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Tadahiko Kawai

ON THE BENDING OF A SECTORIAL PLATE

by
Tadahiko Kawai

I. INTRODUCTION

Recently attention has been paid by aeronautical engineers to the problems of bending of sectorial plates associated with the stress distribution in the neighborhood of the wing-fuselage connection.

As a matter of fact, it has been discovered analytically as well as experimentally that for certain values of the included angle, the bending stress in the corner tends toward extremely high values.

Specifically, it was found that within the limitations of the first order plate theory, the stress tends to infinity as the included angle exceeds 90° with the strength of the singularity increasing with the angle.

M.L. Williams, Jr.[1] has investigated these stress singularities. However, he could only discuss the problems qualitatively. S. Woinowsky-Krieger[2] has shown a general method of solution using the Fourier Integral. His method however, requires laborious computations even though it is applicable to any case of sectorial plates with various boundary conditions. Fortunately, simpler solutions can be obtained in three different ways for the case of a sectorial plate whose radial edges are simply supported. As an example, Green's function of a sectorial plate with simply supported radial edges and a clamped circumferential edge has been obtained recently in two different forms, i.e., in double series form (Fourier-Bessel series) and in single series form.

Using the latter solution, Green's function for bending moments and twisting moment are derived in closed form.

With this solution, a general discussion of the stress singularity can be successfully made and influence surfaces as well as moment surfaces can be easily developed. The later solutions will have direct application in the calculation of influence surfaces for skewed slabs which in turn should be extremely useful in the design of skewed bridge slabs.

II. METHOD OF SOLUTION

Given problem:

$$D\,\Delta\Delta\, W = q(r,\theta)$$

where

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

------ Laplacian

Loading

$$q(r,\theta) = \begin{cases} P & (r=\rho,\ \theta=\varphi) \\ 0 & (r\neq\rho,\ \theta\neq\varphi) \end{cases} \qquad (1)$$

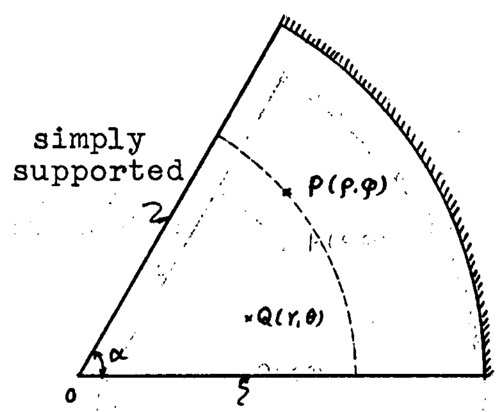


Fig.1

Boundary Conditions:

$$\left.\begin{aligned} &\theta = 0:\ W = 0,\quad M_\theta = -D\left[\nu\left(\frac{\partial^2 W}{\partial r^2}\right) + \frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2}\right] \\ &\theta = \alpha:\ W = 0,\quad M_\theta = 0 \\ &\text{and} \\ &r = a:\quad W = 0,\quad \frac{\partial W}{\partial r} = 0 \end{aligned}\right\} \text{----} (2)$$

(i) Single series solution (Clebsch's method)

Product solution assumed.

$$W(r,\theta) = \sum_{n=1}^{\infty} R_n(r) \sin\frac{n\pi\theta}{\alpha} \qquad \text{--------} (3)$$

$$(\theta \leqq \alpha \leqq 2\pi,\quad \alpha \neq \pi)$$

Substituting equation (3) into equation (1)

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(\frac{n\pi}{\alpha})^2}{r^2} \right)^2 R_n = 0$$

$$\left.\begin{aligned} R_n(r) &= A_n\, r^{\frac{n\pi}{\alpha}} + B_n\, r^{-\frac{n\pi}{\alpha}} + C_n\, r^{\frac{n\pi}{\alpha}+2} + D_n\, r^{-\frac{n\pi}{\alpha}+2} && (\text{for } \rho \leqq r \leqq a) \\ R'_n(r) &= A'_n\, r^{\frac{n\pi}{\alpha}} + C'_n\, r^{\frac{n\pi}{\alpha}+2} && (\text{for } 0 \leqq r \leqq \rho) \end{aligned}\right\} \quad \text{------ (4)}$$

Continuity along r=ρ

$$\left.\begin{aligned} & W' = W, \quad \frac{\partial W'}{\partial r} = \frac{\partial W}{\partial r}, \quad \frac{\partial^2 W'}{\partial r^2} = \frac{\partial^2 W}{\partial r^2} \\ \text{and} \quad & D\frac{\partial}{\partial r}(\Delta W) - D\frac{\partial}{\partial r}(\Delta W') = \frac{2P}{\rho\alpha}\sum_{n=1}^{\infty} \sin\frac{n\pi\varphi}{\alpha} \sin\frac{n\pi\theta}{\alpha} \end{aligned}\right\} \quad \text{--- (5)}$$

Six boundary conditions, $(W)_{r=a}=0$, $\left(\frac{\partial W}{\partial r}\right)_{r=a}=0$ and the four equations (5), determine the six coefficients A_n, B_n, C_n, D_n and A'_n, C'_n.

$$\left.\begin{aligned} A_n &= -\frac{P a^{-\frac{2n\pi}{\alpha}} \rho^{\frac{2n\pi}{\alpha}+2}}{4D((\frac{n\pi}{\alpha})^2-1)} \left(\frac{\alpha^2}{n^2\pi^2} - \frac{1}{\alpha} \right) \sin\frac{n\pi\varphi}{\alpha} \\ B_n &= -\frac{P \rho^{\frac{n\pi}{\alpha}+2}}{4D((\frac{n\pi}{\alpha})^2-1)} \left(\frac{\alpha^2}{n^2\pi^2} - \frac{1}{\alpha} \right) \sin\frac{n\pi\varphi}{\alpha} \\ C_n &= -\frac{P a^{-\frac{2n\pi}{\alpha}} \rho^{\frac{n\pi}{\alpha}}}{4D((\frac{n\pi}{\alpha})^2-1)} \left(\frac{1}{n\pi} + \frac{1}{\alpha} \right) \sin\frac{n\pi\varphi}{\alpha} \\ D_n &= \frac{P \rho^{\frac{n\pi}{\alpha}}}{4n\pi(\frac{n\pi}{\alpha}-1)D} \sin\frac{n\pi\varphi}{\alpha} \\ A'_n &= \frac{P a^{-\frac{2n\pi}{\alpha}} \rho^{\frac{n\pi}{\alpha}+2}}{4D((\frac{n\pi}{\alpha})^2-1)} \left[\frac{\alpha^2}{n^2\pi^2}\left(\left(\frac{a}{\rho}\right)^{\frac{2n\pi}{\alpha}}-1\right) + \frac{1}{\alpha}\left(\left(\frac{a}{\rho}\right)^{\frac{2n\pi}{\alpha}}+1\right) \right] \sin\frac{n\pi\varphi}{\alpha} \\ C'_n &= -\frac{P a^{-\frac{2n\pi}{\alpha}} \rho^{\frac{n\pi}{\alpha}}}{4D((\frac{n\pi}{\alpha})^2-1)} \left[\frac{1}{n\pi}\left(\left(\frac{a}{\rho}\right)^{\frac{2n\pi}{\alpha}}+1\right) - \frac{1}{\alpha}\left(\left(\frac{a}{\rho}\right)^{\frac{2n\pi}{\alpha}}-1\right) \right] \sin\frac{n\pi\varphi}{\alpha} \end{aligned}\right\} \quad \text{--- (6)}$$

