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Robust Performance Attribution Analysis in Investment Management

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Abstract

This dissertation investigates robust optimization models for performance attribution analysis in investment management. Specifically, an investment manager seeks to evaluate the performance of fund managers who manage funds he might invest his clients' money in. A key difficulty for the investment manager is to quantify the fund manager's skill when he may not know the fund manager's allocation precisely. This introduces two main sources of uncertainty for the investment manager: the stock returns and the fund allocations. This dissertation proposes and analyzes robust, quantitative models to address this challenge. We study a robust counterpart to the mean-variance framework when the fund managers' precise allocations are uncertain but belong to known intervals and must sum to one for each manager, present an algorithm to solve the problem efficiently and analyze the investment manager's allocation in the various funds as a function of the benchmark return. Further, we consider the case where the stock returns are also represented as uncertain parameters belonging to a polyhedral set, the size of which is defined by a parameter called the budget of uncertainty, and the investment manager seeks to maximize his worst-case return. We describe how to solve this problem efficiently and analyze how the investment manager's degree of diversification and his specific allocations in the funds vary with the budget of uncertainty.

Chapter 1

Literature Review and Contributions

1.1 Motivation

Traditional performance attribution analysis decomposes the fund return into an *active management component* (reflecting the fund manager's skill) and a *passive management component* (reflecting stock performance) by taking the average of the active-management and passive-management returns, respectively, over a given time period. This approach does not investigate the fund managers' strategies, especially with respect to possible market scenarios, in great detail. In fact, investors could lose important information on fund managers' ability to manage their assets when facing different market conditions, especially in times of crisis, which makes it difficult to evaluate the risks associated with the fund manager's strategy under those circumstances.

From a portfolio construction perspective, investors usually construct their portfolio by considering a trade-off between maximizing the total return and minimizing the total (or downside) risk of the portfolio. The traditional portfolio model does not take into account the uncertainty associate with manager's asset allocation in the portfolio construction procedure. In this research, we will investigate the interaction effect of the active management from multiple managers, and seek to incorporate uncertainty embedded in both asset return and manager's asset allocation in the context of portfolio construction model. This dissertation describes a robust optimization approach to select fund managers by considering two majority risk fund manager bears, with contributions presented in Section 1.2 below.

1.2 Contributions

The high-level contribution we make through this dissertation is to provide financial professionals with robust, quantitative decision tools to help them make manager selection policy in presence of uncertainty, especially taking into account uncertainty on the fund allocation and capturing that stock returns and uncertainty on asset return itself.

We propose two models that build upon the robust optimization framework by considering multi-source uncertainties:

i. Robust Portfolio Management with Uncertainty in Asset Allocation Attribution Analysis,

ii. Robust Portfolio Management with Uncertainty in Asset Allocation and Asset Return.

This dissertation is organized as follows. The remainder of this chapter contains a literature review of performance attribution analysis and robust portfolio management. Chapter 2 presents the model taking account uncertainty in manager's asset allocation, with Section 2.1 describing the motivation of the model, Section 2.2 describing the robust manager selection framework, Section 2.3 proposing two approaches to solve the problem efficiently. Section 2.4 presenting solid numerical results of manager selection policy under our new model and old model, and demonstrating that the robust model provide a strong protection of the downside return, 2.5 concluding the contribution and remarks of the framework. Chapter 3 presents the second model by taking account uncertainty in both manager's asset allocation and asset return, with Section 3.1 describing the motivation of the model, Section 3.2 describing the problem setup and two frameworks, Section 3.3 presenting the upper and lower bound of the robust framework, Section 3.4 exploring the special structure of the problem and proposing two approaches to solve the problem efficiently. Section 3.5 presenting solid numerical results of manager selection policy under our new model, 3.6 concluding the contributions.

1.3 Literature Review

Fund performance attribution analysis is a way to explain the fund manager's performance versus the benchmark. It is very important for investors because it can help them better understand how the return is generated and why the return for a fund differs from a benchmark's. For some of the funds, asset allocation plays the most important role in the total return, while for some other funds, stock selection contributes more. A better understanding of performance attribution will allow investors to make better investment decisions and find the best fund suitable for their purposes and risk tolerance. In addition, in presence of high volatility and high correlation between asset classes – as is the case nowadays – the ability of taking "robust" decisions when making guesses on the market is crucial for a successful fund manager. Comparing with traditional ways of appraising fund managers' performance such as his alpha, Sharpe ratio and information ratio, a dynamic fund attribution analysis provides more information about a fund manager's forecasting power and risk management skills.

1.3.1 Arithmetic excess return

Arithmetic excess return can be expressed as the difference between the portfolio (arithmetic average) return and benchmark return. It captures by how much the manager's fund beat the benchmark as a percentage of the initial investment. The key rules for the arithmetic model are that (i) the sum of different attribution effects must equal the arithmetic excess return, and (ii) for multi-period frameworks, the sum of different attribution effects over time must equal the arithmetic excess return over the whole time horizon considered.

Brinson Model

Brinson et al. [16] have proposed a method to decompose the manager's added value into three factors:

- Asset Allocation,
- Stock Selection,
- Intersection between the two.

The *asset allocation effect* measures the manager's skill in allocating his assets among different industry sectors. It can be expressed as the difference between returns if the benchmark returns are applied to the portfolio weight and the benchmark weights, respectively, i.e. $\sum_{i\in I} R_{bi}(\omega_{fi}-\omega_{bi})$. The *security selection effect* is to measure the manager's skill in picking the high-return stocks in each of these groups. It can be expressed as the difference between returns if the benchmark weights are applied to the difference between the fund return and benchmark return, i.e. $\sum_{i \in I} \omega_{bi}(R_{fi} - R_{bi})$. The *intersection between the two* is the product of the difference in weight and difference in returns, i.e. $\sum_{i \in I}(\omega_{fi} - \omega_{bi})(R_{fi} - R_{bi})$. The summation is performed based on different sectors, for instance, country sector, industry sector, size or momentums.

Drawbacks of Brinson et al. [16]'s analysis are that (i) it does not incorporate the fact that overweighting a portfolio in a negative market that has outperformed the overall benchmark should still be a positive effect, (ii) it fails to distinguish between the static manager and dynamic manager, who is trying to capture the opportunity when the market is up for one sector and over-weigh his portfolio weights on that sector.

The model in Brinson and Fachler [15] solves the first weakness by modifying the asset allocation factor to compare the return against the overall benchmark, as opposed to considering the pure negative or positive return for a sector. The *asset allocation factor* in the Brinson and Fachler model is $\sum_i(\omega_{fi} - \omega_{bi})(r_{bi} - r_b)$, where r_b is the overall return of the benchmark, and i denotes the sector. The fund manager could generate positive asset allocation return in two situations, overweight in an above-average performance sector or underweight in a below-average sector. The *security selection effect* is the same with the original model, which is to measure manager's skill in picking the high-return stocks in each of these groups. It can be expressed by applying the benchmark weights to the difference between the fund return and benchmark return, i.e. $\sum_i \omega_{bi}(r_{fi} - r_{bi})$. The authors argue that this component might not reflect the stock selection effect correctly due to market size effect. A minor outperformance in a large-cap market could result in a larger stock selection component than a substantial outperformance in a small-cap market. This weakness could be overcome by constructing a time series of the differential between the fund return and benchmark return for a sector. The *intersection between the two* is also the same as in the original model, i.e., it is the cross product of the difference in weight and difference in returns: $\sum_i(\omega_{fi} - \omega_{bi})(r_{fi} - r_{bi})$. However, given that the interaction term is not part of the investment decision process, Brinson and Fachler further improved the model by including the interaction effect into the stock selection effect. Actual portfolio weights, as opposed to benchmark portfolio weights, are then used to calculate the stock selection effect. The stock selection factor is therefore $\sum_i \omega_{fi}(r_{fi} - r_{bi})$ in the revised model.

Examples of three different single-period models are illustrated in Tables 1.1 and 1.2.

Table 1.1: Single Period Data [7]

Equity Portfolio	Portfolio Weight	Benchmark Weight	Portfolio Return	Benchmark Return	Excess Return	Allocation Notional Return	Selection Notional Return
UK Equity	40.00%	40.00%	20.00%	10.00%	4.00%	4.00%	8.00%
Japanese equity	30.00%	20.00%	-5.00%	-4.00%	-0.70%	-1.20%	-1.00%
US Equity	30.00%	40.00%	6.00%	8.00%	-1.40%	2.40%	2.40%
Total	100.00%	100.00%	8.30%	6.40%	1.90%	5.20%	9.40%

Table 1.2: Single Period Results [7]

	Arithmetic Model								Geometric Model	
	Brinson Arithmetic			Brinson & Fachler Model 1			Brinson & Fachler Model 2		Geometric	
Equity Portfolio	Asset Allocation	Stock Selection	Intersection	Asset Allocation	Stock Selection	Intersection	Asset Allocation	Stock Selection	Asset Allocation	Security Selection
UK Equity	0.00%	4.00%	0.00%	0.00%	4.00%	0.00%	0.00%	4.00%	0.00%	4.00%
Japanese equity	-0.40%	-0.20%	-0.10%	-0.40%	-0.20%	-0.10%	-0.40%	-0.30%	-0.40%	-0.30%
US Equity	-0.80%	-0.80%	0.20%	-0.80%	-0.80%	0.20%	-0.80%	-0.60%	-0.80%	-0.60%
Total	-1.20%	3.00%	0.10%	-1.20%	3.00%	0.10%	-1.20%	3.10%	-1.20%	3.10%

Multi-period Model

Multi-period arithmetic model suffers from a linking issue of transferring single period attribution factors to total period attribution factors. Since the sum of the excess returns for each period does not equal the total arithmetic excess return, the arithmetic attribution factors should also not be expected to add up to arithmetic attribution factors for the total period. Carino [19] and Menchero [50] both suggested a smoothing algorithm by introducing a factor that could transform single-period performance to its total-period counterpart by summing up single-period performances attribution over time. By multiplying this factor to the asset allocation factor, security selection factor and intersection factor for each period, these revised attribution factors are additive for the total period. It should be noted that the revised attribution factors are different for the length of overall period for analysis, i.e., the revised attribution factor for a quarter is different from the factor for one year and two years. These factors are effective, but also counter-intuitive and cumbersome. Examples are illustrated in Tables 1.3 and 1.4.

Table 1.3: Multi-Period Model [7]

	Portfolio Weight	Benchmark Weight	Portfolio Return	Benchmark Return	Excess Return	Allocation Notional Return	Selection Notional Return
1st Quarter							
UK Equity	40.00%	40.00%	20.00%	10.00%	10.00%	4.00%	8.00%
Japanese equity	30.00%	20.00%	-5.00%	-4.00%	-1.00%	-1.20%	-1.00%
US Equity	30.00%	40.00%	6.00%	8.00%	-2.00%	2.40%	2.40%
Total	100.00%	100.00%	0.00%	5.20%	-5.20%	5.20%	9.40%
2nd Quarter							
UK Equity	70.00%	40.00%	-5.00%	-7.00%	2.00%	-4.90%	-2.00%
Japanese equity	20.00%	30.00%	3.00%	4.00%	-1.00%	0.80%	0.90%
US Equity	10.00%	30.00%	-5.00%	-10.00%	5.00%	-1.00%	-1.50%
Total	100.00%	100.00%	0.00%	-5.10%	5.10%	-5.10%	-2.60%
3rd Quarter							
UK Equity	30.00%	50.00%	-20.00%	-25.00%	5.00%	-7.50%	-10.00%
Japanese equity	50.00%	40.00%	8.00%	5.00%	3.00%	2.50%	3.20%
US Equity	20.00%	10.00%	-15.00%	-20.00%	5.00%	-4.00%	-1.50%
Total	100.00%	100.00%	0.00%	-9.00%	9.00%	-9.00%	-8.30%
4th Quarter							
UK Equity	30.00%	40.00%	10.00%	5.00%	5.00%	1.50%	4.00%
Japanese equity	50.00%	40.00%	-7.00%	-5.00%	-2.00%	-2.50%	-2.80%
US Equity	20.00%	20.00%	25.00%	10.00%	15.00%	2.00%	5.00%
Total	100.00%	100.00%	0.00%	1.00%	-1.00%	1.00%	6.20%

Table 1.4: Multi-Period Model Results [7]

	Carino Revised Attribution (Brinson and Fachler 2)			Menchero Revised Attribution (Brinson and Fachler 2)				Geometric	
	Carino Factor	Asset Allocation	Stock Selection	M	t	Asset Allocation	Stock Selection	Asset Allocation	Stock Selection
1st Quarter				97.81%					
UK Equity	97.49%	0.00%	3.62%		1.28%	0.00%	0.00%	0.00%	3.66%
Japanese equity		-0.94%	-0.27%			0.00%	0.00%	0.13%	-0.27%
US Equity		-0.14%	-0.54%			0.00%	0.00%	-1.39%	-0.55%
Total		-1.09%	2.80%			0.00%	0.00%	-1.27%	2.83%
2nd Quarter									
UK Equity	102.64%	-0.73%	1.42%		0.95%	0.01%	0.00%	-3.45%	1.44%
Japanese equity		-0.87%	-0.20%			0.00%	0.00%	0.10%	-0.21%
US Equity		1.09%	0.51%			-0.01%	0.00%	2.87%	0.51%
Total		-0.51%	1.72%			-0.01%	0.00%	-0.48%	1.75%
3rd Quarter									
UK Equity	104.79%	2.66%	1.60%		5.07%	-0.03%	0.00%	6.24%	1.64%
Japanese equity		1.86%	1.60%			0.05%	0.00%	-0.37%	1.64%
US Equity		-0.80%	1.06%			-0.05%	0.00%	-2.66%	1.09%
Total		3.73%	4.26%			-0.03%	0.00%	3.21%	4.36%
4th Quarter									
UK Equity	99.50%	-0.28%	1.41%		1.69%	-0.01%	0.00%	-0.61%	1.41%
Japanese equity		-0.66%	-0.94%			-0.01%	0.00%	-0.40%	-0.94%
US Equity		0.00%	2.82%			-0.05%	0.00%	0.00%	2.82%
Total		-0.94%	3.29%			-0.07%	0.00%	-1.01%	3.30%
TOTAL		1.19%	12.07%			-0.11%	0.00%	0.40%	12.79%
Four Quarter Total									
UK Equity		1.65%	8.04%			-0.03%	0.00%	1.95%	8.38%
Japanese equity		-0.61%	0.18%			0.03%	0.00%	-0.54%	0.20%
US Equity		0.15%	3.85%			-0.11%	0.00%	-1.26%	3.91%
Total		1.19%	12.07%			-0.11%	0.00%	0.12%	12.83%

Davies and Laker [23] and Kirievsky and Kirievsky [44] suggest applying Brinson's model over multiple periods. They calculate the allocation's notional return and selection's notional return for each time period respectively, then calculate the allocation's notional return and selection's notional return for the total period by compounding the return of each time period. The attribution factors are then the difference between the asset allocation return for compounded notional funds and benchmark return. This method combines the arithmetic and geometric concepts.

1.3.2 Active Performance Measure

The Capital Asset Pricing Model was the first model to decompose the fund return into systematic risk return and unsystematic risk return. The unsystematic return could be considered as the value-added of the manager's performance. Treynor [67], Sharpe [63], Jensen [41] and Jensen [42] developed risk-adjusted performance measures such as the Sharpe Ratio, Treynor Ratio and Information Ratio to evaluate the fund manager's performance. However, the drawbacks are that these measures are all static measures, based on the characteristics of returns of a single time period. Treynor and Mazuy [69] proposed a method to measure the fund managers's ability to capture the up market by introducing the quadratic term $(R_{mt} - R_f)^2$. Arnott et al. [5] and Treynor [68] considered the covariance between portfolio weights and returns but this was only discussed in the context of improving the fundamental indexation. Grinblatt and Titman [35] pointed out that the positive covariance between portfolio weights and returns should bring benefit to investors, and proposed a measure to capture this property.

Dynamic Model

The classic Brinson's analysis is based on a single-period framework and assumes that the portfolio holdings (weights) is static, which is not suitable for active management where weights can be changed over multiple time periods. Dealing with multi-period models by repeating the classic Brinson's method for multiple time periods is subject to several weaknesses as mentioned above.

Lo [47] and Hsu and Myers [38] both proposed approaches to capture the static and dynamic contributions of a fund manager's performance. In their model, weights are considered to be a stochastic process (like asset returns) as opposed to fixed parameters. The dynamic component is measured by the sum of covariances between returns and portfolio weights.

Hsu improved Brinson's model by further decomposing the allocation effect according to the static allocation added-value and dynamic allocation added-value. Considering the cumbersome nature of computing the covariance, the dynamic allocation added-value is computed as the difference between the total allocation added-value and static added-value. In this case, the manager's total added-value is composed of three factors, (i) static allocation added value, (ii) dynamic allocation added value and (iii) security selection added value. Hsu's method could allow investors to easily identify whether a fund is passively managed or actively managed.

In Lo [47]'s model, the sum of covariances between returns and portfolio weights was used to measure the dynamic effect (named Active Component in his paper), and a static weighted-average of the individual securities' expected return was used to measure the static effect (named Passive Component in his paper). Positive covariance between weights and returns implied a successful dynamic management, while a negative one implied a poorly dynamic effect for portfolio's return. Lo further decomposed the covariance matrix as the product of standard deviation of weights, standard deviation of returns and correlation between weights and returns. He argued that both static weight (standard deviation of weights equals to zero) and non-correlation between weights and returns (even weights various over time) could be considered as passive management. Lo [47] also proposed a factor-based Active-Passive decomposition method for the case where the return of each asset satisfied a linear K-factor model.

The significant difference between the two dynamic models is that the definitions of "security selection" and "asset allocation" in Hsu and Myers [38] depend critically on the benchmark, which might be appropriate for managers whose target is to beat certain benchmarks, but will be less natural for certain hedge fund strategies. Lo is trying to capture time-series measures of a manager's forecasting power without considering a specific benchmark.

Examples of Lo's model and Hsu's model are illustrated in Tables 1.5-1.10.

Table 1.5: Active Management Portfolio vs Benchmark

	Portfolio 1: Active Management Portfolio						**Benchmark**					
	Quarterly Portfolio Total Return	Weight	Return	Expected Active Return	Expected Passive Return	Expected Total Return	Quarterly Portfolio Total Return	Weight	Return	Expected Active Return	Expected Passive Return	Expected Total Return
UK Equity												
1st Quarter	13.000%	75%	16%	0.437%	4.470%	4.907%	10.000%	50%	16%	0.000%	4.500%	4.500%
2nd Quarter	9.800%	10%	8%				9.000%	50%	8%			
3rd Quarter	1.200%	64%	3%				0.500%	50%	3%			
Japanese Equity												
1st Quarter	*13.00%*	25%	4%	1.080%	2.013%	3.093%	10.000%	50%	4%	0.000%	2.000%	2.000%
2nd Quarter	*9.80%*	90%	10%				9.000%	50%	10%			
3rd Quarter	*1.20%*	36%	-2%				0.500%	50%	-2%			
Total Portfolio				**1.517%**	**6.483%**	**8.000%**				**0.000%**	**6.500%**	**6.500%**

Table 1.6: Active Management Portfolio vs Benchmark (2)

	Hsu					**Lo**		
	Excess Return Factors Decomposition					Expected Return Analysis For Portfolio 1		
	Stock Selection Factor	Asset Allocation Factor			Total	Return Factors		Total
	Stock Selection	Dynamic Asset Allocation	Static Asset Allocation	Total Asset Allocation	Total Excess Return	Active Component	Active Componen	Expected Total Return
UK Equity								
1st Quarter	0.0000%	0.758%	-0.01%	0.750%	0.7500%	0.437%	4.470%	4.907%
2nd Quarter								
3rd Quarter								
Japanese Equity								
1st Quarter	0.0000%	0.758%	-0.01%	0.750%	0.7500%	1.080%	2.013%	3.093%
2nd Quarter								
3rd Quarter								
Total Portfolio	**0.000%**	**1.517%**	**-0.017%**	**1.500%**	**1.500%**	**1.517%**	**6.483%**	**8.000%**

Table 1.7: Active Management Portfolio vs Passive Management Portfolio

	Portfolio 1: Active Management Portfolio						**Portfolio 2 : Passive Management Portfolio**					
	Quarterly Portfolio Total Return	Weight	Return	Expected Active Return	Expected Passive Return	Expected Total Return	Quarterly Portfolio Total Return	Weight	Return	Expected Active Return	Expected Passive Return	Expected Total Return
UK Equity												
1st Quarter	13.000%	75%	16%	0.437%	4.470%	4.907%	13.600%	80%	16%	0.000%	7.200%	7.200%
2nd Quarter	9.800%	10%	8%				8.400%	80%	8%			
3rd Quarter	1.200%	64%	3%				2.000%	80%	3%			
Japanese Equity												
1st Quarter	*13.00%*	25%	4%	1.080%	2.013%	3.093%	10.000%	20%	4%	0.000%	0.800%	0.800%
2nd Quarter	*9.80%*	90%	10%				9.000%	20%	10%			
3rd Quarter	*1.20%*	36%	-2%				0.500%	20%	-2%			
Total Portfolio				**1.517%**	**6.483%**	**8.000%**				**0.000%**	**8.000%**	**8.000%**

Table 1.8: Active Management Portfolio vs Passive Management Portfolio (2)

	Hsu					**Lo**		
	Excess Return Factors Decomposition					Expected Return Analysis For Portfolio 1		
	Stock Selection Factor	Asset Allocation Factor			Total	Return Factors		Total
	Stock Selection	Dynamic Asset Allocation	Static Asset Allocation	Total Asset Allocation	Total Excess Return	Active Component	Active Component	Expected Total Return
UK Equity								
1st Quarter	0.0000%	0.303%	-0.30%	0.000%	0.0000%	0.437%	4.470%	4.907%
2nd Quarter								
3rd Quarter								
Japanese Equity								
1st Quarter	0.0000%	0.758%	-0.76%	0.000%	0.0000%	1.080%	2.013%	3.093%
2nd Quarter								
3rd Quarter								
Total Portfolio	**0.000%**	**1.062%**	**-1.062%**	**0.000%**	**0.000%**	**1.517%**	**6.483%**	**8.000%**

Table 1.9: Passive Management Portfolio vs Benchmark

	Portfolio 2: Passive Management Portfolio						**Benchmark**					
	Quarterly Portfolio Total Return	Weight	Return	Expected Active Return	Expected Passive Return	Expected Total Return	Quarterly Portfolio Total Return	Weight	Return	Expected Active Return	Expected Passive Return	Expected Total Return
UK Equity												
1st Quarter	13.600%	80%	16%	0.000%	7.200%	7.200%	10.000%	50%	16%	0.000%	4.500%	4.500%
2nd Quarter	8.400%	80%	8%				9.000%	50%	8%			
3rd Quarter	2.000%	80%	3%				0.500%	50%	3%			
Japanese Equity												
1st Quarter	*13.00%*	20%	4%	0.000%	0.800%	0.800%	10.000%	50%	4%	0.000%	2.000%	2.000%
2nd Quarter	*9.80%*	20%	10%				9.000%	50%	10%			
3rd Quarter	*1.20%*	20%	-2%				0.500%	50%	-2%			
Total Portfolio				**0.000%**	**8.000%**	**8.000%**				**0.000%**	**6.500%**	**6.500%**

Table 1.10: Passive Management Portfolio vs Benchmark (2)

	Hsu					**Lo**		
	Excess Return Factors Decomposition					Expected Return Analysis For Portfolio 2		
	Stock Selection Factor	Asset Allocation Factor			Total	Return Factors		Total
	Stock Selection	Dynamic Asset Allocation	Static Asset Allocation	Total Asset Allocation	Total Excess Return	Active Component	Active Component	Expected Total Return
UK Equity								
1st Quarter	0.0000%	0.000%	0.75%	0.750%	0.7500%	0.000%	7.200%	7.200%
2nd Quarter								
3rd Quarter								
Japanese Equity								
1st Quarter	0.0000%	0.000%	0.75%	0.750%	0.7500%	0.000%	0.800%	0.800%
2nd Quarter								
3rd Quarter								
Total Portfolio	**0.000%**	**0.000%**	**1.500%**	**1.500%**	**1.500%**	**0.000%**	**8.000%**	**8.000%**

1.3.3 Geometric excess return

Geometric excess return is the ratio of one plus the portfolio net return divided by one plus the net benchmark return minus one. It measures how much better the manager did than the benchmark as a percentage of the final value of the initial amount invested in the benchmark. The key rule for the geometric excess return model is that the product of different attribution effects must equal the geometric excess return. Further, for multi-periods of time, the product of different attribution effects over time must be equal to the geometric excess return over the total period of time.

Geometric excess return has several advantages compared to arithmetic excess return: (i) Geometric excess returns are easy to do compounding for in multi-period models and do not encounter the linking problem of the arithmetic model, (ii) geometric excess returns are not affected by the currency effect, (iii) since the geometric excess is a percentage relative to benchmark, when the market performs poorly, geometric excess return is much more impressive than when the market is bullish.

A summary on the fund performance attribution analysis can be found in Bacon [7], Morningstar [51] and Morningstar [52].

1.3.4 Factor Attribution Analysis

Factor attribution analysis is another way to decompose and explain the fund return. It could also help the investors to better understand the key factors who drive the fund return and the fund's exposures to different types of risk. Sharpe [62] introduced the Capital Asset Pricing Model which describes the return of a portfolio or a stock as a linear relation between the excess return and its systematic (market) risk. The drawback of CAPM is that it only includes one risk factor, i.e., the risk from the whole market, which may not be the best way to explain the returns, especially when the portfolio is constructed with multiple strategies in mind. Fama and French [28] proposed a three-factor attribution model to describe the stock return with two additional factors: the exposure to small caps and the exposure to high book-to-market ratio. The three-factor attribution model better explains the portfolio returns than the traditional Capital Asset Pricing Model. Carhart [18] added a fourth factor, called the momentum factor, to extend the Fama-French three-factor model in order to capture momentum returns.

