A numerical weather forecast model using bicubic splines to estimate derivatives.

Robert E. Kelly

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A NUMERICAL WEATHER FORECAST MODEL
USING BICUBIC SPLINES
TO ESTIMATE DERIVATIVES

by

Robert E. Kelly

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Professor in Charge

Chairman of Department
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ABSTRACT

The equations of hydrodynamics with the equations of fluid mechanics can be used to forecast the movement of weather systems in the earth's atmosphere. The equations that can be used to obtain a solution of the weather prediction problem are the continuity equation, the momentum equation, the conservation of energy and moisture and the equation of state for an ideal gas.

These equations are written in an \((x,y,\sigma,t)\) coordinate system. The sigma \((\sigma)\) vertical coordinate is used to avoid the problem of using changing surface pressure as the vertical coordinate. The equations in the \((x,y,\sigma,t)\) coordinates are then written in orthogonal spherical curvilinear coordinates. The earth's atmosphere is modeled with a forecast grid that is made up of two special grid forms. One grid is a high resolution grid that is centered over North America which meshes smoothly into a second grid that maintains an equal resolution over the remainder of the globe.

Using initial data obtained from GFDL in Princeton, New Jersey three forecast experiments were attempted. In the first forecast experiment the use of a realistic earth topography caused the growth of noise over the mountainous regions of the earth. In the second forecast experiment the mountain regions were removed. The results of the
second forecast experiment shows a smoothing of all the important weather features. The problems in the second forecast could possibly have been caused by an initial imbalance of the wind vector field. The third forecast experiment was run with a restriction on the calculated divergence for the first simulation hour. The use of the divergence restriction appears to improve the forecast results. In the region of interest, the third experiment shows only a small smoothing of the weather features with a very small amount of phase lag.
1. INTRODUCTION AND HISTORY

The hydrodynamic and thermodynamic equations of fluid mechanics comprise a system that conceivably may be sufficient to represent the large scale motions of the atmosphere. Starting with a known state of the atmosphere the equations can be used to attempt to predict the future configuration of the atmosphere. The basic equations are (a) the equation of motion in three dimensions, (b) the equation of continuity of mass, (c) the equation of state, and (d) the first law of thermodynamics. The non-linear nature of some of the equations makes the likelihood of an analytical solution of the entire system remote, except in some special cases. The physical scales of the motions make the interpretation of laboratory modeling or full-sized experimentation uncertain. One alternative is to use analytic approximations to the equations. Another is numerical methods to approximate the solution of the equations.

The first attempt at numerical weather prediction was made by L. F. Richardson in 1921. The solution required years of work to obtain and the final answers proved to be in error by a large factor. The errors were due to both observational and theoretical problems. Some of the problems involved were (a) a lack of accurate data especially of the upper atmosphere, and (b) insufficient knowledge about finite difference calculus. The observation and theoretical prob-
lems were studied for reasons unrelated to numerical weather forecasting until after World War II, when another attempt was made at numerical weather prediction. The availability of better data gathering techniques, improvements in theoretical knowledge, and most of all the development of the digital computer, helped make the first accurate weather prediction possible by J. Charney in 1949. Since that time there have been many advances and much refinement of technique. Today many computer models with differing degrees of complication exist.

There are presently two basic methods that can be used in numerical weather prediction: (a) the filtered equations method, and (b) the primitive equations method. The major difference in these two techniques involves the handling of the equations of hydrodynamics. The hydrodynamic equations allow for all types of wave motion ranging in frequency from sound waves to Rossby waves, which have great influence on weather, Haltiner and Martin [1]. The high frequency waves motions may be eliminated by using large time steps in the prediction program. Sound waves are easily eliminated by the use of the hydro-static assumption. The FE (filtered equations) models usually try to filter out waves of even higher frequency than sound waves by differentiating and then integrating the hydrodynamical equations. Haltiner [2] outlines one such FE model which combines the two equations of horizontal motion and makes use of the conservation of vor-
ticity to eliminate the gravity waves. PE (primitive equations) methods solve the equations of hydrodynamics without the wave eliminating assumptions used by the FE methods. They generally have smaller time steps and therefore require greater computation time. It is felt that the use of fewer approximations in the PE formulations provides better results. Most of the working models today are of the PE type. Some examples of operational PE models are the eight-layer global PE model used at the National Meteorological Center, Stackpole, Vanderman, and Shuman [3], the Rand Corporation's modified Mintz-Arakawa model [4], and Shuman and Hovermale's [5] six-level PE model.

Outside of the basic type of model and equations chosen, the areas that have the most influence on the weather prediction problem are the size and type of forecasting grid, and the numerical methods used to obtain space and time derivatives. A short survey of these areas will be given.

1.1 Forecasting Grids

The underlying concern in choosing a prediction grid is finding one that will give the type of resolution desired over the area that is needed for the model. To minimize problems caused by changing boundary conditions, most models use either the entire earth or one hemisphere as a basis for computation. Some method is then used to project the chosen
area onto a square or rectangular grid. Although square and rectangular grids are almost universally standard, some experiments have been made with other configurations such as the floating data points used by Mesinger [6].

One widely used square grid scheme is the polar stereographic projection. An example of its use is Shuman and Hovermale's six-layer PE model [5]. The method as outlined by Haltiner and Martin [1] maps an entire hemisphere onto a plane surface with the pole as the center. One drawback of this method is the varying resolution, which is dependent on latitude. Taking the pole as having unit resolution there is an increase in resolution until the equation is reached where the resolution is twice that of the pole.

Another widely used method is the cylindrical map projection. This scheme gives a rectangular grid with linear mapping of longitudinal meridians. It is used in such models as Mintz-Arakawa [4]. The resolution problems with Mercator projection are similar to those of the polar stereographic method. The equator has unit resolution with increasing resolution at higher latitudes, reaching infinite resolution at the pole. One consequence of this is the growth of high frequency, fast traveling waves at higher latitudes. These are usually removed by special smoothing and filtering techniques or there is the possibility these wave patterns will grow and overwhelm the rest of the forecast. A description of some filtering techniques used in the Mintz-Arakawa model
is given by Price [7] and Gates, Batten, Kahle, and Nelson [4].

Much experimentation is being done with equal resolution and other special grids to overcome the problems of cylindrical and polar stereographic projections.


Another method that is used to partially eliminate some of the problems of uneven resolution of the forecast area is not to carry all of the dependent variables in the same spot on a grid. This can take the form of spatial scattering, where the dependent variables can be at the grid point, between grid points, or in the middle of several grid points depending on the program. The Mintz-Arakawa model [4] is an example of spatial scattering of dependent variables. Another form of scattering is time scattering, where the dependent
variables are carried at grid points only at certain times in the program. An example of this is the Phillips model [7].

Equal resolution grids may eliminate the need to carry all of the unneeded points which result from the use of polar stereographic and cylindrical mapping, but a large number of points are still generally needed to obtain a good forecast. Wellek, Kasahara, Washington and DeSanto [15] have found, in experimenting with several different resolutions for the same model, that the smaller the grid the greater the accuracy of the final answer. But at the same time, the smaller and more accurate the grid is made, the greater the computation time is for the overall forecast. One possibility that is proposed to cut down on computation time is to use a high resolution grid only in a restricted area.

Some experiments with high resolution grids in restricted areas have been performed by Gerrity and McPherson [16]. Gerrity and McPherson tried a barotropic PE model with real initial weather data and constant lateral boundary conditions. The results obtained with this high resolution model compare well with presently used himispheric models. The use of constant boundary conditions, although it may work for some cases, is still an artificial restriction of global scale movements to a less than global area and can cause boundary problems as the forecast time increases.
Another possibility for the problem solution is to use the so-called nested grids. In this scheme there is a region of high resolution changing abruptly to a region of a different resolution. This method can cause some problems with the boundary conditions at the point where the resolution changes.

Other experiments have been made with variable grids. A variable grid is one which has a fixed resolution in a certain area and then "telescopes" to give smaller resolution for areas outside of the region of interest. Athens [17] experimented with several variable grids to model a free surface vortex in an incompressible inviscid fluid. Athens reported an increase in total energy of less than 1% after two thousand time steps and also that the more intense the vortex disturbance is the smaller the grid expansion factor must be to obtain good answers.

MacPherson and Price [18] and Price [19] have combined the high resolution telescoping grid with an even resolution grid for use with real weather data on a hemispheric scale. This numerical model has a grid that expands to maintain even resolution and computational stability over the hemisphere. Meshed with the first grid is a second grid that has a high resolution region of interest. The first grid form insures that the problems encountered in both polar stereographic and cylindrical projections are eliminated.
from the forecast model. As a result no high frequency filtering methods are needed for the higher latitudes as in the Mintz-Arakawa model. The second grid form for the high resolution region exists without the noise problems caused by constant boundary conditions or the reflecting of wave motions at the boundary. This smooth expansion (telescoping grid) also gives better results than abrupt changes in grid size, as reported by Koss [20].

Macpherson and Price [18] report that over a 36 hour period a model with a telescoping grid gave results in the region of interest comparable with a uniform resolution grid. The telescoping grid shows a considerable savings in computation time.

1.2 Numerical Methods

The equations of hydrodynamics are highly non-linear in nature. The non-linearity makes a general analytic solution unlikely, so finite difference numerical methods generally must be used. Most of the equations used in weather prediction contain spatial derivatives of the general form \( \frac{\partial f(x)}{\partial x} \). These derivatives are approximated in finite differences by modeling the dependent variable with a polynomial equation and then differentiating to obtain a solution. In the simple case of an assumed linear polynomial on a fixed grid, the differential reduces to the well known first order forward, backward, or central difference. If \( h \)
is taken as the grid spacing, the forward, backward, and central differences are respectively:

\[
\frac{f(x + h) - f(x)}{h} \quad (1.1a)
\]

\[
\frac{f(x - h) - f(x)}{h} \quad (1.1b)
\]

\[
\frac{f(x + h) - f(x - h)}{2h} \quad (1.1c)
\]

Beside the usual finite difference schemes, other methods, such as spline fitted curves, can be used to obtain the approximation of the derivatives. The hydrodynamical equations also involve time differencing terms, eg. \( \frac{\partial f(x)}{\partial t} \) which require special techniques to insure that the solution remains stable.

The penalties that are incurred by approximating the solution of a mathematical problem with a numerical method are the discretization, stability, and truncation errors reported by Haltiner [2]. Only a finite number of digits can be retained after each calculation in a computer. This round-off error combined with the approximate representation of functions in the computer causes the numerical solution \( S_n \) to differ from the exact solution of the difference equation \( S_d \). This is called stability error. The difference between the solution of the differential equation \( S_e \) and the approximate solution of the difference equation is called the discretization error \( S_e - S_d \). Haltiner [2] found that a difference equation is stable when the effect
of the round-off error does not become important after a
great number of steps. In general, stability also implies
convergence of the difference equation to the differential
equation as the time and space steps become smaller. This
means stability requires that the discretization error goes
to zero. The truncation error is defined to be the differ-
ence, expressed as a Taylor series, between the difference
equation and the differential equation. It can be shown
that this error also approaches zero as the time and space
differences approach zero.

Another problem in finite differences can be investi-
gated by considering the equation:

\[ \frac{\partial F}{\partial t} + c \frac{\partial F}{\partial t} = 0 \]  \hspace{1cm} (1.2)

It is a simple first order, linear, partial differential equa-
tion whose analytic solution can be obtained by the separation
of variables. Using separation of variables the general solu-
tion would be a wave of a single harmonic. The numerical
solution that would be obtained using central differencing
in both space and time will consist of two waves rather than
one. The first wave travels in the same direction as the
analytic wave but with a slightly different speed and amplitu-
dee, depending on the ration of \( c\Delta t/\Delta x \). The second wave
travels in the opposite direction of the first wave with
speed, amplitude, and phase dependent on \( c\Delta t/\Delta x \). The first
wave is called the physical mode of the solution and has a counterpart with the analytic wave in the physical world. The second wave, the computational mode, has no physical counterpart in the physical world and is totally a result of the use of finite difference techniques. For all values of $c$, $\Delta t$ and $\Delta x$ if $c\Delta t/\Delta x \geq 1$, the amplitude of one of the waves in the finite difference solution will grow exponentially with time. This departure of the finite difference solution from the analytic solution is called computational instability. It can be shown that for all values of $c$, $\Delta t$ and $\Delta x$, if $c\Delta t/\Delta x \leq 1$, the solution of the equation with finite differences will be computationally stable.