Using equation (6), the following single series soluction is obtained:

$$W(r,\theta;\rho,\varphi) = \begin{cases} \dfrac{\lambda^2 P}{4D} \displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n^2-\lambda^2} \Big[\left(\dfrac{\rho^2}{n^2\pi^2} + \dfrac{r^2}{n^2\pi^2}\right) \left\{ \left(\dfrac{r}{\rho}\right)^{\frac{n}{\lambda}} - \left(\dfrac{\rho r}{a^2}\right)^{\frac{n}{\lambda}} \right\} + \left(\dfrac{\rho^2}{\pi\lambda} - \dfrac{r^2}{\pi\lambda}\right) \times \\ \quad \left\{ \left(\dfrac{r}{\rho}\right)^{\frac{n}{\lambda}} + \left(\dfrac{\rho r}{a^2}\right)^{\frac{n}{\lambda}} \right\} \Big] \sin\dfrac{n\varphi}{\lambda} \sin\dfrac{n\theta}{\lambda} \qquad (0 \leqq r \leqq \rho) \\ \dfrac{\lambda^2 P}{4D} \displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n^2-\lambda^2} \left\{ \left(\dfrac{\rho}{r}\right)^{\frac{n}{\lambda}} - \left(\dfrac{\rho r}{a^2}\right)^{\frac{n}{\lambda}} \right\} \left\{ \rho^2\left(\dfrac{1}{n\pi} - \dfrac{1}{\pi\lambda}\right) + r^2\left(\dfrac{1}{n\pi} + \dfrac{1}{\pi\lambda}\right) \right\} \\ \quad \times \sin\dfrac{n\varphi}{\lambda} \sin\dfrac{n\theta}{\lambda} \qquad (\rho \leqq r \leqq a) \end{cases} \qquad (7)$$

where $\lambda = \frac{\alpha}{\pi}$, $0 \leqq \lambda \leqq 2$ $(\lambda \neq 1)$

(ii) Double series solution (Fourier-Bessel series)

The second solution in Fourier-Bessel series form can be obtained in the following way.

The natural frequencies of the particular plate are determined first leading to an orthogonal function system whose elements represent the modes of vibration corresponding to the particular natural frequencies.

The concentrated load can then be expanded in terms of these orthogonal functions.

Assuming that w(r,e; ρ,φ) can be also expanded into such a series, the unknown coefficients are determined by its substitution into the original differential equation (1).

(a) Set of normal functions for given plate, (Fig.1)

Equation of the free vibration of a plate

$$D\,\Delta\Delta W + \frac{\gamma h}{g}\frac{\partial^2 W}{\partial t^2} = 0 \qquad \text{-----}(8)$$

$$W(r,\theta;t) = F(r)\sin\frac{n\pi\theta}{\alpha}e^{ipt} \qquad \text{------}(9)$$

Substituting (9) into (8)

$$D\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{(\frac{n\pi}{\alpha})^2}{r^2}\right)^2 F-\frac{\gamma h p^2}{g}F=0$$

or $$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{(\frac{n\pi}{\alpha})^2}{r^2}\right)^2 F=k^4 F \qquad \text{----------} \quad (10)$$

where $k^4=\frac{\gamma h p^2}{gD}$

General solution of equation (1))

$$F_n(r)=A_n J_{\frac{n\pi}{\alpha}}(k_s^{(n)}r)+B_n J_{-\frac{n\pi}{\alpha}}(k_s^{(n)}r)+C_n I_{\frac{n\pi}{\alpha}}(k_s^{(n)}r)+D_n I_{-\frac{n\pi}{\alpha}}(k_s^{(n)}r)$$

where $J_{\frac{n\pi}{\alpha}}(k_s^{(n)}r)$: Bessel Function in $\frac{n\pi}{\alpha}$ order

$J_{-\frac{n\pi}{\alpha}}(k_s^{(n)}r)$: " " in $\frac{-n\pi}{\alpha}$ order

$I_{\frac{n\pi}{\alpha}}(k_s^{(n)}r)$: the modified Bessel Function in $\frac{n\pi}{\alpha}$ order

$I_{-\frac{n\pi}{\alpha}}(k_s^{(n)}r)$: " " " " in $\frac{-n\pi}{\alpha}$ order

$(W)_{r=0}$ = finite $\longrightarrow$ $F_n(0)$ = finite

$$\therefore \quad B_n=D_n=0 \quad \left(\because \; J_{-\frac{n\pi}{\alpha}}(0)=\infty,\; I_{-\frac{n\pi}{\alpha}}(0)=\infty\right)$$

$$F_n(r)=A_n J_{\frac{n\pi}{\alpha}}(k_s^{(n)}r)+C_n I_{\frac{n\pi}{\alpha}}(k_s^{(n)}r)$$

Boundary Conditions:

$(W)_{r=a}=0$, $\left(\frac{\partial W}{\partial r}\right)_{r=a}=0$ yield secular equation,

eliminating A_n, C_n

$$J_{\frac{n\pi}{\alpha}}(k_s^{(n)}a)\,I'_{\frac{n\pi}{\alpha}}(k_s^{(n)}a)-J'_{\frac{n\pi}{\alpha}}(k_s^{(n)}a)\,I_{\frac{n\pi}{\alpha}}(k_s^{(n)}a)=0 \qquad \text{------} \quad (12)$$

