2 \frac{\bar{P}}{L} & \frac{\partial V_R}{\partial H} &= 0 \\
\frac{\partial M_L}{\partial R} &= \bar{P} & \frac{\partial M_L}{\partial H} &= -h \\
\frac{\partial M_R}{\partial R} &= -\bar{P} & \frac{\partial M_R}{\partial H} &= -h
\end{align*}
\]

To obtain the relationship between $H$ and $P$, immediately after buckling, the three-moment equation for neutral equilibrium will be used. For members AB and BD, eqn. (6) becomes

$$\frac{h}{EI_c} (C_1 AM_L + AC_M L) + \frac{L}{EI_b} (C_2 AM_L + AC_M L + S_2 AM_R + AC_M R) - \frac{AR}{h} + \Delta F_2 = 0 \quad \ldots \ldots (9)$$

Similarly, the equation for members BD and DE is

$$\frac{L}{EI_b} (S_2 AM_L + AC_M L + C_2 AM_R + AC_M R) + \frac{h}{EI_c} (C_1 AM_R + AC_M R) + \frac{AR}{h} + \Delta F_2 = 0 \quad \ldots \ldots (10)$$

From eq. (8) $\Delta R$, $\Delta H$ are the independent variables of the system. So the variation of axial force $P$ in each member can be expressed in terms of these variables, that is

$$\Delta P = \frac{\partial P}{\partial R} \Delta R + \frac{\partial P}{\partial H} \Delta H \quad \ldots \ldots (11)$$

The variation of $C_i$ for member AB is derived as follows

$$\Delta C_i = \frac{dC_i}{dP} \cdot dP = \frac{dC_i}{dP_1} \left( \frac{\partial P}{\partial R} \Delta R + \frac{\partial P}{\partial H} \Delta H \right) = -C_i ' \cdot 2 \cdot \bar{P} \cdot \frac{\Delta R}{L} \quad \ldots \ldots (12)$$

where $C_i ' = \frac{dC_i}{dP_1}$

In a similar manner, the following expressions for $\Delta C_2$, $\Delta S_2$, $\Delta C_3$ are obtained:

$$\Delta C_2 = C_2 ' \Delta H$$
$$\Delta S_2 = S_2 ' \Delta H$$
$$\Delta C_3 = -\Delta C_1$$
Then eq. (9), and (10) becomes

\[
\left[ \frac{h}{EI_c} c_1 + \frac{L}{EI_b} (c_2 - s_2) + \frac{h}{EI_c} c_1' \frac{2Hh}{L} - \frac{1}{P h} \right] \bar{P} \Delta R
\]

\[
- \left[ \frac{h}{EI_c} c_1 + \frac{L}{EI_b} (c_2' + s_2) + \frac{H L}{EI_b} (s_2' + c_2') \right] h \Delta H + \Delta F = 0 \quad \ldots \quad (14)
\]

\[
- \left[ \frac{h}{EI_c} c_1 + \frac{L}{EI_b} (c_2 - s_2) + \frac{h}{EI_c} c_1' \frac{2Hh}{L} - \frac{1}{P h} \right] \bar{P} \Delta R
\]

\[
- \left[ \frac{h}{EI_c} c_1 + \frac{L}{EI_b} (c_2' + s_2) + \frac{h}{EI_b} (c_2' + s_2') \right] h \Delta H + \Delta F = 0 \quad \ldots \quad (15)
\]

Subtracting Eq. (15) from (14) leads to:

\[
\left[ \frac{h}{EI_c} c_1 + \frac{L}{EI_b} (c_2 - s_2) + \frac{h}{EI_c} c_1' \frac{2Hh}{L} - \frac{1}{P h} \right] \bar{P} \Delta R = 0 \quad \ldots \quad (16)
\]

Since \( \Delta R = 0 \) after buckling, the expression in bracket has to be zero,

\[
\frac{h}{EI_c} c_1 + \frac{L}{EI_b} (c_2 - s_2) + \frac{h}{EI_c} c_1' \frac{2Hh}{L} - \frac{1}{P h} = 0 \quad \ldots \quad (17)
\]

From (3),

\[
(c_2 - s_2) = \frac{1}{(k_i L)^2} (2 - k_i L \cot \frac{k_i L}{2})
\]

\[
c_1' = \frac{dc_1}{dp}, \quad c_1' = \frac{dk_i}{dp}, \quad c_1' = \frac{dk_i}{dp},
\]

\[
= \frac{1}{2P (k_i h)^2} [k_i h \cot k_i h + \frac{(k_i h)^2}{\sin^2(k_i h)} - 2]
\]

These equations substitutes into eq. (17), the stability condition is finally obtained in the form:

\[
P_i h \cot k_i h - \frac{P h}{H L} (2 - k_i L \cot \frac{k_i L}{2}) + \frac{H h}{P L} (2 - k_i h \cot k_i h - \frac{k_i h}{\sin^2 k_i h}) = 0
\]

\[
\ldots \quad (19)
\]

This equation is same one which was developed by Bleich.

See Bleich's book, p. 230, eq. (4:39)
Numerical methods of determining the buckling load.

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STABILITY OF ELASTIC AND PARTIALLY PLASTIC FRAMES

by

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ABSTRACT

The objective of this dissertation is to determine the buckling strength of pinned-base portal frames in the elastic and partially plastic range. The frames are acted upon by a uniform load on the beam and two equal concentrated loads on the columns. It is assumed that failure occurs by sidesway instability in the plane of the applied loads.

It is first pointed out that frame design may be based on one of the following three types of limiting strength: the plastic strength, the inelastic buckling strength and the elastic buckling strength. The importance of developing methods for predicting these strengths is indicated.

An exact solution for elastic buckling of portal frames is obtained. Using this solution, the effect of primary bending moment present in each member at the instant of buckling is clarified. The solution is verified by buckling tests on model steel frames.

A precise method of elastic-plastic analysis for rigid frames is developed. It takes into consideration such effects as axial thrust, yielding, deformations, and residual stresses. The method may be used to determine the stiffness of columns at any stage of loading and also the strength of frames which are prevented from sidesway movement.
Utilizing the results from the elastic-plastic analysis, the problem of sidesway buckling of partially yielded frames is solved. The method of solution is based on a modified moment distribution procedure in which all the necessary constants are modified for the combined effects of axial force and inelastic action. The method is illustrated in connection with the development of a frame buckling curve.

A discussion on the application of the solutions obtained to the design of building frames is included. A method whereby design charts for elastic and inelastic buckling may be derived is also presented.
1. INTRODUCTION

1.1 The Criteria of Structural Design

According to the present-day concept of limit design of engineering structures, three criteria for structural damage are generally considered to be important. One of these is the ultimate or plastic strength criterion. When this is used in designing steel structures the procedure is usually referred to as the "Plastic Method". Valuable advancement has been made in the past twenty years in this field. For a general introduction to the concepts involved in such methods, reference can be made to (1) and (2).*

A second criterion is the buckling strength. This has led to the concept of design for stability. The phenomenon of buckling can be described as the process by which any structure passes from one deformation configuration into another without a change in the load. This phenomenon may occur when the structure is stressed either below or above the elastic limit. Each case requires a different mathematical treatment. Although elastic buckling has been studied extensively in the past century, few solutions exist in the area of inelastic buckling. The knowledge accumulated in this

* The numbers in parentheses refer to references given at the end of the dissertation. They are listed in order of their occurrence.
field has been summarized by Bleich in Ref. (3) and by the Column Research Council in Ref. (4).

In addition to the above mentioned strength limits, the change of shape at any stage of loading could also be considered as the limit of usefulness of a structure. For any structure the deformation under a given load must not exceed a certain specified amount, or its function may be impaired. This is often referred to as a stiffness criterion. At present numerous methods are available for calculating the deformation of structures in the elastic as well as in the plastic range. (5) The allowable deflection of a particular type of structure is usually specified in design codes.

Ideally for any structural design, each member should be selected in such a way that all these criteria are met, and the structure as a whole should also fulfill these requirements. When the plastic method is adopted in design, it is a common practice to proportion the members on the basis of their plastic strength, then perform separate checks to see if the stability and stiffness criteria are satisfied. Thus the member sizes may have to be altered in order to fulfill all these conditions.

Of course there are other criteria that can be used as the design basis, such as design for brittle fracture, fatigue, dynamic energy absorption, etc. However, these conditions are not commonly used in designing civil engineering
structures. This dissertation will only be concerned with the proportioning of statically loaded structures to meet the strength, stability and deformation criteria.

1.2 The Phenomena of Frame Instability

One of the important assumptions made in the plastic theory of structural analysis is that no instability of any type should occur prior to the formation of the failure mechanism. In general three types of instability are encountered; these are:

(1) The local instability of elements which make up the cross section of the member.

(2) The instability of individual members under the action of axial force and bending moment.

(3) The overall instability of the whole structure.

Detailed analytical and experimental work on the first type of instability has clarified the problem and led to simple design recommendations. (6) In the second type of instability, problems involved may be further classified into two categories: (1) Instability of beam-columns due to excessive bending and (2) Lateral-torsional buckling. Extensive studies have been made on the problems in both categories in the past 50 years. Some recent developments in this field can be found in Refs. (7), (8) and (9).

The third type of instability is the subject of this
dissertation. The structure under investigation will be a single story rigid frame. The problem treated here is often referred to as "Frame Instability".

The phenomena of overall instability are illustrated in Fig. 1-1 for a portal frame which is not prevented from side-sway. The following two types of failure may occur:

(1) When a symmetrical frame is loaded by symmetrical vertical forces, it is possible that the frame may pass from a symmetrical stable deformation configuration to an unsymmetrical, unstable configuration (Cases 1 and 2). At this instant the total resistance to any lateral force or lateral movement becomes zero. The load-deflection curve is characterized by a sudden shift from a situation where the applied load can increase as the deformation increases to one where large deflection develops without an increase of load. The behavior is analogous to that of a centrally-loaded column, in which bifurcation of the equilibrium position is possible at a certain critical load.*

(2) When a frame (symmetrical or unsymmetrical) is subjected to a combination of horizontal and vertical forces (Case 3), or when an unsymmetrical frame carries beam loads, the frame deforms laterally upon the application of the first load. The resulting change in geometry may alter the carrying

* In this dissertation, no distinction will be made between buckling load and critical load.
capacity of the individual columns, since the column top is no longer directly over the column base and hence additional bending moments are introduced by the vertical load. The whole structure becomes unstable in this deformed position much like an eccentrically-loaded column. A load-deflection curve for this type of instability is shown as a dot-dashed line in Fig. 1-1. At a certain critical condition, the structure continues to deform with a decrease of load. At the present time only very limited information is available concerning this type of failure.

This dissertation deals only with frame instability problems of the first type. A close examination of Cases (1) and (2) may reveal that the nature of these two cases is quite different, even though they both may be considered as initial motion problems. In Case (1), where the loads are applied directly to the columns, the frame usually carries little or no primary bending moment. Therefore, only the column action alone needs to be considered in the buckling analysis. If the analysis is carried beyond the elastic limit, then the well-known tangent modulus concept for centrally-loaded columns may be used. A brief survey of the presently available methods in solving this type of stability problem is given in Art. 2.1.

However, most rigid frames are primarily designed to support loads by bending action such as Case (2) of Fig. 1-1.
All the members of the frame are subjected to both axial force and bending moment. Therefore, not only the column action should be taken into account in determining the critical loads but also the bending action. The problem becomes more complicated in that yielding may occur in part of the member prior to the occurrence of instability.

To clarify the problem further, a discussion of the load-deformation characteristics of such frames is given here.

Referring to Fig. 1-2, which shows a load-deflection curve of the frame under consideration, it is obvious that at any loading stage before buckling the structure is deformed in a symmetrical manner and that very little horizontal deflection at the column top may be expected. If the frame is properly braced against lateral movement, then the load can increase up to the ultimate load calculated by the plastic theory. For the case of an unbraced frame, there exists the possibility of sidesway buckling. This type of buckling may occur at a load level below the yield (shown as point A in Fig. 1-2), but more frequently would take place when the applied load has caused yielding in some portion of the frame (shown as point B). Since the formation of a plastic mechanism can also be considered as the limit of stability of the whole frame, then it is clear that for unbraced frames the inelastic buckling load should have a value between the yield and the ultimate loads.

The similarity between the buckling of centrally-loaded
columns and that of the frames investigated in this dissertation is illustrated in Fig. 1-3. It can be seen from this figure that for frames with slender columns sidesway buckling would occur before the stress at any point has reached the yield level. This situation is analogous to the elastic buckling of centrally-loaded columns (Euler's problem). For frames with columns of intermediate slenderness ratios buckling takes place when parts of the members have already been yielded. This is similar to the buckling of columns in the inelastic range (problem of Engesser and Shanley).

In obtaining the curves for frames the simple plastic load $w_u$ is taken as the maximum carrying capacity, corresponding to a beam mechanism. Curve (a) of Fig. 1-3b shows the actual load carrying capacity of such frames if sidesway is properly prevented. This curve is determined also for the beam mechanism but with the plastic hinge moment at the column tops reduced for the beam-column action. The inelastic buckling load of unbraced frames is plotted as curve (b). It may be expected that with the column slenderness ratio less than a certain limiting value the unbraced frames would carry approximately the same load as braced frames. Then within this limit the problem of frame stability may be disregarded and the plastic strength can be used as the design basis. If this is not the case the inelastic buckling strength should be considered as the proper criterion. For very tall frames it may be even necessary to design against the elastic
buckling strength (see curve (c) of Fig. 1-3b).

From the above discussion it is evident that in order to design unbraced frames efficiently, methods by which the elastic and the inelastic buckling strength of such frames can be precisely determined are necessary. Unfortunately after an extensive survey of literature, it was found that only very few cases of this type of buckling problem had been studied and that these studies were limited to the elastic range only. No attempt has yet been made to develop methods for predicting the buckling strength in the inelastic range. Some empirical formulas based on experimental results have recently been suggested. These will be discussed in Art. 2.2.

It is the purpose of this dissertation to investigate this type of instability problem extensively and present the results in a form that is suitable for design use.

1.3 Scope of Investigation

This dissertation contains the results of an investigation of the following four phases:

(1) Theoretical and experimental studies on the elastic buckling strength of single story rigid frames. Fig. 1-4 shows the dimensions of the frame and types of loading assumed in the investigation. The parameter \( \alpha \) which relates the magnitude of the concentrated load \( P \) to the intensity of the uniform load \( w \), is kept constant for each case. This
implies that these two types of load are assumed to increase simultaneously with a fixed ratio \( a \) between them. This particular loading condition would simulate a condition that may be expected in the lower stories of a tall building.

(2) A precise method of indeterminate analysis applicable to elastic as well as partially plastic structures.

(3) Development of a numerical method by which the inelastic buckling strength of frames as shown in Fig. 1-4 may be predicted.

(4) Construction of charts to facilitate design by utilizing the results obtained in phases (1), (2) and (3).

Throughout this dissertation, it is assumed that the frames are sufficiently braced in the lateral direction so that buckling can only occur in the plane of the applied loads.
2. THEORETICAL METHODS CURRENTLY AVAILABLE

In this chapter, brief descriptions of some general methods of approach which have been developed for the solution of various frame instability problems are presented. They merely serve to indicate the present knowledge in this field. An extensive survey will be contained in a forthcoming report prepared by the author.\(^{(11)}\)

2.1 Structures Carrying No Primary Bending Moment

This section will outline some important methods for determining the critical load of structures which are stressed principally by axial forces. All the methods resemble very much those ordinarily used in analyzing statically indeterminate structures, except that some modification is made to include the bending effect of the axial thrust. A systematic survey of the existing theory, covering both exact and approximate methods, was given by Kavanagh\(^{(12)}\).

2.1.1 Methods of Analysis and Criteria of Stability

Essentially three different approaches have been introduced to solved stability problems considered here. Those are (1) the analytical methods (2) the convergence methods and (3) the energy method. Each method is based on a particular type of stability criterion.
1) The Analytical Methods

The analytical method is often referred to as the determinant method. It includes all the procedures of analysis which involve the solution of a system of linear simultaneous equations. The unknowns in these equations are the additional forces and moments induced by buckling, and the joint displacements and rotations of the members. The coefficients of these unknowns are usually functions of the dimensions of the members and the axial forces. In general it is possible to set up as many equations as there are unknowns. The trivial solution of this system of equations represents the state of stable equilibrium. Finite values of the unknowns, indicating the existence of additional forces and deformation, can occur only when the determinant $\Delta$ of the system is equal to zero. There are an infinite number of solutions to the equation $\Delta = 0$, corresponding to an infinite number of possible unstable equilibrium configurations. The configuration which gives the smallest value of loading determines the critical load of the structure.

Depending on the unknown quantities selected, there are three different methods that can be used to set up these simultaneous equations, namely:

a) Bleich's four-moment equation method
b) The slope-deflection method, and
c) Mises method
The first two methods are commonly used and will be discussed briefly in the succeeding paragraphs. The procedure involved in using Mises Method can be found in Refs. (13) and (14).

In the four-moment equation method, the unknowns selected are the end moments, additional forces and the rotation of members \( \rho \). Referring to Fig. 2-1, which shows two consecutive members \( L_i \) and \( L_{i+1} \) of a structure before and after buckling, the four-moment equation can be shown to have the following form: \( ^{(3)} \)

\[
\frac{L_i}{E_0 I_i} (s_i M_i^t + c_i M_i^r) + \frac{L_{i+1}}{E_0 I_{i+1}} (c_{i+1} M_{i+1}^t + s_{i+1} M_{i+1}^r) + (p_{i+1} - p_i) = 0 \quad (2.1)
\]

where

- \( M = \) End moment
- \( L = \) Length of members
- \( \rho = \) Rotation of members

\[
S = \frac{1}{(kL)^2} \left[ \frac{kL}{\sin kL} - 1 \right], \quad C = \frac{1}{(kL)^2} \left[ 1 - kL \cot kL \right] \quad (2.2)
\]

and

\[
k = \sqrt{\frac{P}{E I}} \quad (2.3)
\]

\( P = \) Axial force in members

This equation represents a relation between the four end moments of the two adjacent members and the bar rotations \( \rho \). The values of \( s \) and \( c \) are tabulated in Refs. (3) and (15) as functions of \( kL \). Equations of this type form the first group of the stability equations. To solve an actual problem two
additional groups of equations are needed. They are obtained by considering 1) the geometrical relations between rotations and deformations, and 2) the equilibrium conditions of the structural system when it is in the state of unstable equilibrium. The existence of non-zero solutions of the variables in these sets of equations defines the limit of stability.

