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The purpose of this report is the determination of the post-buckling behavior of long rectangular plates subjected to edge compression as they are used in stiffened plate panels. The behavior investigated is the relationship between the average applied edge stress and the strain at the longitudinal edge. Of primary interest here are steel plates having a b/t ratio ranging from 60 to 100, where b is the plate width and t is the thickness.

Included in this report is the evaluation of existing theoretical solutions, a discussion of existing experimental data, and a comparison of the theoretical and experimental results. The experimental work discussed is not limited to steel or the b/t ratios indicated above.

It was concluded that Koiter's theoretical equation adequately describes the post-buckling behavior of simply supported plates with b/t ratios ranging from 60 to 100.

Consideration was also given to the effect of welding residual stresses on the ultimate strength of the plate.
1. INTRODUCTION

This report describes a study undertaken to obtain the load-shortening relationship for long rectangular plates subjected to edge compression in the longitudinal direction. This relationship is intended to be used in the ultimate strength analysis of longitudinally stiffened plate panels subjected to axial compression and transverse hydrostatic pressure.

It is quite possible for the relative dimensions of the plate and stiffener to be such that buckling of the plate will occur before the ultimate strength of the panel is developed. Thus, in order that the ultimate strength be determined, the post-buckling behavior of the plate must be known. This is the problem investigated in this report: the post-buckling behavior of long plates. The solution must, of course, be applicable to the appropriate boundary conditions.

The desired relationship is the average applied stress vs. the axial strain at the longitudinal edge (load-shortening). This relationship is to be used as the effective stress-strain curve for the plate at strains above the critical strain.

The buckling load of a simply supported rectangular plate subjected to edge compression was given by Bryan in a paper published in 1891. At that time it was assumed that the buckling load was the highest load a plate could carry. In 1930 conclusive experimental proof was presented by Schuman and Back to show that the ultimate load was
actually higher than the buckling load. In the same year Schnadel (3) published a paper describing a theoretical investigation of the post-buckling behavior of a simply supported rectangular plate. The "effective width" concept was defined but was not determined.

In 1932 von Karman (4) published a theoretical study of a simply supported rectangular plate. A relationship for the "effective width" was developed. It is shown below:

\[
\frac{b_e}{b} = \sqrt{\frac{\sigma_{cr}}{\sigma_e}}
\]

(1)

where \( b_e \) is the effective width, \( \sigma_{cr} \) is the buckling strength, and \( \sigma_e \) is the axial stress at the longitudinal edges. As no mention was made of Schnadel's work in von Karman's paper, it is possible that each developed the effective width concept independently.

Since the work at Schnadel, von Karman, and Schuman and Back, more rigorous theoretical solutions have been developed and more refined experiments have been conducted. Much of this work is analyzed in this report leading to the choice of a solution best describing the behavior of a plate in a stiffened plate panel.
2. THEORETICAL INVESTIGATIONS

2.1 BACKGROUND

The behavior of an unbuckled plate is described as follows. The out-of-plane deflection \( w \) is zero. The relationship between the applied stress and the edge strain is dependant upon the restraint against in-plane displacement of the longitudinal edges. For the case in which the edges are free to displace the relationship in \( \overline{\sigma} = E \varepsilon_e \) and for the case in which the edges are restrained from displacing the relationship \( \overline{\sigma} = E\varepsilon_e/(1-\nu^2) \) where \( \overline{\sigma} \) is the average applied edge stress, \( \varepsilon_e \) is the axial strain at the longitudinal edges, \( E \) is Young's modulus, and \( \nu \) is Poisson's ratio.

Once the plate has buckled, however, the behavior becomes much more complicated. The magnitude of the out-of-plane deflection is such that the small deflection equation \( \nabla^4 \phi = 0, \nabla^4 w = 0 \) are no longer valid. Von Karman's large deflection equations must instead be used. These partial, simultaneous, non-linear differential equations are:

\[
\nabla^4 \phi = E \left( \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right)
\]

\[
\nabla^4 w = \frac{E}{D} \left( \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial x \partial y} \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 \phi}{\partial y^2} \right)
\]

where \( D = Et^3/12(1-\nu^2) \), \( w \) is the out-of-plane deflection, and \( \phi \) is Airy's stress function. The in-plane stress resultants are computed using equation (4).
\[ N_x = t \frac{\partial^2 \phi}{\partial y^2}, \quad N_y = t \frac{\partial^2 \phi}{\partial x^2}, \quad N_{xy} = t \frac{\partial^2 \phi}{\partial x \partial y} \]  

(4)

\( N_x \) and \( N_y \) are the normal membrane forces per unit width and \( N_{xy} \) is the membrane shearing force per unit width.

The first large deflection equation expresses in-plane equilibrium and compatibility where \( \phi \) identically satisfies the equilibrium equations:

\[ \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \]  

(5a)

\[ \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0 \]  

(5b)

The strain displacement relationships including second order terms are:

\[ \varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \]  

(6a)

\[ \varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \]  

(6b)

\[ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \]  

(6c)

where \( \varepsilon_x \) and \( \varepsilon_y \) are the membrane strains in the \( x \) and \( y \) direction respectively, \( \gamma_{xy} \) is the membrane shearing strain, \( u \) is the displacement in the \( x \) direction, and \( v \) is the displacement in the \( y \) (see Fig. 1 for the plate geometry and coordinate system) direction. The second large deflection equation expresses the out-of-plane equilibrium.
The inherent difficulty in obtaining an exact solution of Equations 2 and 3 necessitates the formulation of approximate solutions. In general an approximate solution incorporates assumed functions for one or all of the three displacements u, v, w. The functions contain a number of arbitrary parameters and the solution then involves the determination of these parameters.

The discussions of the theoretical solutions included in this chapter will describe the method used, the assumed or derived deflection function, the imposed boundary conditions, consideration of secondary buckling, and all the assumptions made.

The following is a brief explanation of some of the terms which are used here.

**Effective Width**

\[ b_e = \left( \frac{\sigma}{a} \right) b \]

- the effective width represents the hypothetical width of a plate subjected to a uniform stress \( \sigma_e \) (axial stress at the longitudinal edge) which would carry the same total load as the plate in question.

**Load-Shortening**

The term refers to the relationship between the applied load and the unit shortening of the plate in the direction of the load. The unit shortening is equivalent to the axial longitudinal edge strain \( \varepsilon_e \).

Given the effective width, the load-shortening relationship can be expressed non-dimensionally as

\[ \frac{\sigma}{\sigma_{cr}} = \left( \frac{b}{b_e} \right) \left( \frac{\varepsilon_e}{\varepsilon_{cr}} \right) \]

where \( \varepsilon_{cr} \) is the buckling strain of a simply supported plate.
Secondary Buckling

The expression secondary buckling refers to a change in the buckle pattern from $n$ half waves to $n + 1$ half waves. It is associated with a rapid jump in either the strain or the load and must be differentiated from the gradual modification of an existing deflection mode.

2.2 BOUNDARY CONDITIONS

In order that the theoretical solutions be applicable to the stiffened plate problem, the boundary conditions used in the analysis must agree with the probable conditions existing in a stiffened plate panel. The following is a brief discussion of these probable boundary conditions.

1) Out-of-plane deflection is equal to zero. Thus at the four edges

$$w = 0$$

2) The edge moments are equal to zero. Thus, at the longitudinal edge (see Fig. 1)

$$\frac{\partial^2 w}{\partial y^2} = 0$$

and at the transverse edge

$$\frac{\partial^2 w}{\partial x^2} = 0$$

With the longitudinal edges supported by stiffeners with small torsional resistance (such as T stiffeners), the edge moments should approach zero. However, for a
torsionally rigid stiffener, the end moments could approach a condition of fixed support. Considering the stiffeners presently used in ship construction, it is more probable that the edge moments may be assumed to equal zero.

3) Restraint against normal in-plane displacement of the longitudinal edges is zero and thus the edges are allowed free expansion. However, the displacement does not vary along the length and, thus, the edges must remain straight. Thus, at the longitudinal edges:

\[ \int_a^0 \sigma_y \, dx = 0 \]

and

\[ \frac{\partial v}{\partial x} = 0 \]

where \( \sigma_y \) is the membrane stress in the y direction and \( a \) is the length of the plate.

Because of the interaction between adjoining panels and the interaction between the stiffener and the plate, it is highly probable that the longitudinal edges are forced to remain straight. The restraint against displacement is much less readily determined. The majority of investigators have assumed zero restraint against in-plane displacement and the same assumption is made here.

Fortunately, restraint against in-plane displacement
affects primarily the buckling load and has only a small effect on the post-buckling behavior and ultimate strength of the plate.

4) The normal in-plane displacement of the transverse edge does not vary with respect to the width and must remain straight. Thus at the transverse edges

\[ \frac{\partial u}{\partial y} = 0 \]

5) The shearing stress is zero at the longitudinal edge. Together with the above statement that \( \frac{\partial v}{\partial x} = 0 \), the condition

\[ \frac{\partial u}{\partial y} = 0 \]

will satisfy the zero shearing stress requirement.

The actual stress condition at the longitudinal edge is quite difficult to determine. For a multi-bay panel without stiffeners (such as that shown in Fig. 5), there are no shearing stresses at the longitudinal edge of each bay. The axial strain in this case would vary along the edge. For a homogeneous column such as the stiffener alone, the axial strain would be constant. In a stiffened panel where the two are joined together the actual condition must fall somewhere between the two conditions stated above. For a given plate size, the smaller the stiffener the closer the condition of a zero.
shearing stress and a variable strain is approached. However, if the stiffener is very large a condition of constant strain is approached and thus shearing stresses are present. The majority of investigators have assumed the condition of zero shearing stresses.

6) The shearing stress is zero at the transverse edge. Thus, at the transverse edge

$$\frac{\partial v}{\partial x} = 0$$

2.3 **MARGUERRE**

Procedure

Marguerre obtains the post-buckling behavior of an elastic, initially flat, square plate. The method of solution incorporates von Karman's large deflection equations and the principle of minimum potential energy. A function containing three arbitrary constants is assumed for \( w \) after which a solution is obtained for \( \phi \) from the first large deflection equation (Eq. 2). The total strain energy is then minimized with respect to the constants, thus resulting in a relationship between the constants and the applied load.

Marguerre imposed the following boundary conditions:

1) \( w \left( \pm \frac{b}{2}, y \right) = w \left( x, \pm \frac{b}{2} \right) = 0 \)

Thus the edges must remain in the original plane of the plate (the coordinate system originates at the center of the plate).
2) \( \frac{\partial^2 w}{\partial y^2} (x, \pm \frac{b}{2}) = \frac{\partial^2 w}{\partial x^2} (\pm \frac{b}{2}, y) = 0 \)

Thus, the edge moments are zero.

3) \( u (\pm \frac{b}{2}, y) \) is independent of \( y \); \( v (x, \pm \frac{b}{2}) \) is independent of \( x \). Thus, the edges are constrained to remain straight.

4) \[ \int_{-\frac{b}{2}}^{\frac{b}{2}} \sigma_y (x, \pm \frac{b}{2}) \, dx = 0 \]

The longitudinal edges are thus free to expand as a straight line.

