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RESIDUAL STRESS AND THE COMPRESSIVE STRENGTH OF STEEL

Progress Report

THE INFLUENCE OF RESIDUAL STRESS ON THE INSTABILITY
OF COLUMNS

A Dissertation

by

Alfons Wilhelm Huber

This work has been carried out as a part of an investigation sponsored jointly by the Column Research Council, the Pennsylvania Department of Highways and Bureau of Public Roads, and the National Science Foundation.

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1. ABSTRACT

The problem of the influence of residual stress on column strength is divided into two parts. First the formation and distribution of residual stresses in beams is studied and then their influence upon axially and eccentrically loaded columns is investigated.

General formulas for residual stresses in beams are developed for both linear and non-linear stress-strain relationships. The stresses in a plate during the cooling process are calculated and typical examples of cooling stresses are given for various cross-sections. The distribution of residual stresses near the end, in particular the length from the free end to the section where the full value of stress is obtained, was derived from considerations of a plane stress problem. The effect of plastic deformations and partial relaxations is also studied.

The general theory of axially loaded columns with symmetrical residual stress patterns is developed and the effect of different variables is investigated. Axially and eccentrically loaded columns containing residual stress patterns with at least one axis of symmetry are treated by both exact and approximate methods.

In order to correlate the theoretical derivations with tests, residual stresses were measured in a number of beams and a number of axially loaded columns were tested. Experimental verification was thus established for the case of axially loaded columns.
2. INTRODUCTION

Nearly every structural member contains residual stresses. By residual stresses are meant those stresses which are present in a member in the absence of external forces. The resultant of internal forces and moments of such members is therefore equal to zero. Some of the more important types of residual stresses are those due to uneven cooling of shapes immediately after hot rolling, due to plastic deformations and due to welding. Then it is desirable to know their influence on structural behavior.

The influence of residual stresses on bending members is usually negligible; it merely results in slightly increased deflections without impairing the carrying capacity of the bending member.\(1, 4\)*

In the case of columns one should expect a rather significant influence, since an increase in deflections during the loading process will lower the strength of a column appreciably.

Therefore, it is desired to find, first, the distribution and magnitude of residual stresses in column members and, secondly, to find a solution to the column problem.

The theoretical results of the study of residual stresses and column instability can be applied to an arbitrary stress-strain relationship. An ideal-plastic material was considered in the majority of specific examples. A parabolic stress-strain curve was used for

* Number in parenthesis refer to the List of References on page 111.
some examples of residual stress determination and axial column instability. All tests were made on A-7 structural steel which resembles closely an ideal-plastic material.

The literature on residual stress measurements and techniques is considerable (Reference 2 gives a partial bibliography of the subject, see also References 3 and 4), but comparatively little has been done to predict such stresses. Timoshenko(5), Melan and Parkus(6) give the solution of the temperature stress problem according to the theory of elasticity. A solution of the problem of the stress distribution in a very short H-beam due to non-uniform cooling was presented by Baud and Inan(7) with the assumption that a certain temperature difference gives rise to the residual stresses. The problem of residual stress distribution near the ends of beams is similar to the shear lag problem of panels with stiffeners.(8) Of particular interest is the length at which the influence of a set of stresses in equilibrium is negligible. St. Venant's principle is only applicable to members of a solid cross-section.

Comparatively little work has been done on the influence of residual stresses on column strength. Earlier experimental work on the influence of residual longitudinal welding stresses showed no lowering of the carrying capacity of short axially loaded columns(9). Because of the shortness of the tested columns and the favorable residual stress pattern no appreciable influence could be expected in this particular case. Important contributions on the strength of axially loaded columns with a symmetrical residual stress distribution were made by Osgood(10), and by Yang, Beedle and Johnston(1). Osgood gave the general derivation
while the latter authors had considered the particular case of steel columns of rectangular shape. Horne\(^{(11)}\) investigated the axial column problem by considering the modified stress-strain relationship of the material when residual stresses are present and by applying the tangent modulus and reduced modulus theories. Axial tests of annealed and as-delivered steel columns have shown that the reduction in carrying capacity is only to a small extent due to accidental eccentricities or initial deflections. A major factor are residual stresses present in as-delivered columns.\(^{(12)}\).

Ketter, Kaminsky and Beedle\(^{(13)}\) have studied the influence of residual stress on the eccentrically loaded column. They have shown how the moment-curvature relationships of columns are influenced by residual stresses and how a solution can be obtained according to the classical inelastic column theory\(^{(14)}\).

In this dissertation, the problem is divided into two parts. First the distribution and formation of residual stresses in beams are studied. Then the influence of residual stresses upon the behavior of axially and eccentrically loaded columns is investigated.

Throughout the dissertation the common assumption is made that bending strains have a linear distribution and that each fiber of a member follows the same stress-strain law.

On the basis of the above assumptions simple solutions for residual stresses in the interior of long beams of any material are obtained. The conditions near the free ends of beams are investigated according to the theory of elasticity. The effects of cooling, plastic
deformations and partial relaxations are also studied.

An axially loaded column of any material with symmetrical residual stresses is investigated by expansion of the theory referred to above. The eccentrically and axially loaded column of ideal-plastic material with a residual stress distribution of only one axis of symmetry is investigated by the classical method by taking proper account of the influence of residual stresses on the variables. An approximate solution is presented which permits the rapid investigation of the influence of different residual stress conditions for an ideal elastic-plastic material. This solution reduces to the one given by Jezek for no residual stresses.(14)

In the experimental part of the investigation only axially loaded columns were tested. Residual stresses were measured on a large number of WF shapes. The stress-strain behavior was measured on column specimens of short length (cross-section tests). Thereby the experimental verification of the theory was established for axially loaded columns.

In summary, the solution to the following problems was obtained in this dissertation:

1. The development of a generalized theory of residual stresses in long beams and the application to specific structural shapes.
2. The magnitude and distribution of cooling residuals in a plate and the residual stress-time relationship.
3. The theoretical and experimental behavior of axially loaded columns containing symmetrical (cooling) residual stresses.
4. The critical stresses of axially and eccentrically loaded steel columns containing residual stresses formed by cold bending.

5. The development of approximate formulas for the ultimate strengths of beam columns of rectangular shape containing residual stresses.

The theoretical study of residual stresses is taken up in the next chapters, which is followed by the study of the instability of axially and eccentrically loaded columns. The final chapters deal with the experimental investigation and remarks on theory and tests.
3. GENERAL DERIVATION OF RESIDUAL STRESSES IN BEAMS

3.1 Linear Stress-Strain Relation

If the fibers of a long beam are subjected to a non-linear strain distribution, an internal stress system is necessary to satisfy both equilibrium and compatibility. These strains, \( \varepsilon \), could be due to such causes as uneven temperature distribution, deformations beyond the elastic limit of the material, also by prestressing of fibers as in a prestressed concrete beam. A solution will be sought only for the interior of the beam.

The assumptions for the derivation are the same as used in the strength of materials i.e. the material is homogeneous and isotropic, the fibers follow the same stress-strain curve as the material exhibits in simple tension and compression, previously plane cross-sections remain plane after deformation of the beam, and the stress-strain relationship is linear and identical in tension and compression.

A beam of arbitrary cross-section, as shown in Figure 1a, is assumed to have a non-linear distribution of internal strains, \( \varepsilon \), which are constant along each individual fiber (Figure 1b). These strains correspond to stresses, \( \sigma \varepsilon \). At the ends of the beam, however, boundary conditions require zero magnitude of stresses. To remove the violation of the boundary conditions, stresses of opposite sign have to be superposed, which is a legitimate procedure in theory of elasticity. According to St. Venant's principle these superimposed boundary stresses will become linearly distributed a sufficient distance away from the ends.

For the residual stress in the interior of the beam we may then write the following equation:
\[ \sigma_r = E \left[ \varepsilon - (a + bx + cy) \right] \]  

(3.1)

where the second term is the expression for a plane strain distribution.

Equilibrium requires that the following set of equations be satisfied:

\[ \int_A \sigma_r dA = 0 \]
\[ \int_A \sigma_r x dA = 0 \]
\[ \int_A \sigma_r y dA = 0 \]

(3.2)

With the aid of the equation (3.2) we can now find the three unknowns \( a, b \) and \( c \) in equation (3.1).

\[ \int_A \sigma_r dA \int_A E \varepsilon (a + bx + cy) dA = 0 \]

\[ \int_A E \varepsilon dA - a \int_A E \varepsilon dA - b \int_A E x dA - c \int_A E y dA = 0 \]

If we choose the coordinate system such that the last two integrals become zero (for \( E \) constant for all fibers the origin would be identical with the center of gravity of the section), \( a \) can be determined directly from the expression

\[ a = \frac{\int_A E \varepsilon dA}{\int_A E dA} \]

and \( O = \int_A E x dA = \int_A E y dA \)

From the second of equations (3.2)

\[ \int_A \sigma_r x dA \int_A E \varepsilon (a + bx + cy) dA = 0 \]
\[ \int_A E \varepsilon x dA - a \int_A E x dA - b \int_A E x^2 dA - c \int_A E y x dA = 0 \]

(3.3)
If \( E \) is constant for all fibers, the last two integrals in equation (3.3) express simply the moment of inertia multiplied by \( E \).

Using the third of equations (3.2), another equation similar to equation (3.3) is obtained. Writing

\[
J_x = \int_{A} E y^2 dA, \quad J_y = \int_{A} E x^2 dA, \quad J_{xy} = \int_{A} E xy dA
\]

the following simultaneous equations can be solved for \( b \) and \( c \).

\[
\int_{A} E e x dA - b J_y - c J_{xy} = 0
\]

\[
\int_{A} E e y dA - b J_{xy} - c J_x = 0
\]

Finally substituting \( a, b, \) and \( c \) back into equation (3.1)

\[
\sigma_T = E \left[ E - \frac{\int_{A} E e dA}{\int_{A} E dA} - \frac{(x J_x - y J_{xy}) \int_{A} E x dA + (y J_y - x J_{xy}) \int_{A} E y dA}{J_x J_y - J_{xy}^2} \right]
\]

(3.4)

Equation (3.4) is very similar to the general beam bending equation. For axial symmetry of the section and of the strain distribution, \( \epsilon \), only the first two terms of equation (3.4) remain, which means that no rotation of the strain plane occurs.

If \( \epsilon \) is linear of the form \( \epsilon = d + ex + fy \) equation (3.1) takes the form

\[
\sigma_T = E \left[ (d-a) + (e-b)x + (f-c)y \right]
\]

substituting into equations (3.2) \( a, b, \) and \( c \) can be determined.

\[
a = d, \quad b = e, \quad c = f
\]
This result is important because it shows that no residual stresses will result from a linear distribution of strains. There must be non-linearity of applied strain in order that residual stresses may form.

An important engineering material, steel, has a stress-strain relation that agrees well with the assumptions made. However, beyond the yield point the strains increase while the stress stays constant (Figure 2). Unloading follows again a linear law at the same modulus of elasticity as in loading.

Next the strain, ε, in equation (3.1) will be assumed to be such that the yield point was exceeded. Then equation (3.1) takes the form

\[ \sigma_r = E \left[ \varepsilon - (a' + bx + c'y) \right] \quad \varepsilon \leq \sigma_y \]
\[ \sigma_r = [\sigma_y - E(a' + bx + c'y)] \quad \varepsilon > \sigma_y \]  (3.5)

Equations (3.5) together with the equilibrium equations (3.2) will be sufficient to solve the problem. A closed general solution cannot be given without specifying the strain, ε, and the shape of the beam.

If ε is assumed to be a linear bending strain, ε to be constant for the elastic region, and the x and y-axes to be axes of symmetry for the section, a relatively simple solution results (Figure 3). Considering the applied stress to be made up of an elastic stress, \( E\varepsilon \), minus the stress that is in excess of the yield point, \( E\varepsilon' \), we can rewrite the equation (3.4) as follows:

\[ \sigma_r = E\varepsilon - \frac{y}{I_x} \left( \int_{A_1} E\varepsilon y dA - \int_{A_2} E\varepsilon' y dA \right) \quad y \leq \gamma_1 \]  (3.6)
\[ \sigma_r = \sigma_y - \frac{y}{I_x} \left( \int_{A_1} E\varepsilon y dA - \int_{A_2} E\varepsilon' y dA \right) \quad y > \gamma_1 \]
where the first of equations (3.6) applies in the area, $A_1$, and the second in the area, $A_2$, of the cross-section. The fibers in the area, $A_2$, have been deformed permanently to the strain, $\varepsilon'$. Equation (3.6) can be simplified further, when it is recognized that:

$$E\varepsilon = \frac{1}{Ix} \int E\varepsilon' y dA$$

Since the left side represents elastic bending stress which is equal to $E\varepsilon$. Then

$$\sigma_r = \frac{1}{Ix} \int E\varepsilon' y dA$$

$$\sigma_r = -E\varepsilon' + \frac{1}{Ix} \int E\varepsilon' y dA$$

Equation (3.7) indicates that only the permanent strain, $\varepsilon'$, causes residual stress. This result is important and will be useful in later developments.