The slope-deflection method for stability analysis was first adopted by Chwalla and Jokisch for analyzing multi-story frames.\(^{(16)}\) In this method, the angular rotation \(\Theta\) of the joints and the bar rotations \(\rho\) are used as the unknowns and the terminal moments are expressed in terms of these unknowns. Analogous to the slope-deflection equations in indeterminate analysis, the pair of equations has the form

\[
M_A = K \left( \Theta_A + CE_B - (1-C)\rho \right)
\]

\[
M_B = K \left( CE_A + \Theta_B - (1-C)\rho \right)
\]

(2.4)

where

\[
K = \frac{c}{c^2-s^2} \frac{EI}{L} = \text{stiffness factor}
\]

\[
C = \frac{s}{c} = \text{carry-over factor}
\]

(2.5)

Two equations of this type can be written for each member, expressing the moments at the two ends in terms \(\Theta\) and \(\rho\). The end moments are then eliminated by substituting these expressions into the equilibrium conditions. In general two types of equilibrium conditions are necessary, one is obtained by considering \(\Sigma M = 0\) at each joint, and the other
by considering the equilibrium of the whole structure. After eliminations of the end moments, a system of equations linear and homogeneous in \( \theta \) and \( \rho \) is obtained. The stability criterion \( \Delta = 0 \) can then be applied to these equations for the determination of the critical load. Tabulations of the constants \( K \) and \( C \) can be found in Refs. (3) and (15).

Except for very simple cases, the direct solution of the equation \( \Delta = 0 \) is not possible because terms involved in these simultaneous equations are transcendental in character. The usual procedure is to solve for the values of \( \Delta \) corresponding to a number of different values of the applied loads and to plot \( \Delta \) vs. the loads. The point where the curve first crosses the load axis gives the lowest buckling load.

2) The Convergence Methods

Because of the difficulties involved in obtaining exact solutions by the analytical approach, various types of numerical methods have been developed. These include the moment distribution procedure adopted by Lundquist\(^{17, 18}\) and the relaxation methods applied by Boley\(^{19}\).

The moment distribution method for stability analysis is based on the same concept as that originated by Cross for analyzing continuous beams and frames, but with all the constants, such as the stiffness factors and carry-over factors, being modified for the effect of axial loads. Values of
these factors have been tabulated in Refs. (15), (20) and (21) as functions of $kL$ for the cases when the far ends of the members are pinned or fixed.

For structures without joint translation, Lundquist has established a stability criterion, known as the Stiffness Criterion, (he also developed the Series Criterion by further reasoning). The physical meaning of this criterion can be interpreted as the following: At any joint of a structure where several members are rigidly connected, a unit moment is applied, causing the joint to rotate through an angle $\theta$, the magnitude of which depends on the sum of the stiffnesses $\Sigma K$ of all these members. As the applied load increases, the value of $\Sigma K$ tends to diminish gradually. The limit of stability is reached when the total stiffness approaches zero, that is $\Sigma K = 0$. This means that at the instant of buckling no external moment is required to rotate any joint in the structure. Lundquist's criterion can be applied quite successfully in analyzing structures such as bridge trusses, roof trusses, and aircraft frames. (18, 21)

The convergence of the moment distribution method and the uniqueness of the results were proven by Hoff (22). His proof has led to the following important theory for the determination of the stability of frameworks: "If a framework is subject to external moments and the Cross method as modified for axial loads is used in the calculation of the distribution of bending moments of the bars, and a finite,
A finite set of values means that the moment distribution method converges, and uniqueness implies that the solution must be independent of the order of balancing.

To analyze structures with joint translation, the moment distribution procedure as described above cannot be applied without modification. Several such modifications have recently been proposed by Winter et al. (15), Perri (23), and Livesley and Chandler (21). In the following paragraphs the method developed by Winter et al is described, since this particular method will be adopted in Chapter 5 to investigate the buckling strength of yielded frames.

For analyzing the sidesway buckling load of the frame shown in Fig. 2-2, an arbitrary lateral displacement \( d \) is first introduced at each column top. Then from Eq. (2.4) the fixed end moments are

\[
M_{F1} = M_{FA} = M_{FB} = K_1 (1 + C_1) \frac{d}{L_1}
\]

\[
M_{F2} = M_{FC} = M_{FD} = K_2 (1 + C_2) \frac{d}{L_2}
\]

The joints are then released and the unbalanced moments are distributed, the resulting column moments will be \( M_A' \), \( M_B' \) etc. The condition of equilibrium of each column requires that the shear force be given by:

\[
Q_i = \frac{1}{L_i} (M_A' + M_B' - P_d)
\]
From Eq. (2.6), the displacement \( d \) can be written as

\[ d = \frac{L}{K_1(1 + C_1)} M_{F1} \quad (2.8) \]

Substitution of Eq. (2.8) into Eq. (2.7) results in the following expressions for the shear forces:

\[
Q_1 = \frac{1}{L_1} \left( M'_A + M'_b - \frac{k_1^2 L_1 E_1}{K_1(1 + C_1)} M_{F1} \right) \\
Q_2 = \frac{1}{L_2} \left( M'_c + M'_d - \frac{k_2^2 L_2 E_2}{K_2(1 + C_2)} M_{F2} \right) \quad (2.9)
\]

where \( k \) is as defined in Eq. (2.3). As the applied load increases the last terms in the parentheses of Eq. (2.9) increase in absolute value relative to the first two terms. Hence, the shear force \( Q \) decreases, indicating a decreasing resistance of the frame to the imposed displacement. The critical condition is reached when

\[ \sum Q = 0 \quad (2.10) \]

that is, when an imposed lateral displacement is no longer counteracted by opposing shear forces.

The relaxation method is fundamentally identical with the moment distribution method. The same stability criterion is used in both cases. For a complete discussion of the mathematical theory involved, reference can be to a paper by Boley\(^{(19)}\) and its discussion by Pullaczck Geiringer\(^{(24)}\).
3) The Energy Method

This method states that when the second variation of the total potential of the system is greater than zero the equilibrium is stable, and when it is equal to zero the equilibrium is neutral (limit of stability). Once the total potential is properly formulated, the problem can then be solved by using variation principles. The application of this method to solved buckling problems has been discussed by Mises\(^{(13)}\), Ratzersdorfer\(^{(14)}\), Kasarnowsky and Zetterholm\(^{(25)}\) and F. Bleich and H. Bleich.\(^{(26)}\)

2.1.2 Buckling in the Inelastic Range

In the previous discussions, it is assumed that all the members of the structure are stressed within the elastic limit. However, in most practical cases instability of the structural system would occur when the stresses in some members have already exceeded that limit. Therefore, the methods outlined in the last article have to be modified in order to take the inelastic effect into consideration.

Since all the members are subjected to axial force only, uniform yielding may be assumed for every section along each member. Then according to the tangent modulus concept (no strain-reversal) the flexural rigidity EI should be replaced by \(E_t I\), where \(E_t\) is the slope of the stress-strain diagram of the particular material determined from a coupon test. But for material like steel which has
discontinuously varying $E_t$ values, this type of modification cannot be applied.

Recently investigations in Fritz Laboratory have shown that the inelastic buckling strength of steel columns is influenced considerably by residual stresses, and that a total stress-strain diagram of the whole section obtained from a stub column test should be used to determine the $E_t$ values.\(^{(27)}\) At the present time considerable information has been obtained concerning the magnitude and distribution of these residual stresses and their effect on column instability.\(^{(28)}\) By utilizing such available information, a numerical procedure is suggested here for analyzing stability problems in the inelastic range.

For a given loading system, the axial load $p$ and the stresses $\sigma$ in all the members are first calculated. Then from Fig. 2-3, the ratio of $\frac{I_e}{I}$ can be determined for each member, where $I_e$ is the effective moment of inertia of the section carrying the axial load $p$ (if no strain-reversal is assumed in the analysis $I_e$ is simply the moment of inertia of the unyielded parts of the section). To obtain $k$ values to be used in Eq. (2.2), the flexural rigidity $EI$ in Eq. (2.3) should be replaced by $EI_e$. The same substitution should also be made in Eqs. (2.1) and (2.5). The buckling analysis can then be carried out by any methods described in Art. 2.1.1.
2.2 Structures Carrying Primary Bending Moment

The discussion on structural stability contained in Art. 2.1 was under the assumption that no bending moments (or flexural deformation) are present in the individual members at the instant of instability. But in most engineering structures this condition is not satisfied. Generally, bending moments and deformations are introduced as soon as external load is applied. Therefore, it is necessary to study buckling problems in the presence of such effects. The frame stability problem investigated in this dissertation represents an important example of such problems.

Unfortunately, the solution of stability problems of this type becomes very involved and so far only a few attempts have been made to solve them. These attempts will be briefly discussed in this section.

2.2.1 Buckling in the Elastic Range

1) Chwalla's Investigations

In Ref. (29) Chwalla presented an exact solution to the problem of sidesway buckling of a rigid frame under transverse loads as shown in Fig. 2-4. He showed analytically that for a symmetrical frame carrying symmetric loads the type of instability is characterized by a bifurcation of the equilibrium positions, with a sudden change of the
load-deformation relationship. By solving a system of second order differential equations defining the equilibrium at the buckled state, he obtained the following two conditions which should be satisfied simultaneously at the limit of stability:

**Equilibrium Condition:**

\[
\frac{H^2}{P^2} \left( -k_n \cot k_n h \right) \cos \frac{k_n l}{2} + \cos \frac{k_n l}{2} - \cos k_n \left( \frac{l}{2} - \frac{L}{n} \right) - \frac{k_n H}{P} \sin \frac{k_n l}{2} = 0 \tag{2.11a}
\]

**Buckling Condition:**

\[
t_n \cot k_n h - \frac{P l}{H L} \left( 2 - k_n \cot \frac{k_n l}{2} \right) + \frac{H l}{H L} \left( 2 - k_n \cot k_n h - \frac{k_n^2 l^2}{5 k_n^2} \right) = 0 \tag{2.11b}
\]

where \( k_n = \sqrt{\frac{P}{E I_c}} \) and \( k_n = \sqrt{\frac{H}{E I_b}} \).

Using these two equations, the buckling loads of a few cases were computed by the author and the results are shown in Table I. Two types of frames were selected, the first type with \( L = h \) and the second with \( L = 3h \). Moments of inertia of the beam and the column were assumed to be the same \( I_b = I_c = I \). The number \( n \) in the table indicates the position of the applied loads \( P \) (see Fig. 2-4a). The loads \( P \) act at the center of the beam when \( n \) equals 2 and at the top of columns when \( n = \infty \). In the latter case, the problem is reduced to the type discussed in Art. 2.1 and its solution can be easily obtained. By comparing the values listed in the first row of Table I, it can be seen that the critical load of such frames is reduced when the loads are applied away from the column tops. For the frame with
L = h the maximum reduction amounts to less than 3%, whereas for the frame with L = 3h the reduction can be as much as 11%.

In the same paper, Chwalla suggested that the presence of the axial force in the beam (horizontal reaction H at the base) is responsible for part of the reductions, because the stiffness of the beam is decreased due to this force. To check this suggestion, the values $P_{cr}$ shown in the second row of Table I were calculated for the loading system shown in Fig. 2-5. The horizontal force H applied to the beam was determined from Eq. (2.10). It is noted that for the case L = 3h the buckling loads $P_{cr}$ determined from this simplified loading condition are very close to the exact values $P_{cr}$. It may be concluded from this study that for practical purposes the actual loading system can be substituted by the loading system assumed here in determining the critical loads.

Experiments were conducted on small model frames by Chwalla and Kollbrunner to check the theoretical solution discussed in this article. Excellent correlation was observed in all the tests.

b) Puwein’s Investigation

Following Chwalla’s suggestion, Puwein obtained an approximate solution to the sidesway buckling of portal frames carrying uniformly distributed load.
used a simplified loading system similar to that of Fig. 2-5. Therefore, the buckling load computed from his solution would be slightly higher than the exact value. In Chapter 3 of this dissertation an exact solution to the problem will be presented.

Puwein also extended his work to portal frames with partial base fixity and to gable frames. A similar substitution of loading system was used in all the cases.

c) Recent Developments by Masur, et al

Very recently, Masur, et al, succeeded in extending Bleich's method, the slope-deflection method, and the moment distribution method, as described in Art. 2.1.2, to the analysis of the stability of frames carrying primary bending moment[32]. By using these extensions, the stability problems of the type under consideration can be solved in a systematic manner.

In the first part of their paper, the original four-moment equation was modified by including the effect of the transverse loads to the following form.

\[
\frac{L_i}{E l_i} (s_i M_i^p + c_i M_i^r) + \frac{L_{i int}}{E I_{i int}} (c_{i int} M_i^{p int} + s_{i int} M_i^{r int})
+ (f_{i int} - f_i) + (F_i^r - F_i) = 0 \quad (2.12)
\]
where $F_{1i}$ and $F_{i+1}$ represent respectively, the loading terms at the left end of the member $i$, and at the right end of the member $i + 1$. These loading terms depend on the applied transverse load and the axial force in the member. They have been derived explicitly in Ref. (3j) for some simple loading conditions.

If only two members meet at the joint $i$, $M_{1i}$ becomes equal to $M_{i+1}$ and Eq. (2.12) simplifies to the three-moment equation.

\[
\frac{L_i}{EI_i} s_i M_i + \left( \frac{L_i}{EI_i} c_i + \frac{L_{i+1}}{EI_{i+1}} c_{i+1} \right) M_{i+1} + \frac{L_{i+1}}{EI_{i+1}} s_{i+1} M_{i+1} + (f^i_{i+1} - \bar{f}_i) + (f^i_{i+1} - \bar{f}_i) = 0 \tag{2.13}
\]

Equations (2.12) and (2.13) can also be used to solve statically indeterminate structures, including the effect of axial forces.

When a structure passes from one deformation configuration into another, there will be changes of reactions and deformations, these changes will in turn, cause variations of moments $\Delta M$, axial forces $\Delta P$, and the member rotations $\Delta \rho$. Consequently, there will be variations of $s$, $c$, and $F$, of each member due to the change of axial force. However, all these variations should satisfy Eq. (2.12) in its variational form.
Eq. (2.14) is called the four-moment equation for neutral equilibrium. The critical load of structures may be obtained by solving simultaneously equations of the types (2.12) and (2.14). This method will be adopted in Chapter 3 to obtain an elastic solution to the frame stability problem under investigation.

Following a similar procedure, Masur, et al, also modified the slope-deflection equations (Eq. (2.4)) to include the effect of transverse load by introducing terms which are commonly called the fixed-end moments. The generalized slope-deflection equations, then, have the form:

\[
\begin{align*}
M_A &= K \left( \theta_A + C \theta_B - (1+C) \rho \right) \pm M_{FA} \\
M_B &= K \left( C \theta_A + \theta_B - (1+C) \rho \right) \pm M_{FB}
\end{align*}
\]

with

\[
M_{FA} = F_A \cdot C F_B \quad \text{and} \quad M_{FB} = C F_A + F_B
\]
where $F_A$ and $F_B$ are the loading terms for the ends A and B.

Equation (2.15) was also derived by Winter, et al, in Ref. (15).

The technique of using variations of deformations, reactions, moments, and axial forces, occurring immediately after buckling, for solving stability problems can be applied equally well in the slope-deflection method. The variational form of Eq. (2.15) may be shown to be:

$$\Delta M_A = K \left[ \Delta \Theta_A + C \Delta \Theta_B - (1+C) \Delta f' \right] + \Delta K \left[ \Theta_A + (1+K \frac{4C}{\Delta K}) \Theta_B \right] = \Delta M_{FA} \quad (2.16)$$

A similar expression can be written for $\Delta M_B$. By separate applications of Eq. (2.15) and Eq. (2.16) two groups of slope-deflection equations are obtained. Equations of the first group lead to the relationships between the reactions and the applied loads when the structure is in stable equilibrium. From equations of the second group, similar relationships may be obtained for the unstable state. The critical load of the structure can be determined by solving these two types of relations simultaneously.

The moment distribution procedure developed by Winter, et al, as described in Art. 2.1.1, was extended further to the investigation of stability problems considered in this section. The extension is principally based on the variational form of the slope-deflection equations as given above.
The criterion \( \Sigma Q = 0 \) at the limit of stability can also be used in the present case. Since the method involves successive numerical approximations, the buckling load can generally be determined without solving simultaneous equations.

It should be pointed out that the methods described by these authors can be successfully applied only to very simple structures. For more complicated cases, the evaluation of those variational terms contained in Eqs. (2.14) and (2.16) becomes very cumbersome.

2.2.2 Buckling in the Inelastic Range

In recent years, because of the rapid development in the plastic method of design, the problem of frame stability in the inelastic range has received special attention. Numerous test programs were carried out at several institutions in England to study the reduction of plastic strength of single and multiple story rigid frames due to instability \((34, 35, 36, 37)\). Based on the results obtained from these tests, several types of empirical formulas have been proposed for estimating the inelastic buckling loads \((38)\). Nevertheless, due to the complexity of the problem, very little theoretical work has been done in this field. The analytical studies contained in Chapter 5 of this dissertation may be considered as the first attempt to solve this type of stability problem.
In the following paragraphs, brief discussions on some empirical methods that are presently available will be presented:

a) Generalized Ranking Formula Proposed by Merchant\(^{(38)}\)

Merchant has recently suggested that the actual load carrying capacity \(W_F\) (or the inelastic buckling strength) of unbraced frames may be assumed to depend linearly on two parameters: 1) the simple plastic strength \(W_u\), and 2) the elastic buckling strength \(W_{cr}\), and that a simple relationship of the following form may be used for estimating \(W_F\):

\[
\frac{1}{W_F} = \frac{1}{W_u} + \frac{1}{W_{cr}} \tag{2.17}
\]

This equation is called the "Generalized Rankine Formula". Tests on model frames of rectangular sections conducted at Manchester University and at Cambridge University, have shown that, in general, the buckling load given by Eq. (2.17) is very much on the safe side\(^{(34, 35, 37)}\). So far, no test has been performed on full scale frames of WF shapes to check the validity of this formula.

b) The AISC Formula

To safeguard against frame instability, the following formula is recommended in the AISC Plastic Design Manual\(^{(39)}\) for proportioning the columns in unbraced frames:

\[
2 \frac{\bar{P}}{P_y} + \frac{1}{10} \frac{h}{r} \leq 1.0 \tag{2.18}
\]
in which $F$ is the axial force in the column when the frame carries its maximum load, $P_y$, the axial yield load of the column, and $\frac{h}{F}$, the slenderness ratio. The justification of this rule can be found in Ref. (2). It has also been partially confirmed by tests (40).

In the preceding sections, currently available methods for the solution of two types of instability problems are discussed. For the problem of the first type, where the members are primarily stressed by axial forces, methods are readily available for determining the buckling loads either in the elastic or in the inelastic range. Whereas for the second type of the stability problem, where bending moment as well as axial force effects have to be considered, only very limited theoretical investigations have been made. These investigations are limited to elastic buckling. At present no method is in existence for solving inelastic buckling problems of this type. Therefore, the main object of the present research is to develop such methods. In the next chapter the elastic buckling problem will be studied in detail. These studies will constitute the first step toward the solution of the problem in the inelastic range.
3. **BUCKLING STRENGTH OF ELASTIC FRAMES**

Theoretical and experimental studies on the buckling of rigid frames with the loading condition shown in Fig. 1-4 will be presented in this chapter. Also included is a discussion concerning the effect of partial base fixity on the buckling strength of such frames.