5) \[ \frac{\partial u}{\partial y} (x, \pm \frac{b}{2}) = \frac{\partial v}{\partial x} (\pm \frac{b}{2}, y) = 0 \]

Combining conditions (3) and (5) requires the shearing stress at the edge to be zero.

Marguerre's deflection function is:

\[ w = f_1 \cos \frac{\pi x}{b} \cos \frac{\pi y}{b} - f_3 \cos \frac{3\pi x}{b} \cos \frac{\pi y}{b} + c f_3 \cos \frac{3\pi x}{b} \cos \frac{3\pi y}{b} \]

(7)

where \( f_1, f_3, \) and \( c \) are unknown constants. The three term trigonometric series was intended to approximate the shape of a plate in which additional buckles have formed near the longitudinal edges. Marguerre states that "intermediate buckling must form near the edges since the edge strips like the whole plate must split into square panels." He noted that this phenomenon had been observed by Lahde. The assumed function also allows transverse flattening to take place. The amount of flattening is dependent upon \( c \) and \( f_3 \).
After substituting the above function into Eq. 2, this equation can be solved for $\xi$. As $\xi$ and $w$ are both known the total stress energy can be computed as a function of the unknown constants. Differentiation of the total strain energy with respect to $f_1$, $f_3$, and $c$ leads to three simultaneous equations. Because of the complexity of these equations Marguerre chose to use the equations

$$\frac{\partial U}{\partial f_1} = 0, \quad \frac{\partial U}{\partial f_3} = 0 \quad (U \text{ is the total strain energy})$$

determine by trial and error the value of $c$ yielding the lowest load capacity of the plate. $C$ was determined to be 1/2.

The resulting equations relating average stress and edge strain are:

$$\frac{\bar{\sigma} - \sigma_{cr}}{\varepsilon - \varepsilon_{cr}} = \frac{E}{2} \frac{4 - 6Z + 18.6Z^2}{4 - 3Z + 31.8Z^2} \frac{f(Z)}{2} \quad (8a)$$

$$\frac{\varepsilon - \varepsilon_{cr}}{\varepsilon - 4.02\varepsilon_{cr}} = 11.25 \frac{20 - 3Z + 31.8Z^2}{31.8 - 1/Z + 350Z^2} \quad (8b)$$

where $Z = f_3/f_1$. The effective width based on the above equation is:

$$\frac{b_e}{b} = \frac{\varepsilon_{cr}}{\varepsilon_e} + \frac{1}{2} f(Z) \left(1 - \frac{\varepsilon_{cr}}{\varepsilon_e}\right) \quad (9)$$

Marguerre refers to the above equation as an "exact" one. As an approximation for it he suggests using the following equation:

$$\frac{b_e}{b} = 0.81 \left(\frac{\varepsilon_{cr}}{\varepsilon_e}\right)^{1/2} + 0.19 \quad (10)$$
A second approximate equation proposed by Marguerre is:

$$\frac{b_e}{b} = \sqrt[3]{\frac{\epsilon_{cr}}{\epsilon_e}}$$  \hspace{1cm} (11)

Equation 11 will be used in this report as it gives a better approximation of the "exact" formula in the range $1 \leq \epsilon_e/\epsilon_{cr} \leq 20$.

In addition to his "exact" solution, Marguerre has obtained a second solution for the affective width using an inexact method in which he assumes that $\tau_{xy} = 0$. If Poisson's ratio is then assumed to be zero, the strain energy computation is greatly simplified. At $\epsilon_e/\epsilon_{cr} = 20$ the inexact method gives a value of $\overline{\sigma}/\sigma_{cr}$ approximately 12 percent less than the "exact" method.

Discussion

Marguerre calls his method of solution a "mixed method" as both the principle of minimum potential energy and the large deflection equations were used to obtain the solution. Had minimum energy principles alone been used, the solution would be an upper bound in that for a given strain the predicted load capacity would be greater than or equal to that given by an exact solution. However, with the use of the "mixed method" this is not necessarily the case. According to van der Neut, the "mixed method" would give an upper bound solution if the solution or Eq. 2 based on the assumed deflection function is exact. (7) In Marguerre's report the calculations involved in obtaining the "exact" solution are not included, making it impossible to determine absolutely that the solution is an upper bound.
One of the important factors determining the accuracy of Marguerre's solution is the adequacy of the three term cosine series to approximate the deflected shape of the plate. The close agreement between Marguerre's solution and Levy's solution (see Sec. 2.4) in which six terms were used indicates that the three term series is quite adequate.

The function chosen by Marguerre limits the longitudinal wave pattern to a single half-wave mode, a three half-wave mode, or a combination of the two. The shape developed in the solution is a single half-wave modified by the three half-wave mode. At $\frac{\epsilon_e}{\epsilon_{cr}} = 20$, $f_3 = 0.23 f_1$ showing the first term to be the dominant term. Only at a very high load would the three half-wave mode dominate the first mode. Thus, secondary buckling as defined in Section 2.1 is not included in Marguerre's solution. Only the first equilibrium mode is investigated. As secondary buckling constitutes a most important factor in a comparison of the solutions, it will be further discussed in Sections 2.4 and 2.5.

2.4 LEVY Procedure

Levy (8) develops the elastic behavior of a rectangular plate subjected to normal pressure and edge compression based on the large deflection theory as defined by von Karman's equations (Eqs. 2 and 3). The deflected shape $w$ is represented by a Fourier series. A solution for $\xi$ is then obtained from the first large deflection equation (Eq. 2). The relationship between the deflection coefficient and the
normal pressure is determined by substituting $\psi$ and $w$ into Eq. 3.

The relationship thus developed is for a rectangular plate subjected to both normal pressure and edge compression. Using this relationship but incorporating only a finite number of terms of the Fourier series, solutions are obtained for various combinations of plate aspect ratio $(a/b)$ and loading. One of the cases investigated was a square plate subjected only to uniaxial edge compression. This case will be discussed here.

The imposed boundary conditions are:

1) $w(o,y) = w(b,y) = w(x,o) = w(x,b) = 0$
   (The coordinate system originates at the intersection of the edges)

2) $\frac{\partial^2 w}{\partial x^2}(o,y) = \frac{\partial^2 w}{\partial x^2}(b,y) = \frac{\partial^2 w}{\partial y^2}(x,o) = \frac{\partial^2 w}{\partial y^2}(x,b) = 0$

3) $\int_{0}^{b} u dx$ is independent of $y$

4) $\int_{0}^{b} v dy$ is independent of $x$

Thus the edges are constrained to remain straight.

4) $\int_{0}^{b} \sigma_y(x,o) dx = \int_{0}^{b} \sigma_y(x,b) dx = 0$
5) \( \tau_{xy}(o,y) = \tau_{xy}(b,y) = \tau_{xy}(x,o) = \tau_{xy}(x,b) = 0 \)

The shear is zero along the edges.

The Fourier series:

\[
w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{m,n} \sin \frac{m\pi x}{b} \sin \frac{n\pi y}{b}
\]  

(12)

satisfies the first and second boundary conditions. After substituting Eq. 12 into Eq. 2, \( \phi \) is determined as

\[
\phi = -\frac{a}{2} y^2 + \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} b p, q \cos \frac{p\pi x}{b} \cos \frac{q\pi y}{b}
\]

(13)

where \( b p, q = \frac{E}{4(p^2+q^2)} (B + \ldots + B) \).

Each "B" is a function of the deflection coefficients expressed as a double summation.

Since the edges must remain straight (third boundary condition) the total shortening in the x direction must be independent of y and the total shortening in the y direction must be independent of x.

Integrating \( u \) over the length of the plate one obtains

\[
\int_{0}^{b} u dx = \int_{0}^{b} \left[ \varepsilon - \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] dx = -\frac{a b}{E} - \frac{\pi^2}{8b} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} m^2 w_{m,n}^2
\]

(14)

Equation 14 shows the total shortening to be independent of y and also gives the relationship between the shortening, the edge stress, and the deflection coefficients. A similar result was obtained for the y direction.

* Levy does not state this as a boundary condition; however, the computed stress function satisfies this condition.
The relationship between the Fourier coefficients for the normal pressure (which are known) and the unknown deflection coefficients is determined by substituting the expressions for \( w \) and \( \phi \) into the second large deflection equation. This relationship is:

\[
pr,s = D \omega_{r,s} \left( \frac{r^2}{b^2} \frac{2}{\pi} \frac{2}{\pi} + \frac{s^2}{a^2} \right) - \frac{\pi}{2} t \omega_{r,s} \frac{r^2}{a^2} \frac{2}{\pi} + \frac{\tau}{4} \left( G_1 + \ldots + G_9 \right)
\]

(15)

where \( pr,s \) is the normal pressure coefficient and each "G" is a function of the deflection coefficients expressed as a series. Equation 15 thus represents an infinite family of cubic equations with an infinite number of unknowns.

Of the infinite number of terms of the Fourier series, six terms were used to determine the post-buckling behavior of a square plate \((\omega_{1,1}, \omega_{1,3}, \omega_{3,1}, \omega_{3,3}, \omega_{1,5}, \omega_{5,1})\). The first six equations of the family represented by Eq. 15 that did not result in \( 0 = 0 \) were selected and the series were expanded to give the six coefficients. The equations were solved by a method of successive approximations for sixteen values of \( \sigma/\sigma_{cr} \) ranging up to 5.95.

Discussion

In order to check the accuracy of the six term approximation, Levy computed the effective width using two term, three term, and four term approximations for the deflection function. The four term approximation gave the same value to three places as did the six term approximation.
Thus, the six term approximation is completely adequate for describing
the load-shortening behavior.

As is the case in Marguerre's solution, the deflected shape
developed in Levy's solution is a single half-wave modified by higher
harmonics. At $\frac{\epsilon e}{\epsilon cr} = 13$, $w_{3,1} = 0.20 w_{1,1}$ where $w_{3,1}$ is the second
largest term. Levy has thus developed a very accurate description of
the first equilibrium mode.

The simultaneous equations from which the solution is obtained are
cubic and thus yield three solutions for each coefficient. According
to Levy the method of successive approximation will yield a solution
corresponding to stable equilibrium. The validity of the above
statement is questionable as there would seem to be no reason for the
method to seek a position of stable rather than an unstable equilibrium.
The mode upon which the solution converges will depend upon the arbi-
trary values of the coefficients which are selected to begin the
iteration.

Even if the validity of Levy's statement is accepted, it would not
necessarily apply to plates that are not square. The second equilibriu
mode for a square plate is two half-waves of length $1/2$ b. For a plate
of aspect ratio four, the equilibrium mode closest to the original
buckling mode is five half-waves of length $4/5$ b. The mode for which
the wave length is $1/2$ b is the fourth equilibrium mode. It is thus
much less likely that secondary buckling would be delayed up to
The close agreement between Marguerre's and Levy's solutions (see Fig. 8) gives much support to the use of Marguerre's approximate equation as a design equation under the assumption that secondary buckling does not occur. As is indicated above this assumption might be adequate for plates of low aspect ratio. However, it is much less likely that it can be applied to plates of high aspect ratio as used in stiffened plating.

