
A. Roesli
LATERAL LOAD DISTRIBUTION IN MULTI-BEAM BRIDGES

by

Alfred Roesli

Progress Report 10 - Prestressed Concrete Bridge Members

Part of an Investigation Sponsored by:

PENNSYLVANIA DEPARTMENT OF HIGHWAYS
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CONCRETE PRODUCTS DIVISION

July 1955
Lehigh University
Institute of Research
Bethlehem, Pennsylvania
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ACKNOWLEDGEMENTS

This report concerns a part of a research program on prestressed concrete, carried out at the Fritz Engineering Laboratory, Lehigh University, Bethlehem, Pennsylvania of which Professor W.J. Eney is Director.

This research program is guided and sponsored by the Lehigh Prestressed Concrete Committee, Mr. A.E. Cummings,* chairman, and is represented by the following organizations:

- Pennsylvania Department of Highways
- U.S. Bureau of Public Roads
- American Steel & Wire Div., U.S. Steel
- John A. Roebling's Sons Corporation
- Reinforced Concrete Research Council
- Concrete Products Company of America

The author is greatly indebted to Dr. C.E. Ekberg, Director of this research program and Professor in charge of the present report. His help and advice, as well as that given by the entire special committee for this doctoral work with Dr. F.W. Schutz, Jr. as chairman, is sincerely appreciated.

Deep appreciation is expressed for the help given by Dr. E. Bareiss who prepared the program for computation of the numerical values. Tabular values were compiled on a contract basis by Remington Rand, Inc., New York on their Univac Computer.

The many suggestions and the untiring help received from the entire staff of Fritz Engineering Laboratory and especially

* Deceased
from the author's friends, Dr. B. Thürlimann, who furnished much technical advice, and Messrs. A.N. Sherbourne and A. Smislova for help on the editing of this paper, are sincerely acknowledged. The cooperation of Mr. I. Scott, who prepared the figures, and Mrs. V. Olanovich, who typed the original manuscript, is gratefully appreciated.
ABSTRACT

A method is derived to analyze multi-beam bridges and especially to determine for what portion of the live load each beam must be designed.

Present design procedure being rather conservative in its scope, this analysis will represent a more realistic solution for the problem and may lead to a more economical design of such bridges.

The method is based on the theory of orthotropic plates and on the main assumption that the interaction of the beams provided by shear keys and lateral prestress exclude any slip between the beams.

Formulas for deflections, moments and forces are derived for bridges with various degrees of lateral prestress. A limiting case was found in the "articulated plate", a bridge with no lateral bending stiffness, but with a beam connection which transmits the full shear force.

Numerical values are presented for the most important loading conditions and for bridges of various sizes. For design purposes the coefficients of lateral load distribution may be used.

Among other results, it was found, that for two standard trucks placed side by side on a 27-ft. wide bridge,
the maximum load carried by a 3-ft. wide beam is 55% of a wheel load, as compared to 80% recommended by the specifications, and is almost independent of the amount of lateral prestress.
I. INTRODUCTION

1. General:

The design of modern structures tends to utilize construction materials to an optimum while an appropriate factor of safety is still maintained. This requires however an accurate investigation of the stresses in the structures. In many cases, the simplified analysis based upon the beam theory has to be replaced by a more exact one, considering the structure as a two or three dimensional one, such as a plate or a shell. It is hoped that the following study is a contribution to this development.

The investigation described herein deals with the lateral load distribution in multi-beam bridges. This type of bridge is constructed from precast beams made of reinforced or prestressed concrete. They are placed side by side on the abutments and joined together laterally by steel rods which may or may not be prestressed. In order to increase the interaction between the beams, continuous longitudinal shear keys are formed at the joints, e.g., by dry packed mortar in a recess formed at the sides of each precast unit. Fig. 1 shows an isometric view and Fig. 2 a cross-section of such a multi-beam bridge.
The problem to be investigated is the interaction of the beams and the determination of the portion of the load each beam must carry should the load be applied to one of the beams. Various degrees of lateral prestress will have to be considered.

In the present design procedure of these bridges a limited interaction of the beams is considered. It is assumed that a loaded beam carries 80% of the applied load and that the remaining portion is distributed among the adjacent beams. This assumption is based on an interpretation of design section 3.3.1b of the 1949 AASHO Specification. (1)*

Past experience has shown, that bridges designed accordingly, are stiffer than expected and that the individual beams are unnecessarily heavy. (2) The application of this bridge system is thus limited to short span lengths. In order to increase the span length and to design such bridges more economically, a more exact analysis is necessary.

Extensive investigations have been made on similar bridge systems. Of most interest for the present study

* Numbers refer to List of References.
are the ones made on bridges formed by two systems of parallel beams spaced at equal intervals in the longitudinal and lateral direction. The beams are rigidly connected at their points of intersection and support the bridge deck. This type of construction will hereafter be referred to as a gridwork and is shown in Fig. 4.

Several methods, some of which are only approximate, have been developed to analyze gridwork systems. Massonnet presents in Reference (3) a very helpful survey of these methods. It appears that the analysis of a gridwork as an orthotropic plate is very efficient. The gridwork with its discontinuous elastic properties is replaced by an equivalent plate, having the same average bending stiffnesses in the two directions as existing in the gridwork. Y. Guyon applied this method to investigate the lateral load distribution of beam and girder prestressed concrete bridges. (4) To simplify the analysis he neglected the torsional resistance of the beams. Based on this work Massonnet extended the investigation and included the torsional resistance of the beams as well as that of the bridge deck. (5)

Both of the above mentioned authors derived general methods to analyze these structures and prepared design tables. The
latter were set up for a live load of sinusoidal nature. They considered this type of loading as a sufficiently close approximation of any loads encountered in the design of bridges.

P.B. Morice and G. Little described some laboratory tests on models of gridwork systems. (6) They concluded from their experiments that the analysis of gridworks can well be based on the theory of the orthotropic plate.

A bridge formed by beams with the sides in continuous contact to each other could be considered as a limiting case of a gridwork and the methods mentioned above could be used for its analysis. However, it is shown in this investigation that the assumptions on which these methods are based do not generally hold for multi-beam bridges. A method is therefore derived in this work which will generally hold for the latter systems, and which is also based on the theory of orthotropic plates. The theory is first described and then modified for multi-beam bridges. The resulting differential equation is solved, considering the structure as a plate with two opposite edges simply supported and the other two edges free. Numerical calculations are presented for the most important loading conditions, for bridges with different spans and widths, and
also for various degrees of lateral prestress and different sizes of beams. Using the coefficients of the lateral load distribution the results are presented in a form suitable for the practical design of such bridges.

2. Introduction to the Theory of Orthotropic Plates:

An orthotropic plate is defined as a plate with different bending stiffnesses $D = EI$ in two orthogonal directions $x$ and $y$ in the plane of the plate. These may result either from different moduli of elasticity $E_x$ and $E_y$ of the material in the two directions, or from different moments of inertia $I_x$ and $I_y$ per unit width of the plate.

An example for the first kind of orthotropic plates is a timber plate. Assuming the $x$-axis parallel and the $y$-axis perpendicular to the grain, the modulus of elasticity in the $y$-direction $E_y$ is, according to E. Seydel, (7) only about $1/10$ of that in the $x$-direction.

A corrugated steel sheet is a typical example of an orthotropic material of the second kind. The modulus of elasticity is the same in every direction; the material itself is isotropic. Here the different bending stiffnesses are functions of the shape of the sheet. The average moment of inertia with respect to the neutral plane in the
direction perpendicular to the generator is much smaller than in the direction parallel to the generator. Fig. 3 shows this case.

The basic assumptions in the theory of the orthotropic plate are identical with those in the theory of the homogeneous plate, namely:

(1) The thickness of the plate is small compared with its other dimensions.

(2) The deflections w are small compared with the thickness of the plate.

(3) The transverse stresses $\sigma_z$ are small and their influence on the deformation can be neglected.

For a right hand coordinate system $(x, y, z)$ where $x$ and $y$ are in the plane of the plate and parallel to the two distinct directions of the orthotropic plate, the differential equation for the deflection $w$ parallel to the $z$-direction is given by:

$$(EI)_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + (EI)_y \frac{\partial^4 w}{\partial y^4} = p(x_1 y)$$  (1a)

$(EI)_x$ and $(EI)_y$ represent the two bending stiffnesses per unit width of the plate. $2H$ is a coefficient containing two parts. The first part is the twisting resistance of the plate and a second, smaller part is a function of the two bending stiffnesses modified by the Poisson's ratio. $p(x_1 y)$
denotes the intensity of the load at the point (xy) parallel to z.

Dividing equation (1a) by \((EI)_x\), the following differential equation results which is mainly used hereafter:

\[
\frac{\partial^4 w}{\partial x^4} + 2\beta \frac{\partial^4 w}{\partial x^2 \partial y^2} + \alpha \frac{\partial^4 w}{\partial y^4} = \frac{p(x_1 y)}{(EI)_x}
\]  

\[(1b)\]

\[
2\beta = \frac{2H}{(EI)_x} \quad (2)
\]

\[
\alpha = \frac{(EI)_y}{(EI)_x} \quad (3)
\]

\(2\beta\) shall be called the coefficient of torsional rigidity and \(\alpha\) the ratio of the bending stiffnesses.

The derivation of equation (1) is attributed to J. Boussinesq (1874) and is given, for example, in Reference 8, page 188. M.T. Huber (9, 10) applied it for various plate structures of technical importance: Concrete slabs reinforced in two directions for cracked cross-sections, beam and slab construction, gridwork systems; corrugated steel sheets and plywood plates.

For the discussion of a multi-beam bridge a coordinate system \((x, y, z)\) is assumed as shown in Fig. 1 with \(x\) and \(y\)-axis respectively parallel and perpendicular to the beam in the middle plane of the bridge. The \(z\)-axis is positively directed downward and the origin is located at midspan of
one support. This type of bridge can be treated as an orthotropic plate of the second kind $E_x = E_y$ if, in addition to the basic assumptions of the general theory, the following assumptions are made:

1. The connections between the beams are such, that the points of contact in two adjacent beams deform equally.

2. The number of beams is large enough, such that the real structure can be replaced by an idealized one with continuous properties, in order that the differential calculus can be applied.

The first assumption implies that the connections prevent the beams from slipping against one another, and that they transmit the full shear force. This can be accomplished either by lateral prestress and/or by shear keys as shown in Fig. 2. With lateral prestress only the shear forces are transmitted by the friction produced by the prestress. If $\mu$ is the coefficient of static friction between two concrete surfaces and $\sigma_p \, dA$ is the lateral prestress applied on the differential area $dA$ a maximum shear force of $\int_A \mu \sigma_p \, dA$ can be transmitted by the friction forces without a slip.
The second assumption requires that the width of the beams is small compared with the width of the bridge such that the actual lateral bending stiffness $EI_y$ and the coefficient $2H$, which are both discontinuous, can be replaced by average values. This will be discussed in the following for $EI_y$ or $\alpha$ and in a later paragraph for $2H$ or $2\beta$.

If, for a sufficiently high lateral prestress and for any given load, the entire beam sides remain in compression; the bridge will behave as a homogenous slab. The bending stiffness $EI_y$ in the lateral direction will be constant and will be the same in the direction of the beams; or $\alpha = 1$.

Assuming a smaller prestress, the bending moments $M_y$ in the loaded slab may produce tensile stresses in the joints between the beams. Since the joints do not have any or only minor tensile strength (from the shear keys) these tensile stresses will have to be carried by the prestressing elements. The joints tend therefore to open up, thus reducing the bending stiffness in this cross-section; or $\alpha \ll 1$.

A limiting case with no lateral bending stiffness in the joints may be reached if no lateral prestress is applied. In this case the joints are unable to transmit bending moments, but according to assumption 1, the shear keys transmit the full shear forces. Such a structure can be thought
of being formed of beams with the adjacent sides connected by continuous hinges along the beams. Hereafter this type of structure will be called an articulated plate.

Fig. 2 shows a schematic cross-section of the bridge with the joints having a smaller lateral bending stiffness than the beams.

The simplest way of obtaining average values for the lateral bending stiffness would be by experiment on a model bridge. This could be accomplished by supporting the bridge along the two edge beams instead of the supports at the ends of the beams. Subjecting the plate to a load which does not vary along the length of the plate, produces a deformed surface of cylindrical shape which can easily be measured. Comparing these deflections with the theoretical ones of an isotropic plate with the same boundary and loading conditions, an average values for \( EI_y \) can be obtained. This test can be done for various magnitudes and locations of the lateral prestressing elements. Dividing \( EI_y \) by the flexural rigidity \( EI_x \) per unit width of the beams gives the coefficient \( \alpha \). This may vary as shown previously between the two limiting values 0 and 1. For design purposes it should be accurate enough to assume
first an appropriate value of $\alpha$ and later to check whether the lateral bending moment can safely be transmitted by the reduced section of the joints.

In the application of the theory of the orthotropic plate the main difficulty is not so much the determination of the coefficient $\alpha$ but the determination of the coefficient of the torsional rigidity $2\beta$.

Various simplified assumptions can be found in the literature for different structures. Most of these can be traced back to Huber (9, 10, 11).

For two-way reinforced concrete slabs, Huber recommended the following approximation:

$$\beta = \sqrt{\alpha}$$

Massonnet (3) used a similar expression in his investigation of gridwork systems viz:

$$\beta = \hat{\beta} \sqrt{\alpha}$$

where $\hat{\beta}$ is a parameter of torsion varying between 0 and 1. This approximation allowed him to consider any proportion of torsional resistance of the beams and the bridge deck.

