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# The Capacitated Reliable Fixed-charge Location Problem: Model and Algorithm

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# The Capacitated Reliable Fixed-charge Location Problem: Model and Algorithm

by

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# Abstract

Many classical facility location models, like the uncapacitated fixed-charge location problem (known as UFLP), implicitly assume that the facilities will operate well all the time. However, in fact, facilities may fail sometimes due to natural or economic factors. The reliability fixed-charge location problem (RFLP) is based on the UFLP, taking failures into account. Both the UFLP and the RFLP assume that facilities are uncapacitated, which is obviously not true in reality. Capacity limits in reality may mean that customers have to be served from facilities much farther than their regularly assigned facilities at different levels. In this thesis, we present a model for choosing facility locations to minimize expected cost, while also taking into account both the possible failures and the capacity of facilities. The goal is to choose facility locations that are both inexpensive under traditional objective functions and also reliable under capacity constraints. This capacitated reliability approach is new in the facility location literature. We formulate a capacitated reliability model based on the RFLP and present an exact Lagrangian relaxation algorithm to solve it.

# Chapter 1

# Introduction

Facility location is one of the oldest classes of supply chain management problems. There are two roles played in the system: facilities and customers. The facilities in the supply chain may be plants, warehouses, or distribution centers and the customers may be consumers or downstream retailers. Usually facility location problems involve two stages: first, deciding which candidate facilities should be open; second, choosing which facility to serve which customer.

The uncapacitated fixed-charge location problem (UFLP) is a classical facility location problem that selects facility locations and customer assignments in order to balance the trade-off between initial setup costs and day-to-day transportation costs. However, some of the constructed facilities may fail due to disruptions caused by natural disasters, terrorist attacks or labor strikes. One notable example is Hurricanes Katrina and Rita in 2005 on the U.S. Gulf Coast, which destroyed facilities at almost all levels of the supply chain.

When a facility failure occurs, some customers have to be reassigned to farther facilities that incur greater transportation costs. Thus some facility location models that minimize fixed and transportation costs as well as hedge against facility failures were developed. Snyder and Daskin [23] first introduced the reliable fixed-charged location problem (RFLP) model based on the UFLP and presented a Lagrangian relaxation algorithm to solve it. The strategy for handling facility disruptions in this model is to assign each customer to a set of facilities in sequence so that each customer would be served by a first backup facility if its primary facility fails, a second backup if its first backup fails, and so on. And the objective function is a weighted sum of the day-to-day transportation cost under normal curcumstances and the expected transportation cost, taking disruptions into account. Realistically, no company would accept a supply network with high normal operating costs just to hedge against very rare facility disruptions. Snyder and Daskin [23] constructed the trade-off curve for the RFLP using a 49-node dataset to demonstrate empirically that substantial improvements in reliability are often possible with minimal increases in operating cost. The steepness of the left part of the curve indicates that large improvements in reliability can be attained without large increases in UFLP cost.

However, the RFLP model in [23] has a significant drawback in that it assumes the facilities are uncapacitated. Although this assumption is quite common in facility location models, it is actually unrealistic. Besides, capacity really has a significant influence on the facility choices and the objective value. We can see how it works from comparisons between the results of the UFLP and the CFLP (the capacitated fixed-charge location problem) with the 49-node set in Table 1.1. In the CFLP case, all facility have a capacity of 300, which is still greater than the highest demand. Obviously, capacity increases the total cost and the CFLP also needs more time to solve in CPLEX. It is clear from classical optimization theory that any minimization problem will have a greater optimal objective function value when extra constraints are added..

Capacity has the same effect on the RFLP. In the reliability context, backup facilities can only serve the customers of failed facilities if they have enough extra capacity to satisfy the customers' demand. That makes the capacitated version different from the uncapacitated model and also more complicated than traditional capacitated facility location models. There are many possible ways to model the capacity constraints and to formulate them mathematically, since the traditional notion of capacity is complicated by the multiple levels of service in reliability models. We will discuss the details of our approach to modeling capacity when we discuss the model in Section 3.2. In any case, the capacitated version of the model will obviously result in higher transportation costs than the uncapacitated model and may also influence the facility locations

| Model | # Opened facilities | Optimal value | CPLEX time | # iterations |
|---|---|---|---|---|
| 49 UFLP | 7 | 1133294.887 | 0.13 | 294 |
| 49 CFLP | 10 | 1205975.537 | 1.03 | 2052 |

Table 1.1: UFLP and CFLP results comparison

themselves. We will confirm these suggestions in our computational study in Chapter 5.

The remainder of this thesis is structured as follows. We review the related literature in Chapter 2. In Chapter 3, we formulate a capacitated model based on the RFLP. In Chapter 4, we propose a Lagrangian relaxation algorithm to solve the problem. We present computational results in Chapter 5 and a summary in Chapter 6.

# Chapter 2

# Literature Review

In this chapter, we present an overview of other studies related to this thesis. We first review the development of research on facility location models. Then we shall focus on the models handling possible system failures due to facility disruptions. Last, we introduce models taking capacity into account.

Facility location models have been extensively studied in the literature. They typically try to determine where to locate the facilities among a set of candidate sites, and how to assign customers to the facilities, so that the total cost can be minimized [7, 14]. One of the first facility location models was formulated in 1909 when A. Weber introduced the famous Weber problem [26]. Most facility location problems are modeled as integer programming problems. Daskin [7] and Drezner [8] give extensive discussions of these models and their solution algorithms.

