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Design Economy by Connection Restraint

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Robert A. Hechman
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Research Fellow

Contents In Brief—Heretofore continuity in building framing has been taken advantage of infrequently because of a lack of information on the restraint values of beam-column connections. Laboratory work has now provided much of this information and a rational and workable design procedure has been developed. Beams are first designed for maximum moment assuming simple supports. Then the ratio of beam stiffness to the sum of column stiffnesses at the joint is calculated. Charts equating this ratio to the percentage rigidity give a reduction factor \( F \), which is applied to the section modulus of the simple beam. The beam corresponding to this reduced section modulus is the one to use. It will be 15 to 20 per cent lighter than if assumed to be simply supported.

The design of the beams in multi-storied steel building frames has usually been based on the simplifying assumption that the ends of the beam are freely supported. While this assumption leads to a safe design, economy is sacrificed since no account is taken of the reduction in maximum positive moment that results from the end restraint that is present even in the most flexible connections. It is notable that slight increases in the stiffness of standard types of end connections provide enough end restraint to reduce the average weight of beams in a building frame by 15 to 20 per cent.

Unfortunately the application to building frames of methods of analyzing continuous structures is exceedingly laborious, and furthermore, there has been considerable uncertainty as to just how dependable and to what degree a semi-rigid connection provides end restraint. Now, however, experimental evaluation of the behavior of various types of beam-column connections has furnished much of the information necessary for the design of rigid and semi-rigidly connected frames, and this article presents such a design method which may be applied to any building frame in which the minimum restraint values of the connections have been determined.

Background of the method

The possibilities of economy are greatest when there is a repetition of similar span lengths, load conditions and connection types. To demonstrate the possible economy, a study of 105 beam sizes for various uniform loads and degrees of restraint has been made by the writers, showing substantial saving in weight for handbook selection of beams, even for cases of small percentage rigidity. The beam sizes ranged from 12-in. 22-lb. sections to 21 in. 63-lb. sections, the spans from 16 to 24 ft., and the loads from 80 to 120 lb. per sq ft. of floor. Fig. 1 shows the average minimum savings for various percentage rigidities. The beams were designed by the method outlined in this article.

Correlated to the problem of beam design are the problems of connection design, column design, and analysis for wind stresses. Experimental work on connections and columns are now in progress at Lehigh with the specific problem of building design in mind. Experimental work on moment-resisting riveted and welded connections that has furnished much of the necessary information for this design method are listed in the bibliography at the end of this article.

The semi-rigid joint

Before discussing the design of beams, it is necessary to have a definition of the term, semi-rigid joint. If the beam-column connections of a building frame transmit bending moment without relative rotation between the end of the beam and the column, the connection and the structure are termed "rigid" (Fig. 2a). In such a case, the connections afford 100 per cent restraint or full continuity, and the maximum bending moments are at the ends of the beam. If the connections transmit bending moment with some relative rotation between the end of the beam and the column, the connections and the structure are termed "semi-rigid" (Fig. 2b). In such a structure, the connections resist bending moment to some degree less than in the case of full continuity, and the moment in the center of the span is always less than if the connection afforded no restraint, as in a simply supported beam (Fig. 2c).

The semi-rigid joint, such as the standard beam web connection, the top and seat angle connection, and the split-l connection thus results in a restraint somewhere between full fixity and full freedom of rotation. It is important to note that 100 per cent restraint does not afford the greatest possible economy in building construction, largely because the cost of making the rigid connection tends to overbalance the saving in beam cost. Maximum economy for beam and connections usually occurs...
at a degree of restraint somewhere between 40 and 75 per cent.

The joint constant

A typical graph of the test of a semi-rigid connection is shown in Fig. 3 in which applied connection moment is plotted against relative-column-beam end rotation. The connection passes through three stages: first, an initial stage where moment is approximately proportional to rotation; second, a yielding of the connection; and third, a stage of accelerated rotation finally resulting in failure or very excessive deformation.

The first stage is the useful design range of the connection. It is especially important that the connection also have a sufficient factor of safety with respect to rotation. The maximum rotation which a semi-rigid connection approaches is the simple-beam end slope, and this occurs well within the rotation at failure for all semi-rigid connections except a few having very high rigidities. In these few cases, the working moment must be based on the ultimate moment.

The experimental determination of one factor is necessary as a basis for the design method. This is the connection constant, $J$, which may be defined as:

$$J = \frac{N}{E\phi} \quad (1)$$

where $\phi$ is the rotation due to an applied moment $M$ and $E$ is Young's modulus. Physically, the joint constant is the slope of the first stage of the moment-rotation curve divided by the modulus of elasticity of the material. It is a measure of the connection stiffness. A connection, therefore, whose joint constant $J$ is large is more rigid, or has more moment-taking ability within its working capacity, than one whose joint constant is small.

The percentage rigidity, $p$, depends on the connection constant $J$ and the stiffness of the beam $K$, which is the gross moment of inertia of the cross section divided by the span length.

$$p = \frac{100}{1 + 2KJ} \quad (2)$$

The percentage rigidity, then, is fixed when the connection and the beam size are chosen.

Fig. 4 is plotted from Eq. 2 and shows the relation between the joint constant and the percentage rigidity for various values of beam stiffness in the case of a beam fastened to rigid walls by semi-rigid joints. Most building beams have a stiffness of 0.5 to 5.0. It may be seen that in the design range of $p$ less than 70 per cent, a considerable variation in the connection constant $J$ has little influence on the percentage rigidity $p$. For instance, in the case of a beam of stiffness $K = 1.5$ and $J = 10$, the reduction of the connection stiffness $J$ by 100 per cent to 5 would change the span-end moment less than 20 per cent.

It follows that while differences in welding or riveting processes may affect the value of $J$ considerably there will be relatively much less variation in the actual beam moment. It also follows that a range of permissible variation in connection behavior, as shown in Fig. 3, should be allowed for any typical connection to take care of variations in fabricating as well as non-uniform relationship between moment and relative angle change.

Proposed design method

The proposed method of design is one that proportions the connection for the semi-fixed end moment which would occur if the columns did not rotate, and proportions the beam for maximum center moment which occurs when the columns do rotate.

In order to develop a direct method of design the beams in spans adjacent to that under consideration will be neglected. The approximation is on the side of safety and also allows the method to be applied to outside panels which have no adjacent beams. Fig. 5a shows the most critical load condition for maximum moment in beam AB, and Fig. 5b shows the same beam with adjacent beams...
omitted. Symmetrical conditions of load, connections, and adjacent columns are assumed to exist. Connections of 50 per cent rigidity are assumed in the following derivation.

The ordinary relation between the moment at the end A of a beam, AB, and the angle changes at its two ends is;

\[ M_{AB} = 2E(K(2\theta_A + \theta_B)) \pm M_J \]  \hspace{1cm} (3)

where \( M_J \) is the fixed end moment in a fully rigid connection.

When semi-rigid connections providing 50 per cent rigidity are introduced the equation becomes:

\[ M_{AB} = EK(1.25\theta_A + 0.25\theta_B) \pm M_J \]  \hspace{1cm} (4)

Due to symmetry, \( \theta_A = -\theta_B = -\theta_C \), hence:

\[ M_{AB} = EK\theta_A - \frac{M_J}{2} \]  \hspace{1cm} (5)

The moments acting on the joint must be in static equilibrium, hence:

\[ M_{AB} + 2M_K = 0 \]  \hspace{1cm} (6)

Substituting (4) and (5) into (6) there results:

\[ \theta_A = \frac{M_J}{2} \left( \frac{1}{4EKc + 2EKa} \right) \]

\[ = \frac{M_J}{2} \left( \frac{1}{2EKc + 8Ka} \right) \]  \hspace{1cm} (7)

Subscripts C and B in Eq. 7 refer to columns and beam, respectively. Substituting (7) into Eq. (5), there results:

\[ M_{AB} = -M_J \left( \frac{1}{2 + \frac{K_a}{K_c}} \right) \]  \hspace{1cm} (8)

The moment at the center of the beam is given by:

\[ M_C = M_J + M_{AB} \]

For rigidities equal to or less than 75 per cent the center moment is maximum and will govern the design of the beam. In the design procedure the beam is first designed as a simple beam, freely supported. The required simple beam section modulus is then multiplied by a reduction factor \( F \) which gives the section modulus required for the worst condition of loading but which takes advantage of the semi-rigid connections.

Section modulus required by proposed method
\[ F = \frac{M_C}{M_J} = \frac{1 - M_J}{M_J} \left( \frac{1}{2 + \frac{K_a}{K_c}} \right) \]  \hspace{1cm} (10)

The reduction factor \( F \), then, is the factor by which the simple-beam section modulus is multiplied to obtain the required section modulus for the beam. The ratio \( M_J/M_C \) depends on the type of load.

Eq. (10) was evaluated for end connections providing 50 per cent rigidity; similar equations have been derived for other rigidities. In Figs. 6, 7, and 8, charts are shown which give the reduction factor for various types of loads and percentage rigidities.

The design procedure

The following design procedure is based upon the assumption that data are available which give the dependable end restraint value, or "percentage rigidity", of any standard connection. Such values have already been evaluated for a limited number of connection types. Tests now in progress at the Fritz Laboratory sponsored by the American Institute of Steel Construction will supplement previous work on riveted connections by J. Charles Rathbun in this country and by J. F. Baker and others in England. The combined results of these tests should establish dependable criteria for riveted connections. In the welding field, highly rigid connections have been tested, but in the semi-rigid class only the seat and top angle type has been studied in detail. Further work is needed on various types of welded connections.

It is important to note that, in spite of the present lack of established standards, the application of this design procedure may be made to any particular building design through the expedient of actually testing typical proposed connections to be used in the structure.

The actual details of the design procedure may be outlined as follows:
1. Design the beams for maximum bending moment assuming simple supports.
2. Calculate \( K_a = \frac{P_a}{E_b} \) for the beam and \( \Sigma K_c = \Sigma E_c \) for the columns above and below one end of the beam.
3. Determine the ratio of \( \frac{K_a}{\Sigma K_c} \).
decide on the percentage rigidity to use in design, and determine from Figs. 6, 7, or 8 the reduction factor \( F \) for the existing load condition.

4. Multiply the section modulus required for simple-beam design by the reduction factor \( F \), and redesign the beam on the basis of the reduced modulus.

5. Calculate the semi-rigid end moment for the condition of all beams loaded by multiplying the fixed end moment by the per cent rigidity assumed.

6. Select a connection on the basis of end reaction, semi-rigid end moment, and percentage rigidity assumed.

In step 2 the stiffness of the beam, \( K_B \), is based on the simple beam design. \( K_B \) could be based upon the reduced \( I_B \) of the final design, but since this is not known the approximation provides a direct design procedure and is on the side of safety. However, if a particular beam size is repeated under identical loading conditions a great number of times a further economy would be introduced by estimating \( K_B \) as 80 to 85 per cent of \( K_B \) for simple beam moment and verifying the estimate after the beam is designed on the basis of reduced moment. One trial design would be the most that might be required.

If the column sizes are not the same at each end the design may be based on the more flexible end with the approximation again on the side of safety. If the loading condition is moderately unsymmetrical, the end moments may be approximated at each end by Eq. 8, and the approximate bending moment diagram for the beam constructed. The severe loading condition assumed in Fig. 5 and the extreme improbability of its occurrence renders meaningless small errors of a few per cent which might be introduced by applying Eq. 8 to unsymmetrical conditions.

In step 3 the decision regarding what per cent rigidity to use in design may be made arbitrarily, but after a little practice its selection will be based on questions of feasibility, economy, and preference for a particular connection type. The final design of the connection in step 6 ultimately may be made simply by reference to standardized connection tables which give safe values of shear, moment, and percentage rigidity. At present the selection must be based on existing experimental data available in the publications listed in items 2, 3, 4 and 7 of the accompanying bibliography.

The design procedure may be illustrated as far as beam selection is concerned by an example. It is desired to select a beam, having 50 per cent rigid connections for a uniform load. For a simple beam, a 16-in. 40-lb. WF beam would be required with \( I = 515.5 \text{ in.}^4 \).

\[
K_B = \frac{515.5}{240} = 2.15 \text{ in.}^4
\]

\[
\frac{K_B}{K_C} = \frac{2.15}{4.55} = 0.47
\]

From Fig. 6, for \( n = 50 \) per cent, \( F = 0.73 \) Required \( W = (0.73)(60 \text{ in.}) = 43.8 \text{ in.}^4 \) Use a 15-in. 33-lb. I-beam

Saving = 40 - 33 = 7 lb. or 17.5 per cent

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FLEXIBLE WELDED ANGLE CONNECTIONS

By BRUCE JOHNSTON and LLOYD F. GREEN

INTRODUCTION

This report presents the results of a series of tests made on welded beam-to-beam or beam-to-column connections as used in standard tier building construction. The connections are similar to some of those proposed as tentative standards in December 1938, by the American Institute of Steel Construction. It was desired that the connections be flexible enough to allow with safety the full end rotation which might be expected in a freely supported simple beam.

Connections of the types shown in Fig. 1 (a), 1 (b) and 1 (c) were tested. The direct pull tests shown in Fig. 1 (a) were to determine the relative flexibility of different lengths of angle legs. On the basis of these direct pull tests the flexible top and seat angle connections shown in Fig. 1 (b) were designed for the simple beam end rotation of typical beam designs. The purpose of the top angle is simply to support the compression flange against twisting at the ends of the beam. Tests had previously been made at the Fritz Laboratory on seat and top angle connections similar to those in Fig. 1 (b). In these previous tests the top angles were much thicker than in the present series and the connections were designed to be "semi-rigid," or moment resisting. There is currently much interest in the possibilities of the economical design made possible by the use of semi-rigid connections; nevertheless, most beams in buildings at present are designed with the assumption of simple supports and in such cases it is essential that the welded connections have the desired degree of flexibility to give full simple beam end rotation.

The present investigation was carried out at the Fritz Engineering Laboratory of Lehigh University, in cooperation with the Welding Research Committee of the American Welding Society. In October 1939, the Committee authorized this work and appropriated a sum of $200 to cover the cost of fabrication of specimens. The investigation was a regular research project of the Fritz Engineering Laboratory, of which Professor Hale Sutherland, Head of the Department of Civil Engineering, is Director. Acknowledgment is made to Mr. Howard J. Godfrey, Engineer of Tests, and to all others on the laboratory staff for their continued assistance in carrying out the program. Helpful suggestions regarding the program were made by Mr. Heath Lawson, Mr. La Motte Grover, Mr. F. H. Dill and others.

TEST PROGRAM AND PROCEDURE

The test program consisted of three groups of tests. Group I consisted of direct pull tests varying the angle leg size to obtain the relative flexibility of different leg lengths; Group II consisted of direct pull tests varying the weld on the outstanding leg and also subjecting the weld to repeated load; and Group III consisted of full-size connection tests designed on the basis of the results of Groups I and II, and subjecting the specimen to repeated loads. The direct pull tests in Groups I and II simulated the action of the top angle and the upper end of the web angle, permitting a selection of the most desirable angle size and welding procedure at a minimum of expense. Subjecting the angle to a direct pull tested it more rigorously than in the case of an actual top and seat, or web angle connection. Details of the size, type and method of fabrication of the test specimens are as follows:

Group I—The direct pull specimens (Fig. 1 (a)) were held during welding so that the welds on one pair of angles were all done with the legs in a vertical position and the bead laid horizontally, simulating the top angle leg welded to the column. The welds on the other pair of angles on the same specimen were all done with the legs in a horizontal position and the bead laid horizontally similar to a top angle leg welded to a beam. The specimen was jigged very carefully so that the two main pull plates were in a straight line. This group consisted of five specimens made up of equal leg angles 1/4 inch thick and 4 inches long, the variable being the length of the leg. Table 1 presents the details of the specimens as well as test results.

Group II—These specimens, consisting of Tests No. 6 to 14 inclusive, were fabricated similarly to those of Group I. Five 4 by 4 by 1/4-inch angle specimens and four 3/4 by 21/2 by 1/4-inch angle specimens were tested. The type of the weld was varied in this group as illustrated in Table 2.

*To be presented at the Annual Meeting, A. W. S., Cleveland, Ohio, Oct. 21 to 23, 1940. Contributions to the Structural Steel Committee.
1 Assistant Director, Fritz Engineering Laboratory, Lehigh University, Bethlehem, Pennsylvania.
2 Garrett Liederman Hoppes Research Fellow in Civil Engineering.