The eigen-function corresponding to the eigen value $k_s^{(n)}$.

$$\Delta_{\frac{n\pi}{\alpha}}(k_s^{(n)}r)\sin\frac{n\pi\theta}{\alpha} \quad (n=1,2,3,\cdots,\; s=1,2,3\cdots) \cdots (13)$$

where $$\Delta_{\frac{n\pi}{\alpha}}(k_s^{(n)}r)=J_{\frac{n\pi}{\alpha}}(k_s^{(n)}r)\,I'_{\frac{n\pi}{\alpha}}(k_s^{(n)}a)-J'_{\frac{n\pi}{\alpha}}(k_s^{(n)}a)\,I_{\frac{n\pi}{\alpha}}(k_s^{(n)}r)\cdots(14)$$

(b) Orthogonality of the system of eigen functions:

$$\int_0^a \int_0^\alpha \left[\Delta_{\frac{m\pi}{\alpha}}(k_i^{(m)} r) \sin \frac{m\pi\theta}{\alpha} \right] \left[\Delta_{\frac{n\pi}{\alpha}}(k_j^{(n)} r) \sin \frac{n\pi\theta}{\alpha} \right] r\, dr\, d\theta = 0 \quad \text{------}(15)$$

$$(m \neq n, \quad i \neq j)$$

or

$$\int_0^a \Delta_{\frac{n\pi}{\alpha}}(k_i^{(m)} r) \Delta_{\frac{n\pi}{\alpha}}(k_j^{(n)} r)\, r\, dr \times \int_0^\alpha \sin \frac{m\pi\theta}{\alpha} \sin \frac{n\pi\theta}{\alpha}\, d\theta = 0$$

since

$$\int_0^\alpha \sin \frac{m\pi\theta}{\alpha} \sin \frac{n\pi\theta}{\alpha}\, d\theta = \begin{cases} 0 & (m \neq n) \\ \frac{\alpha}{2} & (m = n) \end{cases}$$

$\int_0^a \Delta_{\frac{n\pi}{\alpha}}(k_i^{(m)} r)\, \Delta_{\frac{n\pi}{\alpha}}(k_j^{(m)} r)\, r\, dr = 0 \quad (i \neq j)$ can be proved by the following Lommel Integrals of Bessel Functions.

$$\left.\begin{aligned}
\int_0^a r J_N(\lambda r) J_N(\mu r)\, dr &= \frac{a}{\lambda^2 - \mu^2} \left\{ \mu J_N(\lambda a) J_N'(\mu a) - \lambda J_N'(\lambda a) J_N(\mu a) \right\} \\
\int_0^a r J_N(\lambda r) I_N(\mu r)\, dr &= \frac{a}{\lambda^2 + \mu^2} \left\{ \mu J_N(\lambda a) I_N'(\mu a) - \lambda J_N'(\lambda a) I_N(\mu a) \right\} \\
\int_0^a r I_N(\lambda r) I_N(\mu r)\, dr &= \frac{-a}{\lambda^2 + \mu^2} \left\{ \mu I_N(\lambda a) I_N'(\mu a) - \lambda I_N'(\lambda a) I_N(\mu a) \right\}
\end{aligned}\right\} (16)$$

and

$$\left.\begin{aligned}
\int_0^a r \left\{ J_N(\lambda r) \right\}^2 dr &= \frac{a^2}{2} \left[\left\{ J_N'(\lambda a) \right\}^2 + \left(1 - \frac{N^2}{\lambda^2 a^2}\right) \left\{ J_N(\lambda a) \right\}^2 \right] \\
\int_0^a r J_N(\lambda r) I_N(\mu r)\, dr &= \frac{a}{2\lambda} \left[J_N(\lambda a) I_N'(\lambda a) - J_N'(\lambda a) I_N(\lambda a) \right] \\
\int_0^a r \left\{ I_N(\lambda r) \right\}^2 dr &= -\frac{a^2}{2} \left[\left\{ I_N'(\lambda a) \right\}^2 - \left(1 + \frac{N^2}{\lambda^2 a^2}\right) \left\{ I_N(\lambda a) \right\}^2 \right]
\end{aligned}\right\} \text{---}(17)$$

where $N = \frac{n\pi}{\alpha}$; λ, μ represents one of $k_i^{(m)}, k_j^{(m)}$.

Using equation (17)

$$\int_0^a r\{\Delta_N(\lambda r)\}^2 dr = a^2\{J_N(\lambda a)\}^2\{I_N'(\lambda a)\}^2 = a^2\{J_N'(\lambda a)\}^2\{I_N(\lambda a)\}^2$$
$$= a^2 J_N(\lambda a) I_N(\lambda a) J_N'(\lambda a) I_N'(\lambda a)$$

(c) Fourier-Bessel Expansion of an arbitrary function $q(r,\theta)$

assume

$$g(r,\theta) = \sum_{n=1}^{\infty}\sum_{s=1}^{\infty} A_{ns}\, \Delta_{\frac{n\pi}{\alpha}}(k_s^{(n)} r) \sin\frac{n\pi\theta}{\alpha} \qquad \text{-----}(18)$$

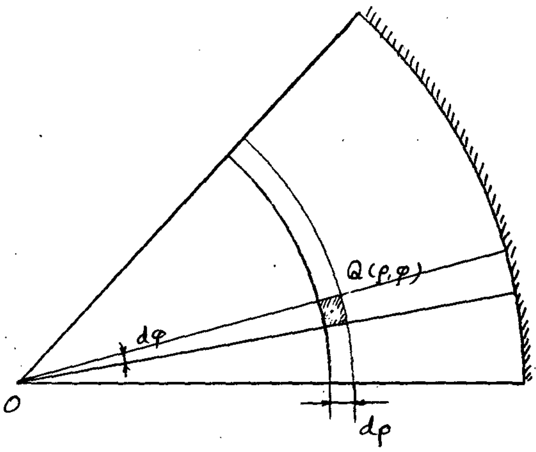
Multiplying both sides of equation (18) by $r\,\Delta_{\frac{n\pi}{\alpha}}(k_s^{(n)} r) \sin\frac{n\pi\theta}{\alpha}$

and integrating over the whole area of the plate.