3.2 Non-Linear Stress-Strain Relation

In the chapter on the axial instability of columns a general stress-strain law will be considered. Therefore residual stress equations will be derived in this section which are also based on a general stress-strain law. The basic assumptions are the same as before. Loading follows a non-linear law, but unloading is again assumed to be under a constant slope, $E$, a behavior typical for materials in engineering use. (Figure 29a) The discussion will be limited to beams where the $y$-axis is an axis of symmetry (Figure 4). Similar to the previous development we can write the residual stresses as the summation of applied stress, $\sigma(\varepsilon)$, axial stress, $\sigma_a$, and bending stress, $\sigma_b$.

$$\sigma_r = \sigma(\varepsilon) - \sigma_a - \sigma_b$$

(3.8)
where the axial stress is given by:

\[ \sigma_a = \frac{1}{A} \int_A \sigma(\varepsilon) \, dA \]

and the bending stress must satisfy the equation:

\[ \int_A \sigma(\varepsilon) y \, dA = \int_A \sigma_b y \, dA \]

In accordance with the assumption of linear bending strain distribution the axial and bending strains are linear.

In the special case of beam bending followed by unloading equation (3.8) can be written in a more specific form because \( \sigma_b \) will be linear, and

\[ \sigma_r = \sigma(\varepsilon) - \frac{1}{A} \int_A \sigma(\varepsilon) \, dA - \frac{V}{I_x} \int_A \sigma(\varepsilon) y \, dA \]  (3.9)

The problem that was solved by equation (3.7) could also be solved by equation (3.9) since the stress-strain relation of Figure 2 is non-linear for strains greater than \( \varepsilon_y \).

Application of the equations developed in this chapter will be made in the following chapters.
4. FORMATION OF COOLING RESIDUAL STRESSES

Hot rolled steel beams and columns are widely used in many types of engineering structures. It is also known that such members contain residual stresses. Some of these stresses are due to non-uniform cooling during the production process. Others are due to fabrication operations, such as the stresses due to cold bending which will be treated in Chapter 6.

The behavior of a plate during cooling will now be analyzed. This will be followed by a presentation of typical examples of cooling residual stresses.

4.1 Residual Stresses Formed During the Cooling Process

As a long plate (cross-section shown in Figure 5) is cooled from a uniform "high" temperature, \( T_0 \), residual stresses will be set-up due to the temperature gradient in the material. These stresses will be a function of time. Of particular interest is the distribution of residual stress after the cooling of the plate to a uniform temperature.

At first the temperature distribution must be known for any time, \( t \). The assumption is made that the thermal diffusivity, \( a^2 \), and thermal conductivity, \( k \), are independent of temperature. It is further assumed that Newton's law of cooling is applicable i.e. the rate of heat emission is proportional to the difference in temperature between the surface and the surroundings\(^6\). For convenience the latter can be taken as zero, since only relative temperatures are required.

The partial differential equation of non-stationary heat flow is:

\[
\frac{\partial T}{\partial t} = \alpha \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)
\]

(4.1)
The boundary conditions are

\[ \frac{\partial T}{\partial n} = -\frac{c}{\kappa} T_s = -h T_s \]  

(4.2)

where \( c \) and \( h \) are constants, and \( n \) is a direction normal to the surface. The subscript \( s \) refers to the surface of the plate. The rate of heat emission per unit area is given by

\[ \frac{\partial q}{\partial t} = c T_s = -\kappa \left( \frac{\partial T}{\partial n} \right)_s \]

For one-dimensional heat flow equation (4.1) can be solved readily(17). For two-dimensional flow a product solution will be used, which is made up of the solutions of one-dimensional flow for the \( x \) and for the \( y \) direction. We can then write

\[ T(x, y, t) = A T_1(x, t) T_2(y, t) \]  

(4.3)

where \( A \) is a constant.

Taking the partial differential of equation (4.3) on the one hand and on the other, substituting back into equation (4.1) we obtain:

\[ \frac{\partial T_1}{\partial t} T_2 + T_1 \frac{\partial T_2}{\partial t} = \alpha^2 \left( \frac{\partial^2 T_1}{\partial y^2} T_2 + T_1 \frac{\partial^2 T_2}{\partial y^2} \right) \]  

(4.4)

Since \( \frac{\partial T_1}{\partial t} = \alpha^2 \frac{\partial T_1}{\partial y^2} \), and \( \frac{\partial T_2}{\partial t} = \alpha^2 \frac{\partial T_2}{\partial y^2} \), equation (4.4) is identically satisfied and equation (4.3) is a solution.

From equation (4.3) for \( t = 0 \): \( T_0 = \alpha T_0^2 \) and \( A = 1/T_0 \)

A solution for one-dimensional flow is given by the series:

\[ T_1 = \sum_{n=1}^{\infty} A_n \cos \omega_n x e^{-\omega_n^2 \alpha^2 t} \]
Then equation (4.3) takes the form:

$$T(y, y, t) = \frac{1}{T_0} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \cos \omega_n x \cos \omega_m y e^{(\omega_n^2 + \omega_m^2) \alpha^2 t} \quad (4.5)$$

The boundary conditions are:

$$\frac{\partial}{\partial x} T(b, y, t) = -h T(b, y, t) \quad (4.6)$$
$$\frac{\partial}{\partial y} T(x, c, t) = -h T(x, c, t)$$

Substituting equation (4.5) into equation (4.6) gives:

$$b \omega_n \tan \omega_n b = bh \quad (4.7)$$
$$c \omega_m \tan \omega_m c = ch$$

$\omega_n$ and $\omega_m$ are determined by the roots of the transcendental equation (4.7).

The coefficients $A_{nm}$ in equation (4.5) are determined from the initial condition:

$$T(x, y, 0) = T_0 = \frac{1}{T_0} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \cos \omega_n x \cos \omega_m y \quad (4.8)$$

Now that the temperature distribution as a function of time has been determined (equation 4.5) the magnitude and distribution of residual stresses due to cooling can be investigated. Due to the temperature differences in the plate non-linear strains will be produced which may be computed from the expression:

$$\varepsilon = \frac{1}{T_0} \int \alpha T \, dT \quad (4.9)$$
Of course, the material properties, such as the modulus of elasticity, E, the coefficient of linear expansion, $\alpha_t$, and the yield point, $\sigma_y$, must be known as functions of temperature. It is also assumed that stresses and strains are proportional up to the yield point and that from here on the material is ideally plastic. For the particular plate that is analyzed below, the temperature relationships of the material properties were based on handbook values (Figures 6b, 6c and 6d). Equation (4.9) is readily evaluated for the linear relationship of $\alpha_t$ shown in Figure 6c. From the temperature distribution, as evaluated by equation (4.5) and plotted for a particular plate in Figure 6e, the strains, $\varepsilon$, can be computed from equation (4.9). Figure 6f shows a plot of the strain distribution over one half of the plate width. The basic equation (3.4) for residual stresses was derived in section 3.1. This equation may now be used to calculate the residual stresses in the plate for any desired time. At first the strain distribution as shown in Figure 6f is expressed with the aid of Figures 6g and 6h in terms of stresses as shown in Figure 6i. Application of equation (3.4) will give a residual stress distribution as shown in Figure 6i where the stress $\sigma$ may exceed the yield value at some points. In this case a correction must be made which will result in a small shift of the base line. The corresponding equation follows directly from equation (3.4). The procedure is the same as the one used for the derivation of equation (3.7). The stress exceeding the yield point is designated by $\Delta \sigma$ and the residual stress is then given by:

$$\sigma_r = (\sigma - \Delta \sigma) - E(x, y, t) \int \frac{\int (\sigma - \Delta \sigma) dA}{\int E dA}$$

(4.10)
The actual evaluation of the integrals for the example was made by a graphical integration of the stresses in Figure 6h resulting in a residual stress distribution as for example shown in Figure 6k.

Because \( \Delta \sigma \) is not known exactly a trial and error procedure is involved, however, it will hardly be necessary to go further than the first approximation. After cooling the maximum values of \( \Delta \sigma \) will determine the residual stress distribution. Then

\[
\sigma_r = -\Delta \sigma_{\text{max}} + \frac{1}{A} \int \Delta \sigma_{\text{max}} dA
\]

(4.11)

We are now in a position to calculate the residual stress distributions in a particular steel plate at different times during cooling. The dimensions of the plate and the assumed heat flow constants are given in Figure 6a. The initial temperature was assumed to be 1300°F. The assumed material properties are given in Figures 6b, 6c and 6d. We can summarize the required steps in the calculations as follows:

1. Evaluate equation (4.5) giving the temperature distribution at any point in the plate. (Figure 6e) Note that the mean temperature through the thickness was used in order to simplify calculations.

2. Evaluate equation (4.9) giving the strain distribution in the plate (Figure 6f).

3. From Figures 6b and 6e the auxiliary curves of Figure 6g are obtained giving the values of the modulus of elasticity in the plate. These values will be necessary for the evaluation of equation (4.10).
4. The strain distribution of Figure 6f is converted with the aid of Figure 6g to a stress distribution as shown in Figure 6h.

5. For each desired time the stress distribution of Figure 6h is used to find the residual stress distribution according to equation (4.10). Graphical integration was used in this example and the resultant residual stress distributions are shown on Figure 7 for one half of the plate width.

The magnitude of residual stress after cooling is quite small (Figure 7, t = ∞). Because of the various assumed values of material properties and the assumptions involved in the heat-flow analysis it cannot be expected that the calculated stresses will equal exactly actual stresses. However, the general process of the formation of residual stresses during cooling was exemplified and the numerical results should allow an estimation of actual conditions.

In the following section the cooling stresses in other shapes will be investigated by proceeding from assumed strain distributions.

4.2 Typical Examples of Residual Stresses

In the previous section the residual stresses in a plate were investigated during cooling to room temperature. A similar analysis of structural shapes would be much more complex. However, it was shown that for the final residual stress distribution only the strains greater than the strain corresponding to yield point is of importance. In this section the effect of various assumed strains on the residual stress distribution will be investigated. It must be kept in mind, however, that the strain distribution is also a function of cross-sectional dimensions and this makes a direct comparison between beams of the same
Stresses have been computed by equation (3.4). The assumed strains and the results of the calculations will now be described.

(a) H-SECTION:

At first a symmetrical strain distribution of parabolic shape with a maximum value of $\varepsilon_0$ as shown in Figure 8a is assumed. The residual stress distribution obtained from equation (3.4) is shown in Figure 8b. In Figure 9 the dimensionless ratio $\frac{\sigma_r}{\varepsilon_0 E}$ is shown as a function of depth-width and flange-web thickness ratios. Also shown on Figure 9 are test results on sections which had a pattern similar to that shown in Figure 8a. As expected the correlation is only approximate, but the general trend is in agreement with calculations. It also must be kept in mind that even the stresses measured in the same section but coming from different heats are often dissimilar.

Now let us consider some other symmetrical strain distributions and their effect on the residuals. Different assumptions are shown in Figures 10, 11, 12 and 13 together with the corresponding residual stress ratios. It would be of interest to compare residual stresses resulting from these assumed strain distributions with actual tests. Suppose residual strain measurements had been made. Then, dividing equation (3.4) by $E$ and omitting the bending term:

$$\varepsilon_r = \varepsilon - \frac{1}{A} \int \varepsilon dA$$
where $\varepsilon_r$ has been measured. The strain is

$$\varepsilon = \varepsilon_r + C$$  \hspace{1cm} (4.12)

where $C$ is an arbitrary constant, since

$$\varepsilon_r = (\varepsilon_r + C) - \frac{1}{A} \int_A (\varepsilon_r + C) \, dA = \varepsilon_r$$

This means then that the strain distribution, $\varepsilon$, must have the same shape as the measured strains and the magnitude differs only by a constant.

Let us now consider some unsymmetrical strain patterns such as those shown in Figures 14, 15 and 16. The resulting residual stresses are also shown in these figures. Of interest is the pattern shown in Figure 14 because the residual stresses are symmetric although the strains were unsymmetric. However, the initially straight beam would take on a permanent curvature after introduction of the residual stresses. (The beams in Figures 15 and 16 would also have a permanent curvature). This curvature is obtained from the bending term of equation (3.4):

$$\phi = \frac{1}{I_y} \int_A \varepsilon x \, dA = \frac{\varepsilon \phi}{b}$$  \hspace{1cm} (4.13)

(b) CHANNEL:

Two distributions of strain, $\varepsilon$, are assumed, and the resulting residual stress distributions are shown in Figures 17 and 18. The channels will also be deformed and, in practice, would have to be straightened.
(c) ANGLE:

Two different leg lengths are assumed and the results are obtained as before from equation (3.4) (Figures 19 and 20). The angles would also have a permanent curvature.

While this indirect method of residual stress analysis gives some qualitative results, we must depend on actual measurements to obtain the residual stress distribution in structural members.
5. DISTRIBUTION OF RESIDUAL STRESSES AT THE ENDS OF BEAMS

In Chapter 3 a solution to the residual stress problem was obtained by a similar method as used in the elementary beam theory. Near the ends of the beam this solution becomes invalid. The theory of elasticity must be used to describe the stresses.