3.1 **Theoretical Analysis**

The modified three-moment equation method as described in Art. 2.2.1 is used to solve the present problem. Two governing equations will be derived by separate applications of Eqs. (2.13) and (2.14). They both define relationships between the applied load and the horizontal reaction at the base, one for the stable state and the other for the unstable state. Instability will occur when the applied load has reached such a value that both of these equations are satisfied.

Figure 3-1 shows the moments and forces acting on a frame before and immediately after buckling. Considering the equilibrium of the frame with symmetric deformation, the following relations are obtained:

\[
\begin{align*}
V_L &= V_R = \bar{P} \\
M_L &= M_R = -Hb
\end{align*}
\]  

(3.1)
On application of Eq. (2.13) to members AB and BD, the threemoment equation for equilibrium reads:

\[
\left( \frac{h}{EI_c} c_1 + \frac{L}{EI_b} c_2 \right) M_L + \frac{L}{EI_b} s_2 M_R + F_2 = 0
\]  
(3.2)

where the subscripts 1 and 2 refer to members AB and BD respectively. The coefficients \( c, s \) and \( F \) are defined as follows:

\[
\begin{align*}
    c_1 &= \frac{1}{(k_1 h)^2} \left( 1 - k_1 h \cot k_1 h \right) \\
    c_2 &= \frac{1}{(k_2 L)^2} \left( 1 - k_2 L \cot k_2 L \right) \\
    s_2 &= \frac{1}{(k_2 L)^2} \left( \frac{\theta_2}{\sin k_2 L} - 1 \right) \\
    F_2 &= \frac{w h}{H} \left( \frac{1}{k_2 L} - \frac{\cos k_2 L}{\sin k_2 L} - \frac{1}{2} \right)
\end{align*}
\]  
(3.3)

with \( k_1 = \sqrt{\frac{p}{EI_c}} \) and \( k_2 = \sqrt{\frac{H}{EI_b}} \).

The expression for \( F_2 \) is taken from Ref. (33). On substitution of these expressions and the expression for \( M_L \) and \( M_R \) from Eq. (3.1) into Eq. (3.2) and rearranging terms, the equilibrium condition becomes:

\[
\frac{H}{p} \left( -k_1 h \cot k_1 h \right) + k_1 h \tan \frac{k_1 L}{2} - \frac{w h}{\tan \frac{k_1 L}{2} - \frac{1}{2}} = 0
\]

For the case with roof load \( w \) only, then \( \bar{p} = \frac{w h}{2} \), and Eq. (3.4) takes the form

\[
\frac{2H}{wL} \left( -k_1 h \cot k_1 h \right) + k_1 h \tan \frac{k_1 L}{2} - \frac{w h}{\tan \frac{k_1 L}{2} - \frac{1}{2}} = 0
\]

(3.5)

Next consider the equilibrium of members when the frame is in the state of unstable equilibrium as shown in Fig. 3-1b, the following equations can be readily obtained:

\[
\begin{align*}
    v_L + \Delta v_L &= \bar{p} - 2 \bar{p} \frac{\Delta R}{l} \\
    v_R + \Delta v_R &= \bar{p} - 2 \bar{p} \frac{\Delta R}{l} \\
    M_L + \Delta M_L &= -(H + \Delta H) h + v_L \Delta R \\
    M_R + \Delta M_R &= -(H + \Delta H) h - v_R \Delta R
\end{align*}
\]  
(3.6)
Combining Eq. (3.6) with Eq. (3.1) leads to:

\[
\begin{align*}
\Delta V_L &= -2 \bar{P} \frac{\Delta R}{L} \\
\Delta V_R &= 2 \bar{P} \frac{\Delta R}{L} \\
\Delta M_L &= -\Delta H + \bar{P} \Delta R \\
\Delta M_R &= -\Delta H - \bar{P} \Delta R
\end{align*}
\]

or

\[
\begin{align*}
\frac{\partial V_L}{\partial R} &= -2 \frac{\bar{P}}{L} & \frac{\partial V_L}{\partial H} &= 0 \\
\frac{\partial V_R}{\partial R} &= 2 \frac{\bar{P}}{L} & \frac{\partial V_R}{\partial H} &= 0 \\
\frac{\partial M_L}{\partial R} &= \bar{P} & \frac{\partial M_L}{\partial H} &= -h \\
\frac{\partial M_R}{\partial R} &= -\bar{P} & \frac{\partial M_R}{\partial H} &= -h
\end{align*}
\]

To obtained the relationship between \( H \) and \( \bar{P} \) immediately after buckling, the three-moment equation for neutral equilibrium will be used. For members AB and BD Eq. (2.14) becomes:

\[
\frac{h}{EI_c} \left( c_1 \Delta M_L + c_2 \Delta M_R + \Delta c_1 M_L + \Delta c_2 M_R + \Delta s_2 M_R \right) + \frac{L}{EI_b} \left( c_1 \Delta M_L + \Delta s_2 M_L + c_2 \Delta M_R + \Delta s_2 M_R + \Delta s_2 M_R \right) - \frac{\sigma R}{h} + \Delta F_2 = 0
\]

Similarly, the equation for members BD and DE is:

\[
\frac{L}{EI_b} \left( c_1 \Delta M_L + \Delta s_2 M_L + c_2 \Delta M_R + \Delta s_2 M_R \right) + \frac{h}{EI_c} \left( c_1 \Delta M_L + \Delta s_2 M_L + c_2 \Delta M_R + \Delta s_2 M_R \right) + \frac{\sigma R}{h} + \Delta F_2 = 0
\]

It can be seen from Eq. (3.8) that after buckling the quantities \( \Delta R \) and \( \Delta H \) may be regarded as the independent variables of the system. So the variation of axial force \( p \) in each member can be expressed in terms of these variables, that is
\[
\Delta p = \frac{\partial p}{\partial R} \Delta R + \frac{\partial p}{\partial H} \Delta H
\]  
(3.11)

The variation of \(c_1\) for member AB is derived as follows:

\[
\Delta c_1 = \frac{dc_1}{dp_1} dp_1
\]

\[
= \frac{dc_1}{dp_1} \left( \frac{\partial p_1}{\partial R} \Delta R + \frac{\partial p_1}{\partial H} \Delta H \right)
\]

\[
= c_1 - \frac{\partial}{\partial p_1} \frac{AR}{L}
\]  
(3.12)

where \(c_1 = \frac{dc_1}{dp_1}\). The last step is obtained by using Eqs. (3.1) and (3.8). In a similar manner, the following expressions for \(\Delta c_2, \Delta s_2\) and \(\Delta c_3\) are obtained:

\[
\Delta c_2 = c_2' \Delta H
\]

\[
\Delta s_2 = s_2' \Delta H
\]

\[
\Delta c_3 = \gamma \Delta c_1
\]  
(3.13)

Substituting Eqs. (3.12) and (3.13) and the expressions for \(\Delta M_L\) and \(\Delta M_R\) from Eq. (3.7) into Eqs. (3.9) and (3.10) the equations of neutral equilibrium become:

\[
\left( \frac{h}{EI_c} c_1 + \frac{L}{EI_b} (c_g-s_g) + \frac{h}{EI_c} c_1' \frac{2h}{L} - \frac{1}{\bar{p} L} \right) \bar{p} \Delta R
\]

\[
- \left( \frac{h}{EI_c} c_1 + \frac{L}{EI_b} (c_g+s_g) + \frac{h}{EI_c} c_1' \frac{2h}{L} - \frac{1}{\bar{p} L} \right) \bar{p} \Delta R = 0
\]  
(3.14)

and

\[
- \left( \frac{h}{EI_c} c_1 + \frac{L}{EI_b} (c_g-s_g) + \frac{h}{EI_c} c_1' \frac{2h}{L} - \frac{1}{\bar{p} L} \right) \bar{p} \Delta R
\]

\[
- \left( \frac{h}{EI_c} c_1 + \frac{L}{EI_b} (c_g+s_g) + \frac{h}{EI_c} c_1' \frac{2h}{L} - \frac{1}{\bar{p} L} \right) \bar{p} \Delta R = 0
\]  
(3.15)

Subtracting Eq. (3.15) from (3.14) leads to:

\[
\left( \frac{h}{EI_c} c_1 + \frac{L}{EI_b} (c_g-s_g) + \frac{h}{EI_c} c_1' \frac{2h}{L} - \frac{1}{\bar{p} L} \right) \bar{p} \Delta R = 0
\]  
(3.16)
Since $\Delta R$ should be finite after buckling, the expression in the bracket has to be zero:

$$\frac{h}{\epsilon_1 c} + \frac{1}{\epsilon_1 b} (c_1 - s_2) + \frac{h}{\epsilon_1 c} c_1' \frac{2uh}{L} - \frac{1}{\bar{P} h} = 0 \quad (3.17)$$

From Eq. (3.3) the following relation may be deduced:

$$(c_1 - s_2) = \frac{1}{(\epsilon_1 L)^2} \left( 2 - \frac{1}{L} \left( \frac{1}{k_1} \frac{d\phi}{dz} \right) \right)$$

$$c_1' = \frac{dc_i}{dP_i} = \frac{dc_i}{d\phi} \frac{d\phi}{dp_i}$$

$$= \frac{1}{Z \bar{P} (\epsilon_1 L)^2} \left( k_{1h} C_{1h} k_{1h} + \frac{(\phi_{1h})^2}{\sin^2 (\phi_{1h})} \right) \quad (3.18)$$

Upon substituting the expression for $c_1$ from Eq. (3.3) and the above expressions for $(c_2 - s_2)$ and $c_1'$ into Eq. (3.17), the stability condition is finally obtained in the form:

$$k_{1h} C_{1h} k_{1h} - \frac{P_h}{hL} \left( 2 - \frac{1}{L} \left( \frac{1}{k_1} \frac{d\phi}{dz} \right) \right) + \frac{H_h}{P_L} \left( 2 - k_{1h} C_{1h} k_{1h} - \frac{\phi_{1h}^2}{\sin^2 (\phi_{1h})} \right) = 0 \quad (3.19)$$

This equation may be seen to be identical with Eq. (2.11b) obtained by Chwalla for the loading condition shown in Fig. 2-4. This implies that the actual condition of load application is immaterial in analyzing Eq. (3.19) as long as the proper value for $\bar{P}$ is used in every case ($\bar{P}$ will always be equal to half of the total loads).

Both Eqs. (3.4) and (3.19) express implicitly the horizontal reaction $H$ as a function of the applied load $\bar{P}$. It would be impossible to obtain simultaneous solution of these two equations for $P_{cr}$ and $H_{cr}$ in explicit forms. Hence, separate computations have to be made for individual cases with the aid of some numerical techniques. The method of solving
these equations is illustrated here for the case of a square frame \((L = h)\) with \(I_c = I_b = I\) subjected to a uniformly distributed load \(w\) on the beam (that is \(a = 0\)). Let \(L = h\) in Eqs. (3.5) and (3.19) and introduce a non-dimensional parameter \(\beta\) which is defined as:

\[
\beta = \sqrt{\frac{H}{\omega_h^2}}
\]

and note the relationships

\[
\kappa_1 = \sqrt{\frac{\omega_h}{2\ell_1}} \quad \text{and} \quad \kappa_2 = \sqrt{\frac{H}{E_1}} - \kappa_1 \beta
\]

The following two equations are obtained:

\[
\beta^2 (-k_h C \tan \kappa_1 h \beta + \kappa_1 h \beta \tan \frac{k_h \beta}{2} - \frac{2}{k_h \beta^3} \tan \frac{k_h \beta}{2} + \frac{1}{\beta^2} = 0 \tag{3.20}
\]

\[
k_h C \tan \kappa_1 h - \frac{1}{\beta^3} (2 - k_h \beta C \tan \frac{k_h \beta}{2}) + \beta^3 (2 - k_h C \tan \kappa_1 h - \frac{k_h \beta}{2}) = 0 \tag{3.21}
\]

These equations are then solved for the value of \(\beta^2\) for each assumed value of \(k_1 h\) by a trial-and-error procedure. For instance from Eq. (3.20) the following values of \(\beta^2\) are obtained 0.0980, 0.09766, 0.0953 corresponding to \(k_1 h = 0.5\), 1.0, 1.35. For \(k_1 h = 0\), the value of \(\beta^2\) can be determined by ordinary structural theory, because in this case the axial force is assumed to have no effect on the reaction \(H\). Using these values of \(k_1 h\) and \(\beta^2\) the applied load \(w\) and the reaction \(H\) can be determined from the equations.

\[
\begin{align*}
\frac{\omega_h}{2} &= (k_h)^2 \frac{H}{h^2} \\
H &= \beta^2 \frac{\omega h}{2} = (k_1 \kappa_1 \beta)^2 \frac{E_1}{h^2}
\end{align*}
\]
The results of the above computation are plotted as curve (I) in Fig. 3-2. It represents the relationship between \( w \) and \( H \) as derived from the non-linear theory. The same relationship based on ordinary structural theory is shown as a dotted line in the figure. A small deviation between these two curves can be noticed, indicating the effect of axial force in indeterminate analysis.

Following the above procedure, Eq. (3.21) is also solved and the final results are plotted as curve (II) in Fig. 3-2. This curve gives the relation between \( w \) and \( H \) when the frame is in the state of unstable equilibrium. The point of interaction of curves (I) and (II) determines the exact buckling load of the structure. As shown in Fig. 3-2, the critical values are:

\[
\left( \frac{wh}{z} \right)_{c} = 1.787 \frac{EI}{h^2} \quad \text{and} \quad H_{cr} = 0.1705 \frac{EI}{h^2}
\]

It can also be noted that the point where curve (II) intersects with the abscissa corresponds to the critical load of the frame when all the loads are assumed to act along the centerline of the columns.

Similar computations were carried out for the case \( \alpha = 2 \) and for the frame with \( L = 3h \). The results are listed in columns (1) and (2) of Table II for \( L = h \) and in columns (6) and (7) for \( L = 3h \). The values of \( P_{cr} \) for the case when all the loads are assumed to act at the column tops are also given in the table and compared with that of the actual loading conditions (\( \alpha = 0 \) and \( \alpha = 2 \)). For the case \( L = h \) the
difference is less than 2% when $a = 0$ and less than 1% when $a = 2$, and for $L = 3h$ the corresponding differences are 6.7% and 2.4%. The reason that the difference becomes smaller for larger values of $a$ can be explained by considering the relative magnitude of the axial forces in the beam at the instant of buckling. For higher values of $a$, the portion of the total load that applies to the beam is reduced, consequently the horizontal reaction $H$ at the base (which is equivalent to the axial force in the beam) is also reduced. The stiffness of the beam is therefore increased slightly. This results in an increase in buckling load.

From this study, it is suggested that for larger column loads with $a = 2$ or greater the influence of axial force $H$ may be ignored in computing the stiffness of the beam and that in practical cases the critical load of such frames can be computed by assuming all the loads acting at the top of columns. Such simplifications will be adopted in subsequent chapters of this dissertation.

3.2 Experimental Investigation

To check the theoretical work presented in Art. 3.1 experiments on model steel frames were conducted. The cross-sectional shape and the loading arrangement of the test frames are as shown in Fig. 3-3. The uniform load $w$ assumed in the analysis was replaced by two concentrated loads $P_1$ each applied at a distance $0.3L$ from the center line of the
columns. Figure 3-4 shows the test setup and the fixtures used for transmitting the loads to their points of application. A dead weight and lever system was used to produce the downward force in the sling. The loading system as a whole could sway freely with the frame at all stages of the test. Details of the test procedure and the experimental techniques employed will be described in a forthcoming report. (40).

All the important information pertaining to these model tests, including the frame dimensions, load ratio α, theoretical predictions and the test results, is summarized in Table III. The "test load" reported in the table is not the buckling load, but the maximum load observed in each test. Because of the unavoidable imperfections of the test specimens and inaccuracy in load application, it was impossible to detect exactly when the test frame started to buckle. However, in general very little increase of load can be expected after the initiation of sidesway movement, so the ultimate load observed from test should be very close to the actual buckling load.

Figure 3-5 shows the load-deflection curves of the test frame P-4. The curve shown in the right is very similar to the load-deflection curve usually obtained from a centrally-loaded column test. Figure 3-6 shows the same frame after unloading, typical sidesway buckling can be seen.

It can be seen from the comparisons given in Table III that satisfactory correlation between the theory and the tests
has been obtained. For both frames P-3 and P-4, the test loads are a few percent lower than the prediction. These discrepancies were mainly due to local yielding at the welded joints, where several yield lines were observed.

3.3 The Effect of Partial Base Fixity

In the above described theoretical and experiment studies, the base of the frame was assumed to be perfectly pinned. However, in actual structures this condition usually does not exist. In most of the so-called "pinned" column bases, the rotational restraint at these bases may be rather appreciable. The actual amount of base restraint depends on the details used in construction and on the foundation soil.

Recently Galambos has shown that the buckling strength of portal frames with small amount of foundation restraint can be considerably higher than that of pinned-base frames. (41) This has also been observed experimentally in some recent tests. (40)

In this section, it is intended to indicate how the theoretical solution discussed in this chapter can be used to determined the buckling load of frames with partial base fixity. As was done in Ref. (41), the base restraint can be simulated by inserting a restraining beam between the two column bases as shown in Fig. 3-7. This beam restrains the column ends in the same way as would be done by an actual base consisting of base plates, the footing and the soil.
With this simplification, the problem can then be solved approximately according to the following steps:

1. Assume a distance $h_1$ from the column top to the inflection point in the column, then $h_2 = h - h_1$. As can be seen from the moment diagram given in Fig. 3-7b, the given frame may be considered as two separate pinned-base frames with their respective column heights equal to $h_1$ and $h_2$.

2. Determine the buckling load $P_{cr}'$ for the upper frame using the theory just presented.

3. Compute the buckling load $P''_{cr}$ for the lower (inverted) frame using the methods discussed in Art. 2.1.1.

4. Compare $P_{cr}'$ with $P''_{cr}$, if they are not equal, a new value of $h_1$ should be assumed and the process repeated. The correct buckling load is obtained when the assumed $h_1$ value gives identical critical loads for both frames, that is $P_{cr}' = P''_{cr}$.

Calculations using the procedure here described have also shown that the effective length of columns in a frame can be reduced appreciably when a small amount of rotational restraint at the base was taken into account. Future research in this field should include the development of methods by which the restraints offered by different types of footing can be evaluated.
3.4 Summary

The investigation presented in this chapter may be summarized as the following:

1. An exact solution to elastic buckling of pinned-base frames was described in detail and some typical results were shown in Table II.

2. The effect of axial force in the beam on the buckling strength of the whole frame was examined. It was found that for practical frames this effect could be ignored for computing the sidesway critical load.

3. Results of model frame tests were presented in Table III. Satisfactory agreement between the theoretical and the experimental elastic buckling loads was found.

