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Geometric Quantization of Classical Metrics on the Moduli Space of Canonical Metrics

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Contents

Abstract

In this thesis, two topics will be studied. In the first part, we investigate the geometric quantization of the Weil-Petersson metric on the moduli space of Fano Kähler-Einstein manifolds. In the second part, we investigate the (weak) pseudo-convexity of the Teichmüller space of Kähler-Einstein manifolds of general type.

In Chapter 1, we review the (infinitesimal) deformation theory of complex structures on compact complex manifolds. Based on Hodge theory, the existence of (infinitesimal) deformations will be discussed in detail. In Chapter 2, we explore the deformation theory of complex structures on compact Fano Kähler-Einstein manifolds with respect to the Kuranishi-divergence gauge. We also give the construction of local canonical sections of the relative tangent bundle. Based on these works, we show that the Weil-Petersson metric can be approximated by the curvatures of the natural L^2 metrics on the direct image of the tensor powers of relative anti-canonical bundles after normalization. In Chapter 3, we look at the Teichmüller space $\mathcal{T}$ of Kähler-Einstein manifolds of general type whose complex structure is unobstructed. Let N be a Riemannian manifold with nonpositive sectional curvature. We prove that the harmonic energy from $\mathcal{T}$ to N is pluri-subharmonic.

Chapter 1

Deformation Theory of Complex Structures

In this chapter, we first review the original idea of Kodaira and Spencer's deformation theory of complex structures on a compact complex manifold. In particular, we take a detailed look at how the Kuranishi gauge is used in the proof of the existence of an analytic family of complex structures. We will also mention the completeness theorem of the analytic family.

Definition 1.0.1 (Complex Manifold)**.** *Suppose X is a second countable, Hausdorff topological space. It is a complex manifold if the following properties are satisfied:*

1 $X = \bigcup\limits_{\alpha\in\Lambda}(U_\alpha, z_\alpha)$*, where for all* $\alpha \in \Lambda$*,* U_α *is open in* X*, and* $z_\alpha : U_\alpha \to z_\alpha(U_\alpha) \subset \mathbb{C}^n$ *is homeomorphism.*

2 $f_{\alpha\beta} = z_\alpha \circ z_\beta^{-1} : z_\beta(U_\alpha \cap U_\beta) \to z_\alpha(U_\alpha \cap U_\beta)$ *is biholomorphic.*

From now on, we only focus on compact complex manifolds. In this case, we can cover X by only finitely many coordinate charts. The complex structure is determined by the holomorphicity of the transition functions. Heuristically, on a given compact complex manifold $X = \{\bigcup\limits_{\alpha\in\Lambda}(U_\alpha, z_\alpha), f_{\alpha\beta}\}$, we fix the underlying smooth structure. If there is another complex structure on it, i.e. $X' = \{\bigcup\limits_{\alpha\in\Lambda}(U'_\alpha, z'_\alpha), f'_{\alpha\beta}\}$, we expect that new complex structure could be obtained by shifting coordinate patchs of the previous one; from the viewpoint of sheaf theory, the deformation lies in the sheaf cohomology $H^1(X, \mathcal{O}(T^{1,0}X))$.

If we consider the complexified tangent bundle $TX^{\mathbb{C}} = TX \otimes \mathbb{C}$, the complex structure is equivalent to the existence of a $J \in End(TX)$ such that $J^2 = -1$ and J satisfies integrability condition, i.e. Nijenhuis tensor vanishes. This J yields to the splitting $T_pX^{\mathbb{C}} = T_p^{1,0}X \oplus T_p^{0,1}X$ for all $p \in X$, with $pr_J^{1,0} = \frac{1}{2}(Id - \sqrt{-1}J)$ and $pr_J^{0,1} = \frac{1}{2}(Id + \sqrt{-1}J)$ the component projections with respect to the complex structure J. Holomorphicity is equivalent to the condition of $pr_J^{0,1}[pr^{1,0}X, pr^{0,1}Y] = 0$, for $X, Y \in TM^{\mathbb{C}}$ where $[X, Y]$ is the Lie bracket for vector fields on TX. Moreover, the complex structure J defines the operators $\partial = pr_J^{1,0}d$ and $\overline{\partial} = pr_J^{0,1}d$ with $\overline{\partial}^2 = 0$. (Here, we abuse notation by applying $pr_J^{1,0}$ and $pr_J^{0,1}$ on the cotangent bundles in the natural way.)

Suppose J' is another complex structure sufficiently close to J such that $TX^{\mathbb{C}} = T^{1,0}X' \oplus T^{0,1}X'$. Then $pr_J^{0,1}$ is an isomorphism between $T^{0,1}X'$ and $T^{0,1}X$. We have a $\varphi \in A^{0,1}(X, T^{1,0}X)$, a vector valued $(0,1)-$form on X which represents the map

$$pr^{1,0} \circ (pr^{0,1})^{-1} : T^{0,1}X \to T^{0,1}X \to T^{1,0}X.$$

In the next section, we will explain how this vector valued one form arises.

Definition 1.0.2 (Analytic family of compact complex manifolds)**.** *We say $\mathscr{X} = \{X_t | t \in B\}$ is an analytic family of compact complex manifolds if*

- *B is a complex manifold which parametrizes complex structures on the given underlying smooth manifold.*
- *there exists holomorphic map $\pi : \mathscr{X} \to B$ such that*
  - *for all t, $\pi^{-1}(t) = X_t$,*
  - *rank $\pi = dimB$.*

Remark 1. *1. $\pi : \mathscr{X} \to B$ is holomorphic, so the total space is a complex manifold.*

2. *For rank* $\pi = dimB$*, this means for* $p \in \mathscr{U}_\alpha \subset \mathscr{X}$ *, if* $z_\alpha(p,t) = (z_\alpha^1, \cdots z_\alpha^n, t_1 \cdots t_m)$ *are suitable local coordinates, then* $\pi(p) = (t_1 \cdots t_m)$*. On the other hand, if* $p \in \mathscr{U}_\alpha \bigcap \mathscr{U}_\beta \subset \mathscr{X}$ *and* $z_\alpha(p,t) = (z_\alpha^1, \cdots z_\alpha^n, t_1 \cdots t_m)$ *,* $z_\beta(p,t) = (z_\beta^1, \cdots z_\beta^n, t_1 \cdots t_m)$*, then the transation function* $f_{\alpha\beta}(t) = z_\alpha(p,t) \circ z_\beta^{-1}(p,t)$ *depends on* t*. Hence different* $t \Longleftrightarrow$ *different transition function* $\Longleftrightarrow$ *different complex structure.*

3. *For a given complex manifold* (X_0, J_0)*, if there exists such an analytic family* $\{X_t | t \in B\}$ *such that* $\pi^0 = (X_0, J_0)$ *we call* (X_t, J_t) *is the deformation of* (X_0, J_0)*.*

Question: For a given compact complex manifold (X_0, J_0), does there exist an analytic family, at least for $B = \{t : |t| < \varepsilon\} \subset \mathbb{C}^n$? If such an analytic family exsits , we call the total space to be an (infinitesimal) deformation family of (X_0, J_0).

1.1 Basic idea of analytic deformation theory

Let X be a compact complex manifold of $dim_{\mathbb{C}} X = n$. Suppose $X = \bigcup\limits_{\alpha \in \Lambda} (U_\alpha, z_\alpha)$, and transition function is $f_{\alpha\beta} = z_\alpha \circ z_\beta^{-1} : z_\beta(U_\alpha \cap U_\beta) \to z_\alpha(U_\alpha \cap U_\beta)$. From Kodaira-Spencer's viewpoint, deforming the complex structure on a given compact complex manifold is equivalent to deforming the transition functions holomorphically, which they describes as

$$\tfrac{dX_t}{dt} = \theta,\ \theta = [\theta_{\alpha\beta}], \text{ where } \theta_{\alpha\beta} = \sum_{i=1}^n \tfrac{\partial f_{\alpha\beta}^i(z_\beta, t)}{\partial t} \tfrac{\partial}{\partial z_\alpha^i} \in H^1(X_t, \mathcal{O}(T^{1,0} X_t)).$$

Remark 2. $[\theta_{\alpha\beta}]$ *is well-defined, moreover*

$$\begin{cases} \theta_{\alpha\beta} + \theta_{\beta\gamma} + \theta_{\gamma\alpha} = 0 \\ \theta_{\alpha\beta} = -\theta_{\beta\alpha}. \end{cases} \tag{1.1.1}$$

Proof. It suffices to check $\theta_{\alpha\beta} = \theta_{\alpha\gamma} + \theta_{\gamma\beta}$.

$$\begin{aligned}\theta_{\alpha\beta} &= \sum_{i=1}^{n} \frac{\partial f^i_{\alpha\beta}(z_\beta, t)}{\partial t} \frac{\partial}{\partial z^i_\alpha} \\ &= \sum_{i=1}^{n} \left(\frac{\partial f^i_{\alpha\gamma}}{\partial z^k_\beta} \frac{\partial z^k_\beta}{\partial t} + \frac{\partial f^i_{\alpha\beta}}{\partial t}\right) \frac{\partial}{\partial z^i_\alpha} \\ &= \sum_{i=1}^{n} \frac{\partial f^i_{\alpha\gamma}(z_\gamma, t)}{\partial t} \frac{\partial}{\partial z^i_\alpha} + \sum_{i=1}^{n} \frac{\partial f^i_{\alpha\beta}}{\partial t} \frac{\partial z^k_\gamma}{\partial z^i_\alpha} \frac{\partial}{\partial z^k_\gamma} \\ &= \theta_{\alpha\gamma} + \theta_{\gamma\beta}\end{aligned}$$

□

At $t = 0$, the Dolbeault theorem tells us that $H^1(X_0, \mathcal{O}(T^{1,0}X_0)) \cong H^{0,1}_{\overline{\partial}}(X_0, T^{1,0}X_0)$. Moreover, Kodaira-Spencer show that for sufficiently small $t \in B$, we can express the deformation by an element $\varphi(t) \in A^{0,1}(X_0, T^{1,0}X_0)$.

Theorem 1.1.1. *On a given compact complex manifold X_0 , if there is an analytic family $\pi : \mathscr{X} = \{X_t | t \in B \subset \mathbb{C}^n, |t| < \epsilon\} \to B$ such that $\pi^{-1}(0) = X_0$ and $X_t = \pi^{-1}(t)$, then the complex structure on X_t is determined by some $\varphi(t) \in A^{0,1}(X_0, T^{1,0}X_0)$ satisfying:*

$$\begin{cases} \overline{\partial}\varphi = \frac{1}{2}[\varphi, \varphi] \\ \varphi(0) = 0. \end{cases} \tag{1.1.2}$$

The $\overline{\partial}$ operator is with respect to the complex structure on (X_0, J_0).

Remark 3. *If we take local holomorphic coordinates $(z_1, \cdots, z_n)$ on X, and let $\varphi = \sum_{i,j} \varphi^i_{\bar{j}} d\bar{z}^j \otimes \frac{\partial}{\partial z_i}$ and $\psi = \sum_{k,l} \psi^k_{\bar{l}} d\bar{z}^l \otimes \frac{\partial}{\partial z_k}$, then $[\varphi, \psi] = \sum_{i,j,k,l} (\varphi^i_{\bar{j}} \partial_i \psi^k_{\bar{l}} + \psi^i_{\bar{j}} \partial_i \varphi^k_{\bar{l}}) d\bar{z}^j \wedge d\bar{z}^l \otimes \frac{\partial}{\partial z_k}$.*

Proof. On $\mathscr{X}$, consider the coordinate charts $\{\mathscr{U}_\alpha, (w_\alpha, t)\}$, such that on $\mathscr{U}_\alpha \cap \mathscr{U}_\beta$, $w_\alpha(t) = \sum_\beta f_{\alpha\beta}(w_\beta(t), t)$, and $f_{\alpha\beta}$ is holomorphic in t.

At $t = 0$, on the central fiber X_0, there are two differentiable coordinate systems: one is inherited from the holomorphic structure, locally, say on U_α, we write it to be $(z^1_\alpha, \cdots, z^n_\alpha)$; another is from the total space $\mathscr{X}$, i.e. $(w^1_\alpha(z, 0), \cdots, w^n_\alpha(z, 0))$. So the Jacobian $\det(\frac{\partial w^i(z,0)}{\partial z^j}) \neq 0$ on U_α. If $|t|$ is sufficiently small, then $\det(\frac{\partial w^i(z,t)}{\partial z^j}) \neq 0$ on U_α. We let $F_i^j = (\frac{\partial w^i(z,t)}{\partial z^j})^{-1}$, and

$$\varphi = \sum_{i,j,k} F_j^i \frac{\partial w^j}{\partial \overline{z}_k} d\overline{z}_k \otimes \frac{\partial}{\partial z_i}.$$

Then it can be checked that φ is well-defined and does not depend on the local coordinates, so it is a global section of $A^{0,1}(X_0, T^{1,0}X_0)$. Let us write $\varphi = \sum_{i,j} \varphi_{\overline{j}}^i(t) d\overline{z}^j \otimes \frac{\partial}{\partial z_i}$, then the holomorphic coordinates $(w^1, \cdots, w^n)$ on X_t satisfies:

$$\frac{\partial w_i}{\partial \overline{z}_k} = \sum_j \varphi_{\overline{k}}^j \frac{\partial w_i}{\partial z_j},$$

for all $i = 1, \cdots, n$ and $k = 1, \cdots, n$.

Under the new complex structure on X_t, a smooth function f is holomorphic if and only if $\big(\overline{\partial} - \sum_i \varphi^i(t)\partial_i\big) f(z) = 0$ on the central fiber, where $\varphi^i(t) = \sum_j \varphi(t)_{\overline{j}}^i d\overline{z}^j$.

When $t = 0$, by the holomorphic structure on X_0, it is easy to see that $\varphi(0) = 0$. To show the first equation in 1.1.2, we use $\overline{\partial} w^i = \sum_j \varphi^j \partial_j w^i$, where $\overline{\partial} = \sum_i \frac{\partial}{\partial \overline{z}_i} d\overline{z}^i$ $\varphi^j = \sum_k \varphi_{\overline{k}}^j d\overline{z}^k$. Applying $\overline{\partial}$ on both sides,

$$\begin{aligned}
0 &= \sum_{p,k,j} \big[\partial_{\overline{p}} \partial_j w^i \varphi_{\overline{k}}^j + \partial_j w^i \partial_{\overline{p}} \varphi_{\overline{k}}^j\big] d\overline{z}^p \wedge d\overline{z}^k \\
&= \sum_{p,k,j,l} \big[\partial_j (\varphi_{\overline{p}}^l \partial_l w^i) \varphi_{\overline{k}}^j + \partial_j w^i \partial_{\overline{p}} \varphi_{\overline{k}}^j\big] d\overline{z}^p \wedge d\overline{z}^k \\
&= \sum_{p,k,j,l} \big[\partial_l \varphi_{\overline{p}}^j \varphi_{\overline{k}}^l + \partial_{\overline{p}} \varphi_{\overline{k}}^j\big] \partial_j w^i d\overline{z}^p \wedge d\overline{z}^k \\
&= \sum_{p,k,j} \big[\partial_{\overline{p}} \varphi_{\overline{k}}^j - \partial_l \varphi_{\overline{k}}^j \varphi_{\overline{p}}^l\big] \partial_j w^i d\overline{z}^p \wedge d\overline{z}^k
\end{aligned} \tag{1.1.3}$$

If t is small enough, $\frac{\partial w^i}{\partial z_j}$ is invertible. We obtain $\overline{\partial} \varphi = \frac{1}{2}[\varphi, \varphi]$. □

However, not every element in $A^{0,1}(X_0, T^{1,0}X_0)$ can express the deformation. The obstruction of deformation lies in $H^2(X_0, \mathcal{O}(T^{1,0}X_0))$.

Theorem 1.1.2. *If (X_0, J_0) is a compact complex manifold, and $\rho \in H^1(X_0, \mathcal{O}(T^{1,0}X_0))$ is an infinitesimal deformation, then $[\rho, \rho] = 0$.*

1.2 Existence of infinitesimal deformations

Theorem 1.2.1 (Kodaira, Kuranishi, Nirenberg, Spencer)**.** *If X_0 is a compact complex manifold, and $H^2(X_0, \mathcal{O}(T^{1,0}X_0)) = 0$, then for every $\eta \in H^1(X_0, \mathcal{O}(T^{1,0}X_0))$, there exists an analytic family $\pi : \mathscr{X} = \{X_t | t \in B, |t| < \varepsilon\} \to B \subset \mathbb{C}^m$ such that*

- $X_0 = \pi^{-1}(0)$,
- $\frac{dX_t}{dt}|_{t=0} = \eta$, *i.e. the Kodaira-Spencer map KS: $T_0(B_\varepsilon) \to H^1(X_0, \mathcal{O}(T^{1,0}X_0))$ is surjective.*

We sketch the proof of the following existence theorem by using **Hodge Theory** to construct the formal power series solution to the deformation equation (1.1.2). To guarantee uniqueness of the solution, we fix **Kuranishi Gauge** i.e. $\overline{\partial}^* \varphi = 0$.

Theorem 1.2.2 (Kodaira, Kuranishi, Nirenberg, Spencer)**.** *Let (X_0, g_0) be a compact complex manifold, and g_0 be Hermitian metric. Suppose $H^2(X_0, \mathcal{O}(T^{1,0}X_0)) = 0$, then*

$$\begin{cases} \overline{\partial}\varphi = \frac{1}{2}[\varphi, \varphi] \\ \overline{\partial}^* \varphi = 0 \\ \varphi(0) = 0 \end{cases} \tag{1.2.1}$$

has a unique power series solution $\varphi(t) = \sum\limits_{i=1}^{\infty} \varphi_i(t)$.

Proof. **Step 1: Construction of the power series**

By Hodge theory, if $\overline{\partial}^* \varphi = 0$, then

$$\begin{aligned} \varphi(t) &= \mathbb{H}(\varphi) + \overline{\partial}\overline{\partial}^* G\varphi + \overline{\partial}^* \overline{\partial} G\varphi \\ &= \mathbb{H}(\varphi) + \frac{1}{2}\overline{\partial}^* G[\varphi, \varphi], \end{aligned} \tag{1.2.2}$$

where $G : A^{0,1}(X_0, T^{1,0}X_0) \to A^{0,1}(X_0, T^{1,0}X_0)$ is the Green operator associated to the Hodge Laplacian $\Box = \overline{\partial}\overline{\partial}^* + \overline{\partial}^* \overline{\partial}$.

Let $\varphi(t) = \sum\limits_{i=1}^{\infty} \varphi_i(t)$, where $\varphi_i(t) = \sum\limits_{|I|=i} t^I \varphi_I$, where I is multi-index.

Let $\mathbb{H}^1(X_0, \mathcal{O}(T^{1,0}X_0)) = span\{\beta_1, \cdots, \beta_k\}$. Then $\varphi_1(t) = \mathbb{H}(\varphi) \in \mathbb{H}^1(X_0, \mathcal{O}(T^{1,0}X_0))$ and there exists $t_1, \cdots, t_k$ such that $\varphi_1(t) = \sum_{j=1}^{k} t_j\beta_j$.

Comparing the power of t in (1.2.2), we get a power series solution to the deformation equation as follows:

$$\begin{cases} \varphi_2 = \frac{1}{2}\overline{\partial}^* G[\varphi_1, \varphi_1] \\ \varphi_3 = \frac{1}{2}\overline{\partial}^* G([\varphi_1, \varphi_2] + [\varphi_2, \varphi_1]) \\ \vdots \\ \varphi_n = \frac{1}{2}\overline{\partial}^* G \sum_{j=1}^{n-1} [\varphi_j, \varphi_{n-j}] \\ \vdots \end{cases} \tag{1.2.3}$$

Step 2: Convergence

First, we show the power series is convergent in $C^{k+\alpha}$ norm $||\cdot||_{k+\alpha}$.

Lemma 1.2.1 (*A priori* estimate)**.**

$$\|\varphi\|_{k+\alpha} \leq C(\|\Box\varphi\|_{k-2+\alpha} + \|\varphi\|_0).$$

Lemma 1.2.2.

$$\|[\varphi, \psi]\|_{k+\alpha} \leq C\|\varphi\|_{k+\alpha+1}\|\psi\|_{k+\alpha+1}.$$

Lemma 1.2.3.

$$\|G\varphi\|_{k+\alpha} \leq C\|\varphi\|_{k-2+\alpha}.$$

Let $\varphi^n(t) = \sum_{i=1}^{n} \varphi_i(t)$, then by (1.2.3),

$$\varphi^n(t) = \frac{1}{2}\overline{\partial}^* G[\varphi^{n-1}(t), \varphi^{n-1}(t)] mod(t^{n+1}).$$

We consider a power series

$$\begin{aligned} A(t) &= \frac{\beta}{16\gamma} \sum_{m=1}^{\infty} \gamma^m (t_1 + \cdots + t_k)^m \\ &= \sum_{|I|\geq 1} A_I t^I \end{aligned} \tag{1.2.4}$$

with β, γ are positive constants, I is multi-index. When $|t| \ll 1$, $A(t)$ is convergent.

Our goal is to find proper β, γ, such that $\|\varphi(t)\|_{k+\alpha} \leq A(t)$, i.e. $\|\varphi_{|I|}\|_{k+\alpha} \leq A_{|I|}$ When $n=1$, $\varphi^1 = \varphi_1 = \sum_{i=1}^{k} t_i \beta_i$, so we can choose β, γ such that $\|\varphi^1\|_{k+\alpha} \leq A(t)$. Assume $\|\varphi^{n-1}(t)\|_{k+\alpha} \leq A(t)$, then

$$\begin{aligned} \|\frac{1}{2}\overline{\partial}^* G[\varphi^{n-1}(t), \varphi^{n-1}(t)]\|_{k+\alpha} &\leq C_1 \|G[\varphi^{n-1}(t), \varphi^{n-1}(t)]\|_{k+\alpha+1} \\ &\leq C_{k,\alpha} C_1 \|[\varphi^{n-1}(t), \varphi^{n-1}(t)]\|_{k+\alpha-1} \\ &\leq C_{k,\alpha} C_1 C \|\varphi^{n-1}(t)\|^2_{k+\alpha} \end{aligned} \tag{1.2.5}$$

By the induction hypothesis, we get

$$\|\varphi^n(t)\|_{k+\alpha} \leq \tilde{C} A(t)^2.$$

But we also have $A(t)^2 \leq \frac{\beta}{\gamma} A(t)$. Now we can further choose β, γ such that $\tilde{C}(\frac{\beta}{\gamma}) \leq 1$, then for all n, $\|\varphi^n(t)\|_{k+\alpha} \leq A(t)$.

Therefore, $\varphi(t)$ is convergent in $C^{k+\alpha}$ norm.

Step 3: We show that under the assumption $H^2(X_0, \mathcal{O}(T^{1,0}X_0)) = 0$, the power series constructed in step 1 and 2 is the solution to $\overline{\partial}\varphi = \frac{1}{2}[\varphi, \varphi]$.

Proposition 1. *If $\varphi(t) = \mathbb{H}(\varphi) + \frac{1}{2}\overline{\partial}^* G[\varphi, \varphi]$, then $\overline{\partial}\varphi = \frac{1}{2}[\varphi, \varphi]$ if and only if $\mathbb{H}[\varphi, \varphi] = 0$.*

Proof. If $\overline{\partial}\varphi = \frac{1}{2}[\varphi, \varphi]$, then $\mathbb{H}[\varphi, \varphi] = 2\mathbb{H}\overline{\partial}\varphi = 0$.

Conversely, if $\mathbb{H}[\varphi, \varphi] = 0$, we will show $\overline{\partial}\varphi = \frac{1}{2}[\varphi, \varphi]$.

Let $\psi = \overline{\partial}\varphi - \frac{1}{2}[\varphi, \varphi]$, by Hodge decompositon, we get

$$
\begin{aligned}
\psi &= \overline{\partial}\varphi - \frac{1}{2}[\varphi,\varphi] \\
&= \frac{1}{2}\overline{\partial}\overline{\partial}^* G[\varphi,\varphi] - \frac{1}{2}[\varphi,\varphi] = -\frac{1}{2}\overline{\partial}^*\overline{\partial} G[\varphi,\varphi] \\
&= -\frac{1}{2}\overline{\partial}^* G([\overline{\partial}\varphi,\varphi] - [\varphi,\overline{\partial}\varphi]) \\
&= -\overline{\partial}^* G[\overline{\partial}\varphi,\varphi] \\
&= -\overline{\partial}^* G([\psi + \frac{1}{2}[\varphi,\varphi],\varphi]) \\
&= -\overline{\partial}^* G[\psi,\varphi]
\end{aligned}
\tag{1.2.6}
$$

By lemma 1.2.1-1.2.3,

$$
\begin{aligned}
\|\psi\|_{k+\alpha} &= \|\overline{\partial}^* G[\psi,\varphi]\|_{k+\alpha} \\
&\le C_1 \|G[\psi,\varphi]\|_{k+\alpha+1} \\
&\le C_2 \|[\psi,\varphi]\|_{k+\alpha-1} \\
&\le C_3 \|\varphi\|_{k+\alpha}\|\psi\|_{k+\alpha}.
\end{aligned}
\tag{1.2.7}
$$

We can choose sufficiently small t such that $C_3\|\varphi\|_{k+\alpha} \ll 1$, which is possible since $\varphi(0) = 0$. Therefore, $\|\psi\|_{k+\alpha} = 0$ and $\psi = 0$. $\square$

Step 4: Regularity of φ

Proposition 2. *$\varphi(z,t)$ is C^∞ in (z,t) and holomorphic in t.*

Proof. By the construction, φ is a power series in t, so it's holomorphic in t and $\sum_{i=1}^{n} \frac{\partial^2\varphi}{\partial t_i \partial \overline{t_i}} = 0$.

On the other hand, we know φ satisfies:

$$
\begin{cases}
\overline{\partial}\varphi = \frac{1}{2}[\varphi,\varphi] \\
\overline{\partial}^*\varphi = 0.
\end{cases}
\tag{1.2.8}
$$

We get, $\square\varphi = \overline{\partial}\overline{\partial}^*\varphi + \overline{\partial}^*\overline{\partial}\varphi = \frac{1}{2}\overline{\partial}^*[\varphi,\varphi]$.

Since as φ satisfies the quasi-linear elliptic equation

$$\sum_{i=1}^{n} \frac{\partial^2 \varphi}{\partial t_i \partial \overline{t_i}} + \Box\varphi - \frac{1}{2}\overline{\partial}^* [\varphi, \varphi] = 0,$$

we have C^∞ regularity is for small t. $\Box$

Step 5: $\{X_t | t \in B_\varepsilon\}$ is a complex analytic family.

The last step follows from Nirenberg-Newlander theorem.

Theorem 1.2.3 (Nirenberg-Newlander)**.** *Locally, let $L_{\bar{i}} = (\frac{\partial}{\partial \bar{z}^i}) - \varphi_{\bar{i}}^{j}(\frac{\partial}{\partial z^j})$, suppose $L_{\bar{i}}$ and $\overline{L_{\bar{i}}}$ are complex linearly independent, $\overline{\partial}\varphi = \frac{1}{2}[\varphi, \varphi]$. Then there are n C^∞ solutions $w_1 \cdots w_n$ to $L_{\bar{i}} = 0$, such that*

$$\det\left(\frac{\partial(w_1, \cdots, w_n, \overline{w_1}, \cdots, \overline{w_n})}{\partial(z_1, \cdots, z_n, \overline{z_1}, \cdots, \overline{z_n})}\right) \neq 0.$$

$\Box$

1.3 Completeness of the analytic family

In this section, we prove Kuranishi's theorem which can be viewed as the completeness of an analytic family of complex structures.

Definition 1.3.1. *Let $\pi : \mathscr{X} = \{X_t | t \in B, |t| < \varepsilon\} \to B \subset \mathbb{C}^m$ be a complex analytic family of compact complex manifolds. The family $(\mathscr{X}, B, \pi)$ is called complete at $t_0 \in B$, if for any complex analytic family $(\mathscr{N}, D, \omega)$ containing 0 and so that $\pi^{-1}(t_0) = \omega^{-1}(0)$, there are a sufficiently small domain E with $0 \in E \subset D$, and a holomorphic map h such that $(\mathscr{N}_E, E, \omega)$ is the complex analytic family induced from $(\mathscr{X}, B, \pi)$ by h, where $\mathscr{N}_E = \omega^{-1}(E)$.*

Theorem 1.3.1. *(Kuranishi)*

(a) Let X be a given compact complex manifold. Let $\{\eta_\nu\}_{\nu=1}^m$ be a base for $\mathbb{H}^1 \cong H^1(X, \mathcal{O}(T^{1,0}X))$. Suppose $\varphi(t)$ is the solution to the Kuranishi equation

$$\varphi(t) = \eta(t) + \frac{1}{2}\overline{\partial}^* G[\varphi(t), \varphi(t)] \tag{1.3.1}$$

where $\eta(t) = \sum_{\nu=1}^{m} t_\nu \eta_\nu$, for sufficiently small $|t| < \epsilon$. Let $B = \{t \in \mathbb{C}^n||t| < \epsilon, \mathbb{H}[\varphi(t), \varphi(t)] = 0\}$. Then for all $t \in B$, $\varphi(t)$ determines a complex structure on X_t.

(b) Let $\psi \in A^{0,1}(X, T^{1,0}X)$ such that $\overline{\partial}\psi = \frac{1}{2}[\psi, \psi]$. Then ψ determines a complex structure on X_ψ. If the sobolev norm $||\psi||_{W^{k,2}}$ is small enough, then there exists a holomorphic vector field $\xi \in H^0(X, \mathcal{O}(T^{1,0}X))^\perp$, and a diffeomorphism $f_\xi : X \to X$ depending on ξ, such that $\varphi(t) = \psi \circ f_\xi$ for some $t \in B$ and X_ψ is biholomorphic to X_t, where $H^0(X, \mathcal{O}(T^{1,0}X))^\perp$ is the orthogonal complement of $H^0(X, \mathcal{O}(T^{1,0}X))$ with respect to the L^2 norm.

