The ultimate strength of welded continuous frames and their components. Progress report no.3: plastic design and the deformation of structures

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The Ultimate Strength of Welded Continuous Frames
and Their Components

Progress Report No. 3

PLASTIC DESIGN AND THE DEFORMATION
OF STRUCTURES

By
Chien-huan Yang, Lynn S. Beadle
and
Bruce G. Johnston

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I - SYNOPSIS

It has been shown by a number of authors that structural design by the plastic design method may effect a considerable saving of materials and also of a simpler procedure than the conventional elastic design method. The deformation of a structure designed by the plastic method is expected to be larger than that of a similar indeterminate structure designed by the elastic method, but is usually less than that of a similar determinate structure. Most of the reserve load-carrying capacity of structures may be utilized without the danger of excessive deformation, the limitation being provided by a convenient method suggested.

II - INTRODUCTION

If a structure designed elastically is loaded beyond its full load* a part of the structure will yield. In most cases yielding will not cause instability. After some yielding, the rotations and deflections of some indeterminate structures may still be within a limit which can be allowed in practical designs. Of course this fact has been mentioned recently by writers on "limit" or "plastic" design (1, 2, 3, 4, 5).

* Full load is defined as the working load multiplied by the factor of safety.
The recent emphasis on this subject may in part be attributed to the development of welding. By this means it is possible to join members with economy and at the same time allow the transmission of the full cross-sectional strength from one member to another. Thus welding can provide a completely continuous frame (often termed a "rigid" frame) and it is sought to take advantage of such continuity through rational design methods.

Baker(5) has suggested the use of plastic design for certain practical engineering structures, and it is interesting to note that the British designer is now allowed to use the plastic method. The British Standard Specification states in part,

"... for the purpose of such design accurate methods of structural analysis shall be employed leading to a load factor of 2.0, based on the calculated or otherwise ascertained failure load of the structure or any of its parts, and due regard shall be paid to the accompanying deformations under working loads, so that deflections and other movements are not in excess of the limits implied in this British Standard."

To illustrate some of the concepts of plastic behavior (and in particular that of deformation) examine the load-deflection relationship of an indeterminate structure "fixed" by welding to a rigid boundary.
For example, consider a beam with uniform section loaded as shown in Fig. 1 with a concentrated load closer to one support than the other.

![Beam Fixed at Both Ends](image1)

![Moment Diagram at Collapse](image2)

Fig. 1

As the load $P$ increases the yield stress is reached first at section $A$ at the support. By simple plastic theory a "plastic hinge" will be formed subsequently at $A$. As the load $P$ increases, the yield stress is next reached at section $B$ and eventually at section $C$ as shown in the diagram.

When the deflection of point $B$ is plotted against load $P$ the curve shown in Fig. 2 results.

![Load vs. Deflection at B](image3)

Fig. 2
The curve consists of three approximately straight line portions joined by short curves. Length OA represents the load-deflection relationship when the whole structure is within the elastic range. The slope of the portion AB in Fig. 2 is equivalent to that of the beam shown in Fig. 3 loaded within the elastic range.

\[ P \]
\[ a \quad b \]

Fig. 3

Likewise the load deflection curve of the beam shown in Fig. 4 loaded within the elastic range will be similar to the portion BC of Fig. 2.

\[ P \]
\[ b \]

Fig. 4

According to conventional elastic design, the beam shown in Fig. 1 will only be expected to carry the load \( P_1 \) although only
one part of the beam reaches the yield stress. This reveals the defect of applying the criterion of stress to this type of structure when elastically designed. The structure will hold the external and internal forces in equilibrium at a load much higher than $P_1$ although the deformation of the structure increases at a higher rate as the load goes higher than $P_1$. However, if the load is kept below $P_2$ in Fig. 2, the rate of deflection will not be any greater than in the perfect elastic structure shown in Fig. 4. Thus, from the viewpoint of strength of the structure, it seems reasonable that the full load should be raised to $P_2$ instead of restricting it to $P_1$ as governed by the criterion of a limiting stress.

When designing a structure, in addition to requiring that all external and internal forces be in equilibrium, engineers usually require that deformations be held within certain limits. Therefore, as long as the structure can hold the loads within an allowable limit of deflection, it will not matter if the flexural stress of the structure exceeds the yield stress.

As a matter of fact, in many conventional bridge and building designs engineers often find the stress-criterion unsatisfactory due to large elastic deformations which current specifications will not allow.

These considerations suggest that the design criterion of structures should be based on deformation instead of stress. (This has also been discussed by Van den Broek(1)). However, a number
of basic questions are posed. If the plastic design method is used and the deformation of structures is limited will a more economical design result? Rolled structural steel members are believed to have reduced resistance to buckling in the plastic range. Would this buckling prevent the wide application of the plastic design method? It is known that the elastic design of indeterminate structures more often than not involves a very lengthy and laborious procedure. Will the procedures in plastic design be more or less complicated?