Reliability issues considering uncertainty in supply chain design arose in the 1980s. The uncertainties can be generally classified into three categories: provider-side uncertainty, receiver-side uncertainty, and in-between uncertainty [28]. Most of the existing literature focuses on receiver-side and in-between uncertainties such as randomness in demand or leadtime [3, 9, 16, 22, 27]. An extensive review is given on these stochastic facility location problems in [22].

However, receiver-side and in-between uncertainties don't typically change the network once the facilities have been built [28]. On the other hand, if a facility fails, customers originally assigned to it have to be reassigned to other (operational) facilities, and thus the transportation cost changes (usually increases). Thus, system failures due to facility disruptions in supply chain design has gained increasing attetion. This topic is related to network reliability theory [4, 19, 20], which aims to improve the probability that a network remains functional after some links are disconnected.

Snyder and Daskin [23] presented the $P$-median and fixed-charge problems based on level assignments, where the candidate sites are subject to random disruptions with uniform failure probability. They formulated their problem as a linear integer program and proposed a Lagrangian relaxation solution method. In contract, Berman et al. [2] and Zhan et al. [28] give a more general model with different failure probabilities for candidate sites. They prove that the solution to the stochastic $P$-median problem coincides with the deterministic problem as the failure probabilities approach zero. They also propose heuristics with bounds on the worst-case performance. Jeon, Snyder, and Shen [11] include inventory cost as another factor that have an effect on the decisions.

[17, 18, 24] also addressed the fortification of existing facilitiesto increase their reliability. These models typically focus on the interdiction-fortification framework based upon the $P$-median facility location problem. They are generally formulated as bilevel programming models. Their main focus is to identify the existing critical facilities to protect under the threat of disruption [28].

Since we plan to combine reliability and capacity together into one model, reviewing capacitated facility location models is necessary. The capacitated version of the UFLP is called the capacitated facility location problem (CFLP). Kuehn and Hamburger [12] gave one of the earliest models and a heuristic procedure for CFLP. Akinc and Khumawala [1] developed branch-and-bound procedures for this problem using the linear programming relaxation and Nauss [15] also did it through Lagrangian relaxation. Snyder and Ülker [25] presented a scenario-based capacitated reliable model. They used scenarios to represent uncertain events and in each scenario, the problem resembles the formulation of the CFLP. Gade [10] presented the scenario-based capacitated reliable model as a two-stage stochastic program with relatively complete recourse and proposed a sampling based procedure known as the Sample Average Approximation (SAA) to approximately solve this model.

# Chapter 3

# The Capacitated Reliable Fixed-charged Location Problem

In this chapter we introduce a capacitated reliable fixed-charge location model. The objective is to minimize the sum of the fixed cost and the expected transportation cost (cumputed using failure probabilities). We assume that each candidate site has the same failure probability and they can fail simultaneously. Capacity constraints are at the heart of this model so we explain how we formulate them and give the reason why we formulate in this way.

## 3.1 Notation

Let $I$ be the set of customers and $J$ the set of potential facility sites. Let $h_i$ be the demand at customer $i$, $c_{ij}$ the transportation cost from facility $j$ to customer $i$, $f_j$ the fixed cost to open facility $j$, and $v_j$ the capacity of facility $j$.

All candiadate facilities in $J$ have a uniform failure probability $q$. But their failures occur independently from each other. Usually the failure probabilty is estimated based on historical statistics, like the percentage of time that the facility is in a disruption status in the long run. Although this assumption is unrealistic, it makes the model easier to solve.

The penalty $\theta_i$ related with each customer $i$ represents the cost of not serving the customer $i$, per unit of demand. $\theta_i$ can be interpreted as a lost-sales cost. It is incurred when all facilities fail or $\theta_i$ is less than some $c_{ij}$ if facility $j$ hasn't failed. In order to model this more easily, we can add an emergency facility $u$ that won't be disrupted and we let $x_u$=1 as a forced constraint. This emergency facility's fixed cost is 0 and the transportation cost $c_{iu}$ equals $\theta_i$ for every customer $i \in I$. From this point forward, the set $J$ is assumed to contain $u$, as well.

The system's reliability is based on the levels assignments strategy. We use $r$ ($r$=0,1...,$|J|$-1) to denote the level at which a facility serves a given customer. When $r$=0, it is a primary assignment. If $r$=1, it is a first backup, and so on. If some customer $i$'s level-$r$ assigned facility failed, the level-($r+1$) assigned facility would serve the customer as backup. The emergency facility will serve customers if all opened facilities failed or the penalty was less than the transportation cost to the remaining facilities.

There are two sets of decision variables in this model:

$$x_j = \begin{cases} 1 & \text{if facility } j \in J \text{ is selected} \\ 0 & \text{otherwise} \end{cases}$$

$$y_{ijr} = \begin{cases} 1 & \text{if customer } i \text{ is assigned to facility } j \text{ as a level-} r \text{ assignment} \\ 0 & \text{otherwise} \end{cases}$$

However, we need to clarify that the level $r$ in CRFLP cannot be interpreted as meaning that there are $r$ closer opened facilities, as in RFLP. That's because we also have to consider the capacity constraints. The closest facility in the remaining ones may not have enough excess capacity, thus we move to the next closest one and check its capacity until some facility satisfies the constraint.