Remark 4. *1. In the Kuranishi theorem, one is not assuming $H^2(X, \mathcal{O}(T^{1,0}X)) = 0$, i.e. the deformation of the complex structure may have obstructions.*

*2. In the previous section, to show the existence of the solution to the deformation equation, we fixed the **Kuranishi Gauge**, i.e. $\overline{\partial}^*\varphi = 0$. However, in general, the Kuranishi gauge may not be fixed. In the following of proof of statement (b), we will see that based on the assumption of sufficiently small solution to the deformation equation, we can find a diffeomorphism to adjust the gauge such that Kuranishi gauge can be achieved.*

3. Statement (b) can be viewed as a completeness theorem of the analytic family.

Proof. (a) follows immediately from Proposition 1.

For (b), we need to show for X_ψ with $||\psi||_{W^{k,2}} < \delta$ for some $\delta \ll 1$, X_ψ is biholomorphic to $X_{\varphi(t)}$ for some $t \in B$. We divide the proof into the following three propositions.

Proposition 3. *For fixed $\eta(t) \in \mathbb{H}^1(X, T^{1,0}X)$, the Kuranishi equation $\varphi(t) = \eta(t) + \frac{1}{2}\overline{\partial}^* G[\varphi(t), \varphi(t)]$ has only one solution with $||\varphi||_k = ||\varphi||_{W^{k,2}} < \delta$.*

Proof. Assume φ and ψ are two solutions to the Kuranishi equation with $\mathbb{H}\varphi =$

$\mathbb{H}\psi = \eta(t)$, and $||\varphi||_k$, $||\psi||_k < \delta$. Then,

$$\begin{aligned}\tau =& \varphi - \psi \\ =& \frac{1}{2}\overline{\partial}^* G[\varphi, \varphi] - \frac{1}{2}\overline{\partial}^* G[\psi, \psi] \\ =& \frac{1}{2}\overline{\partial}^* G([\tau, \tau] + 2[\tau, \psi])\end{aligned} \tag{1.3.2}$$

Hence, $||\tau||_k \leq C||\tau||_k(||\tau||_k + ||\psi||_k)$, and this inequality is true if and only if $||\tau||_k = 0$ due to the fact that $||\varphi||_k$, $||\psi||_k < \delta$. □

According to this proposition, the small enough solution to the Kuranishi equation is uniquely determined by its harmonic part.

The next proposition tells that with respect to Kuranishi gauge, any complex manifold X_ψ with $||\psi||_k \leq \delta$ can be obtained by the solution to the Kuranishi equation.

Proposition 4. *For X_ψ with $||\psi|| < \delta$, if $\overline{\partial}^*\psi = 0$, then there exists a solution $\varphi(t)$ satisfying the Kuranishi equation $\varphi(t) = \eta(t) + \frac{1}{2}\overline{\partial}^* G[\varphi(t), \varphi(t)]$ such that $X_\psi \cong X_{\varphi(t)}$. (Acutally, $\psi = \varphi(t)$).*

Proof. Assume the complex structure on X_ψ is determined by ψ. Then ψ satisfies the deformation equation

$$\begin{cases}\overline{\partial}\psi = \frac{1}{2}[\psi, \psi] \\ \overline{\partial}^*\psi = 0.\end{cases} \tag{1.3.3}$$

Let $\eta = \mathbb{H}(\psi)$, then $\psi = \eta + \frac{1}{2}\overline{\partial}^* G[\psi, \psi]$ with $||\psi||_k < \delta$. By the previous propostion, we know $\psi = \varphi(t)$ for some $|t| < \epsilon$. □

However, in general, the Kuranishi gauge may not hold. The following proposition shows that we can always find a diffeomorphism to adjust the gauge to the Kuranishi gauge.

Proposition 5. *If $\overline{\partial}^*\psi \neq 0$, then for $||\psi||_k < \delta$, there exists a diffeomorphism $f_\xi : X \to X$ determined by a vector $\xi \in H^0(X_0, T^{1,0}X)^\perp$, such that $\overline{\partial}^*(\psi \circ f_\xi) = 0$.*

Proof. **Claim:** For sufficiently small $||\psi||_k$, a diffeomorphism $f : X \to X$, $\varphi = \psi \circ f$ also determines a complex structure on X.

Proof of the Claim: Assume $X_\psi \subset \bigcup_\alpha (U_\alpha, \zeta^\alpha(z))$, where $\zeta^\alpha(z)$ is the local ψ-holomorphic coordinate. Suppose $f : X \to X$ is a diffeomorphism and $X \subset \bigcup_\alpha (f(U_\alpha), \zeta^\alpha(f(z)))$. Then

$$\begin{cases} \overline{\partial}\zeta^\alpha(z) = \psi^\beta \partial_\beta \zeta^\alpha(z) \\ \overline{\partial}\zeta^\alpha(f(z)) = \varphi^\beta \partial_\beta \zeta^\alpha(f(z)). \end{cases} \tag{1.3.4}$$

From the second equation,

$$\frac{\partial \zeta^\alpha}{\partial f^\gamma}\frac{\partial f^\gamma}{\partial \overline{z}^\delta} + \frac{\partial \zeta^\alpha}{\partial \overline{f}^\gamma}\frac{\partial \overline{f}^\gamma}{\partial \overline{z}^\delta} = \varphi^\beta_\delta \left[\frac{\partial \zeta^\alpha}{\partial f^\gamma}\frac{\partial f^\gamma}{\partial z^\beta} + \frac{\partial \zeta^\alpha}{\partial \overline{f}^\gamma}\frac{\partial \overline{f}^\gamma}{\partial z^\beta} \right].$$

Using the first equation, we get

$$\frac{\partial \zeta^\alpha}{\partial f^\gamma}\frac{\partial f^\gamma}{\partial \overline{z}^\delta} + \psi^\beta_{\overline{\gamma}}\frac{\partial \zeta^\alpha}{\partial f^\beta}\frac{\partial \overline{f}^\gamma}{\partial \overline{z}^\delta} = \varphi^\beta_\delta \left[\frac{\partial \zeta^\alpha}{\partial f^\gamma}\frac{\partial f^\gamma}{\partial z^\beta} + \psi^\eta_{\overline{\gamma}}\frac{\partial \zeta^\alpha}{\partial f^\eta}\frac{\partial \overline{f}^\gamma}{\partial z^\beta} \right].$$

Then we obtain

$$\overline{\partial} f^\gamma + \psi^\gamma_{\overline{\beta}} \overline{\partial f}^\beta = \varphi^\beta [\partial_\beta f^\gamma + \psi^\gamma_{\overline{\eta}} \partial_\beta \overline{f}^\eta]. \tag{1.3.5}$$

There exists some $\delta \ll 1$, such that when $||\psi||_k < \delta$, the gauge matrix $[\partial_\beta f^\gamma + \psi^\gamma_{\overline{\eta}} \partial_\beta \overline{f}^\eta]$ is invertible, and the new complex structure φ is determined by ψ and the diffeomorphism f.

Next, we will use geodesics to construct such a diffeomorphism.

Recall that if $z(t) = (z^1(t), \cdots, z^n(t))$ is the geodesic which starts from z_0 with initial velocity ξ, then it satisfies the equation

$$\begin{cases} \dfrac{d^2 z^\alpha(t)}{dt^2} + \Gamma^\alpha_{\beta\gamma} \dfrac{dz^\alpha}{dt}\dfrac{dz^\beta}{dt} = 0 \\ z(0) = z_0 \\ \dfrac{dz(t)}{dt}\Big|_{t=0} = \xi \end{cases} \tag{1.3.6}$$

where $\Gamma^\alpha_{\beta\gamma}$ corresponds to the Chern connection of the Hermitian metric.

$z(t)$ smoothly depends on the initial data $z^\alpha(t) = z^\alpha(t, z_0, \xi)$. Now we let $f^\alpha(z_0, \xi) = z^\alpha(1, z_0, \xi)$. We want to find the Taylor expansion of f with respect to ξ. Notice that

$$f^\alpha(z_0, t\xi) = z^\alpha(1, z_0, t\xi) = z^\alpha(t, z_0, \xi), \tag{1.3.7}$$

take derivative on t, we get

$$\frac{\partial f^\alpha}{\partial \xi^\beta}\xi^\beta + \frac{\partial f^\alpha}{\partial \overline{\xi}^\beta}\overline{\xi}^\beta = \frac{\partial z^\alpha}{\partial t},$$

so,

$$\begin{cases} f^\alpha(z_0, 0) = z_0^\alpha \\ \frac{\partial f^\alpha}{\partial \xi^\beta}(z_0, 0) = \delta_{\alpha\beta} \\ \frac{\partial f^\alpha}{\partial \overline{\xi}^\beta}(z_0, 0) = 0. \end{cases} \tag{1.3.8}$$

Thus, we obtain the expansion

$$f^\alpha(z_0, \xi) = z_0^\alpha + \xi^\alpha + O(|\xi|^2) = z_0^\alpha + \xi^\alpha + h^\alpha.$$

For a given vector ξ, we define the diffeomorphism $f_\xi : z^\alpha \mapsto f^\alpha(z, \xi)$, which satisfies equation (1.3.5), i.e.

$$\overline{\partial}\xi^\alpha + \overline{\partial}h^\alpha + \psi^\alpha_{\overline{\beta}}[d\overline{z}^\beta + \overline{\partial\xi}^\beta + \overline{\partial h}^\beta] = \varphi^\beta[\delta^\alpha_\beta + \partial_\beta\xi^\alpha + \partial_\beta h^\alpha + \psi^\alpha_{\overline{\eta}}(\partial_\beta\overline{z}^\eta_0 + \partial_\beta\overline{\xi}^\eta + \partial_\beta\overline{h}^\eta)],$$

and

$$\varphi^\beta = \overline{\partial}\xi^\beta + \psi^\beta + R(\xi, \psi), \tag{1.3.9}$$

where $R(\xi, \psi)$ smoothly depends on ξ, ψ and their first order derivatives.

We are looking for ξ that the modified complex structure satisfies $\overline{\partial}^*\varphi = \overline{\partial}^*(\psi \circ f_\xi) = 0$, where f_ξ is the diffeomorphism generated by ξ.

Now, take $\xi \in H^0(X, \mathcal{O}(T^{1,0}X))^\perp$, i.e. for all $\eta \in H^0(X, \mathcal{O}(T^{1,0}X))$, $< \xi, \eta >_{L^2} = 0$. By the Hodge decomposition, $\xi = \overline{\partial}^*\overline{\partial}G\xi$, and by equation (1.3.9),

$$\overline{\partial}^*\varphi = \overline{\partial}^*\overline{\partial}\xi + \overline{\partial}^*\psi + \overline{\partial}^*R(\xi, \psi) = 0,$$

and

$$\xi + G\overline{\partial}^*\psi + G\overline{\partial}^* R(\xi, \psi) = 0. \tag{1.3.10}$$

By the implicit function theorem, equation (1.3.10) has solution $\xi = g(\psi)$ for $\|\psi\|_k < \delta$. Since the equation

$$\Box\xi + \overline{\partial}^* R(\xi, \psi) + \overline{\partial}^*\psi = 0$$

is of second order elliptic, ξ is C^∞.

Therefore, with such ξ, the diffeomorphism f_ξ will adjust the gauge into the Kuranishi gauge. $\Box$

We finish the proof of Theorem 1.3.1. $\Box$

Chapter 2

Complex Deformation on Fano Kähler-Einstein Manifolds

In this chapter, we study the analytic family of Fano Kähler-Einstein manifolds. For a Fano Kähler-Einstein manifold, we know that the deformation of complex structure has no obstruction, so the deformation equation with respect to the Kuranishi gauge can be solved and the solution is uniquely determined by its harmonic part. In the following computation, instead of using the Kuranishi gauge, we will use the divergence gauge. In section 1, we show that these two gauges are equivalent on a Fano Kähler-Einstein manifold. As a matter of fact, we prove a more general result. We consider a Fano Kähler manfold with the Ricci potential given by a smooth function f. If we replace the Kuranishi gauge and the divergence gauge by the $f-$Kuranishi gauge and the $f-$divergence gauge, then we conclude that the $f-$Kuranishi gauge is equivalent to the $f-$divergence gauge. This equivalence guarantees that the deformation equation of the complex structure on the Fano manifold is solvable under the $f-$divergence gauge. In particular, on the Kähler-Einstein manifold, the deformation equation of the complex structure is solvable under divergence gauge. In section 2, based on the assumption that the automorphism group is discrete, we compute the Taylor expansion of the Kähler form and volume form on the deformed Fano Kähler-Einstein manifolds. This expansion will be used in the study of the L^2

metric on the direct image sheaf in later sections. In section 3, we investigate the deformation of pluri-anticanonical sections. We establish the deformation equation of these sections and use the Hodge theory to show that the deformation equation is solvable under the Kuranishi-divergence gauge. Furthermore, starting from a pluri-anticanonical section on the central fiber, we can explicitly construct the solution to the deformation equation in terms of the power series. In section 4, by studying the Taylor expansion of L^2 metric on the direct image sheaf, we obtain the quantization of the Weil-Petersson metric on the moduli space. In section 5, we explore the deformation of the holomorphic vector field. Especially, we discuss the solution to the deformation equation of the holomorphic vector field under the assumption that the dimension of the space of holomorphic vector fields is a constant. The results obtained in this chapter is in paper [5].

2.1 Deformation of complex structures on Fano manifolds and gauge equivalence

Definition 2.1.1. *Let (X, J) be a compact complex manifold. X is called Fano if the anticanonical line bundle $K_X^{-1} = \Lambda^n T^{1,0}X$ is ample; equivalently, the first Chern class $c_1(X) > 0$.*

Definition 2.1.2. *Let (X, J, ω) be a Kähler manifold with the Kähler form ω. It is called Kähler-Einstein manifold if the Ricci form satisfies $R_{i\bar{j}} = \rho\omega$, where ρ is a constant. In particular, if $\rho > 0$, X is called the Fano Kähler-Einstein manifold; if $\rho = 0$, X is called the Calabi-Yau manifold; and if $\rho < 0$, X is called Kähler-Einstein manifold of general type.*

Lemma 2.1.1. *Let (X_0, J_0, ω_0) be a compact Fano Kähler manifold with canonical line bundle K_0. Then*

$$H^2(X_0, \mathcal{O}(T^{1,0}X_0)) = 0.$$

Proof. By Serre duality,

$$
\begin{aligned}
H^2(X_0, \mathcal{O}(T^{1,0}X_0)) &\cong H^{n-2}(X_0, \mathcal{O}((T^{1,0}X_0)^* \otimes K_0)) \\
&\cong H^{n-2}(X_0, \Omega^1(K_0)).
\end{aligned}
$$

But on Fano manifold, $c_1(K_0) = -c_1(X_0) < 0$. By the Kodaira vanishing theorem, we know

$$H^2(X_0, \mathcal{O}(T^{1,0}X_0)) = 0.$$

$\square$

Hence, there is no obstruction to the deformation of complex structure on Fano manifolds, and in particular, this is true on compact Fano Kähler-Einstein manifolds. According to Kodaira, Kuranishi, Nirenberg and Spencer's work (see Theroem 1.2.1), there exists an (infinitesimal) analytic family $\pi : \mathscr{X} \to B = \{t = (t_1 \cdots, t_k) \in C^k \big| |t| < \epsilon\}$ of Fano manifolds. Here $k = dim H^1(X_0, T^{1,0}X_0)$. In addition, the deformation equation

$$
\begin{cases}
\overline{\partial}\varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)] \\
\overline{\partial}^* \varphi(t) = 0 \\
\varphi(0) = 0
\end{cases}
\tag{2.1.1}
$$

has a unique power series solution $\varphi(t) = \sum_{i=1}^{\infty} \varphi_i(t)$ with $\varphi_1(t) = \mathbb{H}(\varphi) \in H^1(X_0, T^{1,0}X_0)$, and $\varphi_i(t) = \sum_{\alpha_1 + \cdots \alpha_k = i} t_1^{\alpha_1} \cdots t_k^{\alpha_k} \varphi_{\alpha_1 \cdots \alpha_k}$. Recall that the condition $\overline{\partial}^* \varphi(t) = 0$ is refered to be the **Kuranishi Gauge**.

If (X_0, J_0, ω_0) is a compact Fano Kähler manifold, then by $\partial\overline{\partial}-$Lemma, there is a smooth complex valued function f on X_0 such that $Ric_0 - \omega_0 = \frac{\sqrt{-1}}{2}\partial\overline{\partial} f$. Let $(E, h) \to X_0$ be a complex vector bundle, and $A^{p,q}(X_0, E)$ is the space of smooth E-valued (p,q)-forms. With f, we can define the L^2_f-norm on $A^{p,q}(X_0, E)$. For φ,$\psi \in A^{p,q}(X_0, E)$,

$$< \varphi, \psi >_f = \int_{X_0} (\varphi, \psi)_h e^f \frac{\omega_0^n}{n!}.$$

Moreover, we define $\overline{\partial}^*_f = \overline{\partial}^* - i_{\overline{\nabla} f}$, where $\overline{\nabla} f$ is a $(0,1)-$vector. The $f-$Laplacian is defined to be $\Box_f = \overline{\partial}\overline{\partial}^*_f + \overline{\partial}^*_f\overline{\partial}$. We mention that $\Box_f$ is a second order elliptic self-adjoint operator with respect to the volume form $e^f \frac{\omega_0^n}{n!}$. The E-valued $(p,q)_f-$harmonic form α is defined to be $\Box_f \alpha = 0$, and

$$\mathbb{H}^{p,q}_f(X) = \{\alpha \in A^{p,q}(X_0, E) \big| \Box_f \alpha = 0\}.$$

On a Fano manifold, we know the deformation equation (2.1.1) can be solved under the Kuranishi gauge. In fact, applying the method in the proof of proposition 5, we can adjust the Kuranishi gauge by finding a diffeomorphism $\sigma : X \to X$, such that $\psi = \varphi \circ \sigma$ solves $\overline{\partial}^*_f \psi = 0$.

Proposition 6. *Let (X_0, ω_0) be a compact Fano Kähler manifold. Suppose there is $\delta \ll 1$ such that $\varphi(t)$ solves equation (2.1.1) with $||\varphi(t)||_k < \delta$, for $t \in B$. Then there exists a vector field ξ in the L^2_f orthogonal complement of $\mathbb{H}^0_f(X_0, T^{1,0}X_0)$, such that $\sigma_\xi : X_0 \to X_0$ is a diffeomorphism of X_0 generated by ξ, and $\psi = \varphi \circ \sigma_\xi$ satisfies the following equations*

$$\begin{cases} \overline{\partial}\psi = \frac{1}{2}[\psi(t), \psi(t)] \\ \overline{\partial}^*_f \psi = 0. \end{cases} \tag{2.1.2}$$

Proof. For a diffeomorphism $\sigma : X_0 \to X_0$, if σ and $d\sigma$ are close to the identity map, and if $||\varphi||_k < \delta$, then $\varphi \circ \sigma$ also determines a complex structure ψ on X_0.

Take a vector field ξ in the L^2_f orhtogonal complement of $\mathbb{H}^0_f(X_0, T^{1,0}X_0)$, and consider the geodesic $z(t) = (z_1(t), \ldots, z_n(t))$ starting from z_0 with initial velocity ξ where $z_\alpha(t) = z_\alpha(t, z_0, \xi)$. Let $\sigma_\alpha(z_0, \xi) = z_\alpha(1, z_0, \xi)$, and we define the diffeomorphism $\sigma_\xi : z_\alpha \to \sigma_\alpha(z, \xi)$. Then by Taylor expansion of σ_ξ, by (1.3.9) we get

$$\psi = \overline{\partial}\xi + \varphi + R(\xi, \varphi),$$

where $R(\xi, \varphi)$ smoothly depends on ξ, φ and their derivatives. By the equation $\overline{\partial}^*_f \psi = 0$, we let G_f be the Green operator associated to the f-Laplacian $\Box_f$, then we see ξ satisfies

$$\xi + G_f \overline{\partial}^*_f \varphi + G_f \overline{\partial}^*_f R(\xi, \varphi) = 0. \tag{2.1.3}$$

Notice that $\overline{\partial}^* \varphi = 0$, so the above equation reads as

$$\xi - G_f \overline{\nabla} f \lrcorner \varphi + G_f \overline{\partial}_f^* R(\xi, \varphi) = 0. \tag{2.1.4}$$

Define an operator F from a neighborhood where $R(\xi, \varphi)$ is defined to the L_f^2 orthogonal complement of $\mathbb{H}_f^0(X_0, T^{1,0}X_0)$ by $F(\xi, \varphi) = \xi - G_f \overline{\nabla} f \lrcorner \varphi + G_f \overline{\partial}_f^* R$. Then $\frac{\partial F}{\partial \xi}\Big|_{(0,0)} = Id$, by the implicit function theorem, such a ξ to equation (2.1.4) exists. Moreover, ξ also satisfies

$$\square_f \xi - \overline{\nabla} f \lrcorner \varphi + \overline{\partial}_f^* R(\xi, \varphi) = 0, \tag{2.1.5}$$

which is a second order elliptic equation, so ξ is of class C^∞.

With such a vector field ξ, the new complex structure $\psi = \varphi \circ \sigma_\xi$ satisfies $\overline{\partial}_f^* \psi = 0$. □

Remark 5. *On a Fano Kähler manifold, we refer to the condition $\overline{\partial}_f^* \psi = 0$ as the $f-$Kuranishi gauge.*

Now, we introduce the divergence gauge and the $f-$divergence gauge. The divergence gauge was introduced by X. Sun in his paper [15], where he studied the complex deformation of Kähler-Einstein manifolds of general type. Later, in the paper [16], Sun and Yau also used it to study the complex deformation of the Calabi-Yau manifolds. In the following, we use the notation $\partial_i = \frac{\partial}{\partial z_i}$ and we use Einstein convention from now on i.e. repeated indices mean taking the sum of them.

Definition 2.1.3. *Let $(L, h) \to (X_0, \omega_0)$ be a Hermitian line bundle over a complex manifold. The divergence operator is*

$$\mathrm{div} = Tr \circ \nabla : A^{0,1}(X_0, T^{1,0}X_0 \otimes L) \to A^{0,1}(X_0, L)$$

Locally, if $(z_1 \ldots, z_n)$ are local holomorphic coordinates on X_0 and e is a holomorphic frame of L, for $\eta = \eta_{\bar{j}}^i d\bar{z}_j \otimes \frac{\partial}{\partial z_i} \otimes e \in A^{0,1}(X_0, T^{1,0}X_0 \otimes L)$,

$$\mathrm{div}\eta = (\partial_i \eta_{\bar{j}}^i + \eta_{\bar{j}}^i \partial_i \log(g_0 h)) d\bar{z}_j \otimes e,$$

where g_0 is the determinant of the metric on X_0.

Definition 2.1.4 (X. Sun[15]). *For $\varphi \in A^{0,1}(X_0, T^{1,0}X_0)$, $\mathrm{div}\varphi = 0$ is called the divergence gauge.*

If the underlying manifold is a compact Fano Kähler manifold with the volume density $e^f \frac{\omega^n}{n!}$, we can also define $\mathrm{div_f}$.

Definition 2.1.5. *For $\eta = \eta^i_{\bar{j}} d\bar{z}_j \otimes \frac{\partial}{\partial z_i} \otimes e \in A^{0,1}(X_0, T^{1,0}X_0 \otimes L)$,*

$$\mathrm{div}_f \eta = (\partial_i \eta^i_{\bar{j}} + \eta^i_{\bar{j}} \partial_i \log(g_0 h e^f)) d\bar{z}_j \otimes e;$$

and for $\varphi \in A^{0,1}(X_0, T^{1,0}X_0)$, $\mathrm{div}_f \varphi = 0$ is called the $f-$divergence gauge.

Remark 6. *(1.)* $\mathrm{div}_f \varphi = \mathrm{div}\varphi + \varphi \lrcorner \partial f$.

(2.) In terms of local holomorphic coordinates $(z_1, \dots z_n)$ on X_0, for $\omega_g = \frac{\sqrt{-1}}{2} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$, $\varphi = \varphi^p_{\bar{j}} d\bar{z}^j \otimes \frac{\partial}{\partial z_p}$, we have

$$\begin{aligned}\bar{\partial}^*_f \varphi &= [-(\partial_l \varphi^p_{\bar{j}}) g^{l\bar{j}} + \varphi^l_{\bar{j}} \partial_l g^{p\bar{j}} - \varphi^p_{\bar{j}} g^{l\bar{j}} \partial_l f] \frac{\partial}{\partial z_p} \\ &= [-\partial_l (\varphi^p_{\bar{j}} g_{p\bar{i}}) g^{l\bar{j}} g^{k\bar{i}} - \varphi^k_{\bar{j}} g^{l\bar{j}} \partial_l f] \frac{\partial}{\partial z_k},\end{aligned} \tag{2.1.6}$$

and

$$\begin{aligned}\mathrm{div}_f \varphi &= (\partial_p \varphi^p_{\bar{j}} + \varphi^p_{\bar{j}} \partial_p \log g + \varphi^p_{\bar{j}} \partial_p f) d\bar{z}^j \\ &= [\partial_k (\varphi^p_{\bar{j}} g_{p\bar{l}}) g^{k\bar{l}} + \varphi^p_{\bar{j}} \partial_p f] d\bar{z}^j.\end{aligned} \tag{2.1.7}$$

Next, we will show that on a compact Fano Kähler manifold with volume density $e^f \frac{\omega^n}{n!}$, the $f-$Kuranishi gauge is equivalent to the $f-$divergence gauge. Consequently, under either one of these gauges, the deformation equation of complex structures can be solved, and we still have the analytic family of compact Fano Kähler manifolds.

Firstly, we show the $f-$divergence gauge implies the $f-$Kuranishi gauge.

Lemma 2.1.2. *If (X_0, ω_0) is a Fano Kähler manifold with volume density $e^f \frac{\omega^n}{n!}$, for $\varphi \in A^{0,1}(X_0, T^{1,0}X_0)$, we have*

(1.) $\bar{\partial} \mathrm{div}_f \varphi = \mathrm{div}_f (\bar{\partial} \varphi) - 2\sqrt{-1} \varphi \lrcorner \omega_0$

(2.) $\frac{1}{2}\mathrm{div}_f[\varphi,\varphi] = \varphi\lrcorner\partial(\mathrm{div}_f\varphi)$.

Proof. For the first identity, let $\omega_0 = \frac{\sqrt{-1}}{2}g_{i\bar{j}}dz^i\wedge d\bar{z}^j$ and let $\varphi = \varphi^i_{\bar{j}}d\bar{z}^j\otimes\frac{\partial}{\partial z_i}$. Then $\mathrm{div}_f\varphi = (\partial_i\varphi^i_{\bar{j}} + \varphi^i_{\bar{j}}\partial_i(\log g + f))d\bar{z}^j$. And

$$
\begin{aligned}
\overline{\partial}(\mathrm{div}_f\varphi) &= (\partial_{\bar{l}}\partial_i\varphi^i_{\bar{j}} + \partial_{\bar{l}}\varphi^i_{\bar{j}}\partial_i(\log g + f) + \varphi^i_{\bar{j}}\partial_{\bar{l}}\partial_i(\log g + f))d\bar{z}^l\wedge d\bar{z}^j \\
&= (\partial_i(\partial_{\bar{l}}\varphi^i_{\bar{j}}) + (\partial_{\bar{l}}\varphi^i_{\bar{j}})\partial_i(\log g + f) - \varphi^i_{\bar{j}}(R_{i\bar{l}} - f_{i\bar{l}}))d\bar{z}^l\wedge d\bar{z}^j \\
&= (\partial_i(\partial_{\bar{l}}\varphi^i_{\bar{j}}) + (\partial_{\bar{l}}\varphi^i_{\bar{j}})\partial_i(\log g + f) - \varphi^i_{\bar{j}}g_{i\bar{l}})d\bar{z}^l\wedge d\bar{z}^j \\
&= \mathrm{div}_f(\overline{\partial}\varphi) - 2\sqrt{-1}\varphi\lrcorner\omega_0.
\end{aligned} \tag{2.1.8}
$$

From the second step to the third step, we use the equation $R_{i\bar{j}} - f_{i\bar{j}} = g_{i\bar{j}}$.

For the second equation, we note

$$\frac{1}{2}[\varphi,\varphi] = \varphi^l_{\bar{k}}\partial_l\varphi^i_{\bar{j}}d\bar{z}^k\wedge d\bar{z}^j\otimes\frac{\partial}{\partial z_i},$$

so

$$
\begin{aligned}
\frac{1}{2}\mathrm{div}_f[\varphi,\varphi] &= [\partial_i(\varphi^l_{\bar{k}}\partial_l\varphi^i_{\bar{j}}) + \varphi^l_{\bar{k}}\partial_l\varphi^i_{\bar{j}}\partial_i(\log g + f)]d\bar{z}^k\wedge d\bar{z}^j \\
&= [(\partial_i\varphi^l_{\bar{k}})(\partial_l\varphi^i_{\bar{j}}) + \varphi^l_{\bar{k}}\partial_i\partial_l\varphi^i_{\bar{j}} + \varphi^l_{\bar{k}}\partial_l\varphi^i_{\bar{j}}\partial_i(\log g + f)]d\bar{z}^k\wedge d\bar{z}^j \\
&= \varphi^l_{\bar{k}}(\partial_i\partial_l\varphi^i_{\bar{j}} + \partial_l\varphi^i_{\bar{j}}\partial_i(\log g + f))d\bar{z}^k\wedge d\bar{z}^j \\
&= \varphi\lrcorner\partial(\mathrm{div}_f\varphi).
\end{aligned} \tag{2.1.9}
$$

□

Based on lemma 2.1.2, we can prove the $f-$divergence gauge implies the $f-$Kuranishi gauge.