The program for investigating the plastic behavior of continuous beams carried on at Lehigh University, has thrown further light on the solution to some of these questions. A discussion of the deformation of structures in the plastic range is given in this report, together with a criterion for selecting the full load in the plastic design method. Discussion of some of the remaining questions will be included in a forthcoming report on the results of continuous beam tests.

It is not the purpose of this paper to present rigorous methods for computing deflections. This is being treated separately. Attention is restricted to the first question posed above regarding a limitation of deflection.
III - DEFORMATION OF STRUCTURES IN THE PLASTIC RANGE

If deformation is a logical criterion for satisfactory structural design, then what limitation should be adopted? Actually, a beam made of mild steel as shown in Fig. 1 will carry additional load even after three plastic hinges are developed at A, B, and C. Any additional load will only cause direct tensile stresses in the member, assuming the supports of the beam are prevented from moving longitudinally. The load capacity then is limited by the deformation which can be allowed in actual practice and not by any condition of equilibrium.

In elastic design, although stress is used as a criterion, it is usually found that the deformation of the structure has been limited automatically. A simple example is found in a statically determinate structure. When the yield strength is reached, the deformation will increase very rapidly with only a small increment of additional load. In this case the criterion of stress is also a criterion of deformation in design. However in indeterminate structures one will usually find that the criterion of stress will provide an inherent deformation limitation that is often too far on the safe side. The deformations in indeterminate structures usually do not increase as rapidly as statically determinate structures after reaching their yield strength.

This fact is made more evident in the following example. Here the relation between stress and deflection in statically determinate and indeterminate structures is compared by designing a beam in three ways: simply supported, "fixed-ended", and finally, provided with two plastic hinges.
In the first two examples, although the principles of elastic behavior govern the design, the procedure is somewhat different from practice. It is customary for the engineer to select beams on the basis of a permissible working stress whereas the computations that follow have left the assigning of a safety factor until the end. This was done to facilitate the computation of deflections both at full load and at the working load.

(a) Simply-supported beam designed elastically

![Diagram of simply-supported beam](image)

With the "full load" (working load multiplied by the factor of safety) and span shown in Fig. 5, it is found that the yield moment is 2800 inch kips as shown in the moment diagram of Fig. 6.

![Moment Diagram](image)

Use $\sigma_y = 33 \text{ kips/in}^2$

$$S = \frac{M_y}{\sigma_y} = 85$$

Use 18WF50 ($S = 89, I = 800.6$)
Making use of tables the maximum deflection of the beam loaded with the full load will then be

\[ \delta_{\text{max}} = \frac{F_a b \sqrt{b h_c}}{278 I L} \]

\[ \approx 1.22" \]

(b) Beam fixed at both ends, designed elastically (Fig. 7)

Since the maximum moment is at A,

\[ M_A = \frac{M}{Y} = \frac{F_a b}{L^2} = 1870 \text{ in. kips} \]

\[ S = \frac{M}{Y} = 1870 \frac{1}{33} = 56.6 \text{ in.}^3 \]

Use 16WF36 (S = 56.3, I = 445.3).
The maximum deflection of the beam loaded with the full load will then be

\[ \delta_{\text{max}} = \frac{2Pb^3a^2}{3EI(3b+a)^3} \]

= 0.49"

Note that in case (a) the maximum deflection of the simply-supported beam is three times as large as the maximum deflection of the beam with fixed ends, case (b).

The next logical step, based on the concepts of plastic behavior, is to design the beam with plastic hinges. Fig. 2 shows very clearly that with the formation of three plastic hinges, corresponding to load \( P_3 \), the deflections are indeterminate. However, hinge at load \( P_2 \), where the formation of the last plastic hinge is just commencing at point C, the deflection is determinate. Thus, this case is selected next.

(c) Beam with plastic hinges at points A & B.

The beam is designed on the condition that plastic hinges have developed at A & B and the yield stress is reached at C. According to the simple plastic theory the moment diagram for this condition is shown in Fig. 8.
From the moment diagram, we have

\[ bc = \frac{Pab}{L} = ac - ab \]

\[ \frac{Pab}{L} = 2M_y - \frac{a}{L} (M_p - M_y) \tag{1} \]

There is available a convenient term, \( f \), called the "shape factor",

\[ f = \frac{M_p}{M_y} \]

Thus Eq. (1) becomes

\[ \frac{Pab}{L} = M_y \left( 2f - \frac{a}{L} - \frac{a}{L} \right) \]

and

\[ M_y = \frac{Pb}{\frac{2fl^2}{a} + (1-f)} \tag{2} \]

Substituting in Eq. (2) and assuming a shape factor, \( f \), of 1.15, then

\[ M_y = 1190 \text{ in.kips.} \]

and

\[ S = \frac{M_y}{33} = 36 \text{ in.}^3 \]

Use 14W30 (\( S = 41.8 \quad I = 289.6 \))
The maximum deflection of this beam will be at B (since B is assumed to be a "hinge") and can be conveniently found by the moment area method.