Considering the end of an H-beam (Figure 21), the web will be treated as a two-dimensional plate problem. The stresses can be found from an Airy stress function which satisfies the biharmonic equation

$$\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \quad (5.1)$$

The boundary conditions for the web are:

(a) for \( x = 0 \): \( \sigma_x = \sigma(y) \)
(b) \( \tau_{xy} = 0 \)
(c) for \( y = \pm c \): \( \sigma_y = 0 \)
(d) for \( x = \infty \): stresses must be finite

The stress function is assumed to be a product of a function of \( x \) and a function of \( y \). In order to be able to express \( \sigma(y) \) at \( x = 0 \), the stress function will be assumed as follows:

$$\phi = \sum_{n=0}^{\infty} \lambda_n \cos \alpha_n y \quad (5.2)$$

Then

$$\sigma_{xn} = \frac{\partial^2 \phi}{\partial y^2} = -\alpha_n \lambda_n \cos \alpha_n y$$
$$\sigma_{yn} = \frac{\partial^2 \phi}{\partial x^2} = \lambda_n \cos \alpha_n y$$
$$\tau_{xyn} = -\frac{\partial^2 \phi}{\partial x \partial y} = \alpha_n \lambda_n \sin \alpha_n y \quad (5.3)$$
where \( X_n \) is found by substituting the stress function into the bi-
harmonic equation (equation 5.1). \( X_n \) is then determined by the following
ordinary differential equation

\[
X_n^{IV} - 2 \alpha_n^2 X_n'' + \alpha_n^4 X_n = 0
\]  
(5.4)

The solution of this equation gives under consideration of the boundary
condition (d):

\[
X_n = (c_{1n} + c_{2n} x) e^{-\alpha_n x}
\]

Writing \( c_{1n} \alpha_n^2 = D_n \), the coefficients \( D_n \) are determined from:

\[
\sigma_x \bigg|_{x=0} = \sigma(y) = - \sum_{n=0}^{\infty} D_n \cos \alpha_n y
\]  
(5.5)

c_{2n} is determined from the boundary condition (b)

\[
\tau_{xy} \bigg|_{x=0} = 0 = \sum_{n=0}^{\infty} (c_{2n} - c_{1n} \alpha_n) \alpha_n \sin \alpha_n y
\]  
(5.6)

This is satisfied for \( c_{2n} = c_{1n} \alpha_n = D_n / \alpha_n \). To fulfill
boundary condition (c) at \( y = \pm c \), \( \cos \alpha_n c \) must be zero. This would be
satisfied if \( \alpha_n c = n \pi / 2 \), where \( n = 1, 3, 5, \ldots \). Finally we can write:

\[
\sigma_x = - \sum_{n=0}^{\infty} D_n (1 + \frac{n\pi x}{2c}) e^{-\frac{n\pi x}{2c}} \cos \frac{n\pi y}{2c}
\]

\[
\sigma_y = - \sum_{n=0}^{\infty} D_n (1 - \frac{n\pi x}{2c}) e^{-\frac{n\pi x}{2c}} \cos \frac{n\pi y}{2c}
\]  
(5.7)

\[
\tau_{xy} = - \sum_{n=1}^{\infty} D_n \frac{n\pi x}{2c} e^{-\frac{n\pi x}{2c}} \sin \frac{n\pi y}{2c}
\]
Let us consider an example for further discussion of the problem. Consider a beam web as shown in Figure 22. Taking

\[ \sigma(y) = - \sigma_0 \left(1 - \cos \frac{\pi y}{2c} \right) \]

we can write this expression as a Fourier series of period 4c:

\[ \sigma(y) = - \sigma_0 \left[ \frac{1}{2} + \frac{2}{\pi} - 1 \right] \cos \frac{\pi y}{2c} - \frac{2}{3\pi} \cos \frac{3\pi y}{2c} + \ldots \]  

(5.8)

By comparison of coefficients of \( \sigma_x \)

\[ \left. \sigma_x \right|_{x = 0} = \sigma(y) \]

we obtain:

\[ D_0 = \frac{1}{2} \sigma_0, \quad D_1 = \left(\frac{2}{\pi} - 1\right), \quad D_3 = -\frac{2}{3\pi} \sigma_0, \ldots \quad D_2 = D_4 = \ldots = 0 \]

and by substitution into equations (5.7) the final distribution may be obtained as shown in Figure 22.

At \( y = \pm c \), \( \sigma_y = -1/2 \sigma_0 \). This stress may be removed by applying \( \sigma_y = 1/2 \sigma_0 \). The other stresses will not be changed by this.

The normal stresses rapidly approach uniformity. The distribution of the shearing stresses at the edges \( y = \pm c \) (junction of the web with the flanges) indicate a required length of about \( x = 4c \) to obtain a uniaxial stress condition.

Using the concept of superposition, the residual stress distribution is obtained by removal of the boundary violation (see Chapter 3). Thereby \( \sigma(y) \) is removed from the free end and in the interior we obtain the residual stresses by the superposition of the constant stress with \( \sigma(y) \).
Thus an approximate solution has been found for the distance from a free end where equation (3.4) becomes valid.

A comparison with solutions of the shear-lag problem, where \( s(y) \) is introduced as a stress in the flanges, shows that the required distance for zero shear is about the same. Therefore it may be assumed that for practically all residual stress distributions the required distance is the same.
6. RESIDUAL STRESSES DUE TO PLASTIC DEFORMATION

In this chapter residual stresses due to mechanical deformations (for instance gagging) will be investigated in steel beams. In Chapter 4 it was stated that residual stresses are the result of permanent deformations of parts of a beam section. Consider a rectangular steel beam initially without residual stress (Figure 23). Bending which results in yielding of fibers will leave residuals in the beam (5).

Since the strain distribution is linear, direct use may be made of equation (3.7). We can therefore write:

\[ \varepsilon' = \gamma \left( \frac{y}{\gamma_0} - 1 \right) \]
\[ \gamma \leq \gamma_0 \]

and

\[ \sigma_r = \frac{24y}{d^2} \int_{\gamma_0}^{\gamma} \sigma_y \left( \frac{y}{\gamma_0} - 1 \right) y \, dy \]
\[ \gamma \leq \gamma_0 \]

or after evaluation of the integral:

\[ \sigma_y = \sigma_y \left( \frac{y}{\gamma_0} - 3 \frac{y}{d} + 4 \frac{y^2}{d^2} \right) \]
\[ \gamma \leq \gamma_0 \]  \hspace{1cm} (6.1)

and

\[ \sigma_r = \sigma_y \left( \frac{1}{d} - 3 \frac{y}{d} + 4 \frac{y^2}{d^2} \right) \]
\[ \gamma \geq \gamma_0 \]  \hspace{1cm} (6.2)

Equations (6.1) and (6.2) give the residual stress for any fiber as a function of \( y_0 \), which is a measure of the plastification of the section.

Next let us consider a beam of rectangular cross-section that contains initial residual stresses, \( \sigma_{ri} \), which are symmetrically...
distributed (Figure 24a). If the beam is bent beyond the elastic limit, the applied stresses may be obtained as shown in Figure 24b. In this figure the initial residual stresses have been subtracted from the yield points in tension and compression; thereby the limit at which yielding is reached is determined as indicated by the curved lines in Figure 24b. Due to the initial residual stresses, the yielding process is unsymmetric. The total stresses in the beam before removal of the bending moment are given by the addition of initial residual stresses, $\sigma_\text{ri}$, and applied stresses, $\sigma (\varepsilon)$, (Figure 24c). The bending stresses, $\sigma_p$, due to the removal of the bending moment must be subtracted from the total stresses in the beam before unloading in order to obtain the residual stress. The resulting stresses are hatched in Figure 24c and give the final residual stress distribution which is redrawn in Figure (24d).

In the same way let us consider axial deformation in a rectangular beam containing the same initial stress pattern as before (Figure 25a). The yield limit is indicated in Figure 25b. Superposition of a uniform stress will result in a total stress distribution shown in Figure 25c and the final stresses are shown in Figure 25d. By sufficient plastic deformation it is possible to wipe out completely the initial stress pattern.

Now let us consider another example of an H-beam, an example that was used in section 4.2 (Figure 14). Due to the unsymmetrical residual stress pattern, a bending of the beam resulted and the corresponding curvature amounted to $\theta = \frac{6\phi}{b}$. To obtain a straight beam a permanent curvature of the opposite sign must be introduced. In production mills this is done by "gagging", a straightening process done by application of a
set of concentrated forces. From the $M-\phi$ relationship of the beam with initial residual stresses the required curvature and bending moment to insure straightness after deformation can be obtained (Figure 26a). From the required bending moment the applied stress $\sigma(e)$ may be determined (Figure 26b). The unloading bending stress $\sigma_b$ is elastic. The final residual stress distribution is obtained from the addition of $\sigma(e)$, $\sigma_b$ and the initial residual stress (Figure 26d).

While the previous beam was bent about the weak axis let us consider as a last example the problem of a beam bent about the strong axis. Again the stresses have been determined before in section 4.2. (Figure 16). The procedure is exactly the same as before. The steps and the result are shown in Figure 27.

If an infinite curvature would be applied the cooling stresses would be completely wiped out and only cold bending residuals corresponding to a full plastic moment would remain.\(^{(1)}\)

In this chapter it was shown how residual stresses due to cold-bending can be calculated. It was also shown how the presence of initial curvature and initial residual stresses can be taken into account.
In this chapter some special problems in connection with residual stress distribution will be considered. Reference is made throughout to Chapter 3, in particular equation (3.4).

Residual Stresses in a Plate When \( E \) is Variable:

From an academic point of view it is desirable to know the influence of a variable modulus of elasticity. It is assumed that \( E \) varies according to \( E = E_0(1 - k \frac{y^2}{b^2}) \). The strain distribution is assumed to be parabolic as shown in Figure 28a. Substituting into equation (3.4) we obtain

\[
\sigma_r = E_0 \left(1 - k \frac{y^2}{b^2}\right) \left[1 - 4 \frac{y^2}{b^2}\right] - \frac{40 - 2k}{60 - 5k}
\]

In Figure 28b the stress distribution is plotted for various values of \( k \). Residual stresses become smaller with increase of \( k \), which amounts to a greater rate of change of \( E \).

Residual Stress in a Plate when the Stress-Strain Relation is Non-Linear:

A simple stress-strain law of the form \( \sigma^2 = K\varepsilon \) is assumed (Figure 29a). The strain distribution is the same as in the example above (Figure 28a). Omitting the bending term in equation (3.9) we can write for the residual stress:

\[
\sigma_r = \sigma(\varepsilon) - \frac{1}{A} \int_A \sigma(\varepsilon) \, dA
\]

Expressing \( \sigma(\varepsilon) \) by the given strains and the stress-strain law we obtain

\[
\sigma_r = (K\varepsilon_0)^{\frac{1}{2}} \left[\left(1 - 4 \frac{y^2}{b^2}\right)^{\frac{1}{2}} - \frac{\pi}{4}\right]
\]
In Figure 29 the residual stress distribution is plotted for the parabolic and the linear stress-strain relation. Because of the difference in the stress-strain relations a comparison of the two curves has little meaning.

Now let us consider the same plate under unsymmetrical strain distribution (Figure 30a).

\[ \varepsilon = \varepsilon_o \left( \frac{Y}{d} + \frac{1}{2} \right)^2 \]

and

\[ \sigma(\varepsilon) = (K\varepsilon_o)^{\frac{1}{2}} \left( \frac{Y}{d} + \frac{1}{2} \right) \]

\[ \sigma_a = \frac{1}{A} \int \sigma(\varepsilon) \, dA = \frac{1}{2} (K\varepsilon_o)^{\frac{1}{2}} \]

and

\[ M = \int \sigma(\varepsilon) y \, dA = \frac{1}{12} (K\varepsilon_o)^{\frac{1}{2}} b d^2 \]

where the curvature is

\[ \phi = \frac{M}{EI} = \frac{1}{Ed} (K\varepsilon_o)^{\frac{1}{2}} \]

and the bending stress is given by

\[ \sigma_b = E\phi y = \frac{Y}{d} (K\varepsilon_o)^{\frac{1}{2}} \]

Finally from equation (3.9)

\[ \sigma_f = (K\varepsilon_o)^{\frac{1}{2}} \left( \frac{Y}{d} + \frac{1}{2} - \frac{1}{2} - \frac{Y}{d} \right) = 0 \]
The beam will have no residual stresses but will have a permanent curvature. This result can be anticipated since for a parabolic variation of strains (Figure 30a) the corresponding stress variation is linear (Figure 30b). In order to satisfy equilibrium the section must deform. The corresponding stresses \( \sigma_a \) and \( \sigma_b \) are linear, because they are obtained from the unloading stress-strain relation (Figure 30c). The superposition of Figure 30b and 30c gives zero stresses.

**Change of Residuals Due to Partial Release of Stresses**

Suppose a plate has an initial residual stress distribution as shown in Figure 31a. Then the plate is reduced in width and the problem is to find the residual stress distribution after cutting. Again the general equation (3.4) may be used. A rotation and axial deformation will take place as shown in Figure 31b by the dashed line. To calculate the final stresses (shown in Figure 31c) it is only necessary to substitute the initial stress, \( \sigma_{ri} \), for \( E\varepsilon \) in equation (3.4). By measuring the deformations due to successive cuts it is possible to obtain a picture of the initial stress distribution. This is used in several methods of residual stress measurement (2).