The computational stability example above was worked out with central differences in both space and time but the concept of computational stability extends to other schemes of space and time differences. If forward differencing in time and central space differences are used and a solution is worked out, it would prove to be always computationally unstable. Robert, Shuman, Fredericks, and Gerrity [21] report in building on the work of Courant, Fredericks, and Lewy [22], that the solution of general non-linear equations instability, even if not initially present, will be generated by non-linear interactions during the computation. Robert et al, also postulated that although instabilities will always be a possibility, there are means of obtaining at least relative stability in most finite differencing schemes.
Grammeltvidt [23] has compared ten finite difference schemes for stability. The comparison used a primitive equation approach with a barotropic fluid containing an analytic wave for the initial condition. Grammeltvidt's findings indicate that one very stable difference method is where the advective terms are calculated along nine grid points spatially. Shuman's model [24] attempts to use the long term stability that a nine point advective term difference gives. Grammeltvidt also found the generalized Arakawa scheme exhibited good stability. This method conserves mean vorticity and kinetic energy. Quadratic conservative and total energy conservation schemes were generally found by Grammeltvidt to be more stable than second order conservative methods. The Mintz-Arakawa model [4] takes advantage of quadratic and total energy conservative methods to attempt to maintain computational stability.

Much work has also been undertaken with time differencing schemes. The method of time differencing used in a model has great effect on computational stability and the resultant amplification and phase lag of the physical mode of weather prediction waves. Haltiner [2] discusses a time differencing method which uses central differences and time scattering of data points. The scheme is an implicit method of time stepping and proves to be computationally stable for most cases. One drawback is that, as with most implicit methods, it would require the inversion of a large matrix.
which would be time consuming. Kurihara [25] has had some success with implicit methods and several other iterative schemes of time integration. Kurihara found one method that gives good results on the physical mode, with only a small amount of damping and phase retardation, in a two stage leapfrog-trapezoidal scheme. Grammeltvedt [23] reports though that a certain amount of smoothing is needed in long-term integrations with such leapfrog methods. Matsuno's [26] scheme is a three stage leapfrog method which has a small amplification rate for high frequency waves. Kasahara [27] has modified the original Matsuno scheme to a two stage backward method.

1.3 Spatial Differences with Splines

There are other methods available to approximate the derivatives of the equations of hydrodynamics. One such method is the use of the polynomial spline. Following the work of Price [19] double cubic polynomial spline functions are used for the purpose of approximating the spatial variations of the dependent variables. This scheme eliminates the need to use finite space differences in the calculation of the derivatives. The cubic spline is the approximation of a continuous differentiable function $f(x)$ on a series of non-overlapping subintervals $(a,b)$. The spline is also required to be continuous and possess continuous first and second derivatives (Ahlberg, Nilson, and Walsh
Macpherson and Price [18] give some of the reasons why the spline proves to be such an effective tool in weather prediction programming. The use of splines carries with it an inherent form of curve smoothing, especially in estimates of the first derivative. It is a much simpler scheme than some of the very complex methods used in some numerical weather prediction programs. The smooth nature of the spline follows from Holladay's Theorem (page 3 of reference [29]), sometimes called the minimum curvature property. This theorem states that of all the possible curve fits on the interval the cubic spline is the smoothest with continuous first and second derivatives. Another feature of the cubic polynomial spline is the ease with which it can be adapted for use with an expanding grid, as long as the grid does not have sudden rapid changes in grid point spacing.

Comparison of a model using bicubic spline methods and no special filtering techniques to aid stability with a Mintz-Arakawa model that uses the standard finite difference methods and filtering techniques to promote stability, Price [19] found that the spline model gave a better forecast that the Mintz-Arakawa model.
2. EQUATIONS FOR FORECAST MODEL

In the derivation of the equations of motion used in the computer model three assumptions are made. The first assumption is that the flow in the atmosphere is quasi-horizontal. This assumption can be made because the wind speeds in the vertical directions are generally three orders of magnitude greater than the vertical wind speed, Haltiner and Martin [1]. The second assumption is that on a global basis the atmosphere is in hydrostatic equilibrium so that the net pressure force balances the force of gravity, Haltiner and Martin [1]. This can be expressed as

\[ \frac{\partial p}{\partial z} = -g\rho \]  

(2.1)

where \( p \) is the pressure, \( z \) is the vertical coordinate, \( g \) is the gravity force and \( \rho \) is the density of air. The third assumption is that atmospheric air behaves as a perfect gas, Haltiner and Martin [1]. Thus, the equation of state is

\[ p = \rho RT \]  

(2.2a)

or

\[ pa = RT \]  

(2.2b)

where \( \rho \) is the mass of air per unit volume, \( R \) is the gas constant, \( T \) is the temperature, and \( a \) is the specific volume.

The underlying principle for modeling the motions of the atmosphere is Newton's second law, which states that
the time rate of change of momentum is equal to the sum of
the forces applied. By assuming constant mass this can be
expressed as the product of the mass times the acceleration
being equal to the applied forces. In using nonrelativistic,
classical mechanics, Newton's laws must be applied in an
inertial reference frame. An inertial frame (or non-
accelerating coordinate system) is one in which a body at
rest remains at rest, Owczareck [30]. For classical mech-
anics a frame of reference fixed at the center of the sum
is the usual basis for a coordinate system. In the study
of weather phenomenon a coordinate system fixed in the cen-
ter of the earth suffices. For this system the $n_3$ axis is
the earth's polar axis with the $n_1$ and $n_2$ axes mutually per-
pindicular in the equatorial plane fixed relative to the
stars, Figure 1. At a distance $\rho'$ from the center of the
earth locate the origin $0'$ of a new rectangular coordinate
system $n_1'$, $n_2'$, $n_3'$ with position vector $r$ locating a point
of interest $P$.

The absolute acceleration of point $P$ with respect to
the inertial system in the center of the earth is

$$ a = \frac{D'V}{Dt} + 2\omega \times V + \omega \times (\omega \times r) + \frac{d\omega}{dt} \times r + \ddot{\rho}' \quad (2.3) $$

where $\omega$ is the angular velocity relative to the $n_1'$, $n_2'$, $n_3'$
axes, $V$ is the velocity of $P$ relative to the $n_1'$, $n_2'$, $n_3'$
axes, and $\ddot{\rho}$ is the linear acceleration of the point $0'$.  

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The operator \( \frac{D}{Dt} \) is known as the material time derivative (total derivator) and is expressed as

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + (V \cdot V)
\]

(2.4)

from Owczarek [30]. The term \( D'V/Dt \) represents the acceleration of \( P \) with respect to \( O' \). The Coriolis acceleration is represented by \( 2\omega \times r \). The term \( \omega \times (\omega \times r) \) is called the centripetal acceleration. The term \((d\omega/dt) \times r\) is known as the Euler acceleration.

If \( O' \) is moved to coincide with the center of the earth, \( P \) will represent some point on the surface of the earth at latitude \( \phi \) and longitude \( \theta \), Figure 2. The term \( \phi \) can be dropped and assuming the angular velocity of the earth is constant the Euler acceleration can be seen to be equal to zero. Equation (2.3) can now be written as

\[
a = \frac{DV}{Dt} + 2\omega \times \vec{y} + \omega \times (\omega \times \vec{r})
\]

(2.5)

Since \( \omega \) is now equal to \( \Omega n^3 \), where \( \Omega \) is the angular velocity of the earth, the vector triple product of the centripetal acceleration is now equal to

\[
\omega \times (\omega \times \vec{r}) = -\Omega^2 \vec{R}
\]

(2.6)

where \( \vec{R} \) is the vector perpendicular to the polar axis \( n^3 \) and directed from this axis to \( P \).

The angular velocity \( \omega \) vector can be expressed in terms of a new coordinate system set up at point \( P \).
this rectangular cartesian coordinate system \((x,y,z)\), Figure 2, the z axis points outward from \(P\) in the direction of \(r\). The mutually perpendicular axes \(x\) and \(y\) point respectively in the direction east and north on the earth's surface and are tangent to the surface. In this new coordinate system the angular velocity vector is expressed as

\[
\omega = \Omega \mathbf{n}_3 = \Omega \cos \phi \mathbf{i} + \Omega \sin \phi \mathbf{k} \quad (2.7)
\]

The Coriolis acceleration can be written, using equation (2.7), as

\[
2\omega \times r = (2\omega \Omega \cos \phi - 2\nu \Omega \sin \phi) \mathbf{i} + 2u\Omega \sin \phi \mathbf{j} - 2u\Omega \cos \phi \mathbf{k} \quad (2.8)
\]

where \(u, v, w\) are respectively the components of velocity in the \(x, y\) and \(z\) directions. Using the first assumption of quasi-horizontal flow, the term containing \(w\) is dropped. The term \(2u\Omega \cos \phi\) is also dropped because it is usually negligible compared with the gravity force in the vertical direction. The Coriolis acceleration is now written as

\[
f(\mathbf{k} \times \mathbf{v}) \quad (2.9)
\]

where \(f\) is known as the Coriolis parameter.

\[
f = 2\Omega \sin \phi \quad (2.10)
\]

The velocity vector \(\mathbf{v}\) is written as \(\mathbf{v} = u\mathbf{i} + v\mathbf{j}\)
taking into account assumption number one.

With these modifications equation (2.3) may be written as

\[ a = \frac{D\mathbf{v}}{Dt} + f(k \times \mathbf{v}) - \Omega^2 \mathbf{e} \]  

(2.11a)

or using the definition of the material time derivative equation (2.4)

\[ a = \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla_{H} \right) \mathbf{v} + w \frac{\partial \mathbf{v}}{\partial z} + f(k \times \mathbf{v}) - \Omega^2 \mathbf{e} \]  

(2.11b)

where \( \nabla_{H} \) is the gradient operator in the \( \mathbf{i} \) and \( \mathbf{j} \) directions.

\[ \nabla_{H} = \left( \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} \right) \]  

(2.12a)

Equation (2.11) can also be written using pressure as the vertical coordinate from Price [7] as

\[ a = \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla_{P} \right) \mathbf{v} + \omega \frac{\partial \mathbf{v}}{\partial z} + f(k \times \mathbf{v}) - \Omega^2 \mathbf{e} \]  

(2.11c)

where \( \omega \) is the vertical pressure velocity

\[ \omega = \frac{\partial \mathbf{p}}{\partial t} \]  

(2.13)

and \( \nabla_{P} \) is the gradient operator on a surface of constant pressure

\[ \nabla_{P} = \left( \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} \right) \]  

(2.12b)

A relationship between \( \nabla_{P} \) and \( \nabla_{H} \) can be obtained using the chain rule, Price [7]. For some scalar function \( \gamma(x,y,z,t) \) the chain rule gives a result of
\[
\n\n\partial_a \gamma = \partial_b \gamma + \frac{\partial \gamma}{\partial z} (\partial_a z) \quad (2.14)
\]

where \(a\) and \(b\) can be either \(P\) or \(H\) and \(z\) can be either \(z\) or \(P\).

Euhler's momentum equation, Owczarek [30], can be written as

\[
\alpha = \frac{1}{\rho} \nabla \rho + b + F
\]

where \(b\) is the body force per unit mass and \(F\) is the net frictional force. By combining equations (2.15) and (2.11c) a new equation is generated

\[
(\frac{\partial}{\partial t} + \nabla \cdot \mathbf{v}) \mathbf{v} + \omega \frac{\partial \mathbf{v}}{\partial z} + f(\mathbf{k} \times \mathbf{v}) - \Omega^2 \mathbf{R} = - \frac{1}{\rho} \nabla \rho + b + F
\]

(2.16)

For fluid in the atmosphere, around an idealized spherical earth, the body force \(b\) is according to Newton's law of gravitation \(g\). It can be shown, Haltiner and Martin [1], that \(g\) can be expressed as the gradient of a scalar potential function such that \(g\) is a function of position only and surfaces of constant \(g\) are spherical in shape. In equation (2.16) the term for the centripetal acceleration \(- \Omega^2 \mathbf{R}\) is also a function of position only, and it has cylindrical surfaces of constant potential. In meteorology it is natural to combine these two terms to form one function of position called gravity, Haltiner and Martin [1].

\[
g = g_a + \Omega^2 \mathbf{R}
\]

(2.17)
Since (2.17) is a single valued function of position it can also be expressed as the gradient of a scalar potential function $\phi$

$$g = -\nabla H \phi - \frac{\partial \phi}{\partial z} k$$  \hspace{1cm} (2.18)

Surfaces of constant $g$ differ only slightly from spherical shells so that $\nabla H \phi$ can be set equal to zero and equation (2.18) becomes

$$g = -\frac{\partial \phi}{\partial z} k = -gk$$  \hspace{1cm} (2.19a)

where

$$\phi = gz$$  \hspace{1cm} (2.19b)

and $\phi$ is called the geopotential which represents the potential energy a unit mass has by virtue of its position in the gravity field.