Funds are exposed to different types of risks based on their strategy. For equity mutual funds, the standard style attribution analysis is similar to that in the Fama-French three-factor model or Carhart four-factor model. Fung and Hsieh [29] and Fung and Hsieh [30] proposed an asset-based style analysis for analyzing hedge funds. The asset-based style analysis approach is to find the common sources of risk in the returns and link these common risks to observable prices. If the common sources of risk cannot be linked to marketable prices, the factors are called return-based style factors. The authors studied four different types of hedge funds: Trend-Following funds, Merger Arbitrage Funds, Fixed-Income Hedge Funds and Equity Long-Short Hedge Funds, and summarized seven risk factors. These seven factors are market risk, spread between small-cap stock returns, spread between large-cap stock returns, 10-year Treasury yield, yield spread between 10 year T-bonds and Moody's Baa bonds, trend following bond, currency and commodity factors. Agarwal et al. [3], Agarwal and Naik [2] and Jaeger and Wagner [40] performed similar studies on different types of hedge funds as well. Okunev and White [56] also suggested ten categories of potential market factor pool which investors could consider using as regression factors. The factor attribution analysis approach is highly dependent on choosing the right factors. Researchers usually use principal component analysis or stepwise linear regression to select the factors which have the most explanation power. The K-factor linear model is based on the assumption that the factors are stationary. The K-factor attribution analysis usually achieves high in-sample R^2, but often results in poor out-of-sample fit.

Katsaris et al. [43] proposed an approach to incorporate the investor's qualitative analysis into quantitative factor model. They showed that combining the investor's understanding of a fund's strategy with traditional out of sample statistic criteria could lead to a more robust linear factor model. The market risk model is adjusted with the residue risk and tail risk to better fit the true return distribution. The out of sample explanation power R^2 is used as the weighting parameter in the factor to correct the deviation of the expected return from the actual return. In addition, the author proposed an approach to quantify the factors such as liquidity risk, leverage risk and concentration in the size in the tail risk based on the investor's qualitative analysis.

1.3.5 Portfolio Optimization and Risk Management

Risk measures such as variance, mean absolute deviation, downside standard deviation, Value at Risk (VaR), Conditional Value at Risk (CVaR) and tail conditional expectation (TCE) are powerful risk management tools applied in portfolio optimization.

Markowitz [48] proposed the ground-breaking Modern Portfolio Theory, but it is not extensively used in large scale portfolio construction in its original form because it suffers from several limitations. First, it requires several strong assumptions such as: (a) asset returns are (jointly) normally distributed random variables, (b) correlations between assets are fixed and constant over time. Second, the optimal allocation is very sensitive to the inputs. Third, it encounters computation difficulties for large scale portfolio optimization because it requires solving a large-scale quadratic programming problem with dense covariance matrix.

Comparing to the traditional mean-variance approach which uses a complete set of expected return as inputs, Black and Litterman [14] proposed a scheme to incorporate the investor's view in the equilibrium market to produce a new set of expected returns. This allows the optimal portfolio weights to reflect the investor's views. The degree of uncertainty regarding different views is also considered in the model. The Black-Litterman asset allocation model has gained wide acceptance in financial institutions.

Konno and Yamazaki [45] proposed an alternative Mean-Absolute Deviation (MAD) model which removes the normality assumption of asset return. Mean absolute deviation is used as the risk measure in the model instead of variance. MAD is easier to compute comparing to the original Markowitz model because it uses the L_1 risk function which removes the computational difficulties associated with the covariance matrix in the portfolio variance model.

Value-at-Risk (VaR) is another way to measure downside risk. It was first proposed by J.P. Morgan Chase Co. in RiskMetricsTM [59] as a measure of acceptability for a financial position with random return. It is also part of current regulatory frameworks for banks [65]. However, VaR is subject to several serious limitations as well. In particular, it is not a coherent measure because of its lack of sub-additivity (Artzner et al. [6]). VaR is a nonsmooth, nonconvex, and multi-extreme function [49] and results in intractable non-convex stochastic optimization problems both in the context of minimizing VaR or minimizing a convex function with VaR constraint. Extensive research efforts have focused on portfolio optimization with VaR. Examples related to solving the VaR optimization problem include Basak and Shapiro [8], Benati and Rizzi [12], Campbell et al. [17], Campbell et al. [17], Gaivoronski and Pflug [32], Natarajan et al. [54], Pang and Leyffer [57], Pirvu [58], Tasche and Tibiletti [66] and Wozabal et al. [71]. More generally, in quantile based risk management, Rodriguez [61] showed that the quantile based portfolio optimization problem could be solved by using Brute-Force Method, Greedy Linear and Mixed Integer Programming techniques. Cetinkaya and Thiele [21] proposed a fast-convergent approximation method for the portfolio management problem with a quantile criterion which is computationally tractable.

Rockafellar and Uryasev [60] proposed the Conditional Value at Risk (CVaR) risk measure and showed that it is a coherent risk measure [6]. They also obtained a linear programming formulation to solve the mean-CVaR problem efficiently. Acerbi [1] showed that the portfolio risk which was given by the spectral risk measure could always be formulated as an linear programming problem and CVaR can be viewed as a special case of spectral risk measure. However, Alexander et al. [4] showed that the resulting linear problem was very ill-conditioned when the risk-return had a non-linear structure, and was hard to solve when the number of scenarios became large. Lim et al. [46] pointed out that although CVaR was very important both from a theoretical (coherent measure) and practical perspective, it is fragile in optimization. Ceria et al. [20] showed that one way to solve the fragility was to impose spectral risk constraints with several different risk models at the same time. However, this multi-spectral risk constraints model increased the size of the resulting LP significantly.

Although linear programming formulations can be solved very efficiently in CPLEX and MOSEK even when the problem is of large scale, the efficiency of the algorithms depends significantly on the sparsity of the problem. However, the LP derived from CVaR minimization has a large dense block and the problem is very ill-conditioned. Another drawback is that CVaR optimization usually results in an infinite number of portfolios with the same VaR and CVaR.

Alexander et al. [4] not only showed the ill-conditioning of the CVaR optimization problem, but also proposed a smoothing scheme to minimize CVaR very efficiently by approximating the piecewise linear CVaR objective function with a continuously differentiable piecewise quadratic approximation function. The linear programming CVaR optimization problem then turned out to become a continuous piecewise quadratic convex programming problem. This quadratic convex programming problem typically generated an infinite sequence of approximations converging to a solution. At each iteration, it typically required a function and a gradient evaluation. This smoothing method speeded up solution time by 1187% comparing to the linear programming structure. The relative difference in objective function is less than 1.5%. Iyengar and Ma [39] developed fast gradient descent algorithm based on the smoothing method proposed by Nesterov [55] to iteratively compute approximate solutions for the large scale scenario-based mean-CVaR portfolio selection problem without requiring any linear programming and can guarantee an accuracy with $\epsilon \approx 10^{-3}$. The fast gradient descent algorithm included two steps: (1) smoothing the objective by imposing a strongly convex function and (2) update the weights using a convex combination of two other variables.

1.3.6 Robust Portfolio Optimization and Control

Robust optimization deals with uncertainty by allowing parameters to vary in a certain convex uncertainty set and optimizing the worst case over that set. The robust optimization problem is usually set up in a min-max or max-min framework. For a comprehensive summary of this topic, the reader is referred to Ben-Tal et al. [11], Bertsimas et al. [13] and Ben-Tal et al. [10]. An overview of the most recent developments in robust optimization can be found in Gabrel et al. [31].

Traditional mean-variance models are vulnerable to data errors. Extensive research on protecting portfolios against worst-case estimates have been presented in, for instance, Ben-Tal et al. [10] and Goldfarb and Iyengar [34]. Portfolio optimization with uncertainty over a set of distributions has also been intensively studied. For example, ElGhaoui et al. [27] proposed a robust portfolio framework which assumed that only the bounds on the parameters are known. The author solved the worst-case problem with semi-definite programming and showed dramatic improvement of the worst-case objective of the robust portfolio. DeMiguel and Nogales [24] proposed a class of portfolios that were less sensitive to data error than the Mean-Variance portfolio selection model. A robust estimator was used in the model and a single nonlinear program was solved. Glasserman and Xu [33] considered that the asset returns were driven by market factors that evolved stochastically.

Both the relationship between returns and factors and the evolution of the factors were subject to model error and treated robustly stochastically. The authors developed robust portfolio control rules by applying stochastic factor in model dynamics.

Chapter 2

Robust Portfolio Management with Uncertainty in Asset Allocation

2.1 Motivation

Institutional investors, such as pension funds, university endowments and insurance companies, actively manage their portfolio by investing money in outside fund managers with expectation of generating superior returns while keeping risk in a low level. Fund managers might have quiet different risk and return profiles with regarding to their strategy and investment process. Institutional investor has low tolerance for risk in nature. A thorough understanding of the fund manager's sources of returns and risks inherent in the decision process and selecting superior external managers is critical to the performance of the institutional investor's portfolio.

Manager's asset allocation could significantly change fund's exposures and affect fund's return and risk as a result. Previous research focused on evaluating fund manager's skills by decomposing return into several attributions. For instance, Lo [47] and Hsu and Myers [38] propose to split fund return into active management component (reflecting the fund manager's skill) and a passive management component (reflecting stock performance). Brinson et al. [16] proposed a method to decompose the manager's added value 1) Asset Allocation, 2) Stock Selection and 3) Intersection between the two. These approaches offer investor an effective way to evaluate and select single superior fund manager.

In this Chapter, considering the low risk tolerance for institutional investors, our goal is to construct a portfolio as an institutional investor by selecting multiple superior fund managers, in order to minimize the worst case volatility while keeping the return beating certain benchmark. Asset allocation acts as an important role in fund's performance and may varies from time to time. Uncertainty in return has been well studied in the context of portfolio management, but not for asset allocation in fund management. In this Chapter, we propose a new framework which extends traditional mean-variance model and takes into account the uncertainty in fund's asset allocation in the decision process. We also propose an efficient algorithm to solve the model by either transforming the inner global optimization problem to a series of Mixed Integer linear problem, or deploying the algorithm developed by Chen and Burer [22]. The result shows that uncertainly in manager's asset allocation does affect portfolio's variance and stability. Our robust model provides a consistent and strong protection under the worst case manager's asset allocation.

KeyContribution : In this Chaper we propose a robust framework that takes into account the uncertainty stemming from the asset allocation, in the context of manager selection and portfolio management. We assume that only bounds on fund manager's asset allocation are available. We define the worst-case risk as the largest variance attainable, with limited information on each fund manager's asset allocation. We propose two exact approaches and an hubristic one to solve the problem efficiently. Furthermore, we show that our robust model provides a consistent and strong protection than nominal model under the worst case manager's asset allocation.

2.2 Robust Fund Manager Selection

2.2.1 Problem Setup

We aim to select several fund managers which are different in asset allocation but with same investment strategy or theme, in order to minimize the worst case portfolio risk (variance), while guaranteeing that expected return beats certain benchmark and all the money must be invested. Each manager's asset allocation is under uncertainty and is bounded in certain range. We will use the following notation:

Decision Variable

x_i: allocation to fund manager i;

Parameters under Uncertainty

w_{ij}: allocation to asset j of manager i;

w_{ij}^+: upper bound of allocation to asset j of manager i;

w_{ij}^-: lower bound of allocation to asset j of manager i;

$\overline{w}_{ij}$: nominal allocation to asset j of manager i;

Other Parameters

n: number of candidate fund managers;

m: number of asset classes;

$\bar{r}_j$: expected return from asset j;

τ: portfolio return benchmark.

2.2.2 Problem without Uncertainty

We apply the classical Markowitz portfolio optimization model to fund manager selection problem.

$$\begin{aligned} \min_x \quad & \sum_{i=1}^{n} x_i \sum_{j=1}^{m} \overline{w}_{ij} \sum_{k=1}^{n} x_k \sum_{l=1}^{m} \overline{w}_{kl} \operatorname{cov}(r_j, r_l) \\ \text{s.t.} \quad & \sum_{i=1}^{n} x_i = 1 \\ & \sum_{i=1}^{n} x_i \sum_{j=1}^{m} \overline{w}_{ij} \bar{r}_j \geq \tau \\ & x_i \geq 0, \forall i \end{aligned} \tag{2.1}$$

2.2.3 Problem with Uncertainty

For each fund manager, their allocation to different asset class might change from time to time, but is subject to certain range. Changes of allocation directly impact their return and risk. We aim to minimize the worst case portfolio variance by taking the consideration that asset allocation is under uncertainty for each manager.

$$\begin{aligned} \min_x \quad & \max_\omega \sum_{i=1}^{n} x_i \sum_{j=1}^{m} w_{ij} \sum_{k=1}^{n} x_k \sum_{l=1}^{m} w_{kl} \operatorname{cov}(r_j, r_l) \\ & \text{s.t.} \sum_{j=1}^{m} w_{ij} = 1, \forall i \\ & \qquad w_{ij}^{-} \leq w_{ij} \leq w_{ij}^{+}, \forall i, j \\ \text{s.t.} \quad & \sum_{i=1}^{n} x_i = 1 \\ & \sum_{i=1}^{n} x_i \sum_{j=1}^{m} \overline{w}_{ij} \overline{r}_j \geq \tau \end{aligned} \tag{2.2}$$

2.3 Solution Approach

One of the traditional approaches to solve the robust optimization problem is to reformulate the inner problem as its dual and solve with the outer problem. However, in our case, the inner maximization problem is a non-convex problem which has multiple local maximizers, and leads computational difficulty to solve with its dual formulation.

We propose two algorithms to solve the inner problem, and then solve the outer problem by adding delayed constraints until the solution converge. The first approach to solve the inner problem is to eliminate the quadratic term in the objective function through linearization, and transform the problem into a mixed 0-1 linear program. The second approach is to deploy Chen and Burer [22]'s algorithm to solve the nonconvex quadratic problem globally via completely positive programming.

2.3.1 Inner Problem

Approach 1

Theorem 2.1 *Tuy [70] A convex function attains its maximum on a simplex at a vertex of this simplex.*

Theorem 2.2 *Tuy [70] A vertex of a polyhedron in the hyperplane is a point x^0 such that, for a certain positive integer $\mathcal{I}$, $x_i^0 = x_i^-$ or x_i^+ for $i \neq I$.*

Let J_i be the index of j of the inactive w_{ij} for manager i and denote $w_{ij} = w_{ij}^- + \triangle w_{ij} u_{ij}$, where $\triangle w_{ij} = w_{ij}^+ - w_{ij}^-$. Then for each manager i,

- for $j = J_i, w_{ij} \in (w_{ij}^-, w_{ij}^+)$, i.e. $w_{ij} = w_{ij}^- + \triangle w_{ij} u_{ij}$, where $0 \leq u_{ij} \leq 1$
- for $j \neq J_i, w_{ij} = w_{ij}^-$ or w_{ij}^+, i.e. $w_{ij} = w_{ij}^- + \triangle w_{ij} u_{ij}$, where $u_{ij} \in \{0, 1\}$

In this way, we could enumerate J_i from 1 to m for w_{ij} with regards to each manager i and the original inner nonconvex problem could be reformulated to m^n mixed integer subproblems. For certain subproblem,

$$
\begin{aligned}
\max_\omega \quad & \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{n}\sum_{l=1}^{m} x_i x_k \text{cov}(r_j, r_l)(w_{ij}^- + \triangle w_{ij} u_{ij})(w_{kl}^- + \triangle w_{kl} u_{kl}) \qquad (2.3)\\
\text{s.t.} \quad & u_{ij} \in \{0, 1\}, \text{for } j \neq J_i \\
& 0 \leq u_{ij} \leq 1, \text{for } j = J_i
\end{aligned}
$$

Sherali and Adams [64] proposed RLT inequalities which are derived using a so-called Reformulation-Linearisation Technique. The constraint $y_{ij} = x_i x_j$, together with the bounds $0 \leq x_i \leq 1$ and $0 \leq x_j \leq 1$, implies the following four linear inequalities

$$y_{ij} \geq 0, y_{ij} \leq x_i, y_{ij} \leq x_j, y_{ij} \geq x_i + x_j - 1. \qquad (2.4)$$

By means of linearisation, the problem could be further simplified as m^n linear mixed integer subproblems. Substituting $u_{ij}u_{kl}$ with v_{ijkl}, problem (2.3) could be reformulated as following:

$$\begin{aligned}
\max_{\omega} \quad & \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{n}\sum_{l=1}^{m} x_i x_k \text{cov}(r_j, r_l)\left(w_{ij}^{-}w_{kl}^{-} + 2w_{kl}^{-} \triangle w_{ij}u_{ij} + \triangle w_{ij} \triangle w_{kl}v_{ijkl}\right) \quad (2.5)\\
\text{s.t.} \quad & v_{ijkl} \leq u_{ij}, \forall i, j\\
& v_{ijkl} \leq u_{kl}, \forall k, l\\
& v_{ijkl} \geq u_{ij} + u_{kl} - 1, \forall i, j, k, l\\
& u_{ij} \in \{0, 1\}, \text{for } j \neq J_i\\
& 0 \leq u_{ij} \leq 1, \text{for } j = J_i
\end{aligned}$$

Approach 2

Chen and Burer [22] proposed a new method for solving nonconvex quadratic programming to global optimality via completely positive programming. Their approach is to employ a finite branch-and bound ($B\&B$) scheme, in which branching is based on the first-order KKT conditions and polyhedral-semidominant relaxation are solved at each node of the ($B\&B$) tree. The relaxations are derived from completely positive and doubly nonnegative programs. The original quadratic program is reformulated as a quadratic program with linear equality, nonnegativity and complimentarily constraints. Such problem could be further reformulated as completely positive programming and relaxed in a natural way to a doubly nonnegative program.

2.3.2 Outer Problem

Taking the optimal solution get from the inner problem as parameters for the outer problem and adding delayed constraints, the outer problem could be treated as a traditional convex problem with linear constraints.

For the Sth iteration, the outer problem with two new delayed constraints could be formulated as

$$
\begin{aligned}
\min \quad & z \\
\text{s.t.} \quad & z \geq \sum_{i=1}^{n} x_i \sum_{j=1}^{m} w_{ij}^s \sum_{k=1}^{n} x_k \sum_{l=1}^{m} w_{kl}^s \operatorname{cov}(r_j, r_l), \forall s = 1, 2, ...S \\
& \sum_{i=1}^{n} x_i \sum_{j=1}^{m} w_{ij}^s r_j \geq \tau, \forall s = 1, 2, ...S \\
& \sum_{i=1}^{n} x_i = 1
\end{aligned}
\tag{2.6}
$$

2.3.3 Algorithm

Step 1 Start with a feasible solution $x \in$ X and set iteration number , $s = 0$.

Step 2 Solve the inner problem with candidate solution x^s and obtain a optimal solution w, namely w^{s+1}.

Step 3 Solve the outer problem 2.6 for x^{s+1}, and set $s = s + 1$.

Step 4 Repeat Steps 2 and 3 until the algorithm generates the same two candidate solutions $x \in$ X in two consecutive iterations.

2.4 Numerical Results

In this section, we present three experiments to illustrate our robust solution of manager selection problem with uncertainty in asset allocations. The first set of experiment is to compare the performance of the two approaches proposed in Section 4. The second experiment is to compare our robust approach with the nominal approach from the standpoint of the risk. In the third experiment, we propose an heuristic algorithm by only taking several managers from the large candidate manager pool into the robust manager selection model, and we compare the results of this heuristic method with the two approaches we proposed in Section 4.

2.4.1 Compare the computational results of two approaches

In this set of experiment, we test the efficiency of the two approaches, the MIP approach and Chen and Burer [22] approach. We test three instances: four managers with four assets, six managers with six assets, and twelve managers with six assets. We observe that the first approach is more efficient than the second one, when the problem size is small. However, as the problem size goes large, the computational advantage of the second approach becomes more obvious. As the size goes to twelve managers with six assets, the first approach needs to solve 6^{12} independent mixed integer problem which leads the problem runs forever. The second approach could solve the problem in a reasonable time for this size. Table 2.1 to table 2.5 present the experiment result for comparing interest.

Four Managers with Four Assets

Table 2.1: Four Managers with Four Assets

	worst variance	nominal return	0.15	0.2	0.25	0.3	0.35	0.4	0.45
Manager 1	13.2566	0.0259	0	0	0	0	0	0	0
Manager 2	10.161	0.0227	0	0	0	0	0	0	0
Manager 3	11.2965	0.0457	0	0	0	0.17	0.4368	0.7	0.9632
Manager 4	7.645	0.0267	1	1	1	0.83	0.5632	0.3	0.0368
variance(objective)			7.645	7.645	7.645	7.93	8.6456	9.6935	11.0762
excess return			0.0117	0.0067	0.0017	0	0	0	0
Running Time	(Approach 1)		8.092	7.222	7.294	9.96	16.394	13.791	15.745
Running Time	(Approach 2)		13.274	12.95	15.055	25.6	27.134	26.487	44.493

Six Manager with Six Assets

Table 2.2: Six Manager with Six Assets

	worst variance	nominal return	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5	0.55
Manager 1	7.1842	0.0461	0.4303	0.4303	0.4303	0.4303	0.4303	0.4937	0.3524	0.1778	0.0158
Manager 2	6.4727	0.0341	0.5697	0.5697	0.5697	0.5697	0.5697	0.5063	0.3593	0.2284	0.0915
Manager 3	7.5552	0.0432	0	0	0	0	0	0	0	0	0
Manager 4	9.6658	0.0506	0	0	0	0	0	0	0	0	0
Manager 5	7.8224	0.0573	0	0	0	0	0	0	0.2883	0.5937	0.8927
Manager 6	10.3974	0.0525	0	0	0	0	0	0	0	0	0
variance(objective)			6.2028	6.2028	6.2028	6.2028	6.2028	6.2145	6.4857	6.9147	7.4998
excess return			0.0242	0.0192	0.0142	0.0092	0.0042	0	0	0	0
Running time	(Approach 1)		25768	26467	26218	26423	26487	26407	26672	15859	28566
Running time	(Approach 2)		6848	7167	7152	7242	7258	6870	9594	13233	16236

Table 2.3 and 2.4 shows running time with respect to various range of uncertainty of manager's asset allocation. The first row is the running time of the original setting. The second row and the third row expand the range by 10% and 20% of the nominal manager's asset allocation, respectively. Under both four managers with four assets case and six managers with six assets case, running time increases as the range of the manager's asset allocation gets wider. In addition, running time increases as benchmark return increases as well.

Table 2.3: Running time comparison (in seconds)

	0.01	0.015	0.02	0.025	0.03	0.035	0.04
original bound	13	13	15	26	27	26	44
10% wider	28	46	57	55	59	66	73
20% wider	45	76	75	83	82	97	104

Table 2.4: Running time comparison (in seconds)

	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05	0.055
original bound	6848	7167	7152	7242	7258	6870	9594	13233	16236
10% wider	8470	8566	8512	8559	8700	8471	12124	18470	21184
20% wider	10147	10519	10742	10408	10548	10523	18345	22327	25343

Twelve Managers with Six Assets

Table 2.5: Twelve Managers with Six Assets

	worst variance	nominal return	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5	0.55	0.6
Manager 1	8.921	0.0526	0	0	0	0	0	0	0	0	0	0
Manager 2	9.1132	0.0415	0	0	0	0	0	0	0	0	0	0
Manager 3	11.306	0.0461	0	0	0	0	0	0	0	0	0	0
Manager 4	8.9833	0.059	0	0	0	0	0	0.2253	0.431	0.622	0.8118	0.4923
Manager 5	7.6327	0.0316	0.6421	0.6421	0.6421	0.6421	0.6212	0.4865	0.3427	0.1858	0.0274	0
Manager 6	10.2603	0.0667	0	0	0	0	0	0	0.0005	0	0	0.4042
Manager 7	10.3677	0.029	0	0	0	0	0	0	0	0	0	0
Manager 8	8.8524	0.0201	0	0	0	0	0	0	0	0	0	0
Manager 9	9.9881	0.0145	0	0	0	0	0	0	0	0	0	0
Manager 10	10.031	0.0386	0.3468	0.3469	0.3468	0.3468	0.3051	0.2664	0.2258	0.1922	0.1608	0.1035
Manager 11	7.9499	0.031	0	0	0	0	0	0	0	0	0	0
Manager 12	8.885	0.0494	0.0111	0.011	0.0111	0.0111	0.0736	0.0218	0	0	0	0
variance(objective)			6.8357	6.8357	6.8357	6.8357	6.8387	6.975	7.2184	7.6183	8.1685	8.9515
excess return			0.0192	0.0142	0.0092	0.0042	0	0	0	0	0	0
Running time			30719	30599	29864	30553	31039	54754	55120	57464	48503	27397

2.4.2 Compare the robust model with the nominal model

In this set of experiment, we compare our robust model with the nominal model. We test the performance of the two models under the nominal asset allocation scenario and the worst case asset allocation scenario. We also compare the difference in manager selection policy under the two models. The six managers with six assets case and twelve managers with six assets case are presented for illustration purpose.

Six Managers with Six Assets

Figure 2.1 and figure 2.2 compare the optimal manager selection policy under robust model and nominal model. Under nominal model, Manager 1 is always chosen under all benchmark requirement, but with a decreasing weight as benchmark return increases. Manager 5 is selected and takes an increasing weight in the portfolio as benchmark return exceed 0.045. However, under the robust model, manager 1 takes much less weight in the portfolio comparing with its weight in the nominal model. Manager 2, who is never selected in the nominal model, gains a larger weight under the robust manager selection policy.

Figure 2.3 presents the risk under the nominal model, robust model and the case where the nominal manager allocation is applied when the worst case manager's allocation occurs. The nominal model always gives the lowest risk with manager's nominal asset allocation. The robust model delivers manager selection policy and the minimum risk with the worst case manager's nominal asset allocation. In the scenario that the worst case manager's asset allocation occurs, nominal manager selection policy consistently results in a higher risk than the risk under the robust manager selection policy. From figure 2.3, we could see that the robust model provides a good protection with the worst case scenario.