## 3.2 Formulation

The objective function of the CRFLP is given by

$$\sum_{j \in J} f_j x_j + \sum_{i \in I} \sum_{\substack{j \in J \\ j \neq u}} \sum_{r=0}^{|J|-1} h_i c_{ij} q^r (1-q) y_{ijr} + \sum_{i \in I} \sum_{r=0}^{|J|-1} h_i c_{iu} q^r y_{iur}.$$

This function is the sum of the fixed cost and the expected transportation cost. The first item calculates the total fixed cost. The second item calculates the expected transportation cost for non-emergency facilities. Facility $j$ serves customer $i$ if and only if its lower-level assigned facilities are all disrupted with probability $q^r$, and if $j$ remains available, which occurs with probabilty $1-q$. The third item calculates the expected transportation cost for the emergency facility $u$. Since the emergency facility is unfailable, it serves customers at level $r$ with probabilty $q^r$.

For notational convenience, we define

$$\psi_{ijr} = \begin{cases} h_i c_{ij} q^r, & \text{if } j = u \\ h_i c_{ij} q^r (1-q), & \text{if } j \neq u \end{cases}$$

Then the CRFLP can be formulated as an IP as follows:

(CRFLP)

$$\min \quad \sum_{j \in J} f_j x_j + \sum_{i \in I} \sum_{j \in J} \sum_{r=0}^{|J|-1} \psi_{ijr} y_{ijr} \tag{3.1}$$

$$\text{s.t.} \quad \sum_{j \in J} y_{ijr} + \sum_{s=0}^{r-1} y_{ius} = 1 \qquad \forall i \in I, r = 0, ..., |J|-1 \tag{3.2}$$

$$y_{ijr} \leq x_j \qquad \forall i \in I, j \in J, r = 0, ..., |J|-1 \tag{3.3}$$

$$\sum_{r=0}^{|J|-1} y_{ijr} \leq 1 \qquad \forall i \in I, j \in J \tag{3.4}$$

$$\sum_{i \in I} \sum_{r=0}^{|J|-1} h_i y_{ijr} \leq v_j \qquad \forall j \in J \tag{3.5}$$

$$x_u = 1 \tag{3.6}$$

$$x_j \in \{0, 1\} \qquad \forall j \in J \tag{3.7}$$

$$y_{ijr} \in \{0, 1\} \qquad \forall i \in I, j \in J, r = 0, ..., |J|-1 \tag{3.8}$$

Constraints (3.2) indicate that each customer $i$ must be served by some facility $j$ at level $r$ or by the emergency facility $u$ at a level $s < r$. (By convention we take $\sum_{s=0}^{r-1} y_{ijr} = 0$ if $r = 0$.) Constraints (3.3) prevent customers being assigned to an unconstructed facility. Constraints (3.4) require that a facility can only serve a customer at no more than one level. Constraints (3.6) force the emergency facility $u$ to be opened. Constraints (3.7) and (3.8) are integrality constraints.

It's necessary to discuss constraints (3.5) especially. Obvously the right part of the inequality should be the capacity of facility $j$. Each facility $j$ can serve different customers in different levels. So when we formulate the constraints, the left part of the inequality must take all levels and all customers into account. Initially we formulated the left part in the same way as the transportation cost in the objective function—calculating the expected value of demand that may be assigned to the facility $j$. Then the formulation would be as follows:

$$\sum_{i \in I} \sum_{r=0}^{|J|-1} h_i y_{ijr} q^r (1-q) \leq v_j, \forall j \neq u \tag{3.9}$$

The expected value can be a good estimate of the demand assigned to facilities, however, it is not appropriate to be used to check if the constraints are violated. For example, supose facility 1 just has two assignments: it serves customer 1 at level 0 and serves customer 2 at level 1 in a given solution. Faciliti's capacity is 700 and its failure probability is 0.05. Customer 1's demand is 500 and customer 2's demand is 300. Then the expected demand assigned to facility 1 is 532 because the failure probability is only 0.05. If we use (3.9) as capacity constraints, these assignments for facility 1 are feasible. But once customer 2's primary facility is disrupted, facility 1 must handle the additional demand. Thus facility 1 needs to supply 800 in total which is beyond its capacity. That's why we formulate (3.5) in that way. Then the capacity constraints can guarantee all possible demand can be handled.

# Chapter 4

# Lagrangian Relaxation Algorithm

## 4.1 Lower Bound

We solve (CRFLP) by relaxing constraint(3.2) using Lagrangian relaxation. For given Lagrange multipliers $\lambda_{ir}$, the subproblem is as follows:

(CRFLR-$LR_\lambda$)

$$\min \quad \sum_{j\in J} f_j x_j + \sum_{i\in I}\sum_{j\in J}\sum_{r=0}^{|J|-1} \psi_{ijr} y_{ijr} + \sum_{i\in I}\sum_{r=0}^{|J|-1} \lambda_{ir}\left(1 - \sum_{j\in J} y_{ijr} - \sum_{s=0}^{r-1} y_{ius}\right) \tag{4.1}$$

$$y_{ijr} \le x_j \quad \forall i \in I, j \in J, r = 0, ..., |J| - 1| \tag{4.2}$$

$$\sum_{r=0}^{|J|-1} y_{ijr} \le 1 \quad \forall i \in I, j \in J \tag{4.3}$$

$$\sum_{i\in I}\sum_{r=0}^{|J|-1} h_i y_{ijr} \le v_j \quad \forall j \in J \tag{4.4}$$

$$x_u = 1 \tag{4.5}$$

$$x_j \in \{0,1\} \quad \forall j \in J \tag{4.6}$$

$$y_{ijr} \in \{0,1\} \quad \forall i \in I, j \in J, r = 0, ..., |J| - 1 \tag{4.7}$$