2.5 **STEIN**

Procedure

Stein's solution is for an elastic, initially flat, rectangular plate. Stein expands the displacements \( u, v, \) and \( w \) in power series in terms of an arbitrary parameter. Using the strain displacement relationships of large deflection plate theory (Eq. 6) and the elasticity equations, the stresses can be related to the displacement series. As the stresses and strains are derived from assumed displacement functions, the compatibility requirements are automatically satisfied. The in-plane equilibrium equations (Eq. 5) and the second large deflection equation must, however, still be satisfied. Substituting the stresses into Eqs. 3 and 5 and equating coefficients of like powers of the arbitrary parameter results in an infinite set of linear differential equations. Stein obtains the first approximation by utilizing the first four equations. The second approximation he computes by using the first six equations.

\[ \varepsilon / \varepsilon_{cr} = 16.65 \] as is indicated by Levy for a square plate.
Included in the same report is the post-buckling analysis of a flat plate subjected to a uniform temperature rise.

Stein imposed the following boundary conditions:
1) \( w(0,y) = w(a,y) = w(x,0) = w(x,b) = 0 \)
   (The coordinate system originates at the intersection of the edges).
2) \( \frac{\partial^2 w}{\partial x^2}(0,y) = \frac{\partial^2 w}{\partial x^2}(a,y) = \frac{\partial^2 w}{\partial x^2}(x,0) = \frac{\partial^2 w}{\partial x^2}(x,b) = 0 \)
3) \( \frac{\partial^2 u}{\partial y^2}(0,y) = \frac{\partial^2 u}{\partial y^2}(a,y) = \frac{\partial^2 v}{\partial x^2}(x,0) = \frac{\partial^2 v}{\partial x^2}(x,b) = 0 \)
4) \( \int_0^a \sigma_y(x,0) \, dx = \int_0^a \sigma_y(x,b) \, dx = 0 \)
5) \( \frac{\partial v}{\partial x}(0,y) = \frac{\partial v}{\partial x}(a,y) = \frac{\partial u}{\partial y}(x,0) = \frac{\partial u}{\partial y}(x,b) = 0 \)

Displacements \( u, v, \) and \( w \) are assumed to be in the form of the following series:
\[
\begin{align*}
\mathbf{u} &= \sum_{n=0,2}^{\infty} u(n) \varepsilon^n = u^{(o)} + u^{(2)} \varepsilon^2 + u^{(4)} \varepsilon^4 + \ldots \\
\mathbf{v} &= \sum_{n=0,2}^{\infty} v(n) \varepsilon^n \\
\mathbf{w} &= \sum_{n=1,3}^{\infty} w(n) \varepsilon^n
\end{align*}
\]
where $\epsilon$ is an arbitrary parameter and $(n)$ is a superscript. Utilizing the strain-displacement relationships and the elasticity equations, the stresses can be expressed as follows:

$$N_x = \sum_{n=1,3}^{\infty} N_x^{(n)} (n) + \sum_{m=1,3}^{\infty} \sum_{n=1,3}^{\infty} N_x^{(m,n)} (m+n)$$

(17)

where $N_x^{(n)} = \frac{E_t}{1-\nu^2} \left( \frac{\partial u^{(n)}}{\partial x} + \nu \frac{\partial v^{(n)}}{\partial y} \right)$

$$N_x^{(n,m)} = \frac{E_t}{2(1-\nu^2)} \left( \frac{\partial w^{(m)}}{\partial x} \frac{\partial w^{(n)}}{\partial x} + \nu \frac{\partial w^{(m)}}{\partial y} \frac{\partial w^{(n)}}{\partial y} \right)$$

Similar results were obtained for $N_y$ and $N_{xy}$.

Substitution of the stresses into Eqs. 2 to 4 results in three infinite series each being equal to zero. For these equations to be satisfied, the coefficient of each term must be equal to zero. Thus, an infinite set of differential equations is produced. As an example, the equations shown below are produced by setting the coefficients of $\epsilon^0$ equal to zero.

$$\frac{\partial N_x^{(o)}}{\partial x} + \frac{\partial N_{xy}^{(o)}}{\partial y} = 0$$

(18a)

$$\frac{\partial N_y^{(o)}}{\partial y} + \frac{\partial N_{xy}^{(o)}}{\partial x} = 0$$

(18b)

$$D \nabla^2 w^{(1)} - \left( N_x^{(o)} \frac{\partial^2 w^{(1)}}{\partial x^2} + N_y^{(o)} \frac{\partial^2 w^{(1)}}{\partial y^2} + 2N_{xy}^{(o)} \frac{\partial^2 w^{(1)}}{\partial x \partial y} \right) = 0$$

(18c)
Thus, substituting Eq. 17 into the above equations, the differential equations can be solved for the displacement coefficients. An exact solution of the problem would involve solving the entire set of equations. This is of course impossible. To obtain his second approximation Stein uses the first six equations obtaining values for \( u^{(0)}, u^{(2)}, u^{(4)}, v^{(0)}, v^{(2)}, v^{(4)}, w^{(1)}, \) and \( w^{(3)} \).

The above displacements are functions of \( P^{(o)}, P^{(2)}, P^{(4)} \), where the total edge load \( P = \sum_{n=0,2}^{\infty} p^{(n)} \epsilon^n \) and \( P^{(o)} \) was determined from the solution of Eq. 18 to be equal to \( P_{cr} \) where \( P_{cr} \) is the buckling load of a simply supported plate. Stein chose to express the arbitrary parameter as \( \epsilon^2 = \frac{P-P_{cr}}{P_{cr}} \) thus, \( P = P_{cr} + \epsilon^2 P_{cr} = \sum_{n=0,2}^{\infty} p^{(n)} \epsilon^n \).

Therefore, \( P^{(2)} = P_{cr} \)

\[ p^{(n)} = 0 \quad n \geq 4 \]

The approximate solution is thus complete.

The resulting deflection function is

\[
\begin{align*}
\text{w} &= \left(\frac{2tJ}{\sqrt{3}(1-\nu^2)} + J^2 \omega_3\right) \\
+ \frac{\sin \frac{\pi m x}{a} \sin \frac{\pi n y}{b}}{3} &+ \frac{2tJ}{\sqrt{3}(1-\nu^2)} \frac{\sin 3\frac{\pi m x}{a} \sin \frac{\pi n y}{b}}{3} \\
+ \frac{2tJ}{\sqrt{3}(1-\nu^2)} &\frac{\sin \frac{\pi m x}{a} \sin 3 \frac{\pi n y}{b}}{13} \\
\end{align*}
\]

(19)

where \( J^2 = \beta^2 \frac{Pb}{D\pi^2} - (\beta^2 + n^2)^2 \)

\[
\frac{\beta^4 + n^4}{4} \\
\]

\( \beta = \frac{mb}{a} \)
The load-shortening relationship is given by

\[
\omega = \frac{3}{2} \frac{4 \omega_{13}^{(3)} + 4 \omega_{31}^{(3)}}{\beta^4 + n^4}
\]

\[
\omega_{13}^{(3)} = \frac{4 \beta^4}{(\beta^2 + 9n^2)^2 - (\beta^2 + n^2)^2}
\]

\[
\omega_{31}^{(3)} = \frac{4 n}{(9 \beta^2 + n^2)^2 - 9(\beta^2 + n^2)^2}
\]

The load-shortening relationship is given by

\[
\frac{\varepsilon_e}{\varepsilon_{cr}} = \frac{\sigma}{\sigma_{cr}} + \frac{J^2 \beta^2}{2} + J^4 \omega_{3} \beta^2
\]

and the effective width relationship is

\[
\frac{b_e}{b} = \frac{\sigma}{\sigma_{cr}} \left( \frac{2 \beta^2}{2} + \frac{4 \omega_{3} \beta^2}{2} \right)
\]

**Discussion**

As can be seen in Fig. 7 Stein's solution gives a much lower load capacity for a plate than does Levy's or Marguerre's (Stein's solution was plotted for a plate of aspect ratio four). The reason for this difference in predicted load capacity is the inclusion of secondary buckling in Stein's solution.

The deflection function \( w \) derived by Stein contains the undetermining quantities \( m \) and \( n \) which represent the number of half-waves in the longitudinal and transverse directions, respectively. Each combination of \( m \) and \( n \) represents a deflected position for which the plate is in equilibrium. The load-shortening behavior is of course also dependent
upon \( m \) and \( n \). If it can be assumed that the plate will deform in such a manner that the load is a minimum then secondary buckling will occur at the intersection of the equilibrium curves. The assumption that the lowest load represents the correct load, however, is not necessarily correct.

Stein has analyzed the post-buckling behavior of a column composed of three rigid bars connected by torsional springs and transversely restrained by three non-linear springs (see Fig. 6). Various ratios of \( k_1 \) and \( k_2 \) were considered. According to Stein the behavior of the column is analogous to the behavior of a plate in the post-buckled state.

The possible deflection modes for the column consist of a symmetrical mode, an anti-symmetrical mode, and a mode which is neither symmetrical nor anti-symmetrical. The column would initially buckle into the symmetrical mode. At some point beyond this first buckling load the equilibrium curve for the symmetrical mode will intersect with that for the anti-symmetrical mode, (point A in Fig. 7).

Stein has examined the stability of the symmetrical mode and has shown that the instability will occur at a load in excess of the load given by the intersection of the equilibrium curves (point B in Fig. 7). Depending upon the ratio of \( k_1 \) and \( k_2 \), the instability can occur shortly after the intersection of the equilibrium curves or can occur at a much higher load. At the point of instability the transition from the symmetrical mode to the anti-symmetrical mode will take place.
at a constant load if the loading is being controlled or at a constant strain if the strain is being controlled. It is also possible that a stable equilibrium path (neither symmetrical nor anti-symmetrical) exists between the two modes (dashed line in Fig. 7).

A qualitative description of the behavior of a plate can be based on the analysis of the column. Unfortunately, a quantitative description is as yet not available. To date, no mathematical analysis of plate stability in the post-buckling region has been published.

If instability of the equilibrium occurs shortly after the intersection of the equilibrium curves then Stein's solution should more closely describe the post-buckling behavior than the solutions of Marguerre or Levy. If, however, a large delay occurs between the point of the instability and the intersection of the equilibrium curves, Stein's solution could give misleading results. In addition, for Stein's solution to accurately describe the behavior of a plate, secondary buckling must occur not once but each time the equilibrium curves intersect.

In order for Stein's solution to predict the behavior of a plate in which secondary buckling does occur, the solution must accurately describe the various equilibrium modes. In general there is no way of checking this; however, Stein's solution for the first mode (m, n = 1 for a square plate) can be compared to Levy's solution. Stein's solution gives a higher load capacity and for $e/e_{cr} = 14$. 
the difference is approximately 10 percent, indicating only moderate accuracy. It is difficult to draw any conclusions from this as to the accuracy of the solutions for the higher modes. Even though Stein's solution for the first mode is an upper bound to Levy's essentially exact solution, it can not be concluded that Stein has developed an upper bound solution for the post-buckling behavior in which secondary buckling occurs.