Reprinted from THE WELDING JOURNAL, October 1940
Table I - Direct Pull Tests

<table>
<thead>
<tr>
<th>Test No.</th>
<th>Angle Size</th>
<th>Welding Position</th>
<th>Yield Point Load</th>
<th>Yield Point Defl.</th>
<th>First Crack Load</th>
<th>Max. Defl.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4 x 4 x 1(\frac{1}{2}) x 4</td>
<td>Leg Vertical</td>
<td>4100</td>
<td>0.140&quot;</td>
<td>5800</td>
<td>1.60&quot;</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Leg Horizontal</td>
<td>3700</td>
<td>0.112&quot;</td>
<td>4450</td>
<td>1.30&quot;</td>
</tr>
<tr>
<td>2</td>
<td>3(\frac{1}{2}) x 3(\frac{1}{2}) x 4 x 4</td>
<td>Leg Vertical</td>
<td>4100</td>
<td>0.086&quot;</td>
<td>5500</td>
<td>1.12&quot;</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Leg Horizontal</td>
<td>4150</td>
<td>0.092&quot;</td>
<td>5600</td>
<td>1.05&quot;</td>
</tr>
<tr>
<td>3</td>
<td>3 x 3 x 2(\frac{1}{2}) x 4</td>
<td>Leg Vertical</td>
<td>3750</td>
<td>0.036&quot;</td>
<td>6550</td>
<td>1.02&quot;</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Leg Horizontal</td>
<td>3500</td>
<td>0.034&quot;</td>
<td>5900</td>
<td>1.19&quot;</td>
</tr>
<tr>
<td>4</td>
<td>2(\frac{1}{2}) x 2(\frac{1}{2}) x 2 x 4</td>
<td>Leg Vertical</td>
<td>4575</td>
<td>0.022&quot;</td>
<td>8500</td>
<td>0.83&quot;</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Leg Horizontal</td>
<td>5350</td>
<td>0.040&quot;</td>
<td>7800</td>
<td>0.80&quot;</td>
</tr>
<tr>
<td>5</td>
<td>2 x 2 x 1(\frac{1}{2}) x 4</td>
<td>Leg Vertical</td>
<td>5450</td>
<td>0.016&quot;</td>
<td>9150</td>
<td>0.59&quot;</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Leg Horizontal</td>
<td>4900</td>
<td>0.013&quot;</td>
<td>8200</td>
<td>0.62&quot;</td>
</tr>
</tbody>
</table>

Group III—This group of tests was made on six full size connections, of which three were top and seat angle connections, and three were web angle connections (Table 3). All these connections were fabricated at the Fritz Laboratory, using stub beam ends connected to the web of a 12 WF 65 stub column, as shown in Fig. 1 (b) and 1 (c). The connections were designed for the end reaction rotation corresponding to beam designs for three span lengths. A top and seat angle connection and a web angle connection was designed for each span length. A uniform D. L. + L. L. of 115 lb. per sq. ft. was assumed and calculations were based on the assumption that the beams would carry the entire load of a square floor panel having sides equal to span lengths of 20 ft., 22 ft. and 24 ft. The following beam sizes resulted: 16 WF 36, 18 WF 55, and 21 WF 59, respectively. All end shear was assumed as taken up by the seat angle in the top and seat angle connection. The seat angles were of the minimum size necessary to withstand the end reaction and were designed and welded in accordance with usual practice. In the case of the web angle connections the outstanding leg welds were designed to take the combined shear and bending stresses as in standard practice. It should be noted that the connections were tested with respect to rotation and did not carry end reactions corresponding to the actual design.

All welding was done at the Fritz Laboratory by a qualified welder, using a Grade 10 Electrode. The welds in every case were 1/8-inch fillet, having the same size as the angle thickness. The angles were of stock size and were cut on a power saw to lengths of 4 in. or 1\(\frac{1}{16}\) in. for the direct pull specimens, and 6 in. or 1\(\frac{1}{16}\) in. for the full-size connection specimens. The flexible angle material conformed to A. S. T. M. Standard Specifications A9-36. The longitudinal edges of all connecting angles were welded in the as-rolled condition.

The gages for the direct pull tests in Groups I and II were mounted as shown in Fig. 2. Movement of the heel of the angle from the plate was measured with Ames Dials accurate to 0.001-inch. Beyond the gage range (1.0-inch) the deflection was measured with a steel scale graduated to 0.01 inch. A gage was placed on each end of each pair of angles and the movement of the heel was recorded and averaged.

The relative rotation between the beam end and the column in the Group III tests was measured by means of rotation bars attached to the members as in Fig. 3, which also shows the method of loading. The specimens were inverted and concentrated loads were applied to the beams to produce rotation of the beam ends. The rotations were measured by a 20-in. level bar of the same type used in a previous investigation. The level bar was sensitive to a rotation of \(\frac{1}{20,000}\) of a radian or to \(\pm 10\) seconds, and consisted of a 10-second precision level bubble mounted on an aluminum bar. Two sharpened steel points supported the bar at one end, and the other end was supported by a micrometer screw which was used to bring the bar to level position for each reading. The elevation of the micrometer end of the bar was read by a \(\frac{1}{1000}\) Ames Dial. The relative

Fig. 2—Direct Pull Specimen, Test No. 4, After Yield
dial movement divided by the gage length gave the relative rotation in radians between any two load conditions. For each measurement the level bar rested in an identical location upon the polished surface of the rotation bars which were attached by arms to the beam and column. Ames Dials accurate to a movement of 0.001-inch were mounted on the upper and lower flange of the beam, bearing against the column, thereby enabling the location of the center of rotation.

TEST RESULTS

Group I—Movement of the heel of the angle from the plate was measured at 200-lb. increments until the relation between load and deflection deviated markedly from a straight line. From then on this movement was recorded periodically in order to get the maximum deflection before actual fracture. At every 1000-lb. increment up to just beyond the approximate yield strength the load was dropped to the initial load, and record of the permanent set was obtained. A typical load deflection curve is shown in Fig. 4.

From the load deflection curve an arbitrary yield strength was established. This point was found in all cases by drawing a tangent to the straightest part of the curve in the low load range and a horizontal through the maximum load. The point where a vertical through the point of intersection of the first two lines crossed the curve was considered the yield strength. The relation between the yield strength and the length of outstanding leg is plotted in Fig. 5.

Flexibility was more important than strength and a preliminary study of Group I indicated that the 4 x 4 x 1/4-inch angles would be suitable for a top angle and the 2 1/4 x 3 1/8 x 1/4-inch angles suitable for web angle connections. These sizes were chosen on the basis of 0.10 and 0.08-inch heel deflections, respectively. The limiting heel deflections were below the general yield strength and below any noticeable local yielding or cracking in weld or angle. After making the actual beam connection tests in Group III the preliminary estimates were revised somewhat and limitations as to span length and beam depth are furnished in the summary.

<table>
<thead>
<tr>
<th>TABLE II</th>
<th>DIRECT PULL TESTS UNDER REPEATED LOAD</th>
</tr>
</thead>
<tbody>
<tr>
<td>-----------</td>
<td>--------</td>
</tr>
<tr>
<td>6</td>
<td>4 x 4 x 4</td>
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<tr>
<td>7</td>
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<tr>
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<td>3 x 4 x 4</td>
</tr>
<tr>
<td>14</td>
<td>4 x 4 x 4</td>
</tr>
</tbody>
</table>

Fig. 3—Beam Connection Test Arrangement

Fig. 4—Load Deflection Diagram Test No. 1, in Direct Pull
Table III: Beam Connection Tests

<table>
<thead>
<tr>
<th>Test No.</th>
<th>Beam</th>
<th>Moment at First Repetition</th>
<th>Moment at Second Repetition</th>
<th>Moment at Third Repetition</th>
<th>No. Cycles</th>
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</thead>
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<tr>
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<td>41,600</td>
<td>62,000</td>
<td>72,700</td>
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<tr>
<td>2</td>
<td>West Beam</td>
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<td>66,000</td>
<td>80,540</td>
<td>100</td>
</tr>
<tr>
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<td>East Beam</td>
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<td>92,000</td>
<td>119,500</td>
<td>200</td>
</tr>
<tr>
<td>4</td>
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<td>76,000</td>
<td>102,000</td>
<td>129,000</td>
<td>400</td>
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</tbody>
</table>

Group II—As a result of the tests in Group I two angle sizes were selected for further direct pull tests in which variations in the weld details were tried out. The test procedure also included repetitions of load at 0.8, 1.0, and 1.2 times the desired total deflection corresponding to simple beam end rotation. Two sizes of angles were used in this group, 4 by 4 by 1/4-inch angles to correspond to the top and seat angle connection, and 3 1/2 by 2 1/2 by 1/4-inch angles, with the 3 1/2-inch leg outstanding, to correspond to the beam web connection.

Nine tests were made in all, and the results are tabulated in Table 2. In the first three tests, No. 6 to 8, the repeated load varied from the initial load at the zero increment to the loads necessary to give the previously mentioned total deflections. In these tests the perceptible creep during load repetitions was observed and the welds eventually fractured at 90, 215, and 115 repetitions, respectively. In every case, however, the deflection during the final load repetition was considerably more than necessary for the corresponding simple beam end rotation.

In tests Nos. 9 to 14, inclusive, different lengths of weld return, around the end of outstanding angle leg, were tried out. The repeated load range in these tests varied between the applied load and one-quarter of this amount, on the basis that some dead load is always acting.

Two of these six tests failed as a result of load repetition. In the case of the other four tests, after having at least fifty repetitions at a deflection of twenty per cent or more in excess of the required, the connections withstanded increased static load until total deflections of as high as 1.98 inch were reached at final failure.

A study of these test results indicated that a return of the weld around the ends of the angle toe equal in length to one-quarter of the length of the outstanding leg seemed to produce beneficial results. Tests 10 and 11, (Table 2), in which the 4-inch and 3 1/2-inch angle each have a return of 1 inch around the toe of the outstanding leg, show a very low creep at fifty repetitions of load producing deflections of 38 per cent and 15 per cent, respectively, in excess of the desired deflections. Upon the application of further steady load, initial yielding was found to be induced principally in the angle itself rather than in the root of the toe weld. The total deflection at final fracture was the greatest for this type of direct pull connection. Figures 6 and 7 show the condition at final fracture of Test 10 and Figs. 8 and 9 show corresponding pictures of Test 11.

Group III—Figure 1 shows the general weld details which were chosen for designing the beam connections in this group and the exact details of the six different tests are shown in Table 3. Rotations were measured at successive increments of load to the limit of the level bar's range. The distance between the heel of the angle and the column was measured thereafter with a steel scale. The load was repeated between one-quarter and full load fifty times each at 0.8 simple beam rotation, full simple beam rotation, and 1.2 simple beam rotation. A typical moment-rotation curve is shown in Fig. 11, which shows the behavior which might
have been predicted on the basis of the corresponding direct pull test No. 10.

The results of these tests confirmed the choice of weld details which had been made. At none of the three stages of load repetition was there appreciable creep. After completing the repeated load tests, each of the six connections took rotations far beyond full simple beam rotation to such an extent that it was not practicable in the testing machine to produce complete failure. The welds along the toe of the outstanding leg were not fractured in any of these tests, although partial tearing of the short weld returns took place. Figure 12 shows exposed views of Tests 15, 16 and 17, respectively, after stopping the test. Similar results were obtained in the case of the web angle connections, and these tests were stopped after the lower beam flanges came to bearing on the column web. Exposed views of these three tests are shown in Fig. 13.

**SUMMARY AND CONCLUSIONS**

The use of a short "weld return" on the cut edges of the connection angles as shown in Fig. 10 has been indicated to be beneficial in some respects. It does not follow, however, that connection angles without the weld return would be unsatisfactory. The tests in Group I may be used as a basis for the selection of angles without weld returns as it has been shown that the direct pull tests give a reasonably close prediction of the beam connection behavior. Arguments pro and con with respect to the use of a weld return may be summarized as follows:

*In favor of the "weld return" or "boxing."

1. Initial failure of the material is forced into the angle rather than in the throat of the fillet weld.
2. High concentration of stress along the entire root of the top weld of a top angle is avoided.
3. Comparatively little additional weld metal is used.

*In favor of omitting the "weld return."

1. An over-zealous welder could defeat the whole purpose of the design by running too far down the ends of the angles with the weld return.
2. Additional welding is introduced, increasing the cost.
3. The increased ultimate deflection of this type of connection is unimportant because the beam end could never rotate a fraction of the amount within the working range.

The limited number of repeated loads which were applied to the direct pull specimens and to the beam connections yield information regarding the capacity of the connections to take a limited number of overloads, but should not be construed to represent structural fatigue tests. The following conclusions are intended to apply to cases in which structural fatigue is not a problem.

A. *Direct Pull Tests.*—(1) The greatest strength and largest ultimate deflection prior to fracture was produced in the 3 1/4 and 4-inch outstanding legs when a return weld 1 in. long was carried around the toe of the angle on each side.

(2) Initial yielding and final failure of the connections with no weld return was in the throat of the weld.

(3) Weld returns as described in A-1 relieved the stress concentration in the root of the major portion of

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Fig. 9—Direct Pull Test No. 11, After Final Failure
If the design is governed by a maximum deflection limitation of $L/360$, the maximum depth beam which should be used is 12 inches when the seat and top angle connection is used and 16 inches when the web angle connection is used.

If the design is governed by maximum fiber stress of 20,000 lb. per sq. in. under uniform load, the maximum span length for top and seat angle connections is 19 feet and for beam and web angle connections 25 feet. For beams designed at a unit stress other than 20,000 these limits shall be modified by the ratio of 20,000/$f_{u}$.

In the top and seat angle type of connection the seat angle should be designed to take the full end reaction of the beam. The weld in the web angle connection the fillet weld and caused most of the initial yielding to occur in the angle material. Final failure was by tearing of the weld returns, followed by fracture through the throat of the weld.

The tests were not extensive enough to permit any definite conclusion regarding resistance of the connections to repeated overload.

B. Beam Connection Tests.—(1) The details shown in Fig. 10 are considered suitable for flexible welded angle connections for steel building construction. The top and seat angle type using a 4 by 4 by $\frac{1}{4}$ by 6-inch top angle, and the beam web connection using $3\frac{1}{4}$ by $2\frac{1}{4}$ by $\frac{1}{4}$ by $\frac{3}{4}$ beam depth) will give satisfactory flexible connections within certain limitations of beam depth and span length. These limitations are based on limiting the angles with 4-in. outstanding legs to 0.10-in. heel deflection and the $3\frac{1}{4}$-inch legs to 0.08-inch heel deflection. The limitations may be summarized as follows:

(a) If the design is governed by a maximum deflection limitation of $L/360$, the maximum depth beam which should be used is 12 inches when the seat and top angle connection is used and 16 inches when the web angle connection is used.

(b) If the design is governed by maximum fiber stress of 20,000 lb. per sq. in. under uniform load, the maximum span length for top and seat angle connections is 19 feet and for beam and web angle connections 25 feet. For beams designed at a unit stress other than 20,000 these limits shall be modified by the ratio of 20,000/$f_{u}$.
nection should be designed to take the stress due to combined shear and moment.

3. The centers of connection rotation for the top and seat angle type were near the upper edge of the seat angle. The centers of rotation for the web angle connections were about three-quarters of the beam depth down from the upper edge of the beam.

4. No substantial progressive increase in deflection was observed in the full size connections with a repeated load of one-quarter to maximum, repeating the load fifty times at eighty per cent of the design rotation, at design rotation, and at twenty per cent above design rotation.

5. After completing the cycle of fifty-load repetitions at twenty per cent above design rotation, each connection continued to take increasing moment until rotations of more than three times the full simple beam rotation were reached. None of the connections had completely failed at this maximum rotation.

6. The beam end moments which would be developed through the use of this type of connection would be less than ten per cent of the full fixed end moment in the range considered.

REFERENCES

INTRODUCTION

THE purpose of this investigation is to increase the efficiency of girders having intermediate stiffeners. It has long been known that vertical stiffeners except those at load points do not carry any load and serve only to prevent the web from buckling. By placing the intermediate stiffeners diagonally across each panel, a trussed girder is formed, the stiffeners thereby carrying a portion of the load in addition to performing in an efficient manner their given tasks of preventing buckling. This paper will show that a definite increase in efficiency of steel is obtained, i.e., that the amount of load carried per pound of girder is materially higher than for the conventionally stiffened girders. It will also show how to design this new type girder.

Interest in inclined stiffeners for plate girders was evidenced in the late part of the nineteenth century. In 1895, Johnson, Bryan and Turneaure included a chapter on plate girder design in their book, Modern Framed Structures, in which they advocated inclined stiffeners on the girders because of resultant economy. When this work was criticized on account of the claims for economy, 1 Johnson effectively refuted this criticism. The discussion of Johnson's work in the Engineering News Record brought to light two girders that had been designed and built by Henry Goldmark, 2 Bridge Engineer of the Kansas City, Fort Scott and Memphis Railroad, for use on his railroad. These girders, of fifty and one hundred foot spans, included inclined stiffeners in the first few panels from the supports where the shear was greatest.

In Ireland, at the University of Dublin, in 1904, W. E. Lilly prepared a text on plate girder design, 3 a section being devoted to the design of plate girders with Inclined stiffeners. The contention was that a more economical design would be effected with inclined stiffeners than without them, when the girder span was greater than seventy or eighty feet.

An argument in favor of inclined stiffeners was indirectly advanced by Professor Turneaure of Wisconsin in the Journal of the Western Society of Engineers in 1907. 4 Turneaure investigated the stresses in vertical stiffeners on an actual girder in use. He found that, except at load and reaction points, the vertical stiffeners carried no load and actually served only to prevent web buckling.

In the aircraft industry, the inclined stiffener has been accepted as a means of reducing weight without a corresponding decrease in strength. H. Wagner, in 1929, 5 presented a method for use in airplane design. He estimated that for an angle of inclination of thirty degrees with the vertical, in the direction of the compression stresses in the web, the stiffness of the girder is increased fifty-five per cent with a decrease in weight.

The work at Lehigh was begun on cardboard models and then progressed to steel girders similar to the ones tested in the present investigation. The results of these tests 6 were so favorable that the present extensive investigation was undertaken.