The most commonly applied expression for a gridwork system as shown in Fig. 4 is:
\[ \beta = \frac{1}{2D_x} \left( \frac{C_x}{C_x} + \frac{C_y}{C_y} \right) \]  \hspace{1cm} (6)

where \( C_x \) and \( C_y \) are the torsional rigidities of the individual stringers and beams, respectively, spaced at distances \( c_x \) and \( c_y \) apart. \( D_x \) is the average bending stiffness per unit width of the gridwork in the \( x \)-direction.

The following relation is in general used to determine the torsional rigidity of beams of rectangular cross-section:

\[ C = KGh^3a \]  \hspace{1cm} (7a)

where \( G = \frac{E}{2(1+\nu)} \) is the shearing modulus, \( K \) a cross-section factor depending only on the ratio \( a/h \). This factor is tabulated in several engineering handbooks and textbooks, for example, Reference 12, page 277. In general for concrete structures the effect of Poisson's ratio \( \nu \) is neglected since its influence is small. (13)

Using \( I \), the moment of inertia for rectangular sections, the above formula can then be expressed as:

\[ C = 6KEI \]  \hspace{1cm} (7b)

Substituting this expression in equation (6) and using the average bending stiffnesses per unit width of the gridwork:

\[ D_x = \frac{EI_x}{C_x} \quad \quad D_y = \frac{EI_y}{C_y} \]
one obtains:

$$\beta = 3 \left( K_x + \alpha K_y \right) \quad (8)$$

A multi-beam bridge may be considered as a limiting case of a gridwork system where the beams would correspond to stringers and the transverse members, normally represented by beams in a gridwork system, would be replaced by a continuous plate with average bending stiffness $D_y$. In equation (8) $K_x$ would then assume a value corresponding to a particular $a/h$ ratio of the beams and $K_y$ the value corresponding to a plate. With a later value, $K_y = 0.333$, the above equation would then take the form:

$$\beta = 3 \left( K_x + 0.333 \alpha \right) \quad (9)$$

The two functions (4) and (9) of $\beta$ are plotted in Fig. 6 for $\alpha$ varying between 0 and 1. Equation (4) represents a parabola through the origin and the point $\alpha = 1$, $\beta = 1$, the axis coinciding with the $\alpha$-axis. Equation (9) represents a series of parallel lines, intersecting $\alpha = 0$, such that the intercepts on the $\beta$-axis are three times the $K_x$ values.

Considering now the limiting cases of the multi-beam bridge, it is apparent that for $\alpha = 1$ the differential
equation must be identical with that of an isotropic plate, which is:

\[
\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p(x,y)}{EI}
\]

(10)

This implies that \( \beta \) as well as \( \alpha \) must be 1. For the case of the articulated plate, or \( \alpha = 0 \), the structure maintains the torsional rigidity of the beams, which means that \( \beta \) must be equal to \( 3K_x \).

Each of the two approximations for \( \beta \) given by (4) and (9) satisfies one of the above conditions. Equation (4) yields the correct value for \( \alpha = 1 \) but gives zero for \( \alpha = 0 \). From equation (9), for \( \alpha = 1 \) one obtains a value larger than 1, and the correct value for \( \alpha = 0 \). The true function for \( \alpha \) lies therefore somewhere between these two approximations.

In the following chapter an attempt is made to establish a relation between \( \alpha \) and \( \beta \).
II. DIFFERENTIAL EQUATION FOR THE MULTI-BEAM BRIDGE

1. Derivation of the Differential Equation

The second assumption made previously for the analysis of a multi-beam bridge as an orthotropic plate implies that the number of beams making up the bridge is of such magnitude, that the real structure can be replaced by one of a continuous nature which exhibits equivalent average elastic properties. Hence a differential element, cut out of the latter structure by two pairs of planes parallel to the \( xz \) and \( yx \) planes, will have bending stiffnesses \( E_{Ix} \) and \( E_{Iy} \) per unit width in the directions \( x \) and \( y \) respectively.

Fig. 5 shows the middle plane of such an element with the forces and moments acting per unit width of the element. The moments are designated by arrows with double heads and their directions are defined by the right hand rule. \( p(xy) \) represents the intensity of the applied load.

Three independent equations can be written, expressing the equilibrium of the forces and moments. These are:

\[
Q_x = \frac{\partial M_y}{\partial x} + \frac{\partial M_{yx}}{\partial y} \quad \text{(11a)}
\]

\[
Q_y = \frac{\partial M_x}{\partial y} + \frac{\partial M_{xy}}{\partial x} \quad \text{(11b)}
\]

\[
0 = p(x_1y) + \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} \quad \text{(11c)}
\]
By substituting in equation (11c) the expressions for $Q_x$ and $Q_y$ as given in equations (11a) and (11b) the equilibrium equation for the entire element results:

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2}{\partial x \partial y} \left( M_{xy} + M_{yx} \right) + \frac{\partial^2 M_y}{\partial y^2} = -p(x_1 y) \quad (12)$$

To represent this equation in terms of the deflection of the plate $w$, the relation between the moments and the deformations are used. In the following the effect of Poisson's ratio is neglected. This agrees with general practice for the investigation of reinforced concrete slabs.

This simplification allows the use of the relation between moments and deformations from the elementary beam theory, giving:

$$M_x = -EI_x \frac{\partial^2 w}{\partial x^2} \quad (13a)$$

$$M_y = -EI_y \frac{\partial^2 w}{\partial y^2} = -\alpha EI_x \frac{\partial^2 w}{\partial y^2} \quad (13b)$$

Similar expressions can be written for the twisting moments:

$$M_{xy} = -\beta_x EI_x \frac{\partial^2 w}{\partial x \partial y} \quad (14a)$$

$$M_{yx} = -\beta_y EI_y \frac{\partial^2 w}{\partial x \partial y} = -\alpha \beta_y EI_x \frac{\partial^2 w}{\partial x \partial y} \quad (14b)$$
where $\beta_x$ and $\beta_y$ are two constants to be determined. By substituting these expressions in equation (12) the differential equation for the deflection of the multi-beam bridge is obtained:

$$\frac{\partial^4 w}{\partial x^4} + (\beta_x + \alpha \beta_y) \frac{\partial^4 w}{\partial x^2 \partial y^2} + \alpha \frac{\partial^4 w}{\partial y^4} = \frac{p(x, y)}{EI_x}$$

(15)

with $\beta_x + \alpha \beta_y = 2\beta$ equation (1b) results.

2. Experimental Determination of the Relation Between $\alpha$ and $\beta$

In order to establish an experimental relation between $\alpha$ and $\beta$ the homogenous differential equation is considered:

$$\frac{\partial^4 w}{\partial x^4} + (\beta_x + \alpha \beta_y) \frac{\partial^4 w}{\partial x^2 \partial y^2} + \alpha \frac{\partial^4 w}{\partial y^4} = 0$$

(16)

A particular integral satisfying this equation is (Reference 15, page 164):

$$w = C \cdot x \cdot y$$

(17)

with $C$ an arbitrary constant.

This solution yields from equations (13) and (14):

$$M_x = -\alpha EI_x \frac{\partial^2 w}{\partial x^2} = 0$$

(18a)

$$M_y = -\alpha EI_x \frac{\partial^2 w}{\partial y^2} = 0$$

(18b)
and

\[ M_{xy} = -\beta_x EI_x \frac{2w}{\partial x \partial y} = -\beta_x EI_x \cdot C \quad (19a) \]

\[ M_{yx} = -\alpha \beta_y EI_x \frac{2w}{\partial x \partial y} = -\alpha \beta_y EI_x \cdot C \quad (19b) \]

Adding equations (19a) and (19b) the constant C can be determined as:

\[ C = -\frac{1}{EI_x} \frac{M_{xy} + M_{yx}}{\beta_x + \alpha \beta_y} \quad (20) \]

and the deflection \( w \) results as

\[ w = -\frac{1}{EI_x} \frac{M_{xy} + M_{yx}}{\beta_x + \alpha \beta_y} xy \quad (21a) \]

or

\[ w = -\frac{1}{EI_x} \frac{M_{xy} + M_{yx}}{2\beta} xy \quad (21b) \]

From equations (18) and (19) it is evident that only constant twisting moments are acting on the bridge, deforming the middle plane to the anticlastic surface given by equation (21). This loading and deformation condition is shown in Fig. 7.

In each element \( dy \) of the edges parallel to the \( y \)-axis, the twisting moment \( M_{xy} \) acting on it, can be considered as being formed by two vertical and opposite shearing forces of magnitude \( M_{xy} \) at a distance \( dy \) apart, (see Reference 8, page 47). It is apparent that these
shearing forces balance the ones in the adjacent elements and only the forces in the corners of the plate remain. These are, $\pm M_{xy}$ from the edges parallel to the $y$-axis and, with the same reason $\pm M_{yx}$ from the edges parallel to the $y$-axis, giving equal resulting corner forces of $A = \pm (M_{xy} + M_{yx})$. Thus the deformation as given in equation (21) can be produced by the loading condition shown in Fig. 8.

If such corner loads are applied to a small bridge in the laboratory, an experimental relation between $\alpha$ and $\beta$ can be found from the observed deflections and equation (21b). Also the effect of lateral prestressing on the value of $\beta$ may be established.

3. Assumption for the Relation Between $\alpha$ and $\beta$:

Since these test data are not yet available an assumption for $\beta$ as a function of $\alpha$ must be made. This assumed relation must be chosen so as to satisfy at least the conditions for the limiting cases.

(a) For $\alpha = 0$ the twisting moments are carried only by the beams along their longitudinal axes. With a torsional rigidity of the beams as given in equation (7b), the twisting moments which results from the deformation for this case are:
\[ M_{xy} = -6KEI_x \frac{\partial^2 w}{\partial x \partial y} \]
\[ M_{yx} = 0 \]

(b) For \( \alpha = 1 \) the bridge can be considered as an isotropic plate with \( M_{xy} = M_{yx} \) and \( 2\beta = 2 \) or for \( \alpha = 1 \): \( \beta_x = \beta_y = 1 \).

Assuming that \( \beta_x = 6K(1 - \alpha^{3/2}) + \alpha \) and \( \beta_y = 1 \) and substituting these expressions in \( 2\beta = \beta_x + \alpha \beta_y \) the following approximate relation between \( \alpha \) and \( \beta \) is obtained:

\[ \beta = 3K(1 - \alpha^{3/2}) + \alpha \tag{22} \]

This expression yields exact values for \( \alpha = 0 \) and \( \alpha = 1 \) and reasonably correct values in the intermediate range. It represents a family of curves with \( K \), the constant of torsional rigidity for rectangular beams, as parameter.

In Fig. 6 some of the curves for various \( a/h \) ratios are plotted.

With this assumption and with \( EI_x = EI \) the moments and forces result from equations (11), (13), (14), as follows:

Bending Moments:

\[ M_x = - \frac{2^2 w}{3x^2} EI \tag{23a} \]
\[ M_y = - \frac{2^2 w}{\partial y^2} EI \alpha \tag{23b} \]
Twisting Moments:

\[ M_{xy} = -\frac{2w}{\partial x \partial y} EI (2\beta - \alpha) \]  
\[ M_{yx} = -\frac{2w}{\partial x \partial y} EI \alpha \]  

Shear Forces:

\[ Q_x = -\left[ \frac{\partial^3 w}{\partial x^3} + \alpha \frac{\partial^3 w}{\partial x \partial y^2} \right] EI \]  
\[ Q_y = -\left[ \frac{\partial^3 w}{\partial y^3} + \alpha \frac{\partial^3 w}{\partial x^2 \partial y} (2\beta - \alpha) \right] EI \]

Boundary Shear Forces: (Ref. 15, page 154)

\[ R_x = Q_x + \frac{\partial M_{xy}}{\partial y} = -\left[ \frac{\partial^3 w}{\partial x^3} + 2\beta \frac{\partial^3 w}{\partial x \partial y^2} \right] EI \]  
\[ R_y = Q_y + \frac{\partial M_{yx}}{\partial x} = -\left[ \frac{\partial^3 w}{\partial y^3} + 2\beta \frac{\partial^3 w}{\partial x^2 \partial y} \right] EI \]

4. Limiting Cases of the Differential Equation:

As mentioned before for \( \alpha = 1 \) the differential equation of the orthotropic plate becomes the differential equation of the isotropic plate:

\[ \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p(x_1 y)}{EI} \]  

For the case when \( \alpha \) is zero the differential equation reduces to the one for the articulated plate:

\[ \frac{\partial^4 w}{\partial x^4} + 2 \beta_0 \frac{\partial^4 w}{\partial x^2 \partial y^2} = \frac{p(x_1 y)}{EI} \]
where $\beta_0 = 3 \, K$.

5. Boundary Conditions

In order to solve the differential equation for a given bridge deck, the boundary conditions (hereafter called B.C.) for simple supports and free edges have to be formulated.

(a) Simply Supported Edges

For a simply supported edge parallel to the y-axis the deflection $w$ and the bending moment $M_x$ is zero. The following conditions are resulting:

$$w = 0 \quad (28a)$$

and

$$\frac{\partial^2 w}{\partial x^2} = 0 \quad (28b)$$

(b) Free Edge

Along a free edge parallel to the x-axis the bending moment $M_y$ and the boundary shear

$$R_y = Q_y + \frac{\partial^2 M_{yx}}{\partial x}$$

must be zero. These conditions are fulfilled if

$$\frac{\partial^2 w}{\partial y^2} = 0 \quad (29a)$$

and

$$\frac{\partial^3 w}{\partial y^3} + \beta \frac{\partial^3 w}{\partial x^2 \partial y} = 0 \quad (29b)$$
III. INTEGRATION OF THE DIFFERENTIAL EQUATION

1. General Considerations

In the general theory of linear differential equations, the solution of the given non-homogenous equation (1b) can be obtained by adding to the general solution of the homogenous equation a particular solution of the non-homogenous equation, or (14):

\[ w = w_0 + w_1 \]  

(30)

\( w_1 \) is a particular solution of the complete differential equation taking into consideration the effect of the loading. In general this solution does not satisfy all the B.C. Hence \( w_0 \) has to be superimposed on \( w_1 \) to give the exact solution.