**Residual Stress Distribution in a T-Beam That Has Been Obtained by Splitting of an H-Section:**

The dimensions and assumed residual stress distribution of the H-beam are shown in Figure 32a. As in the previous example the residual stress distribution after cutting is obtained by substitution of the original residual stresses for \( E\varepsilon \) in equation (3.4). Due to the unbalance of residual stress by cutting rotation will take place. The final stress distribution is shown in Figure 32b.
8. COLUMN INSTABILITY - AXIAL SYMMETRY

8.1 General Theory

Let us consider an initially straight steel column of axially symmetric cross-section and likewise symmetric residual stress pattern. If this column is subjected to axial loading certain fibers will reach the yield point earlier than others, thereby reducing the bending stiffness.

When an indefinite number of deflected positions beside the straight position become possible, we say the column is unstable. For axially loaded columns free from residual stress instability occurs when the Euler load, $P_E$, or the tangent modulus load, $P_T$, is reached, depending on the slenderness ratio and the stress-strain curve of the material.

The derivation presented below is for axially loaded columns containing symmetric residuals and is based on the following assumptions:

1. Plane sections remain plane after deformation.
2. Cross-section and residual stress distribution have axial symmetry.*
3. Residual stresses are constant along each fiber.
4. Load is applied axially.
5. Ends of the column are pinned.

The H-column shown in Figure 33a contains residual strains, $\varepsilon_r$ (Figure 33b). Then axial strains are added until a position is reached where the column takes an infinitesimal bent position (Figure 33c). As in the derivation of the Euler differential equation,* unsymmetrical residual stress distributions are treated in Chapter 9.
equilibrium will be considered for the bent position:

\[ P_u = \int_A \sigma x \, dA = \int_A (\sigma - \sigma_a) \, dA \]

\[ P = \int_A \sigma \, dA \]

The stresses, \( \sigma \), in the column are of course a function of the strain distribution (Figures 33b and 33d).

For infinitesimal bending\(^{(10)}\)

\[ P = \int A \Delta \sigma \, dA = \int A E(\epsilon) \, dA = -\int A E(\varepsilon) \phi x^2 \, dA \]

since

\[ \phi = \frac{-\Delta \epsilon}{x} = \frac{u''}{x} \]

we finally obtain

\[ P u + u'' \int A E(x^2) \, dA = 0 \]  \( (8.1) \)

In the case of a linear stress-strain diagram equation (8.1) reduces to the Euler differential equation of elastic buckling. For a general stress-strain relation equation (8.1) is somewhat cumbersome to solve.

Taking the idealized stress-strain curve of steel (Figure 2) the integral in equation (8.1) is readily evaluated:\(^{(1)}\)

\[ \int_A E x^2 \, dA = E \int_{A_e} x^2 \, dA = EI_e \]  \( (8.2) \)

since \( E = 0 \) for the part of the section that has yielded, the integral reduces to \( E \) times the moment of inertia of the part of the section which is elastic.
Knowing the residual stress pattern and the stress-strain curve of a material, the load at which instability will occur is given by

\[ P = \pi^2 \frac{E_i}{L^2} \quad (8.3) \]

or

\[ \sigma_{cr} = \pi^2 \frac{E_i}{(L/F)^2} \quad (8.4) \]

where, in general

\[ E_i = \frac{1}{L} \int_A E \chi^2 dA \quad (8.5) \]

and for steel

\[ E_i = E \frac{I_x}{I} \quad (8.6) \]

In the case of steel another approach to the problem is possible by obtaining the modified stress-strain relationship of a piece containing the same residual stresses as the column (Figure 34A). Due to the presence of residuals the stress-strain curve is no longer linear up to the yield point. The proportional limit is reached when \( \sigma = \sigma_Y - \sigma_{rc} \). Above this stress the strain is given by (Figure 34,B).

\[ \varepsilon = \frac{1}{E} (\sigma_Y - \sigma_{rc}) \quad (8.7) \]

The average stress is obtained from the integration of the applied stress (Figure 34,C)

\[ \sigma_{av} = \frac{1}{A} \int_A \sigma dA \quad (8.8) \]
In a H-section, for instance, when the flange edges have reached the yield point \( \sigma_y \), the stress is:

\[
\sigma_{Al} = \sigma_y - \frac{A_e}{A} \sigma_{rx_o} - \frac{4t}{A} \int_{x_o}^{b/2} \sigma_{rx} \, dx
\]  

(8.9)

Equations (8.7) and (8.9) give the stress-strain curve of an H-specimen containing residual stress. The tangent modulus is obtained by differentiation:

\[
E_t = \frac{d\sigma}{d\varepsilon} = \frac{d\sigma}{dx_o} \frac{dx_o}{d\varepsilon}
\]

where

\[
\frac{d\sigma}{dx_o} = \frac{1}{A} (4tx_o + wd) \frac{d\sigma_{rx_o}}{dx_o} \quad \text{and} \quad \frac{dx_o}{d\varepsilon} = \frac{1}{E} \frac{d\sigma_{rx_o}}{dx_o}
\]

and therefore

\[
E_t = \frac{E}{A} (4tx_o + wd) = E \frac{A_e}{A}
\]  

(8.10)

Since \( E_t \) is a function of \( x_o \), it is now possible to express \( E_t \) in terms of the tangent modulus.

For strong-axis bending of an H-section:

\[
E_{ix} = \frac{AE_t}{A_f} - \frac{3A_w E}{A_f} \]

(8.11)

For weak-axis bending

\[
E_{iy} = E \left[ \frac{A}{A_f} \frac{E_t}{E} - \frac{A_w}{A_f} \right]^{-3}
\]  

(8.12)
For stresses below the proportional limit $E_t = E$, and $E_i = 1$. Thus it is possible to obtain a column curve directly from the stress-strain relationship measured on a short specimen containing residual stress.

For a rectangular beam $A_w = 0$ and since the strong and the weak axes are rotated $90^\circ$ for the rectangle with respect to the H-section:

$$E_iY = E_t$$  \hspace{1cm} (8.13)

$$E_iX = E \left( \frac{E + t}{E} \right)^3$$ \hspace{1cm} (8.14)

Figures 35 and 36 give diagramatic stress-strain relations for H-sections at various stages of loading. The curves depend on the residual stress distribution and consequently on the sequence of yielding. For instance the flanges yield completely before the web starts to yield for the residual stresses shown in Figure 35. On the other hand in Figure 36 the web has reached yielding before the flanges have become completely plastic. The average stresses, however, will reach the yield level at a strain

$$\varepsilon \leq 2 \frac{\sigma_y}{E} = 2 \varepsilon_y$$ \hspace{1cm} (8.15)

according to equation (8.7).

The residual stress magnitude and distribution in the flanges may be reconstructed from the stress-strain curve of a cross-section test. With $\sigma_y$, $E$, $E_t$, and $\varepsilon$ given by the test data, $\sigma_{rxo}$ may be determined from equation (8.7) for discreet values of $\varepsilon$, and the corresponding value of $x_o$ may be obtained from equation (8.10). A number of distributions of residual stress thus calculated are shown in Figure 37 and are compared with measured residual stresses. Their agreement is very close.
8.2 Effect of Different Variables on Axial Instability

The following major variables will be considered in this section:

1. Stress-strain relationship
2. Axis of bending
3. Shape of section
4. Residual stress pattern

All these variables are interrelated with the length of the column as a common variable. The influence of each of these will be investigated and the results will be presented in the usual form of column curves.

Let us consider first a rectangular section with a residual stress distribution as shown in Figure 38a. The stress-strain curve is assumed to be parabolic as shown in Figure 38b and given by the equation: \( \sigma^2 = K\epsilon \), where a value of \( K = 8 \times 10 \text{ ksi}^2 \) was used.

Then

\[
\epsilon = \frac{d\sigma}{d\varepsilon} = \frac{1}{2} \left( \frac{K}{\varepsilon} \right)^{1/2} = \frac{1}{2} \frac{K}{\sigma}
\]

where from Figure 38a

\[
\sigma = \sigma_r + \sigma_{cr} = \sigma_o \left(1 - \frac{Y^2}{d^2} \right) + \sigma_{cr}
\]

The effective modulus for bending about the x-axis is obtained from equation (8.5)

\[
E_{ix} = \frac{d}{dx} \int_{-d/2}^{d/2} E\gamma^2 \, dy
\]
Substituting equation (8.17) into equation (8.16) and the latter into equation (8.18) we obtain by integration of equation (8.18)

\[ E_{ix} = K \left[ -\frac{2}{\sigma_0} + 6 \left( \frac{\sigma_0 + \sigma_{cr}}{3 \sigma_0} \right)^2 \right] \ln \left[ 1 + \frac{1}{2} \left( \frac{3 \sigma_0}{\sigma_0 + \sigma_{cr}} \right) \frac{1}{2} \right] \left( \frac{3 \sigma_0}{\sigma_0 + \sigma_{cr}} \right)^{\frac{1}{2}} \] (8.19)

For bending about the y-axis:

\[ E_{iy} = \frac{12}{K} \int_{-\delta_y}^{\delta_y} \int \mathbf{E}^2 \, dx \, dy \] (8.20)

substituting \( E \) [equation (8.16)] into equation (8.20) and integrating:

\[ E_{iy} = \frac{K}{2 \left[ 3 \sigma_0 (\sigma_0 + \sigma_{cr}) \right]^2} \ln \left[ 1 + \frac{1}{2} \left( \frac{3 \sigma_0}{\sigma_0 + \sigma_{cr}} \right) \frac{1}{2} \right] \left( \frac{3 \sigma_0}{\sigma_0 + \sigma_{cr}} \right)^{\frac{1}{2}} \] (8.21)

If no residual stresses were present, \( E_4 = E \) (equation 8.16). Equations (8.19) and (8.21) are plotted in Figure 38 as a function of \( \sigma_{cr} \).

It is now possible to solve equation (8.4) for \( L/r \). The results are shown in Figure 38d. The difference between strong and weak axis buckling is small and the reason for this is that \( E_{ix} \) differs but slightly from \( E_{iy} \) for the chosen example. However, the residual stresses influence the critical stress in the whole range of slenderness ratios. By reversing the residual stress pattern (tension stresses to compressive stresses and vice versa) higher critical stresses than for the condition of zero residual stresses can be obtained (Figure 38d, reversed pattern).
Using the same section and residual stress distribution as before (Figure 39a) let us find the column curves for steel members with an idealized stress-strain curve (Figure 39b).

Then for bending about the x-axis, from equation (8.6)

\[ E_{ix} = E \left( \frac{2Y_e}{\rho} \right)^3 \]  

(8.22)

and for bending about the y-axis

\[ E_{iy} = E \left( \frac{2Y_e}{\rho} \right) \]  

(8.23)

The average stress in the column above the pseudo proportional limit, \( \sigma_y - 2\sigma_{ro} \), is

\[ \sigma_{cr} = \frac{F_{cr}}{A} = \frac{1}{A} \int_{A} \sigma dA = \sigma_y - 2\sigma_{ro} \left( \frac{2Y_e}{\rho} \right)^3 \]

or

\[ \frac{2Y_e}{\rho} = \left( \frac{\sigma_y - \sigma_{cr}}{2\sigma_{ro}} \right)^\frac{1}{3} \]

Substituting into equations (8.22) and (8.23) we get

\[ E_{ix} = E \frac{\sigma_y - \sigma_{cr}}{2\sigma_{ro}} \]  

(8.24)

and

\[ E_{iy} = E \left( \frac{\sigma_y - \sigma_{cr}}{2\sigma_{ro}} \right)^\frac{1}{3} \]  

(8.25)
Equations (8.24) and (8.25) are plotted in Figure 39c. Finally, as before, the column curves are obtained (Figure 39d). For no residual stresses in the column the Euler curve is the solution until \( \sigma_{cr} = \sigma_y \), which is also the critical stress for shorter columns. Contrary to the previous example, residual stresses influence the critical stress only in the lower and medium range of slenderness ratios. A reversal of the sign of residual stresses would still result in column curves below the curve for zero residual stresses. The latter column curve is therefore the optimum.

Now the influence of variations in the residual stress patterns will be investigated. Let us consider an H-column under different assumptions of residual stresses in the flanges as shown at the top of Figure 40. The web stresses are assumed to be constant and of a magnitude sufficient to satisfy equilibrium. The column curves for the different assumptions are also shown in Figure 40. From these curves it is obvious that a large area of compressive residual stress acting at extreme distances from the bending axis (distribution I and IV, for example) will have a pronounced effect on instability. It will also be noted that critical stresses for strong-axis buckling are higher than for weak axis buckling.

The influence of shape can be demonstrated by comparison of the column curves for a rectangular shape (Figure 39d) with the column curves for the H-column (Figure 40, distribution III). For the rectangular shape the critical stresses for weak-axis buckling are higher than for strong-axis buckling, just reversed from the results on the H-shape.
If loads are applied eccentrically or, on the other hand, if residual stresses do not have axial symmetry and the load is applied axially, a simple solution is not possible. It is necessary to assume deflections in order to arrive at the final deflection curve by successive approximations. In general the procedure is very similar to the classical method of solution by Karman\(^{14}\). Approximate solutions are obtained by working with assumed deflection curves thus reducing the amount of numerical work.