As in [23], the non-fixed-cost portion of the objective function(4.1) can be re-written as follows:

$$\sum_{i\in I}\sum_{j\in J}\sum_{r=0}^{|J|-1}(\psi_{ijr}-\lambda_{ir})y_{ijr}+\sum_{i\in I}\sum_{r=0}^{|J|-1}\lambda_{ir}-\sum_{i\in I}\sum_{r=0}^{|J|-1}\sum_{s=0}^{r-1}\lambda_{ir}y_{ius}$$

$$=\sum_{i\in I}\sum_{j\in J}\sum_{r=0}^{|J|-1}(\psi_{ijr}-\lambda_{ir})y_{ijr}+\sum_{i\in I}\sum_{r=0}^{|J|-1}\lambda_{ir}-\sum_{i\in I}\sum_{s=0}^{|J|-1}\sum_{r=0}^{s-1}\lambda_{ir}y_{ius}$$

(by swapping the indices $r$ and $s$ in the last term)

$$=\sum_{i\in I}\sum_{j\in J}\sum_{r=0}^{|J|-1}(\psi_{ijr}-\lambda_{ir})y_{ijr}+\sum_{i\in I}\sum_{r=0}^{|J|-1}\lambda_{ir}-\sum_{i\in I}\sum_{\substack{r=0,\ldots,|J|-1\\ s=0,\ldots,|J|-1\\ r<s}}\lambda_{is}y_{iur}$$

$$=\sum_{i\in I}\sum_{j\in J}\sum_{r=0}^{|J|-1}(\psi_{ijr}-\lambda_{ir})y_{ijr}+\sum_{i\in I}\sum_{r=0}^{|J|-1}\lambda_{ir}-\sum_{i\in I}\sum_{r=0}^{|J|-1}(\sum_{s=r+1}^{|J|-1}\lambda_{is})y_{iur}$$

Thus, the objective function can be written as

$$\sum_{j\in J}f_jx_j+\sum_{i\in I}\sum_{j\in J}\sum_{r=0}^{|J|-1}\tilde{\psi}_{ijr}y_{ijr}+\sum_{i\in I}\sum_{r=0}^{|J|-1}\lambda_{ir} \tag{4.8}$$

where

$$\hat{\psi}_{ijr}=\begin{cases}\psi_{ijr}-\lambda_{ir}, & \text{if } j\neq u\\ \psi_{ijr}-\lambda_{ir}-\sum_{s=r+1}^{|J|-1}\lambda_{is}, & \text{if } j=u.\end{cases} \tag{4.9}$$

As in the RFLP, this problem separates by $j$, but now computing the benefit $\beta_j$ is a little more complicated because of the capacity constraint. In particular, for each $j\in J$ we need to solve a problem of the form

$$\min \quad \beta_j = \sum_{i \in I} \sum_{r=0}^{|J|-1} \tilde{\psi}_{ijr} y_{ijr} \tag{4.10}$$

$$\text{s.t.} \quad \sum_{i \in I} \sum_{r=0}^{|J|-1} h_i y_{ijr} \leq v_j \tag{4.11}$$

$$\sum_{r=0}^{|J|-1} y_{ijr} \leq 1 \qquad \forall i \in I \tag{4.12}$$

$$y_{ijr} \in \{0, 1\} \qquad \forall i \in I, r = 0, ..., |J| - 1 \tag{4.13}$$

The subproblem for each $j \in J$ is similar to the multiple choice knapsack problem (MCKP [21]). Algorithms based on the linear programming relaxation or a branch-and-bound algorithm [6, 21] can be implemented to solve the MCKP. But there are two differences between the MCKP and the subproblem above. First, the coefficients in the objective function of the MCKP are nonnegative while $\tilde{\psi}_{ijr}$ here may be negative. Second, constraints(4.12) are inequalities in the subproblem above while they are strict equalities in the MCKP. So we instead call CPLEX to solve this subproblem in our MATLAB code.

The solution to every subproblem for each $j$ gives us the beneifits $\beta_j$ and the assignments $y_{ijr}$. If $\beta_j + f_j < 0$ for any $j$, we set $x_j = 1$. And the emergency facility $u$ has been forced open by constraint (3.6). We set $y_{ijr}$ according to the solution to the subproblems above. The optimal objective value for (CRFLR-$LR_\lambda$) is

$$\sum_{j \in J} (\beta_j + f_j) x_j + \sum_{i \in I} \sum_{r=0}^{|J|-1} \lambda_{ir},$$

and this summation gives us a lower bound for the optimal objective value of (CRFLP).

## 4.2 Upper Bound

One cannot obtain feasible solutions as in the RFLP by opening the facilities that are open in the lower bound solution and then assigning customers to level-$r$ facilities in increasing order of distance. It's more tricky for the CRFLP because of the capacity constraints. We need to check the remaining capacity of each facility after every assignment.