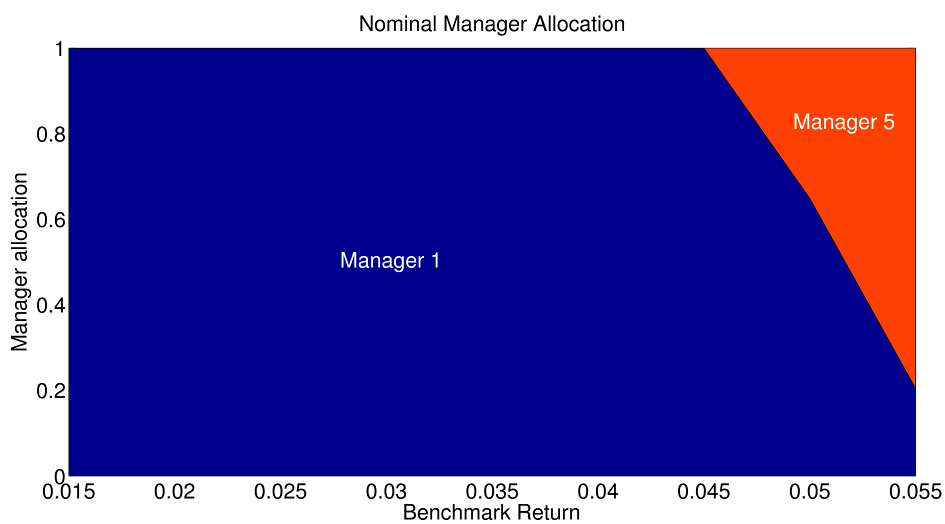

Figure 2.1: Nominal Manager Allocation: Six Managers with Six Assets

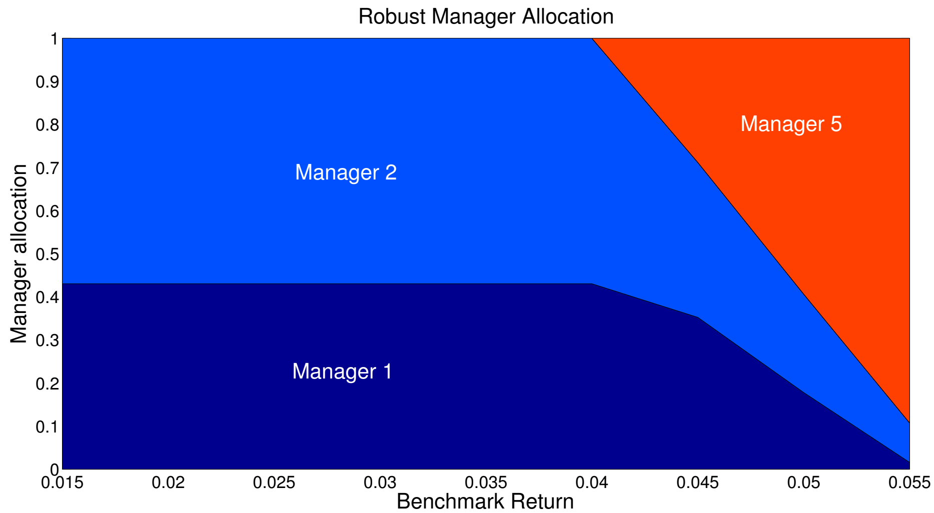

Figure 2.2: Robust Manager Allocation: Six Managers with Six Assets

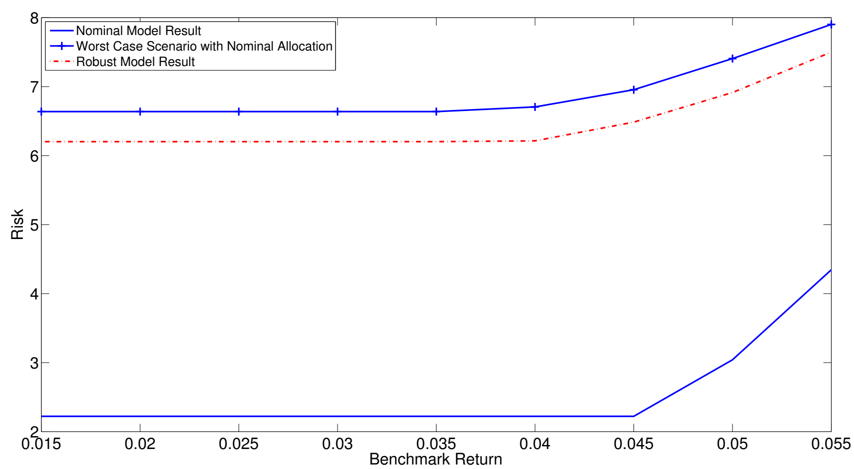

Figure 2.3: Nominal Model v.s. Robust Model: Six Managers with Six Assets

In addition, we test the out of sample return by taking the robust manager selection policy and nominal manager selection policy for the six managers with six assets case. Figure 2.4 and figure 2.5 take the manager selection policy when the benchmark return is set by 0.15% and 0.45% using historical data, but apply the most recent 24 months manager's return. The portfolio expected return of the out of sample data is lower than the benchmark return set in the model, since we are using out of sample data, which is different from the historical one. From the figure, we can see that the robust manager selection policy yields a lower expected return than nominal selection policy, but it also avoid significant downside risk.

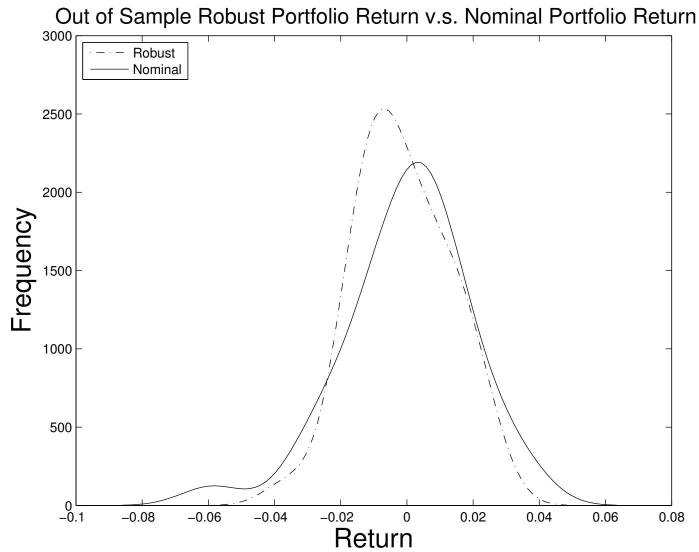

Figure 2.4: Out of Sample Robust Portfolio Return v.s. Nominal Portfolio Return 1

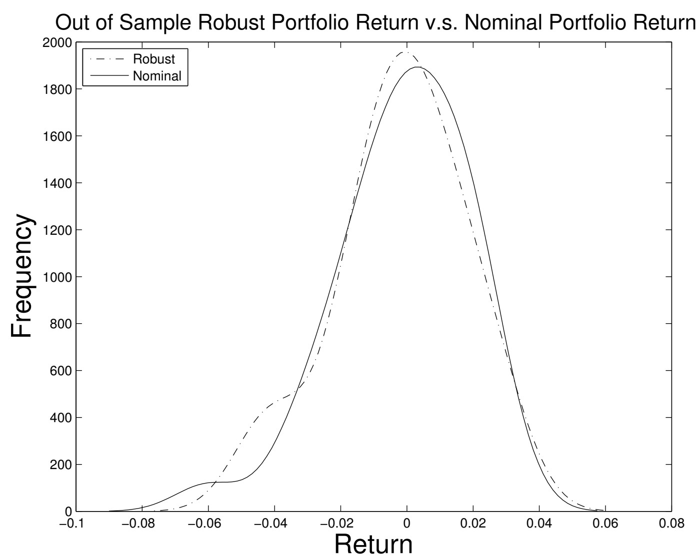

Figure 2.5: Out of Sample Robust Portfolio Return v.s. Nominal Portfolio Return 2

Twelve Managers with six Assets

Figure 2.6 demonstrates that the nominal model offers a lower risk under nominal allocation than the robust model, which is to minimize the worst case risk. However the robust model performs a stronger and consistent protection on the risk given the worst case asset allocation scenario.

From manager selection viewpoint, the robust model and the nominal model also result in very different selection policies. Manager 12, Manager 4 and Manager 10, who take a large weight under robust manager selection policy, are never chosen under the nominal manager selection policy. Meanwhile, Manager 1 and Manager 8 are never selected under robust manager selection policy. Detailed manager allocation information are showed in figure 2.7 and figure 2.8.

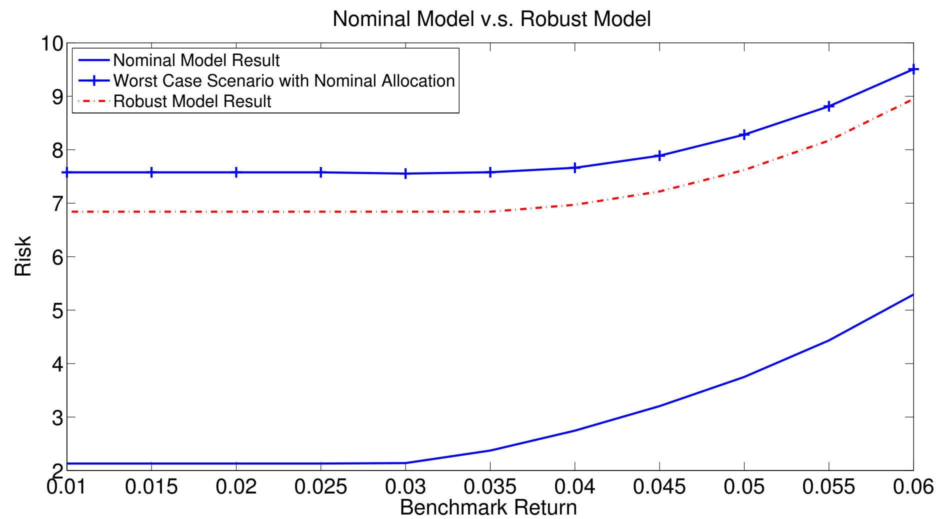

Figure 2.6: Nominal Model v.s. Robust Model: Twelve Managers with Six Assets

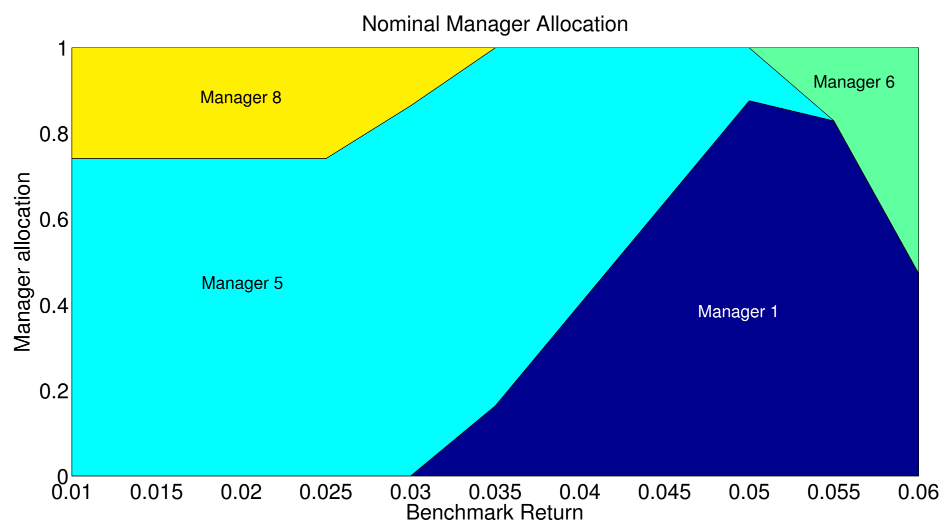

Figure 2.7: Nominal Manager Allocation:Twelve Managers with Six Assets

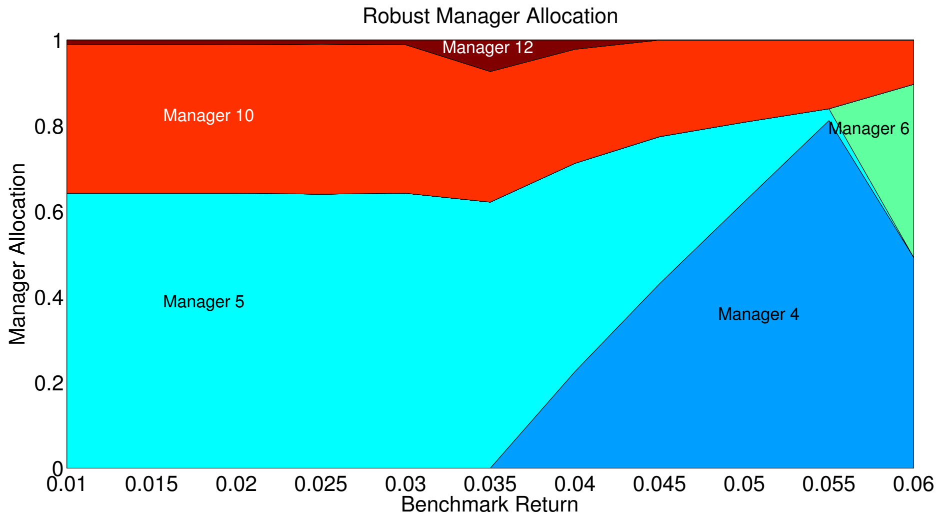

Figure 2.8: Robust Manager Allocation:Twelve Managers with Six Assets

2.4.3 An Heuristic Method

We define ”the worst case efficient managers” as managers whose worst case risk v.s. nominal return dominate other managers and they also lie on the boundary of the convex hull which is compose of all other manager’s worst case risk v.s. nominal return profile. In the heuristic method, only the worst case efficient managers will be selected in model in order to accelerate solving the problem.

In the four managers with four assets case, manager 3 and manager 4 dominate other managers from worst case risk v.s. nominal return viewpoint. In the six managers with six assets case, manager 1, manager 2 and manager 5 are on the boundary of the convex hull including all other manager’s worst case risk v.s. nominal return profile. In the twelve managers with six assets case, manager 4, manager 5 and manager 6 are defined as ”the worst case efficient managers” as they do not only dominate other managers in the standpoint of worst case risk v.s. nominal return, but also lie on the boundary of the convex hull of all other managers. Manager 1 (marked as circle) and manager 12 (marked as cross) are not defined as ”the worst case efficient manager” and are not selected in the heuristic method because they do not lie on the boundary of the convex hull, although no single manager dominates these two from worst case risk v.s. nominal return aspect.

For the cases of four managers with four assets, and six managers with six assets, table 2.6 and table 2.7 compare the result of optimal manager selection policy and the running time which get from the heuristic method with the result gets from the two approaches proposed in Section 4. The heuristic method results in the same optimal result as the other two approaches, but with significantly less time to solve the problem due to less candidate managers, especially for six managers with six assets instance. Efficient frontier got from the heuristic method also overlaps the one with other approaches as shown in figure 2.9 and figure 2.10.

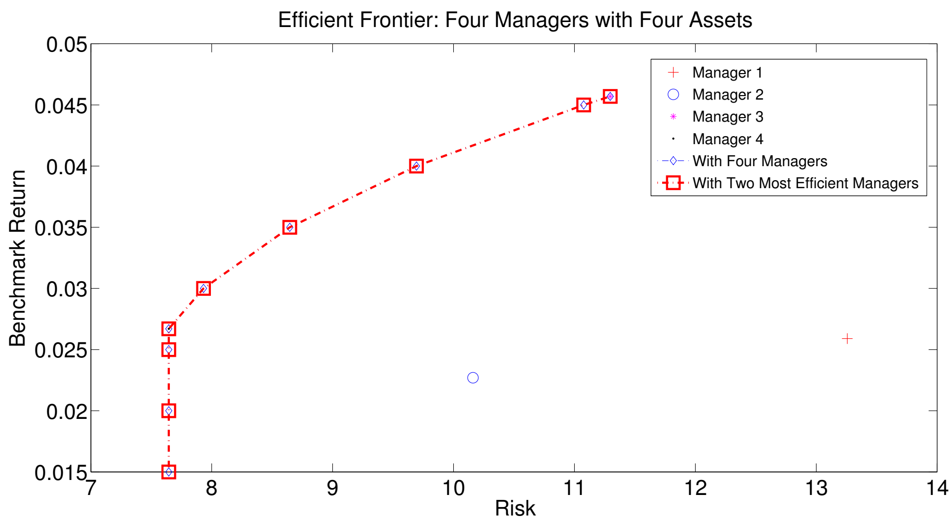

Figure 2.9: Efficient Frontier: Four Managers with Four Assets

Table 2.6: Heuristic Method v.s. Other Methods: Four Managers with Four Assets

	worst variance	nominal return	0.15	0.2	0.25	0.3	0.35	0.4	0.45
Manager 3	11.2965	0.0457	0	0	0	0.1737	0.4368	0.7	0.9632
Manager 4	7.645	0.0267	1	1	1	0.8263	0.5632	0.3	0.0368
variance(objective)			7.645	7.645	7.645	7.9324	8.6456	9.6935	11.0762
Running Time			9.36	10.43	11.07	12.01	12.02	12.82	13.37
					With Four Manager				
variance(objective)			7.645	7.645	7.645	7.9324	8.6456	9.6935	11.0762
Running time	(Approach 1)		8.092	7.222	7.294	9.957	16.394	13.791	15.745
Running time	(Approach 2)		13.274	12.95	15.055	25.636	27.134	26.487	44.493

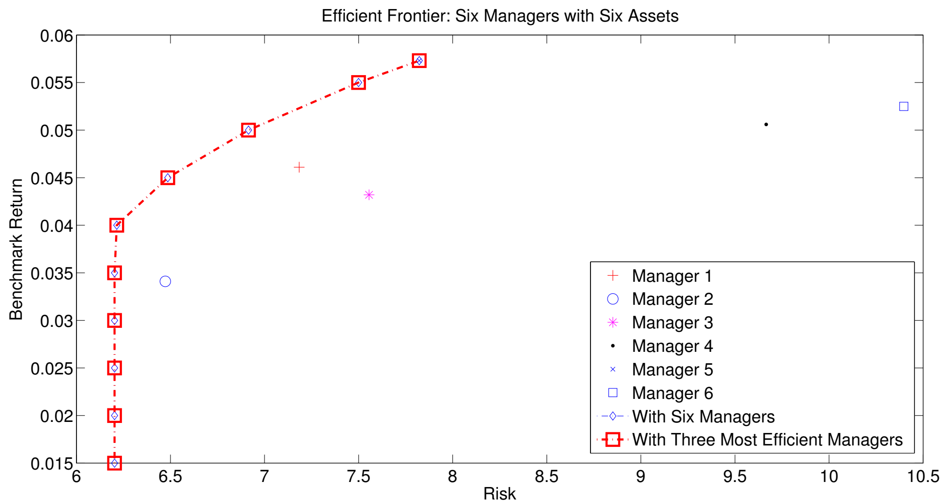

Figure 2.10: Efficient Frontier: Six Managers with Six Assets

Table 2.7: Heuristic Method v.s. Other Methods: Six Managers with Six Assets

	worst variance	nominal return	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5	0.55
Manager 1	7.1842	0.0461	0.4303	0.4303	0.4303	0.4303	0.4303	0.4937	0.3524	0.1778	0.0158
Manager 2	6.4727	0.0341	0.5697	0.5697	0.5697	0.5697	0.5697	0.5063	0.3593	0.2284	0.0915
Manager 5	7.8224	0.0573	0	0	0	0	0	0	0.2883	0.5937	0.8927
variance(objective)			6.2028	6.2028	6.2028	6.2028	6.2028	6.2145	6.4857	6.9147	7.4998
Running time			42	100	38	58	36	44	36	65	99
						With Six Managers					
Running time	(Approach1)		25768	26467	26218	26423	26487	26407	26672	15859	28566
Running time	(Approach2)		6848	7167	7152	7242	7258	6870	9594	13233	16236

For the twelve managers with six assets case, manager 4, manager 5, and manager 6 are selected as the worst case efficient managers in the heuristic algorithm. Candidate managers are reduced from twelve to three, and running time are dropped significantly. However, the heuristic method yields a larger worst case variance comparing to the other two approaches. The difference is diminished as the benchmark return got large, which is as shown in figure 2.12.

Table 2.8: Heuristic Method v.s. Other Methods: Twelve Managers with Six Assets

	worst variance	nominal return	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05	0.055	0.06	0.065
Manager 4	8.9833	0.059	0.8692	0.8693	0.8693	0.8693	0.8698	0.6927	0.5109	0.3291	0.1473	0	0
Manager 5	7.6327	0.0316	0.1308	0.1307	0.1307	0.1307	0.1302	0.3073	0.4891	0.6709	0.8527	0.875	0.2171
Manager 6	10.2603	0.0667	0	0	0	0	0	0	0	0	0	0.125	0.7829
variance(objective)			7.5945	7.5945	7.5945	7.5945	7.5945	7.6518	7.8306	8.1309	8.5538	9.1266	9.9577
Running time			162	74	79	92	90	117	90	152	186	29	17
With Twelve Managers													
variance(objective)			6.8357	6.8357	6.8357	6.8357	6.8387	6.975	7.2184	7.6183	8.1685	8.9515	9.8855
Running time	(Approach2)		30719	30599	29864	30553	31039	54754	55120	57464	48503	27397	1660

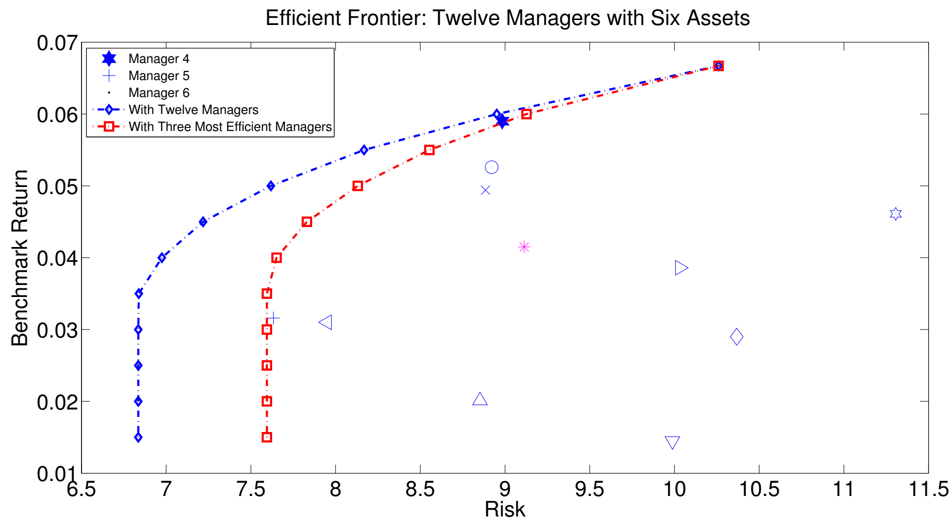

Figure 2.11: Efficient Frontier: Twelve Managers with Six Assets

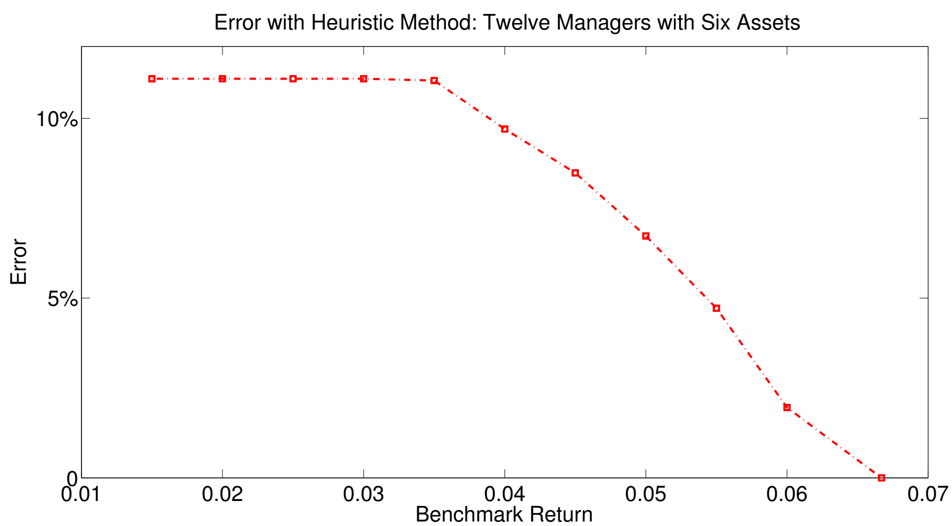

Figure 2.12: Error with Heuristic Method: Twelve Managers with Six Assets

2.5 Conclusions

In this Chapter, we proposed a robust framework that takes into account the uncertainty stemming from the asset allocation, in the context of manager selection and portfolio management. We also proposed two exact approaches and an heuristic one to solve the problem efficiently. In addition, our robust model provides a consistent and strong protection under the worst case manager's asset allocation.

Chapter 3

Robust Portfolio Management with Uncertainty in Asset Allocation and Asset Return

3.1 Motivation

Active management aims to generate superior return by actively changing the weight of securities or asset classes. It has been criticized for delivering lower return than passive management in recent years. Passive management is only subject to uncertainty in assert return, however, active management is subject to uncertainty from two resources: 1) asset allocation, and 2) asset return. In this research, we study how uncertainty in both asset allocation and asset return affect the portfolio return. In addition, we apply robust optimization approach to protect the worst case scenario.

Portfolio optimization problem is developed by Markowitz [48] decades ago to study the trade off between portfolio's return and variance. This framework is decent in theoretical point of view, but lack of practical value. One of the criticism of the model is that the return of securities is assumed to be normal distributed, which is not the case in reality. The second criticism is that the input has to be very accurately estimated, since the output is very sensitive to the input. However, this is very difficult for expected return estimate.

Mordern robust optimization approach is first introduced by El-Ghaoui and Lebret [26] and Ben-Tal and Nemirovski [9], which allow the robust version to be applied to the classical portfolio optimization model. The robust approach uses the distribution from the estimation process to find a robust portfolio in one single optkmizaiotn process. The resulting portfolio is more robust in terms of less sensitive to the input estimate error, and performs better than the classic optimization model.

In Dong and Thiele [25], we studied how uncertainty stemming from manager's asset allocation affects manager selection policy. In this Chapter, we consider the uncertainty in asset allocation and asset return together in our manager selection process and we developed a new robust framework to protect the worst case return.