9.1 General Method of Solution

At first no particular restriction will be made as to shape of column and stress-strain relation. However, it will be assumed that deflections are symmetric with respect to the center and that the ends are pinned. Residual stresses will be assumed to have at least one axis of symmetry and it is also assumed that torsional buckling will not occur. Let us consider an eccentrically loaded column as shown in Figure 41a. Strains in the column can be described by the maximum compressive strain, \(\varepsilon_c\), and bending strains, \(\theta_x\) (Figure 41c). Now follows the most tedious part of the solution that of calculating internal forces, \(P\), and moments, \(M\), for various assumed values of \(\varepsilon_c\) and \(\theta\), for which the stress-strain relation (Figure 41b) has to be used. The results of these calculations can be plotted as \(u-\theta\) and \(P-\theta\) curves where \(\varepsilon_c\) is a common parameter and \(u\) is the deflection plus eccentricity (Figures 41d and e). Eliminating \(\varepsilon_c\), the three variables \(u\), \(P\), and \(\theta\) can be combined into one graph (Figure 41f). It is now desired to find the critical length for a column of given eccentricity, \(e\), and load \(P\).
A deflected shape, $u$, must be assumed (Figure 4lg). Entering Figure 4lf with the value $u$, a value $\phi_2$ is obtained for the given load $P$ (Figure 4lh). By double integration of $\phi_2$ another deflection $u_2$ is obtained, that contains the length, $L$, as parameter. This process can be repeated until $u_n = u_n + 1$ from which $L$ is obtained.

To get the critical length, $L_{cr}$, it is necessary to vary the magnitude of the center deflection, $u_0$, and go through the same process described above until the maximum value of the length has been determined(14).

Residual stresses influence the $u$-$P$-$\phi$ relationship, otherwise the procedure is the same as for a column free from residual stresses.

The procedure of determining the critical column length when the magnitude of the load and residual stresses are given may be summarized as follows:

1. Determine $P$ and $M$ for various values of $\varepsilon_c$ and $\phi$. Calculate $u = M/P$.
2. Plot $P$-$\phi$ and $M$-$\phi$ graphs, where $\varepsilon_c$ is a parameter for both.
3. Combine the two graphs into one $u$-$P$-$\phi$ graph by elimination of $\varepsilon_c$.
4. Assume a deflected shape of the column ($u_1$).
5. From the $u$-$P$-$\phi$ graph determine $\phi_2$ for given $P$.
6. Continue the process until $u_n = u_n + 1$.
7. Determine the critical length by variations of the magnitude of the center deflection. The maximum length obtained on this process is the critical length.
9.2 Buckling of H-Columns

The general method outlined in the previous section will now be applied to the specific solution of an H-column containing cold-bending residual stresses in the flanges. Buckling will only be considered for the weak direction. A practical reason for this limitation is that H-columns that bend about the strong axis fail by a combination of flexural buckling and torsional buckling. An asymmetric residual stress pattern in the flanges would increase the possibility of torsional failure and this subject is beyond the scope of this dissertation.

The column to be investigated has a cross-section as shown in Figure 42a. The flanges are assumed to have a residual stress distribution such as is shown in Figure 42b. These stresses are assumed to be constant along the column. The residual stress distribution was obtained from equations (6.1) and (6.2) by substituting \( b/4 \) for \( y_0 \). In other words, it was assumed that the plastic deformation during prior bending extended halfway to the web. Also shown in Figure 42 are the assumed material properties and the stress-strain curve (Figure 42c).

First, it is necessary to calculate the axial load and the bending moment for various assumed values of compressive strain, \( \varepsilon_C \), and curvature \( \phi \) (Figure 42d). For a large number of combinations of these two parameters \( (x_1/b \approx 1/4) \) we can write after integrating and simplifying:

\[
P - \frac{b}{E} \left[ 0.656 - 2 \frac{x_1}{b} + 1.375 \frac{x_1^2}{b^2} + \left( \frac{\varepsilon_C}{\varepsilon_y} - \frac{\phi b}{2\varepsilon_y} \right) \left( \frac{x_1}{b} + 0.5 \right) - \frac{\phi b}{2\varepsilon_y} \left( 0.25 - \frac{x_1^2}{b^2} \right) + \frac{1}{3} \left( \frac{\varepsilon_C}{\varepsilon_y} - \frac{\phi b}{2\varepsilon_y} \right) \right]
\]

(9.1)
and

\[ \frac{u}{b} = \frac{M}{Pb} = \frac{3}{4} \frac{R_y}{P} \left[ 0.1335 - \frac{v^2}{D^4} + 0.9170 \frac{v^3}{D^3} \right] \left( \frac{\varepsilon_c}{\varepsilon_y} - \frac{b}{2D} \right) 
+ \frac{D}{3} \left( \frac{v^2}{D^2} + 0.1250 \right) \]

where \( \frac{v}{D} = \frac{2(\varepsilon_c - \frac{b}{2D})}{2.15 + \frac{b}{D}} \) is a measure of the distance from the flange center to the beginning of the plastic zone created due to the applied loading. The results of the calculations are presented in Figure 43 where the three essential variables \( u, P, \) and \( \phi \) are plotted with the common parameter \( \varepsilon_c. \) We can eliminate \( \varepsilon_c \) as indicated in Figure 43 by the dashed lines and get a direct relation between \( u \) and \( \phi \) with \( P \) as parameter as shown in Figure 44.

Now we are ready to apply the numerical procedure outlined in the previous chapter. A sample calculation is shown in Figure 45. Finally the results can be presented in the conventional form as column curves with the eccentricity as parameter (Figure 46). Of course these column curves are valid only for the assumed residual stress-distribution. Also shown is the elastic solution of the second order stress problem (secant solution) with the outer fiber at yield point stress.

9.3 Approximate Solutions

The numerical procedure to obtain \((L/r)_{cr}\) is a very tedious one. An approximate solution may be obtained more rapidly by assuming the deflected shape of the column. Then it is only necessary to vary the magnitude of the center line deflection to obtain the critical slenderness ratio. The relationship between the variables, \( P, u \) and \( \phi \) (Figure 44) is of course the basis of the whole method.
Let us assume a sine curve for the deflected shape of the column. Then the curvature at the center is given by

$$\phi_0 = (u_0 - e) \frac{\pi^2}{L^2}$$

and from this

$$L = \pi \sqrt{\frac{u_0 - e}{\phi_0}}$$

or from nondimensional terms of the variables:

$$L = \frac{\pi b}{\sqrt{E_y}} \sqrt{\frac{u_0 - e}{\frac{\phi_0 b}{E_y}}} \quad (9.3)$$

Solving by trial and error or by differentiating equation (9.3) with respect to $u_0$ and setting zero of the derivative we obtain $L_{\text{max}} = L_{\text{cr}}$. From the derivative of equation (9.3) we get:

$$\frac{\phi_0 b}{E_y} = \frac{u_0 - e}{b} \frac{d\left(\frac{\phi_0 b}{E_y}\right)}{d\left(\frac{u_0}{b}\right)} \quad (9.4)$$

Then equation (9.3) can be solved directly for the critical length by substitution of $\phi_0$ and $u_0$ values obtained from the solution of equation (9.4). The results of this approximate solution are shown in Figure 46 by the open circles for particular values of $P/P_y$.

Another curve frequently used is a partial cosine curve (Figure 47). The procedure of course is the same as before. For this curve:

$$L = \frac{2b}{\sqrt{E_y}} \sqrt{\frac{u_0 - e}{\phi_0 b}} \cos^{-1} \frac{e}{u_0} \quad (9.5)$$
The approximate solution according to equation (9.5) is also shown in Figure 44.

In spite of the simplification obtained by assuming the deflection curve, the work required is still considerable. This is especially true if the effect of several residual stress patterns is desired, since the tedious calculations required to plot Figure 44 must be made for each stress pattern. In the following, a relatively simple solution will be obtained for a rectangle containing cold-bending residual stresses. This idealization should give good results for H-sections with an eccentricity or moment in the plane of symmetry of the section normal to the web. The residual stress distribution is assumed to be piecewise linear and constant along the length of the column. The other assumptions are the same basic ones as have been used throughout.

There are two basic types of possible stress distributions. The internal stress distribution due to axial load and bending moment will be either as shown in Figure 48b (Case I) or Figure 48c (Case II) (yielding can also take place on the tension side). The rate of change of the residual stress at the outer fiber is given by the angle, $\varphi$.

By assuming the deflection curve to be a sine curve with the maximum deflection, $v_o$, at the middle we have only to consider the critical section at the center of the column. The total moment at the center is $M_o = P v_o + m_o$ (Figure 48d) where $m_o$ can be due to end moments, to eccentrically applied forces or lateral forces, the only restriction being that the moment diagram be symmetric with respect to the center.
It is possible to express from the internal stress distributions the internal forces and moments as follows.*

Case I

\[ P = P(\phi_0, x_1) \]
\[ M_o = M_0(\phi_0, x_1) \]  \hspace{1cm} (9.6)
\[ \phi_0 = \frac{\pi^2}{L^2} u_o \]

Case II

\[ P = P(\phi_0, x_1, x_2) \]
\[ M_o = M_0(\phi_0, x_1, x_2) \]  \hspace{1cm} (9.7)
\[ x_2 = x_2(\phi_0, x_1) \]

where \( x_1 \) and \( x_2 \) are measures of the distance from the section center to the beginning of plastic zones created due to the applied loading. The other terms have been defined before. The zero subscript refers to the section at the middle of the column. For both cases I and II the equations can be reduced to one equation by combining equations (9.6) respectively (9.7). The resulting expression will be a function, \( F \),

\[ F = F(u_o, L, P) = 0 \]  \hspace{1cm} (9.8)

In order to obtain the critical length, \( L_{cr} \), we must determine the maximum value of \( L = L(u_o) \). The condition for an extreme value of \( L \) is obtained from the implicit form of the variables (equation 9.8):

\[ \frac{dF}{du_o} = \frac{\partial F}{\partial u_o} + \frac{\partial F}{\partial L} \frac{dL}{du_o} = 0 \]  \hspace{1cm} or  \hspace{1cm} \frac{\partial F}{\partial u_o} = 0 \]  \hspace{1cm} (9.9)

which defines the critical deflection. After substituting equation (9.9) into equation (9.8) we obtain finally

\[ F = F(L, P) = 0 \]  \hspace{1cm} (9.10)

* See the Appendix for a detailed derivation
The analytical evaluation of equation (9.10) gives as a result the following equations:

\[
\sigma_{cr} = \frac{P}{A} = \frac{\pi^2 E}{(L/r)^2} \left[ 1 - \frac{E y b - \frac{y b (L/r)^2}{\sigma_{cr}} + \frac{12 m_0}{P b}}{3 E y b + 6 \sigma_{cr}} \right]^3 \quad \text{Case I} \quad (9.11)
\]

\[
\sigma_{cr} = \frac{\left(\sigma_p + \frac{1}{2} E y b\right) \left[(L/r)^2\right]^2}{\left[3 \sigma_p + \frac{4}{3} E y b - \frac{4 m_0}{P b} - \frac{\sigma_{cr}}{\sigma_p + \frac{1}{2} E y b} + \frac{y b (L/r)^2}{3 \pi^2 (r)}\right]^3} \quad \text{Case II} \quad (9.12)
\]

where \(\sigma_p\) is the pseudo proportional limit \(\sigma_y - \sigma_{rc}\).

Equations (9.11) and (9.12) are the solution to the problem. They must be solved by trial and error. If no residual stresses are present we must set \(\varphi = 0\), and \(\sigma_p = \sigma_y\). Then equations (9.11) and (9.12) take the form:

\[
\sigma_{cr} = \frac{\pi^2 E}{(L/r)^2} \left[ 1 - \frac{2 m_0}{P b} \left(\frac{\sigma_y}{\sigma_{cr}} - 1\right) \right]^3 \quad (\frac{2 m_0}{P b} \leq 1) \quad \text{Case I} \quad (9.13)
\]

\[
\sigma_{cr} = \frac{\sigma_y^3 \left[(L/r)^2\right]^2}{\left[\sigma_{cr} - \frac{4 m_0}{P b} - \frac{\sigma_{cr}}{\sigma_y}\right]^3} \quad (\frac{2 m_0}{P b} \geq 1) \quad \text{Case II} \quad (9.14)
\]

After rewriting in a slightly different form equations (9.13) and (9.14) are identical with the solution given by Jezek(14).

A simple limit for the range of applicability cannot be given for equations (9.11) and (9.12) because of their complexity. The following will aid in the estimation of that limit. If case I applies but equation (9.12) for case II was used the actual critical slenderness ratio and critical stress will be larger than the computed
values. If on the other hand case II applies but case I was used the actual critical slenderness ratio and critical stress will be smaller than the computed values.