4. A method for determining the buckling load of partially fixed frames was outlined in Art. 3.3. In general a small amount of base restraint can increase the buckling load appreciably.
Since the portal frame under investigation is statically indeterminate, a method of analysis for partially plastic redundant structures has to be developed before any attempt can be made to solve the buckling problem in the inelastic range. This chapter will present a method by means of which the distribution of moments in a structure can be precisely determined at any stage of loading. In order to obtain results accurate enough for buckling analysis, it is necessary in the present analysis to take into account not only the inelastic action, but also the effects of axial force and the deformations due to bending. The method can also be used to obtain the following information:

1. Load carrying capacity of braced frames (curve (a) of Fig. 1-3b)
2. Deflections of frame at any loading stage.
3. Rotation of the first plastic hinge. *

In the following sections the method will be described specifically for analyzing the type of frame shown in Fig. 1-4, because the results of the analysis will be used in the

* This is often called the hinge angle which is defined as the angle through which a plastic hinge must rotate in order to redistribute the moment and form a plastic mechanism.
next chapter to investigate the stability of frames of this particular type. The approach is equally applicable to the solution of many other types of frames. (42)

4.1 Assumptions and Limitations

The following assumptions are made in developing the present method:

(1) The material possesses the idealized elastic–fully plastic stress–strain relationship in tension and in compression as indicated in Fig. 4–1b.

(2) Plane cross sections remain plane after loading.

(3) The shear force present at any section of the frame is small and its effect on yielding may be neglected.

(4) Throughout the loading history there is no strain reversal of material stressed beyond the elastic limit.

(5) Only deformations (elastic or plastic) due to bending are considered.

(6) The axial force in beam members is small as compared to the thrust in the columns and its effect may be ignored.

As will be seen in the later discussions, the method is subject to the following limitations:

(1) All the members are prismatic.

(2) No lateral (sidesway) displacements at column ends are allowed.
(3) No transverse load is applied to the columns except at the ends.

(4) No instability of any type as discussed in Art. 1.2 should occur prior to the loading stage at which the analysis is performed.

The first step in the development of the method is to obtain separately the moment-end rotation (M-\( \Theta \)) curves of beams and columns. The procedure employed to construct these curves will be described in Art. 4.2 and 4.3.

The first step in the development of the method is to obtain separately the moment-end rotation (M-\( \Theta \)) curves of beams and columns. The procedure employed to construct these curves will be described in Art. 4.2 and 4.3.

4.2 Moment-Rotation Curve of Beam

Referring to the beam in Fig. 4-2, by ignoring the effect of \( H \) the bending moment at distance \( x \) from the center is given by:

\[
M = M_c - \frac{\omega x^2}{2}
\]  

(4.1)

in which \( M_c \) is the center moment. A positive bending moment is one that causes compressive stresses on the outside of the frame. For convenience of numerical computation, Eq. (4.1) may be non-dimensionalized by dividing both sides by \( M_y \), the nominal yield moment of the section (\( M_y = \sigma_y S \)). This results in the expression

\[
\frac{M}{M_y} = \frac{M_c}{M_y} - \frac{\omega x^2}{2 M_y}
\]  

(4.2)
By taking the distance $x$ to be some integer multiplier $n$ of the radius of gyration of the section, $r_b$, then Eq. (4.2) can be written as

$$\frac{M}{M_y} = \frac{M_c}{M_y} - \frac{\omega n^2 r_b^2}{2 M_y} \quad (4.3)$$

From the definitions of $r_b$ and $M_y$, the above equation can be put in the form

$$\frac{M}{M_y} = \frac{M_c}{M_y} - n^2 \omega \frac{d}{4 \sigma_y A} \quad (4.4)$$

in which $d$ is the depth of the section and $A$ is the area.

By letting $\bar{w} = \frac{\sigma_y A}{d}$ and $\frac{w}{\bar{w}} = \lambda$ the following equation may be obtained from Eq. (4.4)

$$\frac{M}{M_y} = \frac{M_c}{M_y} - n^2 \frac{\lambda}{4} \quad (4.5)$$

The parameter $\lambda$ is called the "loading ratio". For practical frames its value is in the range from 0.001 to 0.01. With a fixed value of $\lambda$; Eq. (4.5) determines the bending moment at a distance $nr_b$ from the center of the span, where a value of $M_c/M_y$ is assigned. By knowing the moment at all the sections, the deflected shape of the beam may be established by a numerical integration procedure similar to that developed by Kármán for investigating the strength of eccentrically loaded columns. (43) The procedure is based on the moment-curvature ($M-\phi$) relationship of the beam section. In this dissertation the $M-\phi$ curve for a typical beam section, the 27WF94, is used for all the computations. It was determined on the basis of assumptions (1), (2), (3) and (4) of Art. 4.1 by the method described in Ref. (44) and is plotted in a
non-dimensional form in Fig. 4-3. A symmetrical residual stress pattern as shown in Fig. 4-4 was included in determining this curve. It has been shown in earlier work that the \(M-\phi\) curves for most of the WF sections that are commonly used as beams are approximately the same.\(^{(45)}\) Therefore the results obtained for that particular section, after being properly non-dimensionalized, can also be applied to other sections as well.

The complete procedure of integration is illustrated in Appendix A for the following two cases:

\[
\begin{align*}
(1) \quad & \phi = 0.002 \quad \frac{M_C}{M_y} = 0.5 \\
(2) \quad & \phi = 0.002 \quad \frac{M_C}{M_y} = 1.0
\end{align*}
\]

In these computations it was necessary to consider a specific value of Young's modulus of \(E = 30 \times 10^6\) psi and a value of yield stress \(\sigma_y = 33 \times 10^3\) psi. These values are typical for ordinary structural steel (ASTM A7).

The beam-deflection curve determined by the integration process gives the slope \(\phi\) and the deflection \(y\) of points at distance \(a\) from the center of the beam.* Figure 4-5 shows the deflection curve in non-dimensional form for the two cases listed above. The slope \(\phi\) at all the points along the curve is plotted against the moment \(M/M_y\) at the corresponding points in Fig. 4-6 and against the position of the points in Fig. 4-7.

* The condition that the slope \(\phi\) and the deflection \(y\) are both equal to zero at the center section is assumed as the initial condition in the integration.
The integration process terminates at a section where the bending moment is equal to $M_p$ or where the slope of the deflected shape becomes zero, whichever is reached first. The slope at the point where the moment is equal to $M_p$ (or $M/M_p = 1.14$) becomes discontinuous, indicating the formation of a plastic hinge.

Similar computations can be made for other values of $M_c/M_y$. Thus for a specific value of $\lambda$ (that is at a given load $w$), a set of beam-deflection curves may be obtained. The set of curves for $\lambda = 0.002$ are plotted in Fig. 4-8, the corresponding sets of moment-slope and coordinate-slope curves are shown in Fig. 4-9. The desired moment-rotation ($M-\theta$) curve of the beam shown in Fig. 4-2 may be determined graphically in the following manner: For a given beam member of length $L$ the ratio $L/2r_b$ is first determined. A horizontal line (a) is then drawn on the lower diagram of Fig. 4-9, starting from the particular $L/2r_b$ value. The intersection of the line with each of the curves gives the value of $\theta$ for which the equilibrium condition is satisfied. A vertical line is carried up from each intersection to the corresponding moment-slope curve. This gives the moment at the end of the beam in equilibrium with the assumed center moment $M_c$. By joining a number of points in the upper diagram, curve (b) is obtained. This curve defines the relationship between the moment and rotation at the end when the beam is carrying the assumed load $w = \lambda \bar{w}$. 
Computations were carried out for values ranging from 0.001 to 0.01 to cover most of the practical cases. Results obtained for $\lambda = 0.002, 0.004, 0.006, 0.008$ are presented in Figs. 4-8 to 4-15 inclusive.

4.3 Moment-Rotation Curve of Column

The moment-rotation curve of the column required in the present method of analysis may be determined by the procedure developed by Ojalvo in a recent dissertation. The loading condition investigated is that shown in Fig. 4-2 for the column of a pinned-base portal frame. The procedure involves the construction of a set of column deflection curves for each assumed value of axial load. A numerical integration process which is based on the thrust-moment-curvature relationship of the column section is necessary to construct these curves. In this section the thrust-moment-curvature relationships for the 8WF31 section as shown in Fig. 4-16 are used. They were taken from Ref. (44). A residual stress pattern the same as that shown in Fig. 4-4 was assumed in computing these curves. Strictly speaking the curves obtained can only be applied to this particular section. However it may be shown that the results are equally applicable to those sections which have their radius of gyration to depth ratio $r/d$ close to that of the 8WF31 section. Fortunately this is the case for most of the column sections. (46)

By using the method outlined in Ref. (46) the moment-
rotation curves shown in Figs. 4-18 through 4-21 were determined for columns with slenderness ratios \( h/r_c \) ranging from 30 to 130 subjected to axial loads equal to 0.12, 0.2, 0.3 and 0.4 \( P_y \). The \( M-\theta \) curves for the columns with no axial thrust (Fig. 4-17) were obtained by a method similar to that described in Art. 4.2. The values of Young's modulus and yield stress adopted in constructing these curves were the same as those used for the beam-deflection curves.

After all the \( M-\theta \) curves are made available, the frame shown in Fig. 4-2 may be analyzed by the graphical method presented in Art. 4.4. The completed procedure will be illustrated with examples in Art. 4.5.

4.4 The Method of Analysis

To analyze a given frame (Fig. 1-4) subjected to certain applied loads, the following information is usually known: span length \( L \), column height \( h \), member sizes of the beam and the columns (their \( I \) and \( M_p \) values), material properties \( (E \) and \( \sigma_y \) ) and the loads. From this information the ratios \( L/2r_b \) and \( \lambda \) of the beam and \( h/r_c \) and \( P/P_y \) of the column can be determined. Using the results presented in Art. 4.2 and 4.3 two moment-rotation curves may be obtained, one for the beam and one for the column. When they are plotted together as in Fig. 4-22 the point of intersection gives the moment and rotation at the knee for the equilibrium configuration of the structure. That is, of all the combinations of moment and
rotation at B (or D) that are possible in the column and of all the combinations that are possible for the beam, the correct one is one that is simultaneously compatible with the beam and the column. Once the moment $M$ and rotation $\theta$ at B are determined, it is thus possible to determine the forces in the rest of the frame by statics. The deflection at the center can be obtained immediately from the beam-deflection curve corresponding to a particular value of $M_C$ which is in equilibrium with the applied load $w$ and the corner moment $M_B$. The procedure is indicated in Fig. 4-5.

By successively increasing the applied loads, a complete load-deflection curve of the structure can be obtained with this method. The load carrying capacity of such frames can then be estimated precisely from this curve.

4.5 Illustrative Examples

**Example No. 1** As a first example consider the portal frame shown in Fig. 4-23. From a design handbook the following properties for the sections 24WF120 and 24WF100 are taken (assuming $\sigma_y = 33$ ksi):

- **24 WF 120**  
  $A = 35.29 \text{ in}^2$, $d = 24.31 \text{ in.}$, $r_b = 10.15 \text{ in.}$  
  $P_y = 1165 \text{ kips}$, $M_y = 822 \text{ ft-kips}$, $M_p = 926 \text{ ft-kips}$

- **24 WF 100**  
  $A = 29.43 \text{ in}^2$, $d = 24.00 \text{ in.}$, $r_c = 10.08 \text{ in.}$  
  $P_y = 971 \text{ kips}$, $M_y = 684 \text{ ft-kips}$, $M_p = 765 \text{ ft-kips}$
The parameters needed in the present analysis may be determined as follows:

For beam BD

\[ \bar{w} = \frac{A}{y} \frac{\bar{A}}{d} = 33 \frac{35.29}{24.31} = 47.9 \text{ kips/in.} = 574.9 \text{ kips/ft.} \]

\[ \frac{L}{2r_b} = \frac{50 \times 12}{2 \times 10.15} = 29.6 \]

For column AB (or DE)

\[ \frac{h}{r_c} = \frac{42 \times 12}{10.08} = 50 \]

The values of \( w \) that are to be considered are \( w = 2.30, 3.45 \) and 4.60 kips per ft. The corresponding values of \( \lambda \) are 0.004, 0.006 and 0.008 and the average compressive stresses in the columns are \( P/A = 6.6, 9.9 \) and 13.2 ksi. From Figs. 4-11, 4-13 and 4-15 the moment-rotation curves (\( M_{BD} - \theta \)) for the beam having \( L/2r_b = 29.6 \) are obtained, they are plotted as the dot-dashed lines in Fig. 4-24. Shown as dashed lines are the \( M_{BA} - \theta \) curves for the columns, they are taken from Figs. 4-19, 4-20 and 4-21 for \( h/r_c = 50 \). The intersection of each pair of the curves gives the moment \( M \) and rotation \( \theta \) for each value of \( w \). The solid line in the figure is the complete moment-rotation curve at B (or D). By knowing the value of \( M_B \), the moment at the center \( M_c \) can be easily determined by equilibrium considerations. The deflection at the center \( \delta \) can then be obtained directly from Figs. 4-10, 4-12, and 4-14. The complete load-deflection curve of the frame is shown in
Fig. 4-25. The maximum load that can be carried is seen to be 783 kips. The ultimate load based on simple plastic theory is 920 kips. There is a 15% difference between simple plastic theory and the exact theory.

It may be noticed from Fig. 4-24 that the pair of $M-9$ curves with $\lambda = 0.008$ and $P/P_y = 0.4$ becomes tangent when the applied load reaches the maximum. Any further increase of $w$ would produce no intersection (that is, no equilibrium configuration can exist at this increased load), thus indicating the capacity of the frame has been exhausted.

The example selected here has the special feature that the formation of the plastic hinge at the center and the attainment of the maximum moment capacity of the columns occur at the same time. Therefore no rotation capacity is required for either of the members. This will become evident after comparing Fig. 4-24 with Fig. 4-27 of the next example.

**Example No. 2** The frame to be considered in the second example is that shown in Fig. 4-26. This example is chosen mainly to show how the deflection and hinge angle for such frames can be obtained by using the method developed.

Following the same procedure as that used in Example No. 1 moment-rotation $(M_{BD} - \theta)$ curves of the beam for $\lambda = 0.00315$, 0.00526 and 0.00783 are determined by interpolation from curves shown in Figs. 4-11, 4-13 and 4-15 and other curves of a similar nature which are omitted from this dissertation for
the sake of brevity. The axial forces in the columns corresponding to these selected values of \( \lambda \) are equal to 0.12, 0.2 and 0.3 \( P_y \) respectively. The moment-rotation \((M_{BA} - \theta)\) curves of the column for these values of axial force are then obtained from Figs. 4-18, 4-19 and 4-20. Figure 4-27 shows the two sets of curves and the complete moment-rotation relationship of the columns (solid line).

When the applied load reaches 3.34 kips per ft. \((\lambda = 0.00730)\), the first plastic hinge forms at the center of the beam. At this instant the rotation at the column top is designated as \( \theta' \) (0.0205 radian) and the moment as \( M'_B \) (0.705 \( M_y \)). The corresponding vertical deflection at the hinge is \( \delta' \) (6.53 in.) as shown in Figs. 4-28 and 4-29a. The beam is still bent in a continuous curve at this load. Upon increasing the load further, the column ends would rotate in the manner indicated in Fig. 4-27 by the portion of the solid line beyond the point \( H \). Corresponding to a small increment of rotation \( \Delta \theta \) of the column the increase of center deflection would be \( \Delta \delta = \Delta \theta \cdot L/2 \) and the angle change at the plastic hinge would be \( 2 \Delta \theta \).

It may be noted from Fig. 4-27 that the load reaches its maximum value when the moment carrying capacity of the columns is exhausted (on the flat portion of the \( M-\theta \) curve). To attain this maximum moment, the top of the columns has to rotate through an angle equal to \( \theta'' - \theta' \) (0.0145 radian). Thus the hinge angle at the center is equal to \( 2(\theta'' - \theta') \) and the
deflection at the ultimate load is \( \delta_u = \delta' + (\theta'' - \theta') \frac{L}{2} = 12.62 \text{ inch} \). The complete load-deflection curve and the load-hinge rotation curve of this example are plotted in Fig. 4-29. The total load that can be carried by the frame is 752 kips. The maximum load based on simple plastic theory is 879 kips, indicating a reduction of 14.4\% of the load carrying capacity due to beam-column action.

4.6 Summary

This chapter presents a precise method of analysis for indeterminate frames loaded into the plastic range. The method has been illustrated by reference to pinned-base symmetrical frames. The following summarizes this chapter:

(1) A method for obtaining moment-rotation curves of beams with uniformly distributed load was developed. For the convenience of actual computation a set of non-dimensional nomographs was prepared by which the \( M-\theta \) relationship may be determined graphically for given values of \( \lambda \) and \( L/2r_b \) (Figs. 4-9, 4-11, 4-13 and 4-15).

(2) Following the procedure described in Ref. (4.6), moment-rotation curves for columns having slenderness ratios between 30 to 130 and subjected to axial forces of 0.12, 0.2, 0.3 and 0.4 \( P_y \) were obtained. (Figs. 4-18 through 4-21.)

(3) By utilizing the moment-rotation relationships for beams and columns thus obtained, a method of elastic-plastic
analysis was established. The chief importance of the method is that it provides information with which the inelastic frame stability problem may be investigated. This will be discussed in the next chapter.

(4) Two examples of portal frames were solved to indicate the use of the nomographs and the method developed.

(5) As illustrated in these examples, this type of analysis can be used to determine precisely 1) the load carrying capacity of frames which are restrained against sidesway buckling, 2) the complete load-deflection relationship, and 3) the required rotation capacity at the first plastic hinge.
5. BUCKLING STRENGTH OF PARTIALLY YIELDED STEEL FRAMES

In Chapter 4 a method of analysis for partially yielded steel frames was described and its use was illustrated by reference to portal frames which were assumed to be braced against sidesway movement. The method can be applied to determine exactly the distribution of moments and the variation of yielding along the members at any stage of loading. By incorporating such a technique with the existing method of buckling analysis described in Chapter 2, a procedure for determining the inelastic buckling strength of frames is developed in this chapter. Illustrations will be given to show how the procedure can be used to construct a complete frame buckling curve. (See Fig. 1-3b)

5.1 Assumptions

Because the results obtained from the elastic-plastic analysis of the previous chapter are used directly in the development, the present procedure is also subject to the assumptions and limitations stated in Art. 4.1. In addition to those, the following two assumptions are made:

(1) The axial force in the beam is small and its effect on the bending stiffness may be ignored in the buckling analysis. The justification of this assumption was discussed in Chapter 3 in connection with the elastic buckling problem.
(2) The frame deforms in a perfectly symmetrical shape (no lateral movement should occur at the column tops) up to the instant of buckling. This implies that the method of analysis presented in Chapter 4 can be used to determine the yield configuration at any load level below that which causes sidesway instability. (see limitation 2 of Art. 4.1)

Analogous to the method commonly used in determining the inelastic buckling strength of centrally-loaded columns, the procedure here developed also requires proper evaluation of the reduction of bending stiffness (buckling constant) of all the members due to yielding. By using these reduced stiffnesses in the analysis, the problem of inelastic buckling may be treated in a manner similar to that for the elastic case. Figure 5-1 shows schematically the yield configuration of the beam and the columns at a certain load. The methods of determining the inelastic stiffness of these members will be outlined in the next section.