Key Contribution: Uncertainty in asset return has been well studied in portfolio management. In this Chapter, we investigate two sources of uncertainty stemming from manager's asset allocation and asset return in the context of manager selection. Uncertainty of manager's asset allocation is defined through bounds. Uncertainty in asset return is controlled through bounds and uncertainty budget level. We proposed a robust framework to protect the worst case return. Furthermore, we investigate the property of the lower bound and upper bound of the problem. Two approaches are investigated through the special structure of the problem to solve the problem efficiently.

3.2 Robust Fund Manager Selection

3.2.1 Problem Setup

As an institutional investor, our goal is to select several fund managers to protect against the worst case return scenario. The candidate managers invest in the same asset class but different in asset allocation based on their views of the market. Our portfolio are facing two source of uncertainties varied by each manager: the uncertainty in asset allocation and uncertainty from the asset return. We assume that only limited information of asset allocation are disclosed to the investor. The bounds on manager's asset allocation for each assert class are available. The uncertainty in asset return is also depicted as bounds in our set up. But uncertainty budget for the asset return limits the variation of return, which prevent the return of all asset class go to the worst case, which is highly unlikely.

We consider two cases in our model: 1) no constraint on the benchmark return, 2) with constraint on the benchmark return. We first present the framework of the nominal model without uncertainty in Section 2.2, and then present the two versions robust model in Section 2.3-2.4. We use the following notations for our problem setup:

Decision Variables

x_i: allocation in fund manager i

Parameters related to fund managers' allocations

w_{ij}: (uncertain) allocation of manager i in asset j

w_{ij}^+: upper bound of allocation of manager i in asset j

w_{ij}^-: lower bound of allocation of manager i to asset j

$\overline{w}_{ij}$: nominal allocation of manager i to asset j

r_j: (uncertain) return of asset j

r_j^+: upper bound of return of asset j

r_j^-: lower bound of return of asset j

$\overline{r}_j$: nominal return of asset j

$\hat{r}_j$: deviation of return of asset j

z_j: random variable, $z_j = (r_j - \bar{r}_j)/\hat{r}_j$

Other parameters

n: number of fund managers

m: number of asset classes

cov(r_j, r_l) : covariance between the returns of asset j and asset l

τ: portfolio return benchmark

α: uncertainty budget of asset return

3.2.2 Problem without Uncertainty

We use the simplest portfolio model as the start. In this model, we aim to maximize the portfolio return.

$$
\begin{aligned}
\max_{x\in\mathcal{X}} \quad & \sum_{i=1}^{n} x_i \sum_{j=1}^{m} \bar{w}_{ij}\bar{r}_j \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_i = 1 \\
& x_i \geq 0, \forall i \\
& \text{other constraints}
\end{aligned}
$$

3.2.3 Problem with Uncertainty

We consider two sources of uncertainty from the fund manager asset allocation and asset return together in our framework. We allow uncertainty in assert allocation varies freely because it is fully charged by fund manager. On the other hand, we set up uncertainty budget on asset return, since it is highly unlikely that all asset class go to the worst case. The reason the diversification is to use one asset class to hedge the loss in another asset class when the market is under sever headwinds. in our experiment, we do conclude the case which full uncertainty budget in used to see the pattern of manager selection policy. The robust model is set up as following:

$$
\begin{aligned}
\max_{x\in\mathcal{X}} \quad & \min_{(r,\omega)\in\mathcal{S}} \sum_{i=1}^{n} x_i \sum_{j=1}^{m} w_{ij} r_j \\
& \text{s.t. } \sum_{j=1}^{m} w_{ij} = 1, \forall i \\
& \qquad w_{ij}^{-} \le w_{ij} \le w_{ij}^{+}, \forall i,j \\
& \qquad r_j = \bar{r}_j + \hat{r}_j z_j, \forall j \\
& \qquad \sum_{j=1}^{m} |z_j| \le \alpha \\
& \qquad -1 \le z_j \le 1, \forall j \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_i = 1 \\
& x_i \ge 0, \forall i \\
& \text{other constraints}
\end{aligned}
$$

Considering that short sale is not allowed for fund manager, i.e. $w_{ij}^{-} \ge 0$, the problem be could reformulated as

$$
\begin{aligned}
\max_{x\in\mathcal{X}} \quad & \min_{(r,\omega)\in\mathcal{S}} \sum_{i=1}^{n} x_i \sum_{j=1}^{m} w_{ij} r_j \\
& \text{s.t. } \sum_{j=1}^{m} w_{ij} = 1, \forall i \\
& \qquad w_{ij}^{-} \le w_{ij} \le w_{ij}^{+}, \forall i,j \\
& \qquad r_j = \bar{r}_j - \hat{r}_j z_j, \forall j \\
& \qquad \sum_{j=1}^{m} z_j \le \alpha \\
& \qquad 0 \le z_j \le 1, \forall j \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_i = 1 \\
& x_i \ge 0, \forall i \\
& \text{other constraints}
\end{aligned}
$$

We proposed two problems to test the robust framework in the context of manager selection problem

1) without return benchmark, and

2) with return benchmark constraint

For each of the problem, we compare the result of the robust structure with the nominal model.

3.2.4 Without return benchmark

$$
\begin{aligned}
\max_{x\in\mathcal{X}} \quad & \min_{(r,\omega)\in\mathcal{S}} \sum_{i=1}^{n} x_i \sum_{j=1}^{m} w_{ij} r_j \\
& \text{s.t. } \sum_{j=1}^{m} w_{ij} = 1, \forall i \\
& \qquad w_{ij}^{-} \le w_{ij} \le w_{ij}^{+}, \forall i, j \\
& \qquad r_j = \bar{r}_j - \hat{r}_j z_j, \forall j \\
& \qquad \sum_{j=1}^{m} z_j \le \alpha \\
& \qquad 0 \le z_j \le 1, \forall j \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_i = 1 \\
& x_i \ge 0, \forall i
\end{aligned}
$$

3.2.5 With return benchmark constraint

$$
\begin{aligned}
\max_{x\in\mathcal{X}} \quad & \min_{(r,\omega)\in\mathcal{S}} \sum_{i=1}^{n} x_i \sum_{j=1}^{m} w_{ij} r_j \\
& \text{s.t.} \ \sum_{j=1}^{m} w_{ij} = 1, \forall i \\
& \qquad w_{ij}^{-} \le w_{ij} \le w_{ij}^{+}, \forall i, j \\
& \qquad r_j = \bar{r}_j - \hat{r}_j z_j, \forall j \\
& \qquad \sum_{j=1}^{m} z_j \le \alpha \\
& \qquad 0 \le z_j \le 1, \forall j \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_i = 1 \\
& \sum_{i=1}^{n} x_i \sum_{j=1}^{m} \bar{w}_{ij} \bar{r}_j \ge \tau \\
& x_i \ge 0, \forall i
\end{aligned}
$$

3.3 Benchmark Problem

In this section, we present two sets of benchmark problems. The first set is that only uncertainty in w is considered in the problem, and the second set is that both w and r are takes as their nominal value.

3.3.1 Benchmark Problem 1

In this benchmark problem, we only consider uncertainty in asset allocation and use the estimate expected return as the input in the model.

$$
\begin{aligned}
\max_{x\in\mathcal{X}} \quad & \min_{(r,\omega)\in\mathcal{S}} \sum_{i=1}^{n} x_i \sum_{j=1}^{m} w_{ij}\bar{r}_j \\
& \text{s.t.} \sum_{j=1}^{m} w_{ij} = 1, \forall i \\
& \qquad w_{ij}^{-} \le w_{ij} \le w_{ij}^{+}, \forall i,j \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_i = 1 \\
& x_i \ge 0, \forall i \\
& \text{other constraints}
\end{aligned}
$$

In this case, the inner problem is a linear problem instead of a bilinear problem. In order to solve this problem, we write the dual of the inner problem and cooperate with the outer problem. The problem can be reformulate as

$$
\begin{aligned}
\max_{x,t,u,v} \quad & \sum_{i=1}^{n} t_i + \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}^{+}u_{ij} - \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}^{-}v_{ij} \\
\text{s.t.} \quad & t_i + u_{ij} - v_{ij} \le x_i\bar{r}_j, \forall i,j \\
& \sum_{i=1}^{n} x_i = 1 \\
& x_i \ge 0, \forall i = 1,2,..,n \\
& u_{ij} \le 0, \forall i,j \\
& v_{ij} \le 0, \forall i,j
\end{aligned}
$$

Property: The problem is independent in terms of i, for each sub-problem i, at optimality, we know that $v_{ij} = 0$ or $u_{ij} = 0$, such that the problem could be reformulated as

$$\max_{t,u,v} \quad t_i + \sum_{j=1}^{m} \Big(w_{ij}^{+} \min\{0, x_i\bar{r}_j + v_{ij} - t_i\} - w_{ij}^{-} \min\{0, -x_i\bar{r}_j - u_{ij} + t_i\} \Big)$$

$$\max_{t,u,v} \quad t_i + \sum_{j=1}^{m} \Big(w_{ij}^{+} \min\{0, x_i\bar{r}_j - t_i\} - w_{ij}^{-} \min\{0, -x_i\bar{r}_j + t_i\} \Big)$$

We can rank $x_i\bar{r}_j$, and the problem becomes a piecewise linear problem for each sub-problem i? then the problem can be reformulated as

$$\max_{t,u,v} \quad t_i + \sum_{j=1}^{k} \Big(w_{ij}^{+} (x_i\bar{r}_j - t_i) \Big) - \sum_{j=k+1}^{m} \Big(w_{ij}^{-} (t_i - x_i\bar{r}) \Big)$$

and the slope of t_i is $1 - \sum_{j=1}^{k} w_{ij}^{+} - \sum_{j=k+1}^{m} w_{ij}^{-}$. When $k = 0$, then the slope is $1 - \sum_{j=1}^{m} w_{ij}^{-} \geq 0$, while when $k = m+1$, the slope is $1 - \sum_{j=1}^{m} w_{ij}^{+} \leq 0$. $t_i = \bar{r}_{(k)}$, where k is the smallest integer where slope less than 0. This indicates that

- $w_{ij} = w_{ij}^{-}$ for j ranked strictly above $\bar{r}_{(k)}$ in terms of nominal return;
- $w_{ij} = w_{ij}^{+}$ for j ranked strictly below $\bar{r}_{(k)}$ in terms of nominal return;
- $w_{ij} = 1 - w_{ij}^{-} - w_{ij}^{+}$ for j ranked strictly equal to $\bar{r}_{(k)}$ in terms of nominal return.

3.3.2 Benchmark Problem 2

$$\begin{aligned} \max_{x\in\mathcal{X}} \quad & \sum_{i=1}^{n} x_i \sum_{j=1}^{m} \bar{w}_{ij}\bar{r}_j \\ \text{s.t.} \quad & \sum_{i=1}^{n} x_i = 1 \\ & x_i \geq 0, \forall i \\ & \text{other constraints} \end{aligned}$$

3.4 Solution Approach

One of the traditional method to solve the robust optimization problem to derive the dual of the inner problem and solve with the outer problem. This approach could be applied to our benchmark problem, which has the linear inner problem. However, in other cases, the inner problem is a bilinear problem which has multiple local minimizers, and leads computational difficulty to solve with its dual formulation. We proposed two algorithms to solve the inner problem efficiently.

3.4.1 Inner Problem

The inner problems are the same bilinear problems for two problems described above. The bilinear problem with linear constraints are known as NP hard problem. We first demonstrate several key properties of the bilinear problem, then we proposed two algorithms to solve it based on our framework. The first algorithm is that we first reformulate the bilinear problem to quadratic problem, and deploy Chen and Burer [22]'s algorithm to solve the nonconvex quadratic problem globally via completely positive programming. The second algorithm is that we separate the inner problem into several linear problems by going through the vertexes of one feasible set. We compare all the results and find the minimum one as the optimal solution of the inner problem.

Bilinear Programming

A function $f(x, y)$ is called bilinear if it reduces to a linear one by fixing the vector x or y to a particular value (Nahapetyan [53]). In general, the bilinear function can be represented as follows:

$$f(x, y) = a^T x + x^T Q y + b^T y$$

where $a, x \in \mathbb{R}^n$, $b, y \in \mathbb{R}^m$, and Q is a matrix of dimension $n \times m$.

The bilinear programming problem

$$\begin{aligned} \min \quad & f(x, y) = a^T x + x^T Q y + b^T y \\ \text{s.t.} \quad & x \in X, y \in Y \end{aligned}$$

where X and Y are nonempty polytopes in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

Let $V(X)$ and $V(Y)$ denote the vertex sets of X, Y respectively.

Theorem 3.1 *(Horst and Tuy [36] and [37]) If X and Y are bounded then there is an optimal solution* (x^*, y^*) *of problem 3.1, such that* $x^* \in V(X)$ *and* $y^* \in V(Y)$.

Theorem 3.2 *(Horst and Tuy [36] and [37]) If* (x^*, y^*) *is a solution of problem 3.1, then*

$$min_{x \in X} f(x, y^*) = f(x^*, y^*) = min_{y \in Y} f(x^*, y)$$

Algorithm 1 : Transform the Bilinear Problem to Quadratic Problem

$$\begin{aligned} \min_{(w,r)\in\mathcal{S}} \quad & w^T Q r \\ \text{s.t.} \quad & e^T w_i = 1, \forall i = 1, 2, .., n \\ & w^- \leq w \leq w^+ \\ & r^- \leq r \leq r^+ \end{aligned}$$

$$\text{where } Q = \begin{pmatrix} x_1 & 0 & .. & 0 & x_2 & 0 & .. & 0 & .. & .. & .. & .. & x_n & 0 & .. & 0 \\ 0 & x_1 & .. & 0 & 0 & x_2 & .. & 0 & .. & .. & .. & .. & 0 & x_n & .. & 0 \\ 0 & 0 & .. & 0 & 0 & 0 & .. & 0 & .. & .. & .. & .. & 0 & 0 & .. & 0 \\ 0 & 0 & .. & x_1 & 0 & 0 & .. & x_2 & .. & .. & .. & .. & 0 & 0 & .. & x_n \end{pmatrix}^T_{m \times mn},$$

and $w = (w_1^T, w_2^T, .., w_n^T)$ with $w_i = (w_{i1}, w_{i2}, .., w_{im})^T$.

Thus, $w \in \mathcal{R}^{mn}, r \in \mathcal{R}^m, Q \in \mathcal{R}^{mn \times m}$.

Then this bilinear problem can be expressed as an (indefinite) quadratic problem

$$\begin{aligned} \min_y \quad & \frac{1}{2} y M y^T \\ \text{s.t.} \quad & e^T y_i = 1, \forall i = 1, 2, .., n \\ & y_i^- \leq y_i \leq y_i^+, \forall i = 1, 2, .., n \\ & e^T z \leq \alpha, \\ & r = \bar{r} - \hat{r} z \end{aligned}$$

where $y = (y_1^T, y_2^T, .., y_n^T, r^T, z^T)$ with $y_i = w_i = (w_{i1}, w_{i2}, .., w_{im})^T$, $r = (r_1, r_2, ..., r_m)$, and $z = (z_1, z_2, ..., z_m)$

and $M = \begin{pmatrix} 0 & Q & 0 \\ Q^T & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Then we could use the completely positive programming method developed by Chen and Burer [22] to solve inner non-convex quadratic problem to global optimal.

Algorithm 2 : Mapping the vertexes of one feasible region

In our problem settings, candidate managers could be as many as we need to put into the model. But asset classes are limited to certain amount. The three main asset classes are equities (stocks), fixed-income (bonds) and cash equivalents (money market instruments). As more and more investment strategies are emerging, hedge fund, private equity, commodities, etc are also be classified as an asset class in the portfolio management context. However, the number of asset classes are still with limited and manageable manner comparing to the number of candidate managers.

There are some properties of the feasible set of asset returns: 1) the feasible set of asset returns are separate with the feasible set of asset allocation, 2) the feasible set is under box constraint plus one hyper plane, and 3) limited number of constraints leads to limited number of vertexes. Based on these three attractive properties and Theorem 4, we developed following algorithm to solve the inner problem efficiently:

Step 1: Find all the vertexes of the feasible set in terms of asset return r;

Step 2: For each of the vertex, solve the inner problem in terms of w,

Step 3: Compare the optimal solution of each vertex, and the minimum one is the global optimal solution for the inner problem.

3.4.2 Outer Problem

The outer problem is a quadratic programming problem with fixed w and r get from inner problem,

$$
\begin{aligned}
\max_{x \in \mathcal{X}} \quad & \sum_{i=1}^{n} x_i \sum_{j=1}^{m} w_{ij} r_j \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_i = 1 \\
& x_i \geq 0, \forall i \\
& \text{other constraints}
\end{aligned}
$$

We could solve the problem in finite steps by adding delayed constraints for all previous iterations $s = 1, 2, ...S - 1$, where S is the current iteration.

$$
\begin{aligned}
\max_{z,x} \quad & z \\
\text{s.t.} \quad & z \leq \sum_{i=1}^{n} x_i \sum_{j=1}^{m} w_{ij}^s r_j^s, \forall s = 1, 2, .., S \\
& \sum_{i=1}^{n} x_i = 1 \\
& x_i \geq 0, \forall i \\
& \text{other constraints}
\end{aligned}
$$

3.5 Numerical Results

In this section, we present two sets of experiments to illustrate our robust solution of manager selection problem with uncertainty in asset allocations and asset returns. The first set of experiment does not consider benchmark constraint in the model. The second set of experiment contains the benchmark return return in the nominal model. For each set of the problem, we showed the change of the manager selection policy with respect to the change of uncertainty budget on asset returns. We also compared the robust model with the two benchmark model without uncertainty.

3.5.1 Without return benchmark and variance constraints

In this set of experiment, we do not include return benchmark in our problem. We changed the uncertainty budget of asset returns form 0 to the largest amount of uncertainty.

The upper and lower bound of the problem

We first develop the upper and lower bound of the problem with uncertainty, which gives the portfolio manager a rough bound of the return.

1. Upper Bound: uncertainty budget α is 0 (same with the benchmark 1 problem)

$$
\begin{aligned}
\max_{x\in\mathcal{X}} \quad & \min_{\omega\in\mathcal{W}} \sum_{i=1}^{n} x_i \sum_{j=1}^{m} w_{ij}\bar{r}_j \\
& \text{s.t.} \sum_{j=1}^{m} w_{ij} = 1, \forall i \\
& \qquad w_{ij}^- \le w_{ij} \le w_{ij}^+, \forall i, j \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_i = 1 \\
& x_i \ge 0, \forall i \\
& \text{other constraints}
\end{aligned}
$$

In this case, the problem becomes benchmark 1 problem. From the numerical result, we can see that when the uncertainty budget is very small, the manager selection policy is closed to the benchmark 1.

2. Lower bound: uncertainty budget α is maximum

$$
\begin{aligned}
\max_{x\in\mathcal{X}} \quad & \min_{\omega\in\mathcal{W}} \sum_{i=1}^{n} x_i \sum_{j=1}^{m} w_{ij} r_j^- \\
& \text{s.t.} \sum_{j=1}^{m} w_{ij} = 1, \forall i \\
& \qquad w_{ij}^- \le w_{ij} \le w_{ij}^+, \forall i, j \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_i = 1 \\
& x_i \ge 0, \forall i \\
& \text{other constraints}
\end{aligned}
$$

Four managers with four asset class

For this experiment, we choose four candidate fund managers invest in four same asset classes. We aim to select fund manager's which can protect the worst case return. Table 3.1, figure 3.1 and figure 3.2 demonstrate manager allocation and the robust return. From the result we can see that the allocation is mainly via manager three and manager one. The robust return keep decreasing as uncertainty budget increases. From table 3.2 and figure 3.3, we can see that when uncertainty budget increase, the return of asset 2 goes to its lower bound first, and followed by asset 4, asset 3 and asset 1. Benchmark 2 problem gives the highest return, since no uncertainty is involved in the problem. The one with full uncertainty budget gives the lowest return.

Table 3.1: Manager Allocation: Four Manager with Four Assets

Uncertainty Budget	Manager 1	Manager 2	Manager 3	Manager 4	Objective
0.2	0	0	0.0405	0.9595	0.0237
0.4	0	0	1	0	0.0175
0.6	0.3781	0	0.6219	0	0.0124
0.8	0.3956	0	0.6044	0	0.0077
1	0.4058	0	0.5942	0	0.0029
1.2	0.4142	0	0.5858	0	-0.0004
1.4	0.431	0	0.569	0	-0.0039
1.6	0.2174	0	0.7826	0	-0.008
1.8	0.1552	0	0.8448	0	-0.0113
2	0.169	0	0.831	0	-0.0149
2.2	0.1567	0	0.8433	0	-0.017
2.4	0.1466	0	0.8534	0	-0.0192
2.6	0.1026	0	0.8974	0	-0.022
2.8	0.2632	0	0.7368	0	-0.025
3	0.2821	0	0.7179	0	-0.0276
3.2	0.2727	0	0.7273	0	-0.0294
3.4	0.2566	0	0.7434	0	-0.0312
3.6	0.2222	0	0.7778	0	-0.033
3.8	0.0566	0	0.9434	0	-0.0345
4	0	0	1	0	-0.037
Benchmark 1	0	1	0	0	0.0305
Benchmark 2	0	1	0	0	0.0342

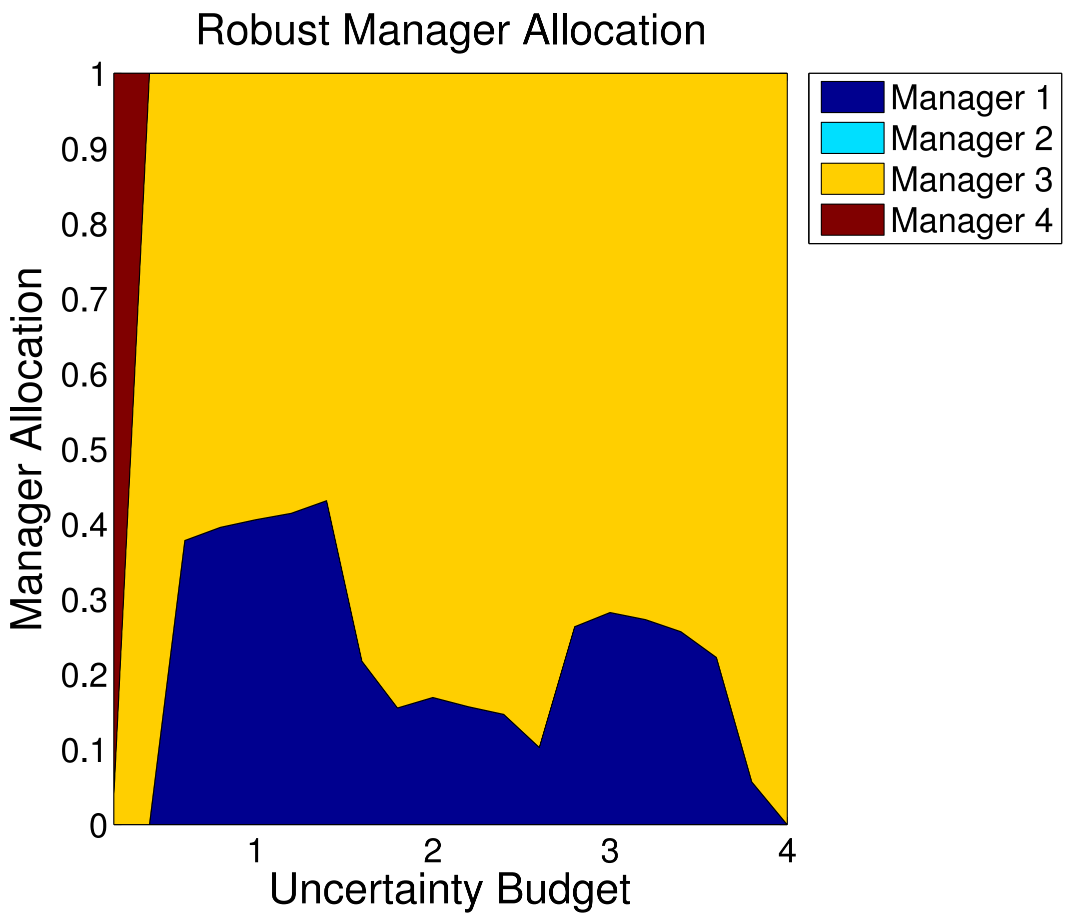

Figure 3.1: ManagerAllocation4A4M

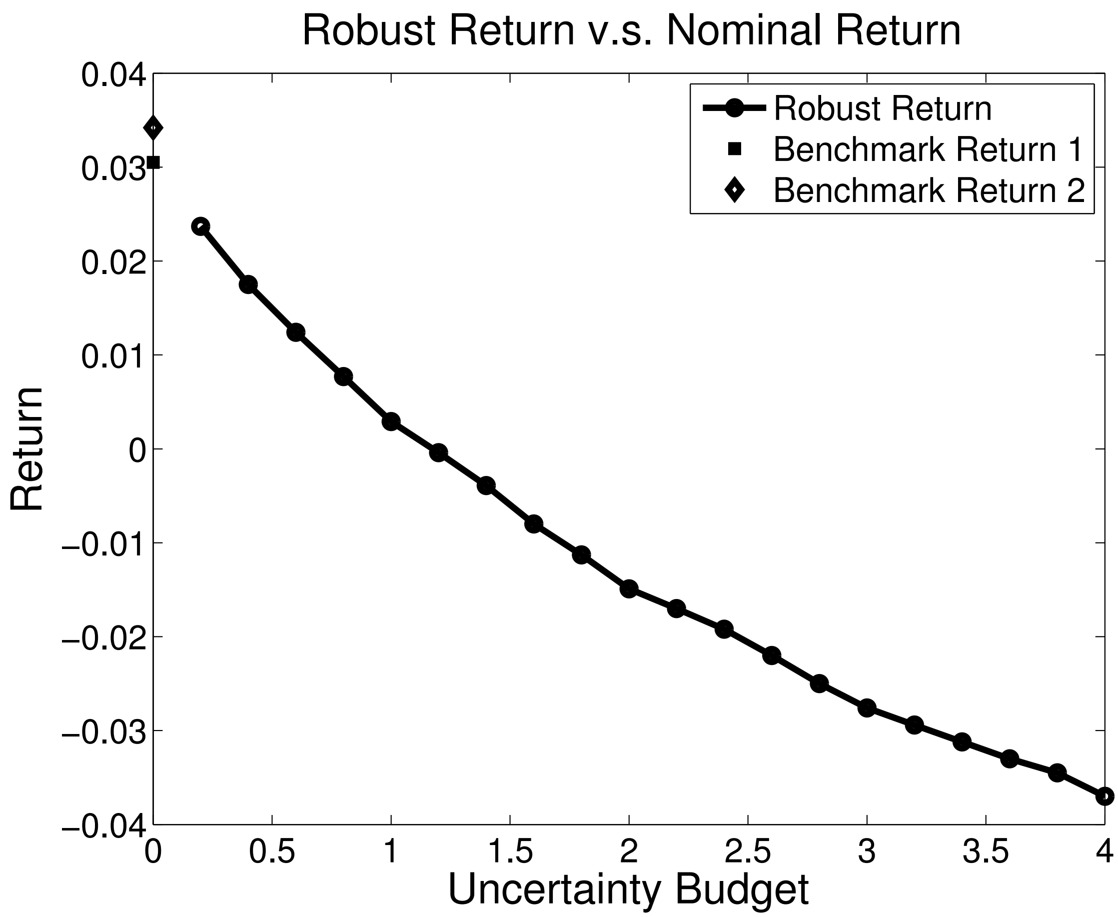

Figure 3.2: PortfolioReturn4A4M

Table 3.2: Uncertainty Allocation

Uncertainty Budget	Asset 1	Asset 2	Asset 3	Asset 4
0.2	0	0.2	0	0
0.4	0	0.4	0	0
0.6	0	0.6	0	0
0.8	0	0.8	0	0
1	0	1	0	0
1.2	0	1	0	0.2
1.4	0	1	0	0.4
1.6	0	1	0	0.6
1.8	0	1	0	0.8
2	0	1	0	1
2.2	0	1	0.2	1
2.4	0	1	0.4	1
2.6	0	1	0.6	1
2.8	0	1	0.8	1
3	0	1	1	1
3.2	0.2	1	1	1
3.4	0.4	1	1	1
3.6	0.6	1	1	1
3.8	0.8	1	1	1
4	1	1	1	1