Proposition 7. *Let (X_0,ω_0) be a compact Fano Kähler manifold with volume density $e^f\frac{\omega^n}{n!}$. Let φ be the Beltrami differential satisfying $\overline{\partial}\varphi = \frac{1}{2}[\varphi,\varphi]$ and $\mathrm{div_f}\varphi = 0$. Then $\varphi\lrcorner\omega_0 = 0$ and $\overline{\partial}^*_f\varphi = 0$.*

Proof.

$$
\begin{aligned}
0 &= \overline{\partial}(\mathrm{div}_f \varphi) \\
&= \mathrm{div}_f(\overline{\partial}\varphi) - 2\sqrt{-1}\varphi \lrcorner \omega_0 \\
&= \frac{1}{2}\mathrm{div}_f[\varphi, \varphi] - 2\sqrt{-1}\varphi \lrcorner \omega_0 \\
&= \varphi \lrcorner \partial(\mathrm{div}_f \varphi) - 2\sqrt{-1}\varphi \lrcorner \omega_0 \\
&= -2\sqrt{-1}\varphi \lrcorner \omega_0.
\end{aligned}
\tag{2.1.10}
$$

Thus, $\varphi \lrcorner \omega_0 = 0$, i.e, $\varphi^p_{\bar{j}} g_{p\bar{l}} = \varphi^p_{\bar{l}} g_{p\bar{j}}$.
Moreover, for $\varphi = \varphi^i_{\bar{j}} d\bar{z}^j \otimes \frac{\partial}{\partial z_i}$

$$
\begin{aligned}
\overline{\partial}^*_f \varphi =& - g^{k\bar{j}} \nabla_k \varphi^i_{\bar{j}} - \varphi \lrcorner \overline{\nabla} f \\
=& \nabla_k (g^{k\bar{j}} \varphi^i_{\bar{j}}) - \varphi \lrcorner \overline{\nabla} f \\
=& - \nabla_k (g^{i\bar{j}} \varphi^k_{\bar{j}}) - \varphi \lrcorner \overline{\nabla} f \\
&= -g^{i\bar{j}} \left[\nabla_k \varphi^k_{\bar{j}} + \varphi \lrcorner \partial f \right] = 0,
\end{aligned}
\tag{2.1.11}
$$

which leads to $\overline{\partial}^*_f \varphi = 0$. $\square$

Remark 7. *Under the symmetry $\varphi \lrcorner \omega_0 = 0$, we have $\mathrm{div}_f \varphi = 2\sqrt{-1}\overline{\partial}^*_f \varphi \lrcorner \omega_0$.*

Before we show the Kuranishi gauge implies the divergence gauge, we need the following lemmas.

Lemma 2.1.3. *Let (X_0, ω_0) be a compact Kähler manifold.*

(1) If $\varphi \in A^{0,1}(X_0, T^{1,0}X_0)$, $\psi \in A^{1,1}(X_0)$, then

$$\overline{\partial}(\varphi \lrcorner \psi) = \overline{\partial}\varphi \lrcorner \psi + \varphi \lrcorner \overline{\partial}\psi.$$

(2) If $\varphi \in A^{0,1}(X_0, T^{1,0}X_0)$, $\overline{\partial}^*_f \varphi = 0$, then

$$\overline{\partial}^*_f (\varphi \lrcorner \omega_0) = \frac{\sqrt{-1}}{2}\mathrm{div}_f \varphi.$$

(3) If $\overline{\partial}(\varphi \lrcorner \omega_0) = 0$, $\overline{\partial}_f^* \varphi = 0$, then

$$\Box_f(\varphi \lrcorner \omega_0) = \frac{\sqrt{-1}}{2}\mathrm{div}_f(\overline{\partial}\varphi) + \varphi \lrcorner (Ric(\omega_0) - \nabla\overline{\nabla} f),$$

where $\Box_f = \overline{\partial}\overline{\partial}_f^* + \overline{\partial}_f^* \overline{\partial}$ is the f-Hodge Laplacian.

(4) For φ, $\psi \in A^{0,1}(X_0, T^{1,0}X_0)$, we have

$$[\varphi, \psi] \lrcorner \omega_0 = \varphi \lrcorner \partial(\psi \lrcorner \omega_0) + \psi \lrcorner \partial(\varphi \lrcorner \omega_0).$$

Proof. (1) Let $\varphi = \varphi^i_{\bar{j}} d\bar{z}^j \otimes \frac{\partial}{\partial z_i}$, $\psi = \psi_{k\bar{l}} dz^k \wedge d\bar{z}^l$. Then

$$\varphi \lrcorner \psi = \varphi^i_{\bar{j}} \psi_{i\bar{l}} d\bar{z}^j \wedge d\bar{z}^l,$$

and

$$\overline{\partial}(\varphi \lrcorner \psi) = \left[\partial_{\overline{p}} \varphi^i_{\bar{j}} \psi_{i\bar{l}} + \varphi^i_{\bar{j}} \partial_{\overline{p}} \psi_{i\bar{l}}\right] d\overline{z}^p \wedge d\bar{z}^j \wedge d\bar{z}^l = \overline{\partial}\varphi \lrcorner \psi + \varphi \lrcorner \overline{\partial}\psi.$$

(2) We adopt the convention of the wedge product as $\eta \wedge \gamma = \frac{1}{\sqrt{2}}(\eta \otimes \gamma - \gamma \otimes \eta)$, then

$$\begin{aligned}
\overline{\partial}_f^*(\varphi \lrcorner \omega_0) =& \frac{\sqrt{-1}}{2}\big\{\partial_l[(\varphi^m_{\overline{p}} g_{m\bar{j}} - \varphi^m_{\bar{j}} g_{m\overline{p}}) g^{k\overline{p}}] g^{l\bar{j}} g_{k\bar{i}} + (\varphi^m_{\overline{p}} g_{m\bar{j}} - \varphi^m_{\bar{j}} g_{m\overline{p}}) g^{l\overline{p}} \partial_l f\big\} d\bar{z}^i \\
=& \frac{\sqrt{-1}}{2}\big\{\partial_l(\varphi^m_{\bar{i}} g_{m\bar{j}} - \varphi^m_{\bar{j}} g_{m\bar{i}}) g^{l\bar{j}} + (\varphi^m_{\overline{p}} g_{m\bar{j}} - \varphi^m_{\bar{j}} g_{m\overline{p}}) \partial_l g^{k\overline{p}} g^{l\bar{j}} g_{k\bar{i}} \\
& + (\varphi^m_{\overline{p}} g_{m\bar{j}} - \varphi^m_{\bar{j}} g_{m\overline{p}}) g^{l\overline{p}} \partial_l f\big\} d\bar{z}^i \\
=& \frac{\sqrt{-1}}{2}\big\{\partial_l(\varphi^m_{\bar{i}} g_{m\bar{j}}) g^{l\bar{j}} + \varphi^m_{\bar{i}} \partial_m f - \partial_l(\varphi^m_{\bar{j}} g_{m\bar{i}}) g^{l\bar{j}} - \varphi^m_{\bar{j}} g_{m\bar{i}} g^{l\bar{j}} \partial_l f \\
& + \varphi^m_{\overline{p}} \partial_m g^{k\overline{p}} g_{k\bar{i}} + \varphi^m_{\bar{j}} \partial_l g_{m\overline{p}} g^{k\overline{p}} g^{l\bar{j}} g_{k\bar{i}}\big\} d\bar{z}^i \\
=& \frac{\sqrt{-1}}{2}\mathrm{div}_f \varphi,
\end{aligned} \tag{2.1.12}$$

where the last equality comes from $\overline{\partial}_f^* \varphi = 0$.

(3)

$$
\begin{aligned}
\Box_f(\varphi\lrcorner\omega_0) &= (\overline{\partial}\overline{\partial}^*_f + \overline{\partial}^*_f\overline{\partial})(\varphi\lrcorner\omega_0)\\
&= \overline{\partial}\overline{\partial}^*_f(\varphi\lrcorner\omega_0) = \overline{\partial}(\frac{\sqrt{-1}}{2}div_f\varphi)\\
&= \frac{\sqrt{-1}}{2}\overline{\partial}[(\partial_i\varphi^i_{\bar{j}} + \varphi^i_{\bar{j}}\partial_i\log g + \varphi^i_{\bar{j}}\partial_i f)d\overline{z}^i]\\
&= \frac{\sqrt{-1}}{2}[\partial_{\bar{k}}(\partial_i\varphi^i_{\bar{j}}) + (\partial_{\bar{k}}\varphi^i_{\bar{j}})\partial_i(\log g + f) + \varphi^i_{\bar{j}}(\partial_{\bar{k}}\partial_i\log g + \partial_{\bar{k}}\partial_i f)]d\overline{z}^k\wedge d\overline{z}^j\\
&= \frac{\sqrt{-1}}{2}\mathrm{div}_f(\overline{\partial}\varphi) + \varphi\lrcorner(Ric(\omega_0) - \nabla\overline{\nabla}f).
\end{aligned}
\tag{2.1.13}
$$

(4) Let $\varphi = \varphi^i_{\bar{j}}d\overline{z}^j\otimes\partial_j$, $\psi = \psi^k_{\bar{l}}d\overline{z}^l\otimes\partial_k$, $\omega_0 = \frac{\sqrt{-1}}{2}g_{s\bar{t}}dz^s\wedge d\overline{z}^t$, then

$$
[\varphi,\psi] = [\varphi^i_{\bar{j}}\partial_i(\psi^k_{\bar{l}}) + \psi^i_{\bar{j}}\partial_i(\varphi^k_{\bar{l}})]d\overline{z}^j\wedge d\overline{z}^l\otimes\partial_k \tag{2.1.14}
$$

So,

$$
\begin{aligned}
[\varphi,\psi]\lrcorner\omega_0 &= \frac{\sqrt{-1}}{2}[\varphi^i_{\bar{j}}\partial_i(\psi^k_{\bar{l}}) + \psi^i_{\bar{j}}\partial_i(\varphi^k_{\bar{l}})]g_{k\bar{t}}d\overline{z}^j\wedge d\overline{z}^l\wedge d\overline{z}^t\\
&= \frac{\sqrt{-1}}{2}[\varphi^i_{\bar{j}}\partial_i(\psi^k_{\bar{l}})g_{k\bar{t}} + \varphi^i_{\bar{j}}\psi^k_{\bar{l}}\partial_i(g_{k\bar{t}}) + \psi^i_{\bar{j}}\partial_i(\varphi^k_{\bar{l}}g_{k\bar{t}}) + \psi^i_{\bar{j}}\varphi^k_{\bar{l}}\partial_i(g_{k\bar{t}})]d\overline{z}^j\wedge d\overline{z}^l\wedge d\overline{z}^t\\
&= \varphi\lrcorner\partial(\psi\lrcorner\omega_0) + \psi\lrcorner\partial(\varphi\lrcorner\omega_0).
\end{aligned}
\tag{2.1.15}
$$

□

Lemma 2.1.4. *Let (X_0,ω_0) be a compact Fano manifold. If $\mu\in A^{0,2}(X_0)$ satisfies $\overline{\partial}\mu = 0$ and $\Box_f\mu = \mu$, then $\mu = 0$.*

Proof. Let $\mu = \mu_{\bar{i}\bar{j}}d\overline{z}^i\wedge d\overline{z}^j$. Then the norm of μ is $|\mu|^2 = \mu_{\bar{i}\bar{j}}\overline{\mu_{\bar{k}\bar{l}}}(g^{k\bar{i}}g^{l\bar{j}} - g^{k\bar{j}}g^{i\bar{l}})$. Let the twisted Hodge Laplacian be $\Box_f = \Box - \overline{\partial}\circ i_{\overline{\nabla}f} - i_{\overline{\nabla}f}\circ\overline{\partial}$ and the $(1,0)$ connection Laplacian $\Delta_f = \Delta + \overline{\nabla}f\lrcorner\overline{\partial}$. Since $\overline{\partial}\mu = 0$, the twisted *Weitzenböck* formula for $(0,2)$-form μ reads as:

$$
\begin{aligned}
\Box_f\mu + \Delta_f\mu &= Ric\circ\mu + \mu\circ Ric - (\nabla\overline{\nabla}f\circ\mu + \mu\circ\nabla\overline{\nabla}f)\\
&= \omega_0\circ\mu + \mu\circ\omega_0.
\end{aligned}
\tag{2.1.16}
$$

By $\Box_f \mu = \mu$, we have

$$\Delta_f \mu = -\mu + \omega_0 \circ \mu + \mu \circ \omega_0. \tag{2.1.17}$$

We also have

$$\begin{aligned} -\int_X |\nabla \mu|^2 e^f dV &= \int_X < \Delta_f \mu, \mu > e^f dV \\ &= \int_X < -\mu + \omega_0 \circ \mu + \mu \circ \omega_0, \mu > e^f dV \\ &= \int_X |\mu|^2 e^f dV. \end{aligned} \tag{2.1.18}$$

Hence $\mu = 0$.

$\Box$

In particular, on a Fano Kähler-Einstein manifold, we also have the following vanishing result.

Corollary 2.1.1. *On a compact Fano Kähler-Einstein manifold* (X_0, ω_0)*, if* $\mu \in A^{0,2}(X_0)$ *satisfies* $\Box \mu = \mu$*, then* $\mu = 0$.

Proof. For the $(0,2)$ form $\mu = \mu_{\overline{k}\overline{l}} d\overline{z}^k \wedge d\overline{z}^l$, by the Weitzenböck identity and the Kähler-Einstein condition,

$$\begin{aligned} \Box \mu + \Delta \mu &= \sum_i R(e_i, e_{\bar{i}})\mu \\ &= \sum_{i,p} -R^{\overline{p}}_{i\bar{i}\overline{k}} \mu_{\overline{p}\bar{l}} - R^{\overline{p}}_{i\bar{i}\bar{l}} \mu_{\overline{k}\overline{p}} d\overline{z}^k \wedge d\overline{z}^l \\ &= \sum_{p,k,l} (R_{p\bar{k}} \mu_{\overline{p}\bar{l}} + R_{p\bar{l}} \mu_{\overline{k}\overline{p}}) d\overline{z}^k \wedge d\overline{z}^l \\ &= \sum_{p,k,l} (g_{p\bar{k}} \mu_{\overline{p}\bar{l}} + g_{p\bar{l}} \mu_{\overline{k}\overline{p}}) d\overline{z}^k \wedge d\overline{z}^l. \end{aligned} \tag{2.1.19}$$

Therefore,

$$\begin{aligned} |\mu|^2 + < \Delta \mu, \mu > &= < \Box \mu + \Delta \mu, \mu > \\ &= (g_{p\bar{k}} \mu_{\overline{p}\bar{l}} + g_{p\bar{l}} \mu_{\overline{k}\overline{p}}) \overline{\mu_{\overline{s}\bar{t}}} g^{s\bar{k}} g^{t\bar{l}} = 2|\mu|^2. \end{aligned} \tag{2.1.20}$$

and

$$-\int_{X_0} |\nabla\mu|^2 = \int_{X_0} < \Delta\mu, \mu >= \int_{X_0} |\mu|^2 \tag{2.1.21}$$

we conclude that $\mu = 0$. $\square$

In the following computation, for the sake of simplifying notations, we assume $B \subset \mathbb{C}$. For the case of $B \subset \mathbb{C}^k$, the computation can be carried out similarly.

Next, we will use induction to show that the $f-$Kuranishi gauge implies $f-$Divergence gauge.

Proposition 8. *Let (X_0, ω_0) be a compact Fano Kähler manifold, $\varphi(t) = \sum\limits_{i=1} t^i\varphi_i \in A^{0,1}(X_0, T^{1,0}X_0)$ is a family of Beltrami differentials satisfying*

$$\begin{cases} \overline{\partial}\varphi_i & = \frac{1}{2}\sum_{j=1}^{i}[\varphi_j, \varphi_{i-j}] \\ \overline{\partial}_f^*\varphi_i & = 0 \end{cases} \tag{2.1.22}$$

for all $i \geq 2$, and φ_1 is harmonic with respect to $\square_f$-Laplacian. Then $\mathrm{div}_f\varphi_i = 0$ *and $\varphi_i \lrcorner\omega_0 = 0$ for all $i \geq 1$.*

Proof. When $i = 1$, $\square_f\varphi_1 = 0$ implies $\overline{\partial}\varphi_1 = 0$, and $\overline{\partial}_f^*\varphi_1 = 0$.

By Lemma 2.1.3, $\overline{\partial}(\varphi_1\lrcorner\omega_0) = \overline{\partial}\varphi_1\lrcorner\omega_0 + \varphi_1\lrcorner\overline{\partial}\omega_0 = 0$, and

$$\square_f(\varphi_1\lrcorner\omega_0) = \frac{\sqrt{-1}}{2}\mathrm{div_f}(\overline{\partial}\varphi_1) + \varphi_1\lrcorner(\mathrm{Ric}(\omega_0) - \nabla\overline{\nabla}\mathrm{f}) = \varphi_1\lrcorner\omega_0.$$

By Lemma 2.1.4, $\varphi_1\lrcorner\omega_0 = 0$, locally, $\varphi^i_{\bar{j}}g_{i\bar{l}} = \varphi^i_{\bar{l}}g_{i\bar{j}}$.

Hence,

$$\mathrm{div}_f\varphi = 2\sqrt{-1}\overline{\partial}_f^*\varphi\lrcorner\omega_0 = 0. \tag{2.1.23}$$

Now assume for $k \leq i-1$, we have $\varphi_k\lrcorner\omega_0 = 0$ and $\mathrm{div}_f\varphi_k = 0$.

Then,

$$\begin{aligned} \overline{\partial}(\varphi_i\lrcorner\omega_0) &= \overline{\partial}\varphi_i\lrcorner\omega_0 \\ &= \frac{1}{2}\sum_{j=1}^{i-1}[\varphi_j, \varphi_{i-j}]\lrcorner\omega_0 \\ &= \sum_{j=1}^{i-1}\varphi_j\lrcorner\partial(\varphi_{i-j}\lrcorner\omega_0) \\ &= 0. \end{aligned} \tag{2.1.24}$$

Therefore, we get

$$
\begin{aligned}
\Box_f(\varphi_i \lrcorner \omega_0) &= \frac{\sqrt{-1}}{2}\operatorname{div}(\overline{\partial}\varphi_i) + \varphi_i \lrcorner (Ric(\omega_0) - \nabla\overline{\nabla} f) \\
&= \frac{\sqrt{-1}}{4}\operatorname{div}(\sum_{j=1}^{i-1}[\varphi_j, \varphi_{i-j}]) + \varphi_i \lrcorner \omega_0 \\
&= \frac{\sqrt{-1}}{2}\sum_{j=1}^{i-1}\varphi_{i-j} \lrcorner \partial(\operatorname{div}\varphi_{\mathrm{i}}) + \varphi_{\mathrm{i}} \lrcorner \omega_0 \\
&= \varphi_i \lrcorner \omega_0.
\end{aligned}
\tag{2.1.25}
$$

By Lemma 2.1.4, $\varphi_i \lrcorner \omega_0 = 0$ and $\operatorname{div}_f \varphi_i = 2\sqrt{-1}\overline{\partial}_f^* \varphi_i \lrcorner \omega_0 = 0$. $\Box$

By proposition 7 and 8, we have proved

Theorem 2.1.1. *On a compact Fano Kähler manifold (X_0, ω_0), if the Beltrami differential $\varphi \in A^{0,1}(X_0, T^{1,0}X_0)$ satisfies $\overline{\partial}\varphi = \frac{1}{2}[\varphi, \varphi]$, then*

$$\overline{\partial}_f^* \varphi = 0 \textit{ is equivalent to } \operatorname{div}_f \varphi = 0.$$

Furthermore, $\varphi \lrcorner \omega_0 = 0$ when either one of these conditions is imposed.

On a Fano Kähler-Einstein manifold, we simply take $f = 0$ to obtain

Corollary 2.1.2. *On a Fano Kähler-Einstein manifolds (X_0, ω_0), if the Beltrami differential $\varphi \in A^{0,1}(X_0, T^{1,0}X_0)$ satisfies $\overline{\partial}\varphi = \frac{1}{2}[\varphi, \varphi]$ then*

$$\overline{\partial}^* \varphi = 0 \textit{ is equivalent to } \operatorname{div}\varphi = 0.$$

Furthermore, $\varphi \lrcorner \omega_0 = 0$ when either one of these conditions is imposed.

Remark 8. *In the paper [13], G. Schumacher used the method of harmonic lift to obtain $\varphi_1 \lrcorner \omega_0 = 0$ on the Kähler-Einstein manifold. We point out that in the paper [14], Siu gave the proof of the existence of harmonic lifting vector fields on the total space of the family of the Kähler-Einstein manifolds of general types. Siu's harmonic lift method has the following properties: Let $\pi : \mathscr{X} \to B$ be an analytic family of complex manifolds. Take a local holomorphic coordinate $(t^1, \cdots, t^n)$ at a* *regular point $t \in B$, and let $(z^1, \cdots, z^n)$ be holomorphic coordinates on $X_t = \pi^{-1}(t)$. For the holomorphic vector fields $\{\frac{\partial}{\partial z^1}, \cdots, \frac{\partial}{\partial z^n}\}$, there exist vector fields on the total space $\{v_1, \cdots, v_n\}$, such that*

1. $\pi(v_i) = \frac{\partial}{\partial t_i}$.

2. *$\overline{\partial}_t v_i$ is harmonic respectively on the fiber manifold.*

Having a harmonic lift is the same as having a canonical smooth trivialization on the total space of the family of deformation manifolds.

2.2 Deformation of the volume form and the Kähler-Einstein form

In this section, we will study the (infinitesimal) deformation of Kähler-Einstein metrics and its volume form under the deformation of the complex structures on the Fano Kähler-Einstein manifold (X_0, ω_0). Our main result is

Theorem 2.2.1. *Let $\pi : \mathscr{X} \to B = \{t \in \mathbb{C} : |t| < \varepsilon\}$ be an analytic family of Fano Kähler-Einstein manifolds. Suppose $H^0(X_0, \mathcal{O}(T^{1,0}X_0)) = 0$. Then the volume form on the nearby fiber $X_t = \pi^{-1}(t)$ is given by*

$$dV_t = \left[1 - |t|^2 \Delta((\Delta + 1)^{-1}|\varphi_1|^2) + O(|t|^3)\right] dV_0,$$

and the deformed Kähler form is

$$\omega_t = \omega_0 - |t|^2 \Big(\frac{\sqrt{-1}}{2} \partial\overline{\partial}(\Delta + 1)^{-1}|\varphi_1|^2\Big) + O(|t|^3),$$

where Δ is the Beltrami-Laplacian on X_0, ∂, $\overline{\partial}$ are operators on X_0, where $\varphi_1 = \mathbb{H}(\varphi) \in H^{0,1}(X_0, T^{1,0}X_0)$. The Beltrami differential φ satisfies

$$\begin{cases} \overline{\partial}\varphi = \frac{1}{2}[\varphi, \varphi] \\ \mathrm{div}\varphi = 0 \\ \varphi(0) = 0. \end{cases} \tag{2.2.1}$$

According to Futaki's result, on a Fano Kähler manifold, the space of holomorphic vector fields is isomorphic to the 1-eigenspace of the twisted Laplacian. This is precisely stated as follows.

Proposition 9. *(Futaki [7]) Let (X_0, ω_0) be a Fano Kähler manifold, such that $R_{i\bar{j}} = g_{i\bar{j}} + \nabla_i \nabla_{\bar{j}} f$. Let $\Delta_f u = \Delta u + \nabla_{\nabla f} u$. Let $T_1 = \{u \in C^\infty(X_0) | \Delta_f u + u = 0\}$, $T_2 = H^0(X_0, \mathcal{O}(T^{1,0}X_0))$. Then $T_1 \cong T_2$.*

For completeness, we include the proof here.

Proof. For all $u \in T_1$,

$$\begin{aligned}
\int_{X_0} |\nabla\nabla u|^2 e^f dV &= \int_X (\nabla_j \nabla_i u \nabla_{\bar{j}} \nabla_{\bar{i}} u) e^f dV \\
&= -\int_{X_0} \nabla_i u \nabla_j (\nabla_{\bar{j}} \nabla_{\bar{i}} u e^f) dV \\
&= -\int_{X_0} \nabla_i u (\nabla_j \nabla_{\bar{j}} \nabla_{\bar{i}} u + \nabla_{\bar{j}} \nabla_{\bar{i}} u \nabla_j f) e^f dV \\
&= -\int_{X_0} \nabla_j u (\nabla_{\bar{i}} \nabla_j \nabla_{\bar{j}} u + R_{j\bar{i}} \nabla_{\bar{j}} u + \nabla_{\bar{j}} \nabla_{\bar{i}} u \nabla_j f) e^f dV \\
&= -\int_{X_0} \nabla_j u \big(\nabla_{\bar{i}} \nabla_j \nabla_{\bar{j}} u + (g_{j\bar{i}} + \nabla_j \nabla_{\bar{i}} f) \nabla_{\bar{j}} u + \nabla_{\bar{j}} \nabla_{\bar{i}} u \nabla_j f\big) e^f dV \\
&= -\int_{X_0} \big[(\Delta_f u)_{\bar{i}} u_i + |\nabla u|^2\big] e^f dV \\
&= -\int_{X_0} (-|\nabla u|^2 + |\nabla u|^2) e^f dV = 0.
\end{aligned} \tag{2.2.2}$$

Therefore, ∇u is a holomorphic vector field; i.e. $\nabla u \in T_2$. And $\nabla : T_1 \to T_2$ is a well-defined operator.

Conversely, let $W \in T_2$. Locally, $W = W^i \frac{\partial}{\partial z^i}$. W is holomorphic; i.e. $\nabla_{\bar{j}} W^i = 0$

for all $j = 1, \cdots, n$, $\mathrm{divW} = \nabla_{\mathrm{i}}\mathrm{W}^{\mathrm{i}}$ and $\mathrm{div_f W} = \mathrm{divW} + \mathrm{W^i}\nabla_{\mathrm{i}}\mathrm{f}$.

$$\begin{aligned}
\Delta_f(\mathrm{div_f W}) =&\Delta(\mathrm{divW} + \mathrm{W^i}\nabla_{\mathrm{i}}\mathrm{f}) + < \nabla \mathrm{f}, \nabla(\mathrm{divW} + \mathrm{W^i}\nabla_{\mathrm{i}}\mathrm{f}) > \\
=&\Delta(\nabla_i W^i + W^i \nabla_i f) + \nabla_j f \nabla_{\bar{j}}(\nabla_i W^i + W^i \nabla_i f) \\
=&\nabla_j \nabla_{\bar{j}} \nabla_i W^i + \nabla_j \nabla_{\bar{j}}(W^i \nabla_i f) + \nabla_j f \nabla_{\bar{j}} \nabla_i W^i + \nabla_j f \nabla_{\bar{j}}(W^i \nabla_i f) \\
=&\nabla_j(\nabla_i \nabla_{\bar{j}} W^i - R_{i\bar{j}} W^i) + W^i \nabla_j \nabla_{\bar{j}} \nabla_i f + \nabla_j W^i \nabla_{\bar{j}} \nabla_i f \\
&+ \nabla_j f(\nabla_i \nabla_{\bar{j}} W^i - R_{i\bar{j}} W^i) + \nabla_j f \nabla_{\bar{j}} \nabla_i f W^i \\
=& - \nabla_j \left[(g_{i\bar{j}} + \nabla_{\bar{j}} \nabla_i f) W^i\right] + W^i \nabla_j \nabla_{\bar{j}} \nabla_i f + \nabla_j W^i \nabla_{\bar{j}} \nabla_i f \\
&- \nabla_j f \left[(g_{i\bar{j}} + \nabla_{\bar{j}} \nabla_i f) W^i\right] + \nabla_j f \nabla_{\bar{j}} \nabla_i f W^i \\
=& - \nabla_i W^i - W^i \nabla_i f \\
=& - \mathrm{div}_f W.
\end{aligned} \tag{2.2.3}$$

Hence, $div : T_2 \to T_1$ is a well-defined operator.

Now, suppose $W = W^i \frac{\partial}{\partial z^i}$ is a holomorphic vector field. Then $\nabla_{\bar{k}} W^i = 0$ for $k = 1, \cdots, n$.

$$\begin{aligned}
\nabla_{\bar{k}} \mathrm{div_f W} &= \nabla_{\bar{k}}(\nabla_i W^i + W^i \nabla_i f) \\
&= \nabla_{\bar{k}} \nabla_i W^i + W^i \nabla_{\bar{k}} \nabla_i f \\
&= -R_{i\bar{k}} W^i + W^i \nabla_{\bar{k}} \nabla_i f = -g_{i\bar{k}} W^i.
\end{aligned} \tag{2.2.4}$$

We conclude $\nabla \mathrm{div}$ is injective.

On the other hand, if u satisfies $\Delta_f u + u = 0$, then

$$\mathrm{div}_f(\nabla u) = \partial_i u^i + u^i \partial_i \log \det g + u^i \partial_i f = \Delta_f u = -f.$$

So $\mathrm{div_f}\nabla$ is also injective.

Therefore, $T_1 \cong T_2$. □

Taking $f = 0$, and the above argument gives

Corollary 2.2.1. *Let (X_0, ω_0) be a Fano Kähler-Einstein manifold, let $T_1 = \{u \in C^\infty(X_0) | \Delta u + u = 0\}$, and $T_2 = H^0(X_0, T^{1,0}X_0)$. Then $T_1 \cong T_2$.*

By this corollary, we know if there are no holomorphic vector fields on the Fano Kähler-Einstein manifold, then the operator $\Delta + 1$ is invertible.

Next we give the proof of the main result of this section.