5.2 **Stiffness of Members After Yielding**

Since the modified moment distribution procedure described in Art. 2.1.1 is adopted in the next section to solve the present problem, it is necessary to obtain the following buckling constants:

1. For beams - stiffness factor $K_b$ (assuming far end fixed) and carry-over factor $C_b$
2. For columns - stiffness factor $K_c$ (assuming far end hinged)

5.2.1 Determination of Stiffness and Carryover Factors for the Beam

For a given set of loads $w$ and $P$ the bending moment at $B$ (or $D$) is first determined by the elastic-plastic analysis of Chapter 4. Knowing the two end moments, the moment diagram of the beam may be easily constructed by statics. Figure 5-2a shows a typical example of such a diagram. According to the elementary theory of strength of materials, the flexural behavior (in the elastic and plastic range) of any section of the beam is governed completely by the moment-curvature relationship of the member. In the elastic range the slope of the moment-curvature diagram is constant and equal to the flexural rigidity of the section $EI_b$. When the applied moment exceeds the elastic limit, the slope (or rigidity) starts to decrease and approaches zero when the moment is near the value $M_p$. The effective flexural rigidity $(EI_b)_{eff}$ of the section can thus be determined as the instantaneous slope on the $M-\phi$ diagram corresponding to the applied moment. In this chapter the moment-curvature relationship of a typical beam section 27WF94 as shown in Fig. 4-3 is adopted. The variation of $EI_b$ of this shape with the applied moment is shown in Fig. 5-3. It may be seen from Fig. 4-3 that the actual yield moment of this section is only 70 percent of the nominal value of $M_y$. Therefore,
yielding will occur at sections where the moment is greater than 0.7 My. This is indicated in Fig. 5-2b for the beam under consideration.

If no strain reversal is assumed to take place at the moment of buckling, the stiffness of the beam can be determined by considering a beam with variable EI. For the elastic part the flexural rigidity of the beam is EI_b while for the plastic part the effective flexural rigidity is reduced as if the yielded portions of the beam were removed. The effective flexural rigidity (EI_b)_eff is plotted as a function of the applied moment in Fig. 5-3. Fig. 5-4a shows a symbolic representation of a beam corresponding to the yield configuration indicated in Fig. 5-2b. It is called the "reduced beam" in this dissertation. The stiffness of this beam can be evaluated by the method of column analogy which is commonly used in indeterminate analysis. (47) The analogous column of the reduced beam is shown in Fig. 5-4b. The width of the column at each section is inversely proportional to the flexural rigidity of that section. To determine the bending moment at B induced by a unit rotation at B, a unit load of one radian is then applied to the analogous column at the end B. The stiffness of the beam at B is equivalent to the stress on the analogous column at that point, that is

$$K_B = M_B = \left( \frac{P}{A} \right) + \left( \frac{M_c}{I} \right) = \frac{1}{A} + \frac{1}{2} \cdot \frac{L}{I}$$

(5.1)
in which \( A \) is the area of the analogous column and \( I \) is the moment of inertia about the centroidal axis \( G-G \). Similarly the moment at \( D \) is equal to:

\[
M_D = \frac{1}{A} - \frac{L}{2} \frac{L}{2} I
\]  
(5.2)

The carry-over factor is simply the ratio of the moment at \( D \) to that at \( B \), or

\[
C_b = \frac{M_D}{M_B}
\]  
(5.3)

If it is known that both ends of the beam would rotate through the same angle and in the same direction at the instant of sidesway, the stiffness may be computed by using the analogous column shown in Fig. 5-5b. The centroidal axis \( G-G \) is now at the right end of the column and the area is assumed to be infinity. The stiffness \( K_b'' \) of the beam at the left end is

\[
K_b'' = \frac{M_c}{I} = \frac{L}{2} \frac{L}{2} I
\]  
(5.4)

where \( I \) is the moment of inertia about axis \( G-G \). The carry-over factor is not needed in this case.

5.2.2 Determination of Column Stiffness

The stiffness factor \( K_c' \) of a column with hinged ends can be determined as the slope of the moment-rotation curve of a beam-column as shown in Fig. 5-6. Within the elastic range the stiffness is given by

\[
K_c' = \frac{1}{C} \frac{EI_c}{h}
\]  
(5.5)

where \( C = \frac{1}{(kh)^2} \left( \frac{1}{2} (1, \ldots, 1, h) \right) \) and \( \kappa = \sqrt{\frac{P}{EI_c}} \)  
(5.6)
As the applied moment increases beyond the elastic limit, the stiffness of the column decreases and approaches zero when the moment is near the maximum value. At this instant the column has lost completely its resistance to any further increase of bending moment. However, if a moment of opposite sense is applied, the column will behave elastically again and its stiffness is equal to that given by Eq. (5.5). This is indicated as "unloading" in Fig. 5-6.

The variation of the stiffness $K_c'$ with the applied end moment for columns having slenderness ratios ranging from 30 to 130 subjected to axial forces of 0.12, 0.2, 0.3, and 0.4 $P_y$ are shown in Figs. 5-7 and 5-8. They are obtained by measuring the slope on the moment-rotation curves presented in Figs. 4-18, 4-19, 4-20, and 4-21. Knowing the axial force $P$ and the end moment $M$ of the column, as determined from the graphic analysis of Art. 4.4, the column stiffness can be obtained by interpolation from these curves.

5.3 Method of Solution

From the results of the investigation presented in Chapters 3 and 4 and the concept of modified stiffness discussed in Art. 5.2, a method for determining the inelastic buckling strength of frames is developed. It is based on the modified moment distribution procedure due to Winter, et al (15), in which the stiffnesses are modified for the effect of axial force present in the members at a given load. In the present method the modification is made not only for the effect of axial force but also for yielding.
When the dimensions and loading condition of a frame are specified, its buckling strength can be determined in the following manner:

1. Carry out a complete elastic-plastic analysis of the frame by assuming that no sidesway instability occurs at all the stages of loading.

2. Select a suitable load level \( w_1 \) and determine the moment at the column tops. The values of \( K_b, C_b, \) and \( K_c' \) can then be obtained by the procedure described in Art. 5.2.

3. Introduce an arbitrary lateral (sidesway) displacement of \( d \) (see Fig. 2-2) and perform a moment distribution computation for the frame. As indicated in Eq. (2.6) the applied fixed end moment can be taken to be proportional to the stiffness \( K_c' \) of each column. Using the end moment values resulting from the distribution process, the horizontal shear \( Q \) of each column may be determined. The sum of these shears \( \Sigma Q \) should be positive if the selected load \( w_1 \) is below the critical value. In other words, a lateral force is required to displace the frame laterally. In the moment distribution procedure, it is required that assumption (1) of Art. 5.1 be valid, i.e., the effect of small axial load on the bending stiffness of the beam is negligible. Therefore, the stability of the frame may be examined by simply considering the loading system shown in
Fig. 5-9c. As explained in Chapter 3, the buckling load thus determined will be very close to the exact value. Although it is possible to obtain more precise results, with the same procedure, by taking the horizontal force into account (Fig. 5-9b) the work involved is formidable.

(4) Repeat steps (2) and (3) for several values of \( w \) that are in the range between the yield load and the ultimate load. By plotting the total shear \( \Sigma Q \) against the load \( w \) for each case, a curve such as that shown in Fig. 5-10 is obtained. The intersection of this curve with the load axis gives the critical load of the frame which will cause it to sway without the application of lateral load.

In determining the stiffness of the members, the following rules are adopted with regard to unloading of the yielded portion:

1. No strain reversal is assumed to take place for the plastic portion of the beam at the instant of sway buckling.

2. For the case when the first plastic hinge forms at the center of the beam, no unloading of the columns is assumed. This is the situation that usually occurs for tall frames or frames with slender columns.
Both (1) and (2) are in agreement with the generally accepted concept of inelastic buckling due to Engesser and Shanley. (10)

(3) When no plastic hinge forms in the beam, one of the columns may be assumed to unload. This assumption is adopted from Ref. (36) and has been checked with experiments.

The procedure outlined above will be illustrated by two complete examples in the following section in connection with the development of a frame buckling curve. The dimensions of the frames in these examples are such that the reduction of the load carrying capacity due to sidesway buckling is appreciable.

5.4 Development of Frame Buckling Curves

As shown in Fig. 1-3b the carrying capacity of frames having the same span length and acted upon by the same loading system can be related to the slenderness ratio of the columns by a frame buckling curve in a manner similar to that for the case of centrally-loaded columns. The curve generally consists of two segments, one segment defines the load-slenderness ratio relationship for buckling in the elastic range, the other segment for the inelastic range. This section is concerned with the determination of such curves by the method presented in Chapter 3 and in Art. 5.3. To illustrate the complete procedure, a buckling curve for the series of frames shown in Fig. 5.11 will be constructed.
The span length $L$ of the frames is assumed to be $80r$ (88.2 ft.), where $r$ is the radius of gyration of the 33WF130 shape. A value of 2.0 is assigned for the loading parameter $a$. The cross sectional properties and the material constants adopted are as follows:

$$
\begin{align*}
A &= 38.26 \text{ in.}^2 \\
I &= 6699.0 \text{ in.}^4 \\
S &= 404.8 \text{ in.}^3 \\
r &= 13.23 \text{ in.} \\
E &= 30 \times 10^3 \text{ ksi} \\
\sigma_Y &= 33 \text{ ksi} \\
P_Y &= 1263 \text{ kips} \\
M_Y &= 1113 \text{ ft-kips} \\
M_p &= 1282 \text{ ft-kips} \\
\frac{L}{2r} &= 40 \\
\bar{w} &= 457.7 \frac{\text{kips}}{\text{ft.}} \\
\bar{w}L &= 40.369 \text{ kips}
\end{align*}
$$

The ultimate load determined by the simple plastic theory (corresponding to a beam mechanism) is $\bar{P}_u = 349 \text{ kips}$ ($\bar{w}L = 232 \text{ kips}$).

In the elastic range, the buckling load of the frames can be determined approximately by the expression:

$$
\bar{P}_{cr} = \frac{\pi^2 EA}{(4r/k)^2} \tag{5.7}
$$

where $k$ is the effective length factor of columns. The value of $k$ can be taken from Fig. 5-12. More accurate results of the buckling load may be determined by the solution presented in Chapter 3. By assuming different values of column height, a set of $\bar{P}_{cr}$ can be obtained. It is plotted non-dimensionally as the dot-dashed line (line C) in Fig. 5-19. This curve is valid only for frames with slenderness ratios greater than that corresponding to point Y shown on the curve. At that point the elastic buckling load is equal to the load which causes initial yielding at the most highly stressed section.
For frames having column slenderness ratios less than that value, inelastic buckling will generally govern their load-carrying capacity.

To obtain the buckling curve applicable in the inelastic range (curve (b) in Fig. 5-19), it is necessary to determine the strength of several frames with various slenderness ratios. In this example two frames having \( \frac{h}{r} = 60 \) and 80 are chosen for illustration. The method of analysis will be explained in detail for these two cases.

**First Case - Frame having \( h = 60 \ r \)**

The procedure presented in Art. 5.3 is applied here to compute the inelastic buckling strength of this frame. The complete analysis consists of the following steps:

**Step 1** Perform an elastic-plastic analysis of the frame, using the method described in Chapter 4, to determine:

1. The exact load-carrying capacity of the frame if sideways buckling is prevented. The ultimate load thus obtained gives one point on curve (a) of Fig. 5-19.

2. The bending moment at the column tops for any applied loads.

The resulting moment-rotation curve from this analysis is shown in Fig. 5-13.

**Step 2** Select a trial load \( \lambda_1 = 0.0048 \) (wL = 194 kips). The intersection of the corresponding moment-rotation curve
of the beam with that of the column is indicated as $0_1$ in Fig. 5-13. The moment $M_B$ at the column top for this load is equal to 0.830 $M_Y$. The axial force in the columns is

$$\bar{P} = \frac{3}{2} \lambda \bar{w}L = 290.7 \text{ kips}$$

and therefore $$\frac{\bar{P}}{P_Y} = \frac{290.7}{1263} = 0.230$$

By interpolation from Figs. 5-7 and 5-8 the stiffness of columns subjected to this combination of bending moment and axial force are determined to be

$$K_c^l = 24.0 \text{ } M_Y \text{ (loading)}$$
$$K_c^u = 48.91 \text{ } M_Y \text{ (unloading)}$$

The computations involved in the evaluation of the stiffness factor $K_b$ and carry-over $C_b$ of the beam are contained in Appendix B. The values of these factors thus obtained are:

$$K_b = 46.63 \text{ } M_Y$$
$$C_b = 0.7125$$

Since there is no plastic hinge forming at the center section of the beam for this trial load (see Appendix B), then according to rule (3) of Art. 5.3 one of the columns may be assumed to unload in the buckling analysis. If the frame is assumed to sway to the right, the left column will be the unloading column.
Step 3 A fixed end moment due to a lateral displacement at the column tops of $M_{FL} = 100$ ft-kips is arbitrarily assigned to act on the unloading column. Then according to Eq. (2.6) the fixed end moment of the loading column is equal to $M_{FR} = \frac{24,000}{48.91} \times 100 = 49.0$ ft-kips. These moments are distributed and balanced as shown on Fig. 5-14.

The resulting shear force may be determined by Eq. (2.9) as follows:

$$Q_l = \frac{1}{h} \left( M'_L - \frac{P}{K_L} M_{FL} h \right)$$

$$= \frac{1}{h} \left( 53.10 - \frac{290.7}{48.91} \cdot 100 \cdot 60 \cdot 1.1025 \right) = \frac{1}{h} \left( 17.79 \right)$$

$$Q_R = \frac{1}{h} \left( M'_R - \frac{P}{K_R} M_{FR} h \right)$$

$$= \frac{1}{h} \left( 43.25 - \frac{290.7}{24.01113} \cdot 49.60 \cdot 1.1025 \right) = \frac{1}{h} \left( 7.99 \right)$$

$$\Sigma Q = \frac{1}{h} \left( 25.78 \right) > 0$$

Consequently, by the criterion of Eq. (2.10), the frame is stable at this selected load level.

Step 4 Select a second trial load with $\lambda_2 = 0.0050$ (wL = 202 kips) and repeat steps 2 and 3. The total resulting shear force for this trial load is $\Sigma Q = \frac{1}{h} \left( 0.61 \right)$. This indicates that the selected load is very close to the actual buckling load. In Fig. 5-15 the total shear $\Sigma Q$ is plotted against the loading ratio $\lambda$ for these two trials, the critical load is determined as the intersection of this
curve with the \( \lambda \)-axis, that is \( \lambda_{cr} = 0.00501 \). The total load corresponding to this value of \( \lambda \) is \( \bar{P}_{cr} = \frac{3}{2} \cdot \lambda_{cr} \bar{W}L = 303 \) kips, therefore, the ratio \( \frac{\bar{P}_{cr}}{\bar{P}_{u}} = \frac{303}{349} = 0.869 \). This furnishes one point on the inelastic buckling curve.

**Second Case - Frame having \( h = 80 \text{ ft} \)**

**Step 1** Carry out a complete analysis the same as that for the first case. The resulting moment-rotation of the column is shown in Fig. 5-16.

**Step 2** As a first try, a load of \( \lambda_1 = 0.0048 \) (\( \bar{W}L = 194 \) kips) is selected. The point of intersection of the moment-rotation curve of the beam for this value of \( \lambda \) with that of the column is marked as \( 01 \) in the figure. The moment \( M_B \) corresponding to point \( 01 \) is 0.770 \( M_y \). The axial force in the column is

\[
P = \frac{3}{2} \lambda_1 \bar{W}L = 290.7 \text{ kips}
\]

and the ratio \( \frac{\bar{P}}{\bar{P}_{u}} = \frac{290.7}{1263} = 0.230 \)

For this combination of axial force and end moment, the column stiffness is

\[
K_c' = 20.96 \text{ } M_y \text{ (loading)}
\]

It may be seen from Fig. 5-16 that the bending moment at the center is equal to \( M_p \) for this trial load. Then according to rule (2) of Art. 5.3, no unloading should be assumed for either of the columns. Therefore, the beam will be bent in an anti-symmetrical form at the instant of buckling. The stiffness factor of the beam may be determined by
the simplified procedure shown in Fig. 5-5. Numerical computations involved are similar to that of the first case. They are also included in Appendix B. The value thus determined for stiffness with anti-symmetrical bending is $K_b'' = 70.87 \text{ M}_y$.

**Step 3** Introduce a fixed end moment due to lateral displacement of $M_F = 100 \text{ ft-kips}$ for each column. These moments can be distributed and balanced in one cycle as indicated on Fig. 5-17. The resulting shear force is:

$$\sum Q = \frac{2}{h} \left[ M'_L - \frac{P}{K_L} M_{FL} \cdot h \right]
= \frac{2}{h} \left[ 77.18 - \frac{299.7}{20.96 \cdot 1.1025} \cdot 100.80 \cdot 1.1025 \right] = -\frac{1}{h} [45.38] \quad (5.11)$$

This indicates that the trial load $\lambda_1$ is higher than the critical load and that a smaller $\lambda$ value should be assumed for the next try.

**Step 4** Use $\lambda_2 = 0.0045$ as the second trial load and repeat steps 2 and 3. The resulting shear force at the column tops is $\sum Q = -\frac{1}{h} [4.90]$, indicating that the selected $\lambda$ is slightly higher than the correct value. Using these results, the inelastic buckling load of the frame can be determined graphically as shown in Fig. 5-18. The value of $\lambda_{cr}$ is equal to 0.0047 and the total load $\bar{P} = \frac{3}{2} \cdot \lambda_{cr} \cdot \bar{W}L = 271 \text{ kips}$, the ratio $\frac{\bar{P}_{cr}}{\bar{P}_u} = \frac{271}{349} = 0.776$. This gives another point on the inelastic buckling curve.
Similar analyses may be performed for frames with different values of $\frac{h}{r}$. These analyses will result a series of points, each of which gives the buckling load of a particular frame. By passing a curve from $Y$ through these points, an inelastic buckling curve is obtained (labelled as curve (b) in Fig. 5-19). At point $T$ this curve becomes tangent to curve (a) which defines the strength of frames if they are restrained against sidesway buckling. For any frame with a slenderness ratio less than that corresponding to point $T$, its load carrying capacity will not be reduced significantly by lateral instability. Therefore within this region the problem of frame stability may be safely ignored and the design can be based on the plastic strength.

5.5 Experimental Work

Two series of model steel frames having column slenderness ratios in the range between 40 and 100 were tested to determine their buckling strength in the inelastic range. The cross sectional shape and the dimensions of these frames are the same as those shown in Fig. 3-3. The following values of column height were adopted: $h = 15, 20, 25, 30, \text{ and } 35$ inches. The corresponding slenderness ratios were $\frac{h}{r} = 41.7, 55.6, 69.4, 83.3, \text{ and } 97.2$.