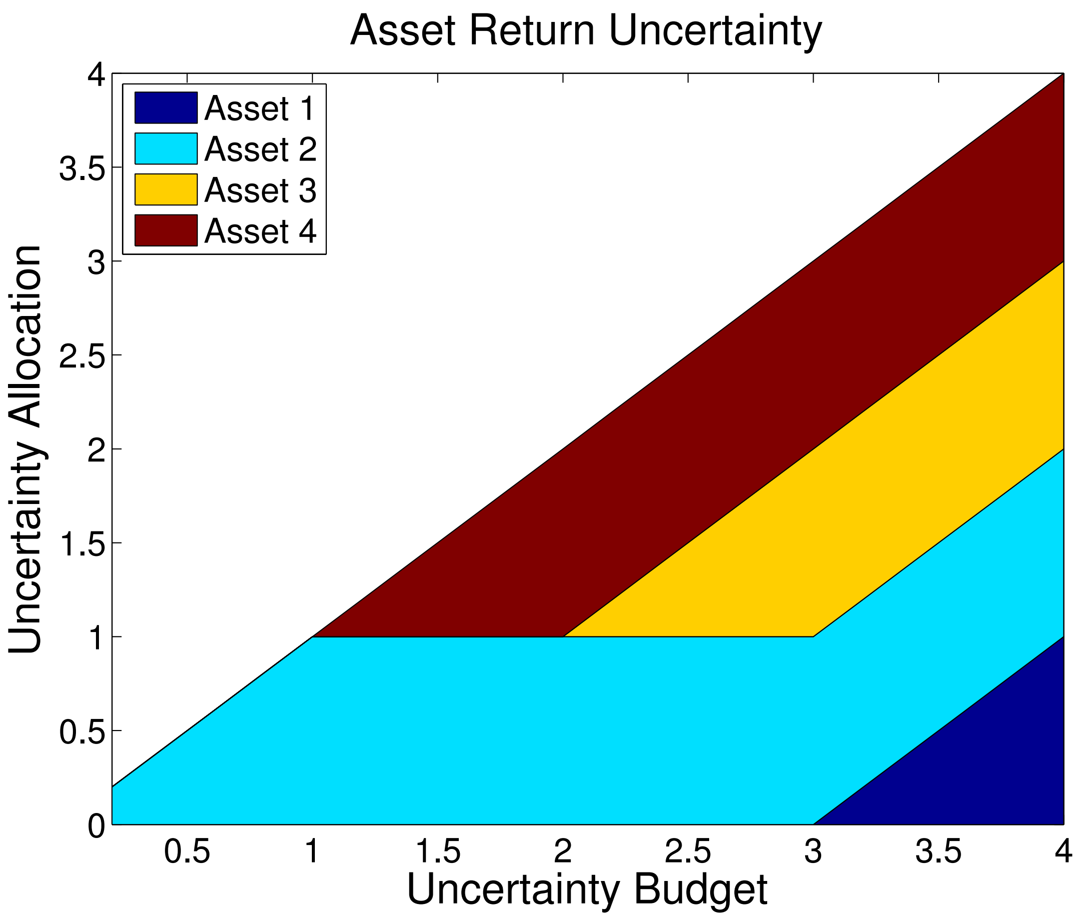

Figure 3.3: AssetReturnUncertainty4A4M

Six managers with six asset class

For this experiment, we consider six candidate fund managers invest in six same asset classes. We aim to select fund manager's which can protect the worst case return. Table 3.3, figure 3.4 and figure 3.5 demonstrate manager allocation and the robust return. From the result we can see that the allocation concentrates on manager six. Table 3.4 and figure 3.6 show that when uncertainty budget increase, the return of asset 5 goes to its lower bound first, and followed by asset 3, asset 1, asset 4, asset 6 and asset 2, which is the same order of the value of the lower bound of each asset class.

Table 3.3: Manager Allocation

Uncertainty Budget	Manager 1	Manager 2	Manager 3	Manager 4	Manager 5	Manager 6	Objective
0.5	0	0	0	0	0	1	-0.0048
1	0	0	0	0	0	1	-0.021
1.5	0	0	0	0	0	1	-0.0316
2	0	0	0	0	0	1	-0.0423
2.5	0	0	0	0	0	1	-0.0505
3	0	0	0	0	0	1	-0.0587
3.5	0	0	0	0	0	1	-0.0607
4	0	0	0	0	0	1	-0.0621
4.5	0	0	0	0	0	1	-0.0643
5	0	0	0	0	0	1	-0.0659
5.5	0	0	0	0	0	1	-0.0663
6	0	0	0	0	0	1	-0.0667
Benchmark 1	0	0	0	0	1	0	0.013
Benchmark 2	1	0	0	0	0	0	0.0371

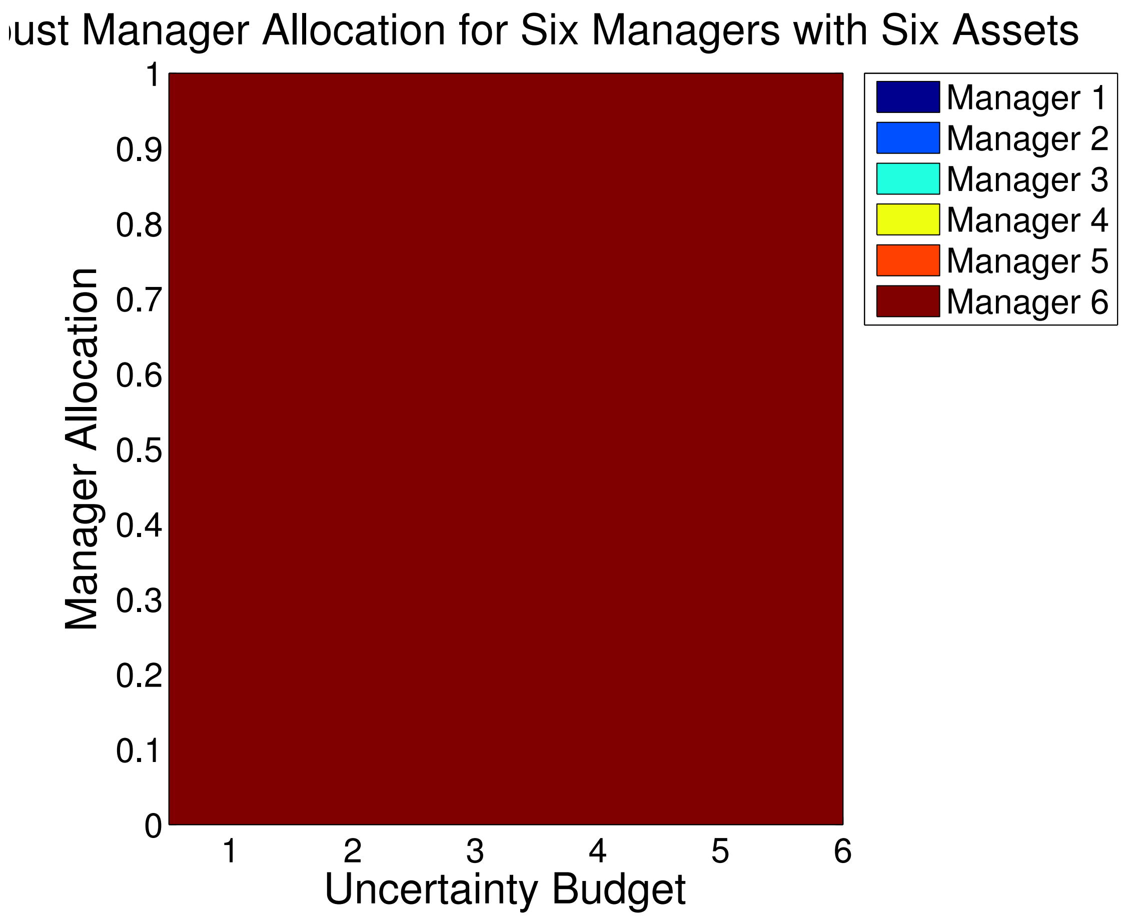

Figure 3.4: ManagerAllocation6A6M

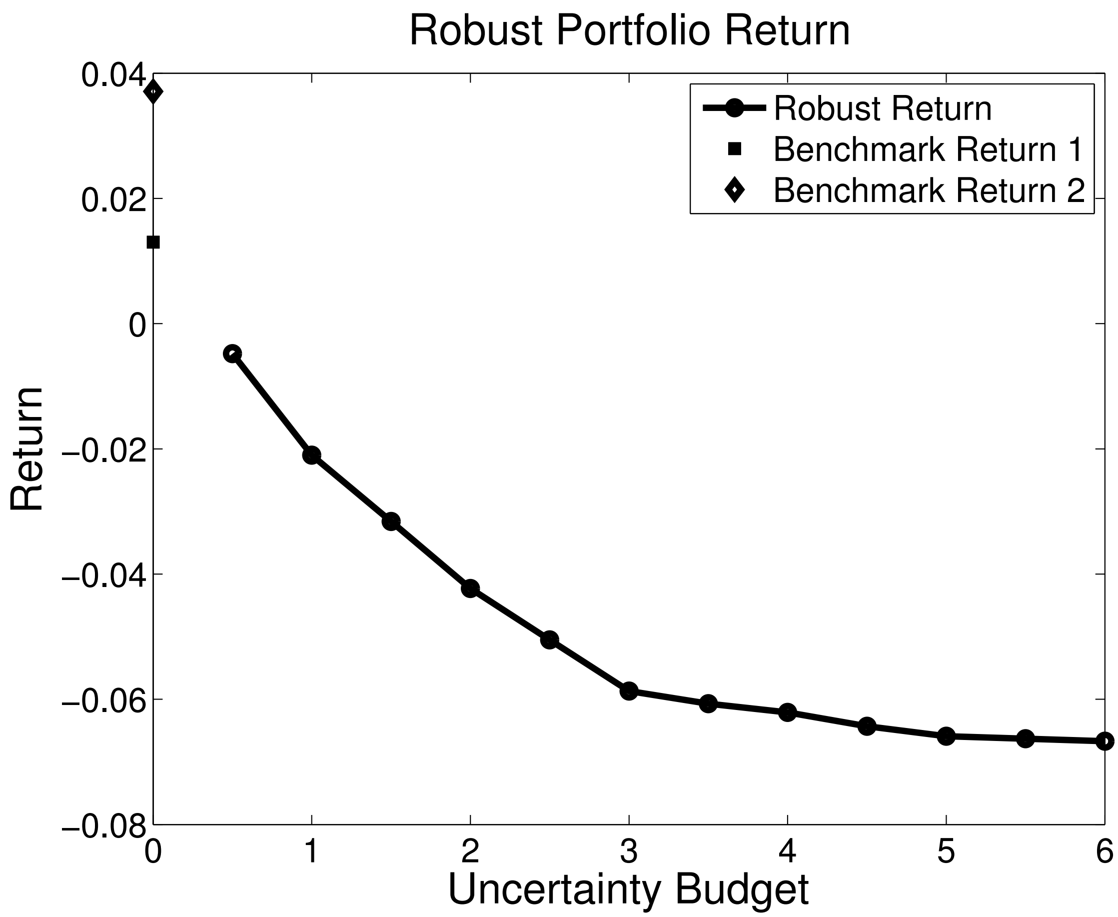

Figure 3.5: PortfolioReturn6A6M

Table 3.4: Uncertainty Allocation

Uncertainty Budget	z1	z2	z3	z4	z5	z6
0.5	0	0	0	0	0.5	0
1	0	0	0	0	1	0
1.5	0	0	0.5	0	1	0
2	0	0	1	0	1	0
2.5	0.5	0	1	0	1	0
3	1	0	1	0	1	0
3.5	1	0	1	0.5	1	0
4	1	0	1	1	1	0
4.5	1	0	1	1	1	0.5
5	1	0	1	1	1	1
5.5	1	0.5	1	1	1	1
6	1	1	1	1	1	1

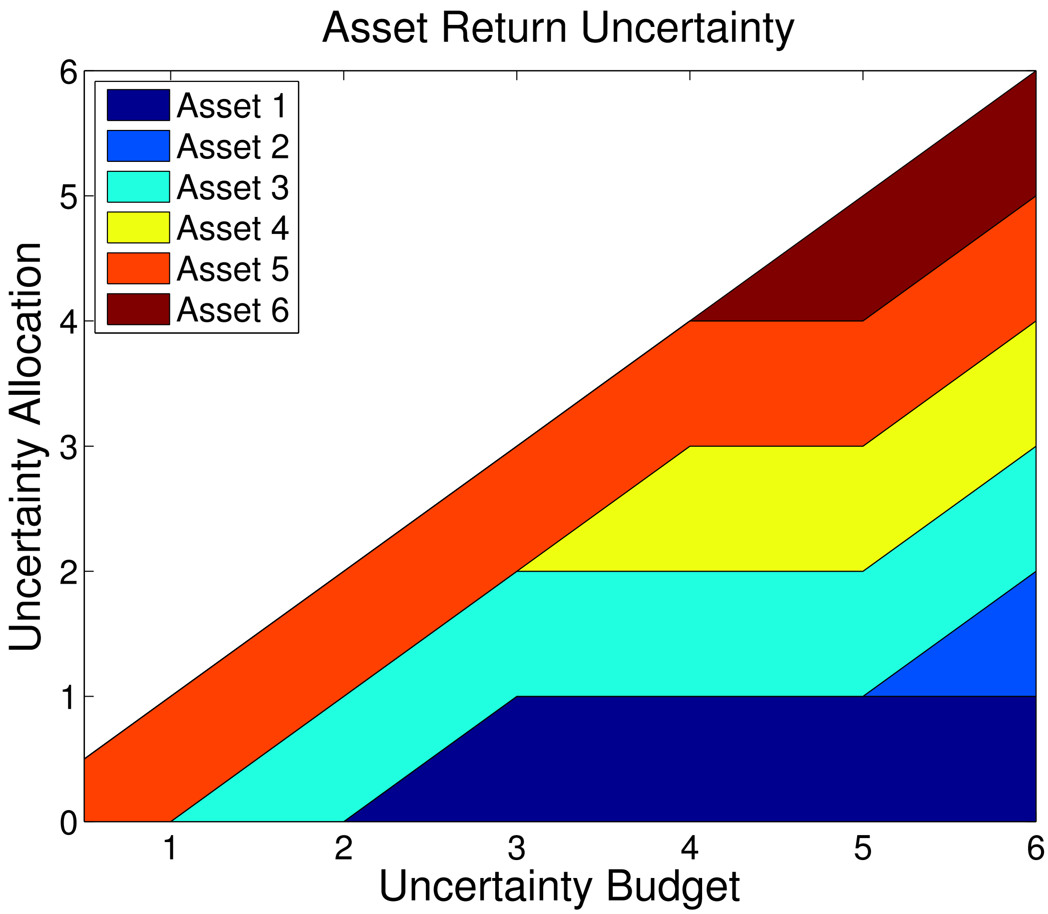

Figure 3.6: AssetReturnUncertainty6A6M

Twelve managers with six asset class

In this experiment, we aim to select fund managers from twelve candidate managers, and each of them invest in six asset classes. As shown in figure 3.8 , robust return decreases as uncertainty budget increases. When uncertainty budget is very small, all the money allocate to manager 7. As the uncertainty increases, money are allocated through manager 7, manager 9, and manager 12. As the uncertainty set increases above 5, all money allocate to manager 7 again.

Table 3.5: Manager Allocation

Uncertainty Budget	Manager 1	Manager 2	Manager 3	Manager 4	Manager 5	Manager 6	
0.5	0	0	0	0	0	0	
1	0	0	0	0	0	0	
1.5	0	0	0	0	0	0	
2	0	0	0	0	0	0	
2.5	0	0	0	0	0	0	
3	0	0	0	0	0	0	
3.5	0.0681	0	0	0	0	0	
4	0.0893	0	0	0	0	0	
4.5	0	0	0	0	0	0	
5	0	0	0	0	0	0	
5.5	0	0	0	0	0	0	
6	0	0	0	0	0	0	
Benchmark 1	0	0	0	0	0	0	
Benchmark 2	0	0	0	0	0	0	
Uncertainty Budget	Manager 7	Manager 8	Manager 9	Manager 10	Manager 11	Manager 12	Objective
0.5	1	0	0	0	0	0	-0.0194
1	0.4527	0	0	0	0	0.5473	-0.0263
1.5	0.4797	0	0	0	0	0.5203	-0.0329
2	0.0746	0	0	0	0	0.9254	-0.0388
2.5	0.9273	0	0	0	0	0.0727	-0.0419
3	0.8172	0	0.0601	0	0	0.1227	-0.0441
3.5	0.8094	0	0.0017	0	0	0.1209	-0.0475
4	0.9107	0	0	0	0	0	-0.0493
4.5	0.9409	0	0.0591	0	0	0	-0.0508
5	1	0	0	0	0	0	-0.0536
5.5	1	0	0	0	0	0	-0.0539
6	1	0	0	0	0	0	-0.0542
Benchmark 1	1	0	0	0	0	0	-0.0081
Benchmark 2	0	0	1	0	0	0	0.0296

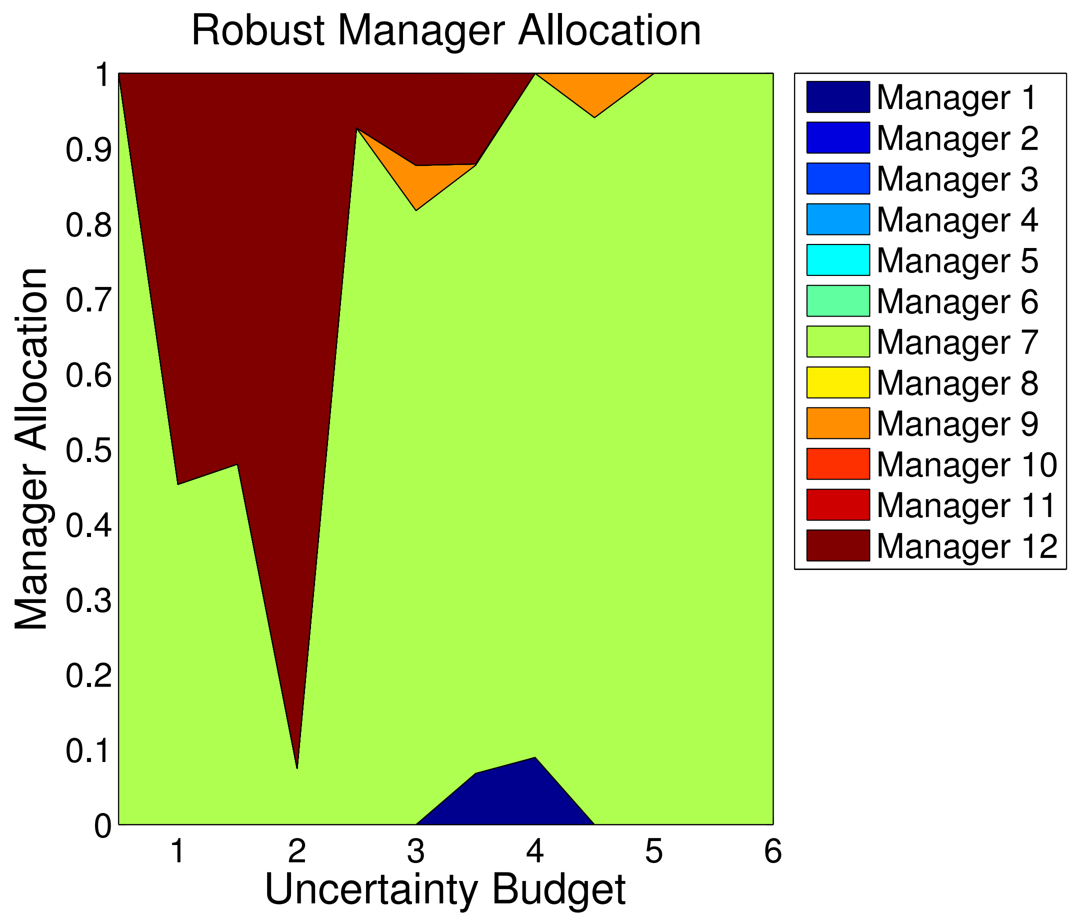

Figure 3.7: ManagerAllocation6A12M

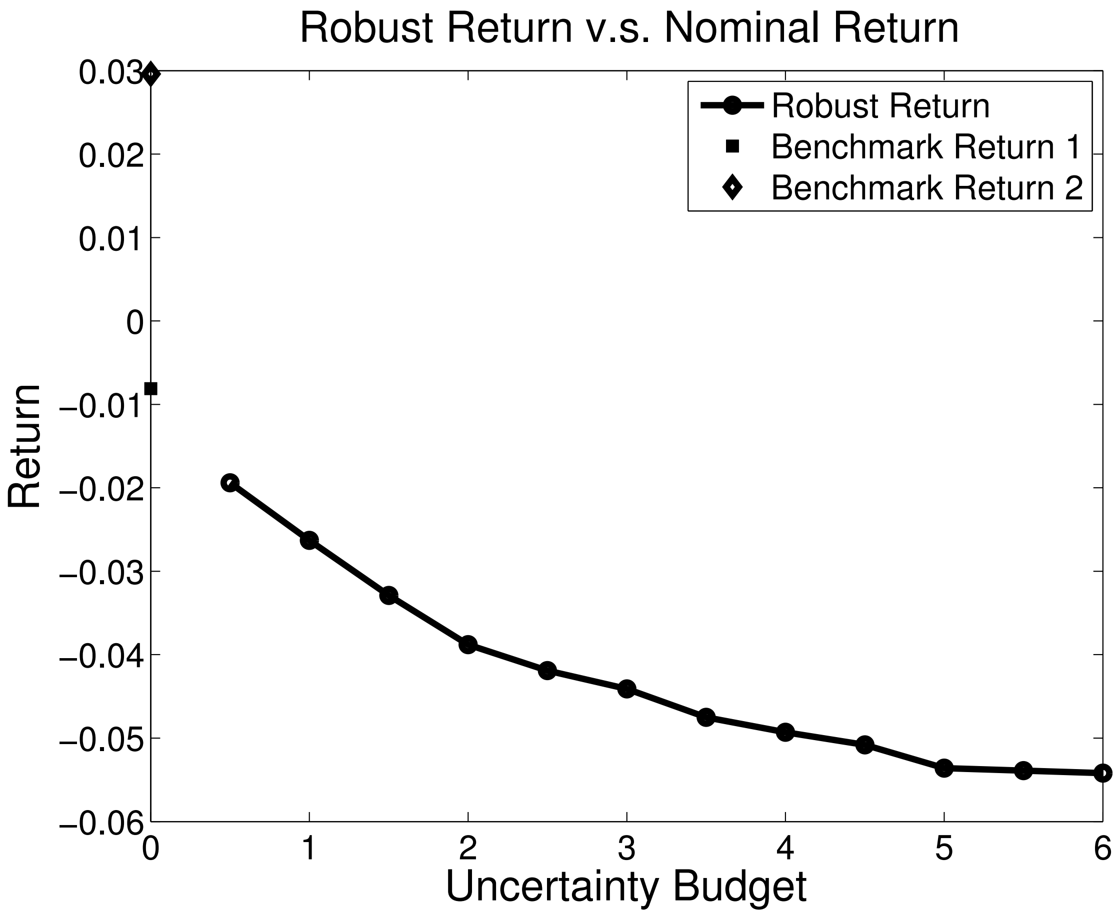

Figure 3.8: PortfolioReturn6A12M

Table 3.6: Uncertainty Allocation

Uncertainty Budget	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5	Asset 6
0.5	0	0.5	0	0	0	0
1	0	1	0	0	0	0
1.5	0	1	0.5	0	0	0
2	0	1	1	0	0	0
2.5	0	1	1	0	0	0.5
3	0	1	1	0	0	1
3.5	0	1	1	0.5	0	1
4	0	1	1	1	0	1
4.5	0	1	1	1	0.5	1
5	0	1	1	1	1	1
5.5	0.5	1	1	1	1	1
6	1	1	1	1	1	1

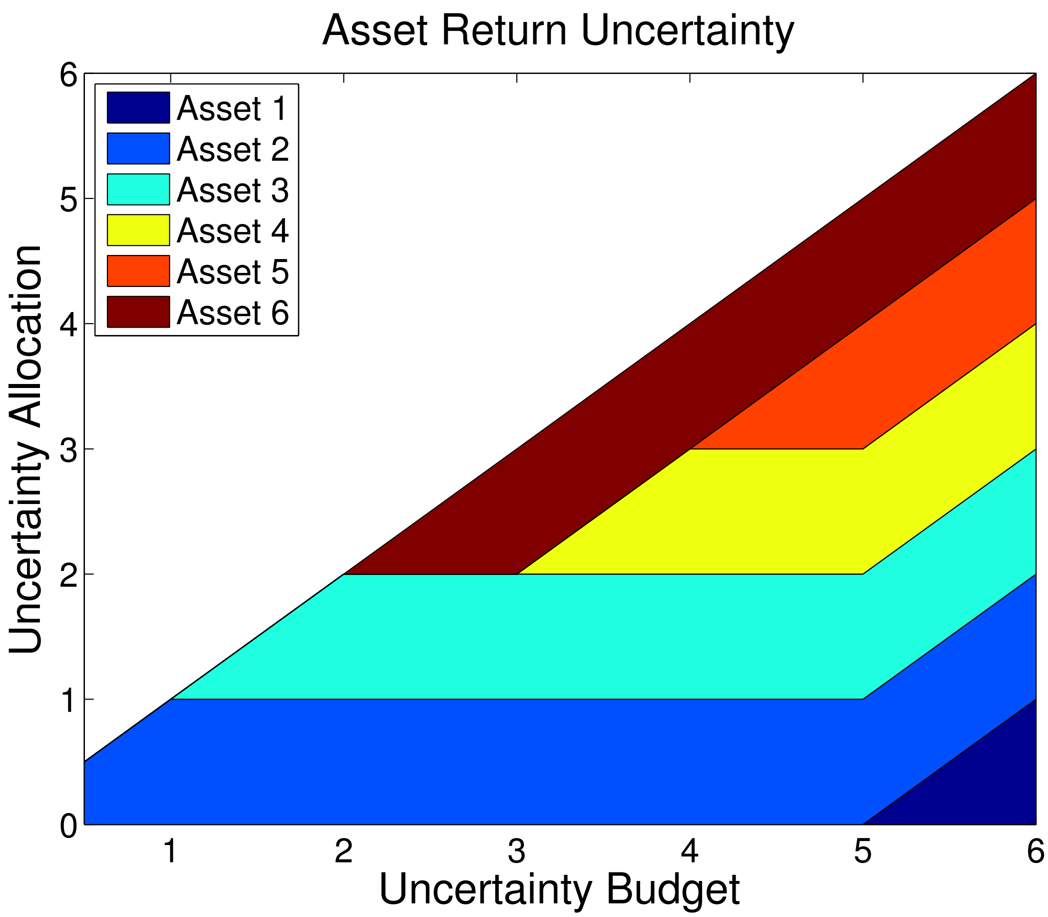

Figure 3.9: AssetReturnUncertainty6A12M

3.5.2 With return benchmark

For this set of experiment, we test how the benchmark return affect manager's selection policy together with the uncertainty in asset allocation and in asset return. From our experiment, as benchmark return increases, manager allocation is more depend on the benchmark return level.