INVESTIGATION

Variables.—In designing a series of plate girders for testing, many variables had to be faced. Depth and thickness of the girder were decided by taking a web for all cases of \( \frac{1}{4} \) inch thickness since it is the thinnest plate that can be arc welded easily, and selecting depths to give depth to thickness ratios of 102, 170 and 220. These ratios resulted in web depths of \( 10^{\frac{1}{4}}, 21^{\frac{1}{4}}, \) and \( 27^{\frac{1}{4}} \) inches, respectively for Series I, II and III. From studying the results of other researches 7 it was decided that spans six times the depth, at least, were needed to get true beam action.

The length of stiffening material was another variable. In Series I (Fig. 1) a constant length of stiffening material was used for all girders. 8 In the other two series each had two girders in which the length of stiffening material was constant—the vertically stiffened girder and the girder having inclined stiffeners inclined at fifty-five degrees to the vertical. The remaining girders of Series II and III had variable lengths of stiffening material dependent on the spacing and arrangement of the stiffeners.

The method of loading was the final variable. As in the previous investigation, 9 a concentrated load at the center line was used. The advantage of using a concentrated load at mid-span lay in the fact that the shear was constant over the entire span and served as a rigorous test of the girders.

OUTLINE OF PROGRAM

As mentioned previously the testing program was divided into three series having \( h/t \) (depth/thickness) ratios of the webs 102, 170 and 220 for Series I, II and III, respectively. The spans and other details for the individual specimens varied somewhat and are given in Fig. 2. The flanges for Series I consisted of two plates welded together—a 6 by \( \frac{1}{4} \) inch plate and a 7 by \( \frac{1}{4} \) inch plate—and represented an overdesign from the usual procedure of about seventy per cent. For Series II and III, the flanges were 10-inch, 25-lb. channels which represented a similar overdesign.

Series I.—There were five girders of this series. Two girders (Nos. 4 and 5) had only vertical stiffeners. Two girders (Nos. 1 and 2) had only inclined stiffeners (inclined 63° and 45°, respectively). One girder (No. 3) had both vertical and inclined stiffeners forming a trussed girder (Fig. 1). To estimate the effect of the web on the
girder properties, a small truss-type girder with no web was also built and tested.

For all of the Series I girders, the length of stiffening material was held constant at 220 inches. It consisted of 3 x 1/4-inch steel plate and the angles of the inclined stiffeners were chosen to give the fixed total of stiffener length. This amount of stiffening material provided four pairs of inclined stiffeners for the sixty-three degree girder (No. 1), six pairs for the forty-five degree girder (No. 2), nine pairs of vertical stiffeners for the conventional girders (Nos. 4 and 5), and finally four pairs of inclined stiffeners and two pairs of vertical stiffeners (at the one-quarter points) for the fifty-five degree girder (No. 3).

Series II.—As first planned this series consisted of only two girders, with a span for each of 10 ft. 6 in. Specimen 7 had thirteen pairs of vertical stiffeners, while specimen 6 had five pairs of vertical stiffeners dividing the web into four panels with a pair of diagonal stiffeners inclined at an angle of fifty-five degrees to the vertical in each panel. For these two the length of stiffening material was held constant at 530 inches.

Due to the superiority of specimen 1-3, having a combination of vertical and inclined stiffeners, over the girders having inclined stiffeners only it was decided to proceed with the former type of design exclusively. As a consequence in three of the remaining girders in Series II (10, 11 and 13), a combination of vertical and inclined stiffeners was used with varying degrees of inclination making the panel length vary. The remaining girder of Series II, No. 12 (as shown in Fig. 1), had stiffeners at thirty degrees' inclination and no vertical stiffeners. The results of this test confirmed the observation made in the Series I tests that a better arrangement of stiffeners results from a truss formation where the stiffeners are alternately vertical and inclined.

Series III.—The final series having a web ratio of 220 should be definitely classed as deep and slender-webbed. The two girders of this series were made with the same total length of stiffening material. Specimen 8 had vertical and inclined stiffeners while specimen 9 had vertical stiffeners only.

Trusses.—A truss was prepared for each of these series. The Series I truss (Fig. 1b) was similar in design to the fifty-five degree girder with two exceptions: it had no web, and it did have center line and end verticals. The details of the truss are given in Fig. 2. The designed ultimate load was 100,000 lb. As for the Series I truss, the Series II and III trusses were similar in design to the corresponding fifty-five degree girder. The designed ultimate load for each was 150,000 lb.

TEST PROCEDURE

A test for similarity was made on most of the girders where fabrication procedure did not interfere. In a preliminary stage of construction, after the flanges had been welded to the web, each girder was given a load-deflection test as a check on the uniformity of the girders. In the first series this test was made when the girder was entirely unstiffened, while in the other series three pairs of vertical stiffeners were added before this test was made. The results showed that there was a close agreement between the girders, indicating that any difference...
in the load-carrying ability of the completed girder could be ascribed to the positioning of the stiffeners.

After each girder was tested for deflection under low loads to ascertain similarity, the fabrication was completed and the girders tested to failure. All tests were carried on in an Olson, four-screw, 300,000-lb. testing machine. The load was applied to the girder at mid-span through a spherical bearing block, a roller and a plate. The reactions were taken by knife edges. Some of the details can be seen in Fig. 1.

Gage lines for a ten-inch Whittemore strain gage were put on every girder at strategic points such as on the top and bottom flanges at mid-span, on all vertical and inclined stiffeners for half a girder. In the case of the vertically stiffened girders, it was felt that sufficient data would be obtained from gage lines on end, center line and quarter-point stiffeners.

Beam deflections were obtained by Ames dials set at mid-span and at points as near the end reactions as practical (about 4 inches). All deflections were read from the under side of the bottom flange, and as a result the load-deflection curves in this paper give the net deflection for the lower flanges. In some cases, due to local crippling, the upper flanges deflected much more than indicated by the load-deflection curves.

The load was applied in increments up to failure. For the smaller girders, the load increment was 8000 lb., but for the larger girders it was taken as 16,000 lb. to a load of about 100,000 lb. and then continued up in 8000-lb. increments. All tests started with an initial load of 1000 lb.

The vertically stiffened girders were subjected to a special series of tests. As a preliminary to the final test to failure, tests were made with but a portion of the vertical stiffeners in place, the purpose being to study the efficacy of vertical stiffeners in preventing web buckling. The first special test was made with vertical stiffeners at reaction points and at mid-span, and when web buckling seemed considerable, as indicated by Ames dials, the test was stopped. Then more stiffeners were welded in, reducing the panel length to one-half or one-third its original length, and another test was made. Finally all the stiffeners were attached and the girder destroyed. In each case, buckling of the web and an increase in the rate of change of deflection were taken as signs of imminent failure.

**RESULTS OF TESTS**

The results of the investigation are given in Table 1. Note that initial web buckling appeared in a variety of places. It was as likely to be found in the end panel as in the middle panels. Note, also, that except for girders I-1 and II-12 (a type of girder not found to be best) all stiffeners either did not buckle or else buckled after high loads were reached. It was noted by the writers that the compressive unit stresses in all cases where stiffeners buckled were in the yield-point range when buckling of the web also took place.

The comparison of the various girders as given by the load-deflection curves in Figs. 3, 5 and 9 shows that the first objective of this research, namely, increasing the load-carrying ability of a girder by inclining alternate stiffeners, was attained. It may also be noted, in studying Fig. 11 that when the work of fitting and welding of the stiffeners in place is considered, it is no harder to put in the inclined stiffeners than to put in the vertical ones.

**Comparative Girder Loads.**—In Table 2 the comparison of load per pound of girder is shown for those girders of each series that had the same spans. In order not to make falsely high claims for the girders with inclined stiffeners the comparison was based on balanced designs, that is, for those girders which failed under relatively low loads, thereby causing low flange stress at ultimate load, the required flange areas were computed that would have given yield-point stresses as the ultimate girder loads were reached. Also, in the vertically stiffened girders, the size of the stiffeners was adjusted. These adjustments in the areas and weights of the girders supposedly would have no influence on the ultimate loads obtained in the tests. In this manner a balanced design was made for each girder and a fairer comparison made.

As mentioned previously, the combination of vertical and inclined stiffeners as typified by specimen 3, Series
I., showed such superior qualities to the girders of Series 1 that had only inclined stiffeners (Nos. 1 and 2), that the former type was adopted as standard. The idea of working up a design method for the inclined stiffeners alone was abandoned. It is noted, however, that specimens 1 and 2, although less strong than 3, showed definitely higher strength than the vertically stiffened girders, 4 and 5.

**Web Buckling.**—Observations of the girder loads causing buckling of the webs were made for three purposes: The primary purpose was to secure data for determining the critical buckling stresses in girders where each panel is divided in two by a diagonally placed pair of stiffeners. This is covered in the appendix. The second purpose was to check up on the Timoshenko formulas for critical buckling stresses of rectangular plates when subjected to shear or flexure or to a combination of the two. The third purpose was to justify the amount of stiffener material used in the vertically stiffened girders (1-4 and 1-5, II-7 and III-9).

In the general case of a plate girder, according to Timoshenko, the web is divided into panels by vertically placed stiffeners, and three conditions of stress are liable to be critical and to cause the web plate to buckle:

1. At the supports, where the shearing force is large, and the plate may be considered as being subjected to uniform shear.
2. At mid-span of the girder, the bending stresses are high, and the plate may be considered as being subjected to bending stresses.
3. At some intermediate point, shear stress and bending assume equal importance and so it becomes necessary to consider them both acting on the plate.

The girders in this research were rather unique in that they were subjected to uniform shear over the entire span and therefore, in the middle panels the third condition obtained, while in the end panels the first condition, or shear only, obtained.

Several points of difference between theory and actuality must be remembered at this point because they undoubtedly explain discrepancies in the results. In the first place, the flanges for all vertically stiffened girders were overdesigned, which would add a bit of strength to the girder that it might not otherwise have. Second, the conditions at the edges of a theoretical simply supported plate are markedly different from the actual edge conditions in a panel of the girder. The edges of the theoretical plate are considered unrestrained while actually the edge conditions in the girder are somewhat different. The edges in contact with the stiffeners would probably be partially restrained, but at the

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**Table 1: Results of Tests**

<table>
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<tr>
<th>GIRDER</th>
<th>INITIAL STRAIN LINES</th>
<th>INITIAL WEB BUCKLING*</th>
<th>INITIAL STIFFENER BUCKLING</th>
<th>ULTIMATE LOAD²</th>
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<tbody>
<tr>
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<tr>
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**Table 2: Efficiency of Girders**

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<tr>
<th>Girder</th>
<th>Actual Load, Lb.</th>
<th>Adjusted Load, Lb.</th>
<th>Ultimate Load, Lb.</th>
<th>Comparative Efficiency, Per Cent</th>
</tr>
</thead>
<tbody>
<tr>
<td>I-1</td>
<td>287</td>
<td>260</td>
<td>89,300</td>
<td>103.1*</td>
</tr>
<tr>
<td>I-2</td>
<td>287</td>
<td>257</td>
<td>89,000</td>
<td>104.4*</td>
</tr>
<tr>
<td>I-3</td>
<td>287</td>
<td>296</td>
<td>115,000</td>
<td>114.9*</td>
</tr>
<tr>
<td>I-4</td>
<td>287</td>
<td>188</td>
<td>56,700</td>
<td>101.3*</td>
</tr>
<tr>
<td>I-5</td>
<td>287</td>
<td>188</td>
<td>61,900</td>
<td>98.7*</td>
</tr>
<tr>
<td>II-6</td>
<td>805</td>
<td>747</td>
<td>148,000</td>
<td>107.6</td>
</tr>
<tr>
<td>II-7</td>
<td>805</td>
<td>483</td>
<td>82,000</td>
<td>100.0</td>
</tr>
<tr>
<td>III-8</td>
<td>1157</td>
<td>1310</td>
<td>206,000</td>
<td>113.8</td>
</tr>
<tr>
<td>III-9</td>
<td>1081</td>
<td>620</td>
<td>86,000</td>
<td>100.0</td>
</tr>
</tbody>
</table>

* as noted by eye
+ Comparison with average of I-4 and I-5 which had vertical stiffeners only.
† Comparison with II-7, which had vertical stiffeners only.
‡ Comparison with III-9, which had vertical stiffeners only.
flanges the conditions more nearly approach fixed edges. Some of the shear was carried by the flanges, but this was counterbalanced by the fact that the thickness of the web plates ran slightly less than 1/8 inch.

For all cases, the critical stress in the plate may be represented by the formula:

\[ S_a = \frac{kE^2}{12h^2(1-v^2)} \]

where \( S_a \) = critical buckling stress
\( k \) = constant dependent on ratio of width to height of plate and is given approximately for rectangular plates by the formula: \( k = 5.35 + 4h'/b' \). For triangular-shaped web plates the values have been worked out by Professor Reynolds in the appendix.

\( E \) = modulus of elasticity (30,000,000 psi for steel)

NOTE: For girders having only vertical stiffeners, if \( b < h \), substitute \( b \) for \( h \) in formula for \( S_a \) and in formula for \( k \) interchange \( b \) and \( h \).

Considering shear as the only force acting on the end panels of the girders with vertical stiffeners only, the critical stresses have been computed and are shown in Table 3. These values should agree with the average shear-stress values in the last column except for the starred values where failure should be by shear rather than by buckling.

It is noted that there is poor agreement between the test results and theory. Niles and Newell8 and also Timoshenko4 in their books indicate that other tests likewise fail to agree well with theory. The supposition

<table>
<thead>
<tr>
<th>Girders</th>
<th>Panel Dimensions</th>
<th>Critical Stress ( S_a )</th>
<th>Observed Buckling Load, ( \frac{P}{2b} ) (K. s. i.)*</th>
</tr>
</thead>
<tbody>
<tr>
<td>I-5 (A)</td>
<td>Length, b = 18 in.</td>
<td>Thickness, ( t = \frac{1}{8} ) in.</td>
<td>k = 6.7</td>
</tr>
<tr>
<td>I-5 (B)</td>
<td>Length, b = 18 in.</td>
<td>Thickness, ( t = \frac{1}{8} ) in.</td>
<td>k = 7.3</td>
</tr>
<tr>
<td>I-5 (C)</td>
<td>Length, b = 9 in.</td>
<td>Thickness, ( t = \frac{1}{8} ) in.</td>
<td>k = 7.34</td>
</tr>
<tr>
<td>I-4 (A)</td>
<td>Length, b = 9 in.</td>
<td>Thickness, ( t = \frac{1}{8} ) in.</td>
<td>k = 7.34</td>
</tr>
<tr>
<td>I-4 (B)</td>
<td>Length, b = 63 in.</td>
<td>Thickness, ( t = \frac{1}{8} ) in.</td>
<td>k = 6.1</td>
</tr>
<tr>
<td>II-7 (A)</td>
<td>Length, b = 21 in.</td>
<td>Thickness, ( t = \frac{1}{8} ) in.</td>
<td>k = 6.1</td>
</tr>
<tr>
<td>II-7 (B)</td>
<td>Length, b = 21 in.</td>
<td>Thickness, ( t = \frac{1}{8} ) in.</td>
<td>k = 6.1</td>
</tr>
<tr>
<td>II-7 (C)</td>
<td>Length, b = 10.5 in.</td>
<td>Thickness, ( t = \frac{1}{8} ) in.</td>
<td>k = 6.6</td>
</tr>
<tr>
<td>III-9 (A)</td>
<td>Length, b = 82.5 in.</td>
<td>Thickness, ( t = \frac{1}{8} ) in.</td>
<td>k = 6.1</td>
</tr>
<tr>
<td>III-9 (B)</td>
<td>Length, b = 41.25 in.</td>
<td>Thickness, ( t = \frac{1}{8} ) in.</td>
<td>k = 6.1</td>
</tr>
<tr>
<td>III-9 (C)</td>
<td>Length, b = 13.75 in.</td>
<td>Thickness, ( t = \frac{1}{8} ) in.</td>
<td>k = 6.9</td>
</tr>
</tbody>
</table>

* K. s. i. = Kips per square inch. † Above yield point in shear, hence mythical value only

NOTE: For girders having only vertical stiffeners, if \( b < h \), substitute \( b \) for \( h \) in formula for \( S_a \) and in formula for \( k \) interchange \( b \) and \( h \).
is that the assumptions made are not valid. A similar study was made for the third condition where combined shear and flexural stress are found at mid-span, but with no better results. It was observed, in making the tests, that buckling was as likely to take place in the end panels as in the middle panels where, in theory, higher web stresses should obtain.

Effect of Stiffeners on Web Buckling.—It is a known fact, and is confirmed in our tests, that intermediate stiffeners do not effect much increase in buckling resistance of a web unless the spacing is made less than the web height. Note in Table 3 (next to last column) the small gains in web strength from I-5(A) to I-5(B) where the panel length in both tests exceeded the web height. Similar small gains are noted for II-7(A) and II-7(B) and for III-9(A) and III-9(B). On the other hand, where the panel length is reduced to less than the web height (compare the (C) test with the corresponding (B) test in each series) a marked gain in strength is obtained. Theory confirms the statements just made. In the formula for critical buckling stress in a plate subjected to shear,

$$S_c = \frac{k^2E^6}{12h^2(1-v^2)}$$

$h$ is assumed to be less than $a$. If the reverse is true, $b$ must be substituted for $h$ in the formula, and in determining the value of $k$ the formula should read:

$$k = 5.35 + 4h^2/h^2$$

Stiffener Stresses.—Formulas for apportioning the shear between the web and the inclined stiffeners and for computing the unit stress in the inclined stiffeners are developed in the appendix. They are as follows:

$$V_{web} = \frac{V}{1 + \frac{a \sin 2\alpha \cos \alpha}{0.8ht}}$$

$$V_s = \frac{V}{1 + \frac{0.8ht}{a \sin 2\alpha \cos \alpha}}$$

$$S_s = \frac{S_c \sin 2\alpha}{0.8} \text{ where } S_c = \frac{V_{web}}{ht}$$

The notation is given in the appendix.