Proof. (of Theorem 2.2.1)

Let $(z_1, \cdots, z_n)$ be local holomorphic coordinates on the central fiber (X_0, ω_0). Then

$$\Omega^{1,0}(X_0) = span\{dz^1, \cdots, dz^n\}.$$

Let $\{e^1, \cdots, e^n\}$ be local holomorphic frames on (X_t, ω_t) obtained by the deformation of complex structure. Then by the Kodaira-Spencer's theory, $e^i = dz^i + \varphi^i_{\bar{j}} d\bar{z}^j$ and

$$\Omega^{1,0}(X_t) = span\{e^1, \cdots, e^n\}.$$

Let $(w_1, \cdots, w_n)$ be local holomorphic coordinates on (X_t, ω_t). Then

$$\begin{aligned} dw^\alpha &= \frac{\partial w_\alpha}{\partial z_i} dz^i + \frac{\partial w_\alpha}{\partial \bar{z}_i} d\bar{z}^i \\ &= \frac{\partial w_\alpha}{\partial z_i} dz^i + \varphi^j_{\bar{i}} \frac{\partial w_\alpha}{\partial z_j} d\bar{z}^i \\ &= \frac{\partial w_\alpha}{\partial z_i}(dz^i + \varphi^i_{\bar{j}} d\bar{z}^j) = \frac{\partial w_\alpha}{\partial z_i} e^i. \end{aligned} \tag{2.2.5}$$

Let $A = (a_{\alpha i})_{n\times n} = (\frac{\partial w_\alpha}{\partial z_i})_{n\times n}$, $|A| = \det A$, $c_n = (-1)^{\frac{n(n-1)}{2}}(\frac{\sqrt{-1}}{2})^n$, and if g_0 is the Kähler metric on the central fiber X_0. Then

$$dV_0 = c_n \det g_0 dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n \tag{2.2.6}$$

Let

$$\begin{aligned} d\tilde{V_t} &= c_n \det g_0 e^1 \wedge \cdots \wedge e^n \wedge \bar{e}^1 \wedge \cdots \wedge \bar{e}^n \\ &= det(I - \varphi\bar{\varphi}) dV_0. \end{aligned} \tag{2.2.7}$$

There is a unique function $f = f(z, \bar{z}, t, \bar{t}) \in C^\infty(X_0 \times B)$ with $f(z, \bar{z}, 0) = 0$ such that the volume form on the deformed manifold X_t is given by

$$\begin{aligned} dV_t &= e^f d\tilde{V_t} = e^f \det(I - \varphi\bar{\varphi}) dV_0 \\ &= c_n e^f \det g_0 |A|^{-2} dw^1 \wedge \cdots \wedge dw^n \wedge d\overline{w}^1 \cdots \wedge d\overline{w}^n \\ &= c_n \det g_t dw^1 \wedge \cdots \wedge dw^n \wedge d\overline{w}^1 \cdots \wedge d\overline{w}^n, \end{aligned} \tag{2.2.8}$$

where g_t is the Kähler metric on the deformed manifold X_t. Hence,

$$detg_t = e^f \det g_0 |A|^{-2}. \tag{2.2.9}$$

On the other hand, as we can see, all terms related to t in the deformed volume form are contained in $e^f \det(I - \varphi\overline{\varphi})$. The expansion of φ about t comes from the power series solution to the deformation equation of the complex structure. To figure out the expansion of e^f about t, we study the deformed $Monge - Ampère$ equation:

$$(-\frac{\partial^2}{\partial w_\alpha \partial \overline{w}_\beta} \log dV_t)^n = dV_t. \tag{2.2.10}$$

Since

$$\frac{\partial}{\partial w_\alpha} = \frac{\partial z_i}{\partial w_\alpha}\frac{\partial}{\partial z_i} + \frac{\partial \overline{z}_i}{\partial w_\alpha}\frac{\partial}{\partial \overline{z}_i}, \tag{2.2.11}$$

and

$$\begin{pmatrix} \frac{\partial z_i}{\partial w_\alpha} & \frac{\partial \overline{z}_i}{\partial w_\alpha} \\ \frac{\partial z_i}{\partial \overline{w}_\alpha} & \frac{\partial \overline{z}_i}{\partial \overline{w}_\alpha} \end{pmatrix} = \begin{pmatrix} \frac{\partial w_\alpha}{\partial z_i} & \frac{\partial \overline{w}_\alpha}{\partial \overline{z}_i} \\ \frac{\partial w_\alpha}{\partial \overline{z}_i} & \frac{\partial \overline{w}_\alpha}{\partial \overline{z}_i} \end{pmatrix}^{-1}.$$

So, $\frac{\partial z_i}{\partial w_\alpha} = (\frac{\partial w_\alpha}{\partial z_j})^{-1}(I - \varphi\overline{\varphi})^{ij}$, and $\frac{\partial \overline{z}_i}{\partial w_\alpha} = -(\frac{\partial w_\alpha}{\partial z_j})^{-1}(I - \varphi\overline{\varphi})^{kj}\overline{\varphi_{\overline{k}}^{i}}$. We get

$$\frac{\partial}{\partial w_\alpha} = (\frac{\partial w_\alpha}{\partial z_j})^{-1}(I - \varphi\overline{\varphi})^{ij}(\frac{\partial}{\partial z_i} - \overline{\varphi_{\overline{i}}^{k}}\frac{\partial}{\partial \overline{z}_k}). \tag{2.2.12}$$

Now, let $a_{i\alpha} = \frac{\partial w_\alpha}{\partial z_i}$, $b^{i\alpha} = (\frac{\partial w_\alpha}{\partial z_i})^{-1}$, we define the operator

$$T = T_t : C^\infty(X_0) \to A^{1,0}(X_0)$$

by $T(f) = \partial f - \overline{\varphi(t)} \lrcorner \overline{\partial} f$. Locally, T is given by $T(f) = \sum_{i=1}^{n} T_i(f) dz^i$ and $T_i f = \partial_i f - \overline{\varphi_{\overline{i}}^{j}} \partial_{\overline{j}} f$.

From (2.2.9),

$$\log\det g_t = f + \log g_0 - \log A - \log \overline{A}.$$

Using $\operatorname{div}\varphi = 0$, we compute,

$$T_i \log A = b^{i\alpha}(I - \overline{\varphi}\varphi)_{il}\partial_j a_{l\alpha} + \overline{\varphi_{\overline{i}}^{k}}\varphi_{\overline{k}}^{l}\partial_l \log g_0. \tag{2.2.13}$$

And

$$\overline{T_i} \log A = -\varphi_{\overline{i}}^{k}\partial_k \log g_0. \tag{2.2.14}$$

Then

$$\frac{\partial}{\partial w_\alpha}\log\det g_t = \frac{\partial z_i}{\partial w_\alpha}T_i(f+\log g_0) - b^{j\alpha}b^{k\alpha}\partial_j a_{k\alpha} - \frac{\partial z_i}{\partial w_\alpha}\big[\overline{\varphi_{\bar i}^k}\varphi_{\bar k}^l\partial_l\log g_0 - \overline{\varphi_{\bar i}^k}\partial_{\bar k}\log g_0\big]. \tag{2.2.15}$$

Now, we compute

$$\frac{\partial}{\partial \overline{w}_\beta}\big(\frac{\partial}{\partial w_\alpha}\log\det g_t\big) = \frac{\partial \overline{z}_p}{\partial \overline{w}_\beta}T_{\overline{p}}\Big\{\frac{\partial z_i}{\partial w_\alpha}T_i(f+\log g_0) - b^{j\alpha}b^{k\alpha}\partial_j a_{k\alpha} - \frac{\partial z_i}{\partial w_\alpha}\big[\overline{\varphi_{\bar i}^k}\varphi_{\bar k}^l\partial_l\log g_0 - \overline{\varphi_{\bar i}^k}\partial_{\bar k}\log g_0\big]\Big\} \tag{2.2.16}$$

Firstly, using Kähler-Einstein condition, we obtain

$$\begin{aligned} T_{\overline{p}}\big(\frac{\partial z_i}{\partial w_\alpha}T_i(f+\log g_0)\big) =& T_{\overline{p}}(b^{j\alpha})(I-\varphi\overline{\varphi})^{ij}T_i(f) + b^{j\alpha}T_{\overline{p}}(I-\varphi\overline{\varphi})^{ij}T_i(f) \\ &+ b^{j\alpha}(I-\varphi\overline{\varphi})^{ij}[-g_{i\overline{p}} - \varphi_{\overline{p}}^q\overline{\varphi_{\bar i}^k}g_{q\bar k} + T_{\overline{p}}T_i f]. \end{aligned} \tag{2.2.17}$$

Secondly,

$$T_{\overline{p}}(b^{k\alpha}b^{j\alpha}\partial_j a_{k\alpha}) = 0. \tag{2.2.18}$$

Thirdly,

$$T_{\overline{p}}\big[\frac{\partial z_i}{\partial w_\alpha}(\overline{\varphi_{\bar i}^k}\varphi_{\bar k}^l\partial_l\log g_0 - \overline{\varphi_{\bar i}^k}\partial_{\bar k}\log g_0)\big] = -\frac{\partial z_i}{\partial w_\alpha}\big[\overline{\varphi_{\bar i}^k}\varphi_{\bar k}^l g_{l\overline{p}} + \varphi_{\overline{p}}^q\overline{\varphi_{\bar i}^k}g_{g\bar k}\big]. \tag{2.2.19}$$

Finally, we obtain

$$-\frac{\partial^2}{\partial w_\alpha\partial\overline{w}_\beta}\log\det g_t = \frac{\partial\overline{z}_p}{\partial\overline{w}_\beta}b^{j\alpha}\big[g_{j\overline{p}} + \partial_j(\varphi_{\overline{p}}^k)(I-\varphi\overline{\varphi})^{ik}T_i(f) - T_{\overline{p}}[(I-\varphi\overline{\varphi})^{ij}T_i(f)]\big]. \tag{2.2.20}$$

We define a local matrix $B = (B_{j\overline{p}})_{n\times n}$ to be

$$B_{j\overline{p}} = g_{j\overline{p}} + (\partial_j\varphi_{\overline{p}}^k)(I-\varphi\overline{\varphi})^{ik}T_i(f) - T_{\overline{p}}[(I-\varphi\overline{\varphi})^{ij}T_i(f)]. \tag{2.2.21}$$

Then the deformed $Monge - Ampère$ equation can be written as

$$\det B_{j\overline{p}} = e^f\det g_0\det(I-\varphi\overline{\varphi}) \tag{2.2.22}$$

Notice that $B_{j\overline{p}}(0) = g_{j\overline{p}}$, $f(0) = 0$, and $\varphi(0) = 0$, so

$$\frac{\partial B_{j\overline{p}}}{\partial t}\big|_{t=0} = -\partial_j\partial_{\overline{p}}\big(\frac{\partial f}{\partial t}\big|_{t=0}\big). \tag{2.2.23}$$

By equation (2.2.22),

$$(\Delta+1)\Big(\frac{\partial f}{\partial t}\Big|_{t=0}\Big)=0. \tag{2.2.24}$$

By corollary 2.2.1, the operator $\Delta+1$ has trivial kernel, which yields $\frac{\partial f}{\partial t}|_{t=0}=0$. Similarly, $\frac{\partial f}{\partial \bar{t}}\big|_{t=0}=0$, and all the first order terms vanish.

For the second order, first we have

$$\frac{\partial^2 B_{j\overline{p}}}{\partial t^2}\Big|_{t=0}=-\partial_j\partial_{\overline{p}}\Big(\frac{\partial^2 f}{\partial t^2}\Big|_{t=0}\Big), \tag{2.2.25}$$

and

$$(\Delta+1)\Big(\frac{\partial^2 f}{\partial t^2}\Big|_{t=0}\Big)=0. \tag{2.2.26}$$

Hence, $\frac{\partial^2 f}{\partial t^2}|_{t=0}=0$. Similarly, $\frac{\partial^2 f}{\partial \bar{t}^2}|_{t=0}=0$.

For the mixed derivative term, we have

$$\frac{\partial B_{j\overline{p}}}{\partial t\partial\bar{t}}\Big|_{t=0}=-\partial_j\partial_{\overline{p}}\Big(\frac{\partial f}{\partial t\partial\bar{t}}\Big|_{t=0}\Big), \tag{2.2.27}$$

and

$$(\Delta+1)\Big(\frac{\partial^2 f}{\partial t\partial\bar{t}}\Big|_{t=0}\Big)=\frac{\partial\varphi}{\partial t}\Big|_{t=0}\overline{\frac{\partial\varphi}{\partial t}}\Big|_{t=0}=|\varphi_1|^2. \tag{2.2.28}$$

$\Delta+1$ is invertible, so

$$\frac{\partial^2 f}{\partial t\partial\bar{t}}\Big|_{t=0}=(\Delta+1)^{-1}|\varphi_1|^2. \tag{2.2.29}$$

Up to the second order, we obtain the expansion for f:

$$f=|t|^2(\Delta+1)^{-1}|\varphi_1|^2+O(|t|^3), \tag{2.2.30}$$

and

$$e^f=1+|t|^2(\Delta+1)^{-1}|\varphi_1|^2+O(|t|^3). \tag{2.2.31}$$

On the other hand, by the deformation equation of the complex structures,

$$\det(I-\varphi\overline{\varphi})=1-|t|^2|\varphi_1|^2+O(|t|^3). \tag{2.2.32}$$

Hence, the expansion of the volume form can be written as (up to the second order),

$$dV_t=(1-|t|^2\Delta(\Delta+1)^{-1}|\varphi_1|^2+O(|t|^3))dV_0. \tag{2.2.33}$$

Finally, we calculate the Taylor expansion of the Kähler form.

$$
\begin{aligned}
\omega_t = Ric(\omega_t) &= -\frac{\sqrt{-1}}{2}\partial_t\overline{\partial}_t \log dV_t \\
&= -\frac{\sqrt{-1}}{2}\frac{\partial^2}{\partial w_\alpha \partial \overline{w}_\beta}(f + \log\det(I-\varphi\overline{\varphi}) + \log\det g_0) dw^\alpha \wedge d\overline{w}^\beta.
\end{aligned} \tag{2.2.34}
$$

Using the above Taylor expansion of f and $\det(I-\varphi\overline{\varphi})$ and

$$
\begin{cases}
dw^\alpha = \frac{\partial w_\alpha}{\partial z_i}(dz^i + \varphi^i_{\overline{j}} d\overline{z}^j) \\
\frac{\partial}{\partial w^\alpha} = \frac{\partial z_i}{\partial w_\alpha} T_i = (\frac{\partial w_\alpha}{\partial z_i})^{-1}(I-\varphi\overline{\varphi})^{ik}(\frac{\partial}{\partial z_i} - \overline{\varphi^j_{\overline{i}}}\frac{\partial}{\partial \overline{z}_j}),
\end{cases} \tag{2.2.35}
$$

we obtain

$$
\omega_t = \omega_0 - |t|^2(\frac{\sqrt{-1}}{2}\partial\overline{\partial}(\Delta+1)^{-1}|\varphi_1|^2) + O(|t|^3). \tag{2.2.36}
$$

Where Δ, ∂, $\overline{\partial}$ are operators on the central fiber (X_0, ω_0).

$\square$

2.3 Deformation of plurianticanonical sections

We assume that the analytic family of Fano Kähler-Einstein manifolds is given by $\pi : \mathscr{X} \to B \subset \mathbb{C}$, and $\sigma : K^{-1}_{\mathscr{X}/B} \to \mathscr{X}$ is the relative anticanonical line bundle on the total space. For any $t \in B$, $K^{-1}_{\mathscr{X}/B}|_t \cong K^{-1}_{X_t}$. For each $m \geq 1$, the direct image sheaf is $R^0\pi_*(K^{-m}_{\mathscr{X}/B}) \to B$. Let $E = \bigcup\limits_{t\in B} E_t \times \{t\}$, where $E_t = H^0(X_t, K^{-m}_{X_t})$. It is well-known that $R^0\pi_*(K^{-m}_{\mathscr{X}/B})$ is isomorphic to E.

In this section, our goal is to establish an L^2-metric on $R^0\pi_*(K^{-m}_{\mathscr{X}/B})$. But, first of all, we will study the holomorphic sections of $K^{-m}_{X_t} \to X_t$ in terms of the deformation of the complex structure.

On the central fiber $\pi^{-1}(0) = (X_0, \omega_0)$ of an analytic family of Fano Kähler-Einstein manifolds $\pi : \mathscr{X} \to B$, let $(z_1, \cdots, z_n)$ be local holomorphic coordinates. Then locally $\Omega^{1,0}(X_0) = span\{dz^1, \cdots, dz^n\}$ and $T^{1,0}X_0 = span\{\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_n}\}$. By the deformation theory, $\Omega^{1,0}(X_t) = span\{e^1, \cdots, e^n\}$ where $e^i = dz^i + \varphi^i_{\overline{j}} d\overline{z}^j$, and $T^{1,0}X_t = span\{e_1, \cdots, e_n\}$ where $e_i = (I-\varphi\overline{\varphi})^{ki}(\frac{\partial}{\partial z_k} - \overline{\varphi^l_{\overline{k}}}\frac{\partial}{\partial \overline{z}_l})$. Furthermore, if

$(w_1, \cdots, w_n)$ are local holomorphic coordinates on X_t, by the formula (2.2.5) and (2.2.12) ,

$$\begin{cases} dw^\alpha = \frac{\partial w_\alpha}{\partial z_i}(dz^i + \varphi^i_{\bar{j}} d\bar{z}^j) = \frac{\partial w_\alpha}{\partial z_i} e^i \\ \frac{\partial}{\partial w^\alpha} = \frac{\partial z_i}{\partial w_\alpha} T_i = (\frac{\partial w_\alpha}{\partial z_i})^{-1}(I - \varphi\overline{\varphi})^{ik}(\partial_i - \overline{\varphi^j_{\bar{i}}}\partial_{\bar{j}}) = (\frac{\partial w_\alpha}{\partial z_i})^{-1} e_i. \end{cases} \tag{2.3.1}$$

Let $s \in A^0(X_0, K^{-m}_{X_0})$ be a smooth section of the plurianticanonical line bundle on the central manifold; locally, $s = \eta(z)(\frac{\partial}{\partial z_1} \wedge \cdots \frac{\partial}{\partial z_n})^m$. Let the deformed section on $K^{-m}_{X_t}$ be given by $s(t) = \eta(z)(e_1 \wedge \cdots \wedge e_n)^m = \eta(z)|A|^m(\frac{\partial}{\partial w_1} \wedge \cdots \wedge \frac{\partial}{\partial w_n})^m$ where $|A| = \det(\frac{\partial w_\alpha}{\partial z_i})$.

Lemma 2.3.1. *Choosing the divergence gauge, i.e.* $\mathrm{div}\varphi = 0$ *for* $\varphi \in A^{0,1}(X_0, T^{1,0}X_0)$, *we have* $s(t) \in H^0(X_t, K^{-m}_{X_t})$ *if and only if*

$$\overline{\partial} s = \varphi \lrcorner \nabla s. \tag{2.3.2}$$

Here, ∇ *is the connection on* $K^{-m}_{X_0} \to X_0$ *and* $\overline{\partial}$ *is the operator on the central fiber* (X_0, ω_0).

Proof. Let $\overline{\partial}_t = \frac{\partial}{\partial \overline{w}_\alpha} \otimes d\overline{w}_\alpha$. Then $s(t) \in H^0(X_t, K^{-m}_{X_t})$ if and only if $\overline{\partial}_t s(t) = 0$ i.e. $\overline{\frac{\partial z_i}{\partial w_\alpha}}\overline{T_i} s(t) = 0$. If $\overline{\frac{\partial z_i}{\partial w_\alpha}}\overline{T_i} s(t) = 0$, then

$$\begin{aligned} 0 = \overline{T_i} s(t) &= (\partial_{\bar{i}} - \varphi^j_{\bar{i}}\partial_j) s(t) = (\partial_{\bar{i}} - \varphi^j_{\bar{i}}\partial_j)\big[\eta(z)|A|^m\big] \\ &= (\partial_{\bar{i}}\eta(z) - \varphi^j_{\bar{i}}\partial_j\eta(z))|A|^m + \eta(z)\big[m|A|^m\partial_{\bar{i}}\log|A| - m|A|^m\varphi^j_{\bar{i}}\partial_j\log|A|\big]. \end{aligned} \tag{2.3.3}$$

Because of $\mathrm{div}\varphi = 0$, we have

$$\begin{aligned} 0 &= (\partial_{\bar{i}}\eta(z) - \varphi^j_{\bar{i}}\partial_j\eta(z)) + m\eta(z)[b^{k\alpha}\partial_{\bar{i}}a_{\alpha k} - b^{k\alpha}\varphi^j_{\bar{i}}\partial_j a_{\alpha k}] \\ &= (\partial_{\bar{i}}\eta(z) - \varphi^j_{\bar{i}}\partial_j\eta(z)) + m\eta(z)[b^{k\alpha}\partial_k(\varphi^j_{\bar{i}}a_{\alpha j}) - b^{k\alpha}\varphi^j_{\bar{i}}\partial_j a_{\alpha k}] \\ &= (\partial_{\bar{i}}\eta(z) - \varphi^j_{\bar{i}}\partial_j\eta(z)) + m\eta(z)\partial_j\varphi^j_{\bar{i}} \\ &= \partial_{\bar{i}}\eta(z) - \varphi^j_{\bar{i}}\partial_j\eta(z) - \eta(z)\varphi^j_{\bar{i}}\partial_j\log g_0^m \\ &= \partial_{\bar{i}}\eta(z) - \varphi^j_{\bar{i}}\nabla_j s. \end{aligned} \tag{2.3.4}$$

Hence, $\overline{\partial} s - \varphi \lrcorner \nabla s = 0$. Conversely, if we trace back the above identities, we can see that $\overline{\partial} s - \varphi \lrcorner \nabla s = 0$ implies $\overline{\partial}_t s(t) = 0$.

□

Equation (2.3.2) is the obstruction equation of deformations of holomorphic sections. Next, we will show equation (2.3.2) is solvable based on Hodge theory.

Lemma 2.3.2. *Let (X, ω_g) be a compact Hermitian manifold. For $\alpha \in A^{p,q}(X)$, $\beta \in A^{p,q+1}(X)$, $\overline{\partial}\alpha = \beta$ is solvable if and only if $\overline{\partial}\beta = 0$ and $\mathbb{H}(\beta) = 0$.*

Proof. If $\overline{\partial}\alpha = \beta$, then $\overline{\partial}\beta = \overline{\partial}^2\alpha = 0$ and $\mathbb{H}(\beta) = \mathbb{H}\overline{\partial}\alpha = 0$.

Conversely, by Hodge decomposition, if $\mathbb{H}\beta = 0$ and $\overline{\partial}\beta = 0$, we have

$$\beta = \mathbb{H}\beta + \overline{\partial}\overline{\partial}^*G\beta + \overline{\partial}^*\overline{\partial}G\beta = \overline{\partial}\overline{\partial}^*G\beta, \tag{2.3.5}$$

and $\alpha = \overline{\partial}^*G\beta$ is such a solution. $\square$

Lemma 2.3.3. *If (X_0, ω_0) is a Fano manifold, then for $\varphi \in A^{0,1}(X_0, T^{1,0}X_0)$ and $s \in A^0(X_0, K_{X_0}^{-m})$, we have $\mathbb{H}(\varphi \lrcorner \nabla s) = 0$*

Proof. (X_0, ω_0) is Fano, so $K_{X_0}^{-m} \to X_0$ is ample. By the Kodaira vanishing theorem,

$$H^q(X_0, K_{X_0}^{-m}) = H^q(X_0, K_{X_0}^{-m-1} \otimes K_{X_0}) = 0 \tag{2.3.6}$$

for all $q \geq 1$. This yields to $\mathbb{H}(\varphi \lrcorner \nabla s) = 0$. $\square$

Next, we will show $\overline{\partial}(\varphi \lrcorner \nabla s) = 0$ by the iteration method. First, we have

Lemma 2.3.4. *On a Kähler manifold (X_0, ω_0), for $\varphi \in A^{0,1}(X_0, T^{1,0}X_0)$ and $s \in A^0(X_0, K_{X_0}^{-m})$, we have*

(1) $\overline{\partial}(\varphi \lrcorner \nabla s) = \overline{\partial}\varphi \lrcorner \nabla s + \varphi \lrcorner \overline{\partial}(\nabla s)$.

(2) $\overline{\partial}(\nabla s) = -\nabla(\overline{\partial}s) - 2\sqrt{-1}\, mRic_0 \otimes s$.

Proof. For (1), let $\varphi = \varphi^i_{\bar{j}} d\bar{z}^j \otimes \frac{\partial}{\partial z_i}$, and $s = \eta(z)e^m$ where $e = \frac{\partial}{\partial z_1} \wedge \cdots \frac{\partial}{\partial z_n}$. Then,

$$\nabla s = (\partial_k \eta + \eta \partial_k \log g_0^m) dz^k \otimes e^m,$$

$$\varphi \lrcorner \nabla s = \varphi^i_{\bar{j}}(\partial_i \eta + \eta \partial_i \log g_0^m) d\bar{z}^j \otimes e^m,$$

and

$$\overline{\partial}(\nabla s) = -[\partial_{\bar{l}}\partial_k\eta + \partial_{\bar{l}}\eta\partial_k \log g_0^m + \eta\partial_{\bar{l}}\partial_k \log g_0^m]dz^k \wedge d\overline{z}^l \otimes e^m.$$

So,

$$\begin{aligned}\overline{\partial}(\varphi\lrcorner\nabla s) &= [\partial_{\overline{k}}\varphi^i_{\bar{j}}(\partial_i\eta + \eta\partial_i \log g_0^m) + \varphi^i_{\bar{j}}(\partial_{\overline{k}}\partial_i\eta + \partial_{\overline{k}}\eta\partial_i \log g_0^m + \eta\partial_{\overline{k}}\partial_i \log g_0^m)]d\overline{z}^k \wedge d\overline{z}^j \otimes e^m \\ &= \overline{\partial}\varphi\lrcorner\nabla s + \varphi\lrcorner\overline{\partial}(\nabla s).\end{aligned}$$

For (2),

$$\overline{\partial}(\nabla s) = (\partial_{\bar{i}}\partial_k\eta + \partial_{\bar{i}}\eta\partial_k \log g_0^m + \eta\partial_{\bar{i}}\partial_k \log g_0^m)d\overline{z}^i \wedge dz^k \otimes e^m. \tag{2.3.7}$$

$$\begin{aligned}\nabla(\overline{\partial}s) &= (\partial_k\partial_{\bar{i}}\eta + \partial_{\bar{i}}\eta\partial_k \log g_0^m)dz^k \wedge d\overline{z}^i \otimes e^m \\ &= -(\partial_k\partial_{\bar{i}}\eta + \partial_{\bar{i}}\eta\partial_k \log g_0^m)d\overline{z}^i \wedge dz^k \otimes e^m.\end{aligned} \tag{2.3.8}$$

Therefore,

$$\overline{\partial}(\nabla s) = -\nabla(\overline{\partial}s) - 2\sqrt{-1}mRic_0 \otimes s.$$

$\square$

Now let $\varphi = \sum_{i=1}^{\infty} t^i\varphi_i$ with $\varphi_1 \in H^1(X_0, T^{1,0}X_0)$, and let $s = \sum_{i=0}^{\infty} t^i s_i$ with s_0 holomorphic. Then $\varphi\lrcorner\nabla s = \sum_{k=1}^{\infty} t^k(\varphi\lrcorner\nabla s)_k$, where $(\varphi\lrcorner\nabla s)_k = \sum_{i=1}^{k} \varphi_i\lrcorner\nabla s_{k-i}$. We point out there is no zeroth order term.

For the first order term,

Lemma 2.3.5. *On a Fano Kähler-Einstein manifold, with* $\mathrm{div}\varphi = 0$*, we have*

$$\overline{\partial}(\varphi_1\lrcorner\nabla s_0) = 0. \tag{2.3.9}$$

Proof. φ_1 is harmonic, s_0 is holomorphic and $\mathrm{div}\varphi = 0$, so

$$\begin{aligned}\overline{\partial}(\varphi_1\lrcorner\nabla s_0) &= \overline{\partial}\varphi_1\lrcorner\nabla s_0 + \varphi_1\lrcorner\overline{\partial}(\nabla s_0) = \varphi_1\lrcorner\overline{\partial}(\nabla s_0) \\ &= \varphi_1\lrcorner(-\nabla\overline{\partial}s_0 - 2\sqrt{-1}\, mRic_0 \otimes s_0)) \\ &= -\varphi_1\lrcorner(2\sqrt{-1}\, m\omega_0 \otimes s_0) \\ &= -2\sqrt{-1}\, m(\varphi_1\lrcorner\omega_0) \otimes s_0 = 0.\end{aligned} \tag{2.3.10}$$

The last equality is from $\varphi\lrcorner\omega_0 = 0$ on a Fano Kähler-Einstein manifold by Corollary 2.1.2. $\square$

To compute higher order terms, firstly, we note the following Lemma.