Four Managers with Four Assets

For the four managers with four asset case, we choose $2\%, 3\%, 3.2\%$ and 3.4% as the benchmark return. Table 3.7 shows the nominal return of each manager. When the benchmark return level is very low, the problem is the same with the problem without benchmark return, since it could always be achieved. As benchmark return increases, manager 3 first takes more weights, and then was substitute by manager 2 and manager 4. As benchmark return becomes as high as 3.4%, all money allocate to manager 2, since other manager has a much lower expect return, which could not meet the benchmark requirement. 3.10 shows the robust portfolio return changes with uncertainty level for each benchmark return level. As benchmark return level increase, the worst case return become more significant low as the concentration effect. All risk concentrate to one manager. Table 3.8 to figure 3.14 show the detailed manager allocation information.

Table 3.7: Nominal Return

	Manager 1	Manager 2	Manager 3	Manager 4
Nominal Return	0.0238	0.0343	0.0305	0.0313

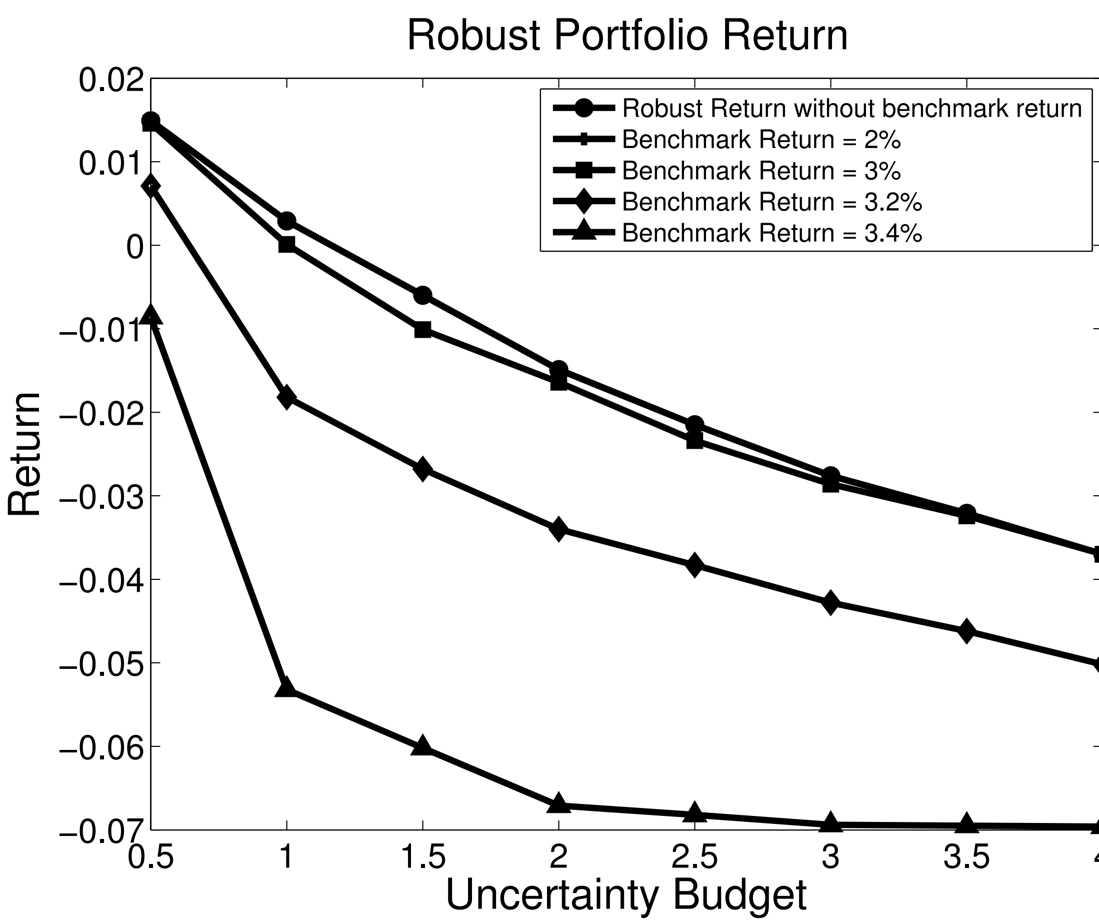

Figure 3.10: Robust Portfolio Return with Benchmark Return

Table 3.8: Manager Allocation: Benchmark Return=2%

Return Benchmark=2%					
Uncertainty Budget	Manager 1	Manager 2	Manager 3	Manager 4	objective
0.5	0.3636	0	0.6364	0	0.0149
1	0.4058	0	0.5942	0	0.0029
1.5	0.0749	0	0.9251	0	-0.006
2	0.169	0	0.831	0	-0.0149
2.5	0.1282	0	0.8718	0	-0.0215
3	0.2821	0	0.7179	0	-0.0276
3.5	0.2432	0	0.7568	0	-0.0321
4	0	0	1	0	-0.037

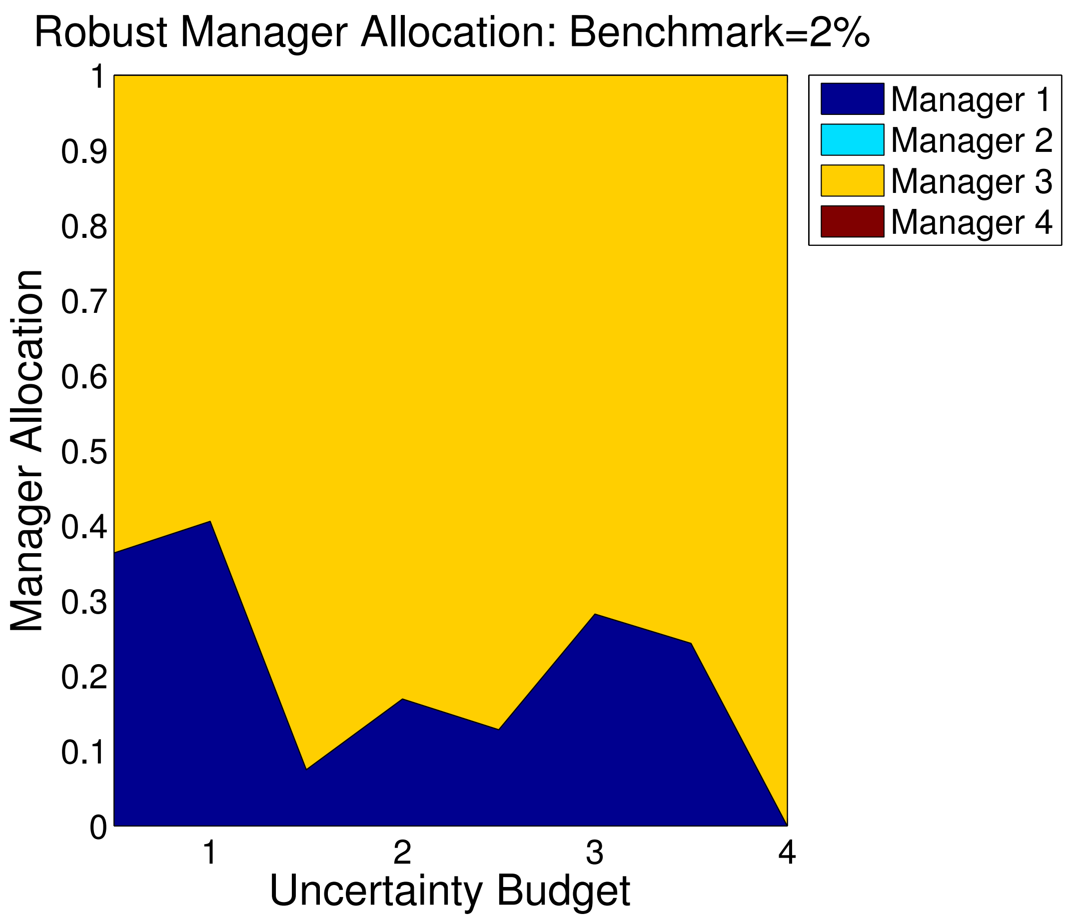

Figure 3.11: Robust Portfolio Manager Allocation with Benchmark Return = 2%

Table 3.9: Manager Allocation: Benchmark Return=3%

Return Benchmark=3%					
Uncertainty Budget	Manager 1	Manager 2	Manager 3	Manager 4	objective
0.5	0.0741	0	0.9259	0	0.0146
1	0.0741	0	0.9259	0	0.0001
1.5	0.0741	0	0.9259	0	-0.0101
2	0.0741	0	0.9259	0	-0.0164
2.5	0.0741	0	0.9259	0	-0.0234
3	0.0741	0	0.9259	0	-0.0286
3.5	0.0741	0	0.9259	0	-0.0324
4	0	0	1	0	-0.037

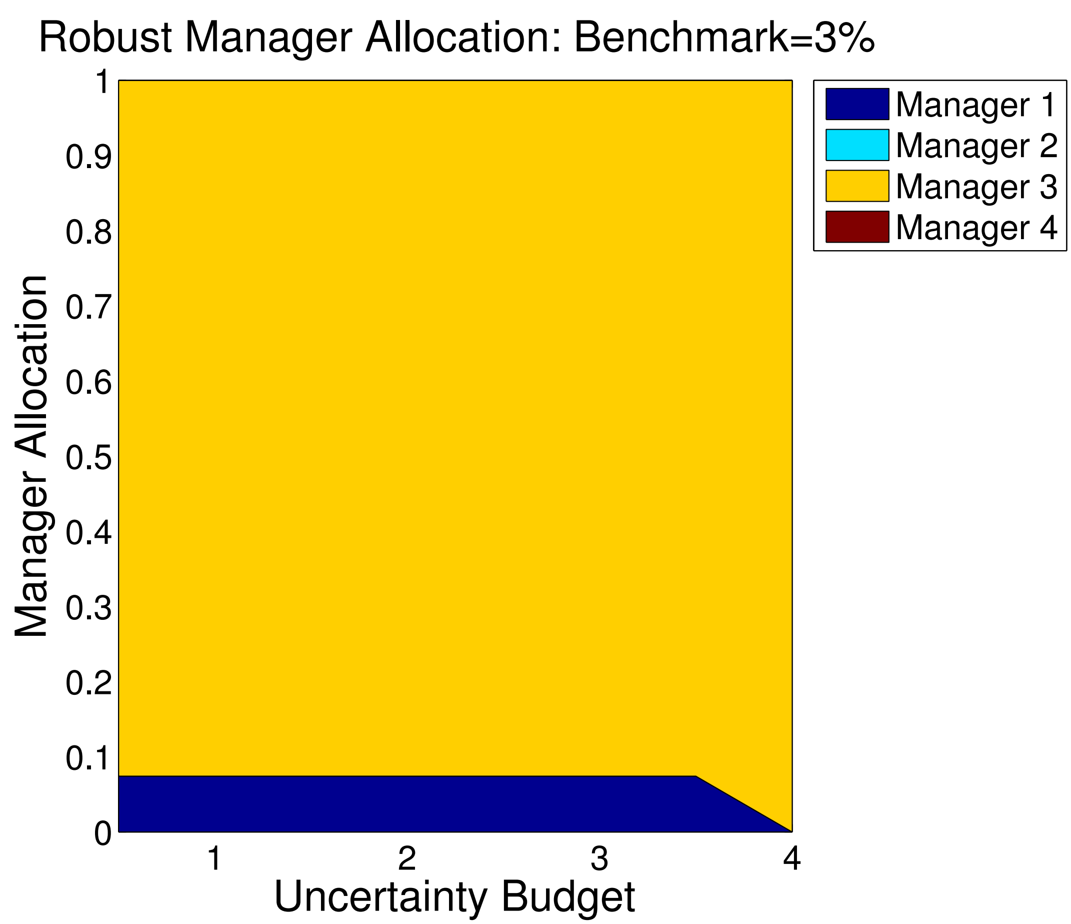

Figure 3.12: Robust Portfolio Manager Allocation with Benchmark Return = 3%

Table 3.10: Manager Allocation: Benchmark Return=3.2%

Return Benchmark=3.2%					
Uncertainty Budget	Manager 1	Manager 2	Manager 3	Manager 4	objective
0.5	0	0.25	0	0.75	0.0071
1	0	0.25	0	0.75	-0.0182
1.5	0	0.3386	0.3545	0.3069	-0.0268
2	0	0.3545	0.418	0.2275	-0.034
2.5	0	0.3677	0.4707	0.1617	-0.0383
3	0	0.3805	0.5219	0.0977	-0.0428
3.5	0	0.4	0.6	0	-0.0462
4	0	0.4	0.6	0	-0.0502

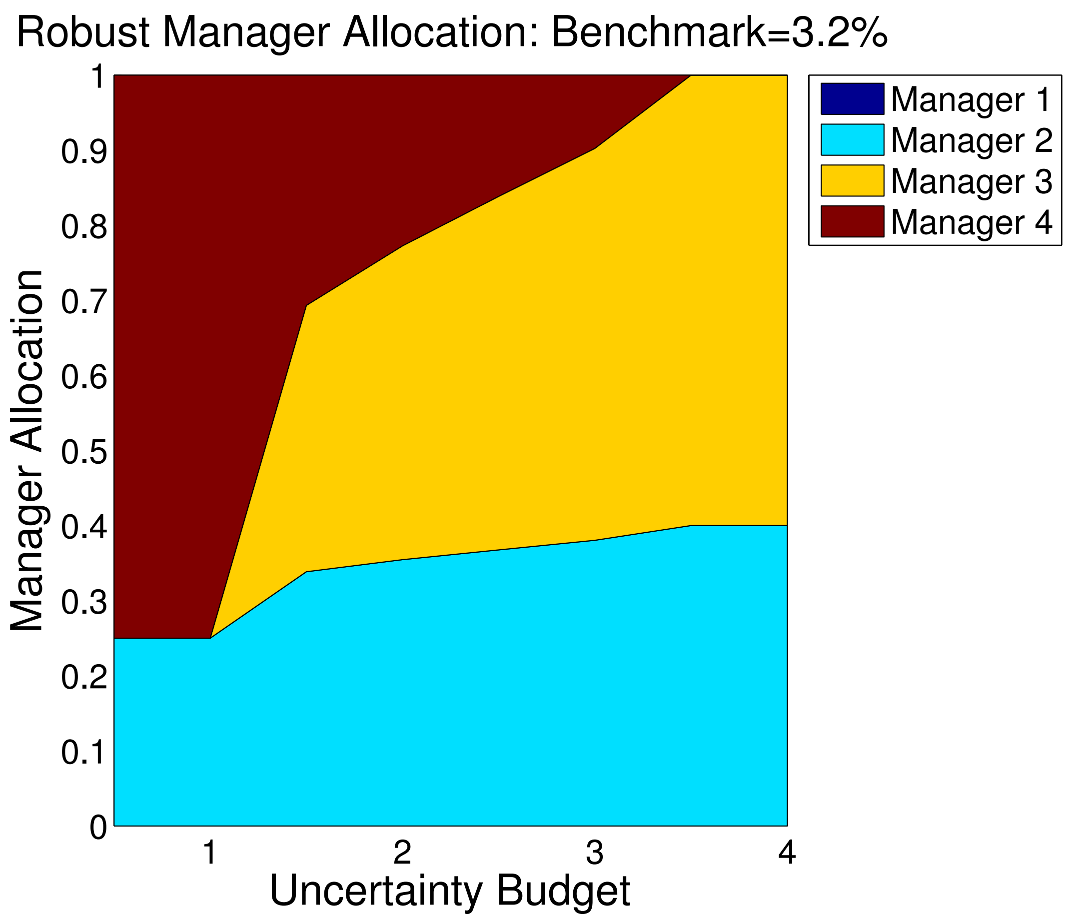

Figure 3.13: Robust Portfolio Manager Allocation with Benchmark Return = 3.2%

Table 3.11: Manager Allocation: Benchmark Return=3.4%

Return Benchmark=3.4%					
Uncertainty Budget	Manager 1	Manager 2	Manager 3	Manager 4	objective
0.5	0	0.9833	0	0.0167	-0.0086
1	0	0.9833	0	0.0167	-0.0532
1.5	0	0.985	0.0067	0.0083	-0.0602
2	0	0.985	0.0065	0.0085	-0.0671
2.5	0	0.9853	0.0078	0.0069	-0.0682
3	0	0.9867	0.0133	0	-0.0694
3.5	0	0.9867	0.0133	0	-0.0695
4	0	0.9867	0.0133	0	-0.0696

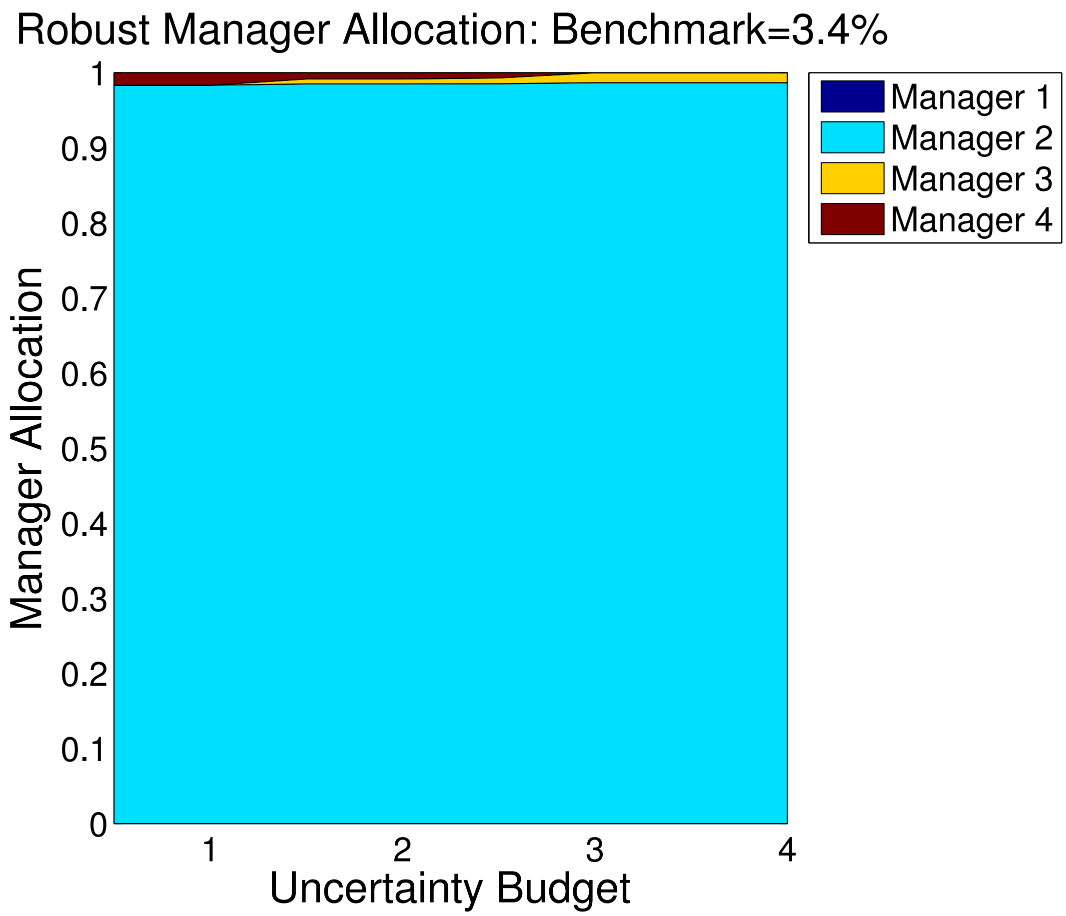

Figure 3.14: Robust Portfolio Manager Allocation with Benchmark Return = 3.4%

Six Managers with Six Assets

For six managers with six assets case, we choose $2\%, 3\%, 3.5\%$ and 3.7% as the benchmark return. Table **??** shows the nominal return of each manager. Again, when the benchmark return level is

$2\%, 3\%$, the problem is the same with no benchmark return problem. All the money concentrate to manager 6. As benchmark return increases, manager 6 takes less weight and manager 3 takes more weight in the portfolio. As benchmark return becomes as high as 3.7%, all money allocate to manager 3, since other manager has a much lower expect return, which could not meet the benchmark requirement. 3.10 shows the robust portfolio return changes with uncertainty level for each benchmark return level. As benchmark return level increase, the worst case return become more significant low as the concentration effect. All risk concentrate to one manager. Table 3.13 to figure **??** show the detailed manager allocation information.