\( w_0 \) is the solution of the homogenous equation:

\[ \frac{\partial^4 w}{\partial x^4} + 2\beta \frac{\partial^4 w}{\partial x^2 \partial y^2} + \alpha \frac{\partial^4 w}{\partial y^4} = 0 \]  

(31)

which represents the differential equation of the unloaded plate with boundary forces and boundary moments acting on it.

For \( w_1 \) the solution of the infinitely long plate strip is used which satisfies the B.C. along the supports. This solution will be derived in the following for the general
case 0 < α < 1. The limiting cases occur with α = 1, the isotropic plate, and with α = 0, the articulate plate.

2. Plate Strip of Infinite Width Under Line Loading:

(a) General Case:

Fig. 9 shows a plate strip of infinite width simply supported along the edges parallel to the y-direction. A line load, applied along the x-axis, and representing the live load, is given by the following sine series:

\[ p(x) = \frac{2P}{\pi c} \sum_{m=1}^{\infty} \frac{1}{m} \sin \frac{\pi u}{l} \sin \frac{\pi c}{l} \sin \frac{\pi x}{l} \quad (32a) \]

or with

\[ \frac{\pi u}{l} = \alpha, \quad \frac{\pi c}{l} = \beta, \quad \frac{\pi x}{l} = \xi \]

\[ p(x) = \frac{2P}{\gamma l} \sum_{m=1}^{\infty} \frac{1}{m} \sin m\alpha \sin m\beta \sin m\xi \] \quad (32b)

where P is the total load uniformly distributed over the length 2c.

Since the deflection is symmetrical with respect to the x-axis, only the portion with positive y will be considered. This is done by cutting the plate along the x-axis. The resulting parts can now be analyzed as unloaded plates on which boundary forces and moments are acting of
such magnitude as to restore the continuity. The deflection $w$ has therefore to satisfy the homogenous differential equation (31) and the B.C.

The Levy solution (Ref. 8, page 125) may be assumed for this equation and is written as follows:

$$w = \sum_{m=1}^{\infty} Y_m \cdot \sin \frac{m \pi x}{l} = \sum_{m=1}^{\infty} w_m$$

(33)

where $Y_m$ is a function of $y$ only. As each term of the series must satisfy the differential equation and all the boundary conditions, it is sufficient to consider only one term.

The B.C. along $x = 0$, and $x = 1$, expressed in equation (28) are satisfied by $\sin \frac{m \pi x}{l}$. In order to determine $Y_m$ the deflection $w_m$ has to be inserted in the differential equation (31), yielding the following fourth order linear differential equation for $Y_m$:

$$\alpha y_m'' - 2\beta \frac{m^2 \pi^2}{l^2} y_m'' + \frac{m^4 \pi^4}{l^4} Y_m = 0$$

(34)

Taking $Y_m = e^{ry}$ the characteristic equation results as:

$$\alpha r^4 - 2\beta \frac{m^2 \pi^2}{l^2} r^2 + \frac{m^4 \pi^4}{l^4} = 0$$

(35)

with the roots:

$$r_{1,2,3,4} = \pm \frac{m \pi}{l} \sqrt{\frac{\beta}{\alpha} \left[ 1 \pm \sqrt{1 - \frac{\alpha}{\beta^2}} \right]}$$

(36)
On the basis of the assumed relation between \( \alpha \) and \( \beta \) these roots will always be real.

Taking:

\[
k_1 = \sqrt{\frac{\beta}{\alpha} \left[ 1 + \sqrt{1 - \frac{\alpha}{\beta^2}} \right]}
\]

\[
k_2 = \sqrt{\frac{\beta}{\alpha} \left[ 1 - \sqrt{1 - \frac{\alpha}{\beta^2}} \right]}
\]

and

\[
\pi \frac{\partial^2 w}{\partial y^2} = \eta 
\]

the general solution of the plate equation becomes:

\[
w_m = (A_m e^{k_1 n} + B_m e^{-k_1 n} + C_m e^{k_2 n} + D_m e^{-k_2 n}) \sin m \xi
\]

where \( A_m, B_m, C_m, D_m \), are arbitrary constants to be determined from the B.C.

Observing that the deflection and its derivatives approach zero at a large distance from the x-axis, it may be concluded that the constants \( A_m \) and \( C_m \) are zero.

From the conditions of symmetry:

\[
\frac{\partial^2 w}{\partial y^2} \bigg|_{y = 0} = 0
\]
yielding:

\[ B_m k_1 + D_m k_2 = 0 \]  \hspace{1cm} (40)

A second equation results from the fact that the load at \( y = 0 \) is equally divided between the two halves of the plate and that these loads have to be in equilibrium with the appropriate shearing forces, or:

\[
\begin{bmatrix}
Q_{ym}
\end{bmatrix}_{y=0} = -\frac{P_m}{2} = -\frac{P}{\gamma} \frac{1}{m} \sin \mu \sin \gamma \sin m \chi
\]  \hspace{1cm} (41)

with the expression (25) for \( Q_y \) this equation reduces to:

\[
B_m \left[ -k_1^3 \alpha + k_1(2\beta - \alpha) \right] + D_m \left[ -k_2^3 \alpha + k_2(2\beta - \alpha) \right]
= \frac{P_1^2}{EI \gamma \tau^3} \frac{1}{m^4} \sin \mu \sin \gamma
\]  \hspace{1cm} (42)

Solving the two equations (40), (42) simultaneously one finds:

\[
B_m = -\frac{P_1^2}{EI \gamma \tau^3} \frac{k_2}{\alpha k_1 k_2 (k_1^2 - k_2^2)} \frac{1}{m^4} \sin \mu \sin \gamma
\]  \hspace{1cm} (43a)

\[
D_m = \frac{P_1^2}{EI \gamma \tau^3} \frac{k_1}{\alpha k_1 k_2 (k_1^2 - k_2^2)} \frac{1}{m^4} \sin \mu \sin \gamma
\]  \hspace{1cm} (43b)

and the deflection results:

\[
w = \frac{P_1^2}{EI \gamma \tau^3} \frac{1}{\alpha (k_1^2 - k_2^2)} \sum_{m=1}^{\infty} \frac{1}{m^4} \left( \frac{e^{-k_2 m \gamma}}{k_2} - \frac{e^{-k_1 m \gamma}}{k_1} \right) \sin \mu \sin \gamma \sin m \chi
\]  \hspace{1cm} (44)
Using equations (23), (26) expressions can be found for the following: $M_x, M_y, M_{xy}, M_{yx}, Q_y$. These are listed in Table (I). These expressions hold for values:

$$0 \leq \alpha \leq 1$$

and

$$\beta_0 \leq \beta \leq 1$$

For the limiting values of $\alpha$ and $\beta$, however, one must proceed to the limit and this will be shown in the following for the deflection $w$.

(b) **Isotropic Plate:**

For the case of the isotropic plate with $\alpha$ and $\beta$ equal to 1, it is easy to see that $k_1 = k_2 = 1$. Substituting these values in equation (44) results in indeterminate expressions $\frac{0}{0}$ for each term of the series. These are examined as follows. Using the identity:

$$\frac{k_1 e^{-k_2 \eta} - k_2 e^{-k_1 \eta}}{\alpha k_1 k_2 (k_1^2 - k_2^2)} = \frac{e^{-k_2 \eta} (k_1 - k_2 e^{(k_1 - k_2) \eta})}{\alpha k_1 k_2 (k_1 + k_2) (k_1 - k_2)}$$

the right hand side can be resolved into the following partial fractions:

$$\frac{e^{-k_2 \eta}}{\alpha k_1 k_2} \left( \frac{1 + e^{-(k_1 - k_2) \eta}}{2 (k_1 + k_2)} + \frac{1 - e^{-(k_1 - k_2) \eta}}{2 (k_1 - k_2)} \right)$$
Observing that for \( \alpha \to 1, \beta \to 1, k_1 \to k_2 = 1 \) and \( k_1 - k_2 \to 0 \), the transition to the limit yields:

\[
\lim_{(k_1-k_2)\to 0} \left( \frac{1+e^{(k_1-k_2)mp}}{2(k_1+k_2)} \right) = \frac{1}{2}
\]

\[
\lim_{(k_1-k_2)\to 0} \left( \frac{1-e^{-(k_1-k_2)mp}}{2(k_1-k_2)} \right) = \frac{m\eta}{2}
\]

and the deflection of the isotropic plate is obtained (Reference 8, page 169):

\[
w = \frac{Pl^2}{2EI} \sum_{m=1}^{\infty} \frac{1}{m^4} \left[ 1 + m\eta \right] e^{-m\eta} \sin m\nu \sin m\gamma \sin m\xi
\]

(45)

The formulas for the moments and forces are given in Reference 16, page 26.

(c) **Articulated Plate:**

For the case of \( \alpha \to 0 \) and \( \beta \to \beta_0 \) it is noted that:

\[
\lim_{\alpha \to 0} k_1 = \lim_{\beta \to \beta_0} \sqrt{\frac{\beta}{2}} \left[ 1 + \sqrt{1 - \frac{\alpha^2}{\beta^2}} \right] = 8
\]

(46a)

For:

\[
\lim_{\alpha \to 0} k_2 = \lim_{\beta \to \beta_0} \sqrt{\frac{\beta}{2}} \left[ 1 - \sqrt{1 - \frac{\alpha^2}{\beta^2}} \right]
\]

the Binomial Expansion is used:

\[
\sqrt{1 - \frac{\alpha^2}{\beta^2}} = 1 - \frac{1}{2} \frac{\alpha}{\beta^2} + \frac{1}{8} \frac{\alpha^2}{\beta^4} - \ldots
\]
and bearing in mind that, since \( \frac{\alpha}{\beta^2} \) is very small, higher order terms can be neglected, the limiting value of \( k_2 \) results as:

\[
\lim_{\alpha \to 0} k_2 = \frac{1}{\sqrt{2\beta_0}} = k_o
\]

Furthermore, one obtains:

\[
\lim_{\alpha \to 0} \alpha (k_1^2 - k_2^2) = 2\beta_0
\]

Now, upon investigating equation (44) the general terms can be partially rewritten as follows:

\[
\frac{k_1 e^{-k_2 \eta} - k_2 e^{-k_1 \eta}}{\alpha k_1 k_2 (k_1^2 - k_2^2)} = \frac{e^{-k_2 \eta}}{\alpha k_2 (k_1^2 - k_2^2)} \frac{e^{-k_1 \eta}}{\alpha k_1 (k_1^2 - k_2^2)}
\]

Using the limiting values as derived above, the second expression becomes zero, the first becomes:

\[
k_o e^{-k_0 \eta}
\]

and the deflection \( w \) of the articulated plate results:

\[
w = \sum_{m=1}^{\infty} \frac{1}{m^4} \frac{P_{12}}{EI \gamma \pi^3 \sqrt{2\beta_0}} \sin m\gamma \sin m\delta \sin m\varepsilon
\]

The expressions for \( M_x \), \( Q_y \), and \( M_{xy} \) are derived from equations (23) to (26) and are given in Table (I). My
for this case is zero. Equations 44, 45, and 48 hold for the complete plate if instead of \( y \) the absolute values, \(|y|\), are used.

3. Solution for the Bridge of Finite Width:

(a) Orthotropic Plate:

The bridge of span \( l \) and width \( 2b \) is loaded with a line loading as shown in Fig. 10.

Two different coordinate systems are used to simplify the numerical calculations.

\((x,y,z)\) is the coordinate system used for the solution of the homogenous equation with the origin at the middle of one support. All the values, such as deflections, forces and moments related to this solution will be marked by an index 0. For the solution of the infinitely wide bridge, the coordinate system \((x_1, y_1, z_1)\) is used with the \( x_1 \)-axis coinciding with the line of the applied load. An index 1 will designate the values in this second system. The distances of the free edges from the two \( x \)-axes are denoted as \( y = \pm b \) and \( y_1 = e'b \), and \( y_1 = -e''b \), where \( e'+e'' = 2 \).

The midpoint of the applied load is given by the coordinates \((u,v)\), where \( v = (1-e^i) \cdot b \) is the distance from the \( x \)-axis. The equations of transformation between the two
coordinate systems are:
\[ x_1 = x \]
\[ y_1 = y - v \]
\[ z_1 = z \]  
(49)

In later equations, similar abbreviations as in the derivation of the particular solution, are used. These abbreviations are given by:

\[ \varepsilon = \frac{y}{b} \]
\[ \lambda = \frac{\gamma b}{l} \]
\[ \eta = \lambda \cdot \varepsilon \]
\[ \eta_1 = \varepsilon + \varepsilon' - \lambda \]  
(50)

As mentioned before, the solution of the non-homogenous differential equation (1b) can be formed by adding the general solution of the homogenous equation \( w_0 \) to the solution \( w_1 \) (equation (44)) of the infinitely wide plate strip in such a way that the B.C. at the free edges are fulfilled.

Considering only one term of the series, the solution of the homogenous equations as given in equation (39) will
be used in the following transformation, using hyperbolic functions:

\[
\omega_m = (A_m \cosh k_1 \eta + B_m \cosh k_2 \eta + C_m \sinh k_1 \eta + D_m \sinh k_2 \eta) \sin \eta
\]

(51)

Where \(A_m, B_m, C_m, \) and \(D_m\) are a new set of constants. The applications of this solution permit simplifications which far outweigh the disadvantage of the more elaborate transitions to the limit.