In calculating this deflection, we will assume the portion BC of the beam is entirely within the elastic range. Actually, the region near point B will yield as shown in Fig. 10.

From Fig. 9b we may calculate the length Δb of the yielded portion.
Thus

\[
\frac{b-\Delta b}{b} = \frac{2M_y}{M_y + M_p} = \frac{2}{1+f}
\]

\[\therefore \Delta b = b(1 - \frac{2}{1+f})\]

For the assumed shape factor, \(f = 1.15\)

\[\Delta b = 0.07b = 1.4 \text{ feet}\]

The \(\frac{M}{EI}\) curve on the conjugate beam at the plastic hinge should be as shown with the dotted line. It is evident that it affects the value of deflection very little.

\[M_y = \sigma_y S = 33 \times 41.8 = 1380 \text{ in.kips}\]

Using the moment-area method with the origin at hinge B:

\[
\frac{\delta}{EI} = \int_b^c M_x dx = \frac{M_y}{2} \left( \frac{1}{1+f} \left( \frac{b}{3} + \frac{2}{1+f} \right) \right)
\]

\[\delta = \frac{M_y b^2}{2(1+f)EI} \left( f + \frac{2}{3} - \frac{f^3}{3} \right) = 1.37''\]

This deflection corresponds to a load of 38.8 kips instead of the 35-kip full load. The 14WF33 beam has a section modulus of 41.8 in\(^3\) rather than the required 37.5 in\(^3\). Thus it will carry a greater load when the moment distribution is as shown in Fig. 8.

Comparing the three different designs (a), (b), and (c), we find the deformation of the third design relatively small. Deflections under the full load have been computed in the above discussion. Since the engineer customarily computes deflections
under the working load, this has also been done using a safety factor of 1.65 for the elastic designs and a load factor of 1.65 for the plastic design. The results are shown in the following table, tabulated with the full load deflections for the three different designs studied.

<table>
<thead>
<tr>
<th>Design Method</th>
<th>Section Used</th>
<th>Max. Possible Deflection under Working Load</th>
<th>Max. Possible Deflection under Full Load</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Elastic design</td>
<td>16WF50</td>
<td>0.74''</td>
<td>1.22''</td>
</tr>
<tr>
<td>using simply-supported beam</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(b) Elastic design</td>
<td>16WF36</td>
<td>0.30''</td>
<td>0.49''</td>
</tr>
<tr>
<td>using fixed-ended beam</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(c) Plastic design</td>
<td>14WF30</td>
<td>0.46''</td>
<td>1.37''</td>
</tr>
<tr>
<td>(limiting the deflection)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Specification limit for deflection \( \left( \frac{1}{360} \times \text{span} \right) = 1.00 \) inches.

In case (c) above, to compute the deflection, refer first to Figure 2. With a load factor of 1.65, the working load will be very close to load \( P_1 \). As a check,

\[
P_W = \frac{P_2}{L^2} = \frac{35}{1.65^2} = 21.2 \text{ kips}
\]

\[
M_A = \frac{P_{ab^2}}{L^2} = 1135 \text{ in.kips}
\]

Now for the 14WF30 section

\[
M_y = 1380 \text{ in.kips}
\]
Thus the beam is within the elastic range under working loads and, as in case (b),

\[ \delta_{\text{max}} = \frac{2Wb^3a^2}{3EI (3b + a)^2} = 0.45'' \]

From the above table it is seen that by use of plastic design a 20\% savings in material is possible over the fully restrained "elastic" beam, the deflection of the "plastic" beam at working loads being less than the simply-supported beam. In each case, deflections are less than the specification limit.

The example chosen (Fig. 1) was selected because its load-deflection relationship (Fig. 2) demonstrates the step-by-step formation of the three plastic hinges. However, in many engineering structures the design is based on uniformly loaded beams. Appendix A has been prepared to show the load-deflection relationships for such loading. Also the case of a single concentrated load at the center is presented. In addition, comparative deflections similar to the above table have been developed.

<table>
<thead>
<tr>
<th>Uniform Load</th>
<th>Centrally Concentrated Load</th>
</tr>
</thead>
<tbody>
<tr>
<td>Design Method</td>
<td>Max. Defl.</td>
</tr>
<tr>
<td></td>
<td>Working Load</td>
</tr>
<tr>
<td>(a) Simple beam, elastic design</td>
<td>16WF47</td>
</tr>
<tr>
<td>(b) Fixed beam, elastic design</td>
<td>16WF36</td>
</tr>
<tr>
<td>(c) Plastic design</td>
<td>16WF30</td>
</tr>
</tbody>
</table>

The results of the deflection computations are shown in the above table.
The approximate load deflection curves for all three designs are shown, non-dimensionally in Fig. 11.

![Diagram of load deflection curves](image)

**Fig. 11**

The necessary information for plotting the load-deflection curves of Fig. 11 is to be found in Appendix B.

The following observations may be made based on Appendices A and B, Fig. 11, and the above tables:

(a) As in the case of a concentrated load off-center, a 20% saving of material may be made using plastic design when the beam is loaded uniformly. Deflections at working load are about one-half those of the simple beam.