For short columns (L/r about 50) equations (9.11) and (9.12) do not hold since the applied stresses would have a different distribution than assumed. However, for the limiting case L/r = 0 the stability problem reduces to a stress problem and the influence of residual stress is wiped out for full yielding of the section. Then for an H-section

\[
\frac{P}{P_y} = \left(1 - \frac{2bt}{A} - \frac{4te}{A}\right) + \sqrt{(1 - \frac{2bt}{A} - \frac{4te}{A})^2 + \frac{4bt}{A} - 1}
\]  

(9.15)

and for a rectangular section

\[
\frac{P}{P_y} = -\frac{2e}{b} + \sqrt{\frac{4e^2}{b^2} + 1}
\]  

(9.16)

The interpolation of critical stresses for short columns between 0 < L/r < 50 can be done without great error.

Of particular interest is the application of this approximate solution to the problem solved previously by the more exact methods. The results of these calculations are also shown in Figure 46. All approximate solutions are in very good agreement with the "exact" solution. At L/r = 0 the approximate solution for the rectangle gives the largest difference. This is due to the difference in column shape, (shape factor of section).
10. EXPERIMENTAL INVESTIGATION

In order to determine how well the various theoretical solutions developed in the previous chapters are confirmed by actual behavior, residual stresses were measured, coupons and "cross-sections" were tested, and experiments on WF columns carried out. A summary of the tests performed is given in Table 1.

All measurements and tests were conducted at the Fritz Engineering Laboratory, Lehigh University. Residual stresses were measured by the sectioning method, cross-sections and columns were tested axially in an 800,000 # capacity Riehle universal testing machine. Only as-delivered and annealed WF sections of steel rolled to ASTM A-7 specifications were used in the experiments.

10.1 Description of Tests

Residual Stresses:

The sectioning method(4) was adopted for the measurement of residual strains because of its simplicity. Longitudinal strains were measured over a 10" gage length by a 1/10,000 Whittemore strain gage on a series of holes. A standard 10 inch mild steel bar was attached to the specimen to observe changes due to temperature between readings. The average error in these measurements corresponds to about ± 600 psi. Following an initial set of readings on drilled and reamed holes serving as gage points, a second set of readings was taken after relaxation due to sawing.
Cross-Section Tests:

In order to predict the axial column strength from the tangent modulus of a stress-strain curve that includes the influence of residual stress, a short length of about 3 to 4 times the depth of the section was used to measure the stress-strain relationship at the middle third. Strains were measured by both bonded electrical resistance gages (SR-4s) and dial gage arrangements. A typical setup for a cross-section is shown in Figure 49. A qualitative picture of the yielding process was obtained by the flaking of the mill scale made clearly visible by whitewashing the specimens with hydrated lime. The load was applied in appropriate increments up to the maximum and the specimen was strained further until reduction of carrying capacity occurred due to local buckling. For a detailed description of test procedures and techniques see Reference (12).

Column Tests:

The columns were tested in the same screw-type testing machine as the cross-sections. In earlier tests the axial load was applied through knife edges(15). Later tests were made on cylindrical bearings of specially treated steel. Figure 50 shows a typical set-up.

Strains were measured by SR-4 gages at the center and near the ends. Deflections were measured with dial gages at the quarter points and at the center of the columns. End rotations were measured with a level bar-dial gage arrangement(13). (Figure 50).

The alignment was made by trial and checked by the SR-4 and deflection gages. Accidental eccentricities were eliminated by relative movement of the column bases with respect to the base fixtures.
Uneven bearing could be partially corrected by the movement of wedges in the base fixture assembly. The effect of initial curvature was partially reduced by application of a slight eccentricity. Most columns had an out-of-straightness of less than 1/16" per 10 feet. (Table 2)

The load was applied in appropriate increments which were determined from a load-deflection graph plotted during the test. The tests were continued beyond the maximum until the load had dropped to about two-thirds of the maximum value. In weak axis tests this drop occurred quite suddenly. Whitewashing of the as-delivered columns gave a qualitative picture of the development of yield lines. The presence of some cold-bending yield lines could not be avoided, but in most cases the center part of the column was substantially free from cold bending as evidenced by an undisturbed mill-scale pattern.

To study the effect of cold-bending, an initially curved member (Figure 62) was subsequently stressed into the plastic range. Loaded as a simple beam with equal loads applied near the ends, the major portion of the member was subjected to a uniform moment (Figure 62). The beam was unloaded when the deflection was sufficiently large to assure a permanent set which would offset the initial deflection. The deflections after unloading had less than one half of the original magnitude of initial deflections. A column was cut out of this beam as indicated in Figure 62 and tested under axial load.

10.2 Test Results

Table 1 lists the types and number of tests performed in the experimental program. The following is a brief presentation of results. The latter are discussed and compared with the theory in
Chapter 11.

Residual Stress:
The results of residual stress measurements on WF shapes are presented in Figure 51. The stresses were calculated from the strains using an elastic modulus, $E$, of $30 \times 10^3$ ksi.

The distribution of the average stresses of the flange edges and the flange centers along a 8WF31 beam is shown in Figure 52a. The same beam was also used to study the length required to retain the residual stress pattern in the center of a cross-section when cut out of a long beam. Figure 52b gives the relationship between the length of the cross-section and the average stresses at the flanged edges and the flange centers.

While the above measurements were made at portions of the beam that showed no evidence of cold-bending (like flaked mill scale), measurements on an 8WF31 shape were made at a section that showed such evidence. The results of these measurements and the lines of flaked mill scale are shown in Figure 53.

Cross-Section Tests:
A typical set of stress-strain curves obtained from a cross-section test made on as-delivered material is shown in Figure 54. The strain at which the section should theoretically reach the yield stress level according to equation (8.7) is also indicated. The average curve for a large number of compression coupons is also shown. The behavior of an annealed cross-section is shown in Figure 55 together with the
average compression coupon curve.

Column Tests:

All columns were tested under an axial loading condition. Test results of as-delivered column specimens are shown in Figures 57 and 59 on a non-dimensional basis. (The "approximation" curves will be described later.) The yield point, \( \sigma_y \), in these curves is based on the value of the corresponding cross-section test. If the slight elastic deflections are attributed to equivalent initial curvature (sine curve assumed) or eccentricity of the columns the values listed in Table 2 result. These values will be important for judging the relative influence of initial column deflections and eccentricities versus the influence of residual stresses.

The influence of cold-bending upon axial column strength was also studied. Figure 62 shows the initial deflection of the 5WF15 1/2 column after cold-bending under a uniform moment. The column test result is presented in Figure 63. A column of the same section was tested in as-delivered condition and this result is also presented in Figure 63 for comparison.
11. REMARKS CONCERNING THEORY AND TESTS

In this chapter some comments will be made on the theory presented, the tests and their correlation.

11.1 Residual Stress:

A linear distribution of strains across the total section of a beam does not produce residual stress. It was shown in Chapter 1 that residual stresses are always due to permanent deformation of the virgin material. These plastic deformations can occur during cooling or they may be introduced by a mechanical process.

The plastic deformations during cooling are dependent on various factors, the most important ones being cooling rate, temperature distribution, material properties and specimen dimensions; and all of these factors are interrelated. Measurements have given a good picture of strain distributions in a large variety of shapes (Figure 51). The majority of shapes exhibited a strain distribution which was approximately a parabola. With this evidence a reasonable strain distribution can be assumed for a qualitative analysis. The influence of dimensions can then be studied if the influence of size on the shape of the strain distribution is negligible. Obviously this condition cannot be assumed for all shapes. However, the agreement of measurements with theoretical predictions is fair for shapes that contained the same type of residual stress distribution (Figures 9 and 51).

The length measured from a free end of a beam (no residual stress) to the section where the full magnitude of residual stresses is
present was found to be approximately twice the depth of a column (Figure 22). This value is larger than one would expect from St. Venant's principle because the column is of open cross-section. The measurements, made along a beam as well as measurements made at one section while the beam was shortened by successive cutting, indicated that for practical purposes a length equal to the depth of the beam is sufficient (Figure 52). Only normal stresses were involved in these measurements. Figure 22 shows that for a length equal to the depth of the section the normal stresses are almost uniform. Therefore the experimental result is in agreement with the theoretical predictions.

This short length is of importance in two respects: (a) The residual stresses due to cooling may be assumed to be constant along the length of the column without appreciable error since the end zones containing variable stresses are short. (b) The length of a cross-section test specimen should be at least two times the larger cross-sectional dimension plus the gage length for which the stress-strain relationship is to be determined.

Residual stresses due to cold-bending about the weak axis of an H-beam geometrically similar to an 8WF81 have been calculated theoretically (Figure 26). Measurements were made on an 8WF31 beam in a region that contained evidence of cold-bending in the form of yield-lines. The original residual stress pattern of the 8WF31 is shown in Figure 51, the stresses in the cold-bent region are shown in Figure 53. The theoretically predicted distribution is in good agreement with the measurements.
11.2 Axial Column Instability

It was shown in Chapter 3 that for axial symmetry of residual stress and loading a relative simple solution is obtained. For a material with a pronounced yield point (steel) the solutions for weak and strong axis buckling showed a pronounced difference for the lower slenderness ratios (Figures 39, 40, 56 and 58). A comparison of as-delivered and annealed cross-section test results shows clearly the influence of residual stresses on the stress-strain relation (Figures 54 and 55). The theoretical predictions of non-linear behavior of as-delivered specimens is in complete agreement with theoretical expectations (Figures 35 and 36).

The stress-strain diagrams of the cross-section tests were used to compute the column curves in Figures 56 and 58 in accordance with the theory developed in section 8.1.

Measured residual stresses were approximated by mathematical expressions. The column curves were then calculated in accordance with the theory developed in section 8.1 and are shown in Figures 60 and 61 for strong and weak axis buckling.

Figures 56 and 60 shows that the column curves for the strong axis (calculated (a) from the stress-strain relations of cross-section tests, and (b) from an idealization of the measured residual stress pattern) are well represented by a parabolic curve. Figures 58 and 61 show that the column curves for the weak axis can be approximated by a straight-line in the inelastic range. Since in the literature both curves have been suggested as empirical column curves,
now, for the first time, a theoretical justification is given. Since the curves were based on test results, they probably have reflected empirically the influence of residual stress (Figures 57 and 59).

Accidental eccentricities also lower axial column strength. Table 2 shows equivalent eccentricities and initial deflections (based on assumed sine-curve) as calculated from the load deflection curves recorded during the test. These values are all very small and would only account for a relatively small reduction in axial column strength for the tests reported here. This is demonstrated in Figure 59 by comparison of annealed and as-delivered columns.

For the non-linear stress-strain relationship considered in Figure 38 the difference between weak and strong axis buckling strength was small. However, the residual stresses influence all slenderness ratios. By reversing the sign of the residual stresses shown in Figure 38a it is possible to increase the column strength above the strength for zero residuals. For steel columns this is not the case. The optimum strength is obtained in the absence of any residual stresses.

The residual stress pattern has a pronounced influence upon the buckling load. In general the larger the area of compressive residual stresses acting at extreme distances from the bending axis the greater the reduction in bending stiffness and thereby in column strength. This is illustrated in Figure 40 by comparison of the hyperbolic with the parabolic residual stress distribution. The
column curve with the hyperbolic pattern is considerably lower. As long as the sum of the applied average stress and the maximum compressive residual stress is lower than the yield point the residual stresses have no influence (Figures 39, 40, and 46). For a rectangular section the effective modulus, $E_{iy}$, reduces to the tangent modulus, $E_t$, for the stress-strain relationship caused by residual stress.

If the residual stress pattern has no axial symmetry, then the load at which the column starts to deflect may be considerably lower than the maximum load. The method of solution for this case is the same as for eccentrically loaded columns. Calculations have been made for a cold-bent pattern assumed to be constant along the length of the column. (Figure 42b). An 5WF18 1/2 column in as-delivered condition (which was cold-bent as indicated in Figure 62), was tested axially. The test results can be seen in Figure 63. Although the residual stress pattern assumed in the calculations does not exactly correspond to the measured pattern in the 5WF18 1/2 column the trend predicted by the theory is also observed in the tests (Figure 63). Namely, that the critical stress for the straightened column was higher than for the as-delivered column for $L/r < 80$.

11.3 Instability of Eccentrically Loaded Columns

The solutions of the eccentric column problem are shown in Figure 46. The secant solution, which does not include residual stress, is unsafe for the eccentricity ratio $\frac{ae}{r} = 1$ within the medium column range ($60 < L/r < 120$). The approximate solutions are in very good agreement with the "exact" solution. The approximate solution based on a rectangle gives satisfactory results rapidly. In reference 13 the
solution of the eccentric column problem is given for a symmetric residual stress pattern. In Figure 64 this solution is reproduced. Also shown are points calculated by the approximate theory. The agreement is very satisfactory. The angle $\varphi$, a measure for the change of residual compressive stress at the outer fibers, is the most important parameter for the strength of the eccentrically loaded column. The rest of the distribution is of importance only if the eccentricity is either very small or very large or if the slenderness ratio is small, (Compare the applied stresses in Figure 42d).

The yield point is an important factor in the inelastic strength of steel columns. In the cross-section tests a variation of the yield point up to $\pm 10\%$ from the average 36 ksi has been observed. Therefore, for short columns ($L/r < 40$) the variation in yield point can amount up to about 20% variation in critical stress. This influence diminishes with increasing slenderness ratio.

* * * * * *

With the aid of the theory presented in this dissertation it is then possible to solve both the axial and the eccentric column problem when residual stresses are present in the member.