The frames were loaded in a manner similar to that shown in Figs. 3-3 and 3-4. In the first series of tests (Series A)
a loading ratio \( \alpha = 7.5 \) was selected. For this selected value of \( \alpha \), the total axial thrust in the columns at the computed ultimate load (simple plastic theory) is approximately equal to \( 0.4 P_y \). The \( \alpha \) value for the frames of the second series (Series B) was kept at 3.35. The corresponding axial force in the columns is \( 0.2 P_y \). In this section, the results obtained from Series B tests are briefly discussed. The complete test results will be presented in a forthcoming report.\(^{(40)}\)

The experimentally observed loads of frames of Series B are plotted nondimensionally against the column slenderness ratios in Fig. 5-20. The curves in this figure can be compared with those shown in Fig. 5-19 which were obtained theoretically for frames of WF section and with an \( \alpha \) value equal to 2.0. Owing to the difference in cross sectional shape of members (different moment-curvature relationships) and loading conditions, the comparison between these two figures can be made only on qualitative basis. It may be seen from Fig. 5-20 that the general trend of the experimental inelastic frame buckling curve (curve (b)) is similar to that of the theoretical buckling curve and that the reduction of load-carrying capacity due to sidesway buckling is about the same order of magnitude as that obtained by theoretical computations.

In order to check more exactly the inelastic buckling theory developed in this chapter, experiments on larger
scale frames of WF section are being planned.

5.6 Buckling of Multi-Story Frames

The theoretical solution contained in this chapter can be used to check approximately the inelastic stability of multi-story frames. The procedure of applying this theory may be illustrated with reference to the single-bay three-story frame shown in Fig. 5-21a. The frame is loaded by a uniform load \( w \) on each floor. Due to the variation of axial thrust in each story, the columns may be made of different member sizes.

In general, three possible modes of sidesway buckling may be expected for this frame, these are illustrated in Fig. 5-21b. Corresponding to each buckling mode a fictitious single-story portal frame may be obtained. Figure 5-21c shows three such frames and their loading conditions for the assumed failure modes. It may be seen in this figure that in obtaining these frames the following simplifications have been made:

1) All the bases are assumed to be pinned.
2) The effect of the bending moments transmitted from the upper and the lower frames is neglected.

The above simplifications may be shown to yield results on the safe side. The buckling loads of the frames can then be computed by the method developed in this chapter. The lowest value among the computed buckling loads of these
frames determines approximately the buckling strength of the multi-story frame.

At the present time there is no experimental evidence to justify this approximate method. Much more extensive investigations on this problem are needed before any conclusive answer can be obtained.

5.7 Summary

The investigations presented in this chapter can be summarized as the following:

1. A method for solving the problem of sidesway buckling of portal frames in the inelastic range was developed. It is a modification of the moment distribution procedure commonly used for analyzing the stability of elastic frames. In the present method, all the required stiffness and carryover constants are modified for the combined effects of axial force and yielding. These effects can be precisely evaluated with the aid of the elastic-plastic analysis described in Chapter 4.

2. To illustrate the procedure presented, two numerical examples were solved in detail and the results were shown in the form of a frame buckling curve. (Fig. 5-19)

3. A qualitative comparison between the frame buckling curve computed by the theoretical procedure and that obtained from model frame tests was described. Although the cross sectional shape and the loading condition of the test frames
differ from those assumed in the theoretical computations, the general trend of those buckling curves was found to be similar. (Fig. 5-20)

(4) Using the results of theoretical solutions, an approximate method for checking the stability of multi-story frames was developed. (Fig. 5-21)
6. APPLICATIONS TO DESIGN

It may be noticed from the buckling curve obtained in Art. 5.4 that for most practical frames overall instability generally takes place in the plastic range and that the reduction of load-carrying capacity due to this type of failure is quite significant for frames with columns of intermediate slenderness ratios. Therefore, for certain frames, it may be necessary to base the design on their inelastic buckling strength. However, the determination of this strength by the procedure proposed in this dissertation requires lengthy computations and tedious graphic technique. It would be impossible to use such procedure in actual practice. There exists a need to develop design charts for practical applications. In this chapter a method for constructing such charts is described. Some typical samples will be presented and their use will be illustrated by solving two design problems.

6.1 Development of Design Charts

In designing any portal frame of the type considered in this investigation the following information is usually given:

1. Loads - w and P (also P and the value of a)
2. Dimensions - Span length L and height h.

It is required that the designer selects a member which
will be sufficient to carry the given loads. For single story building frames, it is usually economical to use the same member size for both beam and column. This is assumed to be the case in developing the design charts presented in this article.

In the following discussion, it is assumed that the design will be based on the limiting strength (plastic strength or buckling strength) and that the frames are allowed to sway only in the plane of the applied loads.

To develop charts for design use, all the above-mentioned factors are to be taken into consideration. After investigating several possible ways of grouping these factors, it was found convenient to fix the following two for each chart:

1. Loading parameter \( a \) (i.e., fixing the ratio between the uniform load \( w \) and the concentrated load \( P \))
2. Span length \( L \)

Therefore the variables involved in each chart were reduced to:

1. Column height \( h \) (or the ratio \( \frac{h}{L} \))
2. Member size
3. Load-carrying capacity \( \frac{P_{cr}}{P_u} \) (or the ratio \( \frac{P_{cr}}{P_u} \))

Then corresponding to each specified member size, a frame buckling curve (\( \frac{P_{cr}}{P_u} \) vs. \( \frac{h}{L} \)) similar to that shown in
Fig. 5-19 may be computed. It will be shown in the subsequent paragraphs that the shape of the non-dimensional buckling curves depends mainly on the radius of gyration \( r \) of the members. Therefore for each chart it is only necessary to compute several of these curves for a few WF shapes having various \( r \) values.

At first, it will be shown that for frames which have

1. same span length \( L \) and height \( h \),
2. same loading condition (constant \( a \)), and
3. uniform member size for both beam and column,

the ratio \( \frac{P_{cr}}{P_u} \) for buckling in the elastic range is dependent only on the radius of gyration of the members. According to Eq. (5.7) the elastic buckling load is given by

\[
\bar{P}_{cr} = \frac{L^2 E I}{(k h)^2} \tag{6.1}
\]

It may be seen from Fig. 5-12 that for all the frames which satisfy the above conditions the value of \( \bar{k} \) is independent of the member size used. On the basis of simple plastic theory \( P_u \) can be computed by the equation

\[
\bar{P}_u = (1+\alpha) \frac{8 M_p}{L} \tag{6.2}
\]

Therefore the ratio \( \frac{P_{cr}}{P_u} \) is

\[
\frac{\bar{P}_{cr}}{\bar{P}_u} = \frac{\frac{L^2 E I}{(k h)^2}}{(1+\alpha) \frac{8 M_p}{L}}
\]
where \( f \) is the shape factor of the member. The quantity in the bracket is a constant for all these frames. The values of \( f \cdot \frac{E}{d} \) for all the commonly used beam sections are in the range from 0.460 to 0.474 and those for all the column sections are between 0.470 and 0.491. These variations may be considered to be small (amount to 3 to 4%) in practical computations. Hence, the ratio \( \frac{P_{cr}}{P_u} \) can be assumed to depend on \( r \) alone.

Next, the same ratio will be examined for buckling in the inelastic range. As illustrated in Art. 5.4, the following two cases are usually encountered in determining the inelastic buckling strength:

1. Buckling takes place when the first plastic hinge has formed at the center of the beam.
2. Buckling takes place before the formation of the plastic hinge in the beam.

For the first case, the buckling load can be expressed by the equation
\[ P_{cr} = (1 + \alpha) \frac{w_{cr} L}{2} \]
\[ = \frac{(1 + \alpha) 4(M_o + M_s)}{L} \]  \hspace{1cm} (6.4)

in which \( M_o \) is the moment applied to the column tops at the instant of sidesway buckling. This moment can be expressed as a fraction of \( M_p \) by the relation \( M_o = \mu M_p \), where \( \mu \) is always less than unity. Substitution of this relation into Eq. (6.4) results in the following equation for the buckling load

\[ \bar{P}_{cr} = (1 + \alpha) \frac{4(1 + \mu) M_p}{L} \]  \hspace{1cm} (6.5)

The ratio \( \frac{P_{cr}}{P_u} \) can be obtained by dividing Eq. (6.5) by Eq. (6.2), thus

\[ \frac{P_{cr}}{P_u} = \frac{(1 + \alpha) \frac{4(1 + \mu) M_p}{L}}{(1 + \alpha) \frac{M_p}{L}} \]
\[ = \frac{1 + \mu}{2} \]  \hspace{1cm} (6.6)

The value of \( \mu \) will be known if the moment \( M_o \) is determined. The determination of this moment requires the elastic-plastic analysis described in Chapter 4 and the inelastic buckling analysis presented in Chapter 5. The parameters involved in the first type of analysis are the following:

For the beam \( \frac{L}{2r} \) and \( \lambda_{cr} = \frac{w_{cr}}{w} \)

For the column \( \frac{h}{r} \) and \( \frac{P_{cr}}{P_y} \)
For frames having the same dimensions, the values of \( \frac{L}{2r} \) and \( \frac{h}{r} \) are inversely proportional to \( r \) and the values of \( \lambda_{cr} \) and \( \frac{P_{cr}}{P_u} \) can be shown to be related to \( r \) as follows:

By definition, \( \bar{w} = \sigma_y \frac{A}{d} \) and \( w_{cr} = \frac{8(1 + \mu)}{L^2} \), the ratio \( \lambda_{cr} \) is then

\[
\lambda_{cr} = \frac{\frac{8(1+\mu)}{L^2}}{\sigma_y \frac{A}{d}} = \frac{16(1+\mu)\cdot f}{L^2} \cdot r^2 \tag{6.7}
\]

and the ratio \( \frac{\bar{P}_{cr}}{P_y} \) is

\[
\frac{\bar{P}_{cr}}{P_y} = \frac{(1+\mu) \cdot \frac{4(1+\mu)M_p}{L}}{\sigma_y \cdot A} = 4(1+\mu)(1+\mu)(4 \cdot \frac{r}{d}) \cdot r \tag{6.8}
\]

As explained before the product \( f \times \frac{r}{d} \) may be considered as constant for most WF shapes. After performing numerous computations of this type it is found that the values of \( \lambda_{cr} \) and \( \frac{P_{cr}}{P_y} \) do not change significantly with \( \mu \) and \( f \). From the above discussions it may be concluded that the results of the elastic-plastic analysis depend mainly on the value of \( r \).

Since these results are used directly to determine the stiffness of members, the buckling strength determined by using these stiffnesses is also dependent mainly on \( r \). Therefore, the ratio \( \frac{P_{cr}}{P_u} \) may be considered approximately as a function
of \( r \) only.

By extending the above explanation further a similar conclusion may be arrived at for the case that buckling occurs before the formation of a plastic hinge in the beam. But in this case the ratio \( \frac{bc}{Pr} \) does not vary appreciably with member sizes. (See the curves shown on Figs. 6-1 and 6-2 in the range of \( \frac{h}{L} = 0.4 \) to 0.8.)

From all the foregoing discussions, it is possible to conclude that the shape of the frame buckling curves (for fixed values of \( L \) and \( \alpha \)) is dependent mainly on the radius of gyration \( r \) of the members and that design charts may be developed by computing several frame buckling curves for some typical WF shapes with various \( r \) values.

In Figs. 6-1 and 6-2 two sample design charts are presented. These charts are computed for the following two cases:

1. \( L = 70 \) ft. \( \alpha = 2.0 \)
2. \( L = 90 \) ft. \( \alpha = 2.0 \)

The structural shapes selected for calculating the buckling curves are 36 WF 260 \((r = 15.00 \) in.), 33 WF 130 \((r = 13.23 \) in.), 27 WF 102 \((r = 10.96 \) in.), 21 WF 73 \((r = 8.64 \) in.) and 18 I 54.7 \((r = 7.07 \) in.). It is expected that curves for members having other radii of gyration may be fitted in by interpolation between these curves. The use of these charts will be illustrated by examples in Art. 6.2.
6.2 **Illustrative Examples**

**Example No. 1**  As a first example consider a portal frame loaded as shown in Fig. 6-3. The following is given:

- **Span length**: \( L = 70 \text{ ft.} \)
- **Column height**: \( h = 56 \text{ ft.} \)
- **Working loads**: \( w = 1.30 \text{ kips/ft.} \)

\[ \begin{align*}
P &= 91 \text{ kips} \\
\alpha &= 2.0 \\
\end{align*} \]

According to current design recommendations,\(^{(2,39)}\) a load factor of 1.85 is adopted. The design ultimate loads are then equal to

\[ \begin{align*}
w_u &= 2.405 \text{ kips/ft.} \\
P_u &= 168.35 \text{ kips}
\end{align*} \]

On the basis of simple plastic theory the required plastic moment is

\[ M_p = \frac{w_u L^2}{16} = \frac{2.405 \times 10^2}{16} = 736.5 \text{ ft.-kips} \]

From the plastic moment table of Ref. (39) the most economical section supplying this value is the 27 WF 94 having \( r = 10.87 \) inch. To check this member size for inelastic frame buckling, the design chart of Fig. 6-1 can be used. Using the curve
corresponding to \( r = 10.96 \) in the chart, the ratio \( \frac{P_{cr}}{P_u} \) is determined to be 0.855. The inelastic buckling strength of the frame then equals

\[
P_{cr} = 0.855 \ P_u
\]

or \( w_{cr} = 2.15 \ \text{kips/ft.} < 2.405 \ \text{kips/ft.} \)

This indicates that the selected member 27 WF 94 is not sufficient to carry the design ultimate load. Therefore, the design has to be revised. This can be done in the following manner: Assume that the same value of \( \frac{P_{cr}}{P_u} \) may be applied to the frame of the new member size and let \( P_{cr} \) be equal to the design ultimate load. Then the required \( M_p \) value is

\[
M_p = \frac{w_u L^2}{16} \ \frac{1}{0.855} = 861.4 \ \text{ft} \cdot \text{kips}
\]

The most economical section is the 30 WF 108 supplying 950 ft-kips. This shape should be adopted for the final design.

**Example No. 2**  The frame of the second example is shown in Fig. 6-4. The following information is given:

\[
\begin{align*}
\text{Span length} & \quad L = 90 \ \text{ft.} \\
& \quad h = 36 \ \text{ft.} \\
& \quad h \ L = 0.4
\end{align*}
\]
Working loads: 

\[ w = 1.20 \text{ kips/ft.} \]
\[ P = 108 \text{ kips} \]
\[ a = 2.0 \]

Using the same load factor as that in Example No. 1 the design ultimate loads are

\[ w_u = 2.22 \text{ kips/ft.} \]
\[ P_u = 199.8 \text{ kips} \]

The required plastic moment value is

\[ M_p = \frac{2.22 \times 90^2}{16} = 1123 \text{ ft-kips} \]

The 33 WF 130 may be selected with \( M_p = 1282 \text{ ft-kips} \) and \( r = 13.23 \text{ in.} \) For the chosen member size, the ratio \( \frac{P_{cr}}{P_u} \) is determined to be 0.92 from the curve \( r = 13.23 \) on Fig. 6-2. Therefore

\[ \omega_{cr} = 0.92 \times \frac{8 \times 1282}{90^2} \]
\[ = 2.33 \text{ kip}^2/ft > 2.22 \text{ kip}^2/ft \]

This shows that the 33 WF 130 section is sufficient.

6.3 Comparison of Theoretical Results with the AISC Design Rule

Since a great deal of computations were carried out for obtaining the design charts presented in Figs. 6-1 and 6-2, it is now possible to check the AISC design rule discussed in Art. 2.2.2 with the theoretical results. The AISC design
rule for frame stability requires that the axial load and the slenderness ratio of columns should be so proportioned that

\[
\frac{2}{P} \cdot \frac{1}{P_0} \cdot \frac{h}{r_c} \leq 1.0
\]  

(6.9)

Within this limit the reduction of load carrying capacity due to sidesway buckling would be negligibly small. In Fig. 6-5 results obtained from theoretical solutions are plotted and compared with the above rule. The computed inelastic buckling loads are expressed as percent of the ultimate load determined by the exact analysis presented in Chapter 4, in which the reduction of strength due to column instability alone is considered (that is the frames are assumed to be prevented from sidesway). The \( \bar{P} \) values used are the axial forces in the columns at this computed ultimate load.

It may be seen from Fig. 6-5 that, within the range of \( \frac{\bar{P}}{P_Y} \) and \( \frac{h}{r_c} \) that has been covered by the theoretical computations, the design rule is somewhat conservative and that the buckling strength of some frames having the combination of \( \frac{\bar{P}}{P_Y} \) and \( \frac{h}{r_c} \) considerably outside of the safe region defined by the rule can be more than .95% of the plastic strength. On the basis of these observations the following rule is tentatively proposed:

\[
\frac{2}{P} \cdot \frac{1}{85} \cdot \frac{h}{r_c} \leq 1.0
\]  

(6.10)

It is plotted as the dotted line on Fig. 6-5. The validity
of this rule for frames subjected to high axial forces and for frames having high column slenderness ratios should be examined by more extensive computations.
7. SUMMARY AND CONCLUSIONS

This dissertation deals with the stability of steel portal frames which are acted upon by a uniform load on the beam and concentrated load on the columns. The frames are assumed to fail by sidesway buckling in the plane of the applied loads. The following problems are included in the investigation:

(1) Buckling of frames in the elastic range.
(2) Method of analysis for structures with non-linear behavior (due to axial forces, deformations and yielding).
(3) Buckling of frames in the inelastic range.

The nature of these problems has been examined in detail in Chapter 1. A survey of the current approaches was presented in Chapter 2.

The contributions contained in this dissertation may be summarized as follows:

(1) An exact solution for determining the elastic buckling strength of pinned-base frames is obtained. Using this solution, the effect of bending moment present in each member at the instant of buckling is clarified.

(2) Results of the elastic solution are compared with those observed from buckling tests. Satisfactory correlation
between theory and experiments has been obtained. (Table III)

(3) A procedure by which the elastic buckling strength of frames with partial base fixity may be determined is presented. Preliminary calculations show that the buckling strength can be increased appreciably due to base restraint.

(4) A precise method of analyzing statically indeterminate frames in the plastic range is established. It takes into consideration such effects as axial thrusts, residual stresses, yielding and deformations due to bending. The method involves the determination of a moment and rotation at a joint by the intersection of moment-end rotation curves for beams and columns.

(5) A numerical integration procedure for obtaining the moment versus end rotation curves of beams is developed. The results of computation are presented in the form of a nomograph which allows the rapid determination of these curves.

(6) Using the above method of analysis, it is possible to determine exactly a) the load-carrying capacity of frames which are restrained from sidesway, b) the complete load-deformation relationship and c) the required rotation capacity at the first plastic hinge.

(7) The problem of sidesway buckling of partially yielded frames is solved. The method of solution consists of the following steps:
1. Perform a precise analysis of the frame under investigation for all stages of loading, assuming that no sidesway instability would take place.

2. Select a suitable trial load and determine the stiffness of members by the method outlined in Chapter 5.

3. Introduce an arbitrary lateral displacement at the column tops and perform a moment distribution computation for the frame. Using the end moment resulting from the distribution process, the horizontal shear of each column may be determined. If the sum of the shears is positive then the trial load is less than the buckling load.