We propose to first open the facilities according to the lower bound solution. Second, at level 0, we first let an opened facility $j$ serve its local customers (customers $i$ for which $c_{ij} = 0$), if any, and then assign the rest of the customers to their nearest facilities which still have enough excess capacity. At the higher levels, we select a customer in a random sequence and assign it to its nearest facility which still can satisfy the additional demand (Please remember to update the capacity for facilities after every assignment). If a customer was assigned to the emergency facility at some level, we don't need to do the assignment for it in higher levels.

## 4.3 Multiplier Updating

Indeed the Lagrangian relaxation algorithm cannot give an optimal objective value exactly. It only provides us a lower bound (LB) and an upper bound (UB). Both the LB and the UB are determined by the Lagrange multiplier $\lambda$. So it's important to find an appropriate value of $\lambda$ to obtain a very tight gap.

The Lagragian relaxation algorithm involves many iterations to update the multipliers $\lambda$. Often, subgradient optimization is used to adjust $\lambda$. In particular, we compute a step-size $\Delta^t$ at each iteration as follows:

$$\Delta^t = \frac{\alpha^t(UB - L^t)}{\sum_{i\in I}\sum_{r=0}^{|J|-1}(1 - \sum_{j\in J} y_{ijr} - \sum_{s=0}^{r-1} y_{ius})}. \tag{4.14}$$

We set $\alpha$ to 2 initially and halve it if the lower bound hasn't been improved after 20 consecutive iterations. $L^t$ denotes the lower bound found at iteration $t$ and UB represents the best known upper bound. The step direction at iteration $t$ is calculated by

$$1 - \sum_{j\in J} y_{ijr} - \sum_{s=0}^{r-1} y_{ius},$$

which equals the violation in the constraint.

Therefore the multipliers $\lambda$ are updated by the following equation:

$$\lambda_{ir}^{t+1} = \lambda_{ir}^{t} + \Delta^{t}\left(1 - \sum_{j \in J} y_{ijr} - \sum_{s=0}^{r-1} y_{ius}\right). \tag{4.15}$$

The Lagrangian process terminates when $(UB - L^t)/L^t < \epsilon$, for some optimality tolerance $\epsilon$ specified by the user.

# Chapter 5

# Computational Results

## 5.1 Effect of Capacity

In Chapter 1, we discuss that capacity constraints may increase the transportation cost and even influence the initial facility location decision. In this section, we use different instances to test the impact of capacity on the solution. We use three datasets (see [7]) derived from 1990 census data: a 49-node dataset, an 88-node dataset and a 150-node dataset. All three datasets consist of cities in the United States mainland.

We set demands $h_i$ to the population of the state in which the city is, divided by $10^5$, for the 49-node set and to the city population divided by $10^4$ for the other two. The 49-node and 88-node sets' fixed cost $f_j$ equals the median home value in the city and the 150-node set's equals $10^5$ for all $j$. The transportation cost $d_{ij}$ is calculated as the great-circle distance on the earth's surface. The emergency cost $\theta_i$ is $10^4$ for all $i$. In both the RFLP and the CRFLP, the failure probability $q$ is set to 0.05 and the set $I$ of customers and the set $J$ of facilities are equal. In

| Model | # Opened facilities | Optimal value | CPLEX time | # iterations |
|---|---|---|---|---|
| 49 RFLP | 9 | 1,195,380 | 35.760 | 18,974 |
| 49 CRFLP | 21 | 2,901,583 | 231.350 | 20,257 |
| 88 RFLP | 13 | 1,707,662 | 793.840 | 60,878 |
| 88 CRFLP | 28 | 4,750,590 | 1,401.940 | 56,031 |
| 150 RFLP | 13 | 2,307,951 | 7,947.140 | 194,008 |
| 150 CRFLP | 45 | 7,908,872 | 27,993.300 | 24,197 |

Table 5.1: RFLP and CRFLP results comparison

our initial experiment on the CRFLP, we chose to use very tight facility capacities in order to highlight the effect of capacity restrictions; in this experiment we set $v_j$ equal to $(X + 1) \times h_j$, where $X$ is drawn from $U[0, 1]$. The emergency facility's capacity is unlimited and fixed cost is 0.

We coded (RFLP) and (CRFLP) in AMPL Version 20070505) on the three sets above and solved them on a machine in Lehigh University's COR@L Lab with 32GB RAM and an AMD Opteron 2.0 GHz (x16) processor, running under Linux. IBM ILOG CPLEX (version 12.6) is used to solve these models in AMPL. The optimality tolerance is set to 5%.

Table 5.1 compares the solution and the CPLEX performance of the RFLP and the CRFLP using the three datasets. The columns marked "# Opened facilities" and "Optimal value" show how many facilities are open and how much the total expected cost is. The columns marked "CPLEX time" and "# iterations" give the CPU time (in seconds) and how many simplex iterations it takes to find the optimal solution using CPLEX. Both the number of opened facilities and the total expected cost increase significantly in the CRFLP. First, that's because more opened facilities occur much more fixed cost. Second, the transportation cost may also increase since customers have to be assigned to farther facilities when the nearer ones don't have enough excess capacity even more facilities are opened. At the same time, CPLEX needs more time to solve the capacitated models. However, the number of simplex iterations may not increase (Especially for the 150-node set, the number of simplex iterations in the capacitated model is much less than in the uncapacitated model). This is probably due to the fact that capacity constraints make each iteration longer but more effective.