Table 3.12: Nominal Return

	Manager 1	Manager 2	Manager 3	Manager 4	Manager 5	Manager 6
Nominal Return	0.0371	0.0118	0.0321	0.0025	0.027	0.0277

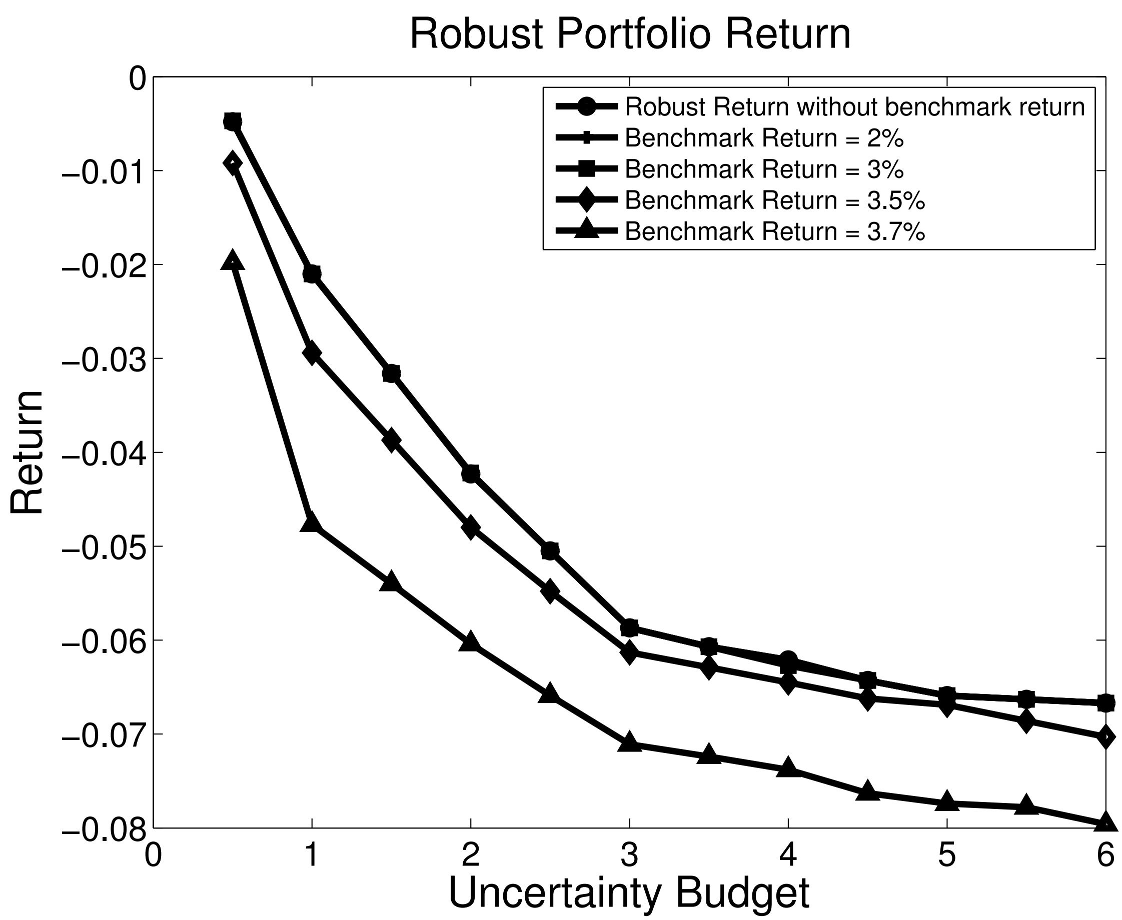

Figure 3.15: Robust Portfolio Return with Benchmark Return

Table 3.13: Manager Allocation: Benchmark Return=2%

Return Benchmark=2%							
Uncertainty Budget	Manager 1	Manager 2	Manager 3	Manager 4	Manager 5	Manager 6	objective
0.5	0	0	0	0	0	1	-0.0048
1	0	0	0	0	0	1	-0.021
1.5	0	0	0	0	0	1	-0.0316
2	0	0	0	0	0	1	-0.0423
2.5	0	0	0	0	0	1	-0.0505
3	0	0	0	0	0	1	-0.0587
3.5	0	0	0	0	0	1	-0.0607
4	0	0	0	0	0	1	-0.0621
4.5	0	0	0	0	0	1	-0.0643
5	0	0	0	0	0	1	-0.0659
5.5	0	0	0	0	0	1	-0.0663
6	0	0	0	0	0	1	-0.0667

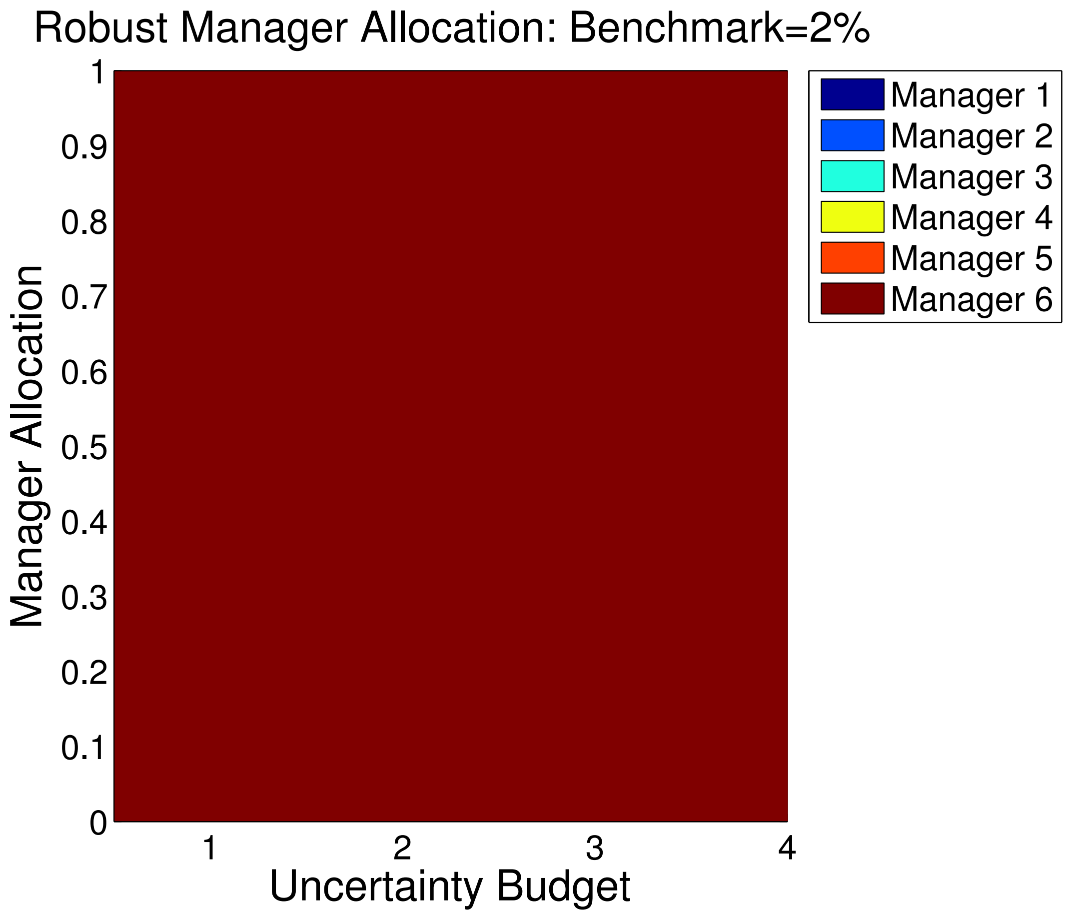

Figure 3.16: Robust Portfolio Manager Allocation with Benchmark Return = 2.0%

Table 3.14: Manager Allocation: Benchmark Return=3%

Return Benchmark=3%							
Uncertainty Budget	Manager 1	Manager 2	Manager 3	Manager 4	Manager 5	Manager 6	objective
0.5	0	0	0.5281	0	0	0.4719	-0.0047
1	0.2468	0	0	0	0	0.7532	-0.021
1.5	0.2468	0	0	0	0	0.7532	-0.0316
2	0.2468	0	0	0	0	0.7532	-0.0422
2.5	0	0	0.5281	0	0	0.4719	-0.0505
3	0	0	0.5281	0	0	0.4719	-0.0587
3.5	0	0	0.5293	0	0.0073	0.4634	-0.0607
4	0	0	0.565	0	0	0.435	-0.0627
4.5	0.0537	0	0.4271	0	0	0.5192	-0.0643
5	0.0106	0	0.5054	0	0	0.4839	-0.0659
5.5	0.0106	0	0.5054	0	0	0.4839	-0.0663
6	0	0	0.5281	0	0	0.4719	-0.0667

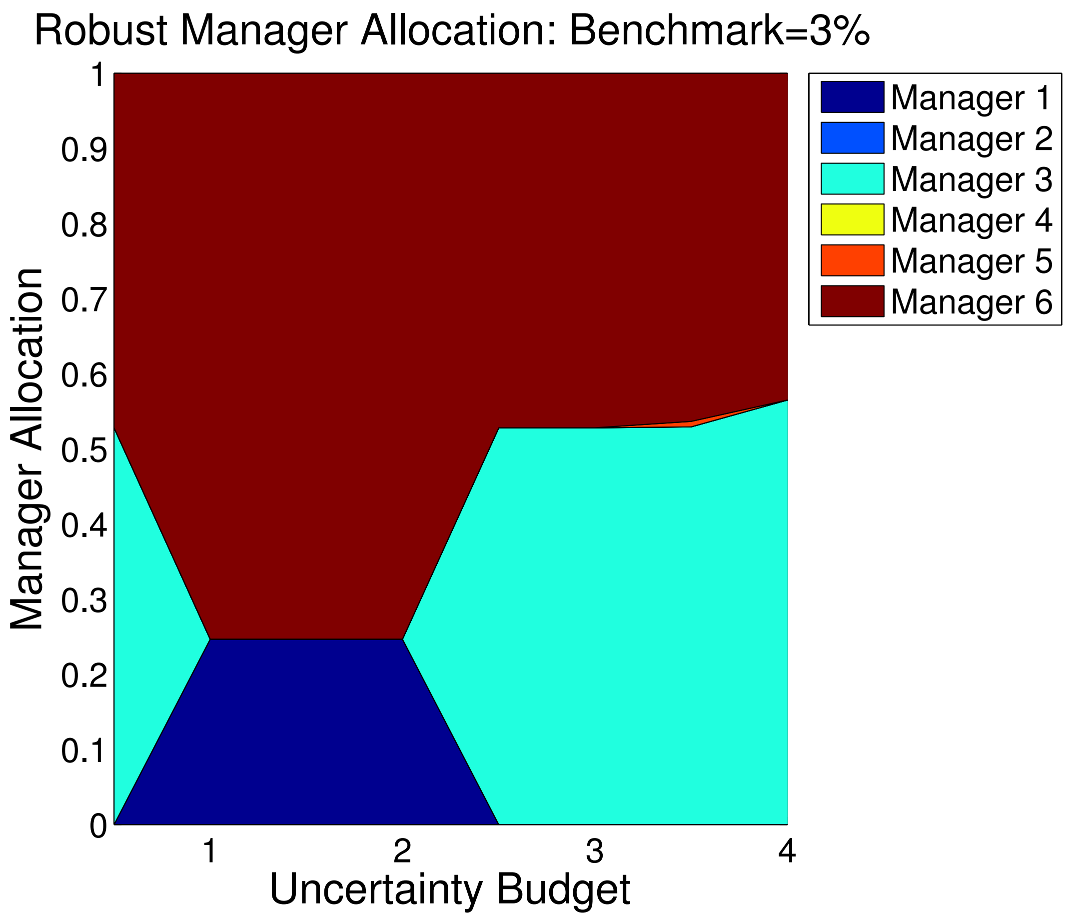

Figure 3.17: Robust Portfolio Manager Allocation with Benchmark Return = 3%

Table 3.15: Manager Allocation: Benchmark Return=3.5%

Return Benchmark=3.5%							
Uncertainty Budget	Manager 1	Manager 2	Manager 3	Manager 4	Manager 5	Manager 6	objective
0.5	0.5878	0	0.4122	0	0	0	-0.0092
1	0.7804	0	0	0	0	0.2196	-0.0294
1.5	0.7804	0	0	0	0	0.2196	-0.0387
2	0.7758	0	0.0098	0	0	0.2144	-0.048
2.5	0.5878	0	0.4122	0	0	0	-0.0548
3	0.5878	0	0.4122	0	0	0	-0.0613
3.5	0.611	0	0.3624	0	0	0.0265	-0.0629
4	0.6536	0	0.2815	0	0	0.0649	-0.0645
4.5	0.7375	0	0.0918	0	0.0004	0.1702	-0.0662
5	0.5878	0	0.4122	0	0	0	-0.0669
5.5	0.6145	0	0.3551	0	0	0.0304	-0.0686
6	0.5878	0	0.4122	0	0	0	-0.0703

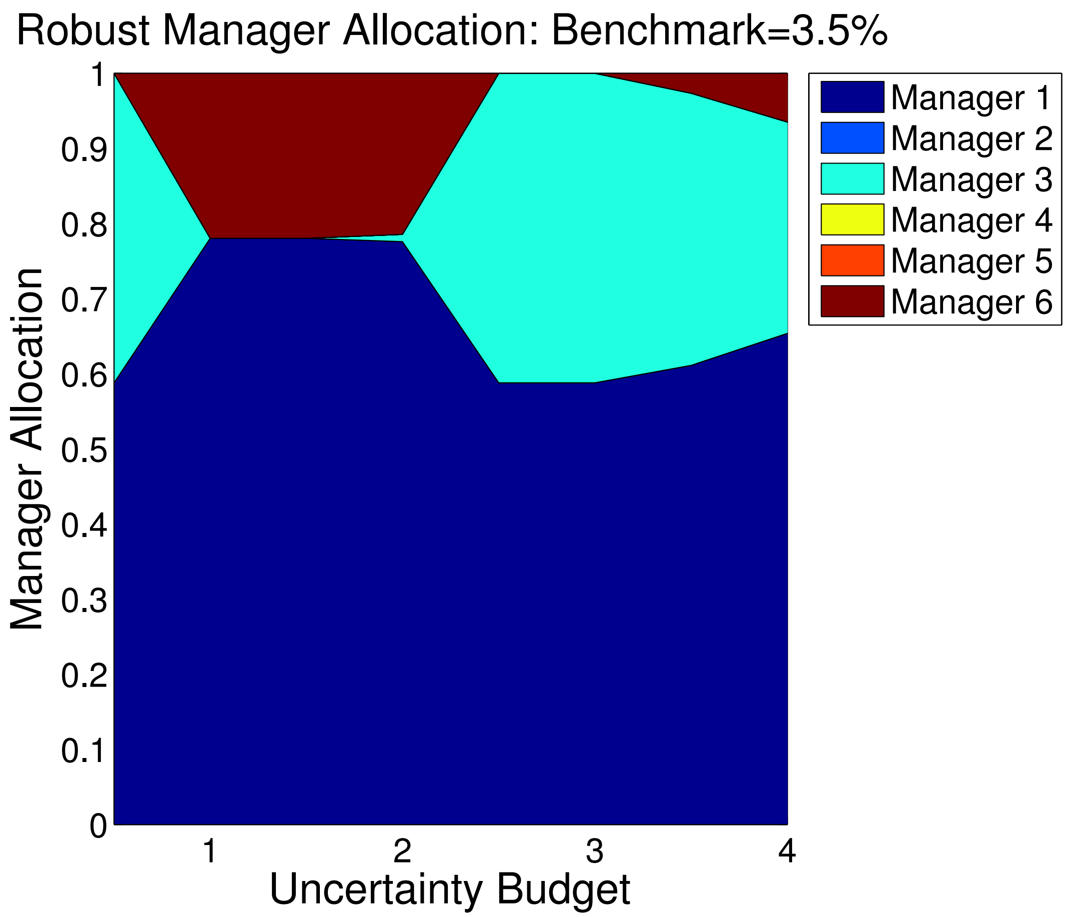

Figure 3.18: Robust Portfolio Manager Allocation with Benchmark Return = 3.5%

Table 3.16: Manager Allocation: Benchmark Return=3.7%

Return Benchmark=3.7%							
Uncertainty Budget	Manager 1	Manager 2	Manager 3	Manager 4	Manager 5	Manager 6	objective
0.5	0.9884	0	0.0116	0	0	0	-0.0198
1	0.9938	0	0	0	0	0.0062	-0.0477
1.5	0.9913	0	0.0054	0	0	0.0033	-0.054
2	0.9884	0	0.0116	0	0	0	-0.0604
2.5	0.9938	0	0	0	0	0.0062	-0.0659
3	0.9884	0	0.0116	0	0	0	-0.0711
3.5	0.9927	0	0.0025	0	0	0.0048	-0.0724
4	0.9923	0	0.0033	0	0	0.0044	-0.0738
4.5	0.9923	0	0.0033	0	0	0.0044	-0.0763
5	0.9923	0	0.0032	0	0	0.0044	-0.0774
5.5	0.9923	0	0.0033	0	0	0.0044	-0.0778
6	0.9884	0	0.0116	0	0	0	-0.0796

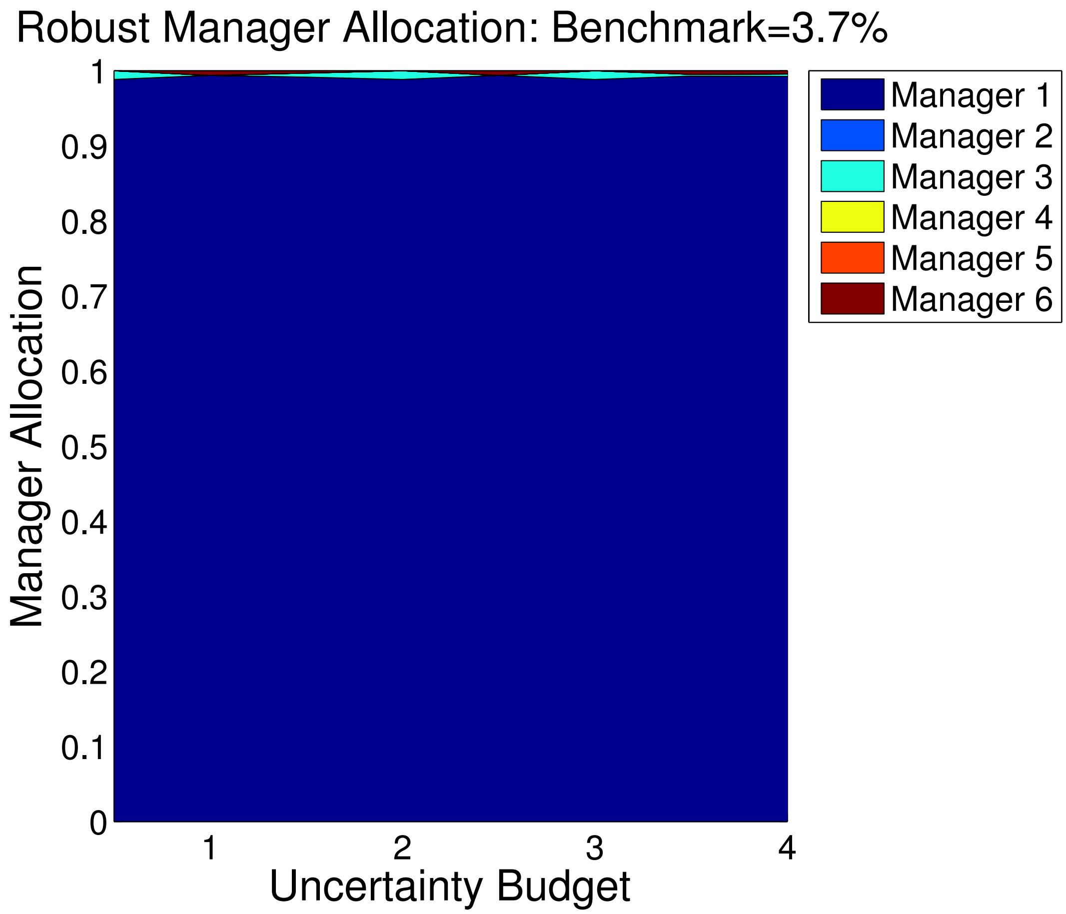

Figure 3.19: Robust Portfolio Manager Allocation with Benchmark Return = 3.7%

Twelve Manager with Six Assets

For the experiment of twelve managers with six assets, we can see the same story here. As benchmark return level increase, managers with high expected return takes over more weights but causes more severe worst return scenarios.

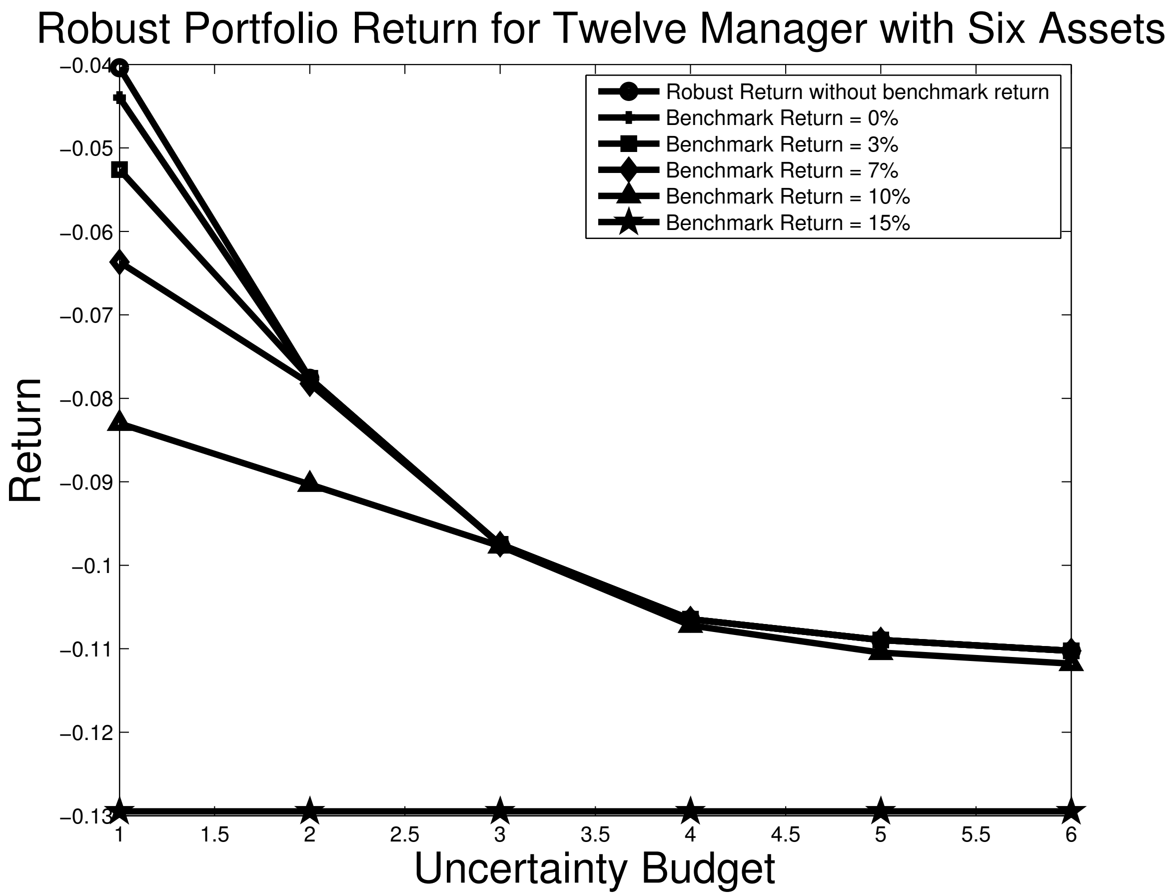

Figure 3.20: Robust Portfolio Return with Benchmark Return

Table 3.17: Manager Allocation: Benchmark Return=0%

Return Benchmark=0%							
Uncertainty Budget	Manager 1	Manager 2	Manager 3	Manager 4	Manager 5	Manager 6	
0.5	0	0	0	0	0	0	
1	0	0	0	0	0	0	
1.5	0	0	0	0	0	0	
2	0	0	0	0	0	0	
2.5	0	0	0	0	0	0	
3	0	0	0	0	0	0	
3.5	0	0	0	0	0	0	
4	0	0	0	0	0	0	
4.5	0	0	0	0	0	0	
5	0	0	0	0	0	0	
5.5	0	0	0	0	0	0	
6	0	0	0	0	0	0	
Uncertainty Budget	Manager 7	Manager 8	Manager 9	Manager 10	Manager 11	Manager 12	Objective
0.5	1	0	0	0	0	0	-0.0194
1	0.4527	0	0	0	0	0.5473	-0.0263
1.5	0.4797	0	0	0	0	0.5203	-0.0329
2	0.5746	0	0	0	0	0.4254	-0.0388
2.5	0.7076	0	0	0	0	0.2924	-0.0421
3	0.8127	0	0.0654	0	0	0.1219	-0.0442
3.5	0.8747	0	0	0	0	0.1253	-0.0475
4	0.8578	0	0	0	0	0.1417	-0.0494
4.5	0.9409	0	0.0591	0	0	0	-0.0508
5	1	0	0	0	0	0	-0.0536
5.5	1	0	0	0	0	0	-0.0539
6	1	0	0	0	0	0	-0.0542

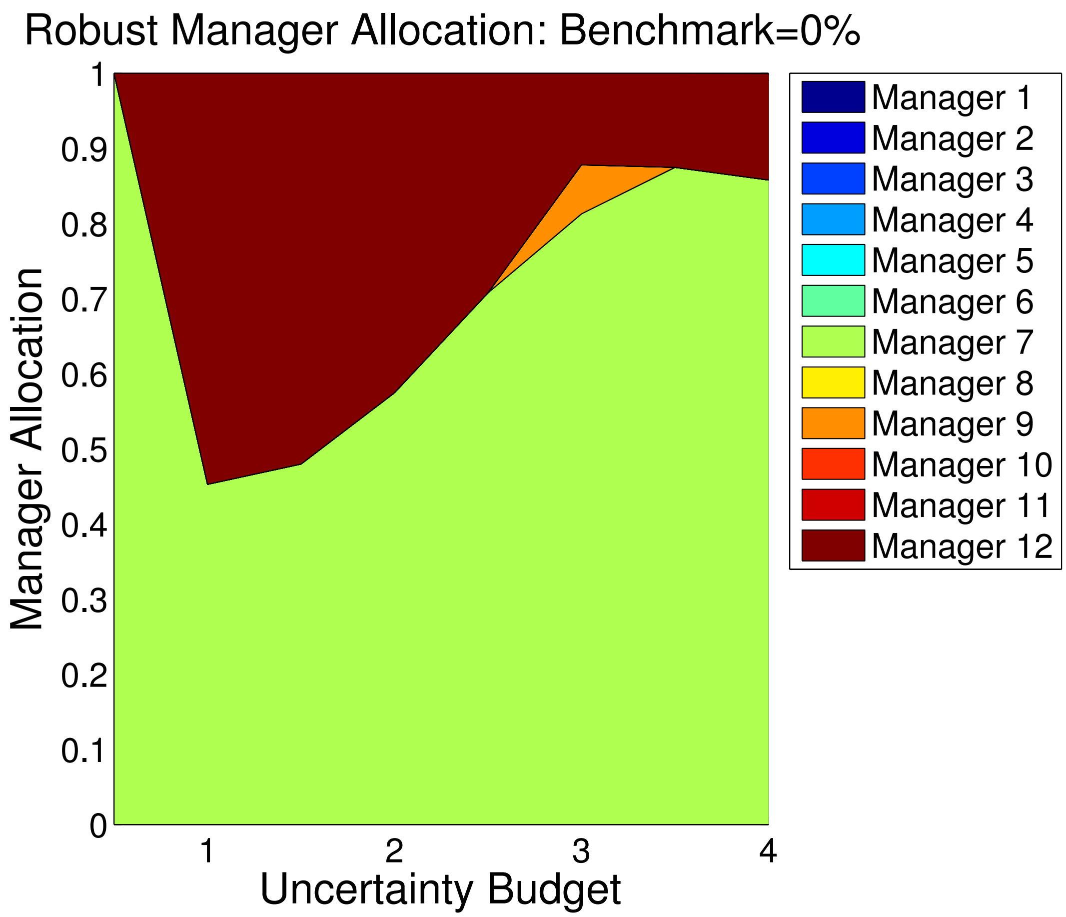

Figure 3.21: Robust Portfolio Manager Allocation with Benchmark Return = 0%

Table 3.18: Manager Allocation: Benchmark Return=2%

Return Benchmark=2%							
Uncertainty Budget	Manager 1	Manager 2	Manager 3	Manager 4	Manager 5	Manager 6	
0.5	0	0	0	0	0	0	
1	0	0	0	0	0	0	
1.5	0	0	0	0	0	0	
2	0	0	0	0	0	0	
2.5	0	0	0	0	0	0	
3	0	0	0	0	0	0	
3.5	0	0	0	0	0	0	
4	0	0	0	0	0	0	
4.5	0	0	0	0	0	0	
5	0	0	0	0	0	0	
5.5	0	0	0	0	0	0	
6	0	0	0	0	0	0	
Uncertainty Budget	Manager 7	Manager 8	Manager 9	Manager 10	Manager 11	Manager 12	Objective
0.5	1	0	0	0	0	0	-0.0194
1	0.701	0	0	0	0	0.299	-0.0283
1.5	0.701	0	0	0	0	0.299	-0.0341
2	0.701	0	0	0	0	0.299	-0.0403
2.5	0.9272	0	0	0	0	0.0728	-0.0419
3	0.8128	0	0.0653	0	0	0.1219	-0.0442
3.5	0.8748	0	0	0	0	0.1252	-0.0475
4	0.9072	0	0	0	0	0.0928	-0.0494
4.5	0.9409	0	0.0591	0	0	0	-0.0508
5	1	0	0	0	0	0	-0.0536
5.5	1	0	0	0	0	0	-0.0539
6	1	0	0	0	0	0	-0.0542