By replacing $-\Omega R$ and $\mathbf{b}$ in equation (2.16) we obtain

$$\left(\frac{3}{3} + \nabla \cdot \nabla\right) \mathbf{v} + \omega \frac{\partial \mathbf{v}}{\partial z} + f(k \times \mathbf{v}) = -\frac{1}{\rho} \nabla p + \mathbf{g} + \mathbf{F}$$  \hspace{1cm} (2.20)

The gravity force can now be eliminated from the momentum equation (2.20) with the use of the hydrostatic approximation, equation (2.1).

In the momentum equation (2.20), the gradient of the pressure may be written as

$$\nabla p = \nabla H p + w \frac{\partial p}{\partial z}$$  \hspace{1cm} (2.21)
By using the chain rule and equation (2.14) replacing $\alpha$ by $P$ and $\beta$ by $H$ equation (2.21) becomes:

$$\nabla P = \nabla_P P - \left( \frac{\partial P}{\partial z} \right) \nabla_P z + w \frac{\partial P}{\partial z}$$  \hspace{1cm} (2.22a)

Since $\nabla_P P = 0$ and $z$ can be replaced from (2.19b), (2.22a) becomes

$$\nabla P = - \frac{\partial P}{\partial z} \frac{1}{g} \nabla_P \phi + w \frac{\partial P}{\partial z}$$  \hspace{1cm} (2.22b)

Now using the hydrostatic equation (2.1) and changing $\frac{\partial P}{\partial z}$ in (2.22b), $\nabla P$ can be returned to equation (2.20) to yield

$$\left( \frac{\partial}{\partial t} + \nabla \cdot \nabla_P \right) \nabla + \omega \frac{\partial \nabla}{\partial P} + f(k \times \nabla) + \nabla_P \phi = - \rho$$  \hspace{1cm} (2.23)

An alternative form of the hydrostatic approximation can be written with the use of equation (2.19b)

$$\frac{\partial \phi}{\partial P} = \frac{\partial \phi}{\partial z} \frac{\partial z}{\partial P} = g\left( - \frac{1}{\rho g} \right) = - \frac{1}{\rho}$$

$$\frac{\partial \phi}{\partial P} + \frac{1}{\rho} = 0$$  \hspace{1cm} (2.24)

The Eulerian continuity equation for a compressible fluid is given by Owczarek [30] as,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \nabla) = 0$$  \hspace{1cm} (2.24a)

$$\frac{1}{\rho} \frac{D\rho}{Dt} + \nabla \cdot \nabla_P \nabla + \frac{\partial w}{\partial z} = 0$$  \hspace{1cm} (2.24b)

Using equation (2.14) $\nabla_P \cdot \nabla$ can be changed to $\nabla_P \cdot \nabla$.  

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\[ \nabla \cdot \mathbf{V} = \nabla_p \cdot \mathbf{V} + \frac{\partial \mathbf{V}}{\partial P} \nabla P \]  
\[ (2.25) \]

The hydrostatic approximation will be used again to convert
\[ \frac{1}{\rho} \frac{Dp}{Dt} \] to
\[ \frac{1}{\rho} \frac{Dp}{Dt} = - \frac{1}{\rho g} \frac{D}{Dt} \left( \frac{\partial p}{\partial z} \right) = - \frac{1}{\rho g} \left[ \left( \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \frac{\partial p}{\partial z} + \mathbf{w} \frac{\partial}{\partial z} \left( \frac{\partial p}{\partial z} \right) \right] \]
\[ (2.26a) \]

using the relation \[ - \frac{1}{\rho g} \frac{\partial}{\partial z} = \frac{\partial}{\partial P} \] equation (2.26a) becomes
\[ \frac{1}{\rho} \frac{Dp}{Dt} = \frac{\partial}{\partial P} \left( \frac{DP}{Dt} \right) - \frac{\partial \mathbf{V}}{\partial P} \nabla P - \frac{\partial \mathbf{W}}{\partial z} \]
\[ (2.26b) \]

Substituting (2.26b) and (2.25) into equation (2.24b) and using the relation (2.13) equation (2.24b) becomes
\[ \frac{\partial \mathbf{w}}{\partial P} + \nabla_p \cdot \mathbf{V} = 0 \]
\[ (2.27) \]

The perfect gas assumption will now be used in obtaining the energy equation.

The energy equation as given by Owczarek [30] is
\[ \frac{DQ}{Dt} = \frac{Dh}{Dt} - \frac{1}{\rho} \frac{DP}{Dt} \]
\[ (2.28) \]

where \( Q \) is the total heat input from all external sources and \( h \) is the total enthalpy. Using the perfect gas assumption the enthalpy \( h \) can be written as
\[ h = c_p T \]
\[ (2.29) \]

Equation (2.29) can be used to write equation (2.28) as
\[ \frac{DQ}{Dt} = \frac{D(c_p T)}{Dt} = \frac{1}{\rho} \frac{DP}{Dt} \]
\[ (2.30) \]
For the equations written in the pressure coordinate the term \( \frac{Dp}{Dt} \) can be set equal to zero since the pressure will be constant over time and position. The temperature \( T \) can also be expressed as the potential temperature \( \theta \), which is defined as the temperature of a parcel of air brought adiabatically from some initial state defined by \( P \) and \( T \) to a final state defined by \( P^* \) and \( \theta \). \( P^* \) is taken to represent a standard pressure of 1000 mb. The equation for potential temperature is

\[
\theta = T \left( \frac{P^*}{P} \right)^{\kappa} = T \left( \frac{P^*}{P} \right)^{\frac{\nu-1}{\nu}}
\]  

(2.31)

where

\[
\kappa = \frac{R}{c_p}
\]

\[
\nu = \frac{c_p}{c_v}
\]

Using the identity \( \frac{Dp}{Dt} = 0 \) and equation (2.31) equation (2.28) can be rewritten as

\[
\frac{DO}{Dt} = c_p \left[ \frac{P^*}{P} \right]^{\kappa} \frac{D\theta}{Dt} = \frac{DQ}{Dt}
\]

(2.32a)

\[
\left( \frac{\partial}{\partial t} + \nu \cdot \nabla P + \omega \frac{\partial}{\partial P} \right)c_P \theta = \left[ \frac{P^*}{P} \right]^{\kappa} \frac{DQ}{Dt}
\]

(2.32b)

Equation (2.31) can be used to change the equation of state of a perfect gas (2.2b) to

\[
\theta - \frac{P^*}{R} P^{1/\nu} a = 0
\]

(2.33)

The mixing ratio \( m \) is defined as the ratio of the mass of water vapour to the mass of dry air. The total change of
mixing ratio per unit time, with pressure as the vertical coordinate, is given by

\[
\frac{D m}{D t} = \frac{\partial m}{\partial t} + \mathbf{V} \cdot \nabla P_m + \omega \frac{\partial m}{\partial P} = M
\]  

(2.34)

where \( M \) is the moisture addition per unit mass.

2.1 Equations in Sigma Coordinates

The hydrostatic approximation, the continuity equation, the equation of motion, the thermodynamic energy equation and the moisture equation which expresses the conservation of water vapour, and the equation of state for a perfect gas (equations (2.24), (2.27), (2.23), (2.34), (2.32b), and (2.33) are written in an \((x,y,P,t)\) coordinate system but this is not the system in which they will be solved. The equations will first be changed so that \( P \) is no longer the vertical coordinate. This change will also result in a change of the expression for a unit mass and a rearrangement of some of the variables. Finally the equations will be written in curvilinear coordinates in the horizontal for solution on the computer.

Because the surface pressure on the earth is not uniform, the surface pressure does not make a good coordinate variable in the vertical direction. To circumvent these difficulties, Phillips [31,32] introduced a new vertical coordinate variable \( \sigma \). This independent variable is defined as
\[ \sigma = \frac{P - P_T}{\pi} \]  
(2.35a)

\[ \pi = P_S - P_T \]  
(2.35b)

\[ P_S = \text{surface pressure} \]

\[ P_T = \text{pressure at the top of the model atmosphere (for this model 200 mb)} \]

This new coordinate now varies monotonically from 0 at the top of the model atmosphere to 1 at the earth's surface, effectively removing the problem of the earth's surface not being a coordinate surface. A set of rules will be derived to change the equations used in the model from \((x, y, P, t)\) to \((x, y, \sigma, t)\) coordinates.

Using the chain rule, the time rate of change of a property on a surface of constant pressure is

\[ \left( \frac{\partial}{\partial t} \right)_P = \left( \frac{\partial}{\partial t} \right)_\sigma + \left( \frac{\partial \sigma}{\partial t} \right)_P \left( \frac{\partial}{\partial \sigma} \right)_t \]  
(2.36a)

Using equation (2.35a)

\[ \left( \frac{\partial \sigma}{\partial t} \right)_P = - \frac{\sigma}{\pi} \left( \frac{\partial \pi}{\partial t} \right)_P \]  
(2.36b)

By using the chain rule in the \(x\) and \(y\) directions and adding the gradient \(\nabla_P\) is obtained as

\[ \nabla_P = \nabla_\sigma - \frac{\sigma}{\pi} \nabla_\pi \left( \frac{\partial}{\partial \sigma} \right)_{xyt} \]  
(2.37)

By applying the chain rule and equation (2.37) to \(\pi\) the time derivative of \(\pi\) is, given from Price [7] as,
\[ \frac{\partial \pi}{\partial t} p_{xy} = \frac{\partial \pi}{\partial t} \sigma_{xy} \]  
(2.38a)

\[ \nabla_\sigma \pi = \nabla_\sigma \pi \]  
(2.38b)

The chain rule can also be used to establish, from Price [7] that

\[ \frac{\partial}{\partial P} = \frac{1}{\pi} \frac{\partial}{\partial \sigma} \]  
(2.39)

Since \( \sigma \) is a function of \( \pi \) and \( P \)

\[ D \sigma = \left( \frac{\partial \sigma}{\partial P} \right) \pi DP + \left( \frac{\partial \sigma}{\partial \pi} \right) P D\pi \]

Since \( \frac{\partial \sigma}{\partial P} \pi = \frac{1}{\pi} \)

and \( \frac{\partial \sigma}{\partial \pi} P = -\frac{\sigma}{\pi} \)

\[ D \sigma = \frac{1}{\pi} DP - \frac{\sigma}{\pi} D\pi \]

dividing by \( Dt \)

\[ \dot{\sigma} = \frac{D\sigma}{Dt} = \frac{1}{\pi} \frac{DP}{Dt} - \frac{\sigma}{\pi} \frac{D\pi}{Dt} \]  
(2.40)

Since \( \frac{DP}{Dt} = \omega \) by using the identities of equations (2.38a) and (2.38b) equation (2.40) becomes

\[ \dot{\sigma} = \frac{1}{\pi} \left\{ \omega - \sigma \frac{\partial \pi}{\partial \sigma} \sigma \nabla_\sigma \nabla_\sigma \right\} \]  
(2.41)

Phillips [31] gives a second method for calculating \( \dot{\sigma} \) which used integration

\[ \dot{\sigma} = -\frac{1}{\pi} \int_0^\sigma \nabla_\sigma \nabla_\sigma \pi d\sigma - \frac{\sigma}{\pi} \frac{\partial \pi}{\partial t} \]  
(2.42)

For notational ease the \( \sigma \) subscript will be omitted
from terms such as \( \nabla \sigma \) or \( \frac{\partial}{\partial \sigma} \) for the remainder of the paper.