| Capacity | # Opened facilities | Optimal value | CPLEX time | # iterations |
|---|---|---|---|---|
| $(X + 1) \times h_j$ | 21 | 3,002,600 | 87.965 | 20,257 |
| $(X + 2) \times h_j$ | 20 | 2,139,062 | 148.657 | 21,526 |
| $(X + 3) \times h_j$ | 15 | 1,532,239 | 159.898 | 29,659 |
| $(X + 4) \times h_j$ | 13 | 1,385,746 | 116.815 | 20,851 |
| $(X + 5) \times h_j$ | 13 | 1,336,669 | 90.466 | 20,452 |
| $(X + 6) \times h_j$ | 12 | 1,319,645 | 86.505 | 20,457 |
| $(X + 7) \times h_j$ | 12 | 1,300,510 | 70.017 | 22,315 |
| $(X + 8) \times h_j$ | 12 | 1,276,014 | 96.058 | 27,071 |
| $(X + 9) \times h_j$ | 11 | 1,242,381 | 88.158 | 25,293 |
| $(X + 10) \times h_j$ | 10 | 1,221,228 | 71.437 | 24,334 |

Table 5.2: CRFLP results using different capacity levels

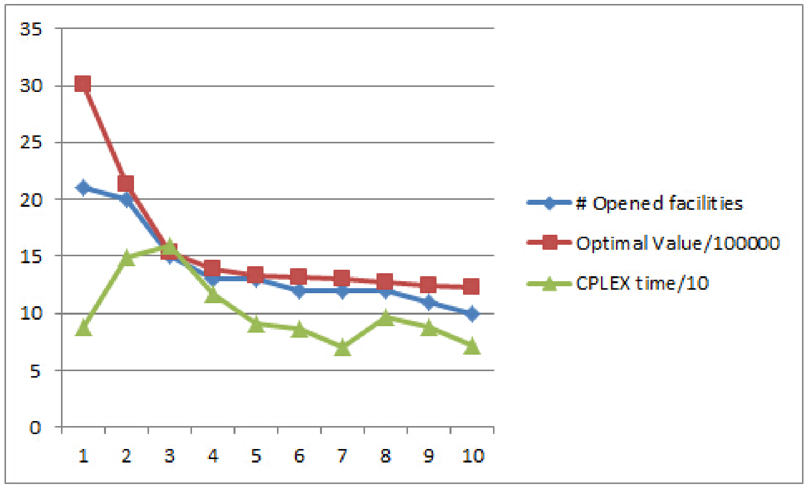


Figure 5.1: CRFLP results using different capacity levels

We are also interested in the effects on the solution and optimal objective value when the capacity changes. Table 5.2 and Figure 5.1 show the results of the CRFLP based on the 49-node dataset with different capacity levels. The other parameters and the CPLEX's settings are the same as above. We use ten sets of data with different capacity values $v_j$. $X$ is drawn from $U[0,1]$; then the ten sets of data are: $(X+1)\times h_j$, $(X+2)\times h_j$, $(X+3)\times h_j$,..., $(X+10)\times h_j$, where $h_j$ is the demand. It's obvious that the number of opened facilities and the optimal value decrease as the capacities increase. But the decrease step size is smaller and smaller while the capacities increase by the same quantity. The number of opened facilities and the optimal value approach those of the uncapacitated model when the capacity is bigger enough. The CPLEX time and the number of simplex iterations don't change in an evident trend but are still more than that in the uncapacitated model.

Therefore, the capacity impacts the solution indeed. The bigger the capacities are, the less the cost is. The uncapacitated model can be regarded as a special case in which each candidate facility's capacity is big enough to cover all demands. However, the comparisons in this section are based on the assumption that the fixed cost is unrelated to the capacity. If the fixed cost was proportional to the capacity, it would be hard to say how the objective value would change. Maybe we can also choose a capacity for the candidate facilities at the first decision stage.

## 5.2 Algorithm Performance

In this section, we test our algorithm on the three datasets we used in Section 5.1 and five randomly generated datasets. The five datasets consist of 50, 75, 100, 125 and 150 nodes. In all datasets, demands $h_i$ were drawn from $U$[0,1000] and rounded to the nearest integer, fixed costs $f_j$ were drawn from $U$[500,1500] and rounded to the nearest integer, and $x$ and $y$ coordinates were drawn from $U$[0,1]. Transportation costs $c_{ij}$ equal to the Euclidean distance between $i$ and $j$. The emergency cost $\theta_i$ was set to 10.

In all eight datasets, the failure probability $q$ was set to 0.05. The capacities $v_j$ were set equal to $(X+1)\times h_j$, where $X$ is drawn from $U[0,1]$. The emergency facility's capacity is unlimited and its fixed cost is 0. We executed the Lagrangian relaxation process to an optimality tolerance of 5%. The algorithm was coded in MATLAB and tested on a machine in Lehigh University's COR@L Lab with 32GB RAM and an AMD Opteron 2.0 GHz (x16) processor, running under Linux. We also coded (CRFLP) based in AMPL (Version 20070505) on the eight sets and solved them in the same hardware environment. IBM ILOG CPLEX (version 12.6) is used to solve these models in AMPL. The optimality tolerance is also set to 5%.