Table 4—Comparison of Computed Stresses with Observed Stresses in Girders with Inclined Stiffeners at Design Loads (Ultimate/2)

<table>
<thead>
<tr>
<th>Girder</th>
<th>$\alpha$ Deg.</th>
<th>$\sin 2\alpha$</th>
<th>$\cos \alpha$</th>
<th>$1 + \frac{a \sin 2\alpha \cos \alpha}{0.8ht}$</th>
<th>Design $V^*$</th>
<th>$V_{web}^*$</th>
<th>$S_c^*$</th>
<th>$V_s^*$</th>
<th>$S_a^*$ from Tests</th>
</tr>
</thead>
<tbody>
<tr>
<td>I-3</td>
<td>55</td>
<td>0.940</td>
<td>0.574</td>
<td>0.612</td>
<td>30.15</td>
<td>18.45</td>
<td>12,100</td>
<td>14,200</td>
<td>15,000</td>
</tr>
<tr>
<td>I-1</td>
<td>63</td>
<td>0.809</td>
<td>0.454</td>
<td>0.690</td>
<td>22.15</td>
<td>15.46</td>
<td>9,700</td>
<td>11,800</td>
<td>12,000</td>
</tr>
<tr>
<td>I-2</td>
<td>45</td>
<td>1.000</td>
<td>0.707</td>
<td>0.546</td>
<td>22.25</td>
<td>12.28</td>
<td>7,700</td>
<td>9,630</td>
<td>8,900</td>
</tr>
<tr>
<td>I-12</td>
<td>30</td>
<td>0.866</td>
<td>0.866</td>
<td>0.867</td>
<td>20.75</td>
<td>15.70</td>
<td>5,910</td>
<td>6,400</td>
<td>6,000</td>
</tr>
</tbody>
</table>

$*$ A Half the Limit Load, $P = 2(V_{web} + V_n)$, from Table V.

$^+$ See appendix for derivation.

$^\dagger$ See appendix for derivation.
safe. In these tests it was found that the plate stiffeners with such a ratio were also reasonably safe; and, although they buckled in a few cases (specimens II-6, II-11) before the ultimate load of the girder was attained, it is noted that they had exceeded the yield-point stress before such buckling took place. In certain cases in Series I (specimens I and 3), where the ratio was twelve, buckling occurred too, but here it must be noted that the stiffeners were immediately under the concentrated center load. In the design of the stiffeners, the intermittent welding was so spaced that an I/r of the stiffeners between the welds of less than forty was maintained.

<table>
<thead>
<tr>
<th>Girder</th>
<th>h</th>
<th>b</th>
<th>h/b</th>
<th>k</th>
<th>h/t</th>
<th>$S_0$ (PsI)</th>
<th>$V_{el}$</th>
<th>$V_{el}$</th>
<th>P = 2($V_0$ + $V_e$)</th>
<th>Test P</th>
</tr>
</thead>
<tbody>
<tr>
<td>I-3</td>
<td>12.75</td>
<td>18.0</td>
<td>0.68</td>
<td>14.6</td>
<td>102</td>
<td>20,000*</td>
<td>20,000*</td>
<td>20,000*</td>
<td>120.6</td>
<td>119**</td>
</tr>
<tr>
<td>II-6</td>
<td>21.25</td>
<td>31.5</td>
<td>0.68</td>
<td>14.6</td>
<td>170</td>
<td>13,700</td>
<td>13,700</td>
<td>13,700</td>
<td>148.6</td>
<td>149**</td>
</tr>
<tr>
<td>II-10</td>
<td>&quot;</td>
<td>35.6</td>
<td>0.58</td>
<td>14.4</td>
<td>&quot;</td>
<td>13,500</td>
<td>13,500</td>
<td>13,500</td>
<td>161.6</td>
<td>160**</td>
</tr>
<tr>
<td>II-11</td>
<td>&quot;</td>
<td>21.2</td>
<td>1.00</td>
<td>19.3</td>
<td>&quot;</td>
<td>18,100</td>
<td>18,100</td>
<td>18,100</td>
<td>180.6</td>
<td>185**</td>
</tr>
<tr>
<td>II-13</td>
<td>&quot;</td>
<td>12.5</td>
<td>1.75</td>
<td>&quot;</td>
<td>&quot;</td>
<td>20,000*</td>
<td>20,000*</td>
<td>20,000*</td>
<td>277.8</td>
<td>265**</td>
</tr>
<tr>
<td>III-8</td>
<td>27.50</td>
<td>41.1</td>
<td>0.68</td>
<td>14.6</td>
<td>220</td>
<td>8,400</td>
<td>8,400</td>
<td>8,400</td>
<td>185.8</td>
<td>206</td>
</tr>
</tbody>
</table>

* Failure by shear, not buckling.
† $V_{el} = S_0 h t$.
‡ $V_{el} = 30,000 a \cos \alpha$.
$V_0 = 1000$.

** Local failure caused early ultimate. On retest with an additional stiffener to prevent local crippling, load reached 127 k.
*** Failing in all parts, i.e., web yielding in shear, web buckling, inclined stiffeners buckling, and flanges yielding.
IV Flanges exceeded yield-point stress. No web buckling.
IV Capacity of machine exceeded—test stopped.

In designing girders having a combination of inclined and vertical stiffeners as already indicated the method

** Flange Stresses. —From the load-stress curves, Figs. 5 to 9, it can be seen that there is good agreement between the computed flange stress and the actual. For the vertically stiffened girders, the flange stresses are low, checking the overdesign already mentioned. In the case of many of the girders with inclined stiffeners (I-3, II-6, II-10, II-11, etc.) the increased strength resulting from the inclination of the stiffeners produced stresses in the flanges above the yield point at ultimate loads.

** Deflections. —The load-deflection curves, Figs. 3 and 5 to 9, show decisively the increase in strength effected by inclining the stiffeners. Note that for specimens II-6 and II-13, Figs. 5 and 8, the load-deflection curves for several other specimens have been included so that a ready comparison may be made between the types of girders. A similar comparison is shown on the curve for III-8, Fig. 9. Since the span for specimen II-13 differed from those for I1-7 and II-12 an adjusted load-deflection curve was constructed on the assumption that the deflection varies as the span. It is noted that the trusses for all three series do not show as much stiffness as the corresponding girders.

** Trusses. —A few words about the trusses are in order. They were designed for ultimate loads of 100,000 lb. for the Series I truss and 150,000 lb. for the Series II and III trusses. The actual ultimate loads were as follows:

Series I truss—92,000 lb.
Series II truss—145,000 lb.
Series III truss—145,000 lb.

In the three trusses tested, there were no weld failures, but it should be observed that some of the welds assume a much greater importance for the truss than they do for the girder. The writers found it easier to fabricate the girders than the trusses.

An attempt was made to compare the fabrication costs of the trusses with the corresponding girders. Difficulty was encountered due to the small sizes of the welds required for the girders (1/4-in. fillet welds), and the results, favorable to the girders, are not to be trusted. For comparison purposes a design was made of a truss with a ninety-foot span and loaded in the same manner as the girders in Example 1 of the Design section. The resulting weight was 38.9 kips as compared with 49.3 kips for the best girder design. More welding was required on the girder. However, in spite of these handicaps girders will probably continue to be popular due to their ease of fabrication and to sundry other reasons.

** DESIGN METHOD**

In designing girders having a combination of inclined and vertical stiffeners as already indicated the method
Girder Design: Example 1

Required: To design one girder of a 90-ft. span through girder railroad bridge for type "E" engine loading. It was decided to have six panels at fifteen feet and to use the 1931 A. R. E. A. specifications. The following is a summary of the girder loads:

<table>
<thead>
<tr>
<th>Total Shears (Dead, Live and Impact)</th>
<th>Total Moments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel</td>
<td>Shear in lb.</td>
</tr>
<tr>
<td>End</td>
<td>368,200</td>
</tr>
<tr>
<td>2nd</td>
<td>241,850</td>
</tr>
<tr>
<td>3rd</td>
<td>135,100</td>
</tr>
</tbody>
</table>

Solution:

Step 1. Try height, \( h = 120 \) in. Panel length, \( b = 180 \) in.

\( k = 0.67 \)

\( k = k_0 + 0.2(k_4 - k_0) \)

Using Table A-11 from the appendix

\( k = 11.3 + 0.2(27.7 - 11.3) = 14.6 \)

Step 2. Multiply given shears by 2 to get limit shears.

Step 3. Try \( t = \frac{1}{4} \) in., then \( h/t = 214 \).

Step 4. Enter chart, Fig. 10, with \( h/t = 214 \), and \( k = 14.6 \) to read \( S_e = 8700 \) psi.

Step 5. Limit shear of web plate \( = V_{wt} = ht S_e = 120 \times \frac{1}{4} \times 8700 = 588,000 \) lb.

This limit shear is greater than the given limit shears for all panels except the first one.

This shows that 120 in. must be very close to the best height as regards efficiency. An earlier trial of \( h = 90 \) in. resulted in the following uneconomical proportions.

| Flanges | 38,320 lb. |
| Web     | 16,650 lb.  |
| Total   | 54,970 lb.  |

The 90-ft. girder was redesigned, for comparison purposes, with vertical stiffeners only for values of \( h = 90 \) in. and 117 in. (best height). The results are summarized in the table. It is noted that for the uneconomical height of 90 in. there is practically no advantage in using the inclined stiffeners, but at the "best heights" the saving in weight is over three tons or 11 per cent of the total weight.

Comparison of Girder Design—Problem 1

<table>
<thead>
<tr>
<th>Vertical Stiffeners Only</th>
<th>Load Limit Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inclined and Vertical</td>
<td></td>
</tr>
<tr>
<td>h = 90 in.</td>
<td>117 in.</td>
</tr>
<tr>
<td>1/4 in.</td>
<td>1/4 in.</td>
</tr>
<tr>
<td>Total Weight</td>
<td>57,600 lb.</td>
</tr>
<tr>
<td></td>
<td>55,400 lb.</td>
</tr>
<tr>
<td>1/4 in.</td>
<td>120 in.</td>
</tr>
<tr>
<td>Total Weight</td>
<td>62,400 lb.</td>
</tr>
</tbody>
</table>
Fig. 9

**GIRDER OF SERIES III**

**NO. 8**

$\theta = 55^\circ$

Fig. 10

**Curves for**

$$E = \frac{210}{1 + \nu},$$

where $\nu$ is dependent on $E$.

$E = 30 	imes 10^6$ psi,

$\nu = 0.3$

**CONCLUSIONS**

1. It has been shown in this investigation that girders having vertical stiffeners at the panel points and diagonally placed stiffeners, placed in the direction of the compressive stresses in the web, are more efficient than girders having vertical stiffeners only. It is acknowledged that the economy is particularly worth while for long girders, say, over eighty feet in span.

2. It is felt that fabrication costs of the new type girder, compared to the old method, are neither raised nor lowered.

3. The design method which has been presented has a good theoretical basis and can be applied with dispatch.

**ACKNOWLEDGMENT**

The writers are indebted to the Bethlehem Steel Company for the steel used in the investigation, to

As a result of these design studies it became evident if real economy is to be obtained that:

1. The designer must be allowed to find the most economical height of girder. If there is a limitation on height of girder little or no saving in weight of girder may be obtained.

2. The span or load must be so great that the height of web will exceed, say, six feet. Greatest economy is obtained when the web thickness of girders having the orthodox vertical stiffeners only is great due to the limitation of the $h/t$ ratio to 160 (A. I. S. C.) or 170 (A. R. E. A.).
In a similar manner it can be shown that
Professor Bruce Johnston, Howard Godfrey, Engineer
of Tests, and former Professor Inge Lyse of the
staff at Fritz Laboratory for criticism and help during the course
of the work, and to Mr. G. E. Claussen of the Welding
Research Committee for valuable library research.

**APPENDIX**

Proof that $V_{web} = \frac{V}{1 + \frac{a \sin 2\alpha \cos \alpha}{0.8h}}$ and $S_n = \frac{S_s \sin 2\alpha}{0.8h}$

Consider cantilever plate of dimensions $h$, $b$ and $t$, subjected to a load, $V$. The deflection due to shear is given by $\Delta = \frac{Vb}{hE_s}$ from which the diagonal length $AB$ becomes $AB_1$. The change in length, $AB - AB_1$, is, very closely,

$\Delta_{diag} = \Delta \cos \alpha = \frac{Vb \cos \alpha}{hE_s} = \frac{V \sin \alpha}{tE_s}$

If there is a pair of diagonal stiffeners, $AB$, in addition to the plate, the shortening of the stiffener will just equal the shortening $\Delta_{diag}$, and the $V$ will be divided into the portion, $V_{web}$, carried by the web and $V_s$, the portion carried by the stiffeners. In this case

$\Delta_{diag} = \frac{V_s b \cos \alpha}{hE_s}$

and for the stiffeners

$\Delta_{diag} = \frac{\text{total stress} \times \text{length} \times \cos \alpha}{E \sin \alpha}$

Now $V = V_s + V_{web} = \frac{\Delta_{diag} h E_s}{b \cos \alpha} + \frac{\Delta_{diag} a E \sin 2\alpha}{2b}$

If $E_s = 0.4E$, $V = \Delta_{diag} E \left[ \frac{0.4h l}{b \cos \alpha} + \frac{a \sin 2\alpha}{2} \right]$

Cancelling out $\Delta_{diag} E / b$,

$V = \frac{0.4h l / \cos \alpha}{a \sin 2\alpha \cos \alpha} = \frac{1}{1 + \frac{a \sin 2\alpha \cos \alpha}{0.8hl}}$

In a similar manner it can be shown that

$V_s = \frac{V}{1 + \frac{a \sin 2\alpha \cos \alpha}{0.8hl}}$

Now the expression, $\Delta_{diag} = \frac{V_s b \cos \alpha}{hE_s}$, may be re-expressed

$\Delta_{diag} = \frac{S_s b \cos \alpha}{E_s}$

where $S_s = V_s / h t$.

Also: $\Delta_{diag} = \frac{bS_s}{E \sin \alpha}$ for the inclined stiffener where

$S_s = \frac{V_s}{a \cos \alpha}$

$\frac{bS_s}{E \sin \alpha} = \frac{S_s b \cos \alpha}{E_s}$ from which $S_s = \frac{S_s \sin 2\alpha}{0.8h}$

Notation:

$V = \text{total shear in pounds.}$

$V_{web} = \text{total shear in pounds carried by the web plate.}$

$V_s = \text{total shear in pounds carried by the inclined}$

$\text{stiffeners.}$

$h = \text{panel height in inches.}$

$t = \text{panel thickness in inches.}$

$a = \text{sectional area of a pair of stiffeners in square}$

$\text{inches.}$

$\alpha = \text{angle of inclination of the inclined stiffeners}$

(see sketch).

**APPENDIX**

(By Prof. Reynolds)

It is the purpose of the present section of this paper to discuss theoretical buckling stresses for the triangular section of a web in the built-up girder bounded by a flange, a vertical stiffener and an inclined stiffener by assuming a shape for the buckling mid-section of the web section. The equations defining the surfaces for the different possible buckling shapes assumed will be such as meet more or less exactly the conditions that exist at the boundary of the triangular section. Provision is made for complete or partial fixity at all edges and for fixity at the flanges with complete or partial fixity along the stiffeners.

Critical stresses are derived upon the assumption that the simpler surfaces assumed are those meeting the requirement that the excess of the work of bending over that of the energy of bending is larger than would be true for any other surface of deformation. This procedure gives values not less than the critical stresses for buckling. The stresses thus derived, for different simple surfaces, are compared with each other and with observed buckling stresses.

Supposing $w$ to represent the deflection measured perpendicularly to any point $(x, y)$ from the unstressed position of the midsection of the web, let $w_x$, $w_{xx}$, $w_{yy}$, etc., represent, respectively, the partial derivative of $w$ with respect to $x$, the second partial derivative of $w$ with respect to $x$, the second partial derivative with

\[ w = \frac{V}{0.8hl} \left( 1 + \frac{a \sin 2\alpha \cos \alpha}{0.8hl} \right) \]

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respect to $x$ and $y$, etc. Then the condition for buckling will be given by

$$-v\int \int (Nw_x^2 + N_w w_x^2 + 2Nw_x w_y) \, dx \, dy = \frac{1}{2} \int \int (w_x + w_y) \, dx \, dy + \frac{1}{2} \left(1 - v\right) \int \int (w_x^2 + w_y^2) \, dx \, dy \tag{A-1}$$

in which $N_x$ and $N_y$ are the stresses parallel, respectively, to the $x$ and $y$ axes, and $N_{xx}$ is the shear per unit length in the web. The value of $v = 1/3$ (Poisson's ratio) is taken to be 0.3. $D = E\beta^2 / 12(1 - v^2)$ in which $E$ is Young's modulus and $t$ the thickness of the web. The value of $E$ used is 30,000,000 psi.

Experiments show that a close approximation to the state of stress in the web is that of constant shear. It is, therefore, first assumed that $N_{xx}$ is constant, $N_y = 0$ and $N_{yy} = 0$ in equation (A-1). This gives a value of $N_{xx}$ taken to be the critical value for the surface of deflection under consideration.