To obtain an idea of the convergence of his solution, Stein compared his first and second approximations. In the first approximation the function developed for \( w \) was a simple sine term and in the second approximation it consisted of three terms. At \( \epsilon_e/\epsilon_{cr} = 6 \) the first approximation gives an average stress approximately 10 percent greater than the second approximation which hardly indicates convergence. It is possible that solution of additional differential equations of the series which would yield higher harmonics would increase the accuracy of the solution such that the behavior in the first mode would agree with that given by Levy's solution.

2.6 KOITER Procedure

Koiter's solution is for an infinitely long, initially flat, elastic plate. To obtain a solution functions were assumed for the three displacements \( u, v, \) and \( w \). The total strain energy computed from the assumed displacements was minimized thus determining the displacements. Solutions were obtained for plates having the three possible rotational
restraint conditions at the longitudinal edges, that is, zero edge moments, elastic restraint, and a fully clamped condition.

The following boundary conditions are applicable in each case:

1) \( \frac{\partial v}{\partial x}(x,0) = \frac{\partial v}{\partial x}(x,b) = 0 \)

The longitudinal edges are thus constrained to remain straight (the y coordinate originates at the edge of the plate).

2) \( \int_0^L \sigma_y(\alpha,y)\,dx = \int_0^L \sigma_y(b,y)\,dx = 0 \)

where \( L \) is the length of the longitudinal half-wave.

For the case in which the edge moments are zero, Koiter has obtained five different solutions each corresponding to a different set of displacement functions. Since the solutions are based on energy principles, the predicted load capacity will approach from above that given by an exact solution and the lowest solution is thus the most nearly correct one. This lowest solution will be discussed here.

A sketch of the transverse out-of-plane deflection profile is shown in Fig. 2. It is composed of two quarter sine waves and a flat portion in the middle.

For \( 0 < y < 1/2 \alpha b \)

\[ w = f \sin \frac{m \pi}{L} \sin \frac{m y}{\alpha b} \]  

(22a)
\[
\begin{align*}
\frac{u}{\alpha b} &= -\varepsilon_e x - \frac{\pi^2}{8L} \sin 2\pi x \sin^2 \frac{\pi y}{a} (22b) \\
\frac{v}{\alpha b} &= v_o(y) - \frac{\pi^2}{8\alpha b} \sin 2\pi x \sin \frac{2\pi y}{a} (22c)
\end{align*}
\]

For \(1/2 \alpha b < y < (1 - 1/2\alpha)b\)
\[
\begin{align*}
w &= f \sin \frac{\pi x}{L} \quad (23a) \\
u &= -\varepsilon_e x - \frac{\pi^2}{8L} \sin 2\pi x \quad (23b) \\
v &= v_1(y) \quad (23c)
\end{align*}
\]

For loads just above the buckling load \(\alpha\) is equal to 1.0 and the transverse shape is a single half-wave. The longitudinal shape is a sine wave.

For the above displacement functions \(\tau_{xy}\) is identically zero throughout the entire plate. Thus an implied boundary condition is that \(\tau_{xy}\) is zero at the longitudinal edges.

Parameters \(f, L, \alpha\) are undetermined--parameter \(f\) is the maximum amplitude of the wave, \(\alpha\) determines the length of the flat portion, and \(L\) is the longitudinal half-wave length. \(v_o(y)\) and \(v_1(y)\) are undetermined functions.

To obtain the solution, the integral for the total potential must be minimized with respect to \(f, \alpha, L, v_o(y),\) and \(v_1(y)\). According to Koiter the total potential will be a minimum with respect to the unknown functions if the portion of \(\varepsilon_y\) which is independent of \(x\) is equal to
\(- \nu \varepsilon_x\). The above fact combined with the direct differentiation of the total strain energy with respect to \(f, \alpha, \) and \(L\) will give the solution.

For the case of clamped longitudinal edges Koiter has obtained two solutions. The solution giving the lowest load capacity for the plate is discussed here.

For \(0 < y < \frac{1}{6} \alpha b\)

\[ w = \frac{1}{3} f \left( 1 - \cos \frac{3\pi y}{\alpha b} \right) \sin \frac{\pi x}{\alpha b} L \quad (24a) \]

\[ u = -\varepsilon_x \left( 1 - \frac{\pi^2 f^2}{L} \right) \left( \frac{1}{48} - \frac{1}{36} \cos \frac{3\pi y}{\alpha b} + \frac{1}{144} \cos \frac{6\pi y}{\alpha b} \right) \sin \frac{2\pi x}{L} \quad (24b) \]

\[ v = v_0(y) + \frac{\pi^2 f^2}{\alpha b} \left( \frac{1}{24} \sin \frac{3\pi y}{\alpha b} - \frac{1}{48} \sin \frac{6\pi y}{\alpha b} \right) \cos \frac{2\pi x}{L} \quad (24c) \]

For \(\frac{1}{6} \alpha b < y < \frac{1}{2} \alpha b\)

\[ w = f \left[ \frac{1}{3} + \frac{2}{3} \sin \left( \frac{3\pi y}{2\alpha b} - \frac{\pi}{4} \right) \right] \sin \frac{\pi x}{L} \quad (25a) \]

\[ u = -\varepsilon_x \left( 1 - \frac{\pi^2 f^2}{L} \right) \left[ \frac{1}{24} + \frac{1}{18} \sin \left( \frac{3\pi y}{2\alpha b} - \frac{\pi}{4} \right) - \frac{1}{36} \sin \frac{3\pi y}{\alpha b} \right] \sin \frac{2\pi x}{L} \quad (25b) \]

\[ v = v_1(y) + \frac{\pi^2 f^2}{\alpha b} \left[ \frac{1}{24} \cos \left( \frac{3\pi y}{2\alpha b} - \frac{\pi}{4} \right) - \frac{1}{24} \cos \frac{2\pi y}{\alpha b} \right] \cos \frac{2\pi x}{L} \quad (25c) \]

For \(\frac{1}{2} \alpha b < y < (1 - 1/2 \alpha) b\)

\[ w = f \sin \frac{\pi x}{L} \quad (26a) \]

\[ u = -\varepsilon_x \left( 1 - \frac{\pi^2 f^2}{8L} \right) \sin \frac{2\pi x}{L} \quad (26b) \]

\[ v = -\varepsilon_2 y \quad (26c) \]
L, α, f, and ε₂ are undetermined parameters and v₀ and v₁ are undetermined functions. The out-of-plane deflection w in the region $0 < y < 1/2 ab$ was chosen such that for $α = 1$ it approximates a cosine curve which is the buckled shape for a plate with clamped edges. As in the case for zero edge moments $τ_{xy}$ is identically zero throughout the plate.

One of the most interesting and important conclusions of Koiter's work is that the effective width is nearly independent of the rotational restraint at the longitudinal edges. The maximum difference between the solutions for a clamped plate and a plate with zero edge moments is 3 percent. The solution for an elastically restrained plate falls between the two solutions as would be expected. Koiter thus proposed that the following equation be used for all three cases:

$$b_e/b = 1.2 \left( \frac{ε_{cr}}{ε_e} \right)^{2/5} - 0.65 \left( \frac{ε_{cr}}{ε_e} \right)^{4/5} + 0.45 \left( \frac{ε_{cr}}{ε_e} \right)^{6/5}$$

(27)

In the range $1 < ε_e/ε_{cr} < 100$ the maximum difference between the above equation and the solution for the case of zero edge moments is 1.5 percent.

**Discussion**

One of the undetermined parameters in Koiter's solution is L, the half-wave length in the longitudinal direction. L is thus determined as a continuous function of the applied load. As the load increases the wave length decreases. This, however, will not be the case for a finite plate. The wave length will be able to change only at finite load
intervals. This is the phenomenon described previously as secondary buckling. Koiter and Stein are thus essentially solving the same problem of the post-buckling behavior of a plate in which secondary buckling is assumed to occur.

Stein's solution as applied to a long plate should agree with Koiter's solution. As can be seen in Fig. 8 this is not the case since Koiter's solution lies approximately half way between Marguerre's and Stein's solutions. Since Koiter employed energy principle, his solution is an upper bound to an exact solution; however, Stein's solution is not necessarily an upper bound. Thus, it is impossible to determine which solution is more exact.

The difference in the two solutions might possible be explained by Koiter's use of a relatively crude shape for the deflection w. The use of a more refined shape in the transverse direction would probably have lowered the curve. If higher harmonies had been added to the longitudinal wave the curve would have again been lowered. It is, however, impossible to determine how close Koiter's solution would then be to Stein's solution.

It thus appears that no definite conclusion can be drawn as to which solution best describes the behavior of a plate in which secondary buckling occurs.
2.7 **HU, LUNDQUIST, AND BATDORF**

Hu, Lundquist, and Batdorf determined the elastic post-buckling behavior of a square plate having initial deviations from flatness. Both the initial deviations and the deflected shape are represented by Fourier series. The method employed to obtain the undetermined coefficients is the same as that used by Levy. The simultaneous equations for the coefficients are developed for a rectangular plate subjected to edge compression and normal pressure. For the specific case of a square plate under uniaxial edge compression, the developed relationships are employed using a finite number of terms of the Fourier series.

The boundary conditions are the same as those imposed by Levy, that is,

1. The edges must remain in the original plane of the plate
2. The edge moments are zero
3. The edges must remain straight
4. Free expansion at the longitudinal edges
5. Zero shearing stresses at the edges.

The assumed deflection function is

$$ w = t \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} K_{m,n} \frac{\sin \frac{m\pi x}{b} \sin \frac{n\pi y}{b}}{b} $$

where $K_{m,n}$ is an undetermined coefficient.
At zero load
\[ w_0 = t \sum_{M=1}^{\infty} \sum_{n=1}^{\infty} K_{mn} \frac{\sin \frac{m \pi x}{b} \sin \frac{n \pi y}{b}}{b} \] (29)

For the solution to the post buckling behavior of a square plate, the coefficients \( K_{1,1} \), \( K_{1,3} \), \( K_{3,1} \), \( K_{3,3} \), \( K_{1,5} \), and \( K_{5,1} \) were used. The initial deflection was approximated by
\[ w_0 = t K_{0,1,1} \frac{\sin \frac{\pi x}{b} \sin \frac{\pi y}{b}}{b} + t K_{0,3,1} \frac{\sin \frac{3\pi x}{b} \sin \frac{\pi y}{b}}{b} \] (30)

Various combinations of \( K_{0,1,1} \) and \( K_{0,3,1} \) were employed as shown below.

<table>
<thead>
<tr>
<th>( K_{0,1,1} )</th>
<th>0</th>
<th>0.01</th>
<th>0.04</th>
<th>0.10</th>
<th>0</th>
<th>0.04</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K_{0,3,1} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.01</td>
<td>0.01</td>
</tr>
</tbody>
</table>

The load shortening plots presented by the authors show the assumed deviations to have only a slight effect on the behavior of the plate. They conclude that the effect of initial deviation from flatness is felt mainly near and at the theoretical flat plate buckling load. For stresses well above and below this point the effect should be negligible.