(b) The same section is chosen in designing the beam for the first two loading conditions (concentrated load off-center and uniformly distributed load).
(c) The uniformly loaded beam has the same reserve capacity above the load at initial yield as that with load off-center.

(d) Considering (b) and (c) above, a portion of this condition is due to the fact that a beam is not available for every section modulus.

(e) For the central concentrated load, the elastic and plastic solutions give identical results.

(f) In each of these plastic designs, the beam is completely elastic at the working loads. It is believed that most other similar structures designed with the same load factor would also be elastic.

The considerations thus far suggest an approximate method for computing deflections which is outlined in Appendix C. Plastic zones are neglected. When compared with computations based on the simple plastic theory, the error is on the safe side, the discrepancy becoming large when the shape factor is large.

Hechtman and Johnston(6) have introduced the semi-rigid connection in building frames in a paper which would bring an economy of materials in building designs. The relief of end moments by semi-rigid connections in a fixed-end beam has at the same time increased the deformation of the member just as in case (c) where the hinge acts as a semi-rigid connection.
When floor beams carry ceilings with plaster, the maximum deflection of the beam under working load is limited by the A.I.S.C. Specifications to $\frac{L}{330}$ where $L$ is the span length. From the above tables, this requirement is satisfied in all of the designs.

However, there are many structures in which such a limitation is probably not necessary since plaster is not used. In this respect, the authors are in agreement with Prof. J. F. Baker who has often expressed the need for suitable deflection limitations for such other structures.

Theoretically, it is possible to design elastically an indeterminate structure to behave like a determinate one; that is, after the yield stress is reached the deformation increases rapidly. The following two illustrations demonstrate this idea.

Neglecting the shearing force, suppose the section modulus of a beam is varied to be in proportion to its moment diagram. Then every section of the beam will reach the yield stress at the same time. Any further increase of load would cause a rather large deformation that can hardly be disregarded in general engineering structures. This design however is usually impractical because of fabrication difficulties, complication of design procedures and the fact that structures so designed do not satisfy other loading systems.

As a second example the beam chosen in Fig. 12 has one end elastically supported and the other end fixed-ended as before. The elastic support length $c$ is arbitrarily chosen as $5'$. 
To find $M_y = 1690$ in.kips by the usual methods and
assuring $C_y = 73$ kips/in$^2$
then,

$$C = \frac{1690}{33} \cdot \frac{E}{I} \cdot 2 \text{ in}^3$$

This design is closer to the plastic design (c) than design (b)
in which both ends were fixed-hinged. The moment diagram of the above
design is:

![Moment Diagram](image)

**Fig. 12**

The maximum moment still occurs at A. But the moments at B and C
are higher than in the case where both ends were fixed.

The ideal design is to make all three sections of the beam (A, B, and C)
reach the yield moment at the same time. It is practically
impossible in this beam. In ordinary design the change of section
modulus along one member is usually avoided and the end supports
can hardly be designed to suit one loading system. Indeterminate
structures cannot be designed practically to behave like determinate
structures, i.e., the possible plastic hinges cannot be designed to
develop at the same time. Plastic design is therefore aimed to make use
of the reserve strength of an indeterminate structure between the initial
yield load and the load at which all possible plastic hinges develop.
IV. PLASTIC DESIGN BASED ON A LIMITATION OF DEFLECTION

On seeking a practical means of plastic design by limiting the deformation of structures the example of Fig. 1 may be examined. The final design, case c, has a good limitation of maximum deflection in the plastic range, namely, the load $P_2$. In such comparatively simple indeterminate structures it is possible to design by investigating the whole sequence of elastic and plastic behavior of the structure until enough plastic hinges develop to make the indeterminate structure act as a determinate one as shown in Fig. 8. Then we design this "plastic determinate" structure on the basis of the initial yield strength of the last hinge.

The difficulty of using this criterion for limiting the deflection would arise from the complication of the design method when the redundancy of the structure increases. It would be necessary to go into a detailed investigation of the elastic-plastic behavior of the structure under gradually increasing loads. This involves a long and laborious procedure in highly redundant structures.

Neglecting the problem of deflection for a moment, the ultimate strength of an indeterminate structure can be obtained by a direct method, discussed in Greenberg's recent paper.[?] For the previous example the procedure is as follows:

![Diagram](image)

Fig. 14
Put plastic hinges at A, B, and C, and apply a virtual displacement at point B. The virtual work done by force P is:

\[ W = P \times \delta \]

Neglecting all work done due to elastic deformation, the work done by the plastic hinges equals the angles of rotation multiplied by the corresponding moment.

\[
(M_p)_A = \frac{\delta}{a} + (M_p)_B \times \frac{\delta}{a} + (M_p)_B \times \frac{\delta}{b} \\
+ (M_p)_C \times \frac{\delta}{b} = W = P \delta
\]

\[ M_p = \frac{PL}{y} = 1400 \text{ in. kips.} \]

Assume \( y = 33 \text{ kips/in.}^2 \)

\[ M_p = y \cdot Z \]

\[ Z = 421.5 \text{ in}^3 \]

Where \( Z \) is the static moment of the section about the neutral axis.