A study of the effect of moment in the web is made by assuming a straight line distribution of stress given by

$$N_y = \frac{1}{2} (2y - h)$$

and combining this with $N_{xx}$, a constant. There is thus obtained some idea of the effect of any existing moment upon the critical buckling stress in the web. In the expression for $N_{xx}$, $h$ is the height of the web and $S$ the stress in lb. per sq. in. at the juncture with the flange.

To simplify the notation we make the following substitutions in which the limits of integration for $y$ are 0 and $h/b$ and for $x$ are 0 and $b$:

$$I_1 = -\int \int N w_x^2 \, dx \, dy \quad I_2 = -\int \int w_x w_y \, dx \, dy \quad I_3 = \int \int w_x w_x \, dx \, dy \quad I_4 = \int \int w_y w_y \, dx \, dy$$

In all the cases considered $I_4 = I_0$ so that equation (A-1) reduces to

$$2IN_{xx} = D(I_1 + 2I_2 + I_4) \tag{A-2}$$

when buckling takes place under constant shear, and to

$$2IN_{xx} = D(I_1 + 2I_2 + I_4) - I_3 \tag{A-3}$$

when moment in the web is taken into account. In any case the critical stress, $S_{cr}$, is given by

$$S_{cr} = N_{xx}/k$$

The eight possible surfaces considered are

$$w_1 = c \sin \pi x/b \sin \pi y/h \sin \pi (x/b - y/h)$$
$$w_2 = c \left(\frac{\pi}{b}\right) y \sin \pi x/b$$
$$w_3 = c \left(\frac{\pi}{b}\right) y \sin \pi (x/b - y/h)$$
$$w_4 = c \sin^2 \pi x/b \sin \pi y/h \sin \pi (x/b - y/h)$$
$$w_5 = c \left(\frac{\pi}{b}\right) y^2 \sin \pi x/b$$
$$w_6 = c \left(\frac{\pi}{b}\right) y^2 \sin \pi (x/b - y/h)$$
$$w_7 = c \sin^2 \pi x/b \sin^2 \pi y/h \sin^2 \pi (x/b - y/h)$$

It will be seen that $w_1$, $w_3$, and $w_5$ provide for hinged edges on all boundaries and one bulge in the deflected surface. The deflections $w_2$, $w_4$, and $w_6$ provide for fixed edges along the flange ($y = 0$) and hinged edges at the other boundaries. Provision is made for fixed edges all around in $w_7$. These deflections also, have one bulge in the deflected surface. The other deflection, $w_1$, is inserted

\[ \text{See "Theory of Elastic Stability," by S. Timoshenko, p. 323.} \]

For comparison when there are two bulges in the deflected surface and hinged edges, we can put the critical stress formula (A-2) in the form

$$S_{cr} = \frac{k^2 E^2}{12(1 - v^2)} \left(\frac{h}{b}\right)^2 \tag{A-4}$$

by making the substitutions:

$$bh_1 = 2\pi h h \delta$$
$$bh_2 = 2\pi h b \delta$$
$$bh_3 = \pi b \delta$$
$$\beta = h/b$$

Then $k$ will have the value

$$k = \beta (p \beta^2 + q + r/\beta^2) \tag{A-5}$$

There will be a minimum value, $k_m$, of $k$ yielding a minimum value of $S_{cr}$, given by

$$S_{cr} = \sqrt{q^2 + 12pr} - q \frac{6p}{6b}$$

The values of $k_m$ for the different surfaces of deflection will be a basis for judging which is most likely to occur. In Table A-1 are given the values found for $p$, $q$, $r$, $k_m$ and $k$ for the eight deflections, $w_i$, to $w_7$.

<table>
<thead>
<tr>
<th>Value of $w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
<th>$w_4$</th>
<th>$w_5$</th>
<th>$w_6$</th>
<th>$w_7$</th>
<th>$w_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$p$</td>
<td>3.49</td>
<td>6.07</td>
<td>6.44</td>
<td>6.08</td>
<td>14.18</td>
<td>11.2</td>
<td>11.2</td>
</tr>
<tr>
<td>$q$</td>
<td>4</td>
<td>4.67</td>
<td>4.49</td>
<td>4.49</td>
<td>10.13</td>
<td>23.64</td>
<td>11.2</td>
</tr>
<tr>
<td>$r$</td>
<td>4.69</td>
<td>6.07</td>
<td>4.67</td>
<td>4.67</td>
<td>8.83</td>
<td>12.16</td>
<td>70.92</td>
</tr>
<tr>
<td>$k_m$</td>
<td>0.698</td>
<td>0.698</td>
<td>0.698</td>
<td>0.698</td>
<td>0.698</td>
<td>0.698</td>
<td>0.698</td>
</tr>
<tr>
<td>$k$</td>
<td>9.85</td>
<td>18.21</td>
<td>14.95</td>
<td>10.97</td>
<td>17.86</td>
<td>25.49</td>
<td>108.3</td>
</tr>
</tbody>
</table>

From the values of $k_m$ in Table A-1 it is obvious that of the deflections with hinged edges $w_1$ is most likely to occur, and of those with the flange edge fixed $w_4$ is most likely to occur. It is assumed that for all edges fixed the deflection $w_4$ involving sine waves will be most likely to occur since this has proved true for the other two groups. The remainder of this discussion concerns itself with these three cases.

In Table A-II are given the values of $k_i, i = 1, 2, 3$ for values of $\beta$ ranging from 0.20 to 1.60.

From this table we can provide for a state of partial fixity lying between being hinged on all edges and totally fixed on all edges by using a value of $k$ given by

$$k = k_1 + f(k_2 - k_1)$$

in which $f$ has the value zero for hinged edges and unity for fixed edges. Since $k_2 = 2.8k_1$, provision for this assumption could be made by allowing $\gamma$ in formula (A-4) to vary from 1 to 2.8.

To make provision for fixed edges along the flanges and partial fixity along the stiffeners, which most likely fits the case in hand, we can let $k = k_1 + f(k_2 - k_1)$ in which $f$ may have values from zero to unity, zero corre-
Table A-II

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$k_1$</th>
<th>$k_2$</th>
<th>$k_3$</th>
<th>$\beta$</th>
<th>$k_1$</th>
<th>$k_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.20</td>
<td>20.8</td>
<td>21.0</td>
<td>58.2</td>
<td>0.44</td>
<td>11.2</td>
<td>11.7</td>
</tr>
<tr>
<td>0.22</td>
<td>19.1</td>
<td>19.3</td>
<td>55.5</td>
<td>0.48</td>
<td>10.9</td>
<td>11.5</td>
</tr>
<tr>
<td>0.24</td>
<td>17.7</td>
<td>17.9</td>
<td>50.6</td>
<td>0.48</td>
<td>10.7</td>
<td>11.4</td>
</tr>
<tr>
<td>0.26</td>
<td>16.5</td>
<td>16.7</td>
<td>46.2</td>
<td>0.50</td>
<td>10.5</td>
<td>11.2</td>
</tr>
<tr>
<td>0.28</td>
<td>15.5</td>
<td>15.8</td>
<td>43.4</td>
<td>0.60</td>
<td>9.9</td>
<td>11.0</td>
</tr>
<tr>
<td>0.30</td>
<td>14.6</td>
<td>14.9</td>
<td>40.9</td>
<td>0.70</td>
<td>9.9</td>
<td>11.4</td>
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<tr>
<td>0.32</td>
<td>13.9</td>
<td>14.2</td>
<td>38.9</td>
<td>0.80</td>
<td>10.2</td>
<td>12.3</td>
</tr>
<tr>
<td>0.34</td>
<td>13.3</td>
<td>13.6</td>
<td>37.2</td>
<td>0.90</td>
<td>10.9</td>
<td>13.2</td>
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<tr>
<td>0.36</td>
<td>12.7</td>
<td>13.1</td>
<td>36.6</td>
<td>1.00</td>
<td>12.0</td>
<td>15.7</td>
</tr>
<tr>
<td>0.38</td>
<td>12.3</td>
<td>12.7</td>
<td>34.4</td>
<td>1.20</td>
<td>15.0</td>
<td>21.0</td>
</tr>
<tr>
<td>0.40</td>
<td>11.9</td>
<td>12.3</td>
<td>33.3</td>
<td>1.40</td>
<td>19.4</td>
<td>26.6</td>
</tr>
<tr>
<td>0.42</td>
<td>11.5</td>
<td>12.0</td>
<td>32.2</td>
<td>1.60</td>
<td>25.3</td>
<td>38.0</td>
</tr>
</tbody>
</table>

By working back from the test results, $f$ was given the value of 0.2. Thus, the equation, $k = k_1 + 0.2(k_2 - k_3)$ is the suggested method of determining $k$ for the design problems. Table 5 shows how well this method of determining $k$ fits the tests of this investigation.

To show the effect of edge restraint in the web panels Table A-III was prepared for the particular case where $h/t = 170$ and for $\zeta = 1$. Equation (A-4) gives $S_c = 938.2k$ from which the theoretical buckling stresses for different values of $\beta$ are worked out. In this table $L_1$, $L_2$ and $L_3$ indicate the lower limits of the critical stress, in kips per sq. in., corresponding to $k_1$, $k_2$ and $k_3$.

Table A-III

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$L_1$</th>
<th>$L_2$</th>
<th>$L_3$</th>
<th>$\beta$</th>
<th>$L_1$</th>
<th>$L_2$</th>
<th>$L_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.20</td>
<td>19.5</td>
<td>19.7</td>
<td>54.5</td>
<td>0.60</td>
<td>9.3</td>
<td>10.3</td>
<td>25.0</td>
</tr>
<tr>
<td>0.24</td>
<td>16.6</td>
<td>16.8</td>
<td>46.4</td>
<td>0.70</td>
<td>9.3</td>
<td>10.7</td>
<td>25.0</td>
</tr>
<tr>
<td>0.28</td>
<td>14.5</td>
<td>14.8</td>
<td>40.6</td>
<td>0.80</td>
<td>9.5</td>
<td>11.6</td>
<td>25.6</td>
</tr>
<tr>
<td>0.32</td>
<td>13.1</td>
<td>13.3</td>
<td>36.6</td>
<td>1.00</td>
<td>11.2</td>
<td>14.7</td>
<td>31.4</td>
</tr>
<tr>
<td>0.36</td>
<td>11.9</td>
<td>12.3</td>
<td>33.4</td>
<td>1.20</td>
<td>14.1</td>
<td>19.7</td>
<td>39.3</td>
</tr>
<tr>
<td>0.40</td>
<td>11.2</td>
<td>11.5</td>
<td>31.2</td>
<td>1.40</td>
<td>18.2</td>
<td>26.8</td>
<td>51.0</td>
</tr>
<tr>
<td>0.50</td>
<td>9.9</td>
<td>10.5</td>
<td>27.6</td>
<td>1.60</td>
<td>22.7</td>
<td>35.6</td>
<td>66.5</td>
</tr>
</tbody>
</table>

When a moment and a shear act upon a panel simultaneously, the stresses caused by the moment reduce the critical buckling stress in the compressive panels. Let $\beta I = 2\lambda$ and let $S_c'$ be the critical shearing stress when the web is carrying a stress due to moment given by $N_s = \frac{2I}{h}$. Then by equation (A-3)

$$S_c' = S_c - \frac{s}{\beta}$$

For $w_1$, $S_c$ = 0.207; thus $S_c' = S_c = 0.207S$ where $S$ is the maximum flexural stress as before. In other words, the critical buckling stress in this case is the critical buckling stress as if shear alone were acting, reduced by $0.207/\beta$ times the maximum bending stress in the web.

Values of $S_1$, $S_2$ and $S_3$ were found to be 0.143, 0.144 and 0.056, respectively, indicating that edge fixity tends to reduce this factor. On account of the mathematical difficulty involved, values of $S_1$ and $S_3$ were not computed, but the above values give some idea of their probable range of values.

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ANALYSIS OF BUILDING FRAMES WITH SEMI-RIGID CONNECTIONS

BY BRUCE JOHNSTON,¹ ASSOC. M. AM. SOC. C. E., AND EDWARD H. MOUNT,² ESQ.

SYNOPSIS

Methods applicable to the analysis of building frames with semi-rigid riveted or welded connections between the beams and columns are presented in this paper. The methods of analysis are too complex for ordinary design use, but the writers have presented simple design procedures, based on these methods of analysis, elsewhere (2) (3),³ and have made them expeditious by the use of charts and diagrams. Such design methods effect permissible economy in the required beam sizes, made possible by considering the partial restraint afforded by standard or near standard connections, particularly riveted or welded connections of the top and seat angle type.

This paper also presents test results of a welded building frame that corroborate the methods of analysis. A study of the effect of neglecting the width of members in the analysis is presented. ("Member width" is used in this paper to indicate column width or beam depth, as the frame is viewed in elevation.) The essential features of the methods of analysis have been presented elsewhere (1) (13) (14), and it is the intention of the writers to modify them slightly so as to simplify the technique by conforming in every way to the usual slope-deflection and moment-distribution procedures.

INTRODUCTION

The design of the steel frames that form the skeleton of multiple-story steel buildings is usually based upon certain simplifying assumptions, chief of which are: (a) For the purpose of beam design the beam-column connections

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³ Numerals in parentheses, thus: (2) (3), refer to corresponding numbers in the Bibliography; see Appendix.

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are assumed to be pin connections, or simple supports; (b) columns are designed without attempting to evaluate the moments introduced by frame action; and (c) the beam-column connections are assumed to be rigid in calculating stresses due to lateral or wind loads.

These assumptions have afforded a means of rapid design calculation. Riveted building frames constructed on the basis of these design assumptions have proved to be safe and reliable, but there remains the possibility of achieving greater economy through the use of more nearly correct design assumptions. According to the British investigations (1), an average saving in the weight of beams of as much as 20% may be expected by taking advantage of the partial end restraint of riveted beam-column connections. Welded construction, with its inherent continuity, also makes similar saving in weight possible.

The basis for the application of more accurate design methods to frames with semi-rigid connections has already been laid in Great Britain (1) and in work of a parallel nature in the United States (2) (3) (4). The comparison of analytical and experimental results presented in this paper was made possible by the construction and test of a welded building frame with three bays and two stories. The beam-column connections used in this frame were of a semi-rigid type previously studied at the Fritz Engineering Laboratory (2) (5) in connection with research programs sponsored by the American Welding Society. Similar tests on riveted building frames have been made in Great Britain (1).

The slope-deflection and moment-distribution methods have as a common purpose the determination of the bending moments at the ends of the individual members of a statically indeterminate frame. Both methods in their usual form are based on the assumption that the deformations of the frames are caused entirely by bending of the members and that the relation between bending moment and distortion is given by the beam formula:

\[ \frac{M}{EI} = \frac{d^2y}{dx^2} \]  

The derivation of the beam theory and the assumptions on which it is based may be found in any text on the strength of materials.

In 1915 the slope-deflection method was applied in the United States (by Wilbur M. Wilson and George A. Maney, Members, Am. Soc. C. E.) to the analysis of wind stresses in tall buildings (6). A more complete treatment followed in 1918 (by Professor Wilson, with F. E. Richart and Camillo Weiss, Members, Am. Soc. C. E.) (7), and in 1931 a modification (8) was introduced by L. T. Evans, Assoc. M. Am. Soc. C. E., to take care of members with varying moments of inertia. The method of moment distribution was first presented in mimeographed form by Hardy Cross, M. Am. Soc. C. E., in 1926 (9) (10) (12). Innumerable variations and short cuts have been applied to the moment-distribution method and, although some of these have merit, the original method remains the outstanding development in recent times for rapid and effective analysis of continuous frames. The application of both the slope-deflection and moment-distribution methods to the analysis of frames with semi-rigid connections was made by John F. Baker, Assoc. M.
Am. Soc. C. E. (13). The width of the members and the semi-rigid nature of the connections are both taken into account, as it is found that the neglect of member width gives rise to considerable error, particularly in the case of analyses of frames with semi-rigid connections.

**Analysis of Frames with Semi-Rigid Joints**

It will be assumed that the reader is already familiar with the usual application of the slope-deflection and moment-distribution methods, for which references are readily available (12) (15) (16) (17). The methods herein presented are identical in mode of application to the usual simple form, with the exception that special coefficients must be used in the slope-deflection equations and for the carry-over and distribution factors in the moment-distribution method. The following method is identical with that previously presented by Professor Baker (13) (14) when the width of member is neglected. When the width of member is considered, the following method differs in two respects from that of Professor Baker: (a) The interior of the joint between connections is assumed infinitely rigid, whereas Professor Baker assumes it to have the same stiffness as the beam; and (b) hypothetical moments are computed at the joint centers by the usual slope-deflection and moment-distribution procedures, whereas in Professor Baker's method separate expressions are given for the moment and shear at the connection, or column face, and are dealt with separately.

The following assumptions apply both to the slope-deflection and to the moment-distribution relations as herein presented: (a) Members are of uniform cross section between their end connections; (b) the semi-rigid connection at the end of a member behaves elastically as defined by the connection constant \( \gamma \); and (c) the interior of the joint between connections is assumed to be infinitely rigid, although free to rotate as a rigid body.

The notation used is shown subsequently in Figs. 5 and 6. The hypothetical moments and shears at the joint centers, when width of column is considered, are designated by the bar above the letter \( M \) or \( V \), thus; \( \bar{M} \) and \( \bar{V} \).