Lemma 2.3.6. *On a Kähler manifold (X_0, ω_0), for φ, $\psi \in A^{0,1}(X_0, T^{1,0}X_0)$, $s \in A^0(X_0, K_{X_0}^{-m})$, we have*

$$\varphi \lrcorner \nabla(\psi \lrcorner \nabla s) + \psi \lrcorner (\varphi \lrcorner \nabla s) = [\varphi, \psi] \lrcorner \nabla s. \tag{2.3.11}$$

Proof. Locally, $\varphi = \varphi^i_{\bar{j}} d\bar{z}^j \otimes \frac{\partial}{\partial z_i}$, $\psi = \psi^k_{\bar{l}} d\bar{z}^l \otimes \frac{\partial}{\partial z_k}$, $s = \eta(z)e^m$, so that

$$\nabla s = (\partial_p \eta + \eta \partial_p \log g_0^m) dz^p \otimes e^m$$

$$\psi \lrcorner \nabla s = \psi^k_{\bar{l}} (\partial_k \eta + \eta \partial_k \log g_0^m) d\bar{z}^l \otimes e^m$$

$$\nabla(\psi \lrcorner \nabla s) = \left[\partial_p[\psi^k_{\bar{l}}(\partial_k \eta + \eta \partial_k \log g_0^m)] + \psi^k_{\bar{l}}(\partial_k \eta + \eta \partial_k \log g_0^m)\partial_p \log g_0^m\right] dz^p \wedge d\bar{z}^l \otimes e^m$$

$$\begin{aligned}\varphi \lrcorner \nabla(\psi \lrcorner \nabla s) =& \left[\varphi^i_{\bar{j}} \partial_i(\psi^k_{\bar{l}}(\partial_k \eta + \eta \partial_k \log g_0^m)) + \varphi^i_{\bar{j}} \psi^k_{\bar{l}}(\partial_k \eta + \eta \partial_k \log g_0^m)\partial_i \log g_0^m\right] d\bar{z}^j \wedge d\bar{z}^l \otimes e^m \\ =& \left[\varphi^i_{\bar{j}} (\partial_i \psi^k_{\bar{l}})(\partial_k \eta + \eta \partial_k \log g_0^m)) + \varphi^i_{\bar{j}} \psi^k_{\bar{l}}(\partial_i \partial_k \eta + \partial_i \eta \partial_k \log g_0^m)\right. \\ &\left. + \varphi^i_{\bar{j}} \psi^k_{\bar{l}}(\partial_k \eta + \eta \partial_k \log g_0^m)\partial_i \log g_0^m\right] d\bar{z}^j \wedge d\bar{z}^l \otimes e^m\end{aligned} \tag{2.3.12}$$

and

$$\begin{aligned}\psi \lrcorner \nabla(\varphi \lrcorner \nabla s) =& \left[\psi^i_{\bar{j}} \partial_i(\varphi^k_{\bar{l}}(\partial_k \eta + \eta \partial_k \log g_0^m)) + \psi^i_{\bar{j}} \varphi^k_{\bar{l}}(\partial_k \eta + \eta \partial_k \log g_0^m)\partial_i \log g^m\right] d\bar{z}^j \wedge d\bar{z}^l \otimes e^m \\ =& \left[\psi^i_{\bar{j}} (\partial_i \varphi^k_{\bar{l}})(\partial_k \eta + \eta \partial_k \log g_0^m)) + \psi^i_{\bar{j}} \varphi^k_{\bar{l}}(\partial_i \partial_k \eta + \partial_i \eta \partial_k \log g_0^m)\right. \\ &\left. + \psi^i_{\bar{j}} \varphi^k_{\bar{l}}(\partial_k \eta + \eta \partial_k \log g_0^m)\partial_i \log g_0^m\right] d\bar{z}^j \wedge d\bar{z}^l \otimes e^m.\end{aligned} \tag{2.3.13}$$

Therefore,

$$\begin{aligned}&\varphi \lrcorner \nabla(\psi \lrcorner \nabla s) + \psi \lrcorner \nabla(\varphi \lrcorner \nabla s) \\ =&[\varphi^i_{\bar{j}}(\partial_i \psi^k_{\bar{l}}) + \psi^i_{\bar{j}}(\partial_i \varphi^k_{\bar{l}})](\partial_k \eta + \eta \partial_k \log g_0^m) d\bar{z}^j \wedge d\bar{z}^l \otimes e^m \\ &+ [\varphi^i_{\bar{j}} \psi^k_{\bar{l}}(\partial_i \eta \partial_k \log g_0^m - \partial_k \eta \partial_i \log g_0^m) + \varphi^i_{\bar{j}} \psi^k_{\bar{l}}(\partial_k \eta \partial_i \log g_0^m - \partial_i \eta \partial_k \log g_0^m)] d\bar{z}^j \wedge d\bar{z}^l \otimes e^m \\ =&[\varphi, \psi] \lrcorner \nabla s.\end{aligned} \tag{2.3.14}$$

$\square$

Lemma 2.3.7. *When $k \geq 2$, for the k-th order term, we have $\overline{\partial}(\sum\limits_{i=1}^{k} \varphi_i \lrcorner \nabla s_{k-i}) = 0$.*

Proof.

$$\begin{aligned}
\overline{\partial}(\sum_{i=1}^{k} \varphi_i \lrcorner \nabla s_{k-i}) &= \sum_{i=1}^{k} \overline{\partial}(\varphi_i \lrcorner \nabla s_{k-i}) \\
&= \sum_{i=1}^{k} \big(\overline{\partial}\varphi_i \lrcorner \nabla s_{k-i} + \varphi_i \lrcorner \overline{\partial}(\nabla s_{k-i})\big) \\
&= \sum_{i=1}^{k} \big(\overline{\partial}\varphi_i \lrcorner \nabla s_{k-i} - \varphi_i \lrcorner \nabla(\overline{\partial} s_{k-i}) - \varphi_i \lrcorner 2\sqrt{-1}\, m\omega_0 \otimes s_{k-i}\big) \\
&= \sum_{i=1}^{k} \big(\overline{\partial}\varphi_i \lrcorner \nabla s_{k-i} - \varphi_i \lrcorner \nabla(\sum_{j=1}^{k-i} \varphi_j \lrcorner \nabla s_{k-i-j})\big) \\
&= \sum_{i=1}^{k} \big(\overline{\partial}\varphi_i - \frac{1}{2}\sum_{j=1}^{i}[\varphi_j, \varphi_{i-j}]\big) \lrcorner \nabla s_{k-i} = 0.
\end{aligned} \tag{2.3.15}$$

The last equality is from the deformation equation. □

Based on Lemma 2.3.5 and Lemma 2.3.7, we have proved

Lemma 2.3.8. *On the Fano Kähler-Einstein manifold, for $\varphi \in A^{0,1}(X_0, T^{1,0}X_0)$, $s \in A^0(X_0, K_{X_0}^{-m})$ and $div\varphi = 0$, we have $\overline{\partial}(\varphi \lrcorner \nabla s) = 0$.*

Proposition 10. *The obstruction equation $\overline{\partial} s = \varphi \lrcorner \nabla s$ for $s \in \Gamma(X_t, K_{X_t}^{-m})$ is solvable.*

Proof. This follows from lemma 2.3.2, lemma 2.3.3 and lemma 2.3.8. □

Next, we construct the power series solution to $\overline{\partial} s = \varphi \lrcorner \nabla s$.

Theorem 2.3.1. *Let $s_0 \in H^0(X_0, K_{X_0}^{-m})$. For $|t|$ small enough, there exists a unique convergent power series $s(t) = \sum\limits_{i=0}^{\infty} t^i s_i \in A^0(X_0, K_{X_0}^{-m})$ such that, $s(0) = s_0$, $\mathbb{H}(s_i) = 0$ for $i \geq 1$ and $s(t)$ satisfies $\overline{\partial} s = \varphi \lrcorner \nabla s$.*

Proof. Let G be the Green's operator on $K_{X_0}^{-m} \to X_0$ associate to the Hodge Laplacian $\Box$. We will construct s_i for $i \geq 1$ so that $s(t) = s_0 + \sum_{i=0}^{\infty} t^i s_i$ is the solution to $\overline{\partial} s = \varphi \lrcorner \nabla s$.

By Hodge theory,

$$\begin{aligned} s &= s_0 + \overline{\partial}\,\overline{\partial}^* G s + \overline{\partial}^* \overline{\partial} G s \\ &= s_0 + \overline{\partial}^* \overline{\partial} G s \\ &= s_0 + \overline{\partial}^* G(\varphi \lrcorner \nabla s). \end{aligned} \tag{2.3.16}$$

Comparing coefficients up to t^k, we obtain a formal power series solution as follows:

$$\begin{cases} s_1 = \overline{\partial}^* G(\varphi_1 \lrcorner \nabla s_0) \\ s_2 = \overline{\partial}^* G(\varphi_1 \lrcorner \nabla s_1 + \varphi_2 \lrcorner \nabla s_0) \\ \quad \vdots \\ s_k = \overline{\partial}^* G(\sum_{i=1}^{k} \varphi_i \lrcorner \nabla s_{k-i}) \\ \quad \vdots \end{cases} \tag{2.3.17}$$

Next, we show the formal power series is convergent in the *Hölder* Space $C^{k,\alpha}$ if $|t|$

is sufficiently small.

$$\begin{aligned}
\|s_i\|_{k+\alpha} &= \|\overline{\partial}^* G(\sum_{j=1}^{i} \varphi_j \lrcorner \nabla s_{i-j})\|_{k+\alpha} \\
&\leq C\|G\sum_{j=1}^{i}(\varphi_j \lrcorner \nabla s_{i-j})\|_{k+\alpha+1} \\
&\leq C\|\sum_{j=1}^{i}(\varphi_j \lrcorner \nabla s_{i-j})\|_{k+\alpha-1} \\
&\leq C\sum_{j=1}^{i}\|\varphi_j \lrcorner \nabla s_{i-j}\|_{k+\alpha-1} \\
&\leq C\sum_{j=1}^{i}\|\varphi_j\|_{k+\alpha-1}\|\nabla s_{i-j}\|_{k+\alpha-1} \\
&\leq C\sum_{j=1}^{i}\|\varphi_j\|_{k+\alpha}\|s_{i-j}\|_{k+\alpha}.
\end{aligned} \tag{2.3.18}$$

There exist $C_1 > 0$ and $0 < \epsilon_1 < \epsilon$, such that $\epsilon_1^j\|\varphi_j\|_{k+\alpha} \leq C_1$. Then,

$$\|s_i\|_{k+\alpha} \leq CC_1(CC_1+1)^{i-1}\epsilon_1^{-i}.$$

Therefore, $s(t)$ converges if $|t| < \frac{\epsilon_1}{CC_1+1}$. □

2.4 L^2-metric on the direct image sheaf

Let $\pi : \mathscr{X} \to B$ be an analytic family of Fano Kähler-Einstein manifolds, and $\sigma : K^{-1}_{\mathscr{X}/B} \to \mathscr{X}$ is the relative anticanonical line bundle over the total space. For any $t \in B$, $K^{-1}_{\mathscr{X}/B}|_t \cong K^{-1}_{X_t}$. For each $m \geq 1$, the direct image sheaf is $R^0\pi_*(K^{-m}_{\mathscr{X}/B}) \to B$. Let $E = \bigcup_{t\in B} E_t \times \{t\}$, where $E_t = H^0(X_t, K^{-m}_{X_t})$. It is known that $R^0\pi_*(K^{-m}_{\mathscr{X}/B})$ is isomorphic to E.

Take a basis $\{S_0^1, \cdots, S_0^{N_m}\} \in H^0(X_0, K^{-m}_{X_0})$, where $N_m = dimH^0(X_0, K^{-m}_{X_0})$.

Let $S^\alpha(t) = S_0^\alpha + \sum_{i=1}^{\infty} t^i S_i^\alpha \in A^0(X_0, K^{-m}_{X_0})$ satisfying $\overline{\partial} S^\alpha(t) = \varphi \lrcorner \nabla S^\alpha(t)$, where $S_0^\alpha \in H^0(X_0, K^{-m}_{X_0})$, $\alpha = 1, \ldots N_m$. Define $\sigma_t : A^0(X_0, K^{-m}_{X_0}) \to A^0(X_t, K^{-m}_{X_t})$ to be

$$\sigma_t(S(t)) = [\det(I - \varphi\overline{\varphi})^{-1} e^{-\overline{\varphi}(t)} \lrcorner (S(t))^{\frac{1}{m}}]^m.$$

Then, σ_t is a well-defined linear isomorphism. Locally, if $S(t)=\eta(z)(\frac{\partial}{\partial z_1}\wedge\cdots\wedge\frac{\partial}{\partial z_n})^m$, then $\sigma_t\big(S(t)\big)=\eta(z)(e_1\wedge\cdots\wedge e_n)^m$. Moreover, $\sigma_t(S^\alpha(t))\in H^0(X_t,K_{X_t}^{-m})$ for $\alpha=1,\dots N_m$. The pointwise L^2-metric on $R^0\pi_*(K_{\mathscr{X}/B}^{-m})\to B$ is defined by $<\sigma_t(S^\alpha(t)),\sigma_t(S^\beta(t))>_{g_t^m}$, where g_t is the determinant of the metric on X_t.

Definition 2.4.1. *The L^2-metric on $E_t=H^0(X_t,K_{X_t}^{-m})$ is defined to be*

$$h_{\alpha\overline{\beta}}(t)=\int_{X_t}<\sigma_t(S^\alpha(t)),\sigma_t(S^\beta(t))>_{g_t^m}dV_t. \tag{2.4.1}$$

Firstly, we will derive the Taylor expansion of the L^2 metric about t, from which the curvature tensor follows immediately.

Lemma 2.4.1.

$$h_{\alpha\overline{\beta}}(t)=\int_{X_0}<S^\alpha(t),S^\beta(t)>_{g_0^m}e^{(m+1)f}\det(I-\varphi\overline{\varphi})dV_0. \tag{2.4.2}$$

Proof. Let $S^\alpha=\eta^\alpha(z)(\frac{\partial}{\partial z_1}\wedge\cdots\wedge\frac{\partial}{\partial z_n})^m$, $S^\beta=\eta^\beta(z)(\frac{\partial}{\partial z_1}\wedge\cdots\wedge\frac{\partial}{\partial z_n})^m$, $\sigma_t(S^\gamma(t))=\eta^\gamma(z)(e_1\wedge\cdots\wedge e_n)^m$ with $e_i=(I-\varphi\overline{\varphi})^{ik}(\frac{\partial}{\partial z_k}-\overline{\varphi_{\overline{k}}^{l}}\frac{\partial}{\partial\overline{z}_l})$. Then

$$e_1\wedge\cdots\wedge e_n\wedge\overline{e}_1\wedge\cdots\wedge\overline{e}_n=\det(I-\varphi\overline{\varphi})^{-1}(\frac{\partial}{\partial z_1}\wedge\cdots\wedge\frac{\partial}{\partial z_n}\wedge\frac{\partial}{\partial\overline{z}_1}\wedge\cdots\wedge\frac{\partial}{\partial\overline{z}_n}), \tag{2.4.3}$$

and

$$dV_t=e^f d\tilde{V_t}=e^f\det(I-\varphi\overline{\varphi})dV_0. \tag{2.4.4}$$

So,

$$<\sigma_t(S^\alpha(t)),\sigma_t(S^\beta(t))>_{g_t^m}=e^{mf}<S^\alpha(t),S^\beta(t)>_{g_0^m}. \tag{2.4.5}$$

And,

$$h_{\alpha\beta}(t)=\int_{X_0}<S^\alpha(t),S^\beta(t)>_{g_0^m}e^{(m+1)f}\det(I-\varphi\overline{\varphi})dV_0. \tag{2.4.6}$$

□

Lemma 2.4.2. *For $\varphi\in A^{0,1}(X_0,T^{1,0}X_0)$, $s\in A^0(X_0,K_{X_0}^{-m})$, we have*

$$\mathrm{div}(\varphi\otimes \mathrm{s})=(\mathrm{div}\varphi)\otimes\mathrm{s}+\varphi\lrcorner\nabla\mathrm{s}. \tag{2.4.7}$$

Proof. Locally, $\varphi = \varphi^i_{\bar{j}} d\bar{z}^j \otimes \partial_i$, $s = \eta(z) e^m$. So $\nabla s = (\partial_k \eta + \eta \partial_k \log g_0^m) dz^k \otimes e^m$ and

$$\begin{aligned} \varphi \lrcorner \nabla s &= \varphi^k_{\bar{j}} (\partial_k \eta + \eta \partial_k \log g_0^m) d\bar{z}^j \otimes e^m \\ &= [\partial_k(\varphi^k_{\bar{j}} \eta) - \eta \partial_k \varphi^k_{\bar{j}} + \eta \varphi^k_{\bar{j}} \partial_k \log g_0^m] d\bar{z}^j \otimes e^m \\ &= [\partial_k(\varphi^k_{\bar{j}} \eta) + \eta \varphi^k_{\bar{j}} \partial_k \log g_0^{m+1} - \eta \varphi^k_{\bar{j}} \partial_k \log g_0 - \eta \partial_k \varphi^k_{\bar{j}}] d\bar{z}^j \otimes e^m \\ &= \mathrm{div}(\varphi \otimes \mathrm{s}) - (\mathrm{div}\varphi) \otimes \mathrm{s}. \end{aligned} \tag{2.4.8}$$

□

Lemma 2.4.3. *If $\eta = \eta_{\bar{k}}\, d\bar{z}^k \otimes e^m \in A^{0,1}(X_0, K^{-m}_{X_0})$ satisfies $\overline{\partial}\eta = 0$, then*

$$\mathrm{div}^* \eta = -\partial_{\bar{p}} (\eta_{\bar{l}} g^{q\bar{l}}) d\bar{z}^p \otimes \partial_q \otimes e^m \tag{2.4.9}$$

and $\overline{\partial}(\mathrm{div}^* \eta) = 0$.

Proof. $\mathrm{div} : \mathrm{A}^{0,1}(\mathrm{X}_0, \mathrm{T}^{1,0}\mathrm{X}_0 \otimes \mathrm{K}^{-\mathrm{m}}_{\mathrm{X}_0}) \to \mathrm{A}^{0,1}(\mathrm{X}_0, \mathrm{K}^{-\mathrm{m}}_{\mathrm{X}_0})$ and $\mathrm{div}^* : \mathrm{A}^{0,1}(\mathrm{X}_0, \mathrm{K}^{-\mathrm{m}}_{\mathrm{X}_0}) \to \mathrm{A}^{0,1}(\mathrm{X}_0, \mathrm{T}^{1,0}\mathrm{X}_0 \otimes \mathrm{K}^{-\mathrm{m}}_{\mathrm{X}_0})$. First, we find a local expression for div^*.

Let $\psi = \psi^i_{\bar{j}} d\bar{z}^j \otimes \partial_i \otimes e^m$ and $\eta = \eta_{\bar{k}} d\bar{z}^k \otimes e^m$. Then,
$\mathrm{div}\psi = (\partial_i \psi^i_{\bar{j}} + \psi^i_{\bar{j}} \partial_i \log g_0^{m+1}) d\bar{z}^j \otimes e^m$. Let $\mathrm{div}^* \eta = A^l_{\bar{k}} d\bar{z}^k \otimes \partial_l \otimes e^m$.

By definition,

$$\int_{X_0} < \mathrm{div}^* \eta, \psi > dV_0 = \int_{X_0} < \eta, \mathrm{div}\psi > dV_0, \tag{2.4.10}$$

$$\begin{aligned} L.H.S. &= \int_{X_0} < A^l_{\bar{k}} d\bar{z}^k \otimes \partial_l \otimes e^m, \psi^i_{\bar{j}} d\bar{z}^j \otimes \partial_i \otimes e^m > dV_0 \\ &= \int_X A^l_{\bar{k}} \overline{\psi^i_{\bar{j}}} g^{j\bar{k}} g_{l\bar{i}} g_0^m dV_0, \\ R.H.S. &= \int_{X_0} < \eta_{\bar{k}} d\bar{z}^k \otimes e^m, (\partial_i \psi^i_{\bar{j}} + \psi^i_{\bar{j}} \partial_i \log g_0^{m+1}) d\bar{z}^j \otimes e^m > dV_0 \\ &= \int_{X_0} \eta_{\bar{k}} \overline{(\partial_i \psi^i_{\bar{j}} + \psi^i_{\bar{j}} \partial_i \log g_0^{m+1})} g^{j\bar{k}} g_0^m dV_0. \end{aligned}$$

So,

$$\begin{aligned}\int_{X_0} A^l_{\bar{k}}\overline{\psi^i_{\bar{j}}} g^{j\bar{k}} g_{l\bar{i}} g_0^m dV_0 &= \int_{X_0} \eta_{\bar{k}}(\partial_{\bar{i}}\overline{\psi^i_{\bar{j}}} + \overline{\psi^i_{\bar{j}}}\partial_{\bar{i}} \log g_0^{m+1}) g^{j\bar{k}} g_0^m dV_0 \\ &= -\int_{X_0} \partial_{\bar{i}}\big(\eta_{\bar{k}} g^{j\bar{k}} g_0^{m+1}\big) g_0^{-1}\overline{\psi^i_{\bar{j}}} dV_0 + \int_{X_0} \big(\partial_{\bar{i}} \log g_0^{m+1}\big) g^{j\bar{k}} g_0^m \eta_{\bar{k}}\overline{\psi^i_{\bar{j}}} dV_0.\end{aligned} \tag{2.4.11}$$

It follows that

$$\begin{aligned}A^l_{\bar{k}} g^{j\bar{k}} g_{l\bar{i}} g_0^m &= \eta_{\bar{k}}(\partial_{\bar{i}} \log g_0^{m+1}) g^{j\bar{k}} g_0^m - \partial_{\bar{i}}(\eta_{\bar{k}} g^{j\bar{k}} g_0^{m+1}) g_0^{-1} \\ &= \eta_{\bar{k}} g^{j\bar{k}} g_0^m \partial_{\bar{i}} \log g_0^{m+1} - \partial_{\bar{i}}(\eta_{\bar{k}} g^{j\bar{k}}) g_0^m - \eta_{\bar{k}} g^{j\bar{k}} \partial_{\bar{i}} g_0^{m+1} g_0^{-m-1} g_0^m \\ &= -\partial_{\bar{i}}(\eta_{\bar{k}} g^{j\bar{k}}) g_0^m.\end{aligned} \tag{2.4.12}$$

By $\overline{\partial}\eta = 0$, we have $\partial_{\overline{p}}\eta_{\bar{i}} = \partial_{\bar{i}}\eta_{\overline{p}}$. So,

$$\begin{aligned}A^l_{\bar{k}} g^{j\bar{k}} g_{l\bar{i}} &= -\partial_{\bar{i}}(\eta_{\bar{k}} g^{j\bar{k}}) \\ A^l_{\overline{p}} g_{l\bar{i}} &= -\partial_{\bar{i}}\eta_{\overline{p}} - \eta_{\bar{k}} g_{j\overline{p}} \partial_{\bar{i}} g^{j\bar{k}} \\ A^q_{\overline{p}} &= -g^{q\bar{i}}\partial_{\bar{i}}\eta_{\overline{p}} - g^{q\bar{i}}\eta_{\bar{k}} g_{j\overline{p}} \partial_{\bar{i}} g^{j\bar{k}} \\ &= -g^{q\bar{i}}\partial_{\overline{p}}\eta_{\bar{i}} + \eta_{\bar{k}} g^{q\bar{i}} \partial_{\overline{p}} g_{j\bar{i}} g^{j\bar{k}} \\ &= -g^{q\bar{i}}\partial_{\overline{p}}\eta_{\bar{i}} - \eta_{\bar{i}}\partial_{\overline{p}} g^{q\bar{i}} = -\partial_{\overline{p}}(\eta_{\bar{i}} g^{q\bar{i}}),\end{aligned}$$

and we get

$$\mathrm{div}^*\eta = -\partial_{\overline{p}}(\eta_{\bar{i}} g^{q\bar{i}}) d\bar{z}^p \otimes \partial_q \otimes e^m, \tag{2.4.13}$$

and,

$$\begin{aligned}\overline{\partial}(\mathrm{div}^*\eta) &= [-\partial_{\bar{j}}\partial_{\bar{k}}(\eta_{\bar{i}} g^{q\bar{i}})] d\bar{z}^j \wedge d\bar{z}^k \otimes \partial_q \otimes e^m \\ &= [\partial_{\bar{j}}\partial_{\bar{k}}(\eta_{\bar{i}} g^{q\bar{i}})] d\bar{z}^j \wedge d\bar{z}^k \otimes \partial_q \otimes e^m.\end{aligned} \tag{2.4.14}$$

The symmetry leads to $\overline{\partial}(\mathrm{div}^*\eta) = 0$. □

Lemma 2.4.4. *For $\eta \in A^{0,1}(X_0, K_{X_0}^{-m})$ and $\overline{\partial}\eta = 0$, we have*

$$\Box\mathrm{div}^*\eta - \mathrm{div}^*\Box\eta = -(\mathrm{m}+1)\mathrm{div}^*\eta. \tag{2.4.15}$$

Proof. Let $\eta = f_{\bar{i}} d\bar{z}^i \otimes e^m$. Then

$$\overline{\partial}^* \eta = -[g^{j\bar{i}} \partial_j f_{\bar{i}} + g^{j\bar{i}} f_{\bar{i}} \partial_j \log g_0^m] e^m,$$

and

$$\overline{\partial}\overline{\partial}^* \eta = -\partial_{\overline{k}} [g^{j\bar{i}} \partial_j f_{\bar{i}} + g^{j\bar{i}} f_{\bar{i}} \partial_j \log g_0^m] d\bar{z}^k \otimes e^m.$$

Because $\overline{\partial}\eta = 0$,

$$\Box \eta = \overline{\partial}\overline{\partial}^* \eta = -\partial_{\overline{k}} [g^{j\bar{i}} \partial_j f_{\bar{i}} + g^{j\bar{i}} f_{\bar{i}} \partial_j \log g_0^m] d\bar{z}^k \otimes e^m.$$

And

$$\operatorname{div}^*(\Box \eta) = \partial_{\overline{k}} \big[\partial_{\overline{p}} (g^{j\bar{i}} \partial_j f_{\bar{i}} + g^{j\bar{i}} f_{\bar{i}} \partial_j \log g_0^m) g^{j\overline{p}} \big] d\bar{z}^k \otimes \partial_i \otimes e^m \tag{2.4.16}$$

On the other hand,

$$\operatorname{div}^* \eta = -\partial_{\overline{p}} (f_{\bar{i}} g^{q\bar{i}}) d\bar{z}^p \otimes \partial_q \otimes e^m := \varphi^q_{\overline{p}} d\bar{z}^p \otimes \partial_q \otimes e^m. \tag{2.4.17}$$

And $\overline{\partial}(\operatorname{div}^* \eta) = 0$.

$$\overline{\partial}^* (\operatorname{div}^* \eta) = [-\partial_l \varphi^i_{\overline{p}} g^{l\overline{p}} + \varphi^q_{\overline{p}} \partial_p g^{i\overline{p}} - \varphi^i_{\overline{p}} g^{l\overline{p}} \partial_l \log g_0^m] \partial_i \otimes e^m, \tag{2.4.18}$$

$$\Box(\operatorname{div}^* \eta) = \partial_{\overline{k}} \big[\partial_l \partial_{\overline{p}} (f_{\bar{j}} g^{i\bar{j}}) g^{l\overline{p}} - \partial_{\overline{p}} (f_{\bar{j}} g^{q\bar{j}}) \partial_q g^{i\overline{p}} + \partial_{\overline{p}} (f_{\bar{j}} g^{i\bar{j}}) g^{l\overline{p}} \partial_l \log g_0^m \big] d\bar{z}^k \otimes \partial_i \otimes e^m. \tag{2.4.19}$$

Hence,

$$\begin{aligned}
&\Box(\operatorname{div}^* \eta) - \operatorname{div}^*(\Box \eta) \\
=&\Big\{ \partial_{\overline{k}} \big[\partial_l \partial_{\overline{p}} (f_{\bar{j}} g^{i\bar{j}}) g^{l\overline{p}} - \partial_{\overline{p}} (f_{\bar{j}} g^{q\bar{j}}) \partial_q g^{i\overline{p}} + \partial_{\overline{p}} (f_{\bar{j}} g^{i\bar{j}}) g^{l\overline{p}} \partial_l \log g_0^m \big] \\
&- \partial_{\overline{k}} \big[(g^{j\bar{l}} \partial_j f_{\bar{l}} + f_{\bar{l}} g^{j\bar{l}} \partial_j \log g_0^m)_{\overline{p}} g^{i\overline{p}} \big] \Big\} d\bar{z}^k \otimes \partial_i \otimes e^m \\
=&\partial_{\overline{k}} \big[\partial_l \partial_{\overline{p}} (f_{\bar{j}} g^{i\bar{j}}) g^{l\overline{p}} - \partial_{\overline{p}} (f_{\bar{j}} g^{q\bar{j}}) \partial_q g^{i\overline{p}} + \partial_{\overline{p}} (f_{\bar{j}} g^{i\bar{j}}) g^{l\overline{p}} \partial_l \log g_0^m \\
&- \partial_{\overline{p}} (g^{l\bar{j}} \partial_l f_{\bar{j}}) g^{i\overline{p}} - \partial_{\overline{p}} (f_{\bar{j}} g^{l\bar{j}}) g^{i\overline{p}} \partial_l \log g_0^m - (f_{\bar{j}} g^{l\bar{j}}) g^{i\overline{p}} \partial_l \partial_{\overline{p}} \log g_0^m \big] d\bar{z}^k \otimes \partial_i \otimes e^m \\
=&\partial_{\overline{k}} \big[\partial_l \partial_{\overline{p}} (f_{\bar{j}} g^{i\bar{j}}) g^{l\overline{p}} - \partial_{\overline{p}} (g^{l\bar{j}} \partial_l f_{\bar{j}}) g^{i\overline{p}} + m (f_{\bar{j}} g^{i\bar{j}}) \big] d\bar{z}^k \otimes \partial_i \otimes e^m \\
=&(m+1) \partial_{\overline{k}} (f_{\bar{j}} g^{i\bar{j}}) d\bar{z}^k \otimes \partial_i \otimes e^m \\
=&-(m+1) \operatorname{div}^* \eta.
\end{aligned} \tag{2.4.20}$$

$\Box$

Lemma 2.4.5. *For $\eta = \eta^i_{\bar{j}} d\bar{z}^j \otimes \partial_i \otimes e^m \in A^{0,1}(X_0, T^{1,0}X_0 \otimes K^{-m}_{X_0})$, if $\overline{\partial}\eta = 0$, then* $\mathrm{div}^*\mathrm{div}(\eta) = \Box\eta$.