Assuming \( f = 1.15 \)

Then \( S = 37.6 \text{ in.}^3 \)
The value of section modulus is even less than design (c) (which has $S = 37.5$). The design is more economical (although the same section, 14WF30, would be required)*. But upon examination of the load-deflection curve (Fig. 15) we find that the deflection corresponding to the load $P_3$ is infinite.

![Diagram of load-deflection curve](image)

**Fig. 15**

It is the load $P_2$ that is recommended as the "full load" of the structure and is selected to limit the deformation. But

*Timoshenko* has formulated a method of computing ultimate strength, based on an analysis of the bending moment diagram. This was used in Appendix A. However, the advantage of the Greenberg method lies in its *general application* to more complicated frames.
only $P_3$ can be found by the direct method of virtual work.

Suppose we divide the load $P_3$ by the shape factor of the member and call the new load $P_4$:

$$P_4 = \frac{P_3}{f}$$

The same result would be obtained if we replaced all the $M_p$ values in the structure by $M_y$. By the principle of virtual work we have

$$\delta P_3 = \alpha A M_p + \alpha B M_p + \alpha C M_p$$

or

$$P_3 = \frac{\alpha A M_p}{\delta} + \frac{\alpha B M_p}{\delta} + \frac{\alpha C M_p}{\delta}$$

and

$$P_4 = \frac{\alpha A M_y}{\delta} + \frac{\alpha B M_y}{\delta} + \frac{\alpha C M_y}{\delta}$$

Where the $\alpha$'s are the angular displacements at each point.

The moment distributions at loads $P_2$ and $P_1$ are shown in Fig. 16.

Moment Diagram Corresponding to $P_2$  
Moment Diagram Corresponding to $P_1$
Using the principle of virtual work and noting that $M_B$ and $M_C$ are less than $M_y$ in Fig. 16b, we have

$$P_1 \leq \frac{\alpha A_y}{\delta} + \frac{\alpha B_M}{\delta} + \frac{\alpha C_M}{\delta}$$

$$P_2 \leq \frac{\alpha A_M}{\delta} + \frac{\alpha B_M}{\delta} + \frac{\alpha C_M}{\delta}$$

Since the geometric dimensions are fixed, the terms $\frac{\alpha}{\delta}$ are constant.

Comparing the above expressions with (4) the following result is obtained

$$P_1 \leq P_4 \leq P_2$$

The shape factor of an "I" section lies between 1.1 and 1.22. Therefore $P_4$ is usually very close to $P_2$.

Let us redesign the beam by this modified method.

Using the virtual work method, expression (3) becomes

$$(M_p) x \frac{\delta}{a} + (M_p) x \frac{\delta}{a} + (M_p) x \frac{\delta}{b} + (M_p) x \frac{\delta}{b} = P x f x \delta \quad (5)$$

$$M_p = \frac{P f L}{9} = M_y f$$

Therefore

$$M_y = \frac{M_p}{9} = 1400 \text{ kip.in.}$$

$$S = \frac{M_y}{\sigma_y} = \text{42.5 in.}^3$$
The section modulus selected in this design is only slightly higher than the one found in design (c) in which we assumed plastic hinges at sections A and B and the yield moment, $M_y$, at section C of the beam. The deflection of design (c) was less than that of a simply supported beam designed within the elastic limit, and since $S$ in this new design is slightly higher, its deformation will be further reduced.

Therefore load $P_4$ is suggested as the full load in case the redundancy of the structure is comparatively high. The load $P_4$ can be determined without a knowledge of the shape factor since the hinge value is reduced from $M_p$ to $M_y$. Thus expression (5) could be written in simpler terms

$$M_y \delta_a + M_y \delta_a + M_y \delta_b + M_y \delta_b = P \delta$$

(6)

Of course, the load $P_2$ may be assumed as the full load in the comparatively simple structures used as examples in this paper.

This design method will give the same result as an elastic design whenever an efficient elastic design can be achieved. One of the examples of Appendix A is an illustration of this, shown in Fig. 17.
By the suggested plastic design method, with plastic hinges at A, B, and C,

\[ P = \frac{M_y}{L} + 2 \frac{M_y}{L} + \frac{M_y}{L} = 3 \frac{M_y}{L} \]

\[ S = \frac{Fl}{6\delta_y} = 47.6 \text{ in.}^3 \]

The identical result is obtained in elastic design. The load deflection curve of this beam is shown in Fig. 18.