**The "Semi-Rigid" Connection.**—The semi-rigid connection may be thought of as a locally weakened section between the end of the beam and the face of the column to which the connection is made. The effect on analysis is the inverse of the effect produced by end haunches or added cover plates. The typical test behavior of a riveted or welded connection of this type is shown in Fig. 1, which presents the plot of the relationship between moment transmitted through the connection and the angle change between the joint center and the end of the beam. In the design range the relationship is assumed linear and the inverse slope is termed the connection constant, \( \gamma \), thus:

\[
\gamma = \frac{\phi}{M} = \frac{\text{Angle change}}{\text{Moment}} \tag{2}
\]

The connection constant \( \gamma \) may be defined as "angle change for unit moment" and may be determined experimentally by testing typical joints. The vertical
line through the origin in Fig. 1 would indicate the behavior of a perfectly fixed connection with $\gamma = 0$, and the horizontal line would represent the behavior of a frictionless pin connection, in which case $\gamma = \infty$. Fig. 2(a) shows the test setup for determining the connection constant at an interior joint of a frame with beam-to-column-flange connections, and Fig. 2(b) shows a similar setup to test the connection between a beam and an exterior column web. These connections correspond to those used on two frames tested by the writers. The relative rotation between the ends of the beam and the center of the column at the joint were measured with a 20-in. level bar which is shown in Fig. 3 in position for measurement of angle changes of the actual
test frame. Fig. 4 is a graph within the test-design range of measured angle change plotted against moment in typical joint tests corresponding to the actual test frame.

**Fig. 3.—Level Bar Used to Measure Rotations**

The Slope-Deflection Equations.—For any individual member of a frame, the relation between its end moments, the angle changes at each end, and the angle change of the member as a whole may be expressed by a pair of slope-deflection equations. For the usually assumed case of uniform beam cross...
section, and with rigid end connections, these equations are written:

\[ M_{AB} = 2EK(2\theta_A + \theta_B - 3R) - M_{RAB} \]  \hspace{1cm} (3a)

and

\[ M_{BA} = 2EK(2\theta_B + \theta_A - 3R) + M_{RBA} \]  \hspace{1cm} (3b)

In Eqs. 3, \( K = \frac{I}{l} \), in which \( I \) is the moment of inertia and \( l = \) the length of the member (distance between joint connections). The angle changes at ends \( A \) and \( B \) are \( \theta_A \) and \( \theta_B \); and \( R = \frac{\Delta}{l} = \) the angle change of the entire member, \( \Delta \) being the relative lateral displacements of ends \( A \) and \( B \). The fixed-end moments for the lateral loads alone are \( M_{RAB} \) and \( M_{RBA} \). The slope-deflection equations may be derived directly from Eq. 1 or by an application of the moment-area principles.

Slope-deflection equations such as Eqs. 3 may be derived by similar methods for members which frame with semi-rigid connections. Fig. 5 shows the notation used and the geometry of the general deflected curve of any such member. Note especially Fig. 5(b), which shows the hypothetical moments at the center of the joint. The slope-deflection equations corresponding to Eqs. 3 for the hypothetical moments \( \tilde{M}_{AB} \) and \( \tilde{M}_{BA} \) at the joint centers for any member \( AB \), as shown in Fig. 5, may be written:

\[ \tilde{M}_{AB} = \frac{1}{1 + 2\alpha + 2\beta + 3\alpha\beta} [2EK(C_{AA}\theta_A + C_{AB}\theta_B - C_{AC}R) \]

\[ - F_{AA}M_{RAB} - F_{AB}M_{RBA} - V_{AB}'b_{AB} \]  \hspace{1cm} (4a)

and

\[ \tilde{M}_{BA} = \frac{1}{1 + 2\alpha + 2\beta + 3\alpha\beta} [2EK(C_{BB}\theta_B + C_{BA}\theta_A - C_{BC}R) \]

\[ + F_{BB}M_{RBA} + F_{BA}M_{RAB} + V_{BA}'b_{BA} \]  \hspace{1cm} (4b)

Except for the fact that new coefficients replace the even integer coefficients in Eqs. 3, the application of Eqs. 4 to any particular problem is identical with the usual slope-deflection procedure.

In Eqs. 4 the new constants \( C_{AA}, C_{AB}, C_{AC}, C_{BB}, C_{BA}, F_{AA}, F_{AB}, F_{BA}, \) and \( F_{BB} \) depend on the dimensions of the members and on the value of the joint constant. Factor \( K \) again is given by \( \frac{I}{l} \), and it should be noted that \( l \) is the length between connections rather than the length between joint centers.
The connection constants \( \gamma_A \) and \( \gamma_B \) for the connections at the two ends of the beam are introduced into new constants \( \alpha \) and \( \beta \) by the relations (1):

\[
\alpha = 2EK \gamma_A. \tag{5a}
\]

and

\[
\beta = 2EK \gamma_B. \tag{5b}
\]

The fixed-end moments for a member with fixed, rigidly connected, ends of span length \( l \) are \( M_{RAB} \) and \( M_{RBA} \); and \( V_{AB}' \) and \( V_{BA}' \) are the shears or reactions at the ends of a member with freely supported ends and span length \( l \).

The constants \( C_{AA}, C_{AB}, C_{AC}, F_{AA}, \) and \( F_{AB} \) in Eq. 4a are given in Table 1 for four different cases. All four cases are for unsymmetrical conditions, the first and the third considering semi-rigid connections, and the second and fourth considering rigid connections. In cases I and II, a finite width of member is considered; and, in cases III and IV, width is ignored. The values in case IV are those commonly assumed—that is, frames with rigid joints and with width of member neglected. In Eq. 4b the constants \( C_{BB}, C_{BA}, C_{BC}, F_{BB}, \) and \( F_{BA} \) for \( M_{BA} \) are obtained from \( C_{AA}, C_{AB}, C_{AC}, F_{AA}, \) and \( F_{AB} \), respectively, by interchanging \( \alpha \) and \( \beta \) and the subscripts A and B.

It is noted that \( C_{BA} \) is equal to \( C_{AB} \).

Eqs. 4 with the preceding coefficients are for moments at the joint centers and therefore are used with exactly the same equilibrium conditions as in the simpler form (2); namely,

For joint equilibrium,

\[
\Sigma \bar{M} = 0. \tag{6a}
\]

and, for story equilibrium,

\[
\Sigma \bar{M} + Vh = 0. \tag{6b}
\]

In Eq. 6b, \( h \) is the story height between neutral axes of two beams.

With Eqs. 4 it is now possible to determine the hypothetical end moments \( \bar{M}_{AB} \) and \( \bar{M}_{BA} \) at the joint centers. Moments and shears are assumed as positive when they act on the ends of the beam with a clockwise sense, or act on the joint with a counterclockwise sense. The hypothetical shears in the joint are constant and equal to the actual shear at the connection. The shears \( \bar{V}_{AB} \) and \( \bar{V}_{BA} \) may be calculated from the moments \( \bar{M}_{AB} \) and \( \bar{M}_{BA} \) by the following:

\[
\bar{V}_{AB} = \bar{V}_{AB} = -\left( \frac{\bar{M}_{AB} + \bar{M}_{BA}}{L} \right) + \bar{V}_{AB}'. \tag{7a}
\]

and

\[
\bar{V}_{BA} = \bar{V}_{BA} = -\left( \frac{\bar{M}_{AB} + \bar{M}_{BA}}{L} \right) - \bar{V}_{BA}'. \tag{7b}
\]

in which \( \bar{V}_{AB}' \) and \( \bar{V}_{BA}' \) are the end shears in a member having span length \( L \) with simply or freely supported ends. The moments at the connections, \( M_{AB} \) and \( M_{BA} \), may now be calculated from the following:

\[
M_{AB} = \bar{M}_{AB} + \bar{V}_{AB} b_{AB}. \tag{8a}
\]

and

\[
M_{BA} = \bar{M}_{BA} + \bar{V}_{BA} b_{BA}. \tag{8b}
\]
### Table 1: Slope-Deflection and Moment-Distribution Constants—Nonsymmetrical Cases

<table>
<thead>
<tr>
<th>Constant</th>
<th>Case I—Nonsymmetrical, semi-rigid connections, finite width of members</th>
<th>Case II—Nonsymmetrical, rigid connections, finite widths</th>
<th>Case III—Nonsymmetrical, rigid connections, zero width</th>
<th>Case IV—Rigid connections, 0 width</th>
</tr>
</thead>
<tbody>
<tr>
<td>C&lt;sub&gt;AA&lt;/sub&gt;</td>
<td>[2 + 3\beta + 6 (1 + \beta) \frac{b_{AB}}{l} + 3 (2 + \alpha + \beta) \frac{b_{AB}^2}{l^2}]</td>
<td>[2 + 3\beta]</td>
<td>[2 + 3\beta]</td>
<td>[2 + 3\beta]</td>
</tr>
<tr>
<td>C&lt;sub&gt;AB&lt;/sub&gt; = C&lt;sub&gt;BA&lt;/sub&gt;</td>
<td>[1 + 3 (1 + \alpha) \frac{b_{AB}}{l} + 3 (1 + \beta) \frac{b_{BA}}{l} + 3 (2 + \alpha + \beta) \frac{b_{AB}b_{BA}}{l^2}]</td>
<td>[1 + \frac{3b_{AB}}{l} + \frac{3b_{BA}}{l} + \frac{6b_{AB}b_{BA}}{l^2}]</td>
<td>[1]</td>
<td>[1]</td>
</tr>
<tr>
<td>C&lt;sub&gt;BB&lt;/sub&gt;</td>
<td>[2 + 3\alpha + 6 (1 + \alpha) \frac{b_{BA}}{l} + 3 (2 + \alpha + \beta) \frac{b_{BA}^2}{l^2}]</td>
<td>[2 + 6 \frac{b_{BA}}{l} + \frac{6b_{BA}^2}{l^2}]</td>
<td>[2 + 3\alpha]</td>
<td>[2]</td>
</tr>
<tr>
<td>C&lt;sub&gt;CAC&lt;/sub&gt;</td>
<td>[3 (1 + \beta) + 3 (2 + \alpha + \beta) \frac{b_{AB}}{l}]</td>
<td>[3 + \frac{6b_{AB}}{l}]</td>
<td>[3 (1 + \beta)]</td>
<td>[3]</td>
</tr>
<tr>
<td>C&lt;sub&gt;BC&lt;/sub&gt;</td>
<td>[3 (1 + \alpha) + 3 (2 + \alpha + \beta) \frac{b_{BA}}{l}]</td>
<td>[3 + \frac{6b_{BA}}{l}]</td>
<td>[3 (1 + \alpha)]</td>
<td>[3]</td>
</tr>
<tr>
<td>F&lt;sub&gt;AA&lt;/sub&gt;</td>
<td>[1 + 2\beta + (1 - \alpha + 2\beta) \frac{b_{AB}}{l}]</td>
<td>[1 + \frac{b_{BA}}{l}]</td>
<td>[1 + 2\beta]</td>
<td>[1]</td>
</tr>
<tr>
<td>F&lt;sub&gt;AB&lt;/sub&gt;</td>
<td>[\beta + (\beta - 2\alpha - 1) \frac{b_{AB}}{l}]</td>
<td>[\frac{b_{BA}}{l}]</td>
<td>[\beta]</td>
<td>[0]</td>
</tr>
<tr>
<td>F&lt;sub&gt;BB&lt;/sub&gt;</td>
<td>[1 + 2\alpha + (1 - \beta + 2\alpha) \frac{b_{BA}}{l}]</td>
<td>[1 + \frac{b_{BA}}{l}]</td>
<td>[1 + 2\alpha]</td>
<td>[1]</td>
</tr>
<tr>
<td>F&lt;sub&gt;FA&lt;/sub&gt;</td>
<td>[\alpha + (\alpha - 2\beta - 1) \frac{b_{BA}}{l}]</td>
<td>[-\frac{b_{BA}}{l}]</td>
<td>[\alpha]</td>
<td>[0]</td>
</tr>
</tbody>
</table>

### (a) Slope Deflection

- **M<sub>SAB</sub>** semi-fixed end moment at joint center: \[\left(1 + \frac{2\beta + 6 \frac{b_{AB}}{l}}{1 + 2\alpha + 2\beta + 3\alpha} \right) M_{RAB} + \left(\beta - \frac{b_{AB}}{l} \left(1 + 2\alpha - \beta\right) M_{RBA}\right) + V_{AB}'b_{AB}\]
- **M<sub>RAB</sub>** carry-over factor between joint centers: \[1 + 3 (1 + \alpha) \frac{b_{AB}}{l} + 3 (1 + \beta) \frac{b_{BA}}{l} + 3 (2 + \alpha + \beta) \frac{b_{AB}b_{BA}}{l^2}\]
- **M<sub>SAB</sub>** end rotation stiffness: \[2 E K \left(2 + 3\beta + 6 (1 + \beta) \frac{b_{AB}}{l} + 3 (2 + \alpha + \beta) \frac{b_{AB}^2}{l^2}\right) / \left(1 + 2\alpha + 2\beta + 3\alpha\right)\]
- **M<sub>YAB</sub>** end moment at A: \[-6 E K \left(1 + \frac{2b_{BA}}{l} \left(2 + \alpha + \beta\right)\right) / \left(1 + 2\alpha + 2\beta + 3\alpha\right)\]

### (b) Moment Distribution

- **M<sub>SAB</sub>** side sway stiffness: \[6 E K \left(2 + \alpha + \beta\right) / \left(1 + 2\alpha + 2\beta + 3\alpha\right)\]
- **M<sub>YAB</sub>** side sway stiffness: \[-6 E K \left(1 + \frac{2b_{BA}}{l} \left(2 + \alpha + \beta\right)\right) / \left(1 + 2\alpha + 2\beta + 3\alpha\right)\]
The slope-deflection equations are simplified when symmetrical conditions of loading and structure exist with respect to any particular member. In such a case \( \alpha = \beta \). Furthermore, \( b_{AB} = b_{BA} = b \); \( M_{RAB} = M_{RBA} = M_{R} \); \( \bar{V}_{AB} = \bar{V}_{BA} = \bar{V}_{AB} = \bar{V}_{BA} = \bar{V} \); and the slope-deflection equations corresponding to case I of Table 1 may be reduced to:

\[
\bar{M}_{AB} = \frac{2 E K}{1 + 3 \alpha} \left[ \left( \frac{2 + 3 \alpha}{1 + \alpha} + \frac{6 b}{l} + \frac{6 b^2}{l^2} \right) \theta_A + \left( \frac{1}{1 + \alpha} + \frac{6 b}{l} + \frac{6 b^2}{l^2} \right) \theta_B - \left( 3 + \frac{6 b}{l} \right) R \right] - \left( \frac{M_{R}}{1 + \alpha} + V' b \right)
\]

\[\text{(9a)}\]

and

\[
\bar{M}_{BA} = \frac{2 E K}{1 + 3 \alpha} \left[ \left( \frac{2 + 3 \alpha}{1 + \alpha} + \frac{6 b}{l} + \frac{6 b^2}{l^2} \right) \theta_B + \left( \frac{1}{1 + \alpha} + \frac{6 b}{l} + \frac{6 b^2}{l^2} \right) \theta_A - \left( 3 + \frac{6 b}{l} \right) R \right] + \left( \frac{M_{R}}{1 + \alpha} + V' b \right)
\]

\[\text{(9b)}\]

For case II of Table 1, considering the semi-rigidity of the joints but neglecting the width of the members, the equations for symmetrical conditions may be reduced to a simple form by letting \( b = 0 \) in Eqs. 9; thus:

\[
\bar{M}_{AB} = M_{AB} = \frac{2 E K}{1 + 3 \alpha} \left[ \left( \frac{2 + 3 \alpha}{1 + \alpha} \right) \theta_A 
\right. 
\left. + \left( \frac{1}{1 + \alpha} \right) \theta_B - 3 R \right] - \frac{M_{R}}{1 + \alpha} 
\]

\[\text{(10a)}\]

and

\[
\bar{M}_{BA} = M_{BA} = \frac{2 E K}{1 + 3 \alpha} \left[ \left( \frac{2 + 3 \alpha}{1 + \alpha} \right) \theta_B 
\right. 
\left. + \left( \frac{1}{1 + \alpha} \right) \theta_A - 3 R \right] + \frac{M_{R}}{1 + \alpha}
\]

\[\text{(10b)}\]

A similar simplification may be made for case III of Table 1 by letting \( \alpha = 0 \) in Eqs. 9, in which case the following equations result:

\[
\bar{M}_{AB} = 2 E K \left[ \left( 2 + \frac{6 b}{l} + \frac{6 b^2}{l^2} \right) \theta_A + \left( 1 + \frac{6 b}{l} + \frac{6 b^2}{l^2} \right) \theta_B - \left( 3 + \frac{6 b}{l} \right) R \right] - (M_{R} + V'b) 
\]

\[\text{(11a)}\]

and

\[
\bar{M}_{BA} = 2 E K \left[ \left( 2 + \frac{6 b}{l} + \frac{6 b^2}{l^2} \right) \theta_B + \left( 1 + \frac{6 b}{l} + \frac{6 b^2}{l^2} \right) \theta_A - \left( 3 + \frac{6 b}{l} \right) R \right] + (M_{R} + V'b)
\]

\[\text{(11b)}\]

**Moment Distribution Applied to Frames with Semi-Rigid Connections.**—The moment-distribution method serves identically the same purpose as the slope-deflection method; that is, the moments at the ends of the individual members
of a frame or continuous beam are determined. It will be assumed, as in the case of the slope-deflection method, that the reader is familiar with the usual moment distribution procedure. The procedure as applied to frames with semi-rigid connections is identical with the usual method, although there are differences in the numerical value of the carry-over, stiffness, and other factors.