Further research is desirable to obtain experimental verification of the theory of eccentrically loaded columns. The development of a solution taking into account combined bending and torsional buckling is also desirable.
12. SUMMARY

1. Residual stresses in columns are formed as a result of permanent deformation of certain fibers during the cooling process, or due to plastic deformation during fabrication, etc.

2. Residual stresses in long beams and their deformations are determined by one fundamental equation (equation 3.4) and modifications thereof. This theory is based on the same assumptions that are used in the elementary theory of bending of beams.

3. At the ends of the beams the theory does not apply within a length of about twice the larger cross-sectional dimension (Figure 22).

4. The magnitude and distribution of cooling residual stresses depend on shape, initial temperature, cooling conditions, and material properties.

5. The influence of residual stresses on the stress-strain relationship can be predicted theoretically and is also observed in tests (Figures 35, 36, 54 and 55).

6. The average value of the static yield point of all column wide-flange shapes is 36,000 psi compression (maximum value 39,800 psi, minimum 31,400 psi).

7. The magnitude of the influence of residual stresses upon column instability is found to depend on various factors:
   (a) the distribution and magnitude of residual stress
   (b) the loading condition (axial-eccentric)
   (c) the shape of the section
   (d) the axis of buckling
   (e) the stress-strain relation of the material.

8. Axially loaded columns made of a material with a definite yield point (mild steel) show a reduction of strength due to residual stresses. However columns whose total stresses do not exceed the
yield point will be unaffected by the presence of residual stresses (Figure 39).

9. The straight-line column curve for the inelastic range of axially-loaded columns, as found empirically almost half a century ago by Tetmajer, has been justified for the first time theoretically by the consideration of residual stresses.

10. Columns of a material without a definite yield level, and with a continuously curving stress-strain diagram, show an influence of residual stress for the whole range of $L/r$. For such materials, and for certain residual stress distributions, a higher buckling strength can be achieved than for zero residuals. (Figure 38).

11. The critical stresses for columns with equal end eccentricities and for axially loaded columns where residual stresses have only one axis of symmetry, can also be predicted (Figure 46). Note that by this theory the secant solution is "unsafe" for $ec/r^2 = 0.1$ and $L/r > 60$.

12. The exact solution of the eccentric column is very cumbersome. An approximate method has been developed which permits a rapid calculation of critical stresses (Figures 46 and 64).
APPENDIX

APPROXIMATE SOLUTION OF RECTANGULAR BEAM COLUMNS

In this appendix the derivation is made of the approximate solution of a rectangular section containing residual stress and which is subjected to axial loading and a symmetric bending moment. The assumptions are stated in section 9.3 and are not repeated here.

Case I:

Axial equilibrium requires that at the center of the column the following equation must be satisfied:

$$\frac{P}{A} = \int_{-b/2}^{b/2} \left[ \sigma_p + \frac{E}{2}(\frac{b}{2} - x) \right] dx + \int_{-b/2}^{b/2} \left[ \sigma_p + \frac{E}{2}b - Ex(x + \phi_0) + E\phi_0 x \right] dx$$

from which follows after integration and combination of terms

$$0 = x_1^2 + \frac{b}{2} - \frac{\sigma_p}{E(\gamma + \phi_0)} \frac{2b}{E} + \frac{b^2}{4} + \frac{2}{E\alpha} \frac{1}{\gamma + \phi_0}$$

and solving for $x_1$

$$x_1 = -\frac{b}{2} + \sqrt{\frac{2\sigma_p b + E\gamma b^2 - \frac{2P}{E\alpha}}{E(\gamma + \phi_0)}} \tag{A.1}$$

Moment equilibrium at the center section requires that

$$\frac{M}{A} = \frac{P}{A} u_0 + \frac{m_0}{A} = \int_{-b/2}^{b/2} \left[ \sigma_p + \frac{1}{2} E\gamma b x - E\gamma x^2 \right] dx + \int_{-b/2}^{b/2} \left[ \sigma_p + \frac{1}{2} E\gamma b - Ex(x + \phi_0) \right] x + E\phi_0 x^2 dx$$
After integration and combination of terms we can write the following function:

\[ F = x_1^3(\varphi + \phi_o) - \frac{3}{4}x_1b_1^3(\varphi + \phi_o) + \frac{b_1^3}{4}(\varphi - \phi_o) + \frac{6Pm_o}{Ev} + \frac{6m_o}{Ea} = 0 \quad (A.2) \]

Where \( \phi_o = -\frac{\pi^2}{L^2} u_o \) is from the assumed sine deflection curve. The partial derivative of \( F \) with respect to \( u_o \) gives:

\[
(x_1 + \frac{b}{2})^3 = \left( \frac{2\sigma_p b + E\varphi b^2 - \frac{2P}{\alpha}}{E(\varphi + \pi^2/L^2 u_o)} \right)^3 = \frac{12P^2L^2}{Ea\pi^2} \quad (A.3)
\]

Expressing \( u_o \) from equation (A.3) and substituting into equation (A.2) we finally get:

\[
0 = \left( \frac{2P^2L^2}{Ea\pi^2} \right)^3 - b + \frac{1}{3} \frac{E\varphi b^3 - 4b^3P^2}{\alpha} + \frac{4m_o}{2\sigma_p b + E\varphi b^2 - \frac{2P}{\alpha}}
\]

from which equation (9.11) was directly obtained.

**Case II (Figure 48c):**

As above we write the equations for axial and moment equilibrium at the column center:

\[
\frac{P}{\alpha} = \int_{-b/2}^{b/2} \left[ \sigma_p + E\varphi (\frac{b}{2} - x) \right] dx + \int_{-b/2}^{b/2} \left[ \sigma_p + \frac{1}{2} E\varphi b - E x (\varphi + \phi_o) + E\varphi x \right] dx + \\
+ \int_{-b/2}^{b/2} \left[ \sigma_p + E\varphi (\frac{b}{2} - x) \right] dx
\]
from which follows after integration:

$$0 = - \frac{2P}{E\alpha} - (\varphi + \phi_0)x_1^2 + (\varphi + \phi_0)x_2^2$$

However, $x_1$ and $x_2$ are related to each other by geometry (Figure 48c):

$$\frac{1}{E}(2\sigma_p + Eyb) = x_1(\varphi + \phi_0) + x_2(\varphi + \phi_0)$$

solving the two equations simultaneously:

$$x_1 = \frac{\sigma_p + \frac{1}{2}Eyb}{E(\varphi + \phi_0)} - \frac{P}{2\sigma_p a + Eyab} \tag{A.4}$$

$$x_2 = \frac{\sigma_p + \frac{1}{2}Eyb}{E(\varphi + \phi_0)} + \frac{P}{2\sigma_p a + Eyab} \tag{A.5}$$

The moment at the center of the column equals to:

$$\frac{M_0}{a} = \frac{P}{a}u_0 + \frac{m_o}{a} = \int_{x_1}^{b/2} \left[ \left( \sigma_p + \frac{1}{2}Eyb \right) x - Eyx^2 \right] dx +$$

$$+ \int_{-x_2}^{-v_2} \left[ \left( \sigma_p + \frac{1}{2}Eyb \right) - E\varphi_1(x + \phi_0)x + E\varphi_0 x^2 \right] dx + \int_{-v_2}^{-x_2} \left[ \left( \sigma_p + \frac{1}{2}Eyb \right) x - Eyx^2 \right] dx$$

from which follows:

$$F = \frac{3}{2} \frac{E}{\alpha}(\sigma_p + \frac{1}{2}Eyb) - \frac{1}{2}\varphi b - x_1^3(\varphi + \phi_0) - x_2^3(\varphi + \phi_0) - \frac{6P}{E\alpha} u_0 - \frac{6m_o}{E\alpha} = 0 \tag{A.6}$$
where as before $\phi_o$ is a linear function of $u_o$. The partial derivative of $\Phi$ gives:

$$
(p + \frac{1}{2} u_o^2)^3 = \frac{2 \left( \frac{\phi_o}{E} + \frac{1}{2} b y \right)^3}{3 P \frac{E}{E_o \pi^2}}
$$

(A.7)

Expressing $u_o$ from equation (4.7) and substituting into equation (A.6) we finally get:

$$
0 = \left[ \frac{2 \left( \frac{\phi_o}{E} + \frac{1}{2} b y \right)^3}{3 P \frac{E}{E_o \pi^2}} \right] \left[ \frac{3}{2} \frac{b}{E_o \pi^2} \left( \frac{\phi_o}{E} + \frac{1}{2} b y \right) - \frac{1}{2} b^3 - \frac{6 \phi_o}{E_o} - \frac{3 P^2}{6 \phi_o + \frac{1}{2} b y} + \frac{6 P^2}{E_o \pi^2} \right] - 6 \left( \frac{\phi_o}{E} + \frac{1}{2} b y \right)^3
$$

from which equation (9.12) was directly obtained.
### TABLE 1

**Test Program**

<table>
<thead>
<tr>
<th>Shape</th>
<th>Coupon Tests</th>
<th>Residual Stresses</th>
<th>Cross-Section Tests</th>
<th>Strong Axis Columns L/r</th>
<th>Weak Axis Columns L/r</th>
</tr>
</thead>
<tbody>
<tr>
<td>4WF13</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>106(^a)</td>
<td>90(^a)</td>
</tr>
<tr>
<td>5WF181/2</td>
<td>1(^b)</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6H15 1/2</td>
<td>10</td>
<td>1</td>
<td>2</td>
<td></td>
<td>82</td>
</tr>
<tr>
<td>8WF24</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8WF31</td>
<td>42</td>
<td>4</td>
<td>4</td>
<td>28(^a), 42(^a), 56(^a)</td>
<td>58, 82</td>
</tr>
<tr>
<td>8WF57</td>
<td>2(^c)</td>
<td></td>
<td>2(^d)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12J14</td>
<td>10</td>
<td>2</td>
<td>1</td>
<td></td>
<td>67</td>
</tr>
<tr>
<td>12WF50</td>
<td>10</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12WF65</td>
<td>10</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14WF43</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14WF426</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>36WF150</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* Indicates number of tests performed.
(a) Results from other programs.
(b) Cold-bent material.
(c) Measurements along one beam.
(d) Annealed material.

### TABLE 2

**Equivalent Eccentricities and Initial Deflections (8WF31)**

<table>
<thead>
<tr>
<th>L, in.</th>
<th>(\delta_e), in.</th>
<th>(e, L/r^2)</th>
<th>(a, L/r)</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>146</td>
<td>0.018</td>
<td>0.097</td>
<td>0.032</td>
<td>0.119</td>
</tr>
<tr>
<td>194</td>
<td>0.002</td>
<td>0.005</td>
<td>0.002</td>
<td>0.007</td>
</tr>
<tr>
<td>164</td>
<td>0.025</td>
<td>0.020</td>
<td>0.020</td>
<td>0.026</td>
</tr>
<tr>
<td>116</td>
<td>0.016</td>
<td>0.030</td>
<td>0.038</td>
<td>0.048</td>
</tr>
<tr>
<td>164</td>
<td>0.045</td>
<td>0.037</td>
<td>0.037</td>
<td>0.046</td>
</tr>
<tr>
<td>116</td>
<td>0.012</td>
<td>0.027</td>
<td>0.027</td>
<td>0.036</td>
</tr>
</tbody>
</table>
FIG. 1: BEAM SECTION & INTERNAL STRAINS

FIG. 2: IDEALIZED STRESS-STRAIN CURVE

FIG. 3: PARTIALLY YIELDED SECTION

FIG. 4: SECTION WITH GENERAL STRESS-STRAIN RELATION

FIG. 5: PLATE DIMENSIONS
Constants for steel

\[ h = 0.1 \]

\[ a^2 = 1.18 \text{ in/min} \]

(a)

(b)

(c)

(d)

(e)

FIG. 6: DEVELOPMENT OF RESIDUAL STRESSES IN A STEEL PLATE

(Continued next page)
FIG. 6: CONTINUED
FIG. 7: RESIDUAL STRESS DISTRIBUTIONS FOR VARIOUS TIMES
FIG. 8: PARABOLIC RESIDUAL STRESS DISTRIBUTION IN A WF

FIG. 9: RESIDUAL STRESSES AS FUNCTION OF WF DIMENSIONS
FIGS. 10 to 12: RESIDUAL STRESSES DUE TO VARIOUS ASSUMED STRAINS
FIGS. 13 to 15: RESIDUAL STRESSES DUE TO VARIOUS ASSUMED STRAINS

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\frac{d}{b} & \frac{w}{t} & \frac{\sigma_{rc}}{E\varepsilon_0} & \frac{\sigma_{ro}}{E\varepsilon_0} & \frac{\sigma_{rt}}{E\varepsilon_0} & \frac{\sigma_{rw}}{E\varepsilon_0} \\
\hline
1.0 & 0.667 & -0.333 & 0.666 & 0.666 & -0.333 \\
\hline
\end{array}
\]
FIGS. 16 to 18: RESIDUAL STRESSES DUE TO VARIOUS ASSUMED STRAINS
FIGS. 19 and 20: RESIDUAL STRESSES DUE TO VARIOUS ASSUMED STRAINS