Proof.

$$
\begin{aligned}
\Box\eta &= \overline{\partial}\overline{\partial}^*\eta \\
&= \partial_{\bar{l}}[-g^{k\bar{j}}\partial_k\eta^i_{\bar{j}} + \eta^k_{\bar{j}}\partial_k g^{i\bar{j}} - \eta^i_{\bar{j}} g^{k\bar{j}}\partial_k \log g_0^m] d\bar{z}^l \otimes \partial_i \otimes e^m \\
&= [-g^{k\bar{j}}\partial_{\bar{l}}\partial_k\eta_{\bar{j}} + \eta^k_{\bar{j}}\partial_{\bar{l}}\partial_k g^{i\bar{j}} - \eta^i_{\bar{j}} g^{k\bar{j}}\partial_{\bar{l}}\partial_k \log g_0^m] d\bar{z}^l \otimes \partial_i \otimes e^m \\
&= \big[- [\partial_{\bar{l}}\partial_k(\eta^i_{\bar{j}} g^{k\bar{j}}) - \eta^i_{\bar{j}}\partial_{\bar{l}}\partial_k(g^{k\bar{j}})] + \eta^k_{\bar{j}}\partial_{\bar{l}}\partial_k(g^{i\bar{j}}) + m\eta^i_{\bar{l}}\big] d\bar{z}^l \otimes \partial_i \otimes e^m \qquad (2.4.21) \\
&= \big[- [\partial_{\bar{l}}\partial_k(\eta^k_{\bar{j}} g^{i\bar{j}}) - \eta^i_{\bar{j}}\partial_{\bar{l}}\partial_k(g^{k\bar{j}})] + \eta^k_{\bar{j}}\partial_{\bar{l}}\partial_k(g^{i\bar{j}}) + m\eta^i_{\bar{l}}\big] d\bar{z}^l \otimes \partial_i \otimes e^m \\
&= [-\partial_i\partial_k\eta^k_{\bar{i}} + (m+1)\eta^i_{\bar{l}}] d\bar{z}^l \otimes \partial_i \otimes e^m \\
&= \mathrm{div}^*\mathrm{div}\eta.
\end{aligned}
$$

□

Based on above lemmas, we obtain

Proposition 11. *Up to the second order, the expansion of the L^2 metric about t at 0 is given by*

$$
\begin{aligned}
h_{\alpha\bar{\beta}}(t) =& h_{\alpha\bar{\beta}}(0) + |t|^2 \int_{X_0} (m+1)((\Delta+1)^{-1}|\varphi_1|^2 < S_0^\alpha, S_0^\beta >_{g_0^m}) dV_0 \\
& - |t|^2 \int_{X_0} (m+1) < (\Box + m + 1)^{-1}(\varphi_1 \otimes S_0^\alpha), \varphi_1 \otimes S_0^\beta > + O(|t|^3).
\end{aligned}
\qquad (2.4.22)
$$

Proof. The 0^{th} **order** term is given by $h_{\alpha\bar{\beta}}(0) = \int_{X_0} < S_0^\alpha, S_0^\beta >_{g_0^m} dV_0$.

For the 1^{st} **order** term:

We first note that $f(0) = 0$, and by the volume form expansion,

$$\frac{\partial f}{\partial t}(0) = \frac{\partial f}{\partial \bar{t}}(0) = 0.$$

Also, by the deformation equation,

$$\frac{\partial}{\partial t}\Big|_{t=0} \det(I - \varphi\overline{\varphi}) = \frac{\partial}{\partial \bar{t}}\Big|_{t=0} \det(I - \varphi\overline{\varphi}) = 0.$$

Let $\frac{\partial}{\partial t}S^{\alpha}(t)|_{t=0} = S_1^{\alpha}$ for $\alpha = 1, \ldots, N_m$. Then,

$$\begin{aligned}
\frac{\partial}{\partial t}\Big|_{t=0} h_{\alpha\overline{\beta}}(t) &= \int_{X_0} \frac{\partial}{\partial t}\Big|_{t=0} < S^{\alpha}(t), S^{\beta}(t) >_{g_0^m} dV_0 \\
&= \int_{X_0} < S_1^{\alpha}, S_0^{\beta} >_{g_0^m} dV_0 \\
&= \int_{X_0} < \overline{\partial}^* G(\varphi_1 \lrcorner S_0^{\alpha}), S_0^{\beta} >_{g_0^m} dV_0 \\
&= \int_{X_0} < G(\varphi_1 \lrcorner S_0^{\alpha}), \overline{\partial} S_0^{\beta} >_{g_0^m} dV_0 = 0.
\end{aligned} \tag{2.4.23}$$

Similarly, $\frac{\partial}{\partial \overline{t}}\Big|_{t=0} h_{\alpha\overline{\beta}}(t) = 0$.

So all first order terms vanish.

For the 2^{nd} **order** terms, we have

by the volume form expansion, $\frac{\partial^2 f}{\partial t^2}(0) = 0$ and $\frac{\partial^2}{\partial t^2}\Big|_{t=0} det(I - \varphi\overline{\varphi}) = 0$, it follows

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2}\Big|_{t=0}\big[e^{(m+1)f} < S^{\alpha}(t), S^{\beta}(t) >_{g_0^m} \det(I - \varphi\overline{\varphi}) dV_0\big] \\
=&\frac{\partial}{\partial t}\Big|_{t=0}\big[(m+1)e^{(m+1)f}\frac{\partial f}{\partial t} < S^{\alpha}(t), S^{\beta}(t) >_{g_0^m} \det(I - \varphi\overline{\varphi}) dV_0 \\
&+ e^{(m+1)f}\frac{\partial}{\partial t} < S^{\alpha}(t), S^{\beta}(t) >_{g_0^m} \det(I - \varphi\overline{\varphi}) dV_0 \\
&+ e^{(m+1)f} < S^{\alpha}(t), S^{\beta}(t) >_{g_0^m} \frac{\partial}{\partial t}\det(I - \varphi\overline{\varphi}) dV_0\big] \\
=&\frac{\partial^2}{\partial t^2}\Big|_{t=0} < S^{\alpha}(t), S^{\beta}(t) >_{g_0^m} \det(I - \varphi\overline{\varphi}) dV_0 \\
=& < S_2^{\alpha}, S_0^{\beta} >_{g_0^m} dV_0 \\
=& < \overline{\partial}^* G(\varphi_1 \lrcorner \nabla S_1^{\alpha} + \varphi_2 \lrcorner \nabla S_0^{\alpha}), S_0^{\beta} >_{g_0^m} dV_0 = 0
\end{aligned} \tag{2.4.24}$$

where $\frac{\partial^2}{\partial t^2}|_{t=0} S^{\alpha}(t) = S_2^{\alpha}$ for any $\alpha = 1, \ldots, N_m$.

Similarly,

$$\frac{\partial^2}{\partial \overline{t}^2}\Big|_{t=0}[e^{(m+1)f} < S^{\alpha}(t), S^{\beta}(t) >_{g_0^m} \det(I - \varphi\overline{\varphi}) dV_0] = 0.$$

For the mixed derivative term, recall equation (2.2.30) and (2.2.31), we have

$$\begin{aligned}
&\frac{\partial^2}{\partial t\partial\bar t}\Big|_{t=0}\big[e^{(m+1)f} < S^\alpha(t), S^\beta(t) >_{g_0^m} \det(I-\varphi\overline{\varphi})dV_0\big]\\
=&\frac{\partial}{\partial\bar t}\Big|_{t=0}\big[(m+1)e^{(m+1)f}\frac{\partial f}{\partial t} < S^\alpha(t), S^\beta(t) >_{g_0^m} \det(I-\varphi\overline{\varphi})dV_0\\
&+e^{(m+1)f}\frac{\partial}{\partial t} < S^\alpha(t), S^\beta(t) >_{g_0^m} \det(I-\varphi\overline{\varphi})dV_0\\
&+e^{(m+1)f} < S^\alpha(t), S^\beta(t) >_{g_0^m} \frac{\partial}{\partial t}\det(I-\varphi\overline{\varphi})dV_0\big]\\
=&(m+1)\frac{\partial^2 f}{\partial t\partial\bar t}\Big|_{t=0} < S_0^\alpha, S_0^\beta >_{g_0^m} dV_0 + \frac{\partial^2}{\partial t\partial\bar t}\Big|_{t=0} < S^\alpha(t), S^\beta(t) >_{g_0^m} dV_0\\
&+ < S_0^\alpha, S_0^\beta >_{g_0^m} \frac{\partial^2}{\partial t\partial\bar t}\Big|_{t=0}\det(I-\varphi\overline{\varphi})dV_0\\
=&(m+1)(\Delta+1)^{-1}(|\varphi_1|^2) < S_0^\alpha, S_0^\beta >_{g_0^m} dV_0 + < S_1^\alpha, S_1^\beta >_{g_0^m} dV_0 - < S_0^\alpha, S_0^\beta >_{g_0^m} |\varphi_1|^2 dV_0\\
=&(m-\Delta)((\Delta+1)^{-1}(|\varphi_1|^2)) < S_0^\alpha, S_0^\beta >_{g_0^m} dV_0 + < S_1^\alpha, S_1^\beta >_{g_0^m} dV_0.
\end{aligned}\tag{2.4.25}$$

Hence, up to second order, the L^2-metric about t can be written as

$$\begin{aligned}
h_{\alpha\overline{\beta}}(t) =& \int_{X_t} < \sigma_t(S^\alpha(t)), \sigma_t(S^\beta(t)) >_{g_t^m} dV_t\\
&= h_{\alpha\overline{\beta}}(0) + |t|^2\int_{X_0}(< S_1^\alpha, S_1^\beta >_{g_0^m} + (m-\Delta)((\Delta+1)^{-1}|\varphi_1|^2) < S_0^\alpha, S_0^\beta >_{g_0^m})dV_0 + O(|t|^3).
\end{aligned}\tag{2.4.26}$$

Next, we will explore further the term $\int_{X_0} < S_1^\alpha, S_1^\beta >_{g_0^m} dV_0$. By $H^1(X_0, K_{X_0}^{-m}) = 0$, we know $\Box G = Id$. Thus we have,

$$\begin{aligned}
< S_1^\alpha, S_1^\beta >_{g_0^m} =& < \overline{\partial}^* G(\varphi_1\lrcorner\nabla S_0^\alpha), \overline{\partial}^* G(\varphi_1\lrcorner\nabla S_0^\beta) >_{g_0^m}\\
=& < G(\varphi_1\lrcorner\nabla S_0^\alpha), \overline{\partial}\overline{\partial}^* G(\varphi_1\lrcorner\nabla S_0^\beta) >_{g_0^m}\\
=& < G(\varphi_1\lrcorner\nabla S_0^\alpha), (\Box - \overline{\partial}^*\overline{\partial}) G(\varphi_1\lrcorner\nabla S_0^\beta) >_{g_0^m}\\
=& < G(\varphi_1\lrcorner\nabla S_0^\alpha), \Box G(\varphi_1\lrcorner\nabla S_0^\beta) >_{g_0^m} - < G(\varphi_1\lrcorner\nabla S_0^\alpha), \overline{\partial}^*\overline{\partial} G(\varphi_1\lrcorner\nabla S_0^\beta) >_{g_0^m} .
\end{aligned}\tag{2.4.27}$$

Moreover,

$$\begin{aligned}\overline{\partial}^*\overline{\partial}G(\varphi_1\lrcorner\nabla S_0^\beta) &= \overline{\partial}^*G\overline{\partial}(\varphi_1\lrcorner\nabla S_0^\beta)\\ &= \overline{\partial}^*G(\overline{\partial}\varphi_1\lrcorner\nabla S_0^\beta+\varphi_1\lrcorner\overline{\partial}(\nabla S_0^\beta))\\ &= \overline{\partial}^*G(\varphi_1\lrcorner(-\nabla(\overline{\partial}S_0^\beta))+\varphi_1\lrcorner 2\sqrt{-1}\,mS_0^\beta\otimes\omega_0)=0.\end{aligned} \tag{2.4.28}$$

We get $< S_1^\alpha, S_1^\beta >_{g_0^m} = < G(\varphi_1\lrcorner\nabla S_0^\alpha), (\varphi_1\lrcorner\nabla S_0^\beta) >_{g_0^m}$.

Based on Lemma 2.4.2 to Lemma 2.4.5, choosing the divergence gauge, we notice that the operator $(\Box+m+1)$ has no kernel, we have

$$\begin{aligned}&\int_{X_0} < G(\varphi_1\lrcorner\nabla S_0^\alpha), \varphi_1\lrcorner\nabla S_0^\beta > dV_0\\ =&\int_{X_0} < G\mathrm{div}(\varphi_1\otimes S_0^\alpha), \mathrm{div}(\varphi_1\otimes S_0^\beta) > dV_0\\ =&\int_{X_0} < \mathrm{div}^*G\mathrm{div}(\varphi_1\otimes S_0^\alpha), \varphi_1\otimes S_0^\beta > dV_0\\ =&\int_{X_0} < (\Box+m+1)^{-1}(\Box+m+1)\mathrm{div}^*G\mathrm{div}(\varphi_1\otimes S_0^\alpha), \varphi_1\otimes S_0^\beta > dV_0\\ =&\int_{X_0} < (\Box+m+1)^{-1}\mathrm{div}^*\Box G\mathrm{div}(\varphi_1\otimes S_0^\alpha), \varphi_1\otimes S_0^\beta > dV_0\\ =&\int_{X_0} < (\Box+m+1)^{-1}\mathrm{div}^*\mathrm{div}(\varphi_1\otimes S_0^\alpha), \varphi_1\otimes S_0^\beta > dV_0\\ =&\int_{X_0} < (\Box+m+1)^{-1}\Box(\varphi_1\otimes S_0^\alpha), \varphi_1\otimes S_0^\beta > dV_0\\ =&\int_{X_0} < \varphi_1\otimes S_0^\alpha-(m+1)(\Box+m+1)^{-1}(\varphi_1\otimes S_0^\alpha), \varphi_1\otimes S_0^\beta > dV_0\\ =&\int_{X_0} < \varphi_1\otimes S_0^\alpha, \varphi_1\otimes S_0^\beta > dV_0-\int_{X_0}(m+1) < (\Box+m+1)^{-1}(\varphi_1\otimes S_0^\alpha), \varphi_1\otimes S_0^\beta > dV_0\\ =&\int_{X_0} |\varphi_1|^2 < S_0^\alpha, S_0^\beta > dV_0-\int_{X_0}(m+1) < (\Box+m+1)^{-1}(\varphi_1\otimes S_0^\alpha), \varphi_1\otimes S_0^\beta > dV_0.\end{aligned} \tag{2.4.29}$$

Therefore,

$$\begin{aligned}
h_{\alpha\overline{\beta}}(t) =& h_{\alpha\overline{\beta}}(0) + |t|^2 \int_{X_0} (m-\Delta)((\Delta+1)^{-1}|\varphi_1|^2 < S_0^\alpha, S_0^\beta >_{g_0^m})dV_0 \\
& + |t|^2 \int_{X_0} |\varphi_1|^2 < S_0^\alpha, S_0^\beta >_{g_0^m} -(m+1) < (\Box + m+1)^{-1}(\varphi_1 \otimes S_0^\alpha), \varphi_1 \otimes S_0^\beta > dV_0 + O(|t|^3) \\
=& h_{\alpha\overline{\beta}}(0) + |t|^2 \int_{X_0} (m+1)((\Delta+1)^{-1}|\varphi_1|^2 < S_0^\alpha, S_0^\beta >_{g_0^m})dV_0 \\
& - |t|^2 \int_{X_0} (m+1) < (\Box + m+1)^{-1}(\varphi_1 \otimes S_0^\alpha), \varphi_1 \otimes S_0^\beta > dV_0 + O(|t|^3).
\end{aligned} \tag{2.4.30}$$

□

Now, we are ready to show

Theorem 2.4.1. *Let $\pi : \mathscr{X} \to B \subset \mathbb{C}$ be an analytic family of Fano Kähler-Einstein manifolds, such that on the central fiber $\pi^{-1}(0) = (X_0, \omega_0)$, we have $H^0(X_0, T^{1,0}X_0)$. Let $K^m_{\mathscr{X}/B} \to B$ be the relative anticanonical line bundle. Then the Ricci curvature of the L^2 metric on the direct image sheaf $R^0\pi_*(K^m_{\mathscr{X}/B})$ has the following asymptotic behavior*

$$\lim_{m\to\infty} \frac{\pi^n}{m^n} R^m_{1\overline{1}} = -\int_{X_0} |\varphi_1|^2 dV_0,$$

where $\int_{X_0} |\varphi_1|^2 dV_0$ is the Weil-Peterson metric on the Moduli space.

Proof. Let $\tau_m = \sum_{\alpha=1}^{N_m} < S_0^\alpha, S_0^\alpha >_{g_0^m}$ be the Bergman Kernel function on the central fiber (X_0, ω_0). By Proposition 12,

$$\begin{aligned}
R^m_{1\overline{1}} =& -\sum_{\alpha=1}^{N_m} \frac{\partial^2 h_{\alpha\overline{\alpha}}(t)}{\partial t \partial \overline{t}}|_{t=0} \\
=& -\int_{X_0} \tau_m (m+1)(\Delta+1)^{-1}|\varphi_1|^2 dV_0 \\
& + \int_{X_0} (m+1) \sum_{\alpha=1}^{N_m} < (\Box + m+1)^{-1}(\varphi_1 \otimes S_0^\alpha), \varphi_1 \otimes S_0^\beta > dV_0.
\end{aligned} \tag{2.4.31}$$

Let $(\Box + m + 1)f = \lambda f$. Then we see

$$\lambda \int_{X_0} f^2 = \int_{X_0} f(\Box + m+1)f = \int_{X_0} f\Box f + (m+1)f^2 \geq (m+1)\int_{X_0} f^2 \tag{2.4.32}$$

so, the eigenvalue of the operator $\Box + m + 1$ is at least $m + 1$.

By Tain-Yau-Zelditch's Bergman kernel expansion formula [6] [17] [20],

$$\tau_m = \frac{m^n}{\pi^n} - \frac{nm^{n-1}}{2\pi^n} + O(m^{n-2}).$$

Then,

$$\begin{aligned} 0 &\leq \int_{X_0} < (\Box + m + 1)^{-1}(\varphi_1 \otimes S_0^\alpha), \varphi_1 \otimes S_0^\alpha > dV_0 \\ &\leq \frac{1}{m+1} \int_{X_0} < \varphi_1 \otimes S_0^\alpha, \varphi_1 \otimes S_0^\alpha > dV_0 \\ &= \frac{1}{m+1} \int_{X_0} |\varphi_1|^2 < S_0^\alpha, S_0^\alpha > dV_0. \end{aligned} \tag{2.4.33}$$

So,

$$R^m_{1\bar{1}} \leq (m+1)\Big[\frac{1}{m+1} \int_{X_0} \tau_m |\varphi_1|^2 - \int_{X_0} \tau_m (\Delta + 1)^{-1} |\varphi_1|^2 dV_0\Big]. \tag{2.4.34}$$

On the other hand,

$$R^m_{1\bar{1}} \geq -(m+1) \int_{X_0} \tau_m (\Delta + 1)^{-1} |\varphi_1|^2 dV_0. \tag{2.4.35}$$

As $m \to \infty$, we get,

$$\lim_{m\to\infty} \frac{\pi^n}{m^n} R^m_{1\bar{1}} = -\int_{X_0} (\Delta + 1)^{-1} |\varphi_1|^2 dV_0 = -\int_{X_0} |\varphi|^2 dV_0. \tag{2.4.36}$$

$\Box$

2.5 Deformation of holomorphic vector fields

On a general Fano Kähler-Einstein manifold, there may exist holomorphic vector fields which are nontrivial. In this section, we will investigate the deformation theory of such vector fields. More specifically, we establish the deformation equation and show that the solution to the deformation equation exists under the assumption that the dimension of the space of holomorphic vector fields is a constant.

Let $\pi : \mathscr{X} \to B \subset \mathbb{C}$ be an analytic family of Fano Kähler-Einstein manifolds, (X_0, ω_0) is the central fiber. Let $V_0 \in H^0(X_0, T^{1,0}X_0)$, $V \in A^0(X_0, T^{1,0}X_0)$ such that $\mathbb{H}(V) = V_0$. Let φ be the Beltrami differential. We define $\sigma : A^0(X_0, T^{1,0}X_0) \to A^0(X_t, T^{1,0}(X_t, T^{1,0}X_t))$ as follows. If $(z_1, \ldots, z_n)$ are local holomorphic coordinates on X_0, $V = V^i(z)\frac{\partial}{\partial z_i}$, then $\sigma(V) = V^i(z)e_i$, where $e_i = (I - \varphi\overline{\varphi})^{ik}(\frac{\partial}{\partial z_k} - \overline{\varphi^j_{\bar{k}}}\frac{\partial}{\partial \bar{z}_j})$. We know σ is an isormorphism. Then,

Proposition 12. *$\sigma(V) \in H^0(X_t.T^{1,0}X_t)$ if and only if $\overline{\partial}V = [\varphi, V]$.*

Proof. Let

$$V = V^i \frac{\partial}{\partial z_i} \in A^0(X_0, T^{1,0}X_0).$$

For $\{e_1, \cdots, e_n\}$ the deformed holomorphic frame, where

$$e_i = (I - \varphi\overline{\varphi})^{ik}(\frac{\partial}{\partial z_k} - \overline{\varphi^j_{\bar{k}}}\frac{\partial}{\partial \overline{z_j}}),$$

the deformed holomorphic coordinate vector field on X_t is given by

$$\begin{aligned} \frac{\partial}{\partial w_\alpha} &= (\frac{\partial w_\alpha}{\partial z_i})^{-1}(I - \varphi\overline{\varphi})^{ik}(\frac{\partial}{\partial z_k} - \overline{\varphi^j_{\bar{k}}}\frac{\partial}{\partial \overline{z_j}}) \\ &= b^{i\alpha}(I - \varphi\overline{\varphi})^{ik}T_k, \end{aligned} \tag{2.5.1}$$

where $(b^{i\alpha})_{n\times n} = (a_{i\alpha})^{-1}_{n\times n}$ and $a_{i\alpha} = \frac{\partial w_\alpha}{\partial z_i}$.

Now, $\sigma(V) = V(t) = V^i e_i = V^i a_{i\alpha}\frac{\partial}{\partial w_\alpha}$, let $\overline{\partial}_t = \frac{\partial}{\partial w_\alpha}dw^\alpha$. Then $V(t) \in H^0(X_t, T^{1,0}X_t)$, if and only if $\overline{\partial}_t V(t) = 0$, if and only if $T_{\bar{i}}(V(t)) = (\partial_{\bar{i}} - \varphi^j_{\bar{i}}\partial_j)(V^k a_{k\alpha}) = 0$. By

$$\partial_{\bar{i}}(a_{k\alpha}) = \partial_{\bar{i}}(\partial_k w_\alpha) = \partial_k(\partial_{\bar{i}} w_\alpha) = \partial_k(\varphi^j_{\bar{i}}\frac{\partial w_\alpha}{\partial z_j}) = a_{j\alpha}\partial_k\varphi^j_{\bar{i}} + \varphi^j_{\bar{i}}\partial_k a_{j\alpha}, \tag{2.5.2}$$

we get

$$(\partial_{\bar{i}} - \varphi^j_{\bar{i}}\partial_j)(a_{k\alpha}) = a_{j\alpha}\partial_k\varphi^j_{\bar{i}},$$

and

$$(\partial_{\bar{i}} - \varphi^j_{\bar{i}}\partial_j)(V^k a_{k\alpha}) = (\partial_{\bar{i}}V^k - \varphi^j_{\bar{i}}\partial_j V^k + V^j\partial_j\varphi^k_{\bar{i}})a_{k\alpha}.$$

Therefore, $\overline{\partial}_t V(t) = 0$, if and only if $\overline{\partial}V = \varphi\lrcorner\partial V - V\lrcorner\partial\varphi$, if and only if $\overline{\partial}V = [\varphi, V]$. $\square$

Lemma 2.5.1. *If $W \in A^0(X_0, T^{1,0}X_0)/H^0(X_0, T^{1,0}X_0)$, we have $(\Box - 1)\mathrm{div}GW = \mathrm{div}W$ where G is the Green operator, and $\Box$ is the Hodge Laplacian.*

Proof. For a vector $U \in A^0(X_0, T^{1,0}X_0)$ we have

$$\begin{aligned}\Box\mathrm{div}U - \mathrm{div}\Box U =&\overline{\partial}^*\overline{\partial}\mathrm{div}U - div\overline{\partial}^*\overline{\partial}U \\ =&-\nabla_k\nabla_{\overline{k}}\nabla_i U^i + \nabla_i\nabla_k\nabla_{\overline{k}}U^i \\ =&\nabla_k(\nabla_i\nabla_{\overline{k}}U^i - \nabla_{\overline{k}}\nabla_i U^i) \\ =&\nabla_k(R_{p\overline{k}}U^p) = g_{p\overline{k}}\nabla_k U^p = \mathrm{div}U.\end{aligned} \tag{2.5.3}$$

Take $W \in A^0(X_0, T^{1,0}X_0)/H^0(X_0, T^{1,0}X_0)$,

$$\begin{aligned}\Box(\mathrm{div}GW - G\mathrm{div}W) =&\Box\mathrm{div}GW - \mathrm{div}W \\ =&\mathrm{div}\Box GW + \mathrm{div}GW - \mathrm{div}W \\ =&\mathrm{div}GW\end{aligned} \tag{2.5.4}$$

i.e. $(\Box - 1)\mathrm{div}GW = \mathrm{div}W$. $\Box$

Lemma 2.5.2. *For $V = V_0 + \sum_{i\geq 1} t^i V_i \in A^0(X_0, T^{1,0}X_0)$ satisfying*

$$\begin{cases} \overline{\partial}V = [\varphi, V] \\ \mathbb{H}[\varphi, V] = 0 \\ V_0 \in H^0(X_0, T^{1,0}X_0) \end{cases}$$

and V_0 is a holomorphic vector field with real potential function. Then $\mathrm{div}V_i = 0$ for all $i \geq 1$.

Remark 9. *On a compact Fano Kähler-Einstein manifold, there always exists a holomorphic vector field with a real potential. The reason is: taking any $V \in H^0(X_0, T^{1,0}X_0)$, the Fano condition tells us $V = \nabla^{1,0}f$ for some complex valued smooth function f and $divV = -\Box f$. Let $f = u + \sqrt{-1}v$, then $divV = -\Box u - \sqrt{-1}\Box v$. By Matsushima's theorem, $\Box divV = divV$, so $\Box(\Box u) + \sqrt{-1}(\Box v) = \Box u + \sqrt{-1}\Box(\Box v)$. Since $\Box$ is a real operator on Kähler manifold, $\Box(\Box u) = \Box u$ and $\Box(\Box v) = \Box v$, again, by Matsushima's theorem, $\nabla^{1,0}(\Box u)$ is such a holomorphic vector field with real potential $\Box u$.*

Proof. (of the lemma 2.5.2)

For $\varphi = \varphi^i_{\bar{j}} d\bar{z}^j \otimes \frac{\partial}{\partial z_i} \in A^{0,1}(X_0, T^{1,0}X_0)$, $V = V^k \frac{\partial}{\partial z_k} \in A^0(X_0, T^{1,0}X_0)$, we have

$$[\varphi, V] = (\varphi^i_{\bar{j}} \partial_i V^k - V^i \partial_i \varphi^k_{\bar{j}}) d\bar{z}^j \otimes \frac{\partial}{\partial z_k}$$

so $\overline{\partial}^* [\varphi, V] = -\nabla_j (\varphi^i_{\bar{j}} \partial_i V^k - V^i \partial_i \varphi^k_{\bar{j}}) \frac{\partial}{\partial z_k}$.

By Hodge decomposition,

$$V = V_0 + \overline{\partial}^* \overline{\partial} G V = V_0 + \overline{\partial}^* G[\varphi, V],$$

then $V_i = \sum_{j=1}^{i} \overline{\partial}^* G[\varphi_j, V_{i-j}]$, for $i \geq 1$.

By Lemma 2.5.1, $(\Box - 1)\mathrm{div} G \overline{\partial}^* [\varphi, V] = \mathrm{div} \overline{\partial}^* [\varphi, V]$. Hence,

$$\begin{aligned}
(\Box - 1)\mathrm{div} V_1 =& (\Box - 1)\mathrm{div} \overline{\partial}^* G[\varphi_1, V_0] = \mathrm{div} \overline{\partial}^* [\varphi_1, V_0] \\
=& -\nabla_k \nabla_j (\varphi^i_{1\bar{j}} \partial_i V^k_0 - V^i_0 \partial_i \varphi^k_{1\bar{j}}) \qquad (2.5.5) \\
=& -\varphi^i_{1\bar{j}} \partial_j \partial_i (\mathrm{div} V_0) = -tr(\varphi_1 \lrcorner \overline{\nabla}^2 \mathrm{div} V_0) = 0
\end{aligned}$$

The last equality holds because $divV_0$ is real and $V_0 \in H^0(X_0, T^{1,0}X_0)$. Now, $V_1 \notin H^0(X_0, T^{1,0}X_0)$ implies the operator $(\Box - 1)$ is invertible, so, $\mathrm{div} V_1 = 0$.