The factors used in the moment-distribution procedure may be derived from the slope-deflection equations or by use of the column analogy (11), as will be described herein. Fig. 6 shows the deformation conditions for determining semi-rigid end moments and the carry-over factor for moment distribution. Table 1 gives the "semi-rigid" end moments, "carry-over" factor, rotation-stiffness factors, sidesway-stiffness factors, and sidesway end moments for the nonsymmetrical cases I, II, III, and IV. Table 2 presents the same factors for the symmetrical cases I, II, and III.

### TABLE 2.—Moment Distribution for Symmetrical Cases

<table>
<thead>
<tr>
<th>Constant</th>
<th>Symmetrical case I, semi-rigid connections, finite width of members</th>
<th>Symmetrical case II, rigid connections, finite member width</th>
<th>Symmetrical case III, semi-rigid connections, zero member width</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_{SAB}$ semi-fixed-end moment at joint center</td>
<td>$M_R \left( 1 + \frac{\alpha}{1 + \alpha} \right) + V'b$</td>
<td>$M_R + V'b$</td>
<td>$M_R \left( 1 + \frac{\alpha}{1 + \alpha} \right)$</td>
</tr>
<tr>
<td>$\gamma_{AB}$ carry-over factor between joint centers</td>
<td>$1 + 6 \left( \frac{1 + \alpha}{1 + \alpha} \right) b$ $\bar{b}$ $\frac{1}{l}$</td>
<td>$1 + \frac{6b}{l} + \frac{6b^2}{\bar{b}^2}$</td>
<td>$1 + \frac{6b}{l} + \frac{6b^2}{\bar{b}^2}$</td>
</tr>
<tr>
<td>$\delta_{MAB}$ end rotation stiffness</td>
<td>$2EK \left( 2 + \frac{3\alpha}{1 + \alpha} + \frac{6b}{l} + \frac{6b^2}{\bar{b}^2} \right) \frac{1}{1 + \alpha}$</td>
<td>$2EK \left( 2 + \frac{6b}{l} + \frac{6b^2}{\bar{b}^2} \right) \frac{1}{1 + \alpha}$</td>
<td>$2EK \left( 2 + \frac{3\alpha}{1 + \alpha} + \frac{6b}{l} + \frac{6b^2}{\bar{b}^2} \right) \frac{1}{1 + \alpha}$</td>
</tr>
<tr>
<td>$\delta_{VAB}$ sidesway stiffness</td>
<td>$\frac{12EK}{l} \frac{b}{1 + \alpha}$</td>
<td>$\frac{12EK}{l} \frac{b}{1 + \alpha}$</td>
<td>$\frac{12EK}{l} \frac{b}{1 + \alpha}$</td>
</tr>
<tr>
<td>$\bar{M}_{VAB}$ sidesway end moment at $A$</td>
<td>$-\frac{6EK}{l} \frac{1 + \frac{2b}{l}}{1 + 3\alpha}$</td>
<td>$-\frac{6EK}{l} \frac{1 + \frac{2b}{l}}{1 + 3\alpha}$</td>
<td>$-\frac{6EK}{l} \frac{1 + \frac{2b}{l}}{1 + \alpha}$</td>
</tr>
</tbody>
</table>

All of the moment-distribution factors relate to hypothetical moments at the centers of the joints, and the general procedure is therefore identical with that used in the simple case. After the hypothetical moments at the joint centers are obtained, the moments and shears at the connections result as before from Eqs. 7 and 8.
In applying the moment-distribution method to frame problems, sidesway may be induced either by lateral loads or by unsymmetrical conditions of loading or frame arrangement. The simplest form of sidesway problem is one involving only lateral loads applied to a one-story frame. The lateral load is distributed to the columns in proportion to their "sidesway stiffness" $S_V$, and semi-fixed or fixed-end moments are distributed to the ends of each column in proportion to the $M_V$ moments for unit sidesway. The next step is the usual moment distribution, but at the conclusion the summation of column shears will not account for the total lateral force. All of the end moments are then multiplied by a constant ratio sufficient to bring the shears into equilibrium with the external lateral force. If the structure is more than one story in height, the procedure is progressively more complicated. Shears are distributed in any one story in proportion to their lateral or sidesway stiffness; but in special cases in which two-story sections adjoin open auditoriums or halls, the combined rigidities of the two stories of columns "in series" must be determined. The stiffness of a two-story group of columns "in series" is given by:

$$S_{VABC} = \frac{S_{VAB} S_{VBC}}{S_{VAB} + S_{VBC}} \quad \text{(12a)}$$

in which $S_{VABC} =$ sidesway stiffness of two columns or groups of columns "in series." The combined rigidity of three tiers of columns "in series" is given by:

$$S_{VABCD} = \frac{S_{VAB} S_{VBC} S_{VCD}}{S_{VAB} S_{VBC} + S_{VBC} S_{VCD} + S_{VCD} S_{VAB}} \quad \text{(12b)}$$

The analysis of two-story and three-story problems of the foregoing type is taken up in texts (12) (16) (17) and the procedure for frames with semi-rigid joints follows exactly the same course. In applying these analytical methods to the development of design methods for multi-storied buildings, under the action of vertical loads alone, it is reasonable to neglect the effect of sidesway.

Certain short cuts for special conditions may be used, provided that their use in simpler forms of moment distribution is already familiar. For example, if the end $B$ of member $A B$ is pin-connected, or freely supported, $\beta$ becomes equal to $\infty$. The "semi-fixed" end moment at $A$ then becomes:

$$\bar{M}_{SAB} = \left( \frac{2 + \frac{2b_{AB}}{l}}{2 + 3\alpha} \right) M_{RAB} \quad \text{(13)}$$

In Eq. 13 $M_{RAB}$ is the fixed-end moment at $A$ due to lateral loads in a beam freely supported at $B$. The rotation stiffness, or distribution factor, for end $A$ of member $A B$ with $B$ freely supported is:

$$S_{MAB} = \frac{2 E K}{2 + 3\alpha} \left( 3 + \frac{6 b_{AB}}{l} + \frac{3 b_{AB^2}}{l^2} \right) \quad \text{(14a)}$$

The sidesway stiffness factor for the same case, with one end freely supported,
is as follows:

\[ S_{VAB} = \frac{6EKL}{F(2+3\alpha)} \]  

Cases II and III in Table 2 may be obtained from Eqs. 13, 14a, and 14b by letting \( \alpha \) and \( b_{AB} \), respectively, be equal to zero.

Another type of special case occurs when the entire frame and loading upon it are symmetrical. If the center line of the frame is on line with a column, there will be no rotation of the column joints and the center line of the center column may be assumed equivalent to a fixed wall. If there are an odd number of panels, the center line of the frame will cut through the center of the beams in the center bay. The rotation of the two ends of each beam in the center panel will be equal in magnitude and opposite in sign. From this condition it follows that the modified moment stiffness or "distribution factor" may be taken as:

\[ S_{MAB} = \frac{2EKL}{1+\alpha} \]

for the ends of beams in the center panel. No carry-over is used in the center panel when the modified stiffness factor is used.

Application of the Column Analogy to Members with Semi-Rigid Connections.

It will be assumed that the reader is familiar with the application of the column analogy, originally developed by Professor Cross (11), to the determination of moment-distribution factors for beams with variable moments of inertia. The width of the "analogous column" is equal to \( \frac{1}{EI} \), and the area of any elemental length \( ds \) of the analogous column is therefore equal to \( \frac{ds}{EI} \). From the fundamental relations of the bent beam,

\[ \frac{d\phi}{M} = \frac{ds}{EI} \]

in which \( d\phi = \) the angle change in any elemental length of beam. At the particular location of the semi-rigid joint, from the definition of the "connection constant,"

\[ \gamma = \frac{d\phi}{M} = \frac{ds}{EI} \]

Hence, the localized area of the analogous column at the semi-rigid connection
is equal to the connection constant $\gamma$. Professor Cross (11) has shown that the area of the analogous column at a pin connection is infinite, and in the region of a completely rigid zone it is equal to zero. The semi-rigid connection obviously is a case somewhere between these two extremes, and the column analogy may readily be used to obtain the moment-distribution factors for a member so connected. Fig. 7 illustrates a cross section through the analogous column of a member with semi-rigid end connections.

**Analysis by Slope-Deflection Method**

An illustrative example will be presented in detail to demonstrate the application of both the slope-deflection and moment-distribution methods to the analysis of a frame with semi-rigid joints, taking into account the width of the members.

The frame shown in Fig. 8 corresponds to one of the frames actually tested (see Fig. 9), and the connection constant used in the analysis was obtained experimentally from tests of a sample joint. All of the connections were identically alike, and each beam, therefore, was individually symmetrical. The results of the connection test gave an experimental value of $\alpha = 0.01775 \times 10^{-3}$ in inch-kip units. The stiffness of the frame members was measured by bending tests preliminary to fabrication of the frame, and the quantity $EI$ was thus found to be $3,550 \times 10^3$ and $3,321 \times 10^3$ for the beams and columns, respectively, in inch-kip units. The net length $l$ of the beams between connections was 168 in. $- 8$ in. $= 160$ in. The columns are continuous, and the beam connections were of the welded seat and top angle type. An approximate correction for column length may be shown to be one third of the beam depth at each end that frames with a beam (see heading "Effect of Width of Member Upon Analysis"). Hence, for the second-story columns, $l = 120 - 6.67 = 113.33$ and, for the first-story columns, $l = 120 - 3.33 = 116.67$. This correction could well be omitted with but little error.

The constant $\alpha$ for the beams was

$$\alpha = \beta = \frac{2EI\gamma}{l} = \frac{2 \times 3,550 \times 10^3 \times 0.01775 \times 10^{-3}}{160} = 0.7877.$$

The fixed-end moment for the loading shown is 221.0 in-kips.

Because of the individual symmetry of the beams, the slope-deflection equations in the form of Eqs. 9 were applicable. The typical equation for any beam is written by substituting the values of $\alpha, EK, b, l$, etc., in Eqs. 9, which for any loaded beam results in the following:

$$M_{AB} = \frac{2 \times 3,550}{160(1 + 3 \times 0.7877)} \left\{ \left( \frac{2 + 3 \times 0.7877}{1 + 0.7877} \right) + \frac{6 \times 4}{160} + \frac{6 \times 4^2}{160^2} \right\} \theta_A$$

$$+ \cdots + \left\{ \left( \frac{1}{1 + 0.7877} \right) + \frac{6 \times 4}{160} + \frac{6 \times 4^2}{160^2} \right\} \theta_B \right\} - \left( \frac{221.0}{1.7877} + 6.5 \times 4 \right) \cdots (17)$$

The right-hand side of this and the following equations has been divided by 1,000 to give more convenient values of $\theta$. The moment $M_{AB} = 34.233 \theta_A$
FIG. 8.—Test Frame and Illustrative Analysis

FIG. 9.—View of Test Frame
+ 9.411 \theta_B - 149.62; \text{ and similarly:} \\
M_{BA} = 34.233 \theta_B + 9.411 \theta_A + 149.62 \ldots \ldots \ldots \ldots \ldots \ldots \ldots (18)

Equations of this type are written for all of the beams in a symmetrical half of the frame, as follows:

\[
\begin{align*}
\bar{M}_{12} &= 34.233 \theta_1 + 9.411 \theta_2 \\
\bar{M}_{21} &= 34.233 \theta_2 + 9.411 \theta_1 \\
\bar{M}_{27} &= 34.233 \theta_2 + 9.411 \theta_7 - 149.62 \quad (- \theta_2 = + \theta_1) \\
\bar{M}_{34} &= 34.233 \theta_3 + 9.411 \theta_4 - 149.62 \\
\bar{M}_{43} &= 34.233 \theta_4 + 9.411 \theta_3 + 149.62 \\
\bar{M}_{48} &= 24.822 \theta_4 \\
\bar{M}_{53} &= 24.822 \theta_2 - 149.62
\end{align*}
\]

Similar equations for the column moments are written by making the proper substitutions in slope-deflection Eqs. 11, as follows:

\[
\begin{align*}
\bar{M}_{13} &= 127.865 \theta_1 + 69.257 \theta_3 \\
\bar{M}_{24} &= 127.865 \theta_2 + 69.257 \theta_4 \\
\bar{M}_{31} &= 127.865 \theta_3 + 69.257 \theta_1 \\
\bar{M}_{42} &= 127.865 \theta_4 + 69.257 \theta_2 \\
\bar{M}_{35} &= 123.910 \theta_2 \\
\bar{M}_{46} &= 123.910 \theta_4 \\
\bar{M}_{53} &= 61.810 \theta_3 \\
\bar{M}_{64} &= 61.810 \theta_4
\end{align*}
\]

The sidesway is obviously zero because of symmetry, and the only unknowns are the four angle changes \(\theta_1, \theta_2, \theta_3, \text{ and } \theta_4\). The necessary and sufficient conditions for the solution are obtained by applying the joint equilibrium equation \(\Sigma M = 0\) to the four joints 1, 2, 3, and 4:

\[
\begin{align*}
\bar{M}_{12} + \bar{M}_{13} &= 0 \\
\bar{M}_{21} + \bar{M}_{24} + \bar{M}_{27} &= 0 \\
\bar{M}_{31} + \bar{M}_{34} + \bar{M}_{35} &= 0 \\
\bar{M}_{43} + \bar{M}_{42} + \bar{M}_{48} + \bar{M}_{46} &= 0
\end{align*}
\]

Rewriting these equations in terms of the unknown \(\theta\)'s:

\[
\begin{align*}
+ 162.052 \theta_1 + 9.411 \theta_2 + 69.214 \theta_3 &= 0 \\
9.411 \theta_1 + 186.874 \theta_2 + 69.214 \theta_4 &= + 149.620 \\
69.214 \theta_1 + 285.934 \theta_3 + 9.411 \theta_4 &= + 149.620 \\
69.214 \theta_2 + 9.411 \theta_3 + 310.756 \theta_4 &= - 149.620
\end{align*}
\]

The solution of these four simultaneous equations may be made by systematic elimination of unknowns (15) (16) or by a method of successive approximations (15). The following solution was obtained by the first method: \(\theta_1 = - 0.33165\);
\( \theta_2 = +1.09286; \theta_3 = +0.62791; \) and \( \theta_4 = -0.74389. \) These values of \( \theta \) are 1,000 times the actual values but will give the correct moments when substituted in the moment equations which previously had been divided by 1,000. The actual moments at the connections may be found from the hypothetical joint center moments by computing the shears with Eqs. 7 and the connection moments with Eqs. 8. An alternate method would be to construct, graphically, the simple beam moment diagram for the full lengths \( L \) upon the joint moment base line. The connection moments then could be scaled off as the ordinate to the moment diagram at the face of the connecting member. Table 3 gives the results by the analytical method.

### TABLE 3.—COMPUTATION BY SLOPE DEFLECTION

<table>
<thead>
<tr>
<th>Location, joint member</th>
<th>Joint moment ( M ) from slope-deflection equation</th>
<th>( P ) by Eqs. 7</th>
<th>Connection moment ( M ) by Eqs. 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-2</td>
<td>- 1.07</td>
<td>-0.198</td>
<td>- 1.86</td>
</tr>
<tr>
<td>2-1</td>
<td>+ 34.20</td>
<td>-0.198</td>
<td>+ 33.50</td>
</tr>
<tr>
<td>2-7</td>
<td>-122.49</td>
<td>+6.500</td>
<td>- 96.49</td>
</tr>
<tr>
<td>3-4</td>
<td>-135.12</td>
<td>+6.531</td>
<td>- 90.00</td>
</tr>
<tr>
<td>4-3</td>
<td>+130.06</td>
<td>-6.489</td>
<td>+104.18</td>
</tr>
<tr>
<td>4-8</td>
<td>-18.46</td>
<td>+6.500</td>
<td>- 7.54</td>
</tr>
<tr>
<td>1-3</td>
<td>+ 1.07</td>
<td>-0.515</td>
<td>- 6.55</td>
</tr>
<tr>
<td>3-1</td>
<td>+ 57.31</td>
<td>-0.515</td>
<td>- 55.59</td>
</tr>
<tr>
<td>3-5</td>
<td>+ 77.79</td>
<td>-1.000</td>
<td>- 74.46</td>
</tr>
<tr>
<td>5-3</td>
<td>+ 38.82</td>
<td>-1.000</td>
<td>- 38.82</td>
</tr>
<tr>
<td>2-4</td>
<td>- 88.20</td>
<td>-0.607</td>
<td>- 86.18</td>
</tr>
<tr>
<td>4-2</td>
<td>- 19.44</td>
<td>-0.607</td>
<td>- 21.46</td>
</tr>
<tr>
<td>4-6</td>
<td>- 92.15</td>
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<td>6-4</td>
<td>- 45.99</td>
<td>+1.186</td>
<td>- 45.99</td>
</tr>
</tbody>
</table>

### ANALYSIS BY MOMENT DISTRIBUTION

The factors required in the moment-distribution procedure have been presented in Tables 1 and 2. Some of the necessary computations in the following have already been made under the heading "Analysis by Slope-Deflection Method":

#### Semi-Fixed End Moment at Joint Center.

\[
M_{SAB} = \frac{221.0}{1.7877} + 6.5 \times 4 = 149.62 \text{ in-kips.}
\]