VALUES ARE RATIO OF $\frac{\sigma_r}{E\epsilon_o}$

FIG. 21: WEB-PLATE OF WF SECTION
3 Term Cosine Series

\[
\frac{\sigma_x}{\sigma_o}
\]

FIG. 22: NORMAL AND SHEARING STRESSES IN THE WEB-PLATE

\[
0.1 \frac{\tau_{xy}}{\sigma_o} \bigg|_{y=c}
\]

FIG. 23: COLD-BENDING STRESSES IN STEEL BEAM
FIGS. 24 to 26: INITIAL RESIDUAL STRESSES AND PLASTIC DEFORMATIONS
FIG. 27: INITIAL RESIDUAL STRESS AND PLASTIC DEFORMATIONS
FIG. 28: RESIDUAL STRESSES IN A PLATE - E VARIABLE

FIG. 29: RESIDUAL STRESSES IN A PLATE - PARABOLIC STRESS-STRAIN CURVE

FIG. 30: RESIDUAL STRESSES IN A PLATE - PARABOLIC STRESS-STRAIN CURVE
FIG. 31: REDISTRIBUTION OF RESIDUAL STRESS DUE TO CUTTING

FIG. 32: REDISTRIBUTION OF RESIDUAL STRESS DUE TO SPLITTING OF H-BEAM
FIG. 33: AXIALLY LOADED COLUMNS - NOTATIONS

(a) STRESS WEB PATTERNS

(b) RESIDUAL STRESS

(c) TOTAL STRAIN

(d) APPLIED STRESS

FIG. 34: STRESS AND STRAIN DISTRIBUTIONS FOR WF SHAPE — FLANGE PARTIALLY YIELDED
FIG. 35: STRESS-STRAIN CURVE FOR WF SHAPE CONTAINING RESIDUAL STRESS - CASE I

FIG. 36: STRESS-STRAIN CURVE FOR WF SHAPE CONTAINING RESIDUAL STRESS - CASE II
FIG. 37: COMPARISON OF MEASURED RESIDUAL STRESSES WITH STRESSES COMPUTED FROM CROSS-SECTION TESTS
FIG. 38: AXIAL INSTABILITY - PARABOLIC STRESS-STRAIN CURVE
FIG. 39: AXIAL INSTABILITY - RECTANGULAR STEEL COLUMNS
FIG. 40: INFLUENCE OF RESIDUAL STRESS DISTRIBUTION UPON COLUMN STRENGTH
FIG. 41: PROCEDURE FOR THE SOLUTION OF ECCENTRICALLY LOADED COLUMNS
(a) 

\[ \sigma_r = \sigma_y (1 - \frac{11}{4} \frac{x}{b}) \quad (\frac{x}{b} \geq 1/4) \]

\[ \sigma_r = \frac{5}{4} \frac{x}{b} \quad (\frac{x}{b} \leq 1/4) \]

\[ \sigma_y = 40 \text{ ksi} \quad \rho_y = \frac{b}{4} \]

\[ E = 30,000 \text{ ksi} \]

(b) 

(c) 

(d) 

FIG. 42: RESIDUAL STRESS AND APPLIED STRESS IN STEEL COLUMN
FIG. 43: RELATIONS BETWEEN $\frac{P}{P_y}$, $\frac{\epsilon_c}{\epsilon_y}$, $\frac{\phi b}{\epsilon_y}$ - $\frac{u}{b}$.
FIG. 44: RELATIONS BETWEEN $\frac{P}{P_y} - \frac{u}{b} - \frac{\phi_b}{\varepsilon_y}$
\[ W_n = \frac{\lambda}{12} (\phi_{n-1} + 10 \phi_n + \phi_{n+1}) \]

\[ \frac{P}{F_y} = 0.8 \quad \frac{e}{b} = 0.0125 \]

\[ \lambda = \frac{L}{8}, \quad b^2 = 16r^2, \quad \epsilon_y = \frac{4}{3} \times 10^{-3} \]

<table>
<thead>
<tr>
<th>POINT</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>( \xi )</th>
<th>FACTOR</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u_1-e/b )</td>
<td>0</td>
<td>0.0273</td>
<td>0.0498</td>
<td>0.0647</td>
<td>0.0700</td>
<td>1.0</td>
</tr>
<tr>
<td>( u_1/b )</td>
<td>0.0125</td>
<td>0.0398</td>
<td>0.0623</td>
<td>0.0772</td>
<td>0.0825</td>
<td>1.0</td>
</tr>
<tr>
<td>( \phi_2 )</td>
<td>0.240</td>
<td>0.805</td>
<td>1.310</td>
<td>1.770</td>
<td>1.930</td>
<td>( \epsilon_y/b )</td>
</tr>
<tr>
<td>( W_n )</td>
<td>9.600</td>
<td>15.675</td>
<td>20.940</td>
<td>11.420</td>
<td>( \epsilon_y/12b )</td>
<td></td>
</tr>
<tr>
<td>SHEAR</td>
<td>57.635</td>
<td>48.035</td>
<td>32.360</td>
<td>11.420</td>
<td>( \epsilon_y/12b )</td>
<td></td>
</tr>
<tr>
<td>( u_2-e/b )</td>
<td>0</td>
<td>57.635</td>
<td>105.670</td>
<td>138.030</td>
<td>149.450</td>
<td>( \epsilon_y^2/12b^2 )</td>
</tr>
<tr>
<td>CHECK</td>
<td>0</td>
<td>0.0270</td>
<td>0.0495</td>
<td>0.0646</td>
<td>0.0700</td>
<td>1.0</td>
</tr>
</tbody>
</table>

\[ 0.07 = 149.45 \frac{\epsilon_y \lambda^2}{12 b^2} \quad \left( \frac{L}{r} \right)^2 = \frac{0.07}{149.45} 9.22 \times 10^6 = (66.4)^2 \]

FIG. 45: EXAMPLE OF NUMERICAL DEFLECTION CALCULATION
FIG. 46: AXIAL AND ECCENTRIC COLUMN CURVES - COLD BENDING RESIDUAL STRESSES
FIG. 47: PARTIAL COSINE DEFORMATION CURVE

FIG. 48: APPROXIMATE ANALYSIS - RECTANGULAR COLUMN
Fig. 49: Cross-Section Test Specimen

Fig. 50: Typical Column Test Set-up
FIG: - 51. RESIDUAL STRESS DISTRIBUTIONS IN WF-SHAPES

(IDEALIZED)

(Continued next page)
FIG: - 51 CONTINUED
FIG: 51 CONCLUDED
FIG. 52: (a) VARIATION OF RESIDUAL STRESS (KSI) ALONG A 8WF31 BEAM

(b) THE VARIATION OF RESIDUAL STRESS (KSI) AT SECTION i-j AS A FUNCTION OF CROSS-SECTION LENGTH
FIG. 53: COLD BENDING RESIDUAL STRESSES IN 8WF31

FIG. 54: CROSS-SECTION STRESS-STRAIN CURVE FOR AS-DELIVERED MATERIAL

FIG. 55: CROSS SECTION STRESS-STRAIN CURVE FOR ANNEALED MATERIAL
FIG. 56: COLUMN CURVES FROM CROSS-SECTION TESTS - STRONG AXIS
FIG. 57: COLUMN TEST RESULTS AND PARABOLIC COLUMN CURVE - STRONG AXIS
FIG. 58: COLUMN CURVES FROM CROSS-SECTION TESTS - WEAK AXIS
FIG. 59: COLUMN TEST RESULTS AND STRAIGHT-LINE COLUMN CURVE - WEAK AXIS
FIG. 60: COLUMN CURVES FROM RESIDUAL STRESSES - STRONG AXIS
FIG. 61 COLUMN CURVE FROM RESIDUAL STRESSES - WEAK AXIS
FIG. 62: COLD BENDING OF 5WF18 1/2 COLUMN AND DEFLECTIONS

FIG. 63: COLUMN CURVES AND TESTS FOR 5WF18 1/2
FIG. 64: ECCENTRIC COLUMN CURVES (13) - COMPARISON WITH APPROXIMATE SOLUTION
# NOTATIONS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a^2)</td>
<td>thermal diffusivity</td>
</tr>
<tr>
<td>(A)</td>
<td>cross-sectional area</td>
</tr>
<tr>
<td>(A_e)</td>
<td>area of the unyielded part of the cross section</td>
</tr>
<tr>
<td>(A_f)</td>
<td>area of both flanges of WF shape</td>
</tr>
<tr>
<td>(A_{\text{num}})</td>
<td>integration constant</td>
</tr>
<tr>
<td>(A_w = wd)</td>
<td>approximated web area</td>
</tr>
<tr>
<td>(b)</td>
<td>flange width</td>
</tr>
<tr>
<td>(c)</td>
<td>plate dimension, a constant</td>
</tr>
<tr>
<td>(d)</td>
<td>depth of H section between center lines of flanges</td>
</tr>
<tr>
<td>(e)</td>
<td>eccentricity of loads</td>
</tr>
<tr>
<td>(E)</td>
<td>Young's modulus of elasticity</td>
</tr>
<tr>
<td>(E_t)</td>
<td>tangent modulus</td>
</tr>
<tr>
<td>(h = c/k)</td>
<td>a constant</td>
</tr>
<tr>
<td>(I)</td>
<td>moment of inertia</td>
</tr>
<tr>
<td>(I_e)</td>
<td>moment of inertia of the unyielded (elastic) part</td>
</tr>
<tr>
<td>(J_x)</td>
<td>(E(y)y^2dA)</td>
</tr>
<tr>
<td>(k)</td>
<td>thermal conductivity</td>
</tr>
<tr>
<td>(L)</td>
<td>total length of a pin-ended column</td>
</tr>
<tr>
<td>(L/r)</td>
<td>slenderness ratio</td>
</tr>
<tr>
<td>(M)</td>
<td>moment</td>
</tr>
<tr>
<td>(m)</td>
<td>moment due to lateral loads on column</td>
</tr>
<tr>
<td>(P)</td>
<td>load on a column</td>
</tr>
<tr>
<td>(P_{\text{cr}})</td>
<td>critical load on a column; in the case of axial load alone it is (P_e) for buckling in the elastic range</td>
</tr>
<tr>
<td>(P_e)</td>
<td>Euler buckling load for a pin-ended column</td>
</tr>
<tr>
<td>(P_t)</td>
<td>tangent modulus load, the load at which bending of a perfectly straight column may commence</td>
</tr>
<tr>
<td>(P_y)</td>
<td>axial load corresponding to yield point stress across entire section</td>
</tr>
<tr>
<td>(q)</td>
<td>heat emission per unit area</td>
</tr>
<tr>
<td>(r)</td>
<td>radius of gyration in the plane of bending</td>
</tr>
<tr>
<td>(t)</td>
<td>flange thickness, time</td>
</tr>
<tr>
<td>(T)</td>
<td>temperature</td>
</tr>
<tr>
<td>(u)</td>
<td>deflection in x-direction</td>
</tr>
<tr>
<td>(w)</td>
<td>web thickness</td>
</tr>
<tr>
<td>(x_0, x_1, y_1)</td>
<td>distances from the center to the beginning of the yielded area</td>
</tr>
<tr>
<td>(\alpha_t)</td>
<td>coefficient of linear expansion</td>
</tr>
<tr>
<td>(\varepsilon)</td>
<td>unit strain</td>
</tr>
<tr>
<td>(\varepsilon_0)</td>
<td>reference strain</td>
</tr>
<tr>
<td>(\varepsilon_c)</td>
<td>maximum compressive stress</td>
</tr>
<tr>
<td>(\varepsilon_y)</td>
<td>strain corresponding to the yield point</td>
</tr>
<tr>
<td>(\sigma)</td>
<td>normal stress</td>
</tr>
<tr>
<td>(\sigma_p)</td>
<td>proportional limit stress</td>
</tr>
<tr>
<td>(\sigma_{\text{cr}} = P/A)</td>
<td>applied average stress on a column</td>
</tr>
<tr>
<td>(\sigma_r)</td>
<td>residual stress</td>
</tr>
<tr>
<td>(\sigma_{\text{rc}})</td>
<td>residual stress at flange edges</td>
</tr>
<tr>
<td>(\sigma_{\text{ro}})</td>
<td>residual stress at flange centers</td>
</tr>
<tr>
<td>(\sigma_{\text{rw}})</td>
<td>residual stress at web center</td>
</tr>
<tr>
<td>(\sigma_{\text{rt}})</td>
<td>residual stress at web edges</td>
</tr>
</tbody>
</table>
$\sigma_y$: yield stress level; average stress in the plastic range
$\gamma$: rate of residual stress variation
$\phi$: curvature, Airy stress function
$\lambda$: panel length
LIST OF REFERENCES


V I T A

The author was born as the second child of Franz and Amalia Huber on May 22, 1926 in Friesach, Austria.

After secondary education in Vienna and brief World War II service he entered the Technische Hochschule in Graz, Austria to pursue a career in Civil Engineering. He received the diploma of the first state examination in June 1949 and was awarded the Garrett-Linderman-Happes Fellowship of Lehigh University in the following year. Since 1952 he has held a position as Research Assistant at Fritz Engineering Laboratory and Instructor in the Division of Mechanics. In June 1952 he received the Master of Science degree in Civil Engineering from Lehigh University. He has been associated with a project on residual stress and column strength during this time.