![Deflection Curve](image)

**Fig. 18**

The deflection curve of this beam has only one straight line part although it is an indeterminate structure. We have shown that the design load \( P_4 \) is between \( P_2 \) and \( P_1 \) in the previous example. In this case, however, \( P_2 \) and \( P_1 \) approach one point,

\[ P_2 = P_4 = P_1 \]
V - DISCUSSION

Full Load and Working Load:

A structure designed to take a certain load may fail before reaching that load due to unavoidable errors that may be introduced in the design. Herdesty (9) has discussed this from an allowable stress point of view. Marin (10) has also tabulated possible sources of error in design. Such discrepancies are:

(a) overrun in computed dead loads,
(b) future increase in live loads,
(c) loss of section due to corrosion
(d) approximations in stress analysis
(e) underrun in dimensions and physical properties
(f) inadequate design theory

(g) errors in distribution of load

(h) errors in fabrication and erection, and

(i) time effects.

Therefore, in actual design, engineers are accustomed to design on the basis of a working load determined by dividing the full load by a certain factor to cover these possible errors. The multiplying factor is called the "Factor of Safety" and its value may be determined by the magnitude of possible errors, estimated statistically.

It is obvious that the chance for all these possible errors to be present simultaneously is small. But the use of the "Factor of Safety" in design does not eliminate all possibility for the structure to reach the full load stresses and deformations. It has only reduced this probability to a certain extent.

Usually in checking the deflection of a structure, no matter what design method is used, it is customary to use the working load. However, the statistical nature of the factor of safety (or load factor) makes it desirable to keep the deflections within some reasonable limit at the full load, and the method recommended provides this limit.
1. In plastic design the criterion of strength of structures should be based upon a limit of deformation rather than upon the ultimate collapse load.

2. According to the criterion of deformation, indeterminate steel structures may have considerable reserve carrying capacity beyond the flexural yield point, whereas for determinate structures this reserve strength is very small.

3. The amount of reserve capacity depends on the structure and the loading.

4. A large portion of this reserve capacity may be used in design without the danger of excessive deformations by a direct method of plastic design assuming that the "hinge value" is $M_y$ instead of $M_p$ as given by the simple plastic theory.

5. For relatively simple structures, further improvement over (4) above is accomplished by using as the full load that at which the formation of the last hinge is just commencing (The full load = $P_2$ as given in the text).

6. The deflections under working loads are less than those of simply-supported beams designed elastically.

7. An approximate method for computing deflections is suggested in Appendix C.

8. The structure will invariably be loaded within the elastic limit in the working range, although this is dependent on the loading, restraint, and the load factor adopted.
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VIII - REFERENCES


Other Reports in the Lehigh series:


APPENDIX A

LOAD-DEFLECTION RELATIONS FOR A UNIFORMLY-LOADED BEAM AND A BEAM WITH CONCENTRATED LOAD

In addition to the example shown in the text, two other conditions are examined as shown in Fig. 19.

(a) Loading Conditions

(b) Moment Diagram

Fig. 19

BEAM WITH UNIFORM LOAD

In the design of the uniformly-loaded beam, comparison is made with the same three design conditions selected in the text. The load of \(2^k/\text{ft}\) is selected since it gives about the same maximum bending moment as case (a) of the previous example.
(a) Simple Beam designed elastically

\[ M = M_y = \frac{Wl}{8} \text{ kips} \]

\[ S = \frac{M_y}{82.0 \text{ in}^3} \]

Use

18WF47 (\( S = 82.3, I = 736.4 \))

\[ \delta_{\text{max}} = \frac{5WL^3}{384EI} = 1.65'' \]

(b) Fixed-ended beam, designed elastically

![Diagram](image)

\[ M = M_y = \frac{WL}{12} = 1800 \text{ in. kips} \]

\[ S = 54.5 \text{ in}^3 \]

Use

16WF35 (\( S = 56.3, I = 446.3 \))

\[ \delta_{\text{max}} = \frac{WL^3}{384EI} = 0.555'' \]
(c) Fixed-ended beam, designed plastically

Selecting the load $P_2$, as the full load as before (Fig.2) and using the moment diagram of Fig. 21a

$$\frac{WL}{8} = M_y + f M_y = (1+f)M_y$$
$$M_y = \frac{WL}{8} (1+f)$$

$$= 1260 \text{ in. kips.}$$
$$S = 38.2 \text{ in.}^3$$

Use

14WF30 ($S = 41.8, I = 289.6$)

Neglecting, again, the plastic zone in the beam, the deflections are computed by moment areas. (Assume $M_y = y S = 33 x 41.8 = 1380 \text{ in. kips}$)

$$\frac{EI}{\delta_{\text{max}}} = \int_{A}^{B} M_x dx = \frac{2}{3} \left( \frac{M_y + M_f L}{y^2} \right) \frac{5L}{8} - M_f \frac{L}{2} \frac{L}{2}$$
DEFLECTION AT WORKING LOADS

\( \delta(a) = \frac{1.65}{1.65} = 1.00" \)

\( \delta(b) = \frac{0.55}{1.65} = 0.33" \)

\( \delta(c) : \) Checking the moment when the load \( \frac{60}{1.65} = 35.4 \) kips, this value is less than \( M_v \), thus the beam is entirely elastic, and