By using the transformation equations derived above, the equation of motion (2.23) can be written as

\[
\frac{\partial}{\partial t} + \nabla \cdot \mathbf{V} + \left[ \frac{1}{\pi} \left[ \omega - \sigma \left( \frac{\partial \pi}{\partial t} - \sigma \nabla \cdot \nabla \pi \right) \frac{\partial}{\partial \sigma} \right] \right] \mathbf{V}
\]

\[+ f(\mathbf{k} \times \mathbf{V}) + (\mathbf{V} - \frac{7\pi}{3} \nabla \pi \frac{\partial}{\partial \sigma}) \phi = - F \] (2.43)

The term \( \frac{1}{7\pi} \left[ \omega - \sigma \left( \frac{\partial \pi}{\partial t} - \sigma \nabla \cdot \nabla \pi \right) \right] \) is exactly equation (2.41), so that (2.43) can be written as

\[
\left[ \frac{\partial}{\partial t} + \nabla \cdot \mathbf{V} + \sigma \frac{\partial}{\partial \sigma} \right] \mathbf{V} + f(\mathbf{k} \times \mathbf{V}) + [\mathbf{V} - \frac{7\pi}{3} (\nabla \pi) \frac{\partial}{\partial \sigma}] \phi
\]

\[= - F \] (2.44)

For the continuity equation an expression for \( \frac{\partial \omega}{\partial \sigma} \) can be obtained by differentiating equation (2.41) with respect to \( \sigma \)

\[
\frac{\partial \sigma}{\partial \sigma} = \frac{\partial \omega}{\partial \sigma} - \frac{1}{7\pi} \left[ \frac{\partial \pi}{\partial t} - \nabla \cdot \nabla \pi - \sigma (\nabla \pi) \frac{\partial \mathbf{v}}{\partial \sigma} \right] \] (2.45)

Using (2.45) to find an expression for \( \frac{\partial \omega}{\partial \sigma} \) along with equation (2.37) the continuity equation can be written as

\[
\frac{\partial \pi}{\partial t} + \nabla \cdot (\pi \mathbf{V}) + \frac{\partial}{\partial \sigma} (\pi \phi) = 0 \] (2.46)

By applying equation (2.39) to equation (2.24) the hydrostatic approximation can be written as

\[
\frac{1}{7\pi} \frac{\partial \phi}{\partial \sigma} + a = 0 \] (2.47)

By using the proper transformation equations (2.37) and
(2.39) and replacing \( \dot{c} \) from equation (2.41) the thermodynamic energy equation (2.32b) can be written as:

\[
\left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \sigma \frac{\partial}{\partial \sigma} \right) c_p \theta = \left( \frac{p^*}{P} \right)^K \frac{\partial Q}{\partial t}
\]  

(2.48)

The moisture (2.34) and perfect gas (2.35) equations now become

\[
\left[ \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \sigma \frac{\partial}{\partial \sigma} \right] m = M
\]

(2.49)

and

\[
\theta - \frac{p^*}{P} \left( \pi \sigma + P_T \right)^{1/\nu} a = 0
\]

(2.50)

In \( x,y,z,t \) coordinates the unit of mass is \( \rho \ dx \ dy \ dz \). By changing from \( x,y,z,t \) to \( x,y,\sigma,t \) coordinates the expression for the unit of mass has also been changed. Since \( dP = \pi d\sigma \) and \( dP = - \rho gdz \) the new expression for the unit of mass in \( x,y,\sigma,t \) coordinates can be written as \( - dx \ dy \ d\sigma \pi g \). In the equations derived so far, momentum, temperature per unit volume or moisture per unit volume are required to be conserved. Therefore, it would be convenient to write the basic equations, in \( x,y,\sigma,t \) coordinates, with regard to the new expression for unit volume.

By replacing \( \frac{\partial \phi}{\partial \sigma} \) from equation (2.49) for \( \frac{\partial \phi}{\partial \sigma} \) in the momentum equation (2.44), multiplying (2.44) by \( \pi \) and adding the result to the continuity equation (2.46) after multiplying by \( \nabla \), collecting terms and replacing \( \pi \nabla \phi \) by \( \nabla (\pi \phi - \phi \nabla \pi) \) the new expression for the momentum equation becomes
\[
\frac{\partial}{\partial t} (\pi \nabla) + \nabla (\nabla \cdot \pi \nabla) + \pi (\nabla \cdot \nabla) + \frac{\partial}{\partial \sigma} (\pi \dot{\sigma} \nabla)
\]
\[
+ \pi f(k \times \nabla) + \nabla (\pi \phi) + (\sigma \pi a - \phi) \nabla \pi = - \pi F
\] 

(2.51)

Replacing for \( \theta \) in equation (2.48) from equation (2.31) and multiplying (2.48) by \( \pi \) the new expression for the thermodynamic energy equation is

\[
\left( \frac{\partial}{\partial t} + \nabla \cdot \nabla + \sigma \frac{\partial}{\partial \sigma} \right) \pi c_p T = \left( \frac{\rho^*}{\rho} \right) \pi \frac{DQ}{Dt}
\] 

(2.52)

Multiplying the continuity equation (2.46) by \( m \) and the conservation of moisture equation (2.49) by \( \pi \) and adding the two equations, equation (2.49) becomes

\[
\frac{\partial}{\partial t} (\pi m) + \nabla \cdot (\pi m \nabla) + \frac{\partial}{\partial \sigma} (\pi m \dot{\sigma}) = \pi M
\] 

(2.53)

2.2 Equations in Curvilinear Coordinates

With the equations in \( x, y, \sigma, t \) coordinates, the only remaining change to be made is to go from a rectangular cartesian coordinate system to spherical orthogonal coordinates. In the spherical coordinate system, surfaces of constant \( \sigma \) will be taken to be approximately spherical in shape. Any \( \sigma \) surface will be taken as being close enough to the earth's surface so that its distance from the center of the earth can be taken as the radius of the earth. The transformation from cartesian to spherical coordinates is taken from Owczarek [30].

In making the transformation to spherical coordinates
It is assumed that the new coordinates are single valued functions of the old rectangular cartesian coordinate system and vice versa.

\[ u_1 = F_1(x, y, z) \]
\[ u_2 = F_2(x, y, z) \]
\[ u_3 = F_3(x, y, z) \] (2.54a)

\[ x_1 = f_1(u_1, u_2, u_3) \]
\[ x_2 = f_2(u_1, u_2, u_3) \]
\[ x_3 = f_3(u_1, u_2, u_3) \] (2.54b)

The spherical system will be assumed to be orthogonal so that

\[ u_1 \cdot u_2 = 1 \quad u_2 \cdot u_2 = 1 \quad u_3 \cdot u_3 = 1 \]
\[ u_1 \cdot u_2 = 0 \quad u_2 \cdot u_3 = 0 \quad u_3 \cdot u_1 = 0 \] (2.55a)

\[ u_1 \times u_2 = 0 \quad u_2 \times u_2 = 0 \quad u_3 \times u_3 = 0 \]
\[ u_1 \times u_1 = u_3 \quad u_2 \times u_3 = u_1 \quad u_3 \times u_1 = u_2 \] (2.55b)

Considering a vector \( \mathbf{dr}_1 \), tangent at some point to the coordinate curve \( u_1 \) it can be shown that

\[ \mathbf{dr}_1 = u_1 ds_1 = h_1 \nabla u_1 ds_1 \] (2.56)

where \( h_1 \) is the proportionality factor between the unit vector \( \nabla u_1 \) which is perpendicular to the surface of
constant $u_1$. The term $ds_1$ represents a differential arc length along the curve of $dr$.

The scalar product of $dr_1$ and $u_1$ is

$$dr_1 \cdot u_1 = ds_1 = h_1 ds_1 u_1 \cdot \nabla u_1 = h_1 ds_1 \frac{\partial u_1}{\partial s_1} = h_1 du_1 \quad (2.57)$$

and therefore

$$ds_1 = h_1 du_1$$
$$ds_2 = h_2 du_2$$
$$ds_3 = h_3 du_3 \quad (2.58)$$

From equation (2.58) it can be seen that

$$dr = u_1 ds_1 + u_2 ds_2 + u_3 ds_3 = h_1 u_1 ds_1 + h_2 u_2 du_2 + h_3 u_3 du_3 \quad (2.59)$$

and

$$(ds)^2 = (ds_1)^2 + (ds_2)^2 + (ds_3)^2 = h_1^2 (du_1)^2 + h_2^2 (du_2)^2 + h_3^2 (du_3)^2 \quad (2.60)$$

The gradient of a scalar point function can be shown to equal

$$\nabla f = \frac{u_1}{h_1} \frac{\partial f}{\partial u_1} + \frac{u_2}{h_2} \frac{\partial f}{\partial u_2} + \frac{u_3}{h_3} \frac{\partial f}{\partial u_3} \quad (2.61)$$

The divergence of a vector function $F$ is expressible as

$$\nabla \cdot F = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial (h_2 h_3 F_1)}{\partial u_1} + \frac{\partial (h_3 h_1 F_2)}{\partial u_2} + \frac{\partial (h_1 h_2 F_3)}{\partial u_3} \right] \quad (2.62)$$
The scalar operator \( \nabla \cdot \nabla \) is
\[
\nabla \cdot \nabla = \frac{V_1}{h_1} \frac{\partial}{\partial u_1} + \frac{V_2}{h_2} \frac{\partial}{\partial u_2} + \frac{V_3}{h_3} \frac{\partial}{\partial u_3} \tag{2.63}
\]
remembering that when this operator is used on a vector function the base unit vectors \( u_1, u_2, u_3 \) are functions of the coordinates.

Using equations (2.61), (2.62) and (2.63) it is possible to change the basic equations to the orthogonal curvilinear coordinate system \((x', y', \sigma, t)\).

The continuity equation (2.46) can be changed by using the expression for the divergence (2.62) and multiplying by \( h_1, h_2, h_3 \).
\[
\frac{\partial (h_1 h_2 h_3 \pi)}{\partial t} + \frac{\partial (h_2 h_3 \pi u)}{\partial x'} + \frac{\partial (h_2 h_3 \pi u)}{\partial y'} + \frac{\partial (h_1 h_2 \sigma \pi)}{\partial \sigma} = 0 \tag{2.64}
\]

By using equations (2.62) and (2.63) and multiplying by \( h_1 h_2 h_3 \) the energy equation becomes
\[
\frac{\partial}{\partial t} (\pi h_1 h_2 h_3 T) + \frac{\partial}{\partial x'} (h_1 h_3 \pi u) + \frac{\partial}{\partial y'} (h_2 h_3 \pi v) + \left( \frac{P}{\rho} \right)^K = 0
\]
\[
\frac{\partial}{\partial \sigma} (h_1 h_2 \pi \sigma \dot{\sigma}) - \frac{\sigma a}{c_p} \frac{\partial (h_1 h_2 h_3)}{\partial t} + h_1 h_3 u \frac{\partial \pi}{\partial x'} + h_2 h_3 v \frac{\partial \pi}{\partial y'} + \frac{h_1 h_2 h_3 H}{c_p} = 0 \tag{2.65}
\]

By similar methods the moisture equation (2.53) becomes
\[ \frac{\partial}{\partial t}(h_1 h_2 h_3 q) + \frac{\partial}{\partial x}(h_1 h_3 u q) + \frac{\partial}{\partial y}(h_2 h_3 u q) = 2 h_1 h_2 h_3 g(e-c) \]  

(2.66)