$$A_{ns} = \frac{2}{\alpha a^2 J_{\frac{n\pi}{\alpha}}(k_s^{(n)}a)\, I_{\frac{n\pi}{\alpha}}(k_s^{(n)}a)\, J_{\frac{n\pi}{\alpha}}'(k_s^{(n)}a)\, I_{\frac{n\pi}{\alpha}}'(k_s^{(n)}a)} \int_0^a\int_0^\alpha g(r,\theta)\, \Delta_{\frac{n\pi}{\alpha}}(k_s^{(n)} r) \sin\frac{n\pi\theta}{\alpha}\, r\, dr\, d\theta \qquad \text{----}(19)$$

(d) Green's function for the deflection of the given plate:

$$g(r,\theta) = \begin{cases} 1 & (r=\rho,\ \theta=\varphi) \\ 0 & (r\neq\rho,\ \theta\neq\varphi) \end{cases}$$



$$\lim_{\substack{d\rho\to 0\\ d\varphi\to 0}} g_0\, \rho\, d\rho\, d\varphi = 1$$

Fig.2

$$g(r,\theta) = \sum_{n=1}^{\infty}\sum_{s=1}^{\infty} \psi_{\frac{n\pi}{\alpha}}(k_s^{(n)}\rho,\varphi)\, \psi_{\frac{n\pi}{\alpha}}(k_s^{(n)} r,\theta)$$

where

$$\psi_{\frac{n\pi}{\alpha}}(k_s^{(n)} r,\theta) = \frac{\left(J_{\frac{n\pi}{\alpha}}(k_s^{(n)} r)\, I_{\frac{n\pi}{\alpha}}'(k_s^{(n)}a) - J_{\frac{n\pi}{\alpha}}'(k_s^{(n)}a)\, I_{\frac{n\pi}{\alpha}}(k_s^{(n)} r)\right)\sin\frac{n\pi\theta}{\alpha}}{\sqrt{\frac{\alpha}{2} a^2 J_{\frac{n\pi}{\alpha}}(k_s^{(n)}a)\, I_{\frac{n\pi}{\alpha}}(k_s^{(n)}a)\, J_{\frac{n\pi}{\alpha}}'(k_s^{(n)}a)\, I_{\frac{n\pi}{\alpha}}'(k_s^{(n)}a)}}$$

$\cdots(20)$

From equation (1) and equation (20)

$$D\Delta\Delta W = \sum_{n=1}^{\infty}\sum_{s=1}^{\infty} \phi_{\frac{n\pi}{\alpha}}(k_s^{(n)}\rho;\varphi)\,\phi_{\frac{n\pi}{\alpha}}(k_s^{(n)}r,\theta) \qquad \text{------- (21)}$$

assume

$$W(r,\theta) = \sum_{n=1}^{\infty}\sum_{s=1}^{\infty} B_{ns}\,\phi_{\frac{n\pi}{\alpha}}(k_s^{(n)}r,\theta) \qquad \text{------- (22)}$$

substituting equation (22) into equation (21)

$$B_{ns} = \frac{1}{k_s^{(n)4}D} \qquad \text{------------ (23)}$$

From equation (22) and equation (23)

$$G(r,\theta;\rho,\varphi) = \sum_{n=1}^{\infty}\sum_{s=1}^{\infty} \frac{\phi_{\frac{n\pi}{\alpha}}(k_s^{(n)}\rho,\varphi)\,\phi_{\frac{n\pi}{\alpha}}(k_s^{(n)}r,\theta)}{\lambda_{ns}^2}$$

where

$$\lambda_{ns}^2 = k_s^{(n)4}D$$

This solution is not suitable for the bending problem of the given plate; however, it is very useful for eigen value problems of the plate as buckling or vibration of the plate. Nevertheless the stress singularity at the corner of the sectional plate can be discussed, using this solution.

(iii) Closed form solution for the case of the opening angle

$$\alpha = \frac{\pi}{n} \quad (n = 1, 2, 3, \cdots).$$

Green function for the deflection of a circular plate with clamped edge (given by J.H. Michell in 1902).

$$G(r,\theta;\rho,\varphi) = \frac{1}{16\pi a^4 D}\Big[a^2(r^2-2\rho r\cos(\theta-\varphi)+\rho^2)\times$$

$$\log\frac{a^2(r^2-2\rho r\cos(\theta-\varphi)+\rho^2)}{a^4-2a^2\rho r\cos(\theta-\varphi)+\rho^2r^2} + (a^2-\rho^2)(a^2-r^2)\Big]$$

$$\text{---------- (25)}$$

Using the "Mirror Method" the solution for a sectional plate can be derived. For example, if $\alpha = \frac{\pi}{2}$, Fig. 3:

$$G(r,\theta;\rho,\varphi) = \frac{1}{16\pi a^4 D}\Big[\, a^2(r^2-2\rho r\cos(\theta-\varphi)+\rho^2)\times \log\frac{a^2(r^2-2\rho r\cos(\theta-\varphi)+\rho^2)}{a^4-2a^2\rho r\cos(\theta-\varphi)+\rho^2r^2}$$

$$-a^2(r^2-2\rho r\cos(\theta+\varphi-\pi)+\rho^2)\log\frac{a^2(r^2-2\rho r\cos(\theta+\varphi-\pi)+\rho^2)}{a^4-2a^2\rho r\cos(\theta+\varphi-\pi)+\rho^2r^2}$$

$$+a^2(r^2-2\rho r\cos(\theta-\varphi-\pi)+\rho^2)\log\frac{a^2(r^2-2\rho r\cos(\theta-\varphi-\pi)+\rho^2)}{a^4-2a^2\rho r\cos(\theta-\varphi-\pi)+\rho^2r^2}$$

$$-a^2(r^2-2\rho r\cos(\theta+\varphi-2\pi)+\rho^2)\log\frac{a^2(r^2-2\rho r\cos(\theta+\varphi-2\pi)+\rho^2)}{a^4-2a^2\rho r\cos(\theta+\varphi-2\pi)+\rho^2r^2}\Big] \quad \text{---- (26)}$$

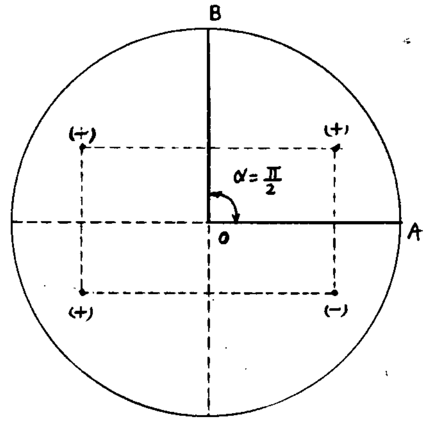


Fig. 3

Although the solution obtained has a closed form, this method is only applicable for the case $\alpha = \frac{\pi}{n}$ $(n = 1, 2, 3, ---)$

(iiii) Green's Function for the Moments M_r, M_θ, $M_{r\theta}$ of an Infinite Wedge-Shaped Plate. (Fig. 4)

The single series solution (7) is used.