One term of the complete solution now takes the form of

\[
\omega_m = (A_m \cosh k_1 \eta + B_m \cosh k_2 \eta + C_m \sinh k_1 \eta + D_m \sinh k_2 \eta) \sin \eta + \omega_m
\]

(52)

The requirement that each term of \(w\) satisfies the B.C. permits the determination of the constants. The B.C. as given by equation (29a and 29b) are:

for \(y = \pm b:\)

\[
M_y = \begin{bmatrix} M_y_0 \end{bmatrix} y = b + \begin{bmatrix} M_y_1 \end{bmatrix} y = e' b = 0
\]

(a)

\[
R_y = \begin{bmatrix} R_y_0 \end{bmatrix} y = b + \begin{bmatrix} R_y_1 \end{bmatrix} y = e' b = 0
\]

(b)

\[
M_y = \begin{bmatrix} M_y_0 \end{bmatrix} y = -b + \begin{bmatrix} M_y_1 \end{bmatrix} y = e'' b = 0
\]

(c)

\[
R_y = \begin{bmatrix} R_y_0 \end{bmatrix} y = -b + \begin{bmatrix} R_y_1 \end{bmatrix} y = e'' b = 0
\]

(d)

(53)
Substituting equation (52) in equation (53) the following simultaneous equations can be formulated for each value of $m$:

From equation (53a):

$$A_m k_1^2 \cosh k_1 m \lambda + B_m k_2^2 \cosh k_2 m \lambda + C_m k_1^2 \sinh k_1 m \lambda + D_m k_2^2 \sinh k_2 m \lambda$$

$$= \frac{1}{m^2 \pi^2} \left[ \frac{M_y}{EI} \right] \left[ \frac{y_1 = e' b}{\sin m \xi} \right] = S_{1m} \quad (54a)$$

From equation (53c):

$$A_m k_1^2 \cosh k_1 m \lambda + B_m k_2^2 \cosh k_2 m \lambda - C_m k_1^2 \sinh k_1 m \lambda - D_m k_2^2 \sinh k_2 m \lambda$$

$$= \frac{1}{m^2 \pi^2} \left[ \frac{M_y}{EI} \right] \left[ \frac{y_1 = -e'' b}{\sin m \xi} \right] = S_{2m} \quad (54b)$$

From equation (53b):

$$A_m k_1^2 \sinh k_1 m \lambda + B_m k_2^2 \sinh k_2 m \lambda + C_m k_1^2 \cosh k_1 m \lambda + D_m k_2^2 \cosh k_2 m \lambda$$

$$= \frac{1}{m^3 \pi^3} \left[ \frac{R_y}{EI} \right] \left[ \frac{y_1 = e' b}{\sin m \xi} \right] = S_{3m} \quad (54c)$$

From equation (53d):

$$-A_m k_1^2 \sinh k_1 m \lambda - B_m k_2^2 \sinh k_2 m \lambda + C_m k_1^2 \cosh k_1 m \lambda + D_m k_2^2 \cosh k_2 m \lambda$$

$$= \frac{1}{m^3 \pi^3} \left[ \frac{R_y}{EI} \right] \left[ \frac{y_1 = -e'' b}{\sin m \xi} \right] = S_{4m} \quad (54d)$$
Determinants will be used to solve this system of equations. Observing the symmetrical and skew-symmetrical properties, the determinant of the denominator reduces to the following: (15)

\[
\begin{vmatrix}
 a_1 & b_1 & c_1 & d_1 \\
 a_1 & b_1 & -c_1 & -d_1 \\
 a_2 & b_2 & c_2 & d_2 \\
 -a_2 & -b_2 & c_2 & d_2 \\
\end{vmatrix} = 4(a_1b_2 - a_2b_1)(c_1d_2 - c_2d_1)
\]

(55)

where:

\[ a_1 = k_1^2 \cosh \gamma_1 m \lambda, \text{ etc.} \]

For the same reason, simple expressions are obtained for the third order determinants \( U_{pg} \), where \( p \) stands for the eliminated column and \( g \) for the eliminated row.

\[ U_{11} = U_{12} = +2b_2(c_1d_2 - c_2d_1) \]

\[ U_{13} = -U_{14} = -2b_1(c_1d_2 - c_2d_1) \]

\[ U_{21} = U_{22} = -2a_2(c_1d_2 - c_2d_1) \]

\[ U_{23} = -U_{24} = +2a_1(c_1d_2 - c_2d_1) \]

\[ U_{31} = -U_{32} = +2d_2(a_1b_2 - a_2b_1) \]

\[ U_{33} = U_{34} = -2d_1(a_1b_2 - a_2b_1) \]

\[ U_{41} = -U_{42} = -2c_2(a_1b_2 - a_2b_1) \]

\[ U_{43} = U_{44} = +2c_1(a_1b_2 - a_2b_1) \] (56)
With these values the constant $A_m$ is a function of the boundary values $S_{1m}$, $S_{2m}$, $S_{3m}$, $S_{4m}$, is found thus:

$$A_m = \frac{U_{11} S_{1m} + U_{12} S_{2m} + U_{13} S_{3m} + U_{14} S_{4m}}{4(a_1 b_2 - a_2 b_1)(c_1 d_2 - c_2 d_1)}$$

or:

$$A_m = \frac{b_2}{a_1 b_2 - a_2 b_1} \frac{S_{1m} + S_{2m}}{2} - \frac{b_1}{a_1 b_2 - a_2 b_1} \frac{S_{3m} - S_{4m}}{2} \quad (57a)$$

and similarly:

$$B_m = \frac{-a_2}{a_1 b_2 - a_2 b_1} \frac{S_{1m} + S_{2m}}{2} + \frac{a_1}{a_1 b_2 - a_2 b_1} \frac{S_{3m} - S_{4m}}{2} \quad (57b)$$

$$C_m = \frac{d_2}{c_1 d_2 - c_2 d_1} \frac{S_{1m} - S_{2m}}{2} - \frac{d_1}{c_1 d_2 - c_2 d_1} \frac{S_{3m} + S_{4m}}{2} \quad (57c)$$

$$D_m = \frac{-c_2}{c_1 d_2 - c_2 d_1} \frac{S_{1m} - S_{2m}}{2} + \frac{c_1}{c_1 d_2 - c_2 d_1} \frac{S_{3m} + S_{4m}}{2} \quad (57d)$$

As is shown below, $A_m$ and $B_m$ contain the symmetrical part of $w_1$, whereas $C_m$ and $D_m$ contain the skew-symmetrical part of $w_1$. If $w_1$ is symmetrical with respect to the x-axis the following identities hold:

$$[M_{y1}] y_1 = b = [M_{yL}] y_1 = -b$$
and

\[
\begin{bmatrix}
R_{y1}
\end{bmatrix} y_1=b = \begin{bmatrix}
-R_{y1}
\end{bmatrix} y_1=-b
\]

or

\[S_{1m} = S_{2m}\]

and

\[S_{m3} = -S_{m4}\]

which gives

\[C_m = D_m = 0.\]

For \(w_1\) being skew-symmetrical with respect to the x-axis and

\[
\begin{bmatrix}
M_{y1}
\end{bmatrix} y_1=b = \begin{bmatrix}
-M_{y1}
\end{bmatrix} y_1=-b
\]

\[
\begin{bmatrix}
R_{y1}
\end{bmatrix} y_1=b = \begin{bmatrix}
R_{y1}
\end{bmatrix} y_1=-b
\]

it follows that \(A_m = B_m = 0.\)

Replacing the coefficients of the determinants by the terms given in equation (57) and using the following abbreviations:

\[i_{1m} = \frac{b_2}{a_1b_2-a_2b_1} = \frac{k_1\sinh k_2 m \lambda}{k_1 \cosh k_1 m \lambda \sinh k_2 m \lambda - k_2^2 \sinh k_1 m \lambda \cosh k_2 m \lambda} \quad (58a)\]

\[i_{2m} = \frac{a_2}{a_1b_2-a_2b_1} = \frac{k_2 \sinh k_1 m \lambda}{k_1 \cosh k_1 m \lambda \sinh k_2 m \lambda - k_2^2 \sinh k_1 m \lambda \cosh k_2 m \lambda} \quad (58b)\]

\[i_{3m} = \frac{d_2}{c_1d_2-c_2d_1} = \frac{k_1 \cosh k_2 m \lambda}{k_1 \sinh k_1 m \lambda \cosh k_2 m \lambda - k_2^2 \cosh k_1 m \lambda \sinh k_2 m \lambda} \quad (58c)\]

\[i_{4m} = \frac{c_2}{c_1d_2-c_2d_1} = \frac{k_2 \cosh k_1 m \lambda}{k_1 \sinh k_1 m \lambda \cosh k_2 m \lambda - k_2^2 \cosh k_1 m \lambda \sinh k_2 m \lambda} \quad (58d)\]
\[
\begin{align*}
\alpha_m &= \frac{b_1}{a_1 b_2 - a_2 b_1} = \frac{k_2 \cosh k_2 m}{k_1 \cosh k_1 m \sinh k_2 m - k_2 \sinh k_1 m \cosh k_2 m} \\
\beta_m &= \frac{a_1}{a_1 b_2 - a_2 b_1} = \frac{k_1 \cosh k_1 m}{k_2 \cosh k_1 m \sinh k_2 m - k_2 \sinh k_1 m \cosh k_2 m} \\
\gamma_m &= \frac{c_1}{c_1 d_2 - c_2 d_1} = \frac{k_2 \sinh k_2 m}{k_1 \sinh k_1 m \cosh k_2 m - k_2 \cosh k_1 m \sinh k_2 m} \\
\delta_m &= \frac{c_1}{c_1 d_2 - c_2 d_1} = \frac{k_1 \sinh k_1 m}{k_1 \sinh k_1 m \cosh k_2 m - k_2 \cosh k_1 m \sinh k_2 m}
\end{align*}
\]

The constants can be written as:

\[
\begin{align*}
A_m &= i_1 m \frac{S_1 m + S_2 m}{2} - j_1 m \frac{S_3 m - S_4 m}{2} \quad \text{(a)} \\
B_m &= -i_2 m \frac{S_1 m + S_2 m}{2} + j_2 m \frac{S_3 m - S_4 m}{2} \quad \text{(b)} \\
C_m &= i_3 m \frac{S_1 m - S_2 m}{2} - j_3 m \frac{S_3 m + S_4 m}{2} \quad \text{(c)} \\
D_m &= -i_4 m \frac{S_1 m - S_2 m}{2} + j_4 m \frac{S_3 m + S_4 m}{2} \quad \text{(d)} \quad \text{(59)}
\end{align*}
\]

And the deflection of the bridge results:

\[
w = \sum_{m=1}^{\infty} (A_m \cosh k_1 m \eta + B_m \cosh k_2 m \eta + C_m \sinh k_1 m \eta + D_m \sinh k_2 m \eta) \sin \phi + w_1
\]

(b) **Isotropic Plate:**

As proved before, the deflections and therefore the moments and shear forces of the infinitely long isotropic
plate result from the equations of the orthotropic plate by a transition to the limit $a \to 1$. In the following the same will be done for the solution of the homogenous equation (32). Since the analysis is the same, the transition will be shown in more detail for the first two terms of equation (51) and only the resulting expressions will be given for the last terms.

The first two terms:

$$A_m \cosh k_1 \eta + B_m \cosh k_2 \eta$$

rewritten with equation (59) as:

$$\left(\frac{i_{1m} S_{1m} + S_{2m}}{2} - j_{1m} \frac{S_{3m} - S_{4m}}{2}\right) \cosh k_1 \eta$$

and with the values given in equation (58) into:

$$\left(1_{m} \cosh k_1 \eta - i_{2m} \cosh k_2 \eta\right) \frac{S_{1m} + S_{2m}}{2} - (j_{1m} \cosh k_1 \eta - j_{2m} \cosh k_2 \eta) \frac{S_{3m} - S_{4m}}{2}$$

and with the values given in equation (58) into:

$$\frac{k_1 \sinh k_2 m \cosh k_1 \eta - k_2 \sinh k_1 m \cosh k_2 \eta}{k_1 \cosh k_1 m \sinh k_1 m - k_2 \sinh k_1 m \cosh k_2 m} \frac{S_{1m} + S_{2m}}{2}$$

$$- \frac{k_2 \cosh k_2 m \cosh k_1 \eta + k_1 \cosh k_1 m \cosh k_2 \eta}{k_1 \cosh k_1 m \sinh k_1 m - k_2 \sinh k_1 m \cosh k_2 m} \frac{S_{3m} - S_{4m}}{2}$$
Using the boundary values $M_{y1}$ and $R_{y1}$ of the isotropic plate it is easy to see that $\frac{S_{1m} + S_{2m}}{2}$ as well as $\frac{S_{3m} - S_{4m}}{2}$ are of finite magnitude. One needs therefore to investigate only the two remaining fractions.

The following steps are necessary:

1. Rewrite the fractions as functions of $(k_1 + k_2)$ and $(k_1 - k_2)$.
2. Divide the numerator and denominator by $(k_1 - k_2)$.
3. Proceed to the limit $(k_1 - k_2) \rightarrow 0$.