For the momentum equation (2.51) it must be remembered that in the term \( \pi(V \cdot V)V \) the base vectors are functions of the coordinates so that

\[ V(V \cdot \pi V) + \pi(V \cdot V)V = \frac{u}{h_1 h_2 h_3} \left[ \frac{\partial (h_3 h_1 \pi u)}{\partial u_2} + \frac{\partial (h_1 h_2 \pi v)}{\partial u_3} \right] \]

\[ + \frac{v}{h_1 h_2 h_3} \left[ \frac{\partial (h_3 h_1 u)}{\partial u_2} + \frac{\partial (h_1 h_2 \pi v)}{\partial u_3} \right] u_3 + \frac{\pi u_2}{h_2} \left[ \frac{\partial}{\partial u_2} (h_3 h_1 v) - \frac{1}{h_2} \frac{dh_2}{du_3} h_2 u \right] u_2 \]

\[ + \frac{\pi v}{h_3} \left[ \frac{\partial}{\partial u_3} (h_2 u) - \frac{1}{h_2} \frac{dm}{dy} h_2 u \right] u_2 + \frac{\pi v}{h_3} \left[ \frac{\partial}{\partial y_3} (h_3 v) - \frac{1}{h_3} \frac{dh_3}{dy} h_3 v \right] u_2 \]  

(2.67)

This simplifies to

\[ V(V \cdot \pi V) + \pi(V \cdot V)V = \frac{1}{h_1 h_3} \left[ \frac{\partial}{\partial u_2} (h_3 \pi u^2) + \frac{\partial}{\partial u_3} (h_2 \pi uv) \right] \]

\[ + \pi uv \frac{dh_2}{du_3} u_2 \]

\[ + \frac{1}{h_1 h_2} \left[ \frac{\partial}{\partial u_2} (h_3 \pi uv) + \frac{\partial}{\partial u_3} (h_2 \pi v^2) u^2 \frac{dh_2}{dy} \right] u_3 \]  

(2.68)

With equations (2.68) and relations (2.61) and (2.62) the momentum equation (2.51) can be written in component
form in the x direction as

\[
\frac{\partial}{\partial t}(h_1 h_2 h_3 \pi u) + \frac{\partial}{\partial x}(uh_1 h_3 u) + \frac{\partial}{\partial y}(vh_1 h_2 u) \\
+ \frac{3}{\mu_0}(h_1 h_2 h_3 \pi \dot{\phi}) + G\pi u + h_1 h_3 \left[ \frac{\partial}{\partial x}(\pi \phi) + \\
(\sigma \pi a - \phi) \frac{\partial \pi}{\partial x} \right] = -h_1 h_2 h_3 \pi F_x
\]  
(2.69a)

and in the y direction as

\[
\frac{\partial}{\partial t}(h_1 h_2 h_3 \pi v) + \frac{\partial}{\partial x}(uh_1 h_3 v) + \frac{\partial}{\partial y}(vh_1 h_2 v) \\
+ \frac{3}{\mu_0}(h_1 h_2 h_3 \pi \dot{\phi}) + G\pi v + h_1 h_2 \left[ \frac{\partial}{\partial y}(\pi \phi) + \\
(\sigma \pi a - \phi) \frac{\partial \pi}{\partial x} \right] = -h_1 h_2 h_3 \pi F_x
\]  
(2.69b)

where G is given by

\[
G = h_1 h_2 h_3 f - u \frac{dm}{dy}
\]  
(2.69c)

It now remains to determine the values of \(h_1\), \(h_2\), and \(h_3\). They are determined in the following manner, an infinitesimal element of volume in the curvilinear coordinate system can be expressed as

\[
dVol = ds_1 ds_2 ds_3 = (h_1 du_1)(h_2 du_2)(h_3 du_3)
\]  
(2.70)

This volume element can also be expressed as the scalar triple product of vectors tangent at a point to the coordinate curves \(u_1\), \(u_2\), \(u_3\) with their arc-lengths differentials of the curves,
\[ d\text{Vol} = (\frac{\partial A}{\partial u_1}) du_1 \left[ (\frac{\partial A}{\partial u_2}) du_2 \times (\frac{\partial A}{\partial u_3}) \right] du_3 \] (2.71)

where \( \mathbf{A} \) is a vector in a cartesian coordinate system at the center of the earth

\[ \mathbf{A} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \] (2.72)

It follows that

\[ \frac{\partial \mathbf{A}}{\partial u_1} du_1 = \left( \frac{\partial x}{\partial u_1} \mathbf{i} + \frac{\partial y}{\partial u_1} \mathbf{j} + \frac{\partial z}{\partial u_1} \mathbf{k} \right) du_1 \] (2.73a)

\[ \frac{\partial \mathbf{A}}{\partial u_2} du_2 = \left( \frac{\partial x}{\partial u_2} \mathbf{i} + \frac{\partial y}{\partial u_2} \mathbf{j} + \frac{\partial z}{\partial u_2} \mathbf{k} \right) du_2 \] (2.73b)

\[ \frac{\partial \mathbf{A}}{\partial u_3} du_3 = \left( \frac{\partial x}{\partial u_3} \mathbf{i} + \frac{\partial y}{\partial u_3} \mathbf{j} + \frac{\partial z}{\partial u_3} \mathbf{k} \right) du_3 \] (2.73c)

The components of the vector \( \mathbf{A} \) can be expressed using \( \theta, \phi \) as \( u_1, u_2, u_3 \) you obtain

\[ x = A \cos\phi \cos\theta \] (2.74a)

\[ y = A \cos\phi \sin\theta \] (2.74b)

\[ z = A \sin\phi \] (2.74c)

Using relations (2.74a,b,c) in equations (2.73a,b,c) and (2.71) you obtain

\[ d\text{Vol} = (du_1)(A \cos\phi du_2)(Adu_3) \] (2.75)

Equating the terms of equation (2.75) with equation (2.70) and changing \( u_1, u_2, u_3 \) to \( \sigma, x, y \) the values of \( h_1, h_2, h_3 \) are found to be
\[ h_1 = 1 \quad (2.76a) \]
\[ h_2 = A \cos \phi \quad (2.76b) \]
\[ h_3 = A \quad (2.76c) \]

Where \( A \) is the mean radius of the earth and \( \phi \) is the latitude of a grid point measured from the equator.

2.3 The Cubic Polynomial Spline

Consider an interval \( a \leq x \leq b \) which is subdivided into subintervals \( a = x_0 < x_1 < \ldots < x_N = b \). This interval may be periodic, in which case \( a = b \), or non-periodic. A cubic polynomial spline is a function \( S(x) \) of ordinates \( y_0, y_1, \ldots, y_N \) which is continuous along with its first and second derivatives on the interval \([a,b]\). This function is identical with a cubic in each subinterval \( x_{j-1} \leq x \leq x_j \) \((j = 1, 2, 3, \ldots N)\). A periodic spline is said to be periodic of period \((b-a)\) when the condition \( S(a+) = S(b-) \) is satisfied.

Since in most applications of numerical methods there is a desire to estimate the first derivative, the cubic polynomial spline can be written in terms of the first derivative \( m_j \) from Ahlberg, et al [28] as

\[
S(x) = m_{j-1} \frac{(x-x) (x-x_{j-1})}{h_j^2} - m_j \frac{(x-x_{j-1}) (x_j-x)}{h_j^2}
\]
\[ y_{j-1} \frac{(x_j-x)^2[2(x-x_{j-1})+h_j]}{h_j^3} + y_j \frac{(x-x_{j-1})^2[2(x_j-x)+h_j]}{h_j^3} \]

(2.77)

where \( m_j \) represents a first derivative and

\[ h_j = x_j - x_{j-1} \]

(2.78)

The first derivative of this equation (2.77) is

\[ S'(x) = m_{j-1} \frac{(x_j - x)(2x_{j-1} + x_j - 3x)}{h_j^2} - m_j \frac{(x-x_{j-1})(2x_j + x_{j-1} - 3x)}{h_j^2} \]

\[ + \frac{y_j - y_{j-1}}{h_j} \frac{6(x_j-x)(x-x_{j-1})}{h_j^3} \]

(2.79)

The second derivative takes the form

\[ S''(x) = -2m_{j-1} \frac{2x_j + x_{j-1} - 3x}{h_j^2} - 2m_j \frac{2x_{j-1} + x_j - 3x}{h_j^2} \]

\[ + 6 \frac{y_j - y_{j-1}}{h_j^3} (x_j + x_{j-1} - 2x) \]

(2.80)

Assuming that the second derivatives must be continuous the limiting values from the sides of \( x_j \) are

\[ S''(x_j-) = \frac{2m_{j-1}}{h_j} + \frac{4m_j}{h_j^2} - 6 \frac{y_j - y_{j-1}}{h_j^2} \]

(2.81a)

\[ S''(x_j+) = -\frac{4m_j}{h_j+1} - \frac{2m_{j+1}}{h_j+1} + 6 \frac{y_{j+1} - y_j}{h_j+1} \]

(2.81b)

When this continuity of first and second derivatives is imposed on \( S''(x) \) at \( x_j \) \((j = 1,2,3,\ldots N-1)\) the results from Ahlberg, et al [28] require,
\[
\frac{1}{h_j} m_{j-1} + 2\left(\frac{1}{h_j} + \frac{1}{h_{j+1}}\right) m_j + \frac{1}{h_{j+1}} m_{j+1} = 3 \frac{Y_j - Y_{j-1}}{h_j^2} + 3 \frac{Y_{j+1} - Y_j}{h_{j+1}^2}
\]

(2.82a)

Or in an abbreviated form

\[
\lambda_j m_{j-1} + 2m_j + \mu_j m_{j+1} = 3\lambda_j \frac{Y_j - Y_{j-1}}{h_j} + 3\mu_j \frac{Y_{j+1} - Y_j}{h_{j+1}}
\]

(2.82b)

where

\[
\lambda_j = \frac{h_{j+1}}{h_j + h_{j+1}} \quad (j = 1, 2, \ldots, N-1)
\]

(2.83a)

\[
\mu_j = 1 - \lambda_j \quad (j = 1, 2, \ldots, N-1)
\]

(2.83b)

The system of equations for the non-periodic spline with general end conditions \(2m_0 + \mu_0 m_1 = c_0\), \(\lambda_N m_{N-1} + 2m_N = c_N\) is

\[
\begin{bmatrix}
2 & \mu_0 & 0 & \cdots & 0 & 0 & 0 \\
\lambda_1 & 2 & \mu_1 & \cdots & 0 & 0 & 0 \\
0 & \lambda_2 & 2 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 2 & \mu_{N-2} & 0 \\
0 & 0 & 0 & \cdots & \lambda_{N-1} & 2 & \mu_{N-1} \\
0 & 0 & 0 & \cdots & 0 & \lambda_N & 2
\end{bmatrix}
\begin{bmatrix}
m_0 \\
m_1 \\
m_2 \\
\vdots \\
m_{N-2} \\
m_{N-1} \\
m_N
\end{bmatrix} =
\begin{bmatrix}
c_0 \\
c_1 \\
c_2 \\
\vdots \\
c_{N-2} \\
c_{N-1} \\
c_N
\end{bmatrix}
\]

(2.84)

where \(c_j (j = 1, 2, 3, \ldots, N-1)\) is the righthand side of equation (2.82b)
For the periodic case, where \( x_0 = x_N, y_0 = y_N \) the equations are

\[
\begin{pmatrix}
2 & \mu_1 & 0 & \cdots & 0 & 0 & \lambda_1 \\
\lambda_2 & 2 & \mu_2 & \cdots & 0 & 0 & 0 \\
0 & \lambda_3 & 2 & \cdots & 0 & 0 & 0 \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & 2 & \mu_{N-2} & 0 \\
0 & 0 & 0 & \cdots & \lambda_{N-1} & 2 & \mu_{N+1} \\
\mu_N & 0 & 0 & \cdots & 0 & \lambda_N & 2
\end{pmatrix}
\begin{pmatrix}
m_1 \\
m_2 \\
m_3 \\
\vdots \\
m_{N-2} \\
m_{N-1} \\
m_N
\end{pmatrix}
= \begin{pmatrix}
c_1 \\
c_2 \\
c_3 \\
\vdots \\
c_{N-2} \\
c_{N-1} \\
c_N
\end{pmatrix}
\]

(2.85)

where \( \lambda_N = h_1/(h_N + h_1), \mu_N = 1 - \lambda_N \). Since the series of equations (2.85) consists of \( M = 1, 2, \ldots N \) equations in \( M = 1, 2, \ldots N \) unknowns and the system is determinate.