\( \delta(c) = \frac{M L^3}{384 EI} = .50" \)

BEAM WITH CENTRAL CONCENTRATED LOAD

(a) Simple beam designed elastically (\( P = 35^k \))

\[ M_y = M = \frac{PL}{4} = 3150 \text{ "k} \]

\[ S = 95.5 \text{ in}^3 \]

USE 18 WF55 (\( S = 98.2, I = 889.9 \))

\( \delta_{\text{max}} = \frac{PL^3}{48EI} = 1.272" \)
APPENDIX A

(b) Fixed-ended beam (elastic design)

\[ M_y = \frac{PL}{6} = 157.5 \text{ kN} \]

\[ S = 47.8 \text{ in.}^3 \]

Use:

16WF36 \( S = 56.3 \ , \ I = 446.3 \)

\[ \delta_{\text{max}} = \frac{P L^3}{192 E I} = 0.636'' \]

(c) Fixed-ended beam (plastic design)

In this case \( P_2 = P_1 \) and the solution is identical with (b) above:

Use:

16WF36

\[ \delta_{\text{max}} = 0.636'' \]
APPENDIX B

LOAD-DEFLECTION RELATIONS FOR BEAMS
UNDER VARIOUS LOAD CONDITIONS

This appendix contains the information necessary for plotting the load-deflection curve of Fig. 11.

The example of Fig. 2 is repeated in Fig. 25:

This curve is computed as follows:

\[ P_1 = \frac{M_y L^2}{ab^2} \quad (a) \]

\( P_a \) is determined from the load and moment diagrams shown in Fig. 26, using superposition.
Again the short curved portions of the $\frac{EI}{EI'}$ diagram are neglected. The changes at C due to the separate loadings (d) (e) and (f) are obtained by the conjugate beam method.

\[
\begin{align*}
\text{EI} \alpha_d &= \frac{M_0}{6} (2a+b) \\
\text{EI} \alpha_e &= \frac{M_0}{3} \\
\text{EI} \alpha_f &= \frac{M_0}{6}
\end{align*}
\]

But

\[\alpha_d + \alpha_e + \alpha_f = 0\]

and

\[M_p = \frac{f}{y}\]

\[\therefore M_c = \frac{M_0}{2L} (2a+b) - \frac{M_y f}{2}\]

Adding moments under the load in (d) (e) and (f),

\[M_o - \frac{bf}{L} M_y - \frac{8M_c}{L} = M_y\]

From the above two equations,

\[M_c = M_y \frac{(L+bf)(2a+b)-fL^2}{2L^2-a(2a+b)}\]

Now,

\[M_0 = P_a \frac{ab}{L}\]

Substituting in the second of the two simultaneous equations,

\[P_a = M_y \frac{L}{ab} \left(1 + \frac{bf}{L} + \frac{8}{L} \left(\frac{(L+bf)(2a+b)-fL^2}{2L^2-a(2a+b)}\right)\right) \quad \text{(b)}\]
$P_2$ is determined from Eq. (2),

\[ P_2 = \frac{M_Y}{b} \left( \frac{2fL}{a} + (1-f) \right). \tag{c} \]

$P_3$, the collapse load, is computed from the moment diagram at collapse:

\[ P_3 \frac{ab}{L} = 2M_p = 2fM_y \]

\[ P_3 = \frac{L}{ab}(2fM_y). \tag{a} \]

Next, the slopes of the various straight-line portions are determined making use of tables such as those in the AISC manual.

For the length OA in Fig. 25,

\[ \text{Slope} = \left. \frac{P}{\delta} \right|_{(OA)} = \frac{3EI(3b^2a^2)}{2b^3a^2}. \tag{e} \]

For the length AB,

\[ \text{Slope} = \left. \frac{P}{\delta} \right|_{(AB)} = \frac{3EI(3L^2-a^2)^2}{a(L^2-a^2)^3}. \tag{f} \]

and for the length BC,

\[ \text{Slope} = \left. \frac{P}{\delta} \right|_{(BC)} = \frac{3EI}{b^3}. \]
It is only necessary to know the slopes with reference to the initial elastic portion. Thus, from (e) and (f),

\[
\frac{E}{\delta}(AB) = \frac{2a b^3 (3L^2 - a^2)^2}{(L^2 - a^2)^2 (3b + a)^2}
\]

and from (e) and (g),

\[
\frac{E}{\delta}(BC) = \frac{2b^2}{(3b + a)^2}
\]

Substituting in expressions (a) through (d) and (h) and (i) the values \(L = 30\), \(a = 10\), \(b = 20\), \(k_y = 1380\) "k", and \(f = 1.15\), we obtain,

\[
\begin{align*}
P_1 &= 25.9 \text{ kips} = 0.65 P_3 \\
P_a &= 35.0 \text{ kips} = 0.88 P_3 \\
P_2 &= 38.8 \text{ kips} = 1.07 P_3 \\
P_3 &= 39.8 \text{ kips} = 1.00 P_3
\end{align*}
\]

\[
\begin{align*}
\frac{E}{\delta}(AB) &= 0.431 \\
\frac{E}{\delta}(BC) &= 0.04
\end{align*}
\]

The non-dimensional values of \(P\) are plotted in Fig 11.
The load-deflection curve for the uniformly loaded beam is computed in the same fashion as the previous example and is sketched in Fig 27.