4. Select another trial load which is higher than the first one and repeat steps 2 and 3. The critical condition will be reached when the sum of the resulting shears becomes zero. The load at which this occurs determines the inelastic buckling strength of the frame.

(8) To facilitate the solution of actual problems, a method whereby design charts may be derived is presented. Sample charts are obtained and their use is illustrated.

The following conclusions can be drawn from the results of the investigation presented in the preceding chapters:
(1) For unbraced portal frames with slender columns, the elastic buckling strength should be used as the basis of design. Methods for predicting such strength have been developed in this dissertation.

(2) In the elastic range, the reduction of buckling strength due to the presence of primary bending moment is small and can be neglected in practical computations.

(3) The ultimate (plastic) strength of frames which are restrained against sideways can be precisely determined. In the determination of this strength the moment-rotation relationship of the columns has to be considered. (Fig. 4-2)

(4) For most practical frames sideways buckling takes place in the inelastic range. This will usually cause a reduction of the load-carrying capacity of the frames.

(5) The prediction of this type of failure can be made by the method presented in this investigation. Due to the complexity of the method, it is recommended in practice to proportion the members such that the possibility of overall may be eliminated. This can be achieved by selecting member sizes to meet the proposed design rule.

The theoretical and experimental studies presented in this dissertation constitute the first steps in obtaining solutions to the buckling of multi-story frames. Further problems that need be studied are:
(1) Experimental verification of the inelastic buckling theory developed in this dissertation.

(2) Inelastic buckling of portal frames with partial base fixity.

(3) Ultimate strength of frames subjected to combined vertical and horizontal loads.

(4) Inelastic instability of multi-story building frames.
8. NOMENCLATURE

A = Cross sectional area
C = Carry-over factor
E = Young's modulus of elasticity
\( E_{st} \) = Strain-hardening modulus
\( E_t \) = Tangent modulus
EI = Flexural rigidity
\((EI)_{eff}\) = Effective flexural rigidity
F = Loading term
H = Horizontal reaction at the base
= Axial force in beam
I = Moment of inertia
\( I_b \) = Moment of inertia of beam section about its strong axis
\( I_c \) = Moment of inertia of column section about its strong axis
\( I_e \) = Effective moment of inertia
\( I_s \) = Moment of inertia of fictitious base restraint beam
K = Stiffness factor
\( K_b \) = Stiffness factor of beam with far end fixed
\( K_b' \) = Stiffness factor of beam having equal end rotations
\( K_c \) = Stiffness factor of column with far end hinged
L = Span length
= Length of member
\( L/2r_b \) = A non-dimensional parameter used in determining moment-rotation curves of beams
M = Moment
\( M' \) = Column moment resulting from moment distribution procedure

\( M_B \) = Moment at column top

\( M_C \) = Moment at the center of beam

\( M_F \) = Fixed end moment

\( M_O \) = Moment at column top at the instant of buckling

\( M_P \) = Full plastic moment = \( \sigma_y Z \)

\( M_Y \) = Nominal yield moment = \( \sigma_y S \)

\( P \) = Concentrated load applied at column top

\( = \) Applied beam load (See Fig. 2-4)

\( P_{cr} \) = Elastic buckling load of frames shown in Fig. 2-4

\( P'_{cr} \) = Elastic buckling load of frames shown in Fig. 2-5

\( P''_{cr} \) = Elastic buckling load of upper frame (See Fig. 3-5b)

\( P_{cr} \) = Elastic buckling load of lower frame (See Fig. 3-5b)

\( \bar{P} \) = Half of the total applied loads = \( (1+\bar{a})wL/2 \)

\( \bar{P}_{cr} \) = Half of the total buckling loads (elastic or inelastic)

\( \bar{P}_{exp} \) = Half of the total buckling loads observed from experiments

\( \bar{P}_u \) = Half of the total ultimate loads based on simple plastic theory

\( \bar{P}_y \) = Half of the total applied loads corresponding to initial yielding

\( P_y \) = Axial yield load = \( \sigma_y A \)

\( Q \) = Shear force at column top

\( R \) = Sidesway deflection

\( = \) Radius of curvature

\( S \) = Section modulus

\( V \) = Vertical reaction

\( W_F \) = Inelastic buckling load of frame (See Eq. (2.17) )
$W_u = \text{Ultimate load of frame by simple plastic theory (See Eq. (2.17) )}$

$W_{cr} = \text{Elastic buckling load of frame (See Eq. (2.17) )}$

$Z = \text{Plastic modulus}$

$b = \text{Flange width}$

$c = \text{Non-dimensional coefficient in four-moment equation defined by Eq. (2.2)}$

$c' = \text{Derivative of } c \text{ with respect to } p$

$d = \text{Depth of section}$

$= \text{Horizontal displacement of column top}$

$f = \text{Shape factor } = Z/S$

$h = \text{Height of frame}$

$h_1, h_2 = \text{Height of fictitious frames (See Fig. 3-7)}$

$h/r_c = \text{Slenderness ratio of column}$

$k = \sqrt{P/EI}$

$n = \text{Integers defining the position of points on a beam deflection curve}$

$= \text{Integers defining the position of the beam loads (See Fig. 2-4)}$

$p = \text{Axial force in member}$

$r = \text{Radius of gyration}$

$r_b = \text{Radius of gyration of the beam section about its strong axis}$

$r_c = \text{Radius of gyration of the column section about its strong axis}$

$s = \text{Non-dimensional coefficient in four-moment equation defined by Eq. (2.2)}$

$s' = \text{Derivative of } s \text{ with respect to } p$

$t = \text{Thickness of flange}$

$w = \text{Intensity of uniformly distributed load}$

$= \text{Thickness of web}$
$w_{cr}$ = Intensity of uniformly distributed load at the limit of stability

$\bar{w}$ = $\sigma_y A / d$

$x$ = Coordinate parallel to beam

$y$ = Coordinate perpendicular to beam

= Deflection of beam

$\alpha$ = Loading parameter which relates the concentrated load $P$ to the uniformly distributed load $w$

$\beta = \frac{H}{w}$

$\Delta$ = Increment of a quantity

= Determinant

$\delta$ = Deflection at a point

$\delta'$ = Deflection at the center section of beam when the first plastic hinge forms at this section

$\delta_u$ = Deflection at ultimate load

$\varepsilon$ = Strain

$\varepsilon_{st}$ = Strain at the onset of strain-hardening

$\varepsilon_y$ = Strain corresponding to initial yield point stress

$\theta$ = Slope of beam deflection curve

= End rotation of member

$\theta'$ = Rotation of column top at the instant when the first plastic hinge forms at the center section of beam

$\theta''$ = Rotation of column top when the ultimate load is reached

$\lambda$ = Loading ratio = $w / \bar{w}$

$\mu$ = Ratio of column end moment at the instant of buckling to plastic moment

$\rho$ = Rotation of member

$\sigma$ = Stress
\[ \sigma_{RC} = \text{Maximum residual compressive stress} \]
\[ \sigma_{RT} = \text{Maximum residual tensile stress} \]
\[ \sigma_y = \text{Yield stress} \]
\[ \phi = \text{Curvature} \]
\[ \phi_y = \frac{2\sigma_y}{E_d} = \text{Curvature corresponding to initial yield} \]
\[ \text{under pure moment} \]
9. **TABLES AND FIGURES**
### Table I  
**ELASTIC BUCKLING LOAD OF FRAMES SHOWN IN FIG. 2-4**

<table>
<thead>
<tr>
<th>Loading Position</th>
<th>Discrepancies</th>
<th>Loading Position</th>
<th>Discrepancies</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) n = 2 Center Point</td>
<td>(4) (3)-(1) (%)</td>
<td>(6) n = 2 Center Point</td>
<td>(9) (8)-(6) (%)</td>
</tr>
<tr>
<td>(2) n = 3 Third Point</td>
<td>(5) (3)-(2) (%)</td>
<td>(7) n = 3 Third Point</td>
<td>(10) (8)-(7) (%)</td>
</tr>
<tr>
<td>(3) n = ∞ Column Tops</td>
<td></td>
<td>(8) n = ∞ Column Tops</td>
<td></td>
</tr>
<tr>
<td>EI</td>
<td>EI</td>
<td>EI</td>
<td>EI</td>
</tr>
<tr>
<td>PcrEIh²</td>
<td>PcrEIh²</td>
<td>PcrEIh²</td>
<td>PcrEIh²</td>
</tr>
<tr>
<td>1.770</td>
<td>1.815</td>
<td>1.775</td>
<td>1.817</td>
</tr>
<tr>
<td>1.821</td>
<td>1.821</td>
<td>2.8</td>
<td>0.3</td>
</tr>
<tr>
<td>2.5</td>
<td>0.2</td>
<td>2.5</td>
<td>0.2</td>
</tr>
<tr>
<td>1.032</td>
<td>1.046</td>
<td>1.058</td>
<td>1.071</td>
</tr>
<tr>
<td>1.160</td>
<td>1.160</td>
<td>11.0</td>
<td>9.8</td>
</tr>
<tr>
<td>8.8</td>
<td>7.7</td>
<td>8.8</td>
<td>7.7</td>
</tr>
</tbody>
</table>

$\frac{P_{cr} - P'_{cr}}{P_{cr}}$ % | 2.6 | 2.4 | 0 | --- |

$P_{cr} = $ Actual buckling load

$P'_{cr} = $ Buckling load determined on the basis of the loading condition as shown in Fig. 2-5.
### Table II - ELASTIC BUCKLING LOAD OF FRAMES SHOWN IN FIG. 1-4

<table>
<thead>
<tr>
<th>Loading Position</th>
<th>Discrepancies</th>
<th>Loading Position</th>
<th>Discrepancies</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha = 0 ) Roof Load Only</td>
<td>( \alpha = 2 ) Column Load Only</td>
<td>( \alpha = 0 ) Roof Load Only</td>
<td>( \alpha = 2 ) Column Load Only</td>
</tr>
<tr>
<td>( (1) )</td>
<td>( (2) )</td>
<td>( (3) )</td>
<td>( (4) )</td>
</tr>
<tr>
<td>( (3) - (1) )</td>
<td>( (3) - (2) )</td>
<td>( (3) - (1) )</td>
<td>( (3) - (2) )</td>
</tr>
<tr>
<td>( \frac{P_{cr}EI}{h^2} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.787</td>
<td>1.811</td>
<td>1.821</td>
<td>1.9</td>
</tr>
</tbody>
</table>

### Table III - ELASTIC BUCKLING TEST RESULTS

<table>
<thead>
<tr>
<th>Test No.</th>
<th>Span Length ( L ) in.</th>
<th>Column Height ( h ) in.</th>
<th>Slenderness Ratio ( h/r )</th>
<th>( \frac{P}{F_1} = \alpha )</th>
<th>Yield Load ( 2F_y ) kip</th>
<th>Predicted Buckling Load ( 2P_{cr} ) kip</th>
<th>Test Load ( 2P_{exp} ) kip</th>
<th>( \frac{P_{exp}}{P_{cr}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>P-3</td>
<td>50</td>
<td>30</td>
<td>83.4</td>
<td>8.5</td>
<td>16.14</td>
<td>8.50</td>
<td>8.15</td>
<td>0.96</td>
</tr>
<tr>
<td>P-4</td>
<td>50</td>
<td>35</td>
<td>97.2</td>
<td>7.7</td>
<td>14.12</td>
<td>6.61</td>
<td>6.46</td>
<td>0.98</td>
</tr>
</tbody>
</table>
Fig. 1-1 TYPES OF FRAME INSTABILITY

Fig. 1-2 LOAD-DEFLECTION RELATIONSHIP OF A SYMMETRICAL FRAME LOADED SYMMETRICALLY
**Fig. 1-3** LOAD-CARRYING CAPACITY OF COLUMNS AND FRAMES
\[ P = a \left( \frac{wL}{2} \right) \]

\[ \bar{P} = (1+a)\frac{wL}{2} \]

Fig. 1-4 FRAME DIMENSIONS AND LOADS
Moments and Member Rotations $\rho$ as shown are assumed to be positive.

Fig. 2-1 NOMENCLATURE USED IN FOUR-MOMENT EQUATION
Fig. 2-2 SIDESWAY BUCKLING OF A RIGID BENT

Fig. 2-3 MOMENT OF INERTIA REDUCTION FACTOR FOR WF MEMBERS ACCORDING TO TANGENT MODULUS THEORY (Taken from Ref. 28)
Stable Equilibrium  
Unstable Equilibrium

(a)  
(b)

Fig. 2-4  FRAME STABILITY PROBLEM STUDIED IN REFS. (29) and (32).

Fig. 2-5  THE EFFECT OF AXIAL FORCE H IN THE BEAM ON THE BUCKLING STRENGTH OF FRAMES
Fig. 3-1 EQUILIBRIUM OF MOMENTS AND FORCES

\[ P = a \left( \frac{wL}{2} \right) \]

\[ \bar{P} = (1 + a) \frac{wL}{2} \]

Fig. 3-2 DETERMINATION OF BUCKLING LOAD BY GRAPHIC METHOD
Fig. 3-3 MODEL FRAME TESTS

Fig. 3-4 ELASTIC BUCKLING TEST
Fig. 3-5 LOAD-DEFLECTION CURVES OF FRAME P-4

Fig. 3-6 FRAME P-4 AFTER TESTING
Fig. 3-7 THE EFFECT OF PARTIAL BASE FIXITY
(a) Typical Stress-strain Curve of Mild Steel

(b) Idealized Stress-Strain Curve

Fig. 4-1 STRESS-STRAIN DIAGRAMS OF STEEL
Fig. 4-2 MOMENTS AND FORCES ON BEAM AND COLUMNS

\[ \frac{M_p}{M_y} = 1.144 \]

Proportional Limit

\[ \frac{M}{M_y} = 0.7 \]

Fig. 4-3 MOMENT-CURVATURE RELATIONSHIP, INCLUDING INFLUENCE OF RESIDUAL STRESSES
\[ \sigma_{RC} = 0.3 \sigma_y \]
\[ \sigma_{RT} = \left( \frac{bt}{bt + w(d - 2t)} \right) \sigma_{RC} \]

\( b = \text{Flange Width} \)
\( d = \text{Depth of section} \)
\( t = \text{Thickness of flange} \)
\( w = \text{Thickness of Web} \)

**Fig. 4-4** ASSUMED COOLING RESIDUAL STRESS PATTERN

**Fig. 4-5** BEAM DEFLECTION CURVES OBTAINED BY NUMERICAL INTEGRATION \((\lambda = 0.002)\)
Fig. 4-6 MOMENT-SLOPE RELATIONSHIPS
($\lambda = 0.002$)

Fig. 4-7 $n$ VS. SLOPE CURVES ($\lambda = 0.002$)
Fig. 4-8  BEAM DEFLECTION CURVES FOR $\lambda = 0.002$
Fig. 4-9 NOMOGRAPH FOR OBTAINING MOMENT-ROTATION CURVE OF BEAM (λ = 0.002)
Fig. 4-10  BEAM DEFLECTION CURVES FOR \( \lambda = 0.004 \)
Fig. 4-11 NOMOGRAPH FOR OBTAINING MOMENT-ROTATION CURVE OF BEAM ($\lambda = 0.004$)
Fig. 4-12: BEAM DEFLECTION CURVES FOR $\lambda = 0.006$
Fig 4-13 NOMOGRAPH FOR OBTAINING MOMENT-ROTATION CURVE OF BEAM (\(\lambda = 0.006\))
Fig. 4-14  BEAM DEFLECTION CURVES FOR $\lambda = 0.008$

$\frac{M_C}{M_y} = 1.1$

$\lambda = 0.008$

$\frac{y}{r_b}$

$L/2r_b$
Fig. 4-15 NOMOGRAPH FOR OBTAINING MOMENT-ROTATION CURVE OF BEAM ($\lambda = 0.008$)
Fig. 4-16 THRUST-MOMENT-CURVATURE RELATIONSHIP INCLUDING INFLUENCE OF RESIDUAL STRESSES
Fig. 4-17 MOMENT-ROTATION CURVES OF COLUMNS ($\bar{P} = 0$)

$30 \ 70 \ 110 = \frac{h}{r_c}$

$\bar{P} = 0$

$\sigma_{rc} = 0.3 \sigma_y$

$\sigma_y = 33 \text{ ksi}$
Fig. 4-18  MOMENT-ROTATION CURVES OF COLUMNS ($\bar{P} = 0.12P_y$)
Fig. 4-19 MOMENT-ROTATION CURVES OF COLUMNS ($\bar{P} = 0.2P_y$)
Fig. 4-20  MOMENT-ROTATION CURVES OF COLUMNS  ($\bar{P} = 0.3P_y$)
Fig. 4-21  MOMENT-ROTATION CURVES OF COLUMNS  \( \bar{P} = 0.4P_y \)
Fig. 4-22 DETERMINATION OF MOMENT AT POINT B

Fig. 4-23 ILLUSTRATIVE EXAMPLE No. 1
Fig. 4-24  MOMENT-ROTATION CURVE  
(EXAMPLE NO. 1)
Fig. 4-25 LOAD-DEFLECTION CURVE OF EXAMPLE NO. 1

Fig. 4-26 ILLUSTRATIVE EXAMPLE NO. 2
Fig. 4-27  MOMENT-ROTATION CURVE (EXAMPLE NO. 2)
Angle Change $= 2\Delta \theta$

Fig. 4-28 DEFLECTION AND HINGE ROTATION DUE TO $\Delta \theta$

Fig. 4-29 LOAD-DEFLECTION AND LOAD-HINGE ROTATION CURVES (EXAMPLE NO. 2)
Fig. 5-1  YIELD CONFIGURATION OF THE FRAME

Fig. 5-2  MOMENT DIAGRAM AND VARIATION OF YIELDING ON THE BEAM
Fig. 5-3 THE EFFECT OF YIELDING ON THE FLEXURAL RIGIDITY OF THE 27WF94 SECTION

\[
\frac{(EI_b)_{\text{eff}}}{EI_b}
\]

\[
\frac{M}{My}
\]

(c) Loads on the Analogous Column

\[
M = 1 \cdot \frac{L}{2}
\]

Fig. 5-4 DETERMINATION OF \(K_b\) and \(C_b\) BY COLUMN ANALOGY
Analogous Column

\[ A = \infty \]

\[ IG \text{ is finite} \]

Fig. 5-5 DETERMINATION OF \( K_d'' \) FOR MEMBER WITH EQUAL END ROTATIONS

Fig. 5-6 STIFFNESS FACTOR OF A BEAM-COLUMN WITH CONSTANT AXIAL FORCE
Variations of Column Stiffness Factor $K_c$ with the Applied Moment.