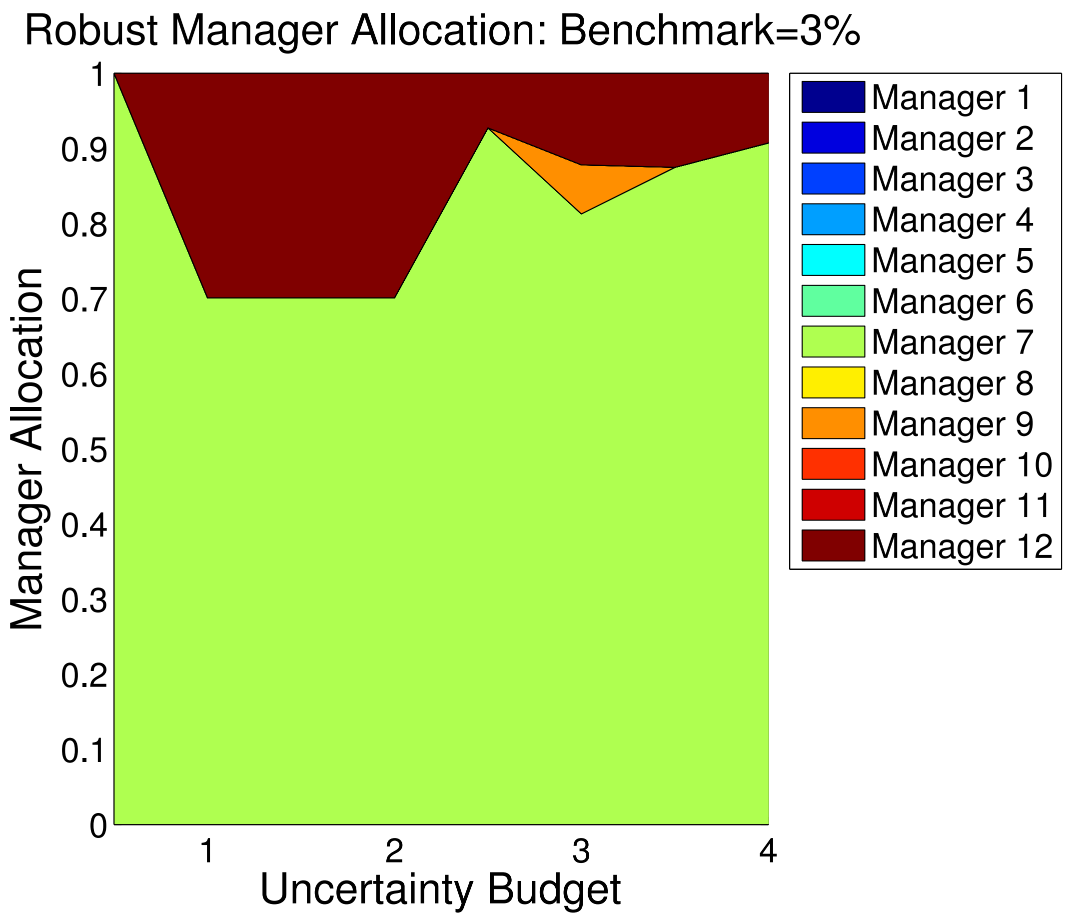

Figure 3.22: Robust Portfolio Manager Allocation with Benchmark Return = 2%

Table 3.19: Manager Allocation: Benchmark Return=2.5%

Return Benchmark=2.5%							
Uncertainty Budget	Manager 1	Manager 2	Manager 3	Manager 4	Manager 5	Manager 6	
0.5	0	0	0	0	0	0	
1	0	0	0	0	0	0	
1.5	0	0	0	0	0	0	
2	0	0	0	0	0	0	
2.5	0	0	0	0	0	0	
3	0	0	0	0	0	0	
3.5	0	0	0	0	0.0522	0	
4	0	0	0	0	0.0537	0	
4.5	0	0	0	0	0	0	
5	0	0	0	0	0	0	
5.5	0	0	0	0	0	0	
6	0	0	0	0	0	0	
Uncertainty Budget	Manager 7	Manager 8	Manager 9	Manager 10	Manager 11	Manager 12	Objective
0.5	0.6295	0	0.3705	0	0	0	-0.0228
1	0.6295	0	0.3705	0	0	0	-0.0342
1.5	0.6295	0	0.3705	0	0	0	-0.0403
2	0.6295	0	0.3705	0	0	0	-0.0463
2.5	0.5441	0.0304	0.4255	0	0	0	-0.0491
3	0.4696	0.0569	0.4735	0	0	0	-0.0517
3.5	0.4775	0	0.4702	0	0	0	-0.0538
4	0.4734	0	0.4729	0	0	0	-0.0557
4.5	0.6295	0	0.3705	0	0	0	-0.0572
5	0.6295	0	0.3705	0	0	0	-0.0583
5.5	0.6295	0	0.3705	0	0	0	-0.0586
6	0.6295	0	0.3705	0	0	0	-0.0589

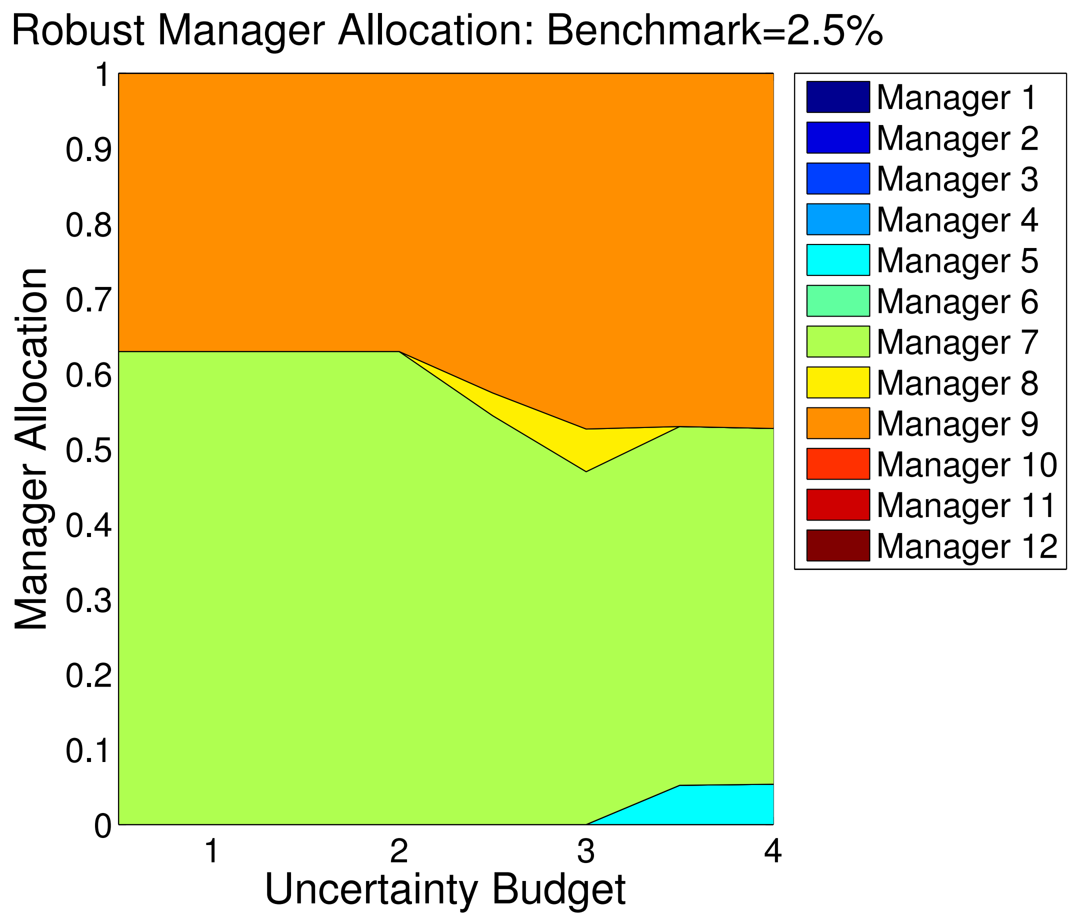

Figure 3.23: Robust Portfolio Manager Allocation with Benchmark Return = 2.5%

Table 3.20: Manager Allocation: Benchmark Return=2.9%

Return Benchmark=2.9%							
Uncertainty Budget	Manager 1	Manager 2	Manager 3	Manager 4	Manager 5	Manager 6	
0.5	0	0	0	0	0	0	
1	0	0	0	0	0	0	
1.5	0	0	0	0	0	0	
2	0	0	0	0	0	0	
2.5	0	0	0	0	0	0	
3	0	0	0	0	0	0	
3.5	0	0	0	0	0.0128	0	
4	0.0089	0	0	0	0	0	
4.5	0	0	0	0	0	0	
5	0	0	0	0	0	0	
5.5	0	0	0	0	0	0	
6	0	0	0	0	0	0	
Uncertainty Budget	Manager 7	Manager 8	Manager 9	Manager 10	Manager 11	Manager 12	Objective
0.5	0.0851	0	0.9149	0	0	0	-0.0278
1	0.0851	0	0.9149	0	0	0	-0.0395
1.5	0.0851	0	0.9149	0	0	0	-0.0468
2	0.0851	0	0.9149	0	0	0	-0.054
2.5	0.0552	0	0.9301	0	0	0.0146	-0.0559
3	0.0851	0	0.9149	0	0	0	-0.0582
3.5	0.0479	0	0.9393	0	0	0	-0.0601
4	0.0584	0	0.9327	0	0	0	-0.0619
4.5	0.0662	0	0.9245	0	0	0.0093	-0.0635
5	0.0851	0	0.9149	0	0	0	-0.0652
5.5	0.0851	0	0.9149	0	0	0	-0.0655
6	0.0851	0	0.9149	0	0	0	-0.0659

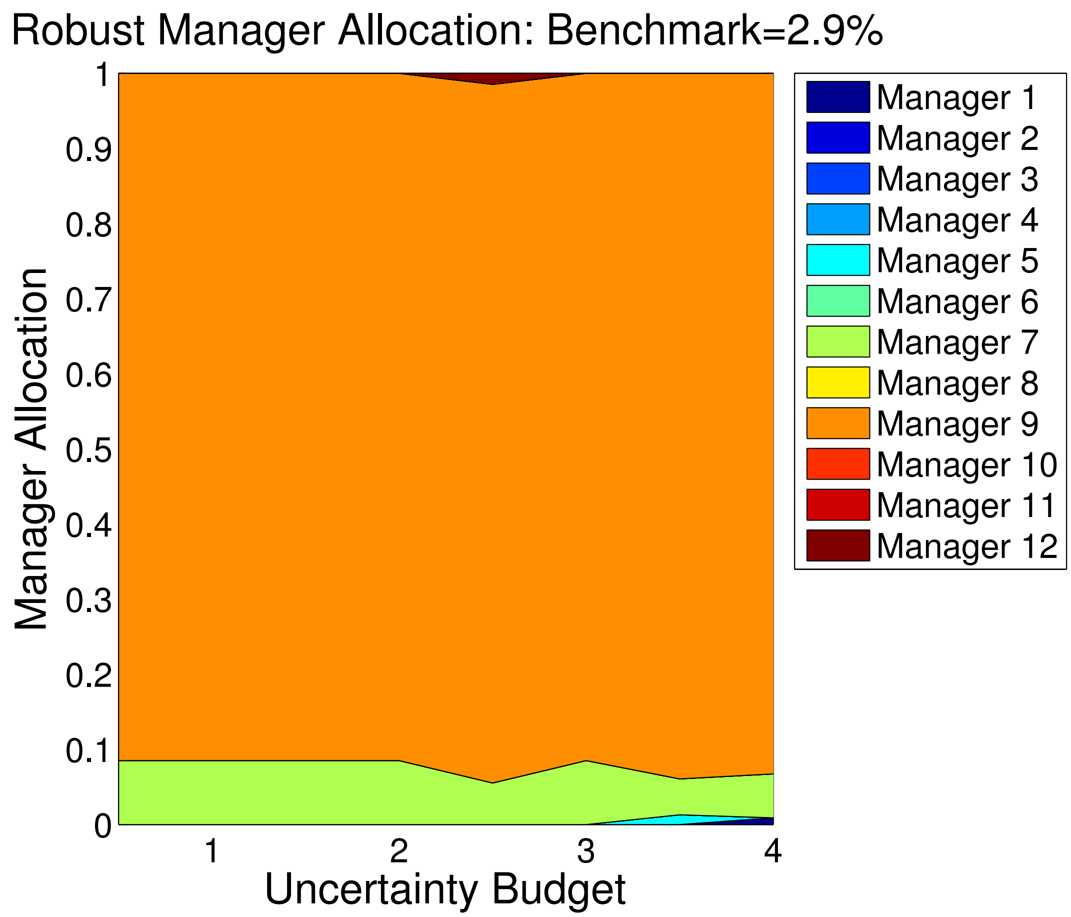

Figure 3.24: Robust Portfolio Manager Allocation with Benchmark Return = 2.9%

3.6 Conclusion

In this Chapter, we proposed a robust framework that takes into account two sources of uncertainty from manager asset allocation and asset return, in the context of manager selection and portfolio management. Upper and lower bound was also provided in our research. We investigated manager allocation pattern in two scenarios of without the return benchmark and without the return benchmark. In addition, we explored the special structure of the problem, and proposed two approaches to solve the problem efficiently. In addition, with modified algorithm, this model could also be applied to hedge fund strategies with lower bound smaller than zero and upper bound larger than one.

Bibliography

[1] C. Acerbi. Spectral measures of risk: a coherent representation of subjective risk aversion. *Journal of Banking and Finance*, 26(7):1505–1518, 2002.

[2] V. Agarwal and N.Y. Naik. Risk and portfolio decisions involving hedge funds. *Review of Financial Studies*, 2004.

[3] V. Agarwal, W. Fung, Y. C. Loon, and N.Y. Naik. Risks in hedge fund strategies: The case of convertible arbitrage. *Working Paper*, Georgia State University and London Business School, 2005.

[4] S. Alexander, T.F. Coleman, and Y. Li. Minimizing cvar and var for a portfolio of derivatives. *Journal of Banking and Finance*, 30(2):583–605, 2006.

[5] R. Arnott, J. Jsu, and P. Moore. Fundamental indexation. *Finanical Analysts Journal*, 61: 83–99, 2005.

[6] P. Artzner, F. Delbean, J.M. Eber, and D. Heath. Coherent measure of risks. *Mathematical Finance*, 9(3):203–228, 1999 July.

[7] R. Bacon. Practical portfolio performance measurement and attribution. *Wiley Finance*, 2004.

[8] S. Basak and A. Shapiro. Value-at-risk managemant: Optimal policies and asset prices. *Value-at-risk managemant: Optimal policies and asset prices*, 14(2):371–405, 2001.

[9] A. Ben-Tal and A.S. Nemirovski. Robust convex optimization. *Mathematics of Operations Research*, 23(4):769–805, 1998.

[10] A. Ben-Tal, T. Margalit, and A. Nemirovski. Robust modeling of multi-stage portfolio problems. *High Performance Optimization*, Kluwer Academic Publishers:125–158, 2000.

[11] A. Ben-Tal, L. ElGhaoui, and A. Nemirovski. Robust optimization. *Princeton University Press*, 2009. Princeton, NJ.

[12] S. Benati and R. Rizzi. A mixed integer linear programming formulation of the optimal mean/value-at-risk portfolio problem. *European Journal of Operational Research*, 176(1): 423–434, 2007.

[13] D. Bertsimas, D.B. Brown, and C. Caramanis. Theory and applications of robust optimization. *SIAM Review*, 53:464–501, 2011a.

[14] F. Black and R. Litterman. Asset allocation: Combining investor views with market equilibrim. *Fixed Income Resarch, Gold man Sachs*, September, 1990.

[15] G. Brinson and N. Fachler. Measuring non-us equity portfolio performance. *Journal of Portfolio Management*, pages 73–76, 1985.

[16] G. Brinson, R. Hood, and G. Beebower. Determinants of portfolio performance. *Financial Analyst Journal*, pages 39–44, 1986.

[17] R. Campbell, R. Huisman, and K. Koedijk. Optimal portfolio selection in a value-at-risk framework. *Journal of Banking and Finance*, 25(9):1789–1804, 2001.

[18] M. Carhart. On persistence in mutual fund performance. *Journal of Finance*, 52:57–82, 1997.

[19] D. Carino. Arithmetic and geometric attribution. *Journal of Performance Mea- surement*, Fall: 59–68, 1999.

[20] S. Ceria, F. Margot, A. Renshaw, and A. Saxena. Novel approaches to portfolio construction: multiple risk models and multisolution generation. *Optimizing Optimization: The Next Generation of Optimization Applications and Theory*, Elsevier Science:23–45, 2009.

[21] E. Cetinkaya and A. Thiele. Data-driven portfolio management with quantile constriants. *Working Paper, Lehigh University*, 2012.

[22] J. Chen and S. Burer. Globally solving nonconvex quadratic programming problems via cm-pletely positive programming. *Manthmatical Programming Computations*, 4:33–52, 2012.

[23] O. Davies and D Laker. Multiple-period performance attribution using the brin- son model. *Journal of Performance Measurement*, Fall:12–22, 2001.

[24] V. DeMiguel and F. J. Nogales. Portfolio selection with robust estimation. *Operations Research*, 57:560–577, 2009.

[25] Y. Dong and A.C. Thiele. Robust portfolio management with uncertainty in asset allocation. *Working paper*, 2014.

[26] L. El-Ghaoui and H. Lebret. Robust solutions to least-squares problems with uncertain data. *SIAM Journal on Matrix Analysis and Applications*, 18:1035–1064, 1997.

[27] L. ElGhaoui, M. Oks, and F Oustry. Worst-case value-at-risk and robust portfolio optimization: a conic programming approach. *Operations Research*, 51(4):543–556, 2003.

[28] E. F. Fama and K. R. French. Common risk factors in the returns on stocks and bonds. *Journal of Financial Economics*, 1:3–56, 1993.

[29] W. Fung and D. A. Hsieh. Asset-based style factors for hedge funds. *Financial Analysts Journal*, 58(5):16–27, 2002a.

[30] W. Fung and D. A. Hsieh. Hedge fund benchmarks: A risk-based approach. *Financial Analysts Journal*, 60(5):65–80, 2004.

[31] V. Gabrel, C. Murat, and A. Thiele. Recent advances in robust optimization an robustness. *Technical report, Universite Paris-Dauphine*, 2012.

[32] A. A. Gaivoronski and G. C. Pflug. Value-at-risk in portfolio optimization: properties and computational approach. *Journal of Risk*, 7(2):1–31, 2005.

[33] P. Glasserman and X. Xu. Robust portfolio control with stochastic factor dynamics. *revised and resubmitted to Operations Research*, 2012.

[34] D. Goldfarb and G.N. Iyengar. Robust portfolio selection problems. *Mathematics of Operations Research*, 28:1–38, 2003.

[35] M. Grinblatt and S. Titman. Performance measurement without benchmarks: An examination of mutual fund returns. *Journal of Business*, 66:47–68, 1993.

[36] R. Horst and H Tuy. Global optimization. *New York:Springer*, 1996.

[37] R. Horst, Pardalos P., and Thoai N. Introduction to global optimization. *Boston:Springer*, 2000.

[38] C. Hsu and W. Myers. Performance attribution: Measuring dynamic allocation skill. *Financial Analysts Journal*, 66(6), 2010.

[39] G. Iyengar and A. K. C. Ma. Fast gradient descent method for mean-cvar optimization. *Under review in Annals OR*, 2009.

[40] L. Jaeger and C. Wagner. Factor modeling and benchmarking of hedge funds: Can passive investments in hedge fund strategies deliver? *The Journal of Alternative Investments*, Winter, 2005.

[41] M. Jensen. The performance of mutual funds in the period 1945-1964. *Journal of Finance*, 23:389–416, 1968.

[42] M. Jensen. Risk, the pricing of capital assets, and the evaluation of investment performance. *Journal of Business*, 42:167–247, 1969.

[43] A. Katsaris, A. Hegazi, and M. Goulet. A total risk measurement framework for hedge funds and funds of funds. *Working Paper*, 2010.

[44] L. Kirievsky and A. Kirievsky. Attribution analysis: Combining attribution effects over time made easy. *The Journal of Performance Measurement*, 4(4):49–59, Summer 2000.

[45] H. Konno and H. Yamazaki. Mean-absolute deviation portfolio optimization and its applications to tokyo stock market. *Management Science*, 37(5):519–531, 1991.

[46] A. E. B. Lim, J. G. Shanthikumar, and G.Y. Vahn. Fragility of cvar in portfolio optimization. *Technical Report, U.C. Berkeley*, 2009.

[47] W. Lo. Where do alphas come from?: A measure of the value of active investment management. *Journal of Investment Management*, 6(22), 2008.

[48] H. Markowitz. Portfolio selection. *The Journal of Finance*, 7(1):77–91, 1952.

[49] H. Mausser and D. Rosen. Beyond var: From measuring risk to managing risk. *Algorithmic Research Quarterly*, 1(2):5–20, 1998.

[50] G. Menchero. Multiperiod arithmetic attribution. *Financial Analysts Journal*, 60(4):76–91, 2004.

[51] Morningstar. Indentigying a sponsor's impact on total returns. *Morningstar*, 2010.

[52] Morningstar. Performance attribution for equity funds and categories methodology. *Morningstar*, 2011.

[53] A.G. Nahapetyan. Bilinear programming. *Working paper*.

[54] K. Natarajan, D. Pachamanova, and M. Sim. Incorporating asymmetric distributional information in robust value-at-risk optimization. *Management Science*, 54(3):573–585, 2008.

[55] Y. Nesterov. Smooth minimization of non-smooth functions. *Mathematical Programming*, 103(1):127–152, 2005.

[56] J. Okunev and D. White. An analysis of the risk factors underlying hedge fund returns. *Intelligent Hedge fund investing*, 2004.

[57] J. Pang and S. Leyffer. On the global minimization of the value-at-risk. *Optimization Methods and Software*, 19(5):611–631, 2004.

[58] T. A. Pirvu. Portfolio optimization under the value-at-risk constraint. *Quantitative Finance*, 7 (2):125–136, 2007.

[59] RiskMetricsTM. third edition. *J.P. Morgan*, 1995.

[60] R.T. Rockafellar and S. Uryasev. Optimization of conditional value-at-risk. *Journal of Risk*, 2 (3):21–41, 2000.

[61] G. J. L. Rodriguez. Portfolio optimization with quantile-based risk measures. *Ph.D. thesis, Massachusetts Institute of Technology*, 1999.

[62] W. Sharpe. Capital asset prices: Atheory of market equilibrim under condistions of risk. *Journal of Finance*, 19:425–442, 1964.

[63] W. Sharpe. Mutual fund performance. *Journal of Business*, 39:119–138, 1966.

[64] H.D. Sherali and W.P. Adams. A reformulation-linearization technique for solving discrete and continuous nonconvex problems. *Kluwer, Dordrecht*, 1998.

[65] Staff. Basel iii: A global regulatory framework for more resilient banks and banking systems. *Bank for International Settlements*, 2011.

[66] D. Tasche and L. Tibiletti. Approximations for the value-at-risk approach to risk-return analysis. *The ICFAI Journal of Financial Risk Management*, 1(2):44–61, 2004.

[67] J. Treynor. How to rate management of investment funds. *Harvard Business Review*, 43: 63–75, 1965.

[68] J. Treynor. Why market-valuation-indifferent indexing works. *Finanical Analysts Journal*, 61: 65–69, 2005.

[69] J. Treynor and K. Mazuy. Can mutual funds outguess the market? *Harvard Business Review*, 44:131–163, 1966.

[70] H. Tuy. Concave minimization under linear constraints with special structure. *Hanoi:Hanoi Inst. Math*, 1983.

[71] D. Wozabal, R. Hochreiter, and G. C. Pflug. A d.c. formulation of value-at-risk constrained optimization. *Optimizationt*, 59(3):377–400, 2010.

Appendix A

Biographical Information

Ms Dong is a doctoral candidate in the Industrial and Systems Engineering at Lehigh University. She graduated from the University of Florida with a Master of Science degree in Industrial and Systems Engineering, where she received an Achievement Award for an Engineering Student. She also holds a Bachelor of Engineering degree in Industrial Engineering from the Beijing Jiaotong University in China. Her professional experience includes internships at the Lehigh University Investment Office, the Walt Disney Company and Siemens IT Solutions and Services, as well as several research and teaching assistantships at Lehigh University. Yang will join JP Morgan in New York, NY as a senior quantitative analyst.