In the system of equations (2.84) there are several possibilities for \( \mu_N, c_O, \lambda_O, c_N \) and therefore more end conditions will need to be specified. The general end conditions for (2.84) can be written as

\[
2m_O + \mu_O m_1 = c_O
\]

(2.86a)

\[
\lambda_N m_{N-1} + 2m_N = c_N
\]

(2.86b)

When suitable values for \( \mu_O, c_O, \lambda_N, c_N \) are specified, the system (2.84) will be determinate. There are three common end conditions imposed. The first is to specify the slopes \( (y'_O \text{ and } y'_N) \) so that
\[ u_0 = \lambda_N = 0, \quad c_0 = 2y_0', \quad c_N = 2y_N' \quad (2.87) \]

A second possibility is to specify the second derivatives \( y_o'' \) and \( y_N'' \) so that

\[ u_0 = \lambda_N = 1 \]

\[ c_0 = \frac{3(y_2-y_1)}{h_1} - \frac{h_1}{2} y_o'' \]

\[ c_N = \frac{3(y_N-y_{N-1})}{h_N} + \frac{h_N}{2} y_N'' \quad (2.88) \]

A third possibility is to specify a zero second derivative at some imaginary end points \( x_{-1} \) and \( x_{N+1} \) which are assumed to be outside the original spline interval \( a = x_0 \leq x \leq x_N = b \). This condition assumes cubic arcs over the intervals \( x_{-1} \leq x \leq x_0 \) and \( x_N \leq x \leq x_{N+1} \) with the location of the points \( x_{-1} \) and \( x_{N+1} \) defined by assuming values for \( \lambda_{-1} \) and \( \mu_{N+1} \)

\[ \lambda_{-1} = \frac{x_0-x_{-1}}{x_2-x_{-1}}, \quad \mu_{N+1} = \frac{x_N-x_{N+1}}{x_{N-1}-x_{N+1}} \quad (2.89) \]

The end conditions now become:

\[ u_0 = \frac{2(2\lambda_{-1}+1)}{\lambda_{-1}+2}, \quad \lambda_N = \frac{2(2\mu_{N+1}+1)}{\mu_{N+1}+2} \]

\[ c_0 = \frac{6(y_2-y_1)(1+\lambda_{-1})}{h_2(2+\lambda_{-1})}, \quad c_N = \frac{6(y_N-y_{N-1})(1+\mu_{N+1})}{h_N(2+\mu_{N+1})} \quad (2.90) \]

To solve the system of equations, for the slopes \( m_j \), in the non-periodic case (2.84), where the matrix part is now a tri-diagonal matrix, there is a simple algorithm.
Consider equations

\[ b_1 x_1 + c_1 x_2 = d_1 \]
\[ a_2 x_1 + b_2 x_2 + c_2 x_3 = c_2 \]
\[ a_{n-1} x_{n-1} + b_{n-1} x_{n-1} + c_{n-1} x_n = d_{n-1} \]
\[ a_n x_{n-1} + b_n x_n = d_n \]  \hspace{1cm} (2.91)

by forming for \((k = 1, 2, \ldots n)\)

\[ p_k = a_k q_{k-1} + b_k (q_0 = 0) \]
\[ q_k = -c_k / p_k \]
\[ u_k = (d_k - a_k u_{k-1}) / p_k (u_0 = 0) \] \hspace{1cm} (2.92a)

These can be solved for \(x_1, x_2, x_3, \ldots x_{n-1}\) to yield

\[ x_k = q_k x_{k+1} + u_k \hspace{0.5cm} (k = 1, \ldots , n-1) \]
\[ x_n = u_n \] \hspace{1cm} (2.92b)

Since \(p_k\) and \(q_k\) stay the same for a given set of coordinates several variables can be calculated on the same mesh without the need to recalculate \(p_k\) and \(q_k\). It can also be shown that for tri-diagonal matrices with a dominant main diagonal, errors at the ends rapidly damp out.

The above procedure can be extended to work for the periodic case, system (2.35). For a series of equations
\[ b_1x_1 + c_1x_2 + a_1x_n = d_1 \]

\[ a_2x_1 + b_2x_2 + c_2x_3 = d_2 \]

\[ a_{n-1}x_{n-1} + b_{n-1}x_n + c_{n-1}x_1 = d_{n-1} \]

\[ c_nx_1 + a_nx_n + b_nx_1 = d_n \]  

(2.93)

Along with the equations for \( p_k \), \( q_k \), and \( u_k \) (2.92a), a new equation is needed

\[ s_k = -a_k s_{k-1}/p_k \quad (s_0 = 1) \]  

(2.94a)

and equation (2.92) is replaced by

\[ x_k = q_k x_{k+1} + s_k x_n + v_k \quad (k = 1, \ldots, n-1) \]  

(2.94b)

or

\[ x_k = t_k x_n + v_n + v_k \quad (k = 1, \ldots, n-1) \]  

(2.94c)

where

\[ t_k = q_k t_{k+1} + s_k \quad (t_n = 1) \]

\[ v_k = q_k v_{k+1} + u_k \quad (v_n = 0) \]  

(2.95)

\( x_n \) is determined from the equation

\[ c_n(t_1x_n + v_1) + a_n(t_{n-1}x_n + v_{n-1}) + b_n x_n = d_n \]  

(2.96)

and then \( x_{n-1}, \ldots, x_1 \) are calculated from (2.94c).

2.4 Double Cubic Polynomial Spline

Consider a rectangular grid \( a = x_0 < x_1 < \ldots < x_{N-1} < x_N = b, \ c = y_0 < y_1 < \ldots < y_{M-1} < y_M = d \) with ordinates \( z_{ij} \)
at grid points $x_i, y_j, i = 0,1,\ldots,N, j = 0,\ldots,M$. The double cubic spline $s(x,y)$ is defined as a piecewise continuous function of polynomials in each rectangle $x_{i-1} \leq x \leq x_i, y_{j-1} < y < y_j, i = 1,2,3,\ldots N, j = 1,2,\ldots M$. The double cubic spline is also assumed to have continuous first and second derivatives in the $x$ and $y$ directions and continuous cross derivatives, Ahlberg, et al [28].

For meteorological fields on a global scale the $x$ direction can be taken to coincide with latitude circles and therefore is periodic since the fields repeat every $360^\circ$. The $y$ direction will be taken to be the longitudinal meridians and will be assumed to have boundaries at the north and south poles so that non-periodic splines can be used.

In the $x$ direction the cubic spline can be written for a line $y = y_j$ with ordinates $z_{ij}$ and slopes $m_{i,j}$ from equation (2.77) as

$$s(x_i, y_j) = m_{i-1,j} \frac{(x_i-x)^2(x-x_{i-1})}{h_i^2} - m_{i,j} \frac{(x-x_{i-1})^2(x_i-x)}{h_i^2} + z_{i-1,j} \frac{(x_i-x)^2[2(x-x_{i-1})+h_i]}{h_i^3} + z_{i,j} \frac{(x-x_{i-1})^2[2(x-x_i)+h_i]}{h_i^3}$$

(2.97)

This reduces to the system of equations (2.85) which can be solved by the previously described methods.

For the $y$ directions along a line $x = x_i$ for ordinates
and slopes $\mathbf{n}_{i,j}$ equation (2.77) gives

$$S(x_i, y_j) = M_{i,j} \frac{(y_j - y)^2 (y_j - y_{j-1})}{k_j^2} - n_{i,j} \frac{(y_{j-1} - y)^2 (y_j - y_{j-1})}{k_j^2}$$

$$+ z_{i,j-1} \frac{(y_j - y)^2 [2(y_j - y_{j-1}) + k_j]}{k_j^3} + z_{i,j} \frac{(y_{j-1} - y)^2 [2(y_{j-1} - y) + k_j]}{k_j^3}$$

(2.98)

where $k_j = y_j - y_{j-1}$. These equations can be expressed as a system of equations similar to equations (2.84) and solved by the method outlined for the non-periodic case.

Methods also outlined in Ahlberg, et al [28] and Price [19] obtain estimates for the derivatives at the mid-points of the grid sides and at the center of grid elements.

### 2.5 Integrating With Splines

The cubic spline can also be used to integrate a function. By taking equation (2.77) on the interval $x_{j-1} \leq x \leq x_j$ and integrating from $x_{j-1}$ to $x_j$ a new equation is obtained

$$\int_{x_{j-1}}^{x_j} S(x) \, dx = \frac{(m_{j-1} - m_j) h_j^2}{12} + \frac{(y_{j-1} + y_j) h_j}{2}$$

(2.99)

By summing equation (2.99) over all $a = x_0 < x_1 < ... < x_{N-1} < x_N = b$ the integral over the entire interval a to b is obtained,

$$\int_{a}^{b} S(x) \, dx = \sum_{j=1}^{N} \frac{(m_{j-1} - m_j) h_j^2}{12} + \sum_{j=1}^{N} \frac{(y_{j-1} + y_j) h_j}{2}$$

(2.100)

where $m_j$ are determined by the methods outlined above for
the periodic and non-periodic cases.

2.6 Forecast Grid

The grid form used in the present model is actually the superposition of two different types of grid on a global scale. One grid gives a fixed resolution in a specified region of interest with a smoothly decreasing resolution outside this region. The second grid insures that a minimum distance between grid points is maintained for different latitude circles. This second grid form relates a real physical distance on the earth's surface to the mathematical mesh in such a way that computational stability is maintained.

For the second grid expansion the number of grid points decreases as latitude circles get closer to the poles because of the decreasing physical distance around a latitude circle. Consider now the grid circle closest to the equator corresponding to a latitude circle \( y_{EQ} \) near the equator. The physical distance \( d_{EQ} \) between grid points is given by

\[
d_{EQ} = \frac{(2\pi A \cos y_{EQ})}{M_{EQ}}
\]

(2.101)

where \( A \) is the mean radius of the earth and \( M_{EQ} \) is the total number of equispaced grid points. Using this physical distance as the basis for a time step, which is computationally stable, it can be seen that at any other grid
latitude circle $y_j$ the physical distance between grid points must be

$$d_j \geq d_{EQ}$$  \hspace{1cm} (2.102)

The distance $d_j$ is given by

$$d_j = \frac{(2\pi A \cos y_j)}{M_j}$$ \hspace{1cm} (2.103)

The number of grid points at the north pole $M_1$ and the south pole $M_N$ are given by

$$M_1 = M_2$$

$$M_N = M_{N-1}$$ \hspace{1cm} (2.104)

This approximation of a physical point, the north or south pole, by a finite number of points is necessary in order to calculate the derivatives in the meridional directions. The polar points themselves are not used as dependent variables but only as boundary points which are updated after time steps by a method which will be described later.

The first type of grid expansion is known as a telescoping grid. By giving constant resolution in a region of interest outside of which the resolution decreases, considerable computation time is saved on generating good resolution forecasts for the primary region of interest. An important factor in generating this grid is that there is a smooth transition from the primary interest region to the outer regions. Ahlberg, et al [28] reported that problems
arose for using splines to estimate derivatives when there are sudden jumps in the physical grid spacings within limited regions. Ahlberg also found that in a one-dimensional region containing four grid points with spacing alternatingly very large, very small, then very large again, the resulting spline in the region of the small spacing exhibited a local maximum and minimum which provides poor estimates of the first and second derivatives. To avoid this problem the telescoping grid is generated in the following way. The latitude and longitude $Y_P, X_P$ of the center of the region of interest is chosen. The number of grid rectangles of constant resolution in the $x$ and $y$ directions around the center of the regions is also chosen, $N_X, N_Y$. Finally the grid spacing in the region of interest $D_X, D_Y$ and the expansion factor to be used outside the region of interest are chosen $E_X, E_Y$.

Some of the factors explained above may need to be changed to enable the grid to expand smoothly around an entire latitude circle. From Figure 4 the parameter $d_o$ is used to determine the changes which must be made in the $x$ direction to insure a smooth expansion. There are four cases of importance which are handled as outlined by Price [19],

1. If $d_o < d_1 - D_X/2$ a new grid must be generated using a new $N_X$ as defined by $N_X^{\text{NEW}} = N_X^{\text{OLD}} + N_N$ with $N_N$
the largest integer less than or equal to \((d_0/\Delta x) + 1\).

If in the new grid as generated \(d_1 < d_2\), the entire process should be recalculated with \(N_{\text{NEW}} = N_{\text{OLD}} + N_N - 1\).