In order to clarify the operation the numerators $N_1$ and $N_2$ and the denominator $D$ will be investigated separately:

The first numerator becomes:

$$N_1 = k_1 \left[ \sinh m \left( \frac{k_1 + k_2}{2} - \frac{k_1 - k_2}{2} \right) \cosh m \left( \frac{k_1 + k_2}{2} + \frac{k_1 - k_2}{2} \right) \right]$$

$$- k_2 \left[ \sinh m \left( \frac{k_1 + k_2}{2} + \frac{k_1 - k_2}{2} \right) \cosh m \left( \frac{k_1 + k_2}{2} - \frac{k_1 - k_2}{2} \right) \right]$$

and with the following identities:

$$\sinh (a + b) = \sinh a \cosh b + \cosh a \sinh b$$

$$\cosh (a + b) = \cosh a \cosh b + \sinh a \sinh b$$

it can be arranged into:

$$(k_1 - k_2) \sinh m \left[ \frac{k_1 + k_2}{2} \cosh m \frac{k_1 - k_2}{2} \cosh m \frac{k_1 + k_2}{2} \cosh m \frac{k_1 - k_2}{2} \right]$$
Dividing by \((k_1 - k_2)\) and proceeding to the limit \((k_1 - k_2) \rightarrow 0\) yields:

\[
N_1: (\sinh m^2 - m^2 \cosh m^2) \cosh \eta + m \eta \sinh m^2 \sinh \eta
\]

Applying exactly the same operations to the second numerator \(N_2\) and to the denominator \(D\) one obtains:

\[
N_2: -(2 \cosh m^2 + m \sinh m^2) \cosh \eta + m \eta \cosh m^2 \sinh \eta
\]
\[
D: 3 \cosh m^2 \sinh m^2 - m^2
\]

Substituting these values in equation (32) and rearranging the expression yields:

\[
A_m \cosh k_1 m \eta + B_m \cosh k_2 m \eta \rightarrow A_m^* \cosh \eta + B_m^* m \eta \sinh \eta
\]

with the values \(A_m^*\) and \(B_m^*\) as given below. Similar expressions are obtained for the later terms.
The deflection of the homogenous bridge thus is obtained as:

\[
w = \sum_{m=1}^{\infty} \left( A_m \cosh \eta + B_m \eta \sinh \eta + C_m \sinh \eta + D_m \cosh \eta \right) \sin m \phi + w_1 \tag{61}
\]

The coefficients \(A^*_m\) to \(D^*_m\) are given by equation (59) if \(i_m\) and \(j_m\) are replaced by \(i^*_m\) and \(j^*_m\) as follows:

\[
i^*_{1m} = \frac{m \lambda \cosh \lambda - \sinh \lambda}{m \lambda - 3 \cosh \lambda \sinh \lambda}
\]

\[
i^*_{2m} = \frac{\sinh \lambda}{m \lambda - 3 \cosh \lambda \sinh \lambda}
\]

\[
i^*_{3m} = \frac{m \lambda \sinh \lambda - \cosh \lambda}{m \lambda + 3 \cosh \lambda \sinh \lambda}
\]

\[
i^*_{4m} = \frac{\cosh \lambda}{m \lambda + 3 \cosh \lambda \sinh \lambda}
\]

\[
j^*_{1m} = \frac{m \lambda \sinh \lambda + 2 \cosh \lambda}{m \lambda - 3 \cosh \lambda \sinh \lambda}
\]

\[
j^*_{2m} = \frac{\cosh \lambda}{m \lambda - 3 \cosh \lambda \sinh \lambda}
\]

\[
j^*_{3m} = \frac{m \lambda \cosh \lambda + 2 \sinh \lambda}{m \lambda + 3 \cosh \lambda \sinh \lambda}
\]

\[
j^*_{4m} = \frac{\sinh \lambda}{m \lambda + 3 \cosh \lambda \sinh \lambda}
\]
and:

\[ S_{1m}^* = \frac{1}{m^2 \gamma^2 E I} \left[ \frac{M_{y1}}{y_1 = +e^{-b}} \right] \frac{\sin m \xi}{ \sin m \xi} \]  

(a)

\[ S_{2m}^* = \frac{1}{m^2 \gamma^2 E I} \left[ \frac{M_{y1}}{y_1 = -e^{+b}} \right] \frac{\sin m \xi}{ \sin m \xi} \]  

(b)

\[ S_{3m}^* = \frac{1}{m^3 \gamma^3 E I} \left[ \frac{R_{y1}}{y_1 = +e^{-b}} \right] \frac{\sin m \xi}{ \sin m \xi} \]  

(c)

\[ S_{4m}^* = \frac{1}{m^3 \gamma^3 E I} \left[ \frac{R_{y1}}{y_1 = -e^{+b}} \right] \frac{\sin m \xi}{ \sin m \xi} \]  

(d)

(63)

(c) Articulated Plate:

For the limiting case \( \alpha \to 0 \) which includes \( k_1 \to \infty \), \( k_2 \to k_0 = \frac{1}{\sqrt{2\beta}} \), and \( M_y = 0 \), it is easy to see from equation (58) that in equation (59) \( A_m \) and \( C_m \) vanish and that \( B_m \) reduces to:

\[ B_m = \frac{k_1^2 \cosh k_1 m \lambda}{k_1^2 \cosh k_1 m \lambda \sinh k_2 m \lambda - k_2^2 \sinh k_1 m \lambda \cosh k_2 m \lambda} \cdot \frac{1^3}{m^3 \gamma^3} \left[ \frac{R_{y1}}{y_1 = +e^{-b}} - \frac{R_{y1}}{y_1 = -e^{+b}} \right] \]

Dividing numerator and denominator by

\[ k_1^2 \cosh k_1 m \lambda \]

and rearranging the expression, we obtain,

\[ B_m = \frac{1}{\alpha k_1^2 k_2 \sinh k_2 m \lambda - \alpha k_2^2 \tanh k_1 m \lambda \cosh k_2 m \lambda} \cdot \frac{1^3}{m^3 \gamma^3} \left[ \frac{R_{y1}}{y_1 = +e^{-b}} - \frac{R_{y1}}{y_1 = -e^{+b}} \right] \]
For $\alpha \to 0$ the second term in the denominator approaches zero and:

$$\lim_{\alpha \to 0} \frac{\alpha k_1^2}{\alpha} = \lim_{\alpha \to 0} \beta \left[ 1 + \sqrt{1 - \frac{\alpha \beta^2}{\beta^2}} \right] = \beta \beta_o$$

It follows that:

$$B_{\text{o}m} = \frac{1}{4\beta_o k_o E_l} \cdot \frac{1}{m^3} \left[ \frac{R_{y1}}{\sinh k_o m^\alpha \sin m^\xi} \right] \frac{y_1 = e^b - [R_{y1}] y_1 = -e^b}{(64a)}$$

and similarly:

$$D_{\text{o}m} = \frac{1}{4\beta_o k_o E_l} \cdot \frac{1}{m^3} \left[ \frac{R_{y1}}{\cosh k_o m^\alpha \sin m^\xi} \right] \frac{y_1 = e^b + [R_{y1}] y_1 = -e^b}{(64b)}$$

The deflection of the articulated plate can now be written as:

$$w = \sum_{m=1}^{\infty} \left( B_{\text{o}m} \cosh k_o m \eta + D_{\text{o}m} \sinh k_o m \eta \right) \sin m^\xi + w_1$$

(65)

4. Explicit Form of Formulas

Equations (60), (61) and (65) can be written in explicit form by replacing the boundary values $M_{y1}$ and $R_{y1}$ by their values obtained from equation (24) and (26) with equation (44). The resulting expression is listed in Table II for the general case of the orthotropic plate together with the important formulas for moments and forces used in the design.
of bridges. Table III contains these expressions for the articulate plate. For the case of the isotropic plate, reference is made to the extensive study by H. Olsen and F. Reinitzhuber of the rectangular plate with two opposite edges simply supported and the two other edges free. (16). This study contains all the formulas, as well as influence surfaces for deflection and moments for plates of various sizes.

5. Influence Surfaces:

In beam statics influence lines are commonly used in studying the effect of concentrated loads or load systems. The influence line for the moment, for example, describes the variation of the moment at a given point of a beam due to the passage of a single load across the beam.

In a similar way influence surfaces can be defined for plate structures. The influence surface, for example, for the moment at a fixed point \((u,v)\) for a unit load is given by a surface with the ordinate \(z\) in the point \((x,y)\) equal to the moment \(M_x\) produced at the point \((u,v)\) by the unit load placed at \((x,y)\).

The following discussion of influence surfaces holds as well for the isotropic as for the orthotropic plate, and covers only those items which will be used in the next
chapter to derive the properties of the lateral load
distribution coefficients. For a more complete investi-
gation including the derivation of the basic rules, one
is referred to K. Girkman, page 225. (11)

The two main rules for the derivation of influence
surfaces are:

1. The deflection surface \( w = w(x,y,u,v) \) due to
P = 1 at the point \((u,v)\) is equal to the influence surface
for the deflection at point \((u,v)\).

2. The influence surfaces for any derivative of the
deflection at the point \((u,v)\) is obtained by differentiating
the deflection surface \( w = w(x,y,u,v) \) due to P = 1 at
\((u,v)\) with respect to the coordinates \(u\) and \(v\).

It can be proved that the above rules hold also for
the case in which the unit load is replaced by a line load,
applied over the distance \(2c\) parallel to the x-axis and
with the mid-point coordinates \(u\) and \(v\). The proof is based
on the fact that, for the given line load at \((u,v)\), the
deflection at \((x,y)\) is equal to the deflection at \((u,v)\)
due to the line load at \((x,y)\). This can be seen from
equation (1), Table II, by interchanging the coordinates
\((x,y)\) and \((u,v)\).
As the expression for the line load in the equation for the deflection is not affected by the differentiation of the latter with respect to \( x \) and \( y \) or to \( u \) and \( v \), it follows that the second rule holds also for the line load. Obviously one obtains in this case the influence surface for this type of loading.

From the first rule it follows that equation (1) in Table II also represents the function of the influence surface for deflection at the point \((u,v)\) due to the given line load. This conclusion holds generally since it follows directly from Maxwell's law of reciprocal deflections.

Applying the second rule to derive the influence line for the moment \( M_x \) at the point \((u,v)\) which is denoted by \([M_x]_{u,v}\) one obtains:

\[
[M_x]_{u,v} = -EI \frac{\partial^2 w}{\partial u^2} = -EI \frac{\partial^2 w}{\partial x^2}
\]

(66)

It is easy to see that the resulting expression is identical with equation (2) in Table II. It follows that this equation also represents the function of the influence surface for the bending moment \( M_x \) at the point \((u,v)\) for the given loading. This result will be used in the following chapter to establish an important property of the coefficients of lateral load distribution.
It must be noted that the above identity holds only for the bending moment $M_x$ and only because of the simple boundary conditions of the plate for $x = 0$, and $x = 1$, and the fact that the effect of Poisson's ratio is neglected. If needed, the functions of the influence surfaces for other moments and forces may be found without difficulty by differentiating equation (1) in Table II according to the second rule. The same results can be obtained from equation (3), (4), and (5) of Table II by considering the coordinate $(x,y)$ as fixed and the coordinates of the midpoint of the loading $(u,v)$ as variable. In this case the equations represent the influence surfaces for the point $(x,y)$. 
IV. LATERAL LOAD DISTRIBUTION

1. General:

The formulas for any location of a live loading derived in the foregoing chapter and summarized in Tables I and II permit an accurate design of multi-beam bridges. To eliminate the extensive amount of calculation, numerical values are given in the later part of this study for deflections, moments and shear forces at points along, and for loading positions in the midspan cross-section of the bridge. They were computed for bridges of different span and width and also for various degrees of lateral prestress.

These loading positions at midspan of the bridge are, in general, the most unfavorable positions encountered in the design of such bridges. However, in many cases, other loading positions have to be considered for which only the principal bending moment $M_x$ would be required. The determination of $M_x$, however, may involve much numerical work. In this case, an approximate determination of the bending moment $M_x$ can be made using coefficients of lateral load distribution, introduced by Y. Guyon. (4)

2. Definition

Considering again the bridge with the live load applied as shown in Fig. 10, the average bending moment in a cross-
section \( x = x_s \) of the bridge is given by:

\[
M_{x \text{av}} = \frac{1}{2b} \int_{-b}^{+b} M_x \, dy
\]  

(67a)

For the case where the effect of Poisson's ratio is neglected this average moment is equal to the bending moment per unit width obtained by replacing the applied live load, which acts over distance \( 2c \), by an equivalent uniformly distributed load, acting over an area \( 2c \cdot 2b \), where \( 2b \) is the width of the bridge.

By integrating equation (2) in Table II the average bending moment is obtained as:

\[
\frac{1}{2b} \int_{-b}^{+b} M_x \, dy = \frac{P}{b} \sum_{m=1}^{\infty} \frac{1}{m^3} \sin mx \sin my \sin m\theta
\]  

(67b)

This expression developed in a sine-series is identical with the average bending moment obtained from conventional beam statics. This is true because the above mentioned loading, which does not vary over the width of the bridge, causes a deformation of cylindrical shape.

The coefficient of the lateral load distribution \( s_{xy} \) for the point \( (x, y) \) subjected to the actual bending moment \( M_x \), may now be defined as:
This is a non-dimensional coefficient and indicates the portion of the average bending moments of the sections \( x = x_s \) which exist at the point \((x_s, y)\).

For the design of a multi-beam bridge one is especially interested in the bending moment associated with each beam incorporated in the bridge. For the beam \( i \), for example, this moment is substantially the bending moment per unit width at the mid-point of the beam, multiplied by its width \( a \); from equation (67a):

\[
a M_x i = s_{xi} \frac{a}{2b} \int_{-b}^{+b} M_x dy = \frac{s_{xi}}{n} \int_{-b}^{+b} M_x dy
\]

where \( n = \frac{2b}{a} \) is the number of beams. Expressing this moment as a percentage of the total cross-sectional moment \( \int_{-b}^{+b} M_x dy \) one obtains:

\[
s_{xi}^2 = \frac{s_{xi}}{n} \times 100
\]

Assuming now that the actual live load is not directly applied upon beam \( i \), the bending moment at beam \( i \) may be obtained from a consideration of the interaction of the beam
as expressed by equation (60). This moment can be thought of as being produced by a proportion of the actual loading, directly applied on an independent beam. It is apparent that the factor of proportionality is identical with the factor given by equation (70). $s^{*}_{x_i}$ also indicates the distribution of the actual load applied on one beam among the other beams. Hence, in the determination of the magnitude of the beam bending moments, each beam is considered as being independent and subjected to a load depending upon its value $s^{*}_{x_i}$.