The greatest reduction in load was caused by the combination of \( K_{0,1,1} = 0.1 \) and \( K_{0,3,1} = 0 \) which represents an initial deviation from flatness of 10 percent of the thickness. At the theoretical critical strain the average stress was reduced by 10 percent. However, at \( \frac{\epsilon_e}{\epsilon_{cr}} = 2 \) the reduction was only 4 percent. The behavior before buckling corresponded closely to the flat plate behavior.
The applicability of the above conclusions to actual plates will depend upon whether the initial deviations fall within the range considered by the authors. Initial deviations considerably higher than 10 percent of the thickness could have an appreciable effect on the load-shortening behavior. Of the experimental investigations included in this report, only Ojalvo and Hull \(^{(13,14)}\) took measurements for initial out-of-flatness. The correlation between the results of these tests and the predictions of Hu, Lundquist, and Batdorf is discussed in Section 3.4.

2.8 ADDITIONAL THEORETICAL SOLUTIONS

The papers of Marguerre, Levy, Stein, and Koiter are rigorous investigations of the post-buckling behavior of elastic flat plates. Other less rigorous solutions have been developed. Some of these are briefly described here.

1. Schnadel \(^{(3)}\)

Schnadel imposes the same boundary conditions and uses the same method that Marguerre employed. His assumed deflection function is

\[
\frac{\sin nx}{a} \sin ny + \frac{\sin 3nx}{a} \sin 3ny
\]

In the derived stress function the terms contributing to the shear are neglected. Thus, \(\tau_{xy}\) is assumed to be equal to zero.

Two simultaneous non-linear algebraic equations describing the equilibrium are obtained by imposing the condition of minimum total strain energy. Schnadel suggests solving these equations by an
iterative method, however, the procedure was not demonstrated in his report.

2. von Karman

As noted in the introduction von Karman was one of the first to investigate the post-buckling behavior of a rectangular plate. His analysis is rather crude as many assumptions were made.

A flat center portion bounded by the quarter sine waves (same as shape used by Koiter) was chosen for the transverse shape. It was then assumed that the edge strips carried the entire load distributed uniformly. The flat portion was thus neglected in the computations and the deflection function reduced to the following:

\[ w = f \sin \frac{m \pi}{L} \sin \frac{m \pi}{b_e} \]  

(32)

Assuming that \( \sigma_y, \tau_{xy} = 0 \) and that the assumed uniform stress is equal to \( \sigma_e, \sigma_e \) was computed from the first large deflection equation (Eq. 2). The stress was then minimized with respect to the half-wave length \( L \). The resulting effective width formula is given by Eq. 1.

von Karman's solution gives a lower load capacity then the solution shown in Fig. 8. This is not surprising considering the number of assumptions made. As \( \sigma_y \) was assumed to be equal to zero, the longitudinal edges are not constrained to remain straight and thus one of the boundary conditions proposed in Section 2.2 is not satisfied. This would lower the predicted load capacity of the plate.
3. **Cox** (15)

Cox employed minimum energy principles. His assumed deflection function is the same as that used by Koiter consisting of a flat center portion bounded by sine waves in the transverse direction and a sine wave in the longitudinal direction. The length of the longitudinal wave was assumed to be equal to that at buckling and did not vary in the post-buckling range as in Koiter's theory.

Cox assumed that both $\tau_{xy}$ and $\sigma_y$ were equal to zero. His solution is thus not necessarily an upper bound to an exact solution. The assumption that $\sigma_y = 0$ implies that the longitudinal edges are not forced to remain straight.

The effective width was determined to be

$$\frac{b_e}{b} = 0.8 \left( \frac{\varepsilon_{cr}}{\varepsilon_e} \right) + 0.09 \quad (33)$$

Equation 33 will give a load-shortening curve which lies between Stein's and Koiter's solutions.

4. **Bengston** (16)

Bengston also employed minimum energy principles. Assuming a sinusoidal deflection function of non-variable wave length, $\phi$ was determined from Eq. 2 after which $u$ and $v$ were computed.

Bengston imposed the condition of constant axial strain at the longitudinal edges. As the function determined for $u$ and $v$ did not satisfy this boundary condition, they were modified to do so. The
total potential was then computed using the assumed deflection function and the modified functions for \( u \) and \( v \). The resulting effective width equation is

\[
\frac{b_e}{b} = 0.483 + 0.517 \left( \frac{\varepsilon_{cr}}{\varepsilon_e} \right)
\]  

(34)

Since a simple sinusoidal function can describe the actual deflected shape only near the buckling load, Eq. 34 is applicable only for loads near the buckling loads. To extend the formula to higher loads, Benston assumed that at nine times the critical strain the highly stressed edge strips would buckle into square panels as did the plate at first buckling. The new effective width would then be the effective width of the edge strips. Thus for \( \varepsilon_e > 9 \varepsilon_{cr} \)

\[
\frac{b_e}{b} = \left[ 0.483 + 0.517 \left( \frac{\varepsilon_{cr}}{\varepsilon_e} - 9 \right) \right] \left[ 0.483 + 0.517 \left( \frac{\varepsilon_{cr}}{\varepsilon_e} \right) \right]
\]  

(35)

An envelope curve for Eqs. 34 and 35 falls approximately 7 percent below Levy's solution.

5. Boley (17)

Boley's solution should not be considered any less rigorous than the solutions of Marguerre, Levy, Stein, or Koiter; however, as it is not considered in detail it is included in this section.

The method employed involves successive solutions of Eqs. 2 and 3 in which the non-linear terms (right hand sides of both equations) are replaced by previously derived functions. The boundary conditions are the same as those employed by Marguerre.
The first step in the solution was the selection of an initial deflection function:

\[ w_1 = f \cos \frac{mnx}{a} \cos \frac{mny}{b} \quad (36) \]

where \( m \) and \( n \) correspond to the half-wave lengths at buckling. Substituting \( w_1 \) into Eq. 2, a solution was determined for \( \phi_1 \). \( \phi_1 \) and \( w_1 \) were then substituted into Eq. 3 and a new deflection function, \( w_2 \), was derived. The process was repeated once more as \( w_2 \) was substituted into Eq. 2 and a new stress function was determined. Two simultaneous equations were thus developed relating \( \sigma_e \) and \( b_e \) and were solved numerically yielding a solution for the effective width. The resulting curve agreed quite well with the solutions of Levy and Marguerre.
3. EXPERIMENTAL INVESTIGATIONS

3.1 GENERAL

The experimental work included in this chapter covers tests of rectangular plates subjected to edge compression in one direction. The discussions consider primarily the boundary conditions provided by the test jigs, the occurrence or non-occurrence of secondary buckling, and the agreement between the test results and the theoretical solutions. A summary of the experimental work is given in Table 1.

3.2 SCHUMAN AND BACK

In 1930 Schuman and Back conducted tests on plates of duralumin, stainless steel, monel metal, and nickel\(^{(2)}\). The thickness at the plates varied from 0.015 in. to 0.095 in. and the b/t ratio varied from 42 to 1600. The aspect ratio ranged from 1.0 to 6.0.

The results shown in Fig. 9 are for tests of duralumin and monel metal plates. The largest b/t ratio is 196.

A schematic diagram of the test jig is shown in Fig. 3. The longitudinal edges of the plate are supported by 45 degree V-grooves. The loading edges were compressed flat end.

The ultimate load was recorded for each plate. However, no measurements were taken for the shortening at the ultimate load or at any intermediate loads. Measurements were recorded for the out-of-plane deflection and profiles were plotted for various loads. The deflection mode at first buckling was retained throughout the entire
range at loading and thus no secondary buckling occurred.

The points shown in Fig. 9 are plotted as $\frac{\sigma_{\text{max}}}{\sigma_{\text{cr}}}$ vs. $\frac{\epsilon_o}{\epsilon_{\text{cr}}}$ where $\epsilon_o$ is the yield strain. $\sigma_o$ (the yield stress) was determined by Schuman and Back as the stress at which the slope of the stress-strain curve was equal to $1/3\ E$. $\epsilon_o$ has been approximated as $E/\sigma_o$ for the points in Fig. 9.

Plotting the maximum load at $\epsilon_e = \epsilon_o$ is incorrect in that the ultimate load is actually reached at a higher edge strain. $\epsilon_o$ approximates the edge strain at which first yielding occurs. By the same reasoning, the actual load supported by the plate at $\epsilon_e = \epsilon_o$ is less than the ultimate load. Thus, the ultimate loads plotted in Fig. 9 should fall above the predicted elastic behavior.

As can be seen in Fig. 9, the points fall considerably lower than the load-shortening curves. There would seem to be two probable reasons for this. First, the restraint necessary to force the longitudinal edges to remain straight is not provided by the test jig. This boundary condition was included in the theoretical investigations of Marguerre, Levy, Koiter, and Stein. The absence of this restraint would reduce the strength of the plates.

A second and probably more important reason is the inability of the V-grooves to retain the longitudinal edges in the original plane of the plate. Once the plate buckles the longitudinal edges begin to warp with the top and bottom of the plate expanding and the middle portion contracting. The middle portion, while remaining in contact
with the groove, will move out of the original plane of the plate. Had no support been provided against out-of-plane movement of the longitudinal edge, the plate would have buckled as a column and would have little post-buckling strength. It can thus be expected that even a small out-of-plane movement of the longitudinal edges would cause a significant drop in the load carried by the plate.

The tests of Sechler (see Sect. 3.3) also had longitudinal edges supported in V-grooves. The results as can be seen in Fig. 9 are very similar.

3.3 SECHLER

In 1933 Sechler reported the results of experiments conducted on plates of Dural, steel, aluminum, and brass (18). The b/t ratio ranged from 15 to 1415 and the aspect ratio ranged from 2/3 to 12.

Only the steel plates are considered here. For the test results plotted in Fig. 9 the maximum b/t ratio is 347.

The plates were tested in a jig which provided V-groove support for all four sides.

As the purpose of the tests was to determine the ultimate strength of the plate, this was the only test data recorded. Neither the shortening nor the out-of-plane deflection was measured. No measurements were taken for initial deviations from flatness; however, plates which visually showed large deviations were not tested.
The longitudinal edges of one of the brass plates were observed to buckle into five half-waves at a load above the plate buckling load. Just before the ultimate was reached the edges buckled into twelve half-waves. This is not, however, the phenomenon described as secondary buckling since the overall wave pattern did not change.

The points shown in Fig. 9 are plotted as \( \frac{\sigma_{\text{max}}}{\sigma_{\text{cr}}} \) vs. \( \frac{\epsilon_o}{\epsilon_{\text{cr}}} \) where \( \sigma_o \) was determining by Sechler as the point at which the stress-strain curve became non-linear. \( \sigma_o \) is thus the proportional limit and \( \epsilon_o \) is equal to \( \sigma_o / E \).

As can be seen from Fig. 9 the results compare well with Schuman and Back's results and are lower than the theoretical curves. The explanation for the results of Schuman and Back's tests can of course be applied here, that is, the V-grooves neither provide sufficient restraint against out-of-plane movement of the longitudinal edges nor enforce straightness of the edges.

3.4 DAVID TAYLOR MODEL BASIN

The David Taylor Model Basin conducted ultimate strength tests on aluminum and steel plates. To date two reports have been issued. The first in 1960, was prepared by Duffy and Allnutt (19) and the second report prepared by Conley, Becker, and Allnutt was issued in 1963 (20).