Fig. 5-7
Fig. 5-8 VARIATIONS OF COLUMN STIFFNESS FACTOR $K_c'$ WITH THE APPLIED MOMENT
Simultaneous Equations

\[ (a) \quad \text{Loading Condition for Determining Buckling Constants} \]

\[ (b) \quad \text{Loading Condition for Analyzing Buckling Load} \]

**Fig. 5-9** SIMPLIFICATION OF LOADING CONDITIONS

**Fig. 5-10** DETERMINATION OF BUCKLING LOAD
Fig. 5-11 EXAMPLE FOR ILLUSTRATING THE CONSTRUCTION OF FRAME BUCKLING CURVE

Fig. 5-12 EFFECTIVE LENGTH FACTOR \( k \) OF PINNED-BASE FRAMES (TAKEN FROM REF. 4)
Fig. 5-13 ELASTIC-PLASTIC ANALYSIS OF FRAME WITH $h = 60 \, r$
Fig. 5-14  MOMENT DISTRIBUTION COMPUTATION FOR THE CASE h = 60 r

\[ M_{FL} = +100.00 \]
\[ M_{L} = +53.10 \]
\[ M_{FR} = +49.00 \]
\[ M_{R} = +43.25 \]

\[ c_b = 0.7125 \]

\[ \Sigma Q \times h \]

\[ \lambda \]

\[ \lambda_{cr} = 0.00501 \]
Fig. 5-16 ELASTIC-PLASTIC ANALYSIS OF FRAME WITH $h = 80 \, r$
Fig. 5-17  MOMENT DISTRIBUTION COMPUTATION
FOR THE CASE h = 80 r

Fig. 5-18  DETERMINATION OF CRITICAL
LOAD (h = 80 r)
Fig. 5-19 ILLUSTRATION OF FRAME BUCKLING CURVE
Fig. 5-20  INELASTIC BUCKLING TESTS ON MODEL STEEL FRAMES (SERIES B)

Fig. 5-21  SIDESWAY BUCKLING OF MULTI-STORY FRAMES
Fig. 6-1  SAMPLE DESIGN CHART FOR L = 70 FEET AND $\alpha = 2.0$
Fig. 6-2  SAMPLE DESIGN CHART FOR L = 90 FEET AND $\alpha$ = 2.0
Fig. 6-3  DESIGN EXAMPLE NO. 1

Working Loads:
\( w = 1.30 \text{ kips/ft.} \)
\( P = 91 \text{ kips} \)

Fig. 6-4  DESIGN EXAMPLE NO. 2

Working Loads:
\( w = 1.20 \text{ kips/ft.} \)
\( P = 108 \text{ kips} \)
Fig. 6-5 - COMPARISON OF THEORETICAL RESULTS WITH THE AISC DESIGN RULE
Appendix A

Typical Numerical Integration Computations for Obtaining Beam Deflection Curves

This appendix will illustrate a numerical integration procedure for determining the slope and deflection of any point in the length CD of the beam pictured in Fig. 4.2.

In all the computations, the moment-curvature relationship of a typical beam section, the 27 WF 94, is used. The section properties and the material constants adopted for the computations are the following:

\[
\begin{align*}
I &= 3266.7 \text{ in}^4 \\
S &= 242.8 \text{ in}^3 \\
E &= 30 \times 10^6 \text{ psi} \\
\gamma &= 33 \times 10^3 \text{ psi} \\
M_y &= 668 \text{ ft-kips} \\
\phi_y &= 81.76 \times 10^{-6} \text{ rad./in.}
\end{align*}
\]

The numerical procedure used to obtain the beam deflection curve is illustrated in tabular form on pages 156 and 157 for the following two cases:

1. \( \lambda = 0.002 \quad M_C/M_y = 0.5 \)
2. \( \lambda = 0.002 \quad M_C/M_y = 1.0 \)

The procedure is basically the same for both cases.

In computing slope and deflection by numerical integration, the beam is broken into a number of short segments of length "arp". (a constant times the radius of gyration of the beam.) Average values of bending moment and curvature
are determined for each segment and are used to obtain values for the change in slope and deflection occurring within that segment. By summing the changes in slope and deflection from an initial starting point, the total slope and deflection at any desired point may be determined. The initial point used here is the center of the beam (point C) at which the slope $\theta$ and the deflection $y$ are assumed to be zero.

An average value of bending moment for any segment is first calculated by statics from Eq. (4.5)

$$\left( \frac{M_1}{M_y} \right)_{\text{ave}} = \frac{M_C}{M_y} - \frac{\lambda}{4}$$

(A.1)

Where $n'$ is a number such that "$n'r_b" is the distance from the center of the beam to the center of any segment. The average curvature $\left( \frac{\partial}{\partial y} \right)_{\text{ave}}$ corresponding to $\frac{M_n}{M_y} \phi_{n_{\text{ave}}}$ is picked from an appropriate $M - \theta$ curve such as Fig. 4.3.

The change in slope is the average curvature times the length of segment (column 6 of Tables A1 and A2)

$$\Delta \theta_n = a r_b \phi_{n_{\text{ave}}}$$

(A.2)

The change in deflection from the properties of the tangent offsets of a small circular arc is (column 12)

$$\Delta y_n = a r_b \theta_n + \frac{(a r_b)^2}{2} \phi_{n_{\text{ave}}}$$

(A.3)

Figure A-1 shows these changes occurring within the segment "arb". The slope at one end of the $(n+1)^{\text{th}}$ segment is (column 7)

$$\theta_{n+1} = \theta_n + a r_b \phi_{n_{\text{ave}}}$$

(A.4)
and the corresponding deflection is (column 8)

$$y_{n+1} = y_n + \alpha r_b \delta_n + \frac{(\alpha r_b)^2}{2} \theta_{ave}$$  \hspace{1cm} (A.5)

As may be seen from Tables A1 and A2, the integration process for the first case ($\lambda = 0.002$ and \frac{M_C}{M_Y} = 0.5) terminates at the point where the slope of the beam deflection curve becomes zero, whereas that for the second case terminates when the moment reaches the $M_p$ value.
Table A1

NUMERICAL INTEGRATION COMPUTATIONS FOR THE CASE $\lambda = 0.002$ AND $\frac{M_C}{M_y} = 0.5$

| $X_n$ | $\theta_n \times 10^{-3}$ | $M_n \over M_y$ | $\overline{\theta}_n$ | $\overline{\theta}_{nave} \times 10^{-3}$ | $\theta_{nave}$ | $\Delta \theta_{nave}$ | $\Delta \theta_{nave}$ | $\Delta \theta_{nave}$ |
|-------|------------------------------|------------------|------------------------|------------------------------------------|-----------------|------------------------|------------------------|
| 0     | 0                            | 0.5000           | 0.4995                 | 0.4995                                   | 0.04084         | 0.8878                 | 0.8878                 |
| $2r_b$| 0.8878                       | 0.4980           | 0.4955                 | 0.4955                                   | 0.04051         | 0.8807                 | 1.7685                 |
| $4r_b$| 1.7685                       | 0.4920           | 0.4875                 | 0.4875                                   | 0.03985         | 0.8665                 | 2.6350                 |
| $6r_b$| 2.6350                       | --               | --                     | --                                       | --              | --                     | --                     |
| $54.4r_b$| 0.2448             | 0.9797           | 0.9851                 | 1.088                                    | 0.08895         | 0.1934                 | 0.0514                 |
| $54.6r_b$| 0.0514*             | 0.9906           |                        |                                          |                 |                        |                        |

__(cont.)__

<table>
<thead>
<tr>
<th>$X_n$</th>
<th>$y_n$</th>
<th>$y_n/r_b$</th>
<th>$\theta_{nave}$</th>
<th>$\Delta y_n$</th>
</tr>
</thead>
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<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.00965</td>
<td>0.00965</td>
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<tr>
<td>$2r_b$</td>
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<td>0.00089</td>
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<td>0.00957</td>
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<td>0.00354</td>
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<td>0.00942</td>
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<td>$6r_b$</td>
<td>0.08639</td>
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<td>--</td>
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<td>$54.4r_b$</td>
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<td>0.33291</td>
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* This is considered to be very close to zero
Table A2
NUMERICAL INTEGRATION COMPUTATION FOR THE CASE $\lambda = 0.002$ AND $M_C/M_y = 1.0$

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<tr>
<th>$X_n$</th>
<th>$\theta_n \times 10^{-3}$</th>
<th>$M_n/M_y$</th>
<th>$M_n/M_y$ave</th>
<th>$\theta_n$/ave</th>
<th>$\theta_{nave} \times 10^{-3}$</th>
<th>$a_r\theta_{nave} \times 10^{-3}$</th>
<th>$(1)+(6)$</th>
</tr>
</thead>
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<tr>
<td>0</td>
<td>0</td>
<td>1.0000</td>
<td>0.9995</td>
<td>1.128</td>
<td>0.09222</td>
<td>2.0049</td>
<td>2.0049</td>
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<td>$2r_b$</td>
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<td>0.9980</td>
<td>0.9955</td>
<td>1.115</td>
<td>0.09116</td>
<td>1.9818</td>
<td>3.9867</td>
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<tr>
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<td>3.9867</td>
<td>0.9920</td>
<td>0.9875</td>
<td>1.095</td>
<td>0.08953</td>
<td>1.9463</td>
<td>5.9330</td>
</tr>
<tr>
<td>$6r_b$</td>
<td>5.9330</td>
<td>--</td>
<td>--</td>
<td>--</td>
<td>--</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$65.2r_b$</td>
<td>17.3236</td>
<td>1.1255</td>
<td>1.1320</td>
<td>2.100</td>
<td>0.17169</td>
<td>0.3732</td>
<td>16.9504</td>
</tr>
<tr>
<td>$65.4r_b$</td>
<td>16.9504</td>
<td>1.1386*</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(continuation)

<table>
<thead>
<tr>
<th>$X_n$</th>
<th>$y_n$</th>
<th>$y_n/r_b$</th>
<th>$a_r \theta_n$</th>
<th>$(a_r)^2 \theta_{nave}$/2</th>
<th>$\Delta Y_n$</th>
<th>$(10)+(11)$</th>
<th>$(8)+(12)$</th>
</tr>
</thead>
<tbody>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0.02179</td>
<td>0.02179</td>
<td>0.02179</td>
<td>0.02179</td>
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<td>0.00200</td>
<td>0.04358</td>
<td>0.02154</td>
<td>0.06512</td>
<td>0.08691</td>
<td></td>
</tr>
<tr>
<td>$4r_b$</td>
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<td>0.00799</td>
<td>0.08667</td>
<td>0.02116</td>
<td>0.10783</td>
<td>0.19474</td>
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<td>$6r_b$</td>
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<td>--</td>
<td>--</td>
<td>--</td>
<td>--</td>
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<tr>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$65.2r_b$</td>
<td>13.97349</td>
<td>1.28551</td>
<td>0.03766</td>
<td>0.00041</td>
<td>0.03725</td>
<td>14.01074</td>
<td></td>
</tr>
<tr>
<td>$65.4r_b$</td>
<td>14.01074</td>
<td>1.28874</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* This value of moment is almost equal to the plastic moment.
\[ R = \frac{1}{\phi_{nave}} \]

\[ \Delta \theta_n = \ar_b \phi_{nave} \]

\[ \Delta y_n = \ar_b \theta_n + \frac{\left(\ar_b\right)^2}{2} \phi_{nave} \]

**Fig. A-1**  INCREMENT OF SLOPE AND DEFLECTION IN THE INTERVAL \( \ar_b \)
Appendix B

Determination of Stiffness of Partially Plastic Beams by the Column Analogy

This appendix contains the numerical computations involved in determining the stiffnesses and carry-over factors for the beams of the two frames considered in Art. 5.4. The dimensions of these frames and the applied loads (selected trial loads) are as follows:

(See Fig. 5.11)

<table>
<thead>
<tr>
<th>Member size</th>
<th>First Case</th>
<th>Second Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>33WF130</td>
<td>33WF130</td>
<td></td>
</tr>
<tr>
<td>Radius of gyration, r</td>
<td>13.23 in.</td>
<td>13.23 in.</td>
</tr>
<tr>
<td>Span length, L</td>
<td>80 r</td>
<td>80 r</td>
</tr>
<tr>
<td>Column height, h</td>
<td>60 r</td>
<td>80 r</td>
</tr>
<tr>
<td>Distributed load, w</td>
<td>2.20 kips/ft.</td>
<td>2.20 kips/ft.</td>
</tr>
<tr>
<td>Loading ratio, λ</td>
<td>0.0048</td>
<td>0.0048</td>
</tr>
</tbody>
</table>

For each case the moments at the ends of the beam are first determined graphically by the elastic-plastic analysis of Chapter 4. The moment at all the sections of the beam can then be computed by statics. From the computed moment values the effective flexural rigidity of all the sections can be determined from the flexural rigidity-moment relationship of the 33WF130 section (similar to that shown in Fig. 5-3). Thus, in effect, a beam of variable EI is obtained. In the buckling analysis, it is required to evaluate the stiffness factor and carry-over factor of this beam. This can be done
conveniently by the method of column analogy. The procedure of applying this method was discussed in Art. 5.2.1. Detailed computations for the beams of the frames considered here will be explained below.

First Case - Frame having $h = 60.0$

Referring to Fig. 5-13, the moment at the column top for $\lambda = 0.0048$ is equal to $0.83 M_y = 924 \text{ ft-kips}$, the moment at the center can then be determined by statics:

$$M_c = \frac{\omega L^4}{24} - M_B = 2136 - 924 = 1212 \text{ ft-kips}$$

or

$$\frac{M_c}{M_y} = 1.089$$

This indicates that the center section of the beam is not fully plastified at the trial load. Figure B-1 shows the distribution of bending moment of the beam and the corresponding yield configuration. To compute the stiffness of this partially yielded beam by the method of column analogy, it is convenient (and also accurate enough) to divide the yielded portion into segments, each having a length of one foot. Within each segment the flexural rigidity may be assumed to be constant. The area and the moment of inertia of the analogous column can then be calculated numerically as tabulated in Table B1.

The moment at the center section of each segment is first computed from the known values of $M_B$ and $M_D$ (listed in column (3) of Table B1). The effective flexural rigidity $\frac{(EI)_{\text{eff}}}{EI}$ of these sections is then determined from the
flexural rigidity - moment curve constructed for the 33WF130. This gives the values shown in column (4). Since the width of the analogous column at any section is inversely proportional to the flexural rigidity of that section, the reciprocal of the values of column (4) gives directly the width of the analogous column at the center sections. For example the width of the first segment is \( \frac{1}{0.134} \frac{1}{EI} = 7.46 \), where EI is the flexural rigidity of the section in the elastic range. The area of each segment of the analogous column can be computed by multiplying its width by its length (listed in column (6)). The moment of inertia of each segment with respect to axis G-G may be computed by using the parallel-axis theorem. The values obtained for all the segments are listed in column (7). As shown in Table B1, the total area and the total moment of inertia thus obtained are \( \frac{187.04}{EI} \) and \( \frac{61,067.08}{EI} \) respectively. The stiffness of the beam or the moment at end B induced by an imposed unit rotation at B is given by Eq. (5.1):

\[
K_b = M_B = \frac{1}{\frac{187.04}{EI}} + \frac{1 \cdot (44.1) \cdot (44.1)}{\frac{61,067.08}{EI}}
\]

\[
= 0.003719 EI
\]

where the unit rotation applied at end B is represented by a unit load applied to the analogous column at point B. The stiffness factor may be expressed in terms of \( M_y/\phi_y \) for EI, that is,

\[
K_b = \frac{0.003719}{\phi_y} M_y = 46.63 M_y
\]
The moment at D is

\[ M_D = \frac{1}{187.04} - \frac{1 \cdot (44.1) \cdot (44.1)}{61,067.08} \]

\[ = -0.002650 \, EI \]

The carry-over factor is therefore equal to \( c_b = \frac{0.002650}{0.003719} = 0.7125 \).

**Second Case - Frame having \( h = 80 \, r \)**

It is required to determine the stiffness of the beam when it is bent into an anti-symmetrical configuration. The bending moment diagram of the beam and the yield configuration are shown in Fig. B-2. In this case only half of the analogous column needs to be considered and its area may be assumed to be infinity. Table B2 contains all the computations involved in determining the moment of inertia of the half column about the axis G-G. The stiffness of the beam, according to Eq. (5.4), is

\[ K_b = M_B = \frac{1 \cdot (44.1) \cdot (44.1)}{34,405.22} = 0.05653 \, EI \]

when expressed in terms of \( M_y \), the stiffness is \( K_b'' = 70.87 \, M_y \).
Table B1

DETERMINATION OF STIFFNESS BY COLUMN ANALOGY \((h = 60 \, r)\)

<table>
<thead>
<tr>
<th>Section</th>
<th>Distance from Axis G-G</th>
<th>M &lt;sup&gt;0&lt;/sup&gt; &lt;br&gt;(M_y)</th>
<th>(\frac{(EI)_{eff}}{EI})</th>
<th>(\frac{1}{(4)})</th>
<th>Area &lt;br&gt;G-G</th>
<th>Moment of Inertia About Axis G-G</th>
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<tr>
<td>0 - 1</td>
<td>0.5</td>
<td>1.088</td>
<td>0.134</td>
<td>7.46</td>
<td>7.46</td>
<td>2.49</td>
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<td>1 - 2</td>
<td>1.5</td>
<td>1.087</td>
<td>0.134</td>
<td>7.46</td>
<td>7.46</td>
<td>17.41</td>
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<td>2 - 3</td>
<td>2.5</td>
<td>1.083</td>
<td>0.140</td>
<td>7.14</td>
<td>7.14</td>
<td>45.22</td>
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<td>3 - 4</td>
<td>3.5</td>
<td>1.077</td>
<td>0.160</td>
<td>6.25</td>
<td>6.25</td>
<td>77.08</td>
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<td>4 - 5</td>
<td>4.5</td>
<td>1.069</td>
<td>0.172</td>
<td>5.81</td>
<td>5.81</td>
<td>118.13</td>
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<td>5.5</td>
<td>1.059</td>
<td>0.190</td>
<td>5.26</td>
<td>5.26</td>
<td>159.56</td>
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<td>6.5</td>
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<td>0.216</td>
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<td>4.63</td>
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<td>1.01</td>
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</tr>
<tr>
<td>43.1 - 44.1</td>
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<td>0.930</td>
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<td>1.08</td>
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\[ \Sigma = 93.52 \quad 30,533.54 \]
Table B2
DETERMINATION OF STIFFNESS BY COLUMN ANALOGY \((h = 80 \, r)\)

<table>
<thead>
<tr>
<th>Section</th>
<th>Distance from Axis G-G</th>
<th>(\frac{M}{M_y})</th>
<th>(\frac{(EI)_{eff}}{EI})</th>
<th>(\frac{1}{(4)})</th>
<th>Moment of Inertia About Axis G-G</th>
</tr>
</thead>
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<td>100.00</td>
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<td>100.00</td>
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<tr>
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<td>0.060</td>
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<tr>
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<td>1.00</td>
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\[ \sum = 34,405.22 \]
DETERMINATION OF BEAM STIFFNESS
FOR THE CASE $h = 60r$

Fig. B-1

DETERMINATION OF BEAM STIFFNESS
FOR THE CASE $h = 80r$

Fig. B-2
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