For $i \geq 2$, $V_i = \sum_{j=1}^{i} \overline{\partial}^* G[\varphi_j, V_{i-j}]$,

$$\begin{aligned}
(\Box - 1)\mathrm{div} V_i =& \sum_{j-1}^{i} (\Box - 1)\mathrm{div} \overline{\partial}^* G[\varphi_j, V_{i-j}] \\
=& \sum_{j=1}^{i} div \overline{\partial}^* [\varphi_j, V_{i-j}] \qquad (2.5.6) \\
=& -tr(\varphi_i \lrcorner \overline{\nabla}^2 \mathrm{div} V_0) - \sum_{j=1}^{i-1} tr(\varphi_j \lrcorner \overline{\nabla}^2 \mathrm{div} V_{i-j}) = 0
\end{aligned}$$

Consequently, $\mathrm{div} V_i = 0$ for all $i \geq 1$. □

Proposition 13. *Suppose $h^0(X_0, T^{1,0}X_0) = h^0(X_t, T^{1,0}X_t) = k =$ constant for $|t| < \epsilon$. Then there exists a unique solution to*

$$\begin{cases} \overline{\partial} V = [\varphi, V] \\ \mathrm{div}(V - \mathbb{H}(V)) = 0. \end{cases}$$

Proof. For each sufficiently small t, $h^0(X_t, T^{1,0}X_t) = k$, it's known that $H^0(X_t, T^{1,0}X_t)$ is a vector bundle. Take a holomorphic basis $\{E^1(t), \dots, E^k(t)\}$, such that $H^0(X_t, T^{1,0}X_t) = span\{E^1(t), \cdots E^k(t)\}$. In particular, $H^0(X_0, T^{1,0}X_0) = span\{E^1(0), \dots E^k(0)\}$. Since

$$\sigma(t)\colon A^0(X_0, T^{1,0}X_0) \to A^0(X_t, T^{1,0}X_t)$$
$$\frac{\partial}{\partial z_i} \mapsto e_i = (I - \varphi\overline{\varphi})^{ij}\left(\frac{\partial}{\partial z_j} - \overline{\varphi^k_{\bar{j}}}\frac{\partial}{\partial \overline{z_k}}\right).$$

is a linear isomorphism, $\sigma^{-1}(E^i(t)) = \tilde{E}^i(t) \in A^0(X_0, T^{1,0}X_0)$ and $\tilde{E}^i(t)$ satisfies $\overline{\partial}\tilde{E}^i(t) = [\varphi, \tilde{E}^i(t)]$. For each $\tilde{E}^i(t)$, we let $\tilde{E}^i(t) = \mathbb{H}(\tilde{E}^i(t)) + \sum_{\alpha\geq 1} t^\alpha \tilde{E}^i_\alpha(t)$ where $\mathbb{H}(\tilde{E}^i(t)) \in H^0(X_0, T^{1,0}X_0)$. By $H^0(X_0, T^{1,0}X_0) = span\{E^1(0), \cdots, E^k(0)\} = span\{\mathbb{H}(\tilde{E}^1(t), \cdots, \tilde{E}^k(t))\}$, there exists a nondegenerate linear map $A_0(t) : H^0(X_0, T^{1,0}X_0) \to H^0(X_0, T^{1,0}X_0)$ such that

$$\begin{bmatrix} E^1(0) \\ \vdots \\ E^k(0) \end{bmatrix} = A_0(t) \begin{bmatrix} \mathbb{H}(\tilde{E}^1(t)) \\ \vdots \\ \mathbb{H}(\tilde{E}^k(t)) \end{bmatrix}.$$

Now, assume $\{e^1(t), \cdots, e^k(t)\}$ is the solution to

$$\begin{cases} \overline{\partial}e^i(t) = [\varphi, e^i(t)] \\ \mathbb{H}[e^i(t)] = E^i(0) \\ div(e^i(t) - E^i(0)) = 0, \end{cases}$$

i.e. $\sigma(e^i(t)) \in H^0(X_t, T^{1,0}X_t)$. Then there is $A(t, \bar{t})$ such that,

$$\begin{bmatrix} e^1(t) \\ \vdots \\ e^k(t) \end{bmatrix} = A(t, \bar{t}) \begin{bmatrix} \tilde{E}^1(t) \\ \vdots \\ \tilde{E}^k(t) \end{bmatrix}$$

However, by the implicit function theorem, for $|t| < \epsilon$, such a linear transformation $A(t, \bar{t})$ always exists.

$\square$

Chapter 3

Pluri-subharmonicity of Harmonic Energy

It is well known that the Teichmüller spaces of hyperbolic Riemann surfaces are contractible. However, little is known about the Teichmüller spaces of higher dimensional Kähler-Einstein manifolds of general type. In this chapter, we will take the first step in studying the (weak) pseudo-convexity of such Teichmüller spaces. Our approach to the problem is in the framework of deformation theory. In particular, based on the deformation theory of Kähler-Einstein manifolds of general type estabilished by X. Sun [15], we compute the first and second variation of harmonic energy. It turns out the first variation of the energy function admits a simple formula depending on the harmonic projection of the Beltrami differential. Based on the second variation, we conclude that with the assumption that the target manifold has Hermitian nonpositive curvature, the energy function is pluri-subharmonic.

In section 1, we review the deformation theory of Kähler-Einstein manifolds of general type, the discussion of which follows from X. Sun's paper [15]. In section 2, we compute the first variation of the energy function and express it in terms of the harmonic Beltrami differentials and Hopf differentials. In section 3, we investigate the second variation and obtain the pluri-subharmonicity of the energy functional.

3.1 Deformation of Kähler-Einstein manifolds of general type

Let M_0 be a compact Kähler manifold with $c_1(M_0) < 0$. By Yau's famous work [19], M_0 admits a unique Kähler-Einstein metric ω_0, such that $Ric(\omega_0) = -\omega_0$. Suppose $H^2(M_0, T^{1,0}M_0) = 0$, i.e. the deformations of complex structures on M_0 have no obstructions. Let $\pi : \mathscr{X} \to B = \{t = (t_1 \cdots, t_k) \in \mathbb{C}^k \big| |t| < \epsilon\}$ be an analytic family of compact Kähler-Einstein manifolds of general types with the central fiber (M_0, ω_0), and for each $t \in B$, $M_t = \pi^{-1}(t)$ is also a Kähler-Einstein manifold such that $Ric(\omega_t) = -\omega_t$. The complex structrure on M_t is represented by $\varphi(t) \in A^{0,1}(M_0, T^{1,0}M_0)$, where $\varphi(t)$ satisfies

$$\begin{cases} \overline{\partial}\varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)] \\ \overline{\partial}^* \varphi(t) = 0, \end{cases} \tag{3.1.1}$$

where $\overline{\partial}$ is on the central fiber (M_0, ω_0), and $\overline{\partial}^*$ depends on the Kähler-Einstein metric ω_0. Let the div operator be defined as in Chapter 2. We first recall the following theorem.

Theorem 3.1.1. *(X. Sun) On a compact Kähler-Einstein manifold of general type (M_0, ω_0), if a Beltrami differential $\varphi \in A^{0,1}(M_0, T^{1,0}M_0)$ satisfies $\overline{\partial}\varphi = \frac{1}{2}[\varphi, \varphi]$, then*

$$\overline{\partial}^* \varphi = 0 \textit{ if and only if } \mathrm{div}\varphi = 0.$$

Furthermore, $\varphi \lrcorner \omega_0 = 0$ when either one of these conditions is imposed.

Therefore, the deformation equation of complex structures on Kähler-Einstein manifolds of general type is solvable under either the Kuranisi gauge $\overline{\partial}^* \varphi = 0$ or the divergence gauge $div\varphi = 0$. In the following discussion, instead of using the Kuranishi gauge, we will take the divergence gauge in the computation. To simplify the notation, we let $B \subset \mathbb{C}$.

For the analytic family $\pi : \mathscr{X} \to B$ of Kähler-Einstein manifolds of general type, the complex structure on each fiber $M_t = \pi^{-1}(t)$ is determined by $\varphi(t) = \varphi_1 + \sum\limits_{i \geq 2} t^i \varphi_i$, where $\varphi_1 \in H^{0,1}(M_0, T^{1,0}M_0)$, and for $i \geq 2$, φ_i satisfies

$$\begin{cases} \overline{\partial}\varphi_i = \frac{1}{2}\sum_{j=1}^{i-1}[\varphi_j, \varphi_{i-j}]. \\ \mathrm{div}\varphi_i = 0. \end{cases} \tag{3.1.2}$$

Let $(z_1, \ldots, z_n)$ be local holomorphic coordinates on (M_0, ω_0), locally

$$\Omega^{1,0}(M_0) = span\{dz^1, \cdots, dz^n\}.$$

Let $\{e^1, \cdots, e^n\}$ be local holomorphic coframes on (M_t, ω_t) obtained by the deformation of the complex structure, i.e. $e^i = dz^i + \varphi^i_{\overline{j}} d\overline{z}^j$. Then,

$$\Omega^{1,0}(M_t) = span\{e^1, \ldots, e^n\}.$$

Let $(w_1, \cdots, w_n)$ be local holomorphic coordinates on (M_t, ω_t), let $a_{j\alpha} = \frac{\partial w_\alpha}{\partial z_j}$, $(b^{j\alpha})_{n\times n} = (a_{j\alpha})^{-1}_{n\times n}$, and $T_i = (\partial_i - \overline{\varphi^k_{\overline{i}}}\partial_{\overline{k}})$. Then

$$\begin{cases} dw^\alpha = \frac{\partial w_\alpha}{\partial z_j} e^j = a_{j\alpha}(dz^j + \varphi^j_{\overline{k}} d\overline{z}^k) \\ \frac{\partial}{\partial w_\alpha} = b^{j\alpha}(I - \varphi\overline{\varphi})^{ij} T_i = b^{j\alpha}(I - \varphi\overline{\varphi})^{ij}(\partial_i - \overline{\varphi^k_{\overline{i}}}\partial_{\overline{k}}). \end{cases} \tag{3.1.3}$$

Now, let $A = (a_{\alpha i})_{n\times n} = (\frac{\partial w_\alpha}{\partial z_i})_{n\times n}$, $|A| = \det A$, $c_n = (-1)^{\frac{n(n-1)}{2}}(\frac{\sqrt{-1}}{2})^n$, and let g_0 be the Kähler metric on the central fiber M_0, then

$$dV_0 = c_n \det g_0 \, dz^1 \wedge \cdots \wedge dz^n \wedge d\overline{z}^1 \wedge \cdots \wedge d\overline{z}^n. \tag{3.1.4}$$

Let

$$\begin{aligned} d\widetilde{V}_t &= c_n \det g_0 \, e^1 \wedge \cdots \wedge e^n \wedge \overline{e}^1 \wedge \cdots \wedge \overline{e}^n \\ &= \det(I - \varphi\overline{\varphi}) \, dV_0. \end{aligned} \tag{3.1.5}$$

There is a unique function $f = f(z, \overline{z}, t, \overline{t}) \in C^\infty(M_0 \times \; B)$ with $f(z, \overline{z}, 0, 0) = 0$ such that the volume form on the deformed manifold M_t is given by

$$\begin{aligned} dV_t &= e^f d\tilde{V}_t = e^f \det(I - \varphi\overline{\varphi}) \, dV_0 \\ &= c_n e^f \det g_0 \, |A|^{-2} dw^1 \wedge \cdots \wedge dw^n \wedge d\overline{w}^1 \cdots \wedge d\overline{w}^n \\ &= c_n \det g_t \, dw^1 \wedge \cdots \wedge dw^n \wedge d\overline{w}^1 \cdots \wedge d\overline{w}^n. \end{aligned} \tag{3.1.6}$$

Hence,

$$\det g_t = e^f \det g_0 |A|^{-2}. \tag{3.1.7}$$

Define the matrix $B = (B_{j\overline{p}})_{n\times n}$ to be

$$B_{j\overline{p}} = g_{j\overline{p}} - (\partial_j \varphi^k_{\overline{p}})(I - \varphi\overline{\varphi})^{ik} T_i(f) + T_{\overline{p}}\big[(I - \varphi\overline{\varphi})^{ij} T_i(f)\big]. \tag{3.1.8}$$

Then the deformed $Monge - Amp\grave{e}re$ equation

$$\big(\frac{\partial^2}{\partial w_\alpha \partial \overline{w}_\beta} \log dV_t\big)^n = dV_t \tag{3.1.9}$$

turns out to be

$$\det B_{j\overline{p}} = e^f \det g_0 \det(I - \varphi\overline{\varphi}), \tag{3.1.10}$$

then

$$\log\det B = f + \log\det g_0 + \log\det(I - \varphi\overline{\varphi}). \tag{3.1.11}$$

On the Kähler-Einstein manifolds of general type, the operator $\Delta - 1$ is automatically invertible for the Beltrami Laplacian Δ. From equation 3.1.8 and 3.1.11, We can derive the Taylor expansion of f; that is

$$f = |t|^2 (1 - \Delta)^{-1}(|\varphi_1|^2) + O(|t|^3). \tag{3.1.12}$$

Also, by the deformation equation of the complex structures,

$$\det(I - \varphi\overline{\varphi}) = 1 - |t|^2|\varphi_1|^2 + O(|t|^3). \tag{3.1.13}$$

We conclude the following theorem.

Theorem 3.1.2. *(X. Sun) Let $\pi : \mathscr{X} \to B = \{t \in \mathbb{C} : |t| < \varepsilon\}$ be an analytic family of Kähler-Einstein manifolds of general type with central fiber $\pi^{-1}(0) = (M_0, \omega_0)$. Then the volume form on the nearby fiber $M_t = \pi^{-1}(t)$ is given by*

$$dV_t = \big[1 + |t|^2 \Delta((1 - \Delta)^{-1}|\varphi_1|^2) + O(|t|^3)\big] dV_0,$$

and the Kähler form is

$$\omega_t = \omega_0 + |t|^2 (\frac{\sqrt{-1}}{2} \partial\overline{\partial}(1 - \Delta)^{-1}|\varphi_1|^2) + O(|t|^3),$$

where Δ is the Beltrami-Laplacian on M_0, ∂, $\overline{\partial}$ are operators on M_0, and $\varphi_1 = \mathbb{H}(\varphi)$ is the harmonic projection of the Beltrami differential φ, satisfying

$$\begin{cases} \overline{\partial}\varphi = \frac{1}{2}[\varphi, \varphi] \\ \operatorname{div}\varphi = 0 \\ \varphi(0) = 0. \end{cases} \tag{3.1.14}$$

3.2 First variation of harmonic energy

Let (N, h) be a Riemannian manifold of dimension $\dim_{\mathbb{R}} N = m$ with nonpositive sectional curvature. Let (M_0, ω_g) be a compact Kähler-Einstein manifold of general type with dimension $\dim_{\mathbb{C}} M_0 = n$ and let $\mathcal{T}$ be its Teichmüller space. We assume that the deformation of the complex structures on M_0 is unobstructed. Let $k = h^{0,1}\left(M_0, T^{1,0}_{M_0}\right) = \dim_{\mathbb{C}} \mathcal{T}$ and let M be the background smooth manifold of M_0. Let A be a fixed homotopy class of maps from M to N.

We consider the functional $E : \mathcal{T} \to \mathbb{R}$. For each point $p \in \mathcal{T}$, we let (M_p, g_p) be the Kähler-Einstein manifold corresponding p. Let $u : M_p \to N$ be a harmonic map in the class A. We let $E(p)$ be the energy of u.

Now let $z_1, \cdots, z_n$ be local holomorphic coordinates on M_0 and let $\omega_g = \frac{\sqrt{-1}}{2} g_{i\bar{j}} dz_i \wedge d\overline{z}_j$. Let $u_1, \cdots, u_m$ be any local coordinates on N and let $h = h_{AB} du_A \otimes du_B$. Let $u_0 : M_0 \to N$ be a harmonic map in the class A.

The energy of u_0 is

$$E(0) = E(u_0) = \int_{M_0} g^{i\bar{j}} h_{AB}(u_0) \partial_i u_0^A \partial_{\bar{j}} u_0^B \; dV_0$$

and the Euler-Lagrange of $E(0)$ is the harmonic map equation

$$\Delta_0 u_0^A + \Gamma^A_{BC}\left(u_0\right) \partial_i u_0^B \partial_{\bar{j}} u_0^C g^{i\bar{j}} = 0$$

for each A, here $\partial_i = \frac{\partial}{\partial z_i}$.

Remark 10. *The Hopf differential $H\left(u_0\right) \in S^2\Omega^{1,0} M_0$ is defined as*

$$H\left(u_0\right) = h_{AB} \partial_i u_0^A \partial_k u_0^B dz_i \otimes dz_k.$$

When M_0 is a Riemann surface we know that $H(u_0)$ is a holomorphic quadratic differential.

Now, assume $\pi : \mathscr{X} \to B = \{t \in \mathbb{C} : |t| < \epsilon\}$ is an analytic family of compact Kähler-Einstein manifolds of general type with the central fiber (M_0, ω_0) and for each $t \in B$, $M_t = \pi^{-1}(t)$ is also a Kähler-Einstein manifold such that $Ric(\omega_t) = -\omega_t$. The target manifold is still (N, h) as above. We define the energy $\mathcal{E}$ on the total space $\mathscr{X}$, and $\mathcal{E}|_t = E_t$, where $E_t = E(t, \bar{t}) = E(u_t)$, and

$$E(u_t) = \int_{M_t} g_t^{i\bar{j}} h_{AB}(u_t) \frac{\partial u_t^A}{\partial w^i} \frac{\partial u_t^B}{\partial \overline{w}^j}\, dV_t.$$

By the deformation equation of the complex structures, let $\varphi_1 \in \mathbb{H}^{0,1}(M_0, T^{1,0}M_0)$. We consider the convergent power series $\varphi(t) = t\varphi_1 + \sum_{i \geq 2} t^i \varphi_i$ where $t \in B \subset \mathbb{C}$ such that

$$\begin{cases} \overline{\partial}_0 \varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)] \\ \overline{\partial}_0^* \varphi(t) = 0 \\ \mathbb{H}(\varphi_i) = 0 \text{ for } i \geq 2. \end{cases}$$

Theorem 3.2.1. *The first variation of $\mathcal{E}$ in the direction φ_1 is given by*

$$\left.\frac{\partial \mathcal{E}}{\partial t}\right|_{t=0} = \int_{M_0} \Lambda\left(\varphi_1 \lrcorner H(u_0)\right) dV_0,$$

where $\Lambda\left(\varphi_1 \lrcorner H(u_0)\right) = g^{i\bar{j}} \varphi^k_{1\bar{j}} H(u_0)_{ik} = g^{i\bar{j}} h_{\alpha\beta} \varphi^k_{1\bar{j}} \partial_i u_0^\alpha \partial_k u_0^\beta$.

Proof. On (M_0, ω_0), locally,

$$\Omega^{1,0}(M_0) = span\{dz^1, \cdots, dz^n\}.$$

Let $\{e^1, \ldots, e^n\}$ be local holomorphic frames on (M_t, ω_{g_t}), then

$$\Omega^{1,0}(M_t) = span\{e^1, \cdots, e^n\}, \text{ where } e^i = dz^i + \varphi^i_{\bar{j}} d\overline{z}^j.$$

Let $\{w_1, \cdots, w_n\}$ be local holomorphic coordinates on M_t, let $a_{j\alpha} = \frac{\partial w^\alpha}{\partial z_j}$ and $(b^{j\alpha})_{n\times n} = (a_{j\alpha})^{-1}_{n\times n}$, $T_i = (\partial_i - \overline{\varphi^k_{\bar{i}}} \partial_{\bar{k}})$, then

$$\begin{cases} dw^\alpha = \frac{\partial w_\alpha}{\partial z_j} e^j = a_{jk}(dz^j + \varphi^j_{\bar{k}} d\overline{z}^k) \\ \frac{\partial}{\partial w^\alpha} = b^{j\alpha}(I - \varphi\overline{\varphi})^{ij} T_i = b^{j\alpha}(I - \varphi\overline{\varphi})^{ij}(\partial_i - \overline{\varphi^k_{\bar{i}}} \partial_{\bar{k}}). \end{cases} \tag{3.2.1}$$

Since the domain is a Kähler-Einstein manifold, as in the previous section in this chapter, we let

$$B_{j\overline{p}} = g_{j\overline{p}} - \partial_j(\varphi^k_{\overline{p}})(I - \varphi\overline{\varphi})^i k T_i(f) + T_{\overline{p}}[(I - \varphi\overline{\varphi})^{ij} T_i(f)].$$

Then

$$\begin{aligned} g_{\alpha\overline{\beta}} &= -R_{\alpha\overline{\beta}} \\ &= \frac{\partial^2}{\partial w_\alpha \partial \overline{w_\beta}} \log\det g_t \\ &= \overline{b^{q\beta}} b^{j\alpha} \overline{(I - \varphi\overline{\varphi})^{pq}} B_{j\overline{p}}. \end{aligned} \tag{3.2.2}$$

and

$$g^{\alpha\overline{\beta}} \frac{\partial u^A}{\partial w_\alpha} \frac{\partial u^B}{\partial \overline{w}_\beta} = B^{j\overline{k}} (I - \varphi\overline{\varphi})^{jm} T_m u^A T_{\overline{k}} u^B. \tag{3.2.3}$$

For the B matrix, we have

$$B_{l\overline{k}}(0) = g_{l\overline{k}}, \text{ and } \tfrac{\partial}{\partial t}|_{t=0} B_{l\overline{k}} = 0,$$

By the power series of $\varphi(t)$,

$$\tfrac{\partial}{\partial t}|_{t=0} (I - \varphi\overline{\varphi})_{ij} = 0,$$

and since φ is holomorphic in t, we have

$$\tfrac{\partial}{\partial t}|_{t=0} T_{\overline{k}} = \tfrac{\partial}{\partial t}|_{t=0}(\partial_{\overline{k}} - \varphi^i_{\overline{k}} \partial_i) = -\varphi^i_{1\overline{k}} \partial_i, \text{ and } \tfrac{\partial}{\partial t}|_{t=0} T_m = 0.$$

Hence,

$$\begin{aligned} \frac{\partial}{\partial t}\bigg|_{t=0} (g^{\alpha\overline{\beta}} \frac{\partial u^A}{\partial w_\alpha} \frac{\partial u^B}{\partial \overline{w}_\beta}) &= g^{j\overline{k}} \frac{\partial}{\partial t}|_{t=0} (T_j u^A T_{\overline{k}} u^B) \\ &= g^{j\overline{k}} \big[\partial_j(\frac{\partial u^A}{\partial t}(0)) \partial_{\overline{k}} u^B_0 - \partial_j u^A_0 \partial_i u^B_0 \varphi^i_{1\overline{k}} + \partial_j u^A_0 \partial_{\overline{k}}(\frac{\partial u^B}{\partial t}(0))\big]. \end{aligned} \tag{3.2.4}$$

Now, $E(u) = \int_{M_t} \frac{\partial u^A}{\partial w_\alpha} \frac{\partial u^B}{\partial \overline{w}_\beta} h_{AB} dV_t$ and $\frac{d}{dt}|_{t=0} dV_t = 0$. Thus

$$
\begin{aligned}
\frac{\partial}{\partial t}\bigg|_{t=0} E(u) &= \int_{M_0} [h_{AB}\frac{\partial}{\partial t}|_{t=0}(g^{\alpha\overline{\beta}}\frac{\partial u^A}{\partial w_\alpha}\frac{\partial u^B}{\partial \overline{w}_\beta}) + (g^{\alpha\overline{\beta}}\frac{\partial u^A}{\partial w_\alpha}\frac{\partial u^B}{\partial \overline{w}_\beta})|_{t=0}\frac{\partial h_{AB}}{\partial t}|_{t=0}]\, dV_0 \\
&= \int_{M_0} h_{AB} g^{j\overline{k}}[\partial_j(\frac{\partial u^A}{\partial t}(0))\partial_{\overline{k}} u_0^B + \partial_j u_0^A \partial_{\overline{k}}(\frac{\partial u^B}{\partial t}(0))]\, dV_0 \\
&\quad - \int_{M_0} [\partial_j u_0^A \partial_i u_0^B \varphi^i_{1\overline{k}}] h_{AB}\, dV_0 + \int_{M_0} g^{j\overline{k}}\frac{\partial u_0^A}{\partial z_j}\frac{\partial u_0^B}{\partial \overline{z_{\overline{k}}}}\frac{\partial h_{AB}}{\partial u_C}\frac{\partial u_0^C}{\partial t}\, dV_0 \\
&= -\int_{M_0} \partial_j(g^{j\overline{k}}\partial_{\overline{k}} u_0^B h_{AB} g)\frac{\partial u_0^A}{\partial t} g^{-1}\, dV_0 - \int_{M_0} \partial_{\overline{k}}(g^{j\overline{k}}\partial_j u_0^A h_{AB} g)\frac{\partial u_0^B}{\partial t} g^{-1}\, dV_0 \\
&\quad - \int_{M_0} g^{j\overline{k}}[\partial_j u_0^A \partial_i u^B \varphi^i_{1\overline{k}}] h_{AB}\, dV_0 + \int_{M_0} g^{j\overline{k}}\frac{\partial u_0^A}{\partial z_j}\frac{\partial u_0^B}{\partial \overline{z_{\overline{k}}}}\frac{\partial h_{AB}}{\partial u_C}\frac{\partial u_0^C}{\partial t}\, dV_0 \\
&= -\int_{M_0} (2g^{j\overline{k}}\partial_j\partial_{\overline{k}} u_0^B h_{AB}\frac{\partial u_0^A}{\partial t} + g^{j\overline{k}}\partial_{\overline{k}} u_0^B \partial_j u_0^C \frac{\partial u_0^A}{\partial t}(\frac{\partial h_{AB}}{\partial u_C} + \frac{\partial h_{AC}}{\partial u_B} - \frac{\partial h_{BC}}{\partial u_A}))\, dV_0 \\
&\quad - \int_{M_0} g^{j\overline{k}}[\partial_j u_0^A \partial_i u_0^B \varphi^i_{1\overline{k}}] h_{AB}\, dV_0 \\
&= -\int_{M_0} g^{j\overline{k}}[\partial_j u_0^A \partial_i u_0^B \varphi^i_{1\overline{k}}] h_{AB}\, dV_0 = -\int_{M_0} \Lambda\left(\varphi_1 \lrcorner H(u_0)\right) dV_0.
\end{aligned}
\tag{3.2.5}
$$

□

Corollary 3.2.1. *If $0 \in \mathcal{T}$ is a critical point of E then $\int_{M_0} \Lambda\left(\varphi_1 \lrcorner H(u_0)\right) dV_0 = 0$ for any $\varphi_1 \in \mathbb{H}^{0,1}\left(M_0, T^{1,0}M_0\right)$. In particular, if M_0 is a Riemann surface and 0 is a critical point of E, then $H(u_0) = 0$.*

3.3 Second variation of harmonic energy

Now we look at the second variation of $\mathcal{E}$ along the base direction. We let $v = \frac{\partial u_t}{\partial t}\Big|_{t=0} \in \Gamma\left(u_0^* TN\right)$. To fix the notation, for any function $\rho \in C^\infty(M_0)$ we let $\Delta\rho = g^{i\overline{j}}\partial_i\partial_{\overline{j}}\rho$. Now we let $K = (1-\Delta)^{-1}\left(|\varphi_1|^2\right)$. To state the second variation formula we need some notation. We let Γ^C_{AB} be the Christoffel symbol of the metric h on N. Let $\nabla^{1,0}$ and $\nabla^{0,1}$ be the connection on $u_0^* TN$ induced by the Levi-Civita connection on N. Precisely, for any section $s = s^A \frac{\partial}{\partial y_A} \in \Gamma\left(u_0^* TN\right)$ we have

$$\nabla^{1,0} s = \left(\partial_i s^A + s^B \partial_i u_0^C \Gamma_{BC}^A\right) dz_i \otimes \frac{\partial}{\partial y_A} \in \Gamma\left(\Omega^{1,0} M_0 \otimes u_0^* TN\right),$$

and

$$\nabla^{0,1} s = \left(\partial_{\bar{j}} s^A + s^B \partial_{\bar{j}} u_0^C \Gamma_{BC}^A\right) d\overline{z}_j \otimes \frac{\partial}{\partial y_A} \in \Gamma\left(\Omega^{0,1} M_0 \otimes u_0^* TN\right).$$

Lemma 3.3.1. *Let R_{ABCD} be the curvature tensor of the metric h on N, the complex Hessian of E is given by*

$$\begin{aligned} \left.\frac{\partial^2 E}{\partial t \partial \bar{t}}\right|_{t=0} = & \int_{M_0} h_{AB} \partial_i u_0^A \partial_{\bar{j}} u_0^B g^{i\bar{j}} \Delta K \ dV_0 \\ & - \int_{M_0} h_{AB} \partial_i u_0^A \partial_{\bar{j}} u_0^B g^{i\bar{q}} g^{p\bar{j}} \partial_p \partial_{\bar{q}} K \ dV_0 \\ & - 2 \int_{M_0} g^{i\bar{j}} R_{ABCD} \partial_i u_0^A \partial_{\bar{j}} u_0^C v^B \overline{v}^D dV_0 \\ & + 2 \int_{M_0} \|\nabla^{1,0} \overline{v} - \overline{\varphi}_1 \lrcorner \overline{\partial} u_0\|^2 dV_0. \end{aligned} \tag{3.3.1}$$

Proof. Let $G^{AB} = g^{\alpha\bar{\beta}} \frac{\partial u^A}{\partial w_\alpha} \frac{\partial u^B}{\partial \overline{w_\beta}} = B^{j\bar{k}} (I - \varphi\overline{\varphi})^{jm} T_m u^A T_{\bar{k}} u^B$, then

$$E(u) = \int_{M_t} g^{\alpha\bar{\beta}} \frac{\partial u^A}{\partial w_\alpha} \frac{\partial u^B}{\partial \overline{w_\beta}} h_{AB} dV = \int_{M_t} G^{AB} h_{AB} dV_t.$$