Three different grades of steel were used (HY-80, \( \sigma_o = 100 \text{ ksi} \); STS, \( \sigma_o = 100 \text{ ksi} \); HTS, \( \sigma_o = 50 \text{ ksi} \)). Three different grades of aluminum were also used (6061-T6, \( \sigma_o = 40 \text{ ksi} \); 5456-H24, \( \sigma_o = 40 \text{ ksi} \); 5456-H321, \( \sigma_o = 35 \text{ ksi} \)).
The b/t ratios of the steel plates varied from 32 to 144 and that of the aluminum plates varied from 48 to 144. The plates had aspect ratios of 2.0 and 3.0. A total of 111 plates were tested, 61 of which had various combinations of longitudinal and transverse welds. (Some unusual residual stress patterns were thus created which unfortunately were not determined). The points shown in Fig. 9 are only for plates without welds.

The four edges of each plate were rounded off and given a radius of one half the plate thickness. The transverse and longitudinal edges were then supported by circular grooves having a radius matching that of the plate edges.

The ultimate load was recorded for each plate. No data was taken for the shortening or the out-of-plane deflections. Strain gages were placed at various locations on the plates and the strains were recorded for each load increment. Very little of this data was included in the reports, however.

The experimental points shown in Fig. 9 are plotted as $\frac{\sigma_{\text{max}}}{\sigma_{\text{cr}}} \text{ vs. } \frac{\varepsilon_{\text{cr}}}{\varepsilon_{\text{cr}}}$. The compressive yield strength was given for each plate; however, the authors did not state how it was determined.

As can be seen in Fig. 9, the test results compare favorably with those of Schuman and Back and those of Sechler. Although the type of support provided at the longitudinal edges is different than that described in the two previous studies, the effect is the same.
The longitudinal edges are not constrained to remain straight and are allowed to move out of the plane of the plate once the edge warps.

Because of the failure of the test set-ups to provide the correct boundary conditions it must be concluded that the tests of Schuman and Back, Sechler, and the David Taylor Model Basin do not adequately simulate the behavior of a plate existing in a stiffened plate panel. These tests can thus have little bearing on the final choice of a load-shortening curve.

3.5 OJALVO AND HULL

Ojalvo and Hull conducted tests on twenty-four 24 S-T3 aluminum plates (13'14'). The plates had aspect ratios of 4.12 and 8.0, and b/t ratios of 71, 91, 138, and 232.

The plates were tested in a jig as shown in Fig. 4. The longitudinal edges extended approximately 1/8 in. into the grooves, and according to the authors this type of support should approximate a simple support condition. Theoretically the jig should not provide the restraint necessary to keep the longitudinal edges straight and will not inhibit lateral movement of the longitudinal edges. It will, however, restrain the longitudinal edges from out-of-plane movement.

The possibility that part of the load might be transferred to the jig was considered by the authors. To minimize any frictional restraint between the longitudinal edges and the rectangular grooves, a lubricant (Molykote type G thinned with fine grease) was applied to the
edges. According to Ojalvo in a letter to the writer, the plate could be moved quite easily in the longitudinal direction at zero load.

An experimental determination of the load transfer from the plate to the jig was carried out. Strain gages were attached to the longitudinal edges in order to measure the vertical strain. The strain at the bottom of the plate was found to be approximately equal to that at the top which would not be the case if some of the load was being transferred to the jig. The maximum load for which the strain was recorded was approximately twice the buckling load and half of the ultimate load. It would then appear that for this range of loading the load transfer was negligible. It was noted by the authors, however, that failure of each specimen was caused by tearing along a longitudinal line between the loading bar and the groove. This would indicate a large shearing stress undoubtedly caused by vertical restraint at the longitudinal edges. Thus it would appear that a fairly substantial portion of the load was transferred to the jig by the time the ultimate load was reached.

Initial out-of-flatness was determined for two of the plates. For a 0.025 in. thick plate (b/t = 232) the maximum deviation was 60 percent of the thickness and for a plate of thickness 0.082 in. (b/t = 71) the maximum deviation was 20 percent.

The out-of-plane deflection was not measured. The plates of aspect ratio 4.12 were observed to buckle into four half-waves and the plates of aspect ratio 8 into eight half-waves. In some of
the tests a gradual change is the wave length was observed upon an increase in the load eventually resulting in an increased number of waves. This increase sometimes occurred with an audible snap.

The actual load-shortening plots for the plates of aspect ratio 4.12 were included in Hull's thesis(13). The curves shown in Fig. 10 were selected as typical examples of the results for the b/t ratios indicated.

The curve plotted for plates of b/t = 71 agrees fairly well with Marguerre's solution. There was a noticeable amount of scatter in the test results with a variation as much as 20 percent near the ultimate load. Koiter's equation would conservatively predict the ultimate load for all the tests in this group.

The curve selected for the plates of b/t = 91 also agrees well with Marguerre's solution. The results for the individual plates differ by as much as 25 percent near the ultimate load. Again Koiter's equation would conservatively predict the ultimate strength of the plates.

The test results for the plates of b/t = 138 deviate considerably from the theoretical curves. The test curves converge towards Marguerre's solution near the ultimate load but for lower loads the agreement is very poor. Except for one plate which had a very low load capacity, the results were very consistent as the maximum difference near the ultimate load was only 10 percent.
The test results for the plates having a b/t ratio of 232 differ spectacularly from the predicted behavior. At the theoretical critical strain the load varied from one-quarter to one-half of the theoretical buckling load. The curves are nearly linear up to a strain of $20 \varepsilon_{cr}$. At the ultimate load the curves approach Koiter's equation. The results were very consistent and varied only about 10 percent near the ultimate load.

There is no obvious answer as to why the results for the thinner plates (b/t = 138, 232) failed to agree with the theoretical predictions. The absence of the restraint necessary to enforce straightness of the longitudinal edge and the possible existence of non-parallelism of the loading edges would cause a reduction in the load. It is unlikely, however, that either is a significant factor in such a large load reduction. It should be noted that these factors would also be present in the thick plates for which the agreement with theory was good.

Probably the most important factor for the low load capacity of the thin plates is the existence of initial deviations from flatness. Hu, Lundquist, and Batdorf (12) (see sect. 2.7) analyzed plates with initial deviation from flatness and found the effect on the load-shortening behavior to be negligible. There are, however, two major differences in this analysis and the situation encountered here.

(1) In the plate for which measurements were taken the initial out-of-flatness was six times as large as that considered by Hu, et al. This would of course increase the deviation from the predicted flat plate behavior.
(2) In the theoretical analysis the assumed wave length for the initial out-of-flatness which caused the greatest effect was equal to the wave length of the buckles of an initially flat plate. Thus once the theoretical buckling load had been passed the plate could be expected to follow the theoretical flat plate behavior. The measurements taken by Ojalvo and Hull showed the initial deviation to be a single half-wave whereas the theoretical buckled shape would consist of four half-waves. For the plate to develop the flat plate buckling mode it must buckle from the single half-wave mode to the four half-wave mode. Theoretically this could not happen until the actual load-shortening curve intersected the theoretical flat plate curve. Assuming the above described behavior is typical, the theoretical flat plate behavior should give a reasonable estimate of the behavior of a long plate containing initial deviations from flatness.

The tests did not, however, follow this pattern as the four half-waves appeared long before the experimental curves intersected the theoretical curves. It is possible that instead of having a buckling phenomenon in which one mode is replaced by another, the four half-wave mode was being superimposed upon the one half-wave mode. Thus both modes existed at the same time. This type of behavior can not be described by any of the present theoretical solutions. As the load is increased the effect of the one half-wave mode is probably lessened as is indicated by the relatively good agreement between the theoretical and experimental curves near the ultimate load.
3.6 Botman and Beseling

Botman and Beseling conducted tests on aluminum plates having thicknesses varying from 1.06 mm. to 4.73 mm. (21, 22, 23, 24). The purpose of the tests was to approximate as closely as possible the behavior of a plate as it exists in a stiffened plate panel. Instead of a stiffened panel multi-bay panels having from one to five bays and supported by knife edges at the stiffener points were tested (see Fig. 5). The load carried by the plate could then be measured directly which would not be the case if a stiffened panel were tested since part of the load would be carried by the stiffeners. The knife edges will provide the same out-of-plane restraint as would the stiffeners. The tests were designed to investigate the effective width in the elastic and inelastic ranges. However, only the tests in which the major portion of the post-buckling strength was developed in the elastic range are considered here.

Careful consideration was given to the design of the test setup. The most important feature was the knife edges which were placed on each side of the plate. The knife edges were designed to provide a minimum amount of vertical frictional restraint and a minimum amount of rotational restraint. A detail of a knife edge is shown in Fig. 5. A brass wire having a 2 mm radius was inserted into a machined slot which was filled with graphite grease. The wire was placed in 5 mm strips separated by 2 mm gaps.

Two other types of knife edges were used in preliminary tests but were discarded in favor of the type described above.
A small amount of play was allowed between the knife edges and the plates as the knife edges were not butted up against the plate.

As in the tests conducted by Ojalvo and Hull, an important unknown factor was the amount of load being transferred to the jig. Boteman and Besseling state that at the higher loads transfer of large load to the knife edges could not be completely prevented. No indication was given as to what percentage of the total load this might be.

The loading edges were fitted in small slots approximately 2 mm in depth. The slots were filled with graphite grease. Strain gages used in one of the preliminary tests indicated that very little horizontal frictional restraint was being exerted on the loading edge. This was the condition desired by the investigators.

The shortening of each specimen was determined by dial gages and the load was read directly from the testing machine. The wave amplitude of the longitudinal centerline of the middle bay was determined for each specimen.

The tests were divided into three groups. The first group consisted of preliminary tests conducted to investigate the test set-up and to determine the number of bays required to provide the restraint necessary to enforce straightness of the longitudinal edges. Because of the close agreement between the results for panels of three and five bays, a three-bay panel was chosen for the second and third groups of tests. The preliminary tests are described in References 21 and 23.
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24S-T aluminum having a yield stress of 50 ksi (determined from a 0.2 percent offset) was used for the second group of tests. The b/t ratio varied from 51 to 124 and \( \varepsilon_{cr}/\varepsilon_{pl} \) varied from 0.0805 to 0.561 where \( \varepsilon_{pl} \) is the strain at the proportional limit. The second group of tests is described in References 22 and 24.

For the third group of tests 75S-T \( (\sigma_0 = 70 \text{ ksi}) \) and 2S-1/2H \( (\sigma_0 = 15 \text{ ksi}) \) aluminum was used. The b/t ratio varied from 31.7 to 122 and \( \varepsilon_{cr}/\varepsilon_{pl} \) varied from 0.0487 to 0.93. The third group of tests is described in Reference 24.

The authors plotted the test results as \( b/e \) vs. \( \sqrt{\varepsilon_{cr}/\varepsilon_e} \). The effective width was determined from the recorded load-shortening data using the equation

\[
\frac{b}{b_e} = \frac{\sigma}{E_s \varepsilon_e}
\]

where \( E_s \) is the secant modulus.