In equation (7)

make $a \longrightarrow \infty$

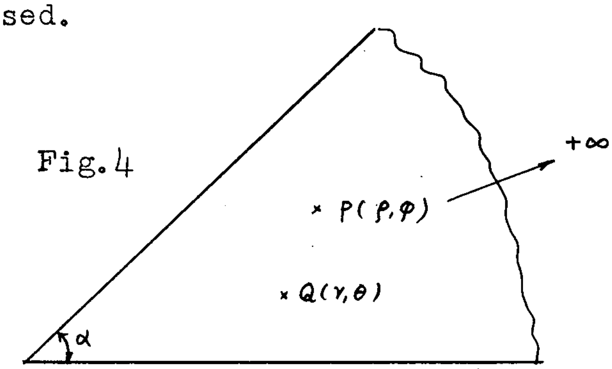


Fig. 4

$$W(r,\theta;\rho,\varphi) = \begin{cases} \dfrac{\lambda P}{4\pi D}\displaystyle\sum_{n=1}^{\infty}\frac{(\frac{r}{\rho})^{\frac{n}{\lambda}}}{n(n^2-\lambda^2)}\left\{(n+\lambda)\rho^2-(n-\lambda)r^2\right\}\sin\frac{n\varphi}{\lambda}\sin\frac{n\theta}{\lambda} & (0 \leqq r \leqq \rho) \\ \dfrac{\lambda P}{4\pi D}\displaystyle\sum_{n=1}^{\infty}\frac{(\frac{r}{\rho})^{\frac{n}{\lambda}}}{n(n^2-\lambda^2)}\left\{-(n-\lambda)\rho^2+(n+\lambda)r^2\right\}\sin\frac{n\varphi}{\lambda}\sin\frac{n\theta}{\lambda} & (\rho \leqq r \lesssim \infty) \end{cases} \quad \text{----- (27)}$$

where $\lambda = \frac{\alpha}{\pi}, \quad 0 \leqq \lambda \leqq 2 \quad (\lambda \neq 1)$

$$\left.\begin{aligned} M_r &= -D\left[\frac{\partial^2 W}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2}\right)\right] \\ M_\theta &= -D\left[\nu\frac{\partial^2 W}{\partial r^2} + \left(\frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2}\right)\right] \\ M_{r\theta} &= -D(1-\nu)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial W}{\partial \theta}\right) \end{aligned}\right\} \quad \text{--------- (28)}$$

The series form expression for M_r, M_θ, $M_{r\theta}$, derived from (27) and (28), can be summed up, resulting into closed form expressions.*

$$\left.\begin{matrix} M_r \\ M_\theta \end{matrix}\right\} = \frac{P}{8\pi}\left[(1+\nu)\log\frac{\cosh(\frac{\pi}{\alpha}\log\frac{r}{\rho}) - \cos\frac{\pi}{\alpha}(\theta+\varphi)}{\cosh(\frac{\pi}{\alpha}\log\frac{r}{\rho}) - \cos\frac{\pi}{\alpha}(\theta-\varphi)} \pm \frac{(1-\nu)}{2}\left(1-\frac{\rho^2}{r^2}\right)\times\right.$$
$$\left.\left(\frac{\sinh(\frac{\pi}{\alpha}\log\frac{r}{\rho})}{\cosh(\frac{\pi}{\alpha}\log\frac{r}{\rho}) - \cos\frac{\pi}{\alpha}(\theta-\varphi)} - \frac{\sinh(\frac{\pi}{\alpha}\log\frac{r}{\rho})}{\cosh(\frac{\pi}{\alpha}\log\frac{r}{\rho}) - \cos\frac{\pi}{\alpha}(\theta+\varphi)}\right)\right]$$

where
upper sign for M_r
lower sign for M_θ (29)

$$M_{r\theta} = \frac{(1-\nu)P}{32\alpha}\left(1-\frac{\rho^2}{r^2}\right)\left[\frac{\sin\frac{\pi}{\alpha}(\theta-\varphi)}{\cosh(\frac{\pi}{\alpha}\log\frac{r}{\rho}) - \cos\frac{\pi}{\alpha}(\theta-\varphi)} - \frac{\sin\frac{\pi}{\alpha}(\theta+\varphi)}{\cosh(\frac{\pi}{\alpha}\log\frac{r}{\rho}) - \cos\frac{\pi}{\alpha}(\theta+\varphi)}\right]$$

These expressions (29) are valid for the entire domain of the plate.

III. ALTERNATIVE METHOD FOR DERIVATION OF M_r, M_θ, (application of conformal mapping)

Moment Sum $\quad M = M_r + M_\theta = -D(1+\nu)\Delta W \quad$ ------- (30)

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* $$\sum_{n=1}^{\infty}\frac{r^n}{n}\cos nx = -\frac{1}{2}\log(1-2r\cos x + r^2)$$ for $|r|<1$

$$\sum_{n=1}^{\infty} r^n \cos nx = \frac{1-r\cos x}{1-2r\cos x + r^2} - 1 = \frac{1}{2}\left(\frac{1-r^2}{1-2r\cos x + r^2} - 1\right)$$