Since $\frac{\partial}{\partial t}|_{t=0} dV_t = \frac{\partial}{\partial \bar{t}}|_{t=0} dV_t = 0$, we see

$$\begin{aligned} \frac{\partial^2 E(u)}{\partial t \partial \bar{t}}\Big|_{t=0} = & \int_{M_0} \left[\left(\frac{\partial^2}{\partial t \partial \bar{t}}\Big|_{t=0} G^{AB}\right) h_{AB}(0) + G^{AB}(0) \frac{\partial^2}{\partial t \partial \bar{t}}\Big|_{t=0} h_{AB} + G^{AB}(0) h_{AB}(0) \frac{\partial^2}{\partial t \partial \bar{t}}\Big|_{t=0} \right] dV_t \\ & + \int_{M_0} \left[\frac{\partial}{\partial t}\Big|_{t=0} G^{AB} \frac{\partial}{\partial \bar{t}}\Big|_{t=0} h_{AB} + \frac{\partial}{\partial \bar{t}}\Big|_{t=0} G^{AB} \frac{\partial}{\partial t}\Big|_{t=0} h_{AB} \right] dV_t. \end{aligned} \tag{3.3.2}$$

Now, let $\frac{\partial u^A}{\partial t}|_{t=0} = v^A$, $\frac{\partial u^A}{\partial \bar{t}}|_{t=0} = \overline{v}^A$ and $\frac{\partial^2 u^A}{\partial t \partial \bar{t}}|_{t=0} = w^A$, we also let

$$G^{AB} = g^{\alpha\bar{\beta}} \frac{\partial u^A}{\partial w_\alpha} \frac{\partial u^B}{\partial \overline{w_\beta}} = B^{j\bar{k}} (I - \varphi\overline{\varphi})^{jm} T_m u^A T_{\bar{k}} u^B, \tag{3.3.3}$$

$$
\begin{aligned}
\frac{\partial}{\partial t}G^{AB} =&(\frac{\partial}{\partial t}B^{j\bar{k}})(I-\varphi\overline{\varphi})^{jm}T_m u^A T_{\bar{k}}u^B \\
&+ B^{j\bar{k}}\frac{\partial}{\partial t}(I-\varphi\overline{\varphi})^{jm}T_m u^A T_{\bar{k}}u^B \\
&+ B^{j\bar{k}}(I-\varphi\overline{\varphi})^{jm}\frac{\partial}{\partial t}(T_m u^A T_{\bar{k}}u^B),
\end{aligned}
\tag{3.3.4}
$$

Thus, we see

$$
\frac{\partial}{\partial t}\Big|_{t=0} G^{AB} = g^{j\bar{k}}[\partial_j v^A \partial_{\bar{k}} u_0^B + \partial_j u_0^A(\partial_{\bar{k}} v^B - \varphi_{1\bar{k}}\partial_m u_0^B)],
\tag{3.3.5}
$$

and,

$$
\frac{\partial}{\partial \bar{t}}\big|_{t=0} G^{AB} = g^{j\bar{k}}[\partial_{\bar{k}} u_0^B(\partial_j \overline{v}^A - \overline{\varphi_{1\bar{j}}^{l}}\partial_{\bar{l}} u_0^A) + \partial_j u_0^A \partial_{\bar{k}}\overline{v}^B],
\tag{3.3.6}
$$

and

$$
\begin{aligned}
\frac{\partial^2}{\partial t\partial\bar{t}}\big|_{t=0} G^{AB} =&\frac{\partial^2}{\partial t\partial\bar{t}}\big|_{t=0}(B^{j\bar{k}})\partial_j u_0^A \partial_{\bar{k}} u_0^B + g^{j\bar{k}}\frac{\partial^2}{\partial t\partial\bar{t}}\big|_{t=0}(I-\varphi\overline{\varphi})^{jm}\partial_m u_0^A\partial_{\bar{k}}u_0^B \\
&+ g^{j\bar{k}}\frac{\partial^2}{\partial t\partial\bar{t}}\big|_{t=0}(T_m u^A T_{\bar{k}} u^B) \\
=& - g^{j\bar{l}}g^{p\bar{k}}\partial_p\partial_{\bar{l}}\big[(1-\Delta)^{-1}|\varphi_1|^2\big]\partial_j u_0^A\partial_{\bar{k}}u_0^B + g^{j\bar{k}}\varphi_{1\bar{l}}^{m}\overline{\varphi_{1\bar{j}}^{l}}\partial_m u_0^A\partial_{\bar{k}}u_0^B \\
&+ g^{j\bar{k}}[(\partial_j w^A - \overline{\varphi_{1\bar{j}}^{l}}\partial_{\bar{l}} v^A)\partial_{\bar{k}} u_0^B + (\partial_{\bar{k}} w^\beta - \varphi_{1\bar{k}}^{m}\partial_m \overline{v}^B)\partial_j u_0^A \\
&+ \partial_j v^A \partial_{\bar{k}}\overline{v}^B + (\partial_j\overline{v}^A - \overline{\varphi_{1\bar{j}}^{l}}\partial_{\bar{l}}u_0^A)(\partial_{\bar{k}} v^\beta - \varphi_{1\bar{k}}^{m}\partial_m u_0^B)\big],
\end{aligned}
\tag{3.3.7}
$$

and

$$
\frac{\partial^2}{\partial t\partial\bar{t}}\Big|_{t=0} h_{AB} = \frac{\partial}{\partial\bar{t}}(\frac{\partial h_{AB}}{\partial u_C}\frac{\partial u^C}{\partial t})\Big|_{t=0} = \frac{\partial^2 h_{AB}}{\partial u_C\partial u_D}v^C\overline{v}^D + \frac{\partial h_{AB}}{\partial u_C}w^C,
\tag{3.3.8}
$$

Let $K = (1-\Delta)^{-1}|\varphi_1|^2$. The volume expansion is written as

$$
\frac{\partial^2}{\partial t\partial\bar{t}}\Big|_{t=0} dV_t = \Delta((1-\Delta)^{-1}|\varphi_1|^2) = \Delta K.
\tag{3.3.9}
$$

Thus,

$$
\begin{aligned}
\frac{\partial^2 E(u)}{\partial t \partial \bar{t}}\bigg|_{t=0} =& \int_{M_0} \big[h_{AB} g^{j\bar{k}} \partial_j u_0^A \partial_{\bar{k}} u_0^B \Delta K - h_{AB} g^{j\bar{l}} g^{p\bar{k}} \partial_j u_0^A \partial_{\bar{k}} u_0^B \partial_p \partial_{\bar{l}} K\big] + h_{AB} g^{j\bar{k}} \varphi^m_{1\bar{l}} \overline{\varphi^l_{1\bar{j}}} \partial_m u_0^A \partial_{\bar{k}} u_0^B \\
&+ h_{AB} g^{j\bar{k}} \big[((\partial_j w^A - \overline{\varphi^l_{1\bar{j}}} \partial_{\bar{l}} v^A) \partial_{\bar{k}} u_0^B + (\partial_{\bar{k}} w^\beta - \varphi^m_{1\bar{k}} \partial_m \overline{v}^B) \partial_j u_0^A) \\
&+ \partial_j v^A \partial_{\bar{k}} \overline{v}^B + (\partial_j \overline{v}^A - \overline{\varphi^l_{1\bar{j}}} \partial_{\bar{l}} u_0^A)(\partial_{\bar{k}} v^\beta - \varphi^m_{1\bar{k}} \partial_m u_0^B)\big] \\
&+ g^{j\bar{k}} \big[\partial_j u_0^A \partial_{\bar{k}} u_0^B (\frac{\partial^2 h_{AB}}{\partial u_C \partial u_D} v^C \overline{v}^D + \frac{h_{AB}}{\partial u_C} w^C)\big] \\
&+ g^{j\bar{k}} \big[\partial_j v^A \partial_{\bar{k}} u_0^B + \partial_j u_0^A (\partial_{\bar{k}} v^B - \varphi^m_{1\bar{k}} \partial_m u_0^B)\big] \frac{\partial h_{AB}}{\partial u_C} \overline{v}^C \\
&+ g^{j\bar{k}} \big[\partial_j u_0^A \partial_{\bar{k}} \overline{v}^B + \partial_{\bar{k}} (u_0^B)(\partial_j \overline{v}^A - \overline{\varphi^l_{1\bar{j}}} \partial_{\bar{l}} u_0^A)\big] \frac{\partial h_{AB}}{\partial u_C} v^C \\
=& \int_{M_0} \big[h_{AB} g^{j\bar{k}} \partial_j u_0^A \partial_{\bar{k}} u_0^B \Delta K - h_{AB} g^{j\bar{l}} g^{p\bar{k}} \partial_j u_0^A \partial_{\bar{k}} u_0^B \partial_p \partial_{\bar{l}} K\big] dV_0 \\
&+ \int_{M_0} \big[h_{AB} g^{j\bar{k}} \partial_j w^A \partial_{\bar{k}} u_0^B + h_{AB} g^{j\bar{k}} \partial_{\bar{k}} w^B \partial_j u_0^A + g^{j\bar{k}} \partial_j u_0^A \partial_{\bar{k}} u_0^B \frac{\partial h_{AB}}{\partial u_C} w^C\big] dV_0 \\
&+ \int_{M_0} \big[h_{AB} g^{j\bar{k}} \varphi^m_{1\bar{l}} \overline{\varphi^l_{1\bar{j}}} \partial_m u_0^A \partial_{\bar{k}} u_0^B - h_{AB} g^{j\bar{k}} \overline{\varphi^l_{1\bar{j}}} \partial_{\bar{l}} v^A \partial_{\bar{k}} u_0^B - h_{AB} g^{j\bar{k}} \varphi^m_{1\bar{k}} \partial_m \overline{v}^B \partial_j u_0^A\big] dV_0 \\
&+ \int_{M_0} h_{AB} g^{j\bar{k}} \big[\partial_j v^A \partial_{\bar{k}} \overline{v}^B + (\partial_j \overline{v}^A - \overline{\varphi^l_{1\bar{j}}} \partial_{\bar{l}} u_0^A)(\partial_{\bar{k}} v^\beta - \varphi^m_{1\bar{k}} \partial_m u_0^B)\big] dV_0 \\
&+ \int_{M_0} g^{j\bar{k}} \big[\partial_j u_0^A \partial_{\bar{k}} u_0^B \frac{\partial^2 h_{AB}}{\partial u_C \partial u_D} v^C \overline{v}^D + \partial_j v^A \partial_{\bar{k}} u_0^B \frac{\partial h_{AB}}{\partial u_C} \overline{v}^C \\
&+ \partial_j u_0^A (\partial_{\bar{k}} v^B - \varphi^m_{1\bar{k}} \partial_m u_0^B) \frac{\partial h_{AB}}{\partial u_C} \overline{v}^C + \partial_j u_0^A \partial_{\bar{k}} \overline{v}^B \frac{\partial h_{AB}}{\partial u_C} v^C \\
&+ \partial_{\bar{k}} u_0^B (\partial_j \overline{v}^A - \overline{\varphi^l_{1\bar{j}}} \partial_{\bar{l}} u_0^A) \frac{\partial h_{AB}}{\partial u_C} v^C\big] dV_0.
\end{aligned}
\tag{3.3.10}
$$

By the harmonic map equation and integration by parts, the second integral in the last equality becomes

$$
\int_{M_0} [h_{AB} g^{j\bar{k}} \partial_j w^A \partial_{\bar{k}} u_0^B + h_{AB} g^{j\bar{k}} \partial_{\bar{k}} w^B \partial_j u_0^A + g^{j\bar{k}} \partial_j u_0^A \partial_{\bar{k}} u_0^B \frac{\partial h_{AB}}{\partial u_C} w^C] dV_0 = 0. \tag{3.3.11}
$$

Using the symmetry $\varphi \lrcorner \omega_g = 0$, the third integral in the last equality is

$$
\begin{aligned}
&\int_{M_0} [h_{AB} g^{j\bar{k}} \varphi^m_{1\bar{l}} \overline{\varphi^l_{1\bar{j}}} \partial_m u_0^A \partial_{\bar{k}} u_0^B - h_{AB} g^{j\bar{k}} \overline{\varphi^l_{1\bar{j}}} \partial_{\bar{l}} v^A \partial_{\bar{k}} u_0^B - h_{AB} g^{j\bar{k}} \varphi^m_{1\bar{k}} \partial_m \overline{v}^B \partial_j u_0^A] dV_0 \\
= &\int_{M_0} h_{AB} g^{j\bar{k}} [(\partial_{\bar{k}} v^A - \varphi^m_{1\bar{k}} \partial_m u_0^A)(\partial_j \overline{v}^B - \overline{\varphi^l_{1\bar{j}}} \partial_{\bar{l}} u_0^B) - \partial_{\bar{k}} v^A \partial_j \overline{v}^B h_{AB} g^{j\bar{k}}] dV_0 \\
= &\int_{M_0} [||\nabla^{1,0} \overline{v} - \overline{\varphi} \lrcorner \overline{\partial} u_0||^2 - \partial_{\bar{k}} v^A \partial_j \overline{v}^B h_{AB} g^{j\bar{k}}] dV_0,
\end{aligned}
\tag{3.3.12}
$$

and the fourth integral in the last equality is,

$$
\begin{aligned}
&\int_{M_0} h_{AB} g^{j\bar{k}} [\partial_j v^A \partial_{\bar{k}} \overline{v}^B + (\partial_j \overline{v}^A - \overline{\varphi^l_{1\bar{j}}} \partial_{\bar{l}} u_0^A)(\partial_{\bar{k}} v^\beta - \varphi^m_{1\bar{k}} \partial_m u_0^B)] dV_0 \\
= &\int_{M_0} [||\nabla^{1,0} \overline{v} - \overline{\varphi} \lrcorner \overline{\partial} u_0||^2 + \partial_j v^A \partial_{\bar{k}} \overline{v}^B h_{AB} g^{j\bar{k}}] dV_0.
\end{aligned}
\tag{3.3.13}
$$

For the last intergral in the last equality, using integration by parts, for two terms of it, we get

$$
\begin{aligned}
&\int_{M_0} [g^{j\bar{k}} \partial_j v^A \partial_{\bar{k}} u_0^B \frac{\partial h_{AB}}{\partial u_C} \overline{v}^C + g^{j\bar{k}} \partial_j u_0^A \partial_{\bar{k}} v^B \frac{\partial h_{AB}}{\partial u_C} \overline{v}^C] dV_0 \\
= &- \int_{M_0} (2\Delta u_0^A v^B \overline{v}^C \frac{h_{AB}}{\partial u_C} + g^{j\bar{k}} \partial_{\bar{k}} u_0^B \partial_j \overline{v}^C v^A \frac{h_{AB}}{\partial u_C} + g^{j\bar{k}} \partial_j u_0^A \partial_{\bar{k}} \overline{v}^B v^C \frac{h_{AC}}{\partial u_B}) dV_0 \\
&- \int_{M_0} [g^{j\bar{k}} \partial_j u_0^A \partial_{\bar{k}} u_0^B \frac{\partial^2 h_{BC}}{\partial u_A \partial u_D} v^C \overline{v}^D + g^{j\bar{k}} \partial_j u_0^A \partial_{\bar{k}} u_0^B \frac{\partial^2 h_{AD}}{\partial u_B \partial u_C} v^C \overline{v}^D] dV_0.
\end{aligned}
\tag{3.3.14}
$$

Thus,

$$\begin{aligned}
\left.\frac{\partial^2 E(u)}{\partial t\partial \bar{t}}\right|_{t=0} =& \int_{M_0} [h_{AB} g^{j\bar{k}} \partial_j u_0^A \partial_{\bar{k}} u_0^B \Delta K - h_{AB} g^{j\bar{l}} g^{p\bar{k}} \partial_j u_0^A \partial_{\bar{k}} u_0^B \partial_p \partial_{\bar{l}} K] dV_0 \\
&+ 2\int_{M_0} ||\nabla^{1,0}\overline{v} - \overline{\varphi}\lrcorner\overline{\partial} u||^2 dV_0 \\
&+ \int_{M_0} (h_{AB} g^{j\bar{k}} \partial_j v^A \partial_{\bar{k}} \overline{v}^B - h_{AB} g^{j\bar{k}} \partial_j \overline{v}^B \partial_{\bar{k}} v^A) dV_0 \\
&+ \int_{M} g^{j\bar{k}} \partial_j u_0^A \partial_{\bar{k}} u_0^B v^C \overline{v}^D [\frac{\partial^2 h_{AB}}{\partial u_C \partial u_D} - \frac{\partial^2 h_{BC}}{\partial u_A \partial u_D} - \frac{\partial^2 h_{AD}}{\partial u_B \partial u_C}] dV_0 \\
&- \int_{M_0} (2\Delta u_0^A v^B \overline{v}^C \frac{h_{AB}}{\partial u_C} + g^{j\bar{k}} \partial_j u_0^A \partial_{\bar{k}} \overline{v}^B v^C \frac{h_{AC}}{\partial u_B} + g^{j\bar{k}} \partial_j \overline{v}^A \partial_{\bar{k}} \overline{u}_0^B v^C \frac{h_{BC}}{\partial u_A}) dV_0 \\
&+ \int_{M_0} g^{j\bar{k}} [(\partial_j u_0^A \partial_{\bar{k}} \overline{v}^B + \partial_{\bar{k}} u_0^B \partial_j \overline{v}^A - \partial_{\bar{k}} u_0^B \overline{\varphi_{1\bar{j}}^{l}} \partial_{\bar{l}} u_0^A) v^C - \partial_j u_0^A \varphi_{1\bar{k}}^m \partial_m u_0^B \overline{v}^C] \frac{\partial h_{AB}}{\partial u_C} dV_0.
\end{aligned} \tag{3.3.15}$$

Performing integration by parts twice on the term $\int_{M_0} (h_{AB} g^{j\bar{k}} \partial_j v^A \partial_{\bar{k}} \overline{v}^B) dV_0$, we get

$$\begin{aligned}
&\int_{M_0} h_{AB} g^{j\bar{k}} \partial_j v^A \partial_{\bar{k}} \overline{v}^B dV_0 \\
=& \int_{M_0} \big[g^{j\bar{k}} \partial_j u_0^A \partial_{\bar{k}} u_0^B v^C \overline{v}^D \frac{\partial^2 h_{DC}}{\partial u_A \partial u_B} + h_{AB} g^{j\bar{k}} \partial_{\bar{k}} v^A \partial_j \overline{v}^B\big] dV_0 \\
+& \int_{M_0} \big[\Delta u_0^A v^B \overline{v}^C \frac{\partial h_{BC}}{\partial u_A} + g^{j\bar{k}} \partial_j \overline{v}^A \partial_{\bar{k}} u_0^B v^C \frac{h_{AC}}{\partial u_B} + g^{j\bar{k}} \partial_j u_0^A \partial_{\bar{k}} v^B \overline{v}^C \frac{h_{BC}}{\partial u_A}\big] dV_0.
\end{aligned} \tag{3.3.16}$$

Thus we obtain

$$\begin{aligned}
\left.\frac{\partial^2 E(u)}{\partial t\partial \bar{t}}\right|_{t=0} =& \int_{M_0} h_{AB} \big[g^{j\bar{k}} \partial_j u_0^A \partial_{\bar{k}} u_0^B \Delta K - h_{AB} g^{j\bar{l}} g^{p\bar{k}} \partial_j u_0^A \partial_{\bar{k}} u_0^B \partial_p \partial_{\bar{l}} K\big] dV_0 \\
&+ 2\int_{M_0} ||\nabla^{1,0}\overline{v} - \overline{\varphi}\lrcorner\overline{\partial} u_0||^2 dV_0 - \int_{M_0} R_{ABCD} \partial_j u_0^A \partial_{\bar{k}} u_0^C v^B \overline{v}^D dV_0.
\end{aligned} \tag{3.3.17}$$

□

Repeatedly applying integration by parts for the first two terms, we obtain the pluri-subharmonicity of the energy.

Theorem 3.3.1. *The second variation of E is*

$$\begin{aligned}\left.\frac{\partial^2 E}{\partial t \partial \bar{t}}\right|_{t=0} = & -\int_{M_0} R_{ABCD} \partial_i u_0^A \partial_{\bar{j}} u_0^C \partial_p u_0^B \partial_{\bar{q}} u_0^D g^{i\bar{j}} g^{p\bar{q}} K \ dV_0 \\ & + \int_{M_0} \|\nabla^{1,0} \overline{\partial} u_0\|^2 K \ dV_0 \\ & - 2\int_{M_0} g^{i\bar{j}} R_{ABCD} \partial_i u_0^A \partial_{\bar{j}} u_0^C v^B \overline{v}^D dV_0 \\ & + 2\int_{M_0} \|\nabla^{1,0} \overline{v} - \overline{\varphi}_1 \lrcorner \overline{\partial} u_0\|^2 dV_0.\end{aligned} \tag{3.3.18}$$

In particular, if we assume that the curvature of (N, h) is Hermitian nonpositive, namely

$$R(X, Y, \overline{X}, \overline{Y}) \leq 0$$

for all $X, Y \in T^{\mathbb{C}}N$. (For example when (N, h) is a Kähler manifold of nonpositive Riemannian sectional curvature.) Then E is a pluri-subharmonic function on the Teichmüller space $\mathcal{T}$.

Proof. In the following proof, we use notation $\partial_i u_0 = u_i$, and so on. We analyze term by term in equation (3.3.17),

$$\begin{aligned} & \int_M h_{AB} g^{i\bar{j}} u_i^A u_{\bar{j}}^B \Delta K dV_0 \\ = & \int_M h_{AB} g^{i\bar{j}} u_i^A u_{\bar{j}}^B g^{p\bar{q}} K_{p\bar{q}} dV_0 \\ = & -\int_M \partial_p (h_{AB} g^{i\bar{j}} u_i^A u_{\bar{j}}^B) g^{p\bar{q}} K_{\bar{q}} dV_0 \\ = & \int_M g^{p\bar{q}} \partial_{\bar{q}} \partial_p (h_{AB} g^{i\bar{j}} u_i^A u_{\bar{j}}^B) K dV_0 \\ = & \int_M [\partial_{\bar{q}} \partial_p (h_{AB}) u_i^A u_{\bar{j}}^B g^{i\bar{j}} g^{p\bar{q}} K + g^{p\bar{q}} h_{AB} \partial_{\bar{q}} \partial_p (g^{i\bar{j}} u_i^A u_{\bar{j}}^B) K] dV_0 \\ = & \int_M \partial_D \partial_C h_{AB} u_p^C u_{\bar{q}}^D u_i^A u_{\bar{j}}^B g^{i\bar{j}} g^{p\bar{q}} K + \int_M h_{AB} \partial_{\bar{q}} \partial_p (g^{i\bar{j}} u_i^A u_{\bar{j}}^B) g^{p\bar{q}} K dV_0. \end{aligned} \tag{3.3.19}$$

$$
\begin{aligned}
&\int_M h_{AB} g^{i\bar{l}} g^{p\bar{j}} u_i^A u_{\bar{j}}^B \partial_p \partial_{\bar{l}} K dV_0 \\
=&-\int_M \partial_p (h_{AB} u_i^A u_{\bar{j}}^B g^{i\bar{l}}) g^{p\bar{j}} K_{\bar{l}} dV_0 \\
=&\int_M \partial_{\bar{l}} [\partial_p (h_{AB} u_i^A u_{\bar{j}}^B g^{i\bar{l}}) g^{p\bar{j}} g] K g^{-1} dV_0 \\
=&\int_M [\partial_{\bar{l}} \partial_p (h_{AB} u_i^A u_{\bar{j}}^B g^{i\bar{l}}) g^{p\bar{j}} K + \partial_p (h_{AB} u_i^A u_{\bar{j}}^B g^{i\bar{l}}) \partial_{\bar{l}} (g g^{p\bar{j}}) g^{-1} K] dV_0 \\
=&\int_M [\partial_D \partial_C h_{AB} u_i^A u_{\bar{j}}^B u_p^C u_{\bar{l}}^D g^{i\bar{l}} g^{p\bar{j}} K + h_{AB} \partial_{\bar{l}} \partial_p (u_i^A u_{\bar{j}}^B g^{i\bar{l}}) g^{p\bar{j}} K \\
&+ h_{AB} \partial_p (u_i^A u_{\bar{j}}^B g^{i\bar{l}}) \partial_{\bar{l}} (g g^{p\bar{j}}) g^{-1} K] dV_0.
\end{aligned}
\tag{3.3.20}
$$

Therefore, the first integral in the equation (3.3.17) is

$$
\begin{aligned}
&\int_M h_{AB} [g^{j\bar{k}} \partial_j u^A \partial_{\bar{k}} u^B \Delta K - h_{AB} g^{j\bar{l}} g^{p\bar{k}} \partial_j u^A \partial_{\bar{k}} u^B \partial_p \partial_{\bar{l}} K] dV_0 \\
=&\int_M \partial_D \partial_C h_{AB} u_i^A u_{\bar{j}}^B u_p^C u_{\bar{q}}^D (g^{i\bar{j}} g^{p\bar{q}} - g^{i\bar{q}} g^{p\bar{j}}) K dV_0 \\
&+ \int_M \left[h_{AB} \partial_{\bar{q}} \partial_p (g^{i\bar{j}} u_i^A u_{\bar{j}}^B) g^{p\bar{q}} K \right] dV_0 \\
&- \int_M \left[h_{AB} \partial_{\bar{l}} (\partial_p (u_i^A u_{\bar{j}}^B g^{i\bar{l}}) g^{p\bar{j}} g) g^{-1} K \right] dV_0.
\end{aligned}
\tag{3.3.21}
$$

Now, we compute the first term in equation (3.3.21),

$$
\begin{aligned}
&\int_M \partial_D \partial_C h_{AB} u_i^A u_{\bar{j}}^B u_p^C u_{\bar{q}}^D (g^{i\bar{j}} g^{p\bar{q}} - g^{i\bar{q}} g^{p\bar{j}}) K dV_0 \\
=&\int_M (\partial_D \partial_C h_{AB} u_i^A u_{\bar{j}}^B u_p^C u_{\bar{q}}^D g^{i\bar{j}} g^{p\bar{q}} - \partial_D \partial_C h_{AB} u_i^A u_{\bar{j}}^B u_p^C u_{\bar{q}}^D g^{i\bar{q}} g^{p\bar{j}}) K dV_0 \\
=&\int_M (\partial_D \partial_C h_{AB} - \partial_B \partial_C h_{AD}) \partial_D \partial_C h_{AB} u_i^A u_{\bar{j}}^B u_p^C u_{\bar{q}}^D g^{i\bar{j}} g^{p\bar{q}} K dV_0 \\
=&\frac{1}{2} \int_M (\partial_D \partial_C h_{AB} + \partial_B \partial_A h_{CD} - \partial_B \partial_C h_{AD} - \partial_A \partial_D h_{BC}) u_i^A u_{\bar{j}}^B u_p^C u_{\bar{q}}^D g^{i\bar{j}} g^{p\bar{q}} K dV_0 \\
=&\int_M R_{ACDB} u_i^A u_{\bar{j}}^B u_p^C u_{\bar{q}}^D g^{i\bar{j}} g^{p\bar{q}} K dV_0 \\
=&-\int_M R_{ABCD} u_i^A u_{\bar{j}}^C u_p^B u_{\bar{q}}^D g^{i\bar{j}} g^{p\bar{q}} K dV_0.
\end{aligned}
\tag{3.3.22}
$$

The second term in equation (3.3.21) is

$$\begin{aligned}
&\int_M \left[h_{AB}\partial_{\overline{q}}\partial_p(g^{i\overline{j}}u_i^A u_{\overline{j}}^B)g^{p\overline{q}}K\right]dV_0\\
=&\sum_A\int_M \nabla_{\overline{p}}\nabla_p(u_i^A u_{\overline{i}}^A)KdV_0\\
=&\sum_A\int_M(\nabla_{\overline{p}}\nabla_p\nabla_i u^A\nabla_{\overline{i}}u^A+|\nabla\nabla u^A|^2+|\nabla\overline{\nabla}u^A|^2+\nabla_{\overline{p}}\nabla p\nabla_{\overline{i}}u^A)KdV_0,
\end{aligned}\tag{3.3.23}$$

and the last integral in equation (3.3.21) is

$$\begin{aligned}
&\int_M \left[h_{AB}\partial_{\overline{l}}(\partial_p(u_i^A u_{\overline{j}}^B g^{i\overline{l}})g^{p\overline{j}}g)g^{-1}K\right]dV_0\\
=&\sum_A\int_M \nabla_{\overline{i}}\nabla_p(u_i^A u_{\overline{p}}^A)KdV_0\\
=&\sum_A\int_M(\nabla_{\overline{i}}\nabla_p\nabla_i u^A\nabla_{\overline{p}}u^A+|\nabla\nabla u^A|^2)KdV_0\\
=&\sum_A\int(\nabla_{\overline{p}}\nabla_i\nabla_p u^A\nabla_{\overline{i}}u^A+|\nabla\nabla u^A|^2)K)dV_0.
\end{aligned}\tag{3.3.24}$$

Hence, by above Lemma 3.3.1,

$$\begin{aligned}
\frac{\partial^2 E(u)}{\partial t\partial\overline{t}}|_{t=0}=&-\sum_{ABCD}\int_M(R_{ABCD}u_i^A u_{\overline{j}}^C u_p^B u_{\overline{q}}^D g^{i\overline{j}}g^{p\overline{q}}+2R_{ABCD}u_i^A u_{\overline{j}}^C v^B\overline{v}^D g^{i\overline{j}})KdV_0\\
&+\sum_A\int_M|\nabla\overline{\nabla}u^A|^2KdV+2\int_M||\nabla^{1,0}\overline{v}-\overline{\varphi_1}\lrcorner\overline{\partial}u||^2dV_0,
\end{aligned}\tag{3.3.25}$$

$K=(1-\Delta)^{-1}|\varphi_1|^2$, so $\Delta K=K-|\varphi_1|^2$. By the maximum principal, we know $K\geq 0$. Combining with the curvature assumption on the target manifold, we conclude that $E(u_t)$ is pluri-subharmonic on the Teichmüller space $\mathcal{T}$. □

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