Table 5.3 compares the algorithm's performance and CPLEX's performance for all the datasets. The "LB", "UB", and "Gap" columns give the lower and upper bounds and the percentage gap. The column marked "# Lag iter." gives the total number of Lagrangian iterations, and "LR time" gives the total number of CPU seconds required in the Lagrangian relaxation algorithm. The column marked "CPLEX time" indicates the CPU time needed in CPLEX and "CPLEX obj" gives the optimal objective value in CPLEX. The last column gives the relative error calculated by (UB-CPLEX obj.)/CPLEX obj.

| # Nodes | LB | UB | Gap (%) | # Lag iter. | LR time | CPLEX time | CPLEX obj. | RE(%) |
|---|---|---|---|---|---|---|---|---|
| 49 | 2,822,900 | 2,962,200 | 4.70 | 146 | 446.100 | 231.350 | 2,901,583 | 2.089 |
| 88 | 4,553,100 | 4,765,400 | 4.66 | 482 | 3,519.100 | 2,401.940 | 4,750,590 | 0.312 |
| 150 | 7,796,739 | 8,165,671 | 4.73 | 1,143 | 9,395.800 | 30,414.700 | 7,908,872 | 3.247 |
| 50 | 33,702 | 35,341 | 4.86 | 205 | 578.800 | 209.920 | 34,942 | 1.143 |
| 75 | 52,937 | 55,574 | 4.98 | 468 | 2,824.700 | 1,616.380 | 53,748 | 3.397 |
| 100 | 69,788 | 72,683 | 4.15 | 643 | 4,969.600 | 4,171.960 | 7,2169 | 0.712 |
| 125 | 83,856 | 87,328 | 4.14 | 951 | 7,519.100 | 13,801.400 | 86,737 | 0.681 |
| 150 | 126,567 | 132,673 | 4.82 | 1,343 | 9,395.800 | 30,414.700 | 129,888 | 0.214 |

Table 5.3: Algorithm Performance

From Table 5.3, it is evident that the Lagrangian relaxation algorithm begins to be faster than CPLEX as the instance becomes larger (contains more nodes). The percentage gap (the column marked Gap) between the upper bound and the lower bound doesn't increase as the number of nodes increases. This is because the termination criterion in our algorithm is that the percentage gap is less than 5%. Thus the percentage gaps here aren't in a trend, some of them are just a little bit less than 5%, some are farther from 5%. If the algorithm terminated after a fixed number of iterations or a CPU time limit, we would expect this gap to increase as the number of nodes continues to increase.

Obviously, our Lagrangian relaxation algorithm took rather more time to solve the CRFLP than the algorithm to solve the RFLP in [23]. That's why we used the percentage gap as the only termination criterion. It's unclear how long the time limit should be for the algorithm to provide an acceptable gap. However, our algorithm wouldn't work well if we narrowed the percentage gap termination criteria. For example, we tried 1% as the criterion but the algorithm took many hours to excute. So it's a disvantage of our algorithm that it can't provide an upper bound very close to the optimal value and a corresponding solution.

# Chapter 6

# Conclusion

In this thesis we presented a new model that introduced capacity into the reliable facility location problem. This model was inspired by the reality that all facilities in a supply chain, like warehouses, distribution centers or plants, have a limited capacity. By adding capacity constraints to the RFLP model, we formulated the CRFLP (capacitated reliable facility location problem) model. Key to our formulation is the form of the capacity constraints, see (3.5). One facility's capacity must be large enough to accomodate the additional customers' demand at all levels as backups. This is because, once a disruption occurs, what was merely a "probability"now becomes an emergency that we must handle. Constraints like (3.9) that only consider the expected value of the assigned demands cannot guarantee that these emergencies can be handled.

This model is solved using Lagrangian relaxation, with better results vs. CPLEX as the number of nodes in the dataset increses. However, the Lagrangian relaxation algorithm in this thesis still can be improved in three aspects. First, the subproblem is just similar but not equivalent to the MCKP; thus, we cannot use existing algorithms to solve it. Calling CPLEX in MATLAB is a way to solve it, but it limits the code to be run in Linux and it may also require more time. Second, the method to find the upper bound is not guaranteed to provide the optimal customer assignments for a given set of opened facilities. This may have an effect on the overall solution time. Third, the optimality tolerance here cannot be set too small, e.g. 1% or less, otherwise the termination criteria will be very hard to achieve. If the first two problems can be solved by implementing new methods, the last one would probably be resolved as well.

Another limitation of our model is the assumption that the facilities all have the same probability $q$ of failing. But this unrealistic assumption makes it easier to compute the probability just knowing that a customer is served by its level-$r$ facility. Otherwise, we need to figure out its lower-level assignments exactly. [2, 5, 13] have proposed several approaches to relax this assumption. $P_{ijr}$, the probability that facility $j$ serves customer $i$ at level $r$, is just the probability $(1 - q_j)$ that $j$ remains open if $r$=0. For other levels, $P_{ijr}$ is equal to

$$(1 - q_j) \sum_{k=0}^{J-1} \frac{q_k}{1 - q_k} P_{i,k,r-1} y_{i,k,r-1}$$

given that facility $k$ serves customer $i$ at level $r - 1$. Future research may focus on how to adapt these methods to apply to our problem.