Substituting equation (30) into equation (1)

$$\Delta M = -(1+\nu)\, q(r,\theta) \qquad \text{-----------} \ (31)$$

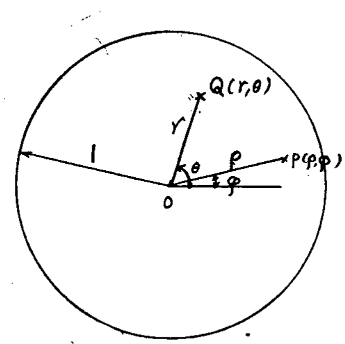
Boundary conditions:

$$(M)_{\theta=0} = (M)_{\theta=\alpha} = (M)_{r=a} = 0$$

(i) Unit Circle

Green's function for M of unit circle

From the theory of potentials



$$M(r,\theta;\rho,\varphi) = \frac{(1+\nu)P}{4\pi} \log \frac{1-2\rho r\cos(\theta-\varphi)+\rho^2 r^2}{r^2-2\rho r\cos(\theta-\varphi)+\rho^2}$$

$$\text{-----------} \ (32)$$

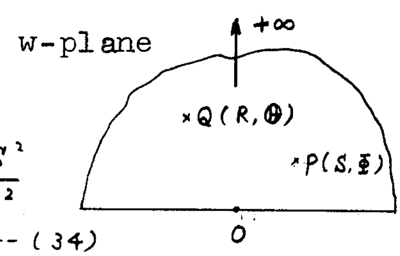
(ii) Semi unit circle

$$M(r,\theta;\rho,\varphi) = \frac{(1+\nu)P}{4\pi} \log \frac{(1-2\rho r\cos(\theta-\varphi)+\rho^2 r^2)(r^2-2\rho r\cos(\theta+\varphi)+\rho^2)}{(r^2-2\rho r\cos(\theta-\varphi)+\rho^2)(1-2\rho r\cos(\theta+\varphi)+\rho^2 r^2)}$$

$$\text{-----} \ (33)$$

For the case of a semi-infinite circle

$r \doteqdot 0$, $\rho \doteqdot 0$, changing notations



$$M(R,\Theta,S,\Phi) = \frac{(1+\nu)P}{4\pi} \log \frac{R^2-2RS\cos(\Theta+\Phi)+S^2}{R^2-2RS\cos(\Theta-\Phi)+S^2} \qquad \text{-----} \ (34)$$

(iii) Infinite Wedge-shaped plate (Fig. 5)

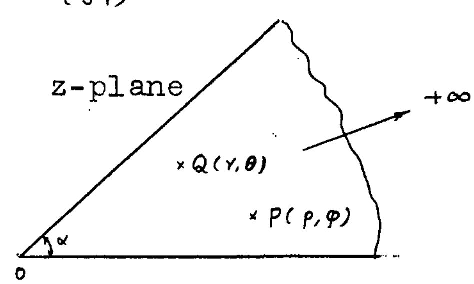


Fig. 5

Mapping function transforming the semi-infinite plane (w-plane) into the infinite wedge plane (z-plane)

$$w = z^{\frac{\pi}{\alpha}} \qquad w = Re^{i\Theta}$$

$$z = re^{i\theta}$$

$$\therefore \begin{cases} R = r^{\frac{\pi}{\alpha}} \\ \Theta = \dfrac{\pi\theta}{\alpha} \end{cases} \qquad \text{-------} \ (35)$$

Substituting equation (35) into equation (34)

$$M(r,\theta;\rho,\varphi) = \frac{(1+\nu)P}{4\pi}\log\frac{r^{\frac{2\pi}{\alpha}} - 2r^{\frac{\pi}{\alpha}}\rho^{\frac{\pi}{\alpha}}\cos\frac{\pi}{\alpha}(\theta+\varphi) + \rho^{\frac{\pi}{\alpha}}}{r^{\frac{2\pi}{\alpha}} - 2r^{\frac{\pi}{\alpha}}\rho^{\frac{\pi}{\alpha}}\cos\frac{\pi}{\alpha}(\theta-\varphi) + \rho^{\frac{\pi}{\alpha}}}$$

$$= \frac{(1+\nu)P}{4\pi}\log\frac{\cosh\left(\frac{\pi}{\alpha}\log\frac{r}{\rho}\right) - \cos\frac{\pi}{\alpha}(\theta+\varphi)}{\cosh\left(\frac{\pi}{\alpha}\log\frac{r}{\rho}\right) - \cos\frac{\pi}{\alpha}(\theta-\varphi)} \quad \text{--- (36)}$$

$$\left.\begin{aligned} M_r &= \frac{1}{2}M + \frac{(1-\nu)}{4(1+\nu)}\left(\frac{r}{\rho} - \frac{\rho}{r}\right)\frac{\partial M}{\partial\left(\frac{r}{\rho}\right)} \\ M_\theta &= \frac{1}{2}M - \frac{(1-\nu)}{4(1+\nu)}\left(\frac{r}{\rho} - \frac{\rho}{r}\right)\frac{\partial M}{\partial\left(\frac{r}{\rho}\right)} \end{aligned}\right\} \quad \text{--- (37)}$$

From equation (36) and equation (37), M_r, M_θ given in equation (29) can be easily derived.

IV. DISCUSSION OF STRESS SINGULARITIES

(i) Influence surfaces and moment surfaces of M_r, M_θ and $M_{r\theta}$

loading point $P(\rho,\varphi)$ fixed ⟶ moment surfaces $M_r(\rho,\varphi), M_\theta(\rho,\varphi), M_{r\theta}(\rho,\varphi)$

influence point $Q(r,\theta)$ fixed ⟶ influence surfaces $m_r(r,\theta), m_\theta(r,\theta), m_{r\theta}(r,\theta)$

(ii) Stress singularities in the neighborhood of the corner.

In the neighborhood of the corner

$$r \doteqdot 0 \qquad \log\frac{r}{\rho} \longrightarrow -\infty$$

$$\cosh\left(\frac{\pi}{\alpha}\log\frac{r}{\rho}\right) \sim \frac{1}{2}\left(\frac{\rho}{r}\right)^{\frac{\pi}{\alpha}}, \quad \sinh\left(\frac{\pi}{\alpha}\log\frac{r}{\rho}\right) \sim -\frac{1}{2}\left(\frac{\rho}{r}\right)^{\frac{\pi}{\alpha}}$$

$$M_r \sim -\frac{(1-\nu)P}{4\pi}\left(\frac{r}{\rho}\right)^{\frac{\pi}{\alpha}-2} \sin\frac{\pi\varphi}{\alpha}\sin\frac{\pi\theta}{\alpha}$$

$$M_\theta \sim +\frac{(1-\nu)P}{4\pi}\left(\frac{r}{\rho}\right)^{\frac{\pi}{\alpha}-2} \sin\frac{\pi\varphi}{\alpha}\sin\frac{\pi\theta}{\alpha}$$

$$M_{r\theta} \sim +\frac{(1-\nu)P}{8\alpha}\left(\frac{r}{\rho}\right)^{\frac{\pi}{\alpha}-2} \sin\frac{\pi\varphi}{\alpha}\sin\frac{\pi\theta}{\alpha} \qquad \text{--------- (38)}$$