Before performing the above calculation the authors corrected the load-shortening curves to account for two effects:

(1) Deviations from the flat plate behavior near the origin of the load-shortening curves and, (2) Deviation from the flat plate behavior for loads near the buckling load.

The first effect was caused by non-parallelism of the loading edges which produced a non-uniform stress distribution across the width of the plate. The second effect is normally assumed to be caused by
initial deviations from flatness; however, the authors feel this is not the case in their tests. One of the plates was strain gaged at the center and the membrane strain was compared to the unit shortening. The two values agreed quite closely which would not be the case if initial deviations from flatness were causing the reduction in the load. The authors feel that the play allowed at the knife edges permitted the plate to buckle as a column before plate buckling occurred. The unit shortening was thus increased above the value given by $\sigma/E$.

The values used for $\epsilon_{cr}$ in the authors effective width plots are experimentally determined values found by plotting load vs. $(w/a)$. The intercept of the load axis is considered the buckling load. For the plots included in this report, the effective width curves were reconverted to load-shortening curves. However, the theoretical value of $\epsilon_{cr}$ was used to make the plots consistent with other data included here.

The experimental buckling load was normally higher than the theoretical value and in at least one case was 20 percent higher. The discrepancy between the experimental and theoretical values could possibly be caused by inaccuracies in the method of determining the experimental value. It is also possible that the buckling load was increased by rotational restraint at the longitudinal edges.

Wave patterns for loads up to two to three times the buckling load were included in the reports. In general the mode at first buckling (five half-waves) was retained for this range of loading.
It is unfortunate that a continuous record was not made for the entire range of loading.

Changes in the wave pattern at loads well above the buckling load were visually observed by the authors. For some of the preliminary tests, the change occurred suddenly and was accompanied by a bang. However, when the sliding pieces were used on the knife edges, a gradual transition to a smaller wave-length took place.

The test results for the 24S-T and 75S-T aluminum plates were remarkably consistent. The results for the 2S-1/2H plates were fairly consistent within themselves but did not agree with the results of 75S-T and 24S-T plates. The proportional limit for 2S-1/2H aluminum is located well below the ultimate strength, thus the major portion of the post-buckling strength was developed in the inelastic range. For this reason the behavior of the 2S-1/2H plates can not be predicted by any of the theoretical elastic solutions.

The experimental curves shown in Fig. 11 are tests of 75S-T plates. These curves can be considered typical of all the tests on 75S-T and 24S-T plates. The results agree very well with Koiter's theoretical elastic curve for the entire loading range which encompasses both elastic and inelastic behavior.

The test set-up of Botman and Besseling satisfies the boundary condition of stiffened panels to a greater extent than do the other experiments considered here. Thus these tests give the best indication,
of the behavior of a plate existing in a stiffened plate panel. The consistent agreement of the results with Koiter's theoretical curve gives then strong support to the use of Koiter's equation as a design formula.

3.7 STEIN

Included in Stein's report is a description of a single test conducted on a 2024-T3 aluminum-alloy plate\(^{(9)}\). The plate was subdivided by knife edges into eleven bays each having an aspect ratio of 5.4 and a b/t ratio of 65.4.

A continuous record of the load-shortening behavior was kept. Until yielding occurred the load carried by the plate was greater than that which would be predicted by any of the theoretical elastic solutions. The ultimate load was approximately \(2.5 \sigma_{cr}\). The agreement or disagreement of a single test with the theoretical solution is not really important.

The significant fact in this test was the occurrence of secondary buckling which took place three times after initial buckling. The original buckled shape of the plate was five half-waves. At an average stress of 1.75 \(\sigma_{cr}\) the plate buckled into six half-waves. It later changed to seven half-waves and still later buckled into an eight half-wave mode. Stein stated in the report that the changes occurred in a violent manner. At each point secondary buckling occurred, a drop in the load was recorded.

The fact that repeated secondary buckling was observed to occur
in one test does not, of course, prove that this behavior is characteristic of a plate in a stiffened plate panel. It is important, however, that the test has shown that repeated secondary buckling can occur. Since it occurred in the test on a plate, it is possible that it might occur in a stiffened panel also.
4. CONCLUSIONS

The theoretical work presented in Chapter 2 was primarily concerned with the work of Marguerre, Levy, Stein, and Koiter. The discussion contained there led to a classification of the solutions into two groups: (1) Marguerre's and Levy's solutions in which only the first equilibrium mode is investigated and thus secondary buckling is not considered and, (2) Stein's and Koiter's solutions in which secondary buckling is considered.

No conclusive evidence has been presented, either theoretical or experimental, to prove that repeated secondary buckling is characteristic of the behavior of long plates. However, as a conservative solution is needed this in fact not the proof required; rather, it must be proved that repeated secondary buckling does not occur in plates existing in a stiffened panel. Since this has not been accomplished only the solutions of Stein and Koiter will be considered for the choice of a design formula. The choice between these two solutions can, however, not be based on theoretical considerations as explained in Section 2.6.

The experimental results of Botman and Besseling, considered by the writer to be the most reliable tests covered in this report, agree extremely well with Koiter's equation. The b/t ratio for these tests ranged from 31 to 122. It was also found that Koiter's equation conservatively predicts the ultimate load of the tests of Ojalvo and Hull for plates having a low b/t ratio.
It is thus the opinion of this writer that Koiter's equation can be used with confidence for long plates having a b/t ratio less than 120. From the tests of Ojalvo and Hull, it appears that for higher b/t ratios, initial deviations from flatness are beginning to have an appreciable effect on the load-shortening behavior.

As the primary interest in this report is in plates having a b/t ratio less than 100, it is proposed that Koiter's equation be used for the post-buckling portion of the effective stress-strain curve of the plate.
5. **APPLICATION**

As indicated in the introduction, the load-shortening behavior will be used as an effective stress-strain curve for the plate. The effective stress-strain curve will then consist of three parts: (1) before the buckling stress, (2) between the buckling stress and the ultimate stress, and, (3) after the ultimate stress.

The first part of the curve consists of a straight line of unit slope. The second part between the buckling stress and the ultimate stress is defined by Koiter's equation:

\[
\frac{\varepsilon}{\sigma} = 1.2 \left( \frac{\varepsilon}{\varepsilon_{cr}} \right)^{0.6} - 0.65 \left( \frac{\varepsilon}{\varepsilon_{cr}} \right)^{0.2} + 0.45 \left( \frac{\varepsilon}{\varepsilon_{cr}} \right)^{-0.2}
\]

(37)

It will be assumed that the plate will continue to carry the ultimate load once it has been attained. Thus the third part of the curve will consist of a straight line of zero slope. Unfortunately, no experimental justification for this assumption exists. However, the ultimate load of the stiffened panel will occur before or soon after the ultimate strength of the plate. The assumption is thus not critical.

No attempt has as yet been made here to define the ultimate load of the plate. The actual ultimate capacity of the plate would be extremely difficult to compute. Because of this, the ultimate average stress will be defined as the stress at which first yielding takes place due to membrane forces. A further simplification will be made in that the strain at which first yielding occurs can be computed by \( \frac{\sigma}{E} \) as if a uniaxial state of stress existed.
For a plate without residual stresses, the maximum membrane stress would be at the longitudinal edge and thus the maximum load would be developed when $\varepsilon_e = \varepsilon_0$. The solid lines in Fig. 15 are stress-strain curves for plates of A-36 steel having b/t ratios of 60 and 100 and having no residual stresses.

A typical residual stress pattern for a plate with stiffeners attached by welding is shown in Fig. 12. In general the compressive residual stress ($\sigma_{rc}$) is quite small as compared to the tensile residual stress ($\sigma_{rt}$) and will be distributed across nearly the entire width of the plate. Because of this it will be assumed that the plate will buckle when $\sigma + \sigma_{rc} = \sigma_{cr}$. The stress distribution after buckling is shown in Fig. 13. The maximum stress no longer occurs at the edge but rather at the edge of the compressive residual stress zone, point A.

The effect of residual stresses on the effective plate stress-strain curve is illustrated in Fig. 14. The solid line represents the stress-strain curve for a plate without residual stresses. As shown by the dotted line the stress at which the plate buckles is reduced by $\sigma_{rc}$. If the maximum stress was assumed to occur at the longitudinal edge, dotted line (d) would represent the ultimate load of the plate. However, as the maximum stress is developed at the edge of the compressive residual stress zone (point A), dotted line (e) actually represents the ultimate load. This load has a higher value than in the previous case.
A numerical example of the effect of residual stresses is shown in Fig. 15 by the dotted line. The plate has a b/t ratio of 60 and assumed residual stresses of $\sigma_{rt} = \sigma_0$ and $\sigma_{rc} = 0.1 \sigma_0$. The stress distribution after buckling was assumed to be parabolic. As can be seen from the figure, the ultimate load has been reduced by about 10 percent.
6. RECOMMENDATIONS FOR FUTURE RESEARCH

6.1 THEORETICAL WORK

The most important area yet to be investigated is that of stability in the post-buckled state. Until stability criteria have been determined, no theoretical basis exists for determining which solution, that which considers secondary buckling or that which does not, correctly describes the post-buckling behavior of long plates. Unfortunately, an investigation of the stability of the equilibrium will require a very precise determination of the equilibrium configurations. As indicated in Chapter 2 this is not easy to do.

An area which needs further investigation is the effect of initial deviations from flatness. Of specific interest is the effect of the various possible initial out-of-flatness modes on the behavior of long plates. In order that the investigation apply to actual plates, larger deviations than were considered by Hu, Lundquist, and Batdorf should be investigated.

Finite difference techniques could be used to make an approximate investigation of both the problems mentioned above.

6.2 EXPERIMENTAL WORK

Very few additional questions can be answered by more experiments unless such experiments are planned and executed in a very exacting manner.

The following items should be considered of prime importance
in any future tests:

(1) A wide range of b/t ratios should be included. Of special interest is the effect of initial deviations on the behavior of plates having a high b/t ratio.

(2) The initial out-of-flatness should be measured.

(3) A complete record of the load-shortening behavior should be kept up to and well beyond the ultimate load.

(4) Out-of-plane deflections should be measured for the entire loading range.