2. If \(d_1 - \Delta x/2 < d_0 < d_1 + \Delta x/2\) by calculating new \(X_{-1}\) and \(X_1\) from \(X_{-1}^{(\text{NEW})} = x_2 + X_M\) and \(X_1^{(\text{NEW})} = X_2 - X_M\), where \(X_M = (X_2 - X_{-2})/3\) a new grid is generated which must satisfy the relation \(d_1 < d_2\) or else Case 1 has to be applied to change \(N_X\).

3. If \(d_1 + \Delta x/2 < d_0 < 2d_2\) a new grid is generated by setting \(N_{\text{NEW}} = N_{\text{OLD}} + M_M. M_M\) is the largest integer which is less than or equal to \((d_0 - d_1)/\Delta x + 1/2\). The new grid must be checked against Case 2.

4. If \(2d_1 < d_0 < (2d_1)\times 10\) make \(XXP\) a grid point.

In the \(y\) direction the grid must be smoothly joined to the northern and southern boundaries, the north and south poles respectively. The entire grid is first shifted until the nearest grid line coincides with the north pole, which slightly changes the \(y\) center of the region of interest. From Figure 4 the problem remains of adjusting \(y_1\) and \(y_2\) to make the southern boundary match. Two cases of importance are considered.

1. If \(D/3 > d\), where \(D\) is the distance from \(y_3\) to the south pole given by \(D = a + b + c = y_3 - \text{south pole}\), when \(a > 0\). By decreasing the expansion factor \(E_Y\) but keeping
it greater than 1 for the first interval and then adjusting EY by a small factor δ so that the lengths c, b, and a have values (EY - δ)d, (EY - 2δ)(EY - δ)d and (EY - 3δ)(EY - 2δ)(EY - δ)d respectively. The factor δ can be calculated by using

\[ a + b + c = 0 = 6\delta^3 - (2 + 11EY)\delta^2 + (1 + 3EY + 6EY^2)\delta + (D/d - EY(1 + EY + EY^2)) = 0 \]

This equation gives one real and two complex roots. The real root is used to calculate a, b and c.

2. If D/3 < d, a < 0 the grid must be generated over again using a new expression for NY_{NEW} = NY_{OLD} + NW. NW is the largest integer less than or equal to a/DY + 1. The new grid is then adjusted using Case 1.

2.7 Grid Description

For the present model the forecast grid was generated using the following values:

\[ \begin{align*}
XP &= -100.0^\circ \\
YP &= 40.0^\circ \\
NX &= 8^\circ \\
NY &= 3^\circ \\
DX &= 5.0^\circ \\
DY &= 5.0^\circ \\
EX &= 1.10
\end{align*} \]
The meridional degrees are measured from zero at the equator with positive values in the northern hemisphere and negative values in the southern hemisphere. The longitudinal degrees are measured from zero at the Greenwich Meridian, positive in an easterly direction and negative in a westerly direction. The number of grid points $M_j$ on each grid longitude line $j$ is, in order from the north pole to the south pole, $J_m = 11, 11, 18, 23, 22, 29, 31, 33, 34, 35, 36, 37, 38, 39, 38, 38, 37, 35, 32, 26, 17, 17$. A representation of the final grid form used for all the forecast experiments is given in Figure 4.

3. NUMERICAL PROCEDURE

The governing equations used for a numerical forecast as previously derived are (2.64), (2.65), (2.66), and (2.69a,b). These equations are of the general form

$$\frac{\partial}{\partial t} \psi = A_{\psi x} + B_{\psi y} \quad (3.1)$$

where $\psi$ is defined as being equal to $h_1 h_2 h_3 \pi$ and $\psi$ takes on values of 1, $u$, $v$, $q$, T. The term $A_{\psi x}$ represents all first derivatives in the $x$ direction and $B_{\psi y}$ represents all first derivatives in the $y$ direction. In the $x$ direction for rows $j = 2, 3, \ldots, N-2, N-1$ the first derivatives are calculated using the periodic spline as outlined above for
the system of equations (2.85). For the $y$ direction the non-periodic spline is used, equations (2.84). For the $y$ direction derivatives the two end conditions at the north and south pole need to be examined.

As stated above, there are three possibilities for the end conditions on the non-periodic spline which are given by (2.87), (2.88) and (2.90). Some estimate for the end condition is needed so that there is a degree of coupling between the slopes of the various longitudinal grid lines that meet at the poles. Since the temperature and pressure fields are radially symmetric in an imaginary standard atmosphere one choice for possible end conditions is to set the first derivative equal to zero. A second choice is to set the second derivatives equal to zero and use equation (2.87) or (2.88) as end conditions.

A third choice would be to set the second derivative equal to some value at an imaginary grid point past the end, equation (2.90). It is felt that equation (2.90) although it offers some advantages might exert too strong an influence on the end slopes without really coupling them any better than the first two possibilities.

Specifying the first derivative equal to zero is also rejected since it does not give a good coupling between all the meridional slopes. For the present model, therefore, equation (2.88) is used with the second derivative set equal to zero.
Since the grid as used for the present model contains a varying number of grid points on any latitude circle the slopes in the $y$ direction cannot be directly calculated as they can in the $x$ direction. First some method of interpolation must be provided to make a regular series of longitude grids in the $y$ direction. To calculate the $y$ direction splines the following method is used. First, for every latitude circle the grid is interpolated to contain the same number of points at the same locations as the equatorial latitude circle $j = 15 M_{15} = 39$. Second, the slopes $B_{\psi y}$ are calculated using the standard non-periodic spline method outlined above. Third, the slopes $B_{\psi y}$ are back-interpolated to obtain an estimate of the slopes on their original grid for their respective latitude circle. Two interpolation algorithms are therefore required to facilitate this procedure.

For interpolating from the original grid to the new grid based on the equatorial latitude circle a simple Newtonian interpolation formula with divided differences is used, with the known ordinates at the nodes $x_1, x_2, x_3$ and $x_4$ being used for estimating the value of a new ordinate at $x^*$ which is between $x_2$ and $x_3$. A linear interpolation is used to calculate points $x_{1.5}, x_{2.5}$ and $x_{3.5}$ at the midpoint of the grid intervals.

$$
\psi_{i+.5} = \frac{1}{2} (\psi_i + \psi_{i+1}) \quad (3.2)
$$
By using the new ordinates at the points $x_{1.5}$, $x_{2.5}$ and $x_{3.5}$ the interpolation formula as given by Price [7] is

$$
\psi(x^*) = \psi_{1.5} + (x^* - x_{1.5})f_{1.5, 2.5} + (x^* - x_{1.5})(x^* - x_{2.5})f_{1.5, 2.5, 2.3} \quad (3.3)
$$

where

$$
f_{i,j} = (\psi_i - \psi_j)/(x_i - x_j) \quad (3.4a)
$$

$$
f_{i,j,k} = (f_{j,k} - f_{i,j})/(x_k - x_i) \quad (3.4b)
$$

These equations (3.3), (3.4a,b) are used to obtain

$$
\psi(x^*) = a\psi_{1.5} + b\psi_{2.5} + c\psi_{3.5} \quad (3.5)
$$

where

$$
a = \xi h^2 (\xi - 1)/(1 + \bar{h})
$$

$$
b = (\xi \bar{h} + 1)(1 - \xi)
$$

$$
c = \xi (1 + \xi \bar{h})/(1 + \bar{h})
$$

$$
\xi = \frac{x^* - (x_2 + h_3/2)}{.5(h_3 + h_4)}
$$

$$
\bar{h} = (h_3 + h_4)/(h_3 + h_2)
$$

$$
h_i = x_i - x_{i-1} \quad (3.6a-f)
$$

The back interpolation of the calculated slopes to the original grid is executed using a weighted averaging interpolation method. To back interpolate to an ordinate $\psi_{i^*}$ on the original grid, equal contributions are used from all nodes within a range $p\|$ with smaller contribution from points
\[ x_{i1-1} \text{ and } x_{i2+1} \text{ outside of } R. \text{ The method is given by:} \]

\[
\psi^*_i = \frac{\sum_{k=1}^{i2} \psi_k + f_2 \psi_{i2+1} + f_1 \psi_{i1-1}}{i_2 - il + f_1 + f_2} \tag{3.7}
\]

where

\[
f_1 = \frac{d_1}{h_{il}} \quad f_2 = \frac{d_2}{h_{i2+1}} \tag{3.8a}
\]

\[
h_i = x_i - x_{i-1} \tag{3.8b}
\]

### 3.1 Time Advancing

In the present model a modified version of the Matsuno two-stage time differencing scheme (Matsuno [28], Gates, Batten, Kahle, and Nelson [4], and Price [7]) is used. This method is the same as a two-stage Euler backward scheme and is discussed by Kurihara [8].

In the first stage of each time step the forward difference yields

\[
\psi^* = \frac{\Pi \psi(t) + F \psi(t) \Delta t}{z^*} \tag{3.9a}
\]

\[
\Pi^* = \Pi(t) + F_1(t) \tag{3.9b}
\]

where \( \psi \) takes on values of \( T, u, v, \) and \( q \) and \( F \) represents all the terms of the general equation (3.1), except for the source terms which will be explained later. The second stage, a backward difference approximation, gives a revised estimate for all the dependent variables, \( \psi \)
3.3 Source Terms

At the end of every simulation hour of the terms in equations (2.64), (2.65), (2.66) and (2.69a,b), known as the source terms, are also updated. The source terms are the heating function \( H \), the moisture source \( E \), which is evaporation, the moisture sink \( C \), which is condensation, and the two friction terms \( F_x \) and \( F_y \). For the heating function, incoming solar radiation, long wave radiation from the different model layers and from the earth's surface, upward transport of sensible heat from the earth's surface and the latent heat release on account of precipitation are taken into account. Ocean water evaporation, large scale precipitation and convective precipitation along with three types of cloud covers are also modelled. An extensive description of the entire modelling procedure for all the terms is provided in [7] and Gates, Batten, Kahle and Nelson [4].

3.4 Setting Polar Points

At the end of every step in the time differencing procedure new values must be generated for the two polar points since these are not independently updated. For latitude circles \( j = 1 \) and \( j = 23 \) the new values of pressure \( (\Pi) \), temperature \( (T) \), and mixing ratio \( (m) \) are calculated by
averaging

\[ j = 1 \quad \psi_{i,1} = \frac{1}{M_2} \sum_{k=1}^{M_2} \psi_{k,2} \quad i = 1, \ldots, M_1 \]  

(3.11a)

\[ j = 23 \quad \psi_{i,23} = \frac{1}{M_{22}} \sum_{k=1}^{M_{22}} \psi_{k,22} \quad i = 1, \ldots, M_{22} \]  

(3.11b)

The winds for the polar points are first related to the Greenwich meridian and then averaged to find a mean component.

\[ u_k = \frac{1}{M_j} \sum_{k=1}^{M_j} V_{k,j} \cos \theta_{k,j} \]  

(3.12a)

\[ v_k = \frac{1}{M_j} \sum_{k=1}^{M_j} V_{k,j} \sin \theta_{k,j} \]  

(3.12b)

\[ V_{i,j} = \left( u_{i,j}^2 + v_{i,j}^2 \right)^{1/2}, \quad i = 1, 2, \ldots, M_j \]  

(3.13)

\[ \theta_{i,j} = \tan^{-1} \left( \frac{v_{i,j}}{u_{i,j}} \right) + X_{i,j}, \quad i = 1, 2, \ldots, M_j \]  

(3.14)

where \( j = 2 \) or \( 22 \).

Using the calculated \( v_p \) and \( u_p \), the new winds for the poles are given by

\[ u_{i,j} = V_p \cos (\theta_p - x_{i,j}), \quad i = 1, \ldots, M_j \]  

(3.15a)

\[ v_{i,j} = V_p \sin (\theta_p - x_{i,j}), \quad i = 1, \ldots, M_j \]  

(3.15b)

\[ V_p = \left( u_p^2 + v_p^2 \right)^{1/2} \]  

(3.16)

\[ \theta_p = \tan^{-1} \left( \frac{v_p}{u_p} \right) \]  

(3.17)

where \( j = 1 \) or \( 23 \).
4. INITIAL DATA AND FORECAST RESULTS

There are several data parameters which need to be initialized before the forecast model can be run. These parameters are divided into two groups, constant parameters which do not change during forecast calculations and dependent variables.