Corner reaction $R = M_{r\theta}(0,0;\rho,\varphi) + M_{\theta r}(0,0;\rho,\varphi) = 2M_{r\theta}(0,0;\rho,\varphi)$

$$R \sim \frac{(1-\nu)P}{4\alpha}\left(\frac{r}{\rho}\right)^{\frac{\pi}{\alpha}-2}\sin\frac{\pi\varphi}{\alpha}\sin\frac{\pi\theta}{\alpha}$$

From equation (38), it is easily seen that M_r, M_θ and R have a singularity at the corner and their intensity is governed by the term $\left(\frac{r}{\rho}\right)^{\frac{\pi}{\alpha}-2}$. That is,

(a) $0 < \alpha < \frac{\pi}{2}$ $\quad \frac{\pi}{\alpha} > 2$ $\quad \therefore \lim_{r\to 0}\left(\frac{r}{\rho}\right)^{\frac{\pi}{\alpha}-2} = 0$

$$M_r(0,\theta;\rho,\varphi) = M_\theta(0,\theta;\rho,\varphi) = R(0,0;\rho,\varphi) = 0$$

(b) $\alpha = \frac{\pi}{2}$ $\quad \frac{\pi}{\alpha} = 2$ $\quad \lim_{r\to 0}\left(\frac{r}{\rho}\right)^{\frac{\pi}{\alpha}-2} = 1$

$$\left.\begin{array}{l} M_r(0,\theta;\rho,\varphi) \\ M_\theta(0,\theta;\rho,\varphi)\end{array}\right\} = \mp\frac{(1-\nu)P}{4\pi}\sin 2\theta\sin 2\varphi$$

$$R = \frac{(1-\nu)P}{2\pi}\cos 2\theta\sin 2\varphi$$

(c) $\frac{\pi}{2} < \alpha < \pi$ $\quad 1 < \frac{\pi}{\alpha} < 2$ $\quad \lim_{r\to 0}\left(\frac{r}{\rho}\right)^{\frac{\pi}{\alpha}-2} = +\infty$

$$\lim_{r\to 0} M_r = -\infty, \quad \lim_{r\to 0} M_\theta = +\infty \quad \lim_{r\to 0} R = +\infty$$

(d) $\pi < \alpha \leqq 2\pi$ $\quad 0 < \frac{\pi}{\alpha} < 1$

$$\lim_{r\to 0} M_r = -\infty \quad \lim_{r\to 0} M_\theta = +\infty, \quad \lim_{r\to 0} R = +\infty$$

when $\alpha = \pi$ it is obvious that $(M_r)_{r=0} = (M_\theta)_{r=0} = 0$ and R is finite.

(iii) Comparison with the Two Dimensional Flow of an Ideal Fluid Around a Wedge.

Velocity potential

$$\phi = C r^{\frac{\pi}{\alpha}} \cos \frac{\pi\theta}{\alpha}$$

Velocity components

$$\left.\begin{aligned} q_r &= -\frac{\partial \phi}{\partial r} = -C\left(\frac{\pi}{\alpha}\right) r^{\frac{\pi}{\alpha}-1} \cos\frac{\pi\theta}{\alpha} \\ q_\theta &= -\frac{1}{r}\frac{\partial \phi}{\partial \theta} = +C\left(\frac{\pi}{\alpha}\right) r^{\frac{\pi}{\alpha}-1} \sin\frac{\pi\theta}{\alpha} \end{aligned}\right\} \quad \text{---(39)}$$

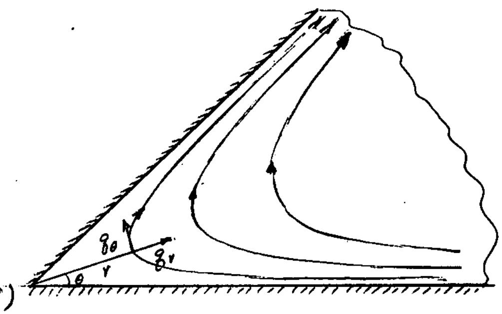
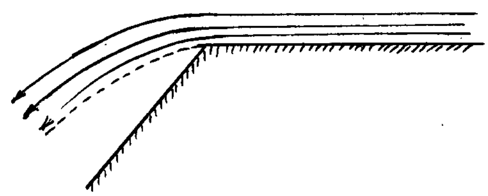


Fig. 6

$q = \sqrt{q_r^2 + q_\theta^2}$ is finite if $\alpha \leq \pi$

However, $\alpha > \pi$ $\quad q \longrightarrow \infty \; (r \longrightarrow 0)$



From Bernoulli's equation

$$p \longrightarrow -\infty$$

such a flow is physically impossible.

discontinuous flow (vortex motion is induced.)

(iv) Stress Singularity at the Corner of a Wedge Plate

$$M_r \sim -\frac{(1-\nu)P}{4\pi}\left(\frac{r}{\rho}\right)^{\frac{\pi}{\alpha}-2} \sin\frac{\pi\varphi}{\alpha} \sin\frac{\pi\theta}{\alpha}$$

(v) Schematic appearance of moment surfaces and influence surfaces of moments.