Similarly the coefficient $s_{x_i}$ may be considered as a measure of the lateral load distribution, and since it is independent of the number of beams $n$, it is more suitable for non-dimensional representation.

It is to be understood that the coefficient $s_{xy}$ for the point $(xy)$ varies with the type and the location of the applied live loading. Its introduction, therefore, does not yield any advantage in the accurate determination of the bending moments. However, it is very useful in approximating the bending moments.

3. Approximation for the Lateral Load Distribution:

Y. Guyon (4) and Massonnet (5) computed the coefficients of the lateral load distribution for a live load of sinusoidal
type given by:

$$p(x) = p_0 \sin \frac{\pi x}{l}$$  \hspace{1cm} (71)$$

where $p_0$ is the load intensity at midspan of the bridge. This expression is identical with the first term of the sine-series of equation (32b) where $P$ is replaced by:

$$p_0 \frac{\gamma l}{2 \sin \psi \sin \gamma}$$

For this loading the moments and forces are given by the formulas derived in Chapter III with $m = 1$.

With equation (2) in Table II and equation (67b) the coefficient of the lateral load distribution for the above loading results as:

$$s_{xy} = \frac{M_x}{M_{x,av}} = \frac{b \pi}{2 l \rho \sqrt{1 - \frac{\rho}{\rho^2}}} \left[ \left( \eta_1 \right) + g (\eta) \right] = s_y$$  \hspace{1cm} (72)$$

where the expression in the bracket above is identical with the expression in the bracket of equation (2) in Table II.

It is easy to see that this coefficient of the lateral load distribution depends only on the eccentricity of the line live load with respect to the bridge axis and on the
ordinate $y$ of the reference point $(xy)$. It is independent of the abscissa $x$ of the cross-section under consideration. It follows that for a particular loading position, the moment curves of all beams are similar in shape, with the moment ordinates depending upon $s_y$. The same relation can be derived for the deflection curves and it follows that the coefficient $s_y$ also represents the ratio of the deflection at a point $y$ of any cross-section to the average deflection of the cross-section.

Because of these considerable advantages and the fact that their values can be more readily computed, Guyon and Massonnet considered the coefficients for the sinusoidal loading as a practical approximation to be used for any type and location of loading encountered in the design of bridges.

The application of this approximation may be justified for loading conditions prescribed in some European specification. According to the American specification for highway bridges (1), however, one must consider the effect of concentrated wheel loads or truck axle loads. For such types of loading which differ considerably from the sinusoidal load, the above approximation coefficients lead to a very favorable distribution of the load but induce a rather small moment in the directly loaded beam.
For a safer design, especially for concentrated loads and wheel loads, the following approximation is proposed. This is based on the exact distribution coefficients computed for the midspan cross-section of the bridge corresponding to a wheel load placed at midspan. It is considered that this position is generally the most unfavorable encountered in bridge design. It may therefore be assumed that this lateral load distribution is representative for all cross-sections of the bridge subject to a similar load application in each cross-section. It is apparent that variations might exist in the curve of lateral load distribution from section to section along the bridge. However, it may also be assumed that, having derived the curve for the worst condition of loading, that is, at midspan, it may be applied to any section to yield a moment which would be higher than the actual moment existing at that section.

Coefficients for lateral load distribution corresponding to the most important loading conditions are computed and given in the following chapter.

4. Properties of the Coefficient of Lateral Load Distribution:

It was stated in Chapter III that equation (2) in Table II representing the moment $M_k$ at point $(x,y)$ due to a load at $(u,v)$ also described the function of the influence surface
for the moment $M_x$ at the point $(u,v)$. From this and using equation (68) an important property of the lateral load distribution coefficients follows: "The curve of the coefficients of the lateral load distribution in the midspan cross-section for a load applied at the point $v$ of the same section, is also the influence line for the coefficient at midspan and for the point $v$." This means that the reciprocal relation holds:

$$s_{yv} = s_{vy} \quad (73)$$

or that the coefficient of the lateral load distribution at $y$ due to a load at $v$ is equal to the coefficient at $v$ due to the load at $y$.

The second property of these coefficients, though obvious, is still worth mentioning, since it is useful for checking purposes.

Keeping the definition of the coefficient in mind (equation (68) ) and integrating $s_{yv}$ over the cross-section one obtains:

$$\int_{-b}^{b} s_{yv} \, dy = \int_{-b}^{b} \frac{M_y}{2b} dy = 2b \quad (74)$$
In performing this operation one may replace the integral by the value of the area under the $s_{yv}$-curve determined using Simpson's rule. In the next chapter the coefficients are computed for the following points:

$$(-b, \frac{-3b}{4}, \frac{-b}{2}, \frac{-b}{4}, 0, \frac{b}{4}, \frac{b}{2}, \frac{3b}{4}, 4b)$$

Applying Simpson's rule to each point and designating the coefficients for the various points as $s_1, s_2 \ldots s_9$ yields:

$$\int_{-b}^{+b} s_{yv} \, dy = 2b \approx \frac{2b}{24} \left[ 2 \sum s_{odd} - (s_1 + s_9) + 4 \sum s_{even} \right]$$

or:

$$\frac{1}{24} \left[ 2 \sum s_{odd} - (s_1 + s_9) + 4 \sum s_{even} \right] = 1$$

This condition may be used to check the accuracy of the distribution coefficients.
V. NUMERICAL CALCULATIONS

1. General:

The extensive amount of numerical work to compute exact values for all formulas listed in Tables II and III was reduced by making use of an electronic high speed computer. Numerical values were computed for 9 points at equal intervals along the midspan cross-section. The values for $M_{xy}$ were determined for points along the support, $x = 0$. Two loading positions in the same cross-section were investigated, namely the load placed in the bridge axis with $v = 0$, or $e' = 1$ (Fig. 10) and the load placed at one edge with $v = -b$ or $e' = 2$.

The following parameters were considered:

(a) The ratio of the bending stiffnesses $\alpha$.

The computations were made for $\alpha = 0$, the case of the articulated plate, and for $\alpha = 0.1$ and $\alpha = 0.5$. With these several $\alpha$-values and the values given by Olsen and Reinitzhuber (16) for the isotropic plate $\alpha = 1$, an interpolation for any intermediate $\alpha$ is possible.

(b) The size of the beams.

The size of the beams, or more precisely the width to depth ratio $a/h$ of the beam making up the bridge
affects the coefficient of the torsional rigidity of the orthotropic plate, (equation 22). The following two ratios which are used in present bridge design were considered in the computation:

\[ \frac{a}{h} = 1.00 \text{ and } \frac{a}{h} = 1.7 \]

(c) The size of the bridge.

With span \( l \) and \( b \) the half width of the bridge, computations were made for:

\[ \frac{b}{l} = 0.5 \ 0.375 \ 0.25 \ 0.125 \]

For any intermediate \( b/l \) ratio, moments and forces may be obtained with sufficient practical accuracy by interpolating between the given values.

(d) Line Loading

For the representation of a wheel load in the form of a line load, the distribution of the load in the longitudinal direction is shown in Fig. 11. It is assumed that the length of contact between the wheel and the wearing surface is 4 inches in the longitudinal direction and that the load is then distributed at an angle of 45° through a 2-inch wearing surface plus the half depth \( h/2 \) of the beams. Thus a total longitudinal distribution of \( 2c = 8 \text{ in.} + h \) results.
This distribution, depending on the depth of the beams, is in turn a function of the span of the bridge.

The present computations were made for one \( c/l \) ratio. This was chosen as \( c/l = 0.0318 \) or \( \pi c/l = 0.1 \), which corresponds approximately to the value of \( c \) obtained from the above distribution and for beam depths and spans currently used for multi-beam bridges.

The writer is aware of the fact that the above assumption for the distribution of a wheel load is not included in the American Specifications for Highway Bridges\(^1\). These specifications do not consider any longitudinal distribution of wheel loads. In bridges analyzed according to the derived method however, a concentrated load would cause infinitely large moments directly under the load. To avoid these infinitely large moments, which are of a theoretical nature only, the above longitudinal distribution over the distance \( 2c \) is assumed.

Any distribution of the load in the lateral direction, caused by the width of the tires and the wearing surface, as well as the beam depth may be taken into account by evaluating the prepared influence lines.
2. Accuracy of the Results:

The calculation of the values for the series expression includes the terms up to and including \( m = 19 \). This limitation was set by the capacity of the computing machine in handling certain large parts of the series.

The accuracy of the results is therefore a function of the convergence of the series in Table II, which in turn depends mainly on the power of the factor directly after the summation sign. Furthermore, it also depends on the location of the points, for which series were computed, with respect to the applied load.

Check calculations have revealed that for points not coinciding with the load location, the computed values, with the exception of those for \( Q_y \), are exact to within three significant figures. For points directly under the load, \( \xi = 0 \) (location of point) for \( e' = 1 \) (location of load) and \( \xi = -1.00 \) for \( e' = 2 \) (see Tables IV - XXVII) the accuracy is as follows:

(a) Deflections:

The convergence of the series with \( 1/m^4 \) is excellent, the calculated values check at least to within three significant figures.
(b) Moments \( (M_x, M_y, M_{xy}) \):

The series converge satisfactorily with \( 1/m^2 \) and the given values are accurate to two significant figures.

(c) Shear Force \( Q_y \):

Since the power of \( m \) in the series expression for \( Q_y \) is 1, the series converges very poorly. For points not directly under the load, the values may be accurate to within two significant figures. For the peak values, occurring in the points directly under the load, only the first figure is correct. A better value may be obtained by considering equilibrium between the applied load and the shear force (neglecting the influence of twisting moments) yielding:

for \( e' = 1 \):

\[
Q_y = - \frac{P}{2.2c}
\]

\[
\frac{Q_y}{P} = - 7.85
\]

for \( e' = 2 \):

\[
Q_y = - \frac{P}{2c}
\]

\[
\frac{Q_y}{P} = - 15.7
\]
In general, the accuracy of the results obtained may be considered sufficient for design purposes, since only slide rule precision is required.

3. Description of the Tables:

The results of the numerical calculations, are given in Tables IV-XXVII (as obtained from the computer). Each table includes values determined for one combination of parameters $\alpha$, $a/h$, $b/l$, as listed in the heading of the table. The left hand column shows the quantities $w$, $M_x$, $M_y$, $Q_y$ and $M_{xy}$, where each one is multiplied by a coefficient which renders it dimensionless; the negative sign for $M_y$ and $Q_y$ should also be noted. The second column indicates the position of the load: $e' = 1$ ($E' = 1$) represents the load placed in the bridge axis and $e' = 2$ ($E' = 2$) the load placed at the edge.

The top line shows the coefficient $\xi = \frac{Y}{b}$ determining the distance of the points from the bridge axis for which values were computed. The sign of $\xi$ is indicated in the third column. For $e' = 1$ the values are given only for $\xi$ positive, since under this loading the values $w$, $M_x$, $M_y$ are symmetrical with respect to the bridge axis; similarly $Q_y$

* Note: Only capital letters are printed by the computer...
and $\mathbf{M}_{xy}$ are skew-symmetrical to this axis.

For $e' = 2$ the values are given in the second and third line for each quantity. The second line includes, as indicated by the sign in the third column, the values for $\xi$ positive, whereas in the third line the values for $\xi$ negative are listed. The values for $\xi = 0$ are given in both lines.

The computed values are represented in the following form:

$$\pm x \cdot 10^n$$

where

$$1 \leq x \leq 0.1$$

and the exponent is a positive or negative integer.

The tables include the numbers for $x$ and $n$ only and have to be read as shown by the following example.

The value for $\frac{wEI}{pl^2}$ for $e' = 1$ and $\xi = 0$ in Table IV is given by:

$$0.33822 \cdot 10^{-1}$$

which stands for:

$$+0.33822 \cdot 10^{-1} = +0.033822$$

4. Coefficients of Lateral Load Distribution:

The coefficients of lateral load distribution, as defined by equation (68), were calculated and are given in Tables XXVIII
and XXIX for bridges composed of beams with a ratio \(a/h = 1.7\) and \(a/h = 1.00\) respectively. Included are the coefficients for the isotropic plate or \(a = 1.0\), which were calculated from the bending moments given by Olsen and Reintzhuber (16).

The same results are plotted in Figs. 12-19. Figs. 12-15 show the coefficients for the load applied in the bridge axis, or \(e' = 1\). The upper graph includes the values for \(a/h = 1.00\), whereas the lower graph shows those for \(a/h = 1.70\). For the case of the isotropic plate \((a = 1.0)\) the ratio \(a/h\) is meaningless.

Figs. 16-19 give the coefficients for \(a/h = 1.7\) with the load applied at the edge \(\xi = -1.00\), and those for \(a/h = 1.00\) with the load at \(\xi = +1.00\).

Each graph includes coefficient values for one \(a\)-value and shows the effect of the various bridge sizes, described by the parameter \(b/\ell\). This representation had to be chosen, because of the unexpectedly small differences in the coefficients for a fixed \(b/\ell\)-ratio and variable \(a\)-values. With the exception of the peak values directly under the load the coefficients are only slightly affected by a variation of \(a\). The effect of this variation is somewhat larger in wide bridges \((b/\ell = 0.500)\) than in narrow and long bridges \((b/\ell = 0.125)\).
Similarly one may conclude that the coefficients calculated for the two a/h ratios differ only by a small amount. Significant differences are obtained for small $\alpha$-values and then only for the points directly under the load.

It follows therefore that the size of the beams and the amount of lateral prestress in a multi-beam bridge affects the coefficients of lateral load distribution only in the points near the applied load.