The two constant parameters are the ocean temperature $T_o$ and the surface geopotential $\phi$. The ocean temperature is used for the calculation of heating functions in the source term model. The surface geopotential is used in the calculation of geopotentials at the two primary vertical grid layers. Both of these parameter values are obtained from the Gates et al (4) versions of the Mintz-Arakawa model. The ocean temperatures are the mean values of ocean temperature readings for January and June on a global basis. The surface elevations used to calculate surface geopotentials are the area averaged surface elevations taken from a 2 1/2 degree cylindrical projection of the earth's surface.

The initial dependent variables are the parameter $P_s - P_t$, the mixing ratio $q$ for the sigma 3/4 layer, the temperature $T$ for the sigma 1/4 and 3/4 layers and the horizontal wind components $u$ and $v$ for the sigma 1/4 and 3/4 layers, figure (5). The data was obtained from the Geophysical Fluid Dynamics Lab (GFDL).
of the National Oceanic and Atmospheric Administration in Princeton, New Jersey. The GFDL data is for 0000 GMT for the days March 1, 2, 3, 4 of the year 1965. The GFDL data is presented for the real and imaginary constant pressure surfaces 1150, 1000, 850, 700, 500, 350, 200, 100, 75, 50, 25, 12, 2 mb, on a polar stereographic projection for the northern and southern hemispheres with a cylindrical projection containing the tropical data from 30° north latitude to 30° south latitude. The initial values for the dependent variables are obtained by first calculating the real surface pressure from the GFDL data. Using the initial surface pressure the values for the temperature, wind components, and mixing ratio for each grid point on the two sigma surfaces are obtained by interpolating and taking an area average value from the GFDL data.

Contour plots of the initial reduced to sea level pressure, the temperature on the 400 and 800 mb levels, the wind components on the sigma 1/4 and 3/4 layers, and mixing ratio for the sigma 3/4 layer are presented in figures 6 thru 13. Contour plots of the initial geopotential heights for the 400 and 800 mb layers are given in figures 14 and 15.

3.1 **TWO PRELIMINARY EXPERIMENTS**

Two preliminary attempts at obtaining a stable
forecast were tried before a satisfactory configuration of initial conditions was generated. In the first attempt with real weather data the detailed earth topography of the model caused the growth of unrealistic regions of warm air over the mountainous areas of the earth which had to be eliminated. The second experiment revealed an initial inbalance of the wind fields and the geopotential heights.

Contour plots of the geopotential heights of the 400 and 800 mb surfaces for a 24 hour forecast using realistic topography are presented as figures 16 and 17. The plots reveal areas of unusual high geopotential over the major mountainous areas of the earth. A further study of this calculated result reveals that one possible cause of these regions of abnormal geopotential height is a steep increase in calculated surface pressure caused by a changing of wind patterns on the sigma 1/4 and 3/4 layers. The occurrence of this phenomenon has also been noted by Phillips (32). Phillips postulates that the problem is caused by the horizontal truncation effects in the calculation of the changes of the horizontal pressure forces for the momentum equation. This truncation effect is a result of writing the equations of hydrodynamics in a sigma coordinate system. Phillips gives some possible solutions to the problem but further
work still needs to be done before it will be possible to include a realistic earth topography in a weather forecast model with a sigma vertical coordinate.

As a result of the investigation into the cause of the problem encountered in the first experiment of the global forecast model the second experiment was run with greatly reduced values for surface geopotential. The new surface heights were set to be one-hundredth of the original surface elevations.

A second experiment was run using the new data set calculated with the smoothed values for surface elevations. The resulting contour plots for the geopotential heights on the 400 and 800 mb constant pressure surfaces after 24 hours of simulation run are shown in figures 18 and 19. The plots are to be compared with the real geopotential heights as given for 0000 GMT for March 2 from the GFDL data in figures 20 and 21.

The comparison of the two sets of contour plots of geopotential heights at 24 hours shows that although the model forecast results appear stable there is a lack of distinguishing features in the calculated results. The calculated results appear to show the reduction of the original data to a quasi-steady state atmosphere with a great reduction of the amplitudes of the wave patterns which are features of the confirmation data
for March 2. In particular the strong ridge off the west coast of North America has been totally damped out on the 400 mb contour plot. On the 800 mb surface the same ridge appears to have been damped and moved in an easterly direction. Also to be noted is the disappearance of the Hudson Bay, Greenland and Siberian highs. The southern hemisphere shows a similar smoothing of distinguishing features.

A further investigation of the second experimental data set shows that the majority of the truncation of the wave motions takes place within the first hour of the simulation run. This rapid generation by the program of the new atmospheric configuration suggests some initial inbalance in the data set. Figures 22 and 23 show plots of the surface pressure and temperature for 45° north latitude from 180° west to 90° west longitude. These two figures show the pressure and temperature both initially and at the end of one simulation hour. The plots suggest that the temperature field changes very little in the first hour of simulation. During the first hour of simulation the pressure field, unlike the temperature, shows large changes. It is this large change of surface pressure on a global scale which probably causes the smoothing seen in the geopotential plots. Further study of the data shows that the fields change very little after the first simulation.
The calculation of the surface pressure changes involves only the divergence of the wind vector field, equation (2.64). If the initial wind velocity field used for the program is not in balance with the initial surface pressure or geopotential fields there could be a large correction made in the surface pressure field by the divergence of the wind velocity. Because of the smoothing nature of the splines used to estimate the derivatives any initial imbalance would quickly dissipate, but the initial data would no longer reflect the true configuration of the atmosphere.

In an attempt to circumvent the problem of an imbalance in the initial wind velocity field the divergence is restricted to be 1/10 of the calculated divergence for the first hour of a simulation run. After the first hour this restriction is removed and the divergence is allowed to take on any calculated value. The results of the calculations using the divergence restraint are presented in the next section.

4.2 FINAL FORECAST EXPERIMENT

The third forecast experiment was attempted using the divergence restraint outlined above. The initial data for this experiment is the same as the initial data for the second test. A presentation of this data is given in figures 6 through 15.
The contour plots of the geopotential heights for the 400 and 800 mb constant pressure surfaces for a 24 hour simulation with the third data set are presented in figures 24 and 25. These calculated geopotentials are to be compared with the actual geopotentials obtained for March 2 from the GFDL tapes, figures 20 and 21. The contour plots of the temperatures for the 400 and 800 mb surfaces as calculated for March 2 are presented in figures 26 and 27. The temperatures for the 400 and 800 mb surfaces obtained from the GFDL tapes for March 2 are presented in figures 28 and 29.

Studying the contour plots for the geopotential and temperature fields it appears that the use of the divergence restraint has improved the forecast results. It is noted that in the third experiment the ridge located off the west coast of North America has intensified in the calculated results. The strengthening of this ridge is not as intense in the calculated results as in the verification data but is much better than the results given by the second experiment. Associated with the strengthening of the ridge off the California coast is the development of a trough over the central plains of Canada and America. There is also a slight intensification of the Greenland and Hudson Bay highs. The calculated results show in both the temperature and geopotential fields the development, though not as strong, of the trough over the central plains areas of North America. The
calculated results do not predict the intensification of the Greenland and Hudson Bay highs. The plots show a weakening of the Greenland and Hudson Bay highs by the model. This same weakening of high pressure systems is also observed in the plots of calculated results for the combined Siberian and Himalayan highs. In the southern hemisphere the plotted results of the calculated data show a general dissipation of all weather systems when compared with the confirmation data.

A study of the calculated southern hemisphere results shows a phase lag for the weather systems compared to the same systems in the confirmation data. One possible explanation, besides model errors, for the dissipation and phase lag that the model experiences in the southern hemisphere is the relatively low resolution of the southern hemisphere by the model. A study of figure 4 shows that the present model grid is sparse in the southern hemisphere.

The problem of phase lag usually improves in areas of high resolution. A study of the calculated results in the high resolution region of interest over North America appears to show a complete absence of phase lag. The systems developing in the region of interest, the ridge off the Pacific coast of North America, the trough through the plains states of America and the Greenland high show no discernable phase lag. The systems further away from the region of interest
for example the intense high over the Himalayas, show a small amount of phase lag but nowhere is the phase lag in the Northern hemisphere as great as in the southern hemisphere. It appears for the present model that there is some correlation between the phase lag and the grid resolution with the region of high resolution showing very small phase lag.

Several meteorological variables were integrated around latitude circles to further investigate the results of the third model experiment. These zonal averages were taken to quantitatively analyze the surface pressure, eddy kinetic energy and zonal kinetic energy. The zonal kinetic energy is defined as \( \frac{1}{2} \rho (\bar{u}^2 + \bar{v}^2) \) where \( \bar{u} = \frac{\bar{p}u}{\bar{p}} \) and \( \bar{v} = \frac{\bar{p}u}{\bar{p}} \). The zonal average eddy kinetic energy is defined as \( \frac{1}{2} \rho (\bar{u}'^2 + \bar{v}'^2) \) where \( \bar{u}' = \bar{u} - \bar{u} \) and \( \bar{v}' = \bar{v} - \bar{v} \). The eddy kinetic energy is a measure of velocity perturbations about a mean average velocity for a latitude circle. The zonal average surface pressure is presented in figure 30. The zonal kinetic energy for the 400 and 800 mb constant pressure surfaces is shown by figures 31 and 32. The zonal eddy kinetic energy for the 400 mb constant pressure surface is shown in figure 33.

A study of the average pressure in the region of interest shows that the calculated results for the third experiment at 24 hours differ by as much as 4 mb compared with the true zonal average as obtained from the GFDL data.
One possible explanation of this result is that the initial imbalance of the wind field has not been totally corrected after one hour of divergence restriction. The small truncation of weather patterns in the region of interest appears to give some evidence that the initial imbalance has not completely dissipated. A study of the plots for the zonal kinetic energy shows that the regions where the calculated results are high compared with the verification data on one layer in the atmosphere, in the region of the other layer the results are lower by approximately the same maximum difference. This also could be caused by the initial imbalance of the wind field. A study of the plots for the eddy kinetic energy seems to show that the model has a dampening effect on the velocity perturbations on a global scale. This smoothing out of the fluctuations about the mean of the wind patterns could possibly be caused by the use of splines to estimate the derivatives and the Euler backward differencing method for time stepping.
4.3 Conclusion

The equations of hydrodynamics and fluid mechanics are solved using bicubic splines on a telescoping grid. The telescoping grid contains a high resolution region of interest over North America. The use of a realistic earth topography is prevented by the generation of noise over the mountainous regions of the earth. Using a smooth earth surface and a divergence restraint for the first simulation hour real weather data is run for a 24 hour time period with the results showing little truncation and phase lag for weather systems in the region of interest.
Figure 1. Newtonian Coordinate System

Figure 2. Earth Centered Coordinate System
Figure 4. Preliminary Grid
Figure 5. Vertical Grid

\begin{align*}
\sigma &= 0 & P &= P_T \\
\sigma &= \frac{1}{4} & T, U, V \\
\sigma &= \frac{3}{4} & q, T, U, V \\
\sigma &= 1 & P &= P_S
\end{align*}
Figure 10. Zonal Wind 800 mb 0000 GMT
Figure 14. Geopotential height 400 mb 0000 GMT (meters × 10^2)
Figure 15. Geopotential Height 800 mb 0000 GMT (meters x 10^2)
Figure 19. Geopotential Height 800 mb 2400GMT (meters x 10^2)
Figure 22. Surface Pressure
Figure 23. Temperature 400 mb
Figure 29. Temperature 800 mb 2400 GMT
Figure 30. Zonal Average Pressure
Figure 31. Zonal Kinetic Energy 400 mb
Figure 32. Zonal Kinetic Energy 300 mb
Figure 33. Zonal Eddy Kinetic Energy 400 mb
REFERENCES


Robert E. Kelly was born on February 16, 1951 in Abington, Pa. He grew up and attended primary and secondary school in Doylestown, Pa. He graduated from the University of Rhode Island with a degree in Mechanical Engineering and Applied Mechanics in 1973. He started graduate work at Lehigh University in the fall of 1973 and expects to graduate with a degree of Master of Sciences in June 1975.