(a) moment surface $M_r(\rho,\varphi)$ (Fig. 7)

As an example, $M_r(\rho,\varphi)$ is considered

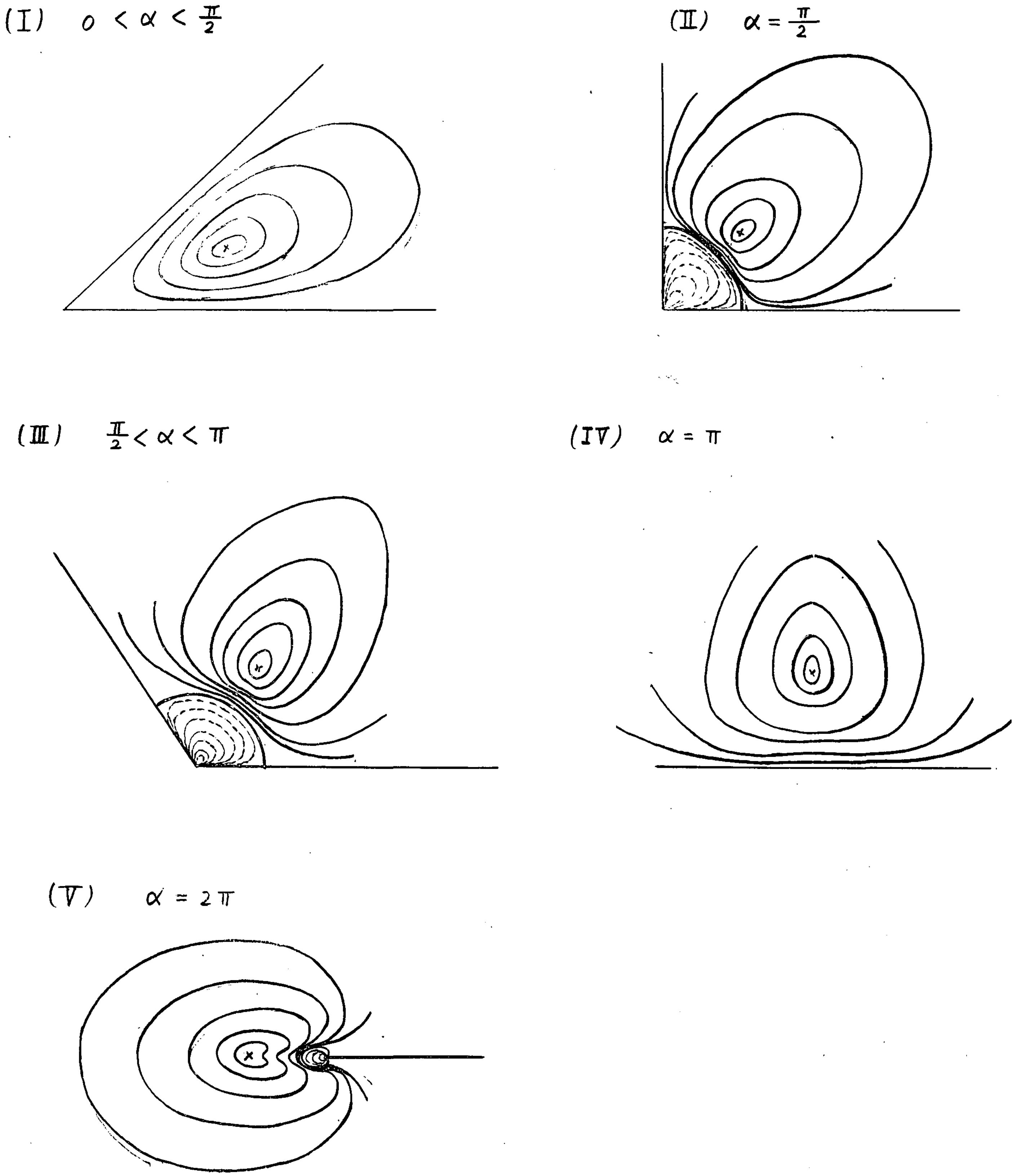

(I) $0 < \alpha < \frac{\pi}{2}$
(II) $\alpha = \frac{\pi}{2}$
(III) $\frac{\pi}{2} < \alpha < \pi$
(IV) $\alpha = \pi$
(V) $\alpha = 2\pi$

Fig. 7

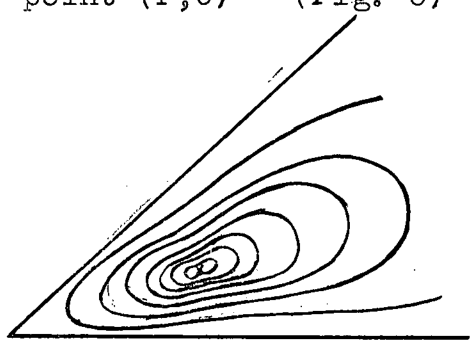
In the case where $\pi < \alpha < 2\pi$

appearance is same as (III)

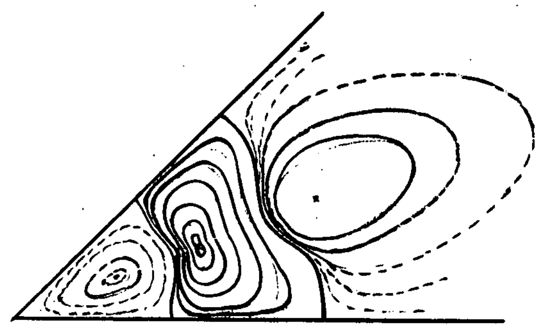
though intensity of stress

singularity increases.

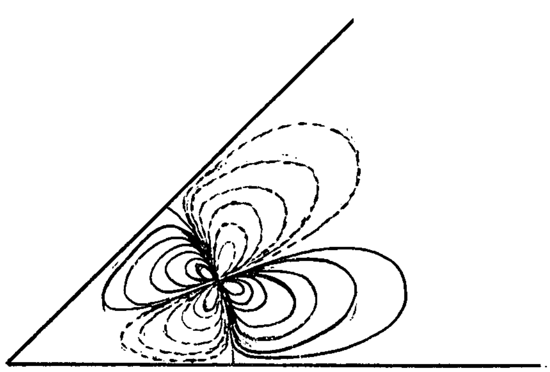
(b) Influence surfaces $m_r(r,\theta)$, $m_\theta(r,\theta)$, $m_{r\theta}(r,\theta)$ for interior point (r,θ) (Fig. 8)



$m_r(r,\theta)$



$m_\theta(r,\theta)$



$m_{r\theta}(r,\theta)$

Fig. 8

at the corner

For any loading point except boundary.

(i) $M_r = M_\theta = R = 0$ $(0 < \alpha < \frac{\pi}{2})$

(ii) $\alpha = \frac{\pi}{2}$

$$\left.\begin{matrix} M_r \\ M_\theta \end{matrix}\right\} = \mp \frac{(1-\nu)P}{4\pi} \sin 2\varphi \sin 2\theta$$

$$R = \frac{(1-\nu)P}{2\pi} \sin 2\varphi \cos 2\theta$$

(iii) $\alpha > \frac{\pi}{2}$ $(\alpha \neq \pi)$

$M_r = -\infty$, $M_\theta = +\infty$

$R = +\infty$

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