For the design of multi-beam bridges, it is recalled, that Figs. 12-15 represent also the influence lines for the coefficients of lateral load distribution at the midpoint $\xi = 0$ and Figs. 16-19 those at the edge points of the bridge. The application will be shown in Chapter VII.
VI. COMPARISON OF THEORETICAL AND EXPERIMENTAL RESULTS

1. General

A field test on a multi-beam bridge is described in Reference 2. The following data is taken from this report.

The tested highway bridge with a span of 34 ft. and a width of 27 ft. is composed of 9 prefabricated, pretensioned concrete beams 36 in. wide and 21 in. deep. Placed side by side the beams are connected together by a steel bolt at midspan (without significant prestress) and dry-packed shear keys. The geometrical parameters for this bridge are:

\[
\frac{a}{h} = \frac{36}{21} = 1.7
\]

\[
\frac{b}{l} = \frac{27}{2(34)} = 0.4
\]

Among other tests a single axle load of 47,000 lbs. was concentrated over the width of one beam, first in the bridge axis and secondly, 4 ft. 9 1/2 in. from the edge. The results for these loading positions will be compared with the theoretical values.

2. Deflections:

Fig. 20 shows the measured and theoretical deflection curves for the load placed in the bridge axis. The cross-
section of the bridge and the loading position are schematically indicated in the top of the graph.

The theoretical curves for \( \alpha = 0, 0.1, \) and 0.5 were obtained for \( b/l = 0.4 \) by interpolating between the values for \( b/l = 0.5 \) and \( b/l = 0.375 \). The deflections were computed with a modulus of Elasticity of \( 6.68 \times 10^6 \) psi, the approximate value determined in the field test. Similarly Fig. 21 shows the measured deflections for the second actual loading and the theoretical deflections for the edge loading.

Comparing the experimental and theoretical deflection curves, one cannot completely exclude the possibility that slip may have occurred between the beams in the tested bridge. Thus the main assumption for the theoretical analysis (no slip between the beams) would have been violated. From observation of the curves this fact is more apparent for the load applied in the bridge axis. On the other hand, the maximum measured deflection, for the same loading position is approached by the theoretical one, for \( \alpha = 0 \), to within 12.5%.

With these facts in mind, one may conclude that the results of the theoretical study are in satisfactory agreement with the experimental ones.
3. Coefficients of Lateral Load Distribution:

In Reference 2, the difficulties arising for determining the exact coefficients of lateral load distribution from the results of the field tests are discussed. For the reasons indicated, only the approximate coefficients based on the distribution of the deflections could be obtained.

These approximate coefficients are defined by equations (68) and (70) when $M_x$ is replaced by the deflection $w$ of the same point and $M_{xav}$ by the average deflection $w_{av}$ of the cross-section under consideration. The $S_{xi}^*$ coefficients so determined are given again in Fig. 22 for the load applied in the centerline and in Fig. 23 for the load applied on the edge. The same figures include the approximate coefficients computed from the theoretical deflections as well as the exact $S_{xi}^*$ coefficients obtained from the theoretical moment distribution. The given theoretical results correspond to a value of $\alpha = 0$.

A comparison of the curves plotted in Fig. 22 reveals a satisfactory agreement between the approximate coefficients resulting from the experimental and theoretical investigations. Both of these curves show a similar trend. The curve representing the exact coefficients deviates from the trend mentioned above to a larger extent. The deviation is especially significant for the sections of the curves corresponding to the directly loaded beam.
Similarly in the case of the edge loading (Fig. 23) one observes a close agreement between the two approximate values for the loaded edge-beams. Again, the corresponding exact curve shows a substantial difference.

To permit a better comparison the following table includes the three different $S^*_{xi}$ coefficients for the loaded beam in both loading cases.

<table>
<thead>
<tr>
<th></th>
<th>Approximate Load Distribution</th>
<th>Exact Load Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Experimental</td>
<td>Theoretical</td>
</tr>
<tr>
<td>Center Loading</td>
<td>16.6%</td>
<td>15.2%</td>
</tr>
<tr>
<td>Edge Loading</td>
<td>20.8%</td>
<td>22.6%</td>
</tr>
</tbody>
</table>

First in the case of center loading, according to the approximate coefficient obtained from the field tests, the center-beam would have to be designed for 16.6% of the applied load. However the exact coefficient indicates that this beam carries 19.6% of the load. The beam would therefore be under-designed, whereby approximately one-fifth of the load would have been neglected.

Secondly in the case of edge loading, the experimentally determined portion of the load to be carried by the edge beam was found to be 20.8%. The corresponding exact value
amounts to 28.3% or one-third higher than the approximate value.

The results of these comparisons may be summarized as follows:

1. The approximate distribution coefficients obtained from the theoretical deflections are satisfactorily verified by the results of the field tests.

2. From this agreement one may conclude that the assumptions and methods used in this theoretical study are justified.

3. The approximate distribution coefficients are especially insufficient for the design of the directly loaded beam. A safe and economical design should be based on the exact distribution coefficient.
VII. USE OF THE COEFFICIENTS OF LATERAL LOAD DISTRIBUTION

The following is an example showing how the coefficients of lateral load distribution may be used in the design of a multi-beam bridge.

It is assumed that the bridge has a span of 36 ft. and otherwise is identical with the bridge described in Chapter VI and in Reference 2. The overall width of the bridge is 27 ft. and includes two 8 in. curbs cast to the edge beams. The nominal roadway is therefore 25 ft. 8 in. The geometrical parameters for this bridge are:

\[
\frac{b}{l} = \frac{27}{2(36)} = 0.375
\]

\[
\frac{a}{h} = \frac{36}{21} = 1.7
\]

The live load to be considered in the design of the bridge is given by the AASHO Specifications (1) and consists of H20-S16-44 truck loadings.

For the determination of the design load of the edge beam it is assumed that \(\alpha = 0\) or that no lateral prestress is applied. The coefficients of lateral load distribution for this case can be taken from Fig. 16 or Table XXVIII. They are plotted to the correct scale in Fig. 24 below the schematic cross-section of the bridge.
On the latter, two truck axles are placed side by side with the minimum spacing between, as given by the specifications and such that the sum or the areas $A_1, A_2, A_3, A_4$ in Fig. 24 is a maximum.

Utilizing now the properties of the curve plotted in Fig. 24 as an influence line for the coefficient of lateral load distribution of the edge beam, the portion of the load $P$ carried by this beam is given by equation (70) with $n = 9$, the number of the beams of width $a$, as:

$$s^* - b = \frac{1}{3\pi} \left( \frac{A_1 + A_2 + A_3 + A_4}{n} \right) = 50.3\%$$

This means that the edge beam carries 50.3% of a wheel load. From the assumption that the same ratio may be used for other cross-sections of the bridge it follows, that the edge beam has to be designed for 50.3% of the wheel loads positioned for maximum effect.

Similarly it was found that the maximum portion of the wheel loads carried by the center beam is 50.2% or practically the same as for the edge beams.

The same evaluation of the coefficient of lateral load distribution was made for other $\alpha$-values, as well as for different $b/h$ ratios and also for $a/h = 1.00$. In all cases it was assumed that the width of the bridge would be 27 ft.
The various \( \frac{b}{h} \) ratios would then correspond to the following bridge spans:

\[
\begin{align*}
\frac{b}{h} &= 0.5 & l &= 27 \text{ ft.} \\
\frac{b}{h} &= 0.375 & l &= 36 \text{ ft.} \\
\frac{b}{h} &= 0.25 & l &= 54 \text{ ft.} \\
\frac{b}{h} &= 0.125 & l &= 108 \text{ ft.}
\end{align*}
\]

For the case of \( \frac{a}{h} = 1.00 \) it was assumed that the bridge would be formed by 15 beams, 21 inches wide. The results of this evaluation are summarized in Fig. 25.

The graph on the left-hand side shows the portion of the wheel load carried by the center beam as a function of \( \alpha \) and with \( \frac{b}{h} \) as parameter. The upper group of curves includes the values obtained for a bridge formed by 9 beams, whereas the lower graph shows those for a bridge composed of 15 beams.

In the right-hand graph the portion of the wheel load carried by the edge beam is plotted in a similar manner.

Both figures confirm the interesting result, that for the loading positions under consideration, the portions of the load carried by the center beam and the edge beam are practically the same. They vary only slightly with the degree of lateral prestress, described by the coefficient \( \alpha \).
9 beams, the largest percentage of a wheel load carried by one beam amounts to 53.2%. For the same bridge composed of 15 beams it amounts to 32.9%. The percentages are smaller for longer bridges and approach in the limits the values corresponding to a uniform load distribution, which are 44.5% and 26.7%.

A comparison of these results with the 80% prescribed by the present specification reveals the obvious advantage of an exact investigation.
VIII. CONCLUSIONS AND RECOMMENDATIONS

The main purpose of this theoretical study has been to provide information for the more economical design of multi-beam bridges. The main results are summarized below:

1. There was found to be a satisfactory agreement between the theoretical and the available experimental results. This may be considered as a confirmation of the established assumptions and the applied methods of analyzing this type of structure.

2. The approximate distribution coefficients based on the deflections are especially insufficient for the design of the directly loaded beam. A safe design for wheel and axle-loads must be based on the given exact load distribution coefficients.

3. The size of the beams and the amount of lateral pre-stress affect the coefficient of lateral load distribution only in the points near the applied load.

4. The curves representing the coefficients of lateral load distribution for the midpoint - and edge - loading can be considered as influence lines for the load-portion to be carried by the center - and the edge beam. Their evaluation, similar to those used in the beam -
statics yield the design load for one beam expressed as the percentage of the left or right wheel loads of the standard truck.

5. For a 27 ft. wide bridge with two trucks placed side by side, it is shown that the maximum load carried by a 3 ft. wide beam is 55% of the right or left wheel loads of one truck. This value is almost independent of the amount of lateral prestress and varies only slightly with the span of the bridge. It has to be compared with the 80% recommended by the present specification.

6. This significant reduction of the design load can fully be utilized only if the connections completely prevent the beams from slipping against one another. The condition mentioned is mainly a technological one to be examined by tests, which are strongly recommended.

7. In the meantime, the results of the theoretical study may be used for the design of bridges if the interaction of the beam is guaranteed by sufficiently high lateral prestress.

8. The relatively small influence on the load distribution which is caused by a variation of $\alpha$ shows clearly that
the main advantage of laterally prestressing a multi-beam bridge is an increase in the shear resistance of the connections between the beams.

9. Further experimental investigations on multi-beam bridges by means of field and laboratory test are recommended. The scope of these tests should incorporate the following points:

(a) Investigation of the influence of the magnitude and the locations of the lateral prestressing elements on the coefficient $\alpha$.

(b) Verification of the assumptions, especially the one concerning the relation between $\alpha$ and $\beta$.

(c) Investigation of the behavior of multi-beam bridges under higher loads, determination of the collapse load and the factor of safety.
a) General Case

\[
\frac{wEI}{P L^2} = \frac{1}{2\pi^3 \gamma \sqrt{1 - \frac{\alpha^2}{\beta^2}}} \sum_{m=1}^{\infty} \frac{1}{m^4} \left( e^{-k_2 m \eta} - e^{k_1 m \eta} \right) \sin m \nu \sin m \gamma \sin m \xi
\]  

(1)

\[
\frac{M_x}{P} = \frac{1}{2\pi \gamma \sqrt{1 - \frac{\alpha^2}{\beta^2}}} \sum_{m=1}^{\infty} \frac{1}{m^2} \left( e^{-k_2 m \eta} - e^{k_1 m \eta} \right) \sin m \nu \sin m \gamma \sin m \xi
\]  

(2)

\[
\frac{M_y}{P} = -\frac{\alpha}{2\pi \gamma \sqrt{1 - \frac{\alpha^2}{\beta^2}}} \sum_{m=1}^{\infty} \frac{1}{m^2} \left( k_2 e^{-k_2 m \eta} - k_1 e^{k_1 m \eta} \right) \sin m \nu \sin m \gamma \sin m \xi
\]  

(3)

\[
\frac{Q_{y1}}{P} = -\frac{1}{2\gamma \sqrt{1 - \frac{\alpha^2}{\beta^2}}} \sum_{m=1}^{\infty} \frac{1}{m^2 \beta_m} \left( \beta_m^{1 - \sqrt{1 - \frac{\alpha^2}{\beta^2}}} - \beta_m^{1 + \sqrt{1 - \frac{\alpha^2}{\beta^2}}} \right) k_2 e^{-k_2 m \eta} \sin m \nu \sin m \gamma \sin m \xi
\]  

(4)

\[
\frac{M_{xy}}{P} = \frac{2\beta - \alpha}{2\pi \gamma \sqrt{1 - \frac{\alpha^2}{\beta^2}}} \sum_{m=1}^{\infty} \frac{1}{m^2} \left( e^{-k_2 m \eta} - e^{k_1 m \eta} \right) \sin m \nu \sin m \gamma \cos m \xi
\]  

(5)

b) Articulated Plate

\[
\frac{wEI}{P L^2} = \frac{1}{\pi^3 \gamma \sqrt{2 \beta_0}} \sum_{m=1}^{\infty} \frac{1}{m^4} e^{-k_0 m \eta} \sin m \nu \sin m \gamma \sin m \xi
\]  

(6)

\[
\frac{M_x}{P} = \frac{1}{\pi \gamma \beta_0 \sqrt{2 \beta_0}} \sum_{m=1}^{\infty} \frac{1}{m^2} e^{-k_0 m \eta} \sin m \nu \sin m \gamma \sin m \xi
\]  

(7)

\[
\frac{Q_{y1}}{P} = -\frac{1}{\gamma} \sum_{m=1}^{\infty} \frac{1}{m} e^{-k_0 m \eta} \sin m \nu \sin m \gamma \sin m \xi
\]  

(8)

\[
\frac{M_{xy}}{P} = \frac{1}{\pi \gamma} \sum_{m=1}^{\infty} \frac{1}{m^2} e^{-k_0 m \eta} \sin m \nu \sin m \gamma \cos m \xi
\]  