#### Carry-Over Factors Between Joint Centers.

- **Beams (Both Directions).**

\[
\tilde{r}_{ab} = \tilde{r}_{ba} = \frac{1 + 6 (1.7877) \frac{4}{160} + 6 (1.7877) \frac{4^2}{160^2}}{2 + 3 (0.7877) + 6 (1.7877) \frac{4}{160} + 6 (1.7877) \frac{4^2}{160^2}} = 0.275.
\]

- **Second-Story Columns (Both Directions).**

\[
r = \frac{1 + 6 \left( \frac{3.33}{113.33} \right) + 6 \left( \frac{3.33^2}{113.33^2} \right)}{2 + 6 \left( \frac{3.33}{113.33} \right) + 6 \left( \frac{3.33^2}{113.33^2} \right)} = 0.542.
\]

- **First-Story Columns (Top to Bottom) (See Table 1).**

\[
r = \frac{1 + 3 \left( \frac{3.33}{116.67} \right)}{2 + 6 \left( \frac{3.33}{116.67} \right) + 6 \left( \frac{3.33}{116.67} \right)^2} = 0.499.
\]
End Rotation Stiffness at Joint Centers.—

Beams (End Bay).—

\[ S_M = \frac{2 (3,550) \times 10^6}{160} \left[ \frac{2 + 3 (0.7877)}{1.7877} + 6 \left( \frac{4}{160} \right) + 6 \left( \frac{4^2}{160^2} \right) \right] = 34.2 \times 10^6. \]

Beams (Modified Stiffness in Center Bay Due to Symmetry Requiring Analysis of Only One Half of Frame—Eq. 14c).—

\[ S_M = \frac{2 (3,550) \times 10^6}{160 (1 + 0.7877)} = 24.8 \times 10^6. \]

Second-Story Columns.—

\[ S_M = \frac{2 (3,321) \times 10^6}{113.33} \left[ 2 + 6 \left( \frac{3.33}{113.33} \right) + 6 \left( \frac{3.33^2}{113.33^2} \right) \right] = 127.9 \times 10^6. \]

First-Story Columns (Upper End).—

\[ S_M = \frac{2 (3,321) \times 10^6}{116.67} \left[ 2 + 6 \left( \frac{3.33}{116.67} \right) + 6 \left( \frac{3.33^2}{116.67^2} \right) \right] = 123.9 \times 10^6. \]

Proportional factors for distributing moments to ends of members are given in Table 4. The distribution may be done either directly on a diagram of the frame in the manner frequently followed or in tabular form. The solution is herein presented in tabular form (see Fig. 10), through five cycles after the initial distribution. Each cycle consists successively of: (a) The carry-over of moments from the previously distributed moments; and (b) the distribution of the new unbalanced moment at each joint to the ends of the members. The final summation of moments may be compared with the results of the solution by the slope-deflection method, and the results are seen to check with a maximum error of two in the third significant figure, or a fraction of 1%, except in the case of the smallest moment of 1.07 in-kips.
when the error is about 2%. The moments resulting from the distribution procedure are hypothetical moments at the joint center, and the actual moments at the connection may be found by the method previously described.

### Table 5

<table>
<thead>
<tr>
<th>Location</th>
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<tbody>
<tr>
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<tr>
<td>1.2</td>
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### Location Factors

<table>
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</thead>
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<tr>
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### Semi-Rigid End Moments

<table>
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<th>Distribution Factors</th>
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<tr>
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<tr>
<td>3.4</td>
<td>0.047</td>
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<tr>
<td>4.3</td>
<td>0.110</td>
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<tr>
<td>4.2</td>
<td>0.041</td>
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<td>4.8</td>
<td>0.080</td>
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<tr>
<td>4.6</td>
<td>0.399</td>
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### Fixed Bases

<table>
<thead>
<tr>
<th>Carry-Over to Fixed Bases. (One Process)</th>
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<tbody>
<tr>
<td>4.6</td>
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</table>

### Moments at Column Bases

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<tr>
<th>Location</th>
<th>Distribution Factors</th>
</tr>
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<tbody>
<tr>
<td>3.5</td>
<td>0.0433</td>
</tr>
<tr>
<td>3.1</td>
<td>0.0468</td>
</tr>
<tr>
<td>3.4</td>
<td>0.0479</td>
</tr>
<tr>
<td>4.3</td>
<td>0.110</td>
</tr>
<tr>
<td>4.2</td>
<td>0.041</td>
</tr>
<tr>
<td>4.8</td>
<td>0.080</td>
</tr>
<tr>
<td>4.6</td>
<td>0.399</td>
</tr>
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</table>

### Summation

<table>
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<tr>
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<th>Carry Over to Fixed Bases. (One Process)</th>
</tr>
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<tr>
<td>3.5-3.1</td>
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<tr>
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<td>+66.88</td>
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<td>3.4-4.2</td>
<td>+1.96</td>
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<td>4.0-4.9</td>
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### Summation of Moments

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</thead>
<tbody>
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### Location Factors

<table>
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<tr>
<th>Location</th>
<th>Distribution Factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.3</td>
<td>0.0462</td>
</tr>
</tbody>
</table>

### Moments at Column Bases

<table>
<thead>
<tr>
<th>Location</th>
<th>Distribution Factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.5</td>
<td>0.0433</td>
</tr>
<tr>
<td>3.1</td>
<td>0.0468</td>
</tr>
<tr>
<td>3.4</td>
<td>0.0479</td>
</tr>
<tr>
<td>4.3</td>
<td>0.110</td>
</tr>
<tr>
<td>4.2</td>
<td>0.041</td>
</tr>
<tr>
<td>4.8</td>
<td>0.080</td>
</tr>
<tr>
<td>4.6</td>
<td>0.399</td>
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### Summation

<table>
<thead>
<tr>
<th>Summation</th>
<th>Carry Over to Fixed Bases. (One Process)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.5-3.1</td>
<td>0.0433</td>
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<tr>
<td>3.4-4.3</td>
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<td>3.1-4.4</td>
<td>+66.88</td>
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<td>3.4-4.2</td>
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<td>4.8-4.4</td>
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<tr>
<td>4.0-4.9</td>
<td>+77.60</td>
</tr>
</tbody>
</table>

### Summation of Moments

<table>
<thead>
<tr>
<th>Summation</th>
<th>Carry-Over Factor</th>
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</tr>
</thead>
<tbody>
<tr>
<td>6.4</td>
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<td></td>
</tr>
</tbody>
</table>

**Fig. 10.—Solution by Moment Distribution**

### Sidesway Induced by Unsymmetrical Vertical Loads

The writers have analyzed the frame shown in Fig. 11 by both slope deflection and moment distribution in order to study the effect of neglecting sidesway as induced by unsymmetrical vertical loads. Space does not permit the details of the analysis, which follows usual procedures, however. The dimensions of the frame and size of members are shown in Fig. 11 and the beam $a b$ is assumed to carry a uniformly distributed load of one kip per foot. As in the previous case, it was assumed that the columns were fixed at the base, but the beam-column connections were assumed to have "50% rigidity," which corresponds to $\alpha = 1$. The results are presented in Table 5.

Although no general conclusions should be drawn from this single case, it is seen that in Table 5 sidesway could have been neglected without great
error in the end moments. Sidesway due to vertical loads usually will be less in frames with semi-rigid connections as compared with the same frames rigidly connected, and will be further decreased in the actual structure by walls and concrete incasement.

**Table 5.—Sidesway Induced by Unsymmetrical Vertical Loads**

<table>
<thead>
<tr>
<th>Joint and member</th>
<th>BY MOMENT DISTRIBUTION</th>
<th>Connection Moments</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Moments at joint centers, neglecting sidesway</td>
<td>Sidesway moment to balance shear</td>
</tr>
<tr>
<td></td>
<td>b, in.</td>
<td>b, in.</td>
</tr>
<tr>
<td>a d</td>
<td>+119.02</td>
<td>-6.74</td>
</tr>
<tr>
<td>d a</td>
<td>+69.32</td>
<td>-10.53</td>
</tr>
<tr>
<td>a b</td>
<td>-119.02</td>
<td>+6.74</td>
</tr>
<tr>
<td>b a</td>
<td>+141.71</td>
<td>+5.10</td>
</tr>
<tr>
<td>b e</td>
<td>-85.61</td>
<td>-10.20</td>
</tr>
<tr>
<td>e b</td>
<td>-44.16</td>
<td>-12.25</td>
</tr>
<tr>
<td>b c</td>
<td>-53.10</td>
<td>+5.10</td>
</tr>
<tr>
<td>c b</td>
<td>-7.63</td>
<td>+6.74</td>
</tr>
<tr>
<td>c f</td>
<td>-7.63</td>
<td>+6.74</td>
</tr>
<tr>
<td>c e</td>
<td>+5.80</td>
<td>-10.53</td>
</tr>
</tbody>
</table>

**Effect of Width of Member Upon Analysis**

In the analysis of frames, the length of each member is often assumed to be equal to the distance center-to-center of joints. The moments thus computed at the joint centers will usually be higher than the actual moment at the connection at the end of the member. This method of computation is usually on the safe side in determining end-connection moments but generally will be on the unsafe side in determining the maximum positive moment near the center of the beam.

An approximate correction is sometimes made for the effect of member width. The end moments computed in the foregoing manner are used to construct the moment diagram. The actual end-connection moment to be used in design is then taken as the ordinate to the moment diagram at the face of the column or connecting member. This method usually gives values of end-connection moments that are too low.

The error by either of the foregoing methods becomes greater as the ratio between the width of the joint and the length of the member increases. The
average errors are also greater for frames with semi-rigid connections than for frames with rigid connections.

In order to develop criteria to determine when, and when not, to consider member width in analysis, the behavior of the frame shown in Fig. 12 was studied for various ratios of joint width to member length. The load was assumed to act uniformly on beams 1-2 and 3-4, and the analyses were made for four different ratios of joint width to member length—namely, \( \frac{1}{15}, \frac{1}{12}, \frac{1}{9}, \) and \( \frac{1}{6} \). Analyses also were made neglecting member width entirely, and the method of arbitrary correction previously outlined was tried also. The analyses were made both for a frame with rigid connections and for a frame with continuous columns but with semi-rigid beam-to-column connections. All of the analyses were made by the method of moment distribution.

Special note should be made of the length of the columns in relation to beam depth. The methods herein presented to take account of width of joint are based on the assumption that the interior of the joint may be considered infinitely rigid in comparison with the bending stiffness of the member. In the case of a beam framing into a column, this assumption seems reasonable, particularly if the column runs through the joint without a splice. In the case of the continuous column, however, the connection moments are introduced by concentrated lateral forces acting at the top and bottom of the beam in the type of connection shown in Fig. 2(b). In such a case it may be shown that nearly correct results may be obtained for the moments in the column by assuming a length correction for the column of one third the beam depth at each end instead of one half the beam depth. This correction was made in the analyses under consideration and was found to give good results in actual frame tests.

The results of these studies are shown in Fig. 13 for the frames with rigid and semi-rigid connections, respectively. The solid lines give the percentage of error of moments determined with a neglect of joint width as compared with
corresponding moments correctly computed at the face of the joint. The broken lines give the percentage of error resulting from the arbitrary correction for joint width by neglecting it in the analysis but using the ordinate of the moment diagram at the face of the joint.

It is noted in Fig. 13 that the maximum percentage of error occurs in the case of the large end moments in the loaded beams at joints 1 and 4. It also may be seen that the errors are usually larger in the frame with semi-rigid connections than in the frame with rigid connections. The errors are appreciable even for low ratios of joint width to beam length. In the case of a one-to-twenty ratio, for example, the error may be as high as 20%, with the average error about 5% for the rigid frame and as high as 25%, and with an average error close to 10% for the frame with semi-rigid connections. As the ratio of joint width to beam length increases, the errors become increasingly larger.

A fairly close approximation for the moment at the connection is obtained by neglecting joint width in the analysis and using, as the connection moment, the moment halfway between the connection and joint center.

**Comparison Between Theoretical Analyses and Test Results**

In order to compare the results of analyses with the actual behavior of building frames, two full-size, all-welded, model building frames were con-
Fig. 14.—Comparison of Computed and
(c) FRAME 1: CRITICAL LOADING (BEAMS B2L, B1C, AND B2R)

Theoretical Analysis, Exterior Joints Assumed Equal to Interior in Rigidity

(d) FRAME 2: CRITICAL LOADING (BEAMS B2L, B1C, AND B2R)

OBSERVED MOMENTS, FOR CRITICAL CONDITIONS
structed. Details of these tests have already been presented in another paper by the writers (2).

Frame No. 1 was made with beam-to-column flange connections, whereas frame No. 2 had beam-to-column web connections. The general dimensions and size of members of frame No. 1 are shown in Fig. 8 in connection with the illustrative example (see heading "Analysis by Slope-Deflection Method"), and a photograph of the same frame is shown in Fig. 9. The beam-to-column connection used in these frames consisted of welded seat and top angles, the details and semi-rigid properties of which have been described elsewhere (2). Vertical loads were applied to the frames by means of water tanks, which are shown in Fig. 9 in one of the loading positions. Each frame was braced laterally near each joint by means of flexible ties welded between columns of the frame and columns of the laboratory. These ties had reduced sections near each end that allowed the frame full freedom to bend or move laterally in its own plane but that prevented movement out of its own plane.

The computation of the moments developed during tests of the frames was made by measuring the rotation at the ends of each beam and at the joint centers by means of the 20-in. level bar which was illustrated in Fig. 3. Then the moments at the end of each beam and column could be calculated by the slope-deflection equations (see Eqs. 3).

The connection constants for typical joints in the frame were determined by means of the setup shown in Fig. 2(b). The experimentally determined values of these connection constants, as determined by Fig. 4, have been used in the theoretical analyses. The method of moment distribution was used and a typical analysis, taking account of the width of member, has been presented in the illustrative example.

Fig. 14 shows both the computed and experimentally determined moments for several of the critical conditions of load that were applied to the two frames. Fig. 14(a) shows moment diagrams for frame No. 1 with only first-floor beam loaded. Fig. 14(b) is for frame No. 1 with unsymmetrical loading in which only one outside second-story beam was loaded. Sidesway was neglected in the analysis but the agreement between analysis and experimental result is excellent. A comparison is made in this case with an analysis assuming completely rigid joints. The actual test results agree well with the analysis for semi-rigid joints but are widely divergent from the analysis for rigid points. It should be noted that the moments "taper out" much more rapidly in a frame with semi-rigid connections than in one with rigid joints. Fig. 14(c) is for a critical condition of loading. In applying the test load for this case, the order of loading was purposely unbalanced but the moments by test are in fairly good agreement with the theoretical analysis. Fig. 14(d) is for frame No. 2, with beam-to-column web connections, and is for the same critical loading condition as Fig. 14(c). The outside column connections in frame No. 2 has less rigidity than the inside, and this was taken into account in the analysis. The analysis based on the assumption that the outside joints are as rigid as
the interior joints is also given, and it is seen that the test results usually fall between the two different analyses.

In general, the test results agree well with the methods of analysis which have been presented. The results also show that the test of a single joint to determine the connection constant gives a satisfactory measure of the behavior of the same type of joint used in an actual frame.

The Design of Frames for Partial Rigidity

The methods of analysis which have been presented in this paper obviously are not directly applicable to design. Any method of statically indeterminate analysis requires an assumed structure as a preliminary to design. To assume a building design, and then to analyze such a highly redundant structure by the methods that have been presented, would be an impractical design procedure, warranted only for very special problems.

For routine building design, a suitable method must be direct and simple in application. Such design methods have been presented by the writers in conjunction with a particular type of all-welded, beam-to-column connections (2), and in a more general article covering the application to any semi-rigidly connected structure (3). The British Steel Structures Research Committee (1) has developed design procedures for frames with semi-rigid riveted connections. In a letter dated December 26, 1940, S. D. Lash, secretary of the Subcommittee on Steel Construction, National Building Code, National Research Council of Canada, stated that simplifications in the original design method have been made in Great Britain and that similar steps are in progress in Canada.

The design procedure for the beams in welded building frames with semi-rigid connections that has been developed by the writers (2) may be outlined as follows:

1. The beams are designed by the usual procedure of computing the required section modulus for maximum simple beam moment.
2. The section modulus for maximum simple beam moment is multiplied by a reduction factor that depends on the distribution of load and relative stiffness of the simple beam and adjacent column sections. This reduction factor is obtained from a graph or simple formula and is based on the most critical combination of load possible.
3. The final beam selection is determined by the reduced section modulus found by step 2.

Although the method was developed for designing beams with welded connections, it is applicable to any frame having connections with the desired semi-rigid properties.

In computing the reduction factor, the stiffening effect of adjacent beams was neglected, and the same formula applies to exterior and interior bays. By this procedure, with end connections designed for 50% end restraint, an
average saving in the weight of beams of between 15% and 20% was found possible. If greater refinement and complexity are introduced into the design procedure, the average saving in weight of beams might be raised to more than 20%.

CONCLUSION

The methods presented and corroborated by test in this paper represent a refinement in the analysis of building frames. It may be questioned whether such refinement is warranted. The concrete encasement of beams, columns, walls, and partitions, and the uncertainties of applied load, all represent indeterminate quantities which, undoubtedly, may have as great, or greater, effect upon frame behavior as does the semi-rigidity of the bare steel connection. Nevertheless, discounting these uncertainties as assets that cannot be counted upon definitely, there remains the certain dependable bare connection end-restraint. This influence can be determined and applied to the development of improved and more economical methods of design.

ACKNOWLEDGMENT

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APPENDIX

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