As we discussed in Section 5.1, we can also introduce new variables that decide how big the capacity is when we construct the facility. The bigger it is, the more it costs. For example, we set different capacity levels $c_1$, $c_2$,..., $c_n$,..., $c_m$. Each level $n$ corresponds to a fixed cost $f_n$. The variable $w_{jn}$ is binary. It equals 1 if we choose $c_n$ for facility $j$. Otherwise, it equals 0. Thus, we would have $\sum_{j \in J} \sum_{n=1}^{m} w_{jn} f_n$ in the objective function and $\sum_{n=1}^{m} w_{jn} = 1, \forall j \in J$ in the constraints. Obviously, bigger capacities produce more fixed cost for each facility while decreasing the number of open facilities and saving some tranportation cost because customers don't need to find farther facilities due to insufficient capacities. This trade-off may be worth further study in the future.

# Bibliography


[1] U. Akinc and B.M. Khumawala. An efficient branch and bound algorithm for the capacitated warehouse location problem. *Management Science*, 23:585–594, 1977.

[2] O. Berman, D. Krass, and M.B.C. Menezes. Facility reliability issues in network $p$-median problems: Strategic centralization and co-location effects. *Operations Research*, 55(2):332–350, 2007.

[3] G. Chen, M.S. Daskin, Z.J.M. Shen, and S Uryasev. A new model for stochastic facility locations. *Naval Research Logistics*, 53(7):617–626, 2006.

[4] C. Colbourn. *The Combinatorics of Network Reliability.* Oxford University Press, New York, 1987.

[5] T. Cui, Y. Ouyang, and Z.-J. M.Shen. Reliable facility location design under the risk of disruptions. *Operations Research*, 58(4):998–1011, 2010.

[6] R. D. Armstrong, D. S. Kung, P. Sinha, and A. A. Zoltners. A computational study of a multiple-choice knapsack algorithm. *ACM Transactions on Mathemaucal Software*, 9(2):184–198, 1983.

[7] M. S. Daskin. *Network and Discrete Location: Models, Algorithms, and Applications.* Wiley, New York, 1995.

[8] Z. Drezner. *Facility Location: A Survey of Applications and Methods.* Springer, New York, 1995.

[9] P.M. Franca and H.P.L. Luna. Soving stochastic transportation-location problems by generalized Benders decomposition. *Transportaion Science*, 16(2):113–126, 1982.

[10] D. Gade. Capacitated facilities location problems with unreliable facilities. Working Paper, University of Arkansas, Department of Industrial Engineering, Fayetteville, AR, USA, 2007.

[11] H.M. Jeon, L.V. Snyder, and Shen Z.J.M. Joint location-inventory optimization with unreliable facilities. Working Paper, Lehigh University, Department of Industrial and Systems Engineering, Bethelehm, PA, USA, 2006.

[12] A.A. Kuehn and M.J. Hamburger. A heuristic program for locating warehouses. *Management Science*, 9:643–666, 1963.

[13] X. Li and Y. Ouyang. A continuum approxiamation approach to reliable facility location design under correlated probabilstic disruptions. *Transportaion Research-Part B*, 44(4):535–548, 2007.

[14] R. Mirchandani and R. Francis. *Discrete Location Theory.* Wiley, New York, 1990.

[15] R.M. Nauss. An improved algorithm for the capacitated facility location problem. *Journal of Operation Research Society*, 29:11951201, 1978.

[16] T. Santoso, S. Ahmed, M. Goetshalckx, and A. Shapriro. A stochastic programming approach for supply chain network design under uncertainty. *European Journal of Operations Research*, 167:96–115, 2005.

[17] P.M. Scaparra and R.L. Church. An optimal approach for the interdiction median problem with fortication. Working Paper 78, Kent Business School, Canterbury, England, UK, 2005.

[18] P.M. Scaparra and R.L. Church. A bilevel mixed integer program for critical infrastructure protection planning. *Computers and Operations Research*, 35:1905–1923, 2008.

[19] D. Shier. *Network Reliability and Algebraic Structures.* Clarendon Press, Oxford, 1991.

[20] M.L. Shooman. *Reliability of Computer Systems and Networks: Fault Tolerance, Analysis, and Design.* Wiley, New York, 2002.

[21] Prabhakant Sinha and Andris A. Zoltners. The multiple-choice knapsack problem. *Operations Research*, 27(3):503–515, 1979.

[22] L.V. Snyder. Facility location under uncertainty: A review. *IIE Transactions*, 38:537–554, 2006.

[23] L.V. Snyder and M.S. Daskin. Reliability models for facility location: The expected failure cost case. *Transportation Science*, 39(3):400–416, 2005.

[24] L.V. Snyder, M.S. Daskin, P.M. Scaparra, and R.L. Church. Planning for disruptions in supply chain networks. *Tutorials in Operations Research*, 35 (INFORMS), 2006.

[25] L.V. Snyder and N.S. Ülker. A model for locating capacitated, unreliable facilities. In *IERC Conference*, Atlanta, GA, USA, 2005.

[26] A. Weber. *Theory of the Location of Industries.* University of Chicago Press, IL, 1957.

[27] T. Zeng and J.E. Ward. The stochastic location-assignment problem on a tree. *Annals of Operations Research*, 136:81–97, 2005.

[28] R. L. Zhan and Shen Z.J.M. Models and algorithms for reliable facility location problems and system reliability optimization. Working Paper, University of Florida, Department of Industrial and Systems Engineering, Gainesville, FL, USA, 2007.

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