(9)
\[
\begin{align*}
\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{\beta^2} \psi &= \frac{1}{\beta^2} \sin \mu \sin \eta \sin \phi \left( \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y^2} \right) + \frac{k}{\beta^2} \left[ U_1 \cosh k_1 \psi + U_2 \cosh k_2 \psi - V_1 \sinh k_1 \psi + V_2 \sinh k_2 \psi \right] \\
\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{\beta^2} \psi &= \frac{1}{\beta^2} \sin \mu \sin \eta \sin \phi \left( \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y^2} \right) + \frac{k}{\beta^2} \left[ U_1 \cosh k_1 \psi + U_2 \cosh k_2 \psi - V_1 \sinh k_1 \psi + V_2 \sinh k_2 \psi \right] \\
\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{\beta^2} \psi &= \frac{1}{\beta^2} \sin \mu \sin \eta \sin \phi \left( \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y^2} \right) + \frac{k}{\beta^2} \left[ U_1 \cosh k_1 \psi + U_2 \cosh k_2 \psi - V_1 \sinh k_1 \psi + V_2 \sinh k_2 \psi \right] \\
\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{\beta^2} \psi &= \frac{1}{\beta^2} \sin \mu \sin \eta \sin \phi \left( \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y^2} \right) + \frac{k}{\beta^2} \left[ U_1 \cosh k_1 \psi + U_2 \cosh k_2 \psi - V_1 \sinh k_1 \psi + V_2 \sinh k_2 \psi \right] \\
\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{\beta^2} \psi &= \frac{1}{\beta^2} \sin \mu \sin \eta \sin \phi \left( \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y^2} \right) + \frac{k}{\beta^2} \left[ U_1 \cosh k_1 \psi + U_2 \cosh k_2 \psi - V_1 \sinh k_1 \psi + V_2 \sinh k_2 \psi \right]
\end{align*}
\]

Note the lower sign is valid for \( \epsilon \geq 0 \), the upper sign is valid for \( \epsilon < 0 \).

\[ U_1 = \frac{k}{\kappa_1} \left[ a \sinh \kappa_1 \psi + b \cosh \kappa_1 \psi \right] \quad V_1 = \frac{k}{\kappa_1} \left[ b \cosh \kappa_1 \psi + \frac{k}{\kappa_1} b_2 \sinh \kappa_1 \psi \right] \quad \psi_1 = \frac{k}{\kappa_1} \left[ a \sinh \kappa_1 \psi + b \cosh \kappa_1 \psi \right] \quad \psi_2 = \frac{k}{\kappa_1} \left[ b \cosh \kappa_1 \psi + \frac{k}{\kappa_1} b_2 \sinh \kappa_1 \psi \right] \]

Table II Formulas for General Case

\[ a = \frac{k}{\kappa_1} \left( e^\kappa_1 \psi + e^{-\kappa_1} \psi \right) - \left( e^{-\kappa_1} \psi + e^{\kappa_1} \psi \right) \quad b_1 = \frac{k}{\kappa_1} \left( e^\kappa_1 \psi + e^{-\kappa_1} \psi \right) - \left( e^{-\kappa_1} \psi + e^{\kappa_1} \psi \right) \quad \gamma = \frac{1}{\beta} \left( 1 + i \frac{-\kappa_1}{\beta} \right) - \infty \]

Notation: see Table III & Fig. 10
\[
\frac{wEI}{P_0^2} = \frac{1}{\pi^3 \sqrt{2 \beta_0}} \sum_{m=1}^{\infty} \frac{1}{m^4} \sin \mu \sin \gamma \sin m \xi \left[ e^{-k_{0m}m\eta} + \frac{1}{2} \left( U_0 \cosh k_{0m}m\eta + V_0 \sinh k_{0m}m\eta \right) \right]
\]
(1)

\[
\frac{M_x}{P} = \frac{1}{\pi \gamma \sqrt{2 \beta_0}} \sum_{m=1}^{\infty} \frac{1}{m^3} \sin \mu \sin \gamma \sin m \xi \left[ e^{-k_{0m}m\eta} + \frac{1}{2} \left( U_0 \cosh k_{0m}m\eta + V_0 \sinh k_{0m}m\eta \right) \right]
\]
(2)

\[
\frac{Q_y}{P} = -\frac{1}{\gamma} \sum_{m=1}^{\infty} \frac{1}{m^2} \sin \mu \sin \gamma \sin m \xi \left[ \pm e^{-k_{0m}m\eta} - \frac{1}{2} \left( U_0 \sinh k_{0m}m\eta + V_0 \cosh k_{0m}m\eta \right) \right]
\]
(3)

\[
\frac{M_{xy}}{P} = \frac{1}{\pi \gamma} \sum_{m=1}^{\infty} \frac{1}{m^2} \sin \mu \sin \gamma \cos m \xi \left[ \pm e^{-k_{0m}m\eta} - \frac{1}{2} \left( U_0 \sinh k_{0m}m\eta + V_0 \cosh k_{0m}m\eta \right) \right]
\]
(4)

\[M_y = 0\]

* Note: the upper sign is valid for \( \varepsilon > (1 - \varepsilon') \)

the lower sign is valid for \( \varepsilon < (1 - \varepsilon') \)

with

\[U_0 = \left(e^{-k_2m\lambda\varepsilon'} + e^{-k_2m\lambda\varepsilon''}\right) \frac{1}{\sinh k_2m\lambda}
\]

\[V_0 = \left(e^{-k_2m\lambda\varepsilon'} - e^{-k_2m\lambda\varepsilon''}\right) \frac{1}{\cosh k_2m\lambda}
\]

\[\lambda = \pi \frac{b}{l}\]
\[\eta = \pi \frac{y}{b}\]
\[\varepsilon = \frac{y}{b}\]
\[\eta = \lambda |\varepsilon + \varepsilon' - 1|
\]

\[\nu = \pi \frac{u}{l}\]
\[\gamma = \pi \frac{z}{l}\]
\[\varepsilon = \frac{\pi x}{l}\]
\[\frac{v}{b} = 1 - \varepsilon'
\]

**Table III**  Formulas for Articulated Plate
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Table XXVIII  Coefficients of Lateral Load Distribution  
(a/h = 1.7)
### Table XXIX  Coefficients of Lateral Load Distribution  
\( \alpha/h = 1.0 \)

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**Table XXIX  Coefficients of Lateral Load Distribution  
\( \alpha/h = 1.0 \)**
FIG. 1  Multi-Beam Bridge With Lateral Prestress

FIG. 2  Schematic Cross-Section of a Multi-Beam Bridge

FIG. 3  Corrugated Steel Sheet as Orthotropic Plate
Fig. 4 Gridwork System

Fig. 5 Forces and Moments on Differential Element
Fig. 6 Assumed Relations Between $\alpha$ and $\beta$
FIG. 7 Deformation Produced by Constant Twisting Moments

\[ M_{xy} \]

\[ M_{yx} \]

FIG. 8 Equivalent Corner Loading

\[ A = + (M_{xy} + M_{yx}) \]

\[ A = -(M_{xy} + M_{yx}) \]

\[ A = +(M_{xy} + M_{yx}) \]
FIG. 9 Plate Strip of Infinite Width Under Line Loading

FIG. 10 Bridge of Finite Width

\[ e' + e'' = 2 \]
\[ v = b(1 - e') \]
FIG. 11 Assumption for the Longitudinal Distribution of a Wheel Load
FIG. 12 Distribution of Load Applied at Center for $\alpha = 0$
FIG. 13  Distribution of Load Applied at Center for $\alpha = 0.1$
FIG. 14  Distribution of Load Applied at Center for $\alpha = 0.5$
FIG. 15 Distribution of Load Applied at Center for Isotropic Plate $\alpha = 1.0$
FIG. 16 Distribution of Load Applied at Edge for $\alpha = 0$
FIG. 17  Distribution of Load Applied at Edge for $\alpha = 0.1$
FIG. 18  Distribution of Load Applied at Edge for $\alpha = 0.5$
FIG. 19  Distribution of Load Applied at Edge for Isotropic Plate $\alpha = 1.0$

Coeficient of Lateral Load Distribution

$b/l = 0.50,
0.25,
0.125$
FIG. 20 Comparison Between Experimental and Theoretical Deflections for Load Applied in Bridge Axis
P = 47.700# at Midspan

FIG. 21 Comparison Between Experimental and Theoretical Deflections for Load Applied at Edge
Experimental Results (based on Deflections)

Theoretical Results for $\alpha = 0$:
(based on Deflections)
(based on Moments)

FIG. 22 Comparison Between Experimental and Theoretical Load Distribution Coefficients for Load at Bridge Axis
FIG. 23 Comparison Between Experimental and Theoretical Load Distribution Coefficients for Load at Edge
FIG. 24 Determination of Load Carried by Edge Beam

Influence Line for Load Distribution
Coefficient for Edge Beam
\( \alpha = 0; \quad \frac{b}{L} = 0.375; \quad \frac{a}{h} = 1.70 \)

Areas:
\[ A_1 = \frac{2.73 + 1.60}{2} \cdot 3 = 6.50 \]
\[ A_2 = \frac{1.18 + 0.93}{2} \cdot 3 = 3.16 \]
\[ A_3 = \frac{0.85 + 0.69}{2} \cdot 3 = 2.31 \]
\[ A_4 = \frac{0.58 + 0.51}{2} \cdot 3 = 1.64 \]
\[ S^* = \frac{13.61}{3.9} \cdot 100 = 50.3\% \text{ of } P \]

\( P = \) wheel load
\[ 8'' \quad 2' \quad 6' \quad 4' \quad 6' \quad 3' \]
FIG. 25  Percent of Wheel Loads Carried by Center- and Edge Beams in a 27ft. Wide Bridge of Variable Span
NOTATIONS

Roman Alphabet

\( a \) Width of the beams
\( A_m, A_n \) Constants of integration
\( 2b \) Width of the bridge
\( B_m, B_n, B_m \) Constants of integration
\( 2c \) Length of the applied line load (Fig. 10)
\( C_x, C_y \) Spacing of stringer and beams (Fig. 4)
\( C_{x}, C_{y} \) Torsional rigidity of stringers and beams (Fig. 6)
\( C_x, C_y \) Constants of integration
\( D_x, D_y, D_{m}, D_{n} \) Flexural rigidities of isotropic & orthotropic plates
\( e \) Base of natural logarithm
\( e', e'' \) Eccentricity of applied load (Fig. 10)
\( E \) Modulus of elasticity
\( E_x \) Modulus of elasticity of material parallel to x-direction
\( E_y \) Modulus of elasticity of material parallel to y-direction
\( G \) Shearing modulus
\( h \) Depth of plate or beams
\( H \) Coefficient in equation 1
\( i \) Beam number i or n beams
\( i_{m}, i_{4m} \) Functions defined by equation 58
\( i_{m}, i_{4m} \) Functions defined by equation 62
\( I \) Moment of inertia per unit width of plate
\( I_{x} \) Moment of inertia per unit width for a section of the orthotropic plate perpendicular to the x-direction and with respect to the y-axis
\( I_{y} \) Moment of inertia per unit width for a section of the orthotropic plate perpendicular to the y-direction and with respect to the x-axis.
\( j_{m}, j_{4m} \) Functions defined by equation 58
\( j_{m}, j_{4m} \) Functions defined by equation 62
\( k_0, k_1, k_2 \) Modified roots of characteristic equation, defined by equations 37,46.
\( K \) Cross-section factor for torsional rigidity of rectangular beams.
\( l \) Span of bridge
\( m_{th} \) Term of Fourier-series
\( M_x, M_y \) Bending moments per unit length of sections of a plate perpendicular to x- and y-axes, respectively
NOTATIONS (continued)

\( M_{xy}, M_{yx} \)  
Twisting moments per unit length of sections of a plate perpendicular to x- and y-axes, respectively.

\( n \)  
Number of beams in a multi-beam bridge

\( p(x,y) \)  
Intensity of the load at the point \((x,y)\) parallel to \(z\)

\( P \)  
Total load uniformly distributed over the length \(2c\)

\( Q_x, Q_y \)  
Shear forces parallel to \(z\)-axis per unit length of sections of a plate perpendicular to x- and y-axes, respectively.

\( r \)  
Roots of characteristic equations given by equation 36

\( \alpha \)  
Coefficient of lateral load distribution defined by equation 68

\( \alpha_{xi}, \alpha_{y4m} \)  
Percent of load carried by beam \(i\)

\( \alpha_{1m}, \alpha_{4m} \)  
Functions defined by equation 54

\( \alpha_{2m}, \alpha_{3m} \)  
Functions defined by equation 63

\( u, v \)  
Coordinates of applied line load (Fig. 10)

\( w \)  
Vertical deflection of plate

\( \omega_0 \)  
Solution of homogenous differential equation

\( \omega_1 \)  
Solution of infinitely wide plate strip

\( x, y, z \)  
Rectangular coordinates

\( Y_m \)  
Function of \(y\), equation 33

\( \gamma \)  
Ratio of bending stiffnesses (equation 3)

\( \beta \)  
Coefficient of torsional rigidity (equation 2)

\( \beta_x, \beta_y \)  
Constants in equation 14

\( \beta_o \)  
Coefficient of torsional rigidity for \(\alpha = 0\)

\( \alpha \)  
Abbreviation defined by equation 32

\( \epsilon \)  
Abbreviation defined by equation 50

\( \eta, \iota \)  
Abbreviation defined by equation 50

\( \lambda \)  
Abbreviation defined by equation 50
NOTATIONS (continued)

μ  Coefficient of static friction
ν  Poisson's ratio. Abbreviation defined by equation 32.
ξ  Abbreviation defined by equation 32
φ  Parameter of torsion
σ  Normal stress
τ  Shear stress
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