(5) Special attention should be given to the boundary conditions at the longitudinal edges. A multi-bay test arrangement appears to be the only acceptable approach.
7. N O M E N C L A T U R E

a  plate length
b  plate width
de  effective width
D  plate rigidity; \( \frac{Et^3}{12(1-\nu^2)} \)
E  Young's modulus
L  half-wave length in longitudinal direction
N_x  normal membrane force per unit width in the x direction
N_y  normal membrane force per unit length in y direction
N_{xy}  membrane shearing force per unit width
t  plate thickness
u  displacement in the x direction
v  displacement in the y direction
w  out-of-plane displacement
x,y  cartesian coordinate axes
\gamma_{xy}  membrane shearing strain
\varepsilon  arbitrary parameter
\varepsilon_x  membrane strain in x direction
\varepsilon_y  membrane strain in y direction
\varepsilon_e  axial strain at the longitudinal edge
\varepsilon_{cr}  buckling strain of a simply supported plate; \( \frac{\pi^2(t/b)^2}{3(1-\nu^2)} \)
\varepsilon_y  yield strain
\nu  Poisson's ratio
\sigma_x  membrane stress in x direction
\sigma_y  membrane stress in y direction
\[ \sigma \quad \text{average applied edge stress} \]
\[ \sigma_{\text{max}} \quad \text{maximum average applied edge stress} \]
\[ \sigma_e \quad \text{axial stress at the longitudinal edge} \]
\[ \sigma_{cr} \quad \text{buckling stress of a simply supported plate}; E \varepsilon_{cr} \]
\[ \sigma_{rc} \quad \text{compressive residual stress} \]
\[ \sigma_{rt} \quad \text{tensile residual stress} \]
\[ \sigma_0 \quad \text{yield stress} \]
\[ \tau_{xy} \quad \text{membrane shearing stress} \]
\[ \phi \quad \text{Airy's stress function} \]
\[ \nabla = \frac{\partial^4}{\partial x^4} + \frac{2}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \]
8. TABLES AND FIGURES
<table>
<thead>
<tr>
<th>Investigators and Back (Ref. 2)</th>
<th>Material</th>
<th>b/t</th>
<th>a/b</th>
<th>No. of Bays</th>
<th>Support at Longitudinal Edge</th>
<th>Support at Transverse Edge</th>
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<td>David Taylor Model Basin (Ref. 19,20)</td>
<td>Steel Aluminum</td>
<td>32-144</td>
<td>2,3</td>
<td>1</td>
<td>Circular Groove</td>
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<td>71-232</td>
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<td>Botman and Besseling (Ref. 21,22,23,24)</td>
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<td>51-124</td>
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<td>1-5</td>
<td>Knite Edge</td>
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Fig. 1 Rectangular Plate Subjected to Edge Compression in One Direction

Fig. 2 Koiter's Transverse Profile for Simply Supported Plate
Fig. 3 Test Jig Used By Schuman and Back (2)

Fig. 4 Test Jig Used By Ojelvo and Hull (13,14)
Fig. 5 Multi-Bay Panel Used By Botman and Besseling (21, 22, 23, 24)
Fig. 6 Rigid Bar Column Analyzed By Stein

Fig. 7 Load-Shortening Behavior of Rigid Bar Column
Fig. 8 Theoretical Solutions
Fig. 9 Ultimate Strength Tests \((2,18,19,20)\)
Fig. 10 Test Results of Ojalvo and Hull
Fig. 11 Test Results of Botman and Besseling

\[ \frac{\overline{\epsilon}}{\epsilon_{CR}} \]

\[ \frac{\epsilon_e}{\epsilon_{CR}} \]

Marguerre
Koiter
Stein

--- b/t = 122
--- b/t = 119
--- b/t = 99

(21, 22, 23, 24)
Fig. 12 Typical Residual Stress Pattern

Fig. 13 Stress Distribution After Buckling

Fig. 14 Stress-Strain Curve Including Effect of Residual Stresses
2.4

Without Residual Stresses

With Residual Stresses

Fig. 15: Effective Stress-Strain Curve of the Plate
In addition to the references listed in Chapter 10, many other publications pertaining to the subject matter have been studied. To assist the reader in further study, the more significant publications are listed below.

1. Yamanoto, M., and Kondo, K.
   BUCKLING AND FAILURE OF THIN RECTANGULAR PLATES IN COMPRESSION, Tokyo Imperial Univ., Aeronautical Research Institute, 10, No. 1, Report No. 119 (1935).

2. Kromm, A., and Marguerre, K.
   BEHAVIOR OF A PLATE STRIP UNDER SHEAR AND COMRESSIVE STRESSES BEYOND THE BUCKLING LIMIT, NACA TM 870, (1938) (Translation)

3. Ramberg, W., McPherson, A. E., and Levy, S.
   EXPERIMENTAL STUDY OF DEFORMATION AND OF EFFECTIVE WIDTH IN AXIALLY LOADED SHEET-STRINGER PANELS, NACA TN 684 (1939)

4. Levy, S., Goldenburg, D., and Zibritosky, G.
   SIMPLY SUPPORTED LONG RECTANGULAR PLATE UNDER COMBINED AXIAL LOAD AND NORMAL PRESSURE, NACA TN 949, (Oct. 1944)

5. Cox, H. L.
   THE BUCKLING OF A FLAT RECTANGULAR PLATE UNDER AXIAL COMPRESSION AND ITS BEHAVIOR AFTER BUCKLING, Aeronautical Research Committee (Great Britain), III, No. 2041, (1945)

6. Cox, H. L.
   THE BUCKLING OF A FLAT RECTANGULAR PLATE UNDER AXIAL COMPRESSION AND ITS BEHAVIOR AFTER BUCKLING, II-CONDITIONS FOR PERMANENT BUCKLES, Aeronautical Research Committee (Great Britain) III, No. 2175, (1945)

7. Hemp, W. S.
   THE THEORY OF FLAT PANELS BUCKLED IN COMPRESSION, Aeronautical Research Committee (Great Britain), III, No. 2178, (1945)

   THE DEVELOPMENT OF A TECHNIQUE FOR TESTING STIFF PANELS IN EDGWISE COMPRESSION, Proceedings of the Society for Experimental Stress Analysis, No. II (1948)
9. Bijlaard, P. P.
   Determination of the Effective Width of Plate with Small Deviations from Flatness by the Method of Split Rigidities, Proc. 1st United States Congress of Applied Mechanics, p. 357-362, (1951)

10. Coan, J. M.
   Large-Deflection Theory for Plates with Small Initial Curvature Loaded in Edge Compression, Journal of Applied Mechanics, 18, No. 2, (1951)

11. Argyris, J. H., and Dunne, P. C.

12. Bleich, F.
   Buckling Strength of Metal Structures, p. 459, McGraw-Hill Book Company, New Yor, 1952

13. Mayers, J., and Budiansky, B.
   Analysis and Behavior of Simply Supported Flat Plates Compressed Beyond the Buckling Load into the Plastic Range, NACA TN 3368, (1955)

14. Alecseev, S. A.

15. Gerard, G.
   Handbook of Structural Stability, Part IV - Failure of Plates and Composite Elements, NACA TN 3784 (1957)

16. Masur, E. F.

17. Timoshenko, S. P., and Gere, J. M.

18. Yamaki, N.
   Experiments on the Post-Buckling Behavior of Square Plates Loaded in Edge Compression, Trans. ASME, Series E, 83 (June, 1961)

   Post-Buckling Behavior of Flat Plates, Trans. ASCE, 127 (II) p. 227 (1962)
10. REFERENCES

1. Bryan, G. H.
ON THE STABILITY OF A PLANE PLATE UNDER THRUST IN ITS OWN PLANE, Proc. London Mathematical Society, 22, p. 54, (1890)

2. Schuman, L., and Back, G.
STRENGTH OF RECTANGULAR PLATES UNDER EDGE COMPRESSION, 16th Annual Report NACA, No. 356 (1930)

3. Schnadel, G.
EXCEEDING THE BUCKLING LIMIT IN THIN PLATES, David Taylor Model Basin, No. 64, (1949), (Translation).

4. von Karman, T., Sechler, E. E., and Donnel, L. H.
The Strength of Thin Plates in Compression, Trans. ASME, 54, (2)(January, 1932)

5. Marguerre, K.
The Apparent Width of Plate in Compression, NACA TM 833, (1937)(Translation)

Experimental Studies of the Effective Width of Buckled Sheets, NACA TM 814 (1936)(Translation)

7. van der Neut, A.
POST-BUCKLING BEHAVIOR OF STRUCTURES, N.A.T.O. Advisory Group for Aeronautical Research and Development, No. 60 (1956)

8. Levy, S.
BENDING OF RECTANGULAR PLATES WITH LARGE DEFORMATIONS, 28th Annual Report NACA, No. 737, (1942)

9. Stein, M.
LOADS AND DEFORMATIONS OF BUCKLED RECTANGULAR PLATES, NASA TR R-40 (1959)

10. Stein, M.
The Phenomenon of Change in Buckle Pattern in Elastic Structures, NASA TR R-39 (1959)

11. Koiter, W. T.
The Effective Width of Flat Plates for Various Longitudinal Edge Conditions at Loads Far Beyond Buckling Load, National Luchtvaartlaboratorium (Netherlands), Rep. S287 (1943), (In Dutch)
   EFFECT OF SMALL DEVIATIONS FROM FLATNESS ON EFFECTIVE WIDTH
   AND BUCKLING OF PLATES IN COMPRESSION, NACA TN 1124 (1946)

13. Hull, F. H.
   LOAD-SHORTENING CHARACTERISTICS OF THIN PLATES IN COMPRESSION,

14. Ojavo, M., and Hull, F. H.
   EFFECTIVE WIDTH OF THIN RECTANGULAR PLATES, Proc. ASCE, 84
   (EM-3) Paper 1718, (July 1958)

15. Cox, H. L.
   THE BUCKLING OF THIN PLATES IN COMPRESSION, Aeronautical Research
   Committee (Great Britain), II, No. 1554, (1933)

16. Bengston, H. W.
   SHIP PLATING UNDER COMPRESSION AND HYDROSTATIC PRESSURE, The
   Society of Naval Architects and Marine Engineers, Trans., 47,
   (1939)

17. Boley, S.
   A PROCEDURE FOR THE APPROXIMATE ANALYSIS OF BUCKLED PLATES,
   Journal of the Aeronautical Science, 22, p. 337 (May 1955)

18. Sechler, E. E.
   THE ULTIMATE STRENGTH OF THIN FLAT PLATES IN COMPRESSION,
   Guggenheim Aeronautics Laboratory, California Institute of
   Technology, Pub. No. 27 (1933)

19. Duffy, D. J., and Allnutt, R. B.
   BUCKLING AND ULTIMATE STRENGTH OF PLATING LOADED IN EDGE
   COMPRESSION, PROGRESS REPORT 1-6061-T6 Aluminum Plates,
   David Taylor Model Basin, No. 1419, (1960)

   BUCKLING AND ULTIMATE STRENGTH OF PLATING LOADED IN EDGE
   COMPRESSION, PROGRESS REPORT 2 - Unstiffened Panels, David
   Taylor Model Basin, No. 1682, (1963)

   THE EXPERIMENTAL DETERMINATION OF THE EFFECTIVE WIDTH OF FLAT
   PLATES IN THE ELASTIC AND PLASTIC RANGE, Nationaal Luchtaart-
   laboratorium (Netherlands), Rep. S414 (1953), (In Dutch)

22. Botman, M.
   THE EXPERIMENTAL DETERMINATION OF THE EFFECTIVE WIDTH OF FLAT
   PLATES IN THE ELASTIC AND PLASTIC RANGE (Part II), Nationaal
   Luchtaartlaboratorium (Netherlands), Rep. S438, (1954) (In
   Dutch)
23. Botman, M., and Besseling, J. F.

24. Botman, M.
THE EFFECTIVE WIDTH IN THE PLASTIC RANGE OF FLAT PLATES UNDER COMPRESSION (Part III), Nationaal Luchtvaartlaboratorium (Netherlands), Rep. S465 (1955)
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