![Diagram](image)

\[ W_1 = \frac{12 M_2}{L} = 46 \text{ kips} = 3.65 W_3 \]

\[ W_2 = \frac{3 M_2 (1+f)}{L} = 66 \text{ kips} = 3.93 W_3 \]

\[ W_3 = \frac{16 f M_2}{L} = 76.5 \text{ kips} = 1.00 W_3 \]

The two slopes are determined to be

\[ \frac{W}{\delta(0A)} = \frac{384 E I}{L^3} \]

\[ \frac{W}{\delta(AB)} = \frac{384 E I}{5 L^3} = 0.2 \frac{W_j}{\delta_j(0A)} \cdot \]

Comparing with equation (f), the slope of the previous example,

\[ \frac{W}{\delta(0A)} = \frac{252 a^2 b^3}{L^3 (3b+a)^2} = 1.52 \]
The load - deflection relation for the beam with central concentrated load is sketched in Fig. 28.

\[ M_y = C \cdot S = 1860 \text{ in.kips.} \]

\[ P_1 = \frac{8M_y}{L} = 41.3 \text{ kips} = 0.37 P_3 \]

\[ P_2 = P_1 \]

\[ P_3 = P_1 + 47.5 \text{ kips} = 1.00 P_3 \]

The slope of the elastic straight-line portion is

\[ \frac{E}{\delta}(a) = \frac{192 EI}{L^3} \]

Thus, comparing with the first example, taking into account the two different I-values,

\[ \frac{E}{\delta}(III) = \frac{128 I_1 \mu a^2 b^3}{L^3 I_1 (3b+a)^2} = 1.2 \]

This information is plotted in Fig 11.

In each case the short curved portions of the load-deflection plots have been sketched as an approximation.
APPENDIX C

APPROXIMATE METHODS FOR COMPUTING DEFLECTIONS

As an example we shall use the beam shown in Fig. 1. The procedure is indicated in Fig. 29.

Instead of attempting to deal with the curved portions, the straight-line portions are used as shown dotted. The dot-dash relation is the result of assumptions made in previous computations when we neglected the dotted portion of the $M/EI$ diagram (Fig. 9b).

The steps are as follows:

(1) Compute the loads $P_1$, $P_a$, and $P_2$, making use of bending moment diagrams and the formulas given in the AISC beam tables.
From Appendix B expressions (a) through (c)

\[ P_1 = \frac{M_o L^2}{ab^2} \]

\[ P_a = \frac{M_o L}{ab} \left( 1 + \frac{bf}{L} + \frac{a}{L} \left( \frac{(L+bf)(2a+b)-fL^2}{2L^2 - a(2a+b)} \right) \right) \]

\[ P_2 = \frac{M_o}{b} \left( \frac{2fl}{a} + (1-f) \right) \]

(1) Solve the deflection at load \( P_1 \):

\[ \delta_A = \frac{2Pb^3a^2}{3EI(3b+a)^2} \]

(2) Determine the slopes AB and BC from tables.

\[ \frac{P}{\delta} \mid (AB) = \frac{3EI(3L^2-a^2)^2}{a(L^2-a^2)^3} \]

\[ \frac{P}{\delta} \mid (BC) = \frac{2a^2}{(3b+a)^2} \]

(4) The deflections at points B and C may then be determined.

\[ \delta_B = \delta_A + (P_a-P_1) \times \frac{a(L^2-a^2)^3}{3EI(3L^2-a^2)^2} \]  \( (j) \)

\[ \delta_C = \delta_B + (P_2-P_a) \times \frac{2a^2}{(3b+a)^2} \]

Usually the working load will be below \( P_1 \). However, should it be greater, say in the range AB, then expression \( (j) \) may be used, replacing \( P_a \) by the working load.
Although it appears from Fig. 29 that a better approximation might be made by assuming $M_0 = \frac{M_u + M_p}{2}$, such a recommendation is not made at the present time.

However, it appears from the discussion that beam tables could be expanded to include the above formulae, should the concepts of plastic design and analysis become common among structural engineers.
APPENDIX D

NOMENCLATURE

a, b  Segments of length
E  Modulus of Elasticity
f  Shape factor
I  Moment of inertia of a beam section about its neutral axis
L  Length of span between beam supports
M  Moment at a beam section
M_y  Yield moment at a beam section
M_p  Plastic hinge moment at a beam section
P  Applied load
P_w  Working load on beams
S  Section modulus
Z  Static moment of area of a beam section about its neutral axis (also known as the Plastic Modulus)
α  Angle change
δ  Deflection
σ_y  Yield point stress