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Analyzing multiple internal rates of return

Ingrid Christine Schafrick
Lehigh University

Schafrick, Ingrid Christine

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Ingrid Christine Schafrick

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Abstract

In this paper we present a new method for determining project acceptability when multiple internal rates of return are present. An internal rate of return is an interest rate which causes the present worth of a cash flow stream to equal zero. Many internal rates of return can occur when there are both cash inflows and outflows in a single cash flow stream. Unfortunately, this makes the project selection problem difficult. Lorie and Savage initially present this problem in their seminal paper in 1955. Since its publication, several authors have produced solutions to this issue. In this paper, we present a new method for determining suitability when multiple rates are calculated. This method is unique because it considers the minimum attractive rate of return when determining which internal rate of return should be used to determine project acceptability. We illustrate that this method can be used whenever there exists at least one real internal rate of return and show that it is consistent with present worth.

Chapter 1

Introduction

1.1 Project Acceptability

In today's corporate world, one goal of executives is to maximize the return on investment of capital projects. To do this, investors must determine the potential for growth of their capital investments. If the potential is adequate, the project is suitable. Currently, there are several methods that management uses to determine whether a project is acceptable. The most popular analysis tools are present worth, internal rate of return (IRR), and for non-profit firms, cost-benefit analysis.

To understand these methods, we must first define two rates, the interest rate, *i*, and the minimum attractive rate of return (MARR). The "interest rate, or the rate of capital growth, is the rate of gain received from an investment." [9, p. 28] The minimum attractive rate of return is a "cut-off rate representing a yield on investments that is considered minimally acceptable." [9, p. 199] It is also known as the "cutoff rate, hurdle rate, and marginal growth rate (MGR)." [7, p. 303] We will refer to these two rates throughout the paper.

1.2 Analysis Tools

The present worth is a "net equivalent amount at the present that represents the difference between the equivalent disbursements and the equivalent receipts of an investment's cash flow for a selected interest rate." [9, p. 153] It is an absolute measure of worth, which is used in total investment analysis, as projects do not have to be analyzed incrementally. The present worth, with the interest rate in the interval (–1, ∞), x_n the cash flow at time period n, and n ranging from zero to infinity, is defined as:

$$PW(i) = x_0 + \frac{x_1}{(1+i)} + \ldots + \frac{x_n}{(1+i)^n}$$

The decision rules are rather simple:

If $PW(i) > 0$ Accept the project

If $PW(i) = 0$ The project is indifferent

If $PW(i) < 0$ Reject the project

Since present worth is easy to calculate and the decision rules are simple to follow, it is appealing as an investment criterion. There are also two other features that make it attractive. The first is that it considers the time value of money with respect to the interest rate. Second, it determines a single unique value of worth for each interest rate. [9, p. 153] These three characteristics make the present worth attractive when determining project acceptability.

The internal rate of return is the interest rate at which the present worth of a stream of cash flows is equal to zero. In economic terms, the IRR "is the interest rate earned on the unrecovered balance over an investment's life so that the unrecovered balance at the end of that time is zero." [9, p. 161] Thus, it is the rate of return of the investment, often viewed as a measure of efficiency. It is defined as follows, where x_t is the cash flow at time *t* and *i** is the interest rate, which makes the following statement true:

$$PW(i) = \sum_{t=0}^{n} \frac{x_t}{(1+i^*)^t} \equiv 0$$

In order for a firm to accept an investment decision, the IRR must be greater than the minimum attractive rate of return, and to accept a borrowing decision the IRR must be less than the MARR. Considering the economic meaning of the internal rate of return, it is also popular when determining project suitability.

1.3 Issues with the Internal Rate of Return

This paper will focus on the internal rate of return as a measure of product acceptability. However, when using the internal rate of return as a tool for analysis, several issues need to be mentioned before progressing. The first concern occurs when mutually exclusive projects are ranked based on their internal rate of return. The trouble arises when the internal rate of return provides a different conclusion than ranking based on present worth. Lohmann [6] explains, in order to arrive at the correct rank, it must be assumed "that capital that remains invested (or borrowed) in an opportunity grows at the IRR and cash released by a decision about the opportunity grows at the MGR." [7, p. 304] The common mistake that is made when using the IRR is that investors assume that the cash released from an investment will continue to grow at the IRR rather than the marginal growth rate. When this assumption is made, an incorrect rank may result. Another issue is what to do when no IRR exists. This commonly occurs when a cash flow is made up of either all receipts or all disbursements. [9, p. 161] The IRR does not exist because for no real interest rate does the present worth, as a function of the interest rate, equal zero. Therefore, since no IRR exists, it cannot be used to determine project acceptability. Finally, the last issue with the IRR is that multiple internal rates of return may exist, clouding the decision of determining which IRR is applicable, if any at all. The following authors have looked at this scenario in depth: Lorie and Savage [6], Teichroew, Robichek, and

Montalbano [8], Cannaday, Colwell, and Paley [3], Hajdasinski [4], and Hazen [5]. Each has created a different solution to the multiple internal rates of return scenario. The challenge of determining which, if not all internal rates of return is pertinent, is the focus of this paper.

1.4 Literature Review

Since the mid 1950's when Lorie and Savage published their paper, "Three Problems in Rationing Capital" in *The Journal of Business*, the quest to determine whether to accept a project or not when multiple internal rates of return occur began. In their seminal work, Lorie and Savage point out for multiple internal rates of return to occur, there must be "initial cash outlays, subsequent net cash inflows, and final cash outlays." [6, p. 237] To explain why this happens, Lorie and Savage refer to an investment in an oil pump. Initially, there must be cash outlays to purchase the pump to extract oil; while there is oil to extract cash inflows take place. However, once the oil is gone, cash outlays take place to remove the oil pump and clean up the location. Since there are many internal rates of return, it is unclear which IRR or if all the internal rates of return should be used to determine whether to invest in the oil pump. Therefore, based on this scenario, Lorie and Savage claim, "the rate-of-return criterion for judging the acceptability of investment proposals is ambiguous or anomalous." [6,

p. 237] Since the publication of this paper, many have published solutions regarding the use of the internal rate of return when many occur.

Teichroew, Robichek, and Montalbano [8] define the different types of projects that exist and which will expect to see multiple internal rates of return. In their paper, they also explain an approach for determining whether to accept or reject a project that contains many internal rates of return. Their method includes the following steps.

(1) Calculate the internal rates of return and choose the largest IRR.

(2) Solve for the future worth at each period using the largest internal rate of return.

(3) Choose the least negative/most positive value of future worth.

(4) Truncate the future worth function to the period with the least negative/most positive future worth.

(5) Solve for i_{min} using the truncated cash flow set equal to zero.

(6) Depending on whether the initial cash flow is positive or negative, determine whether k_{min} or r_{min} exists.

(7) Use the chart given by Teichroew, Robichek, and Montalbano to determine whether to invest in the project or not. [8, pp. 163-173]

This is the first adopted procedure for determining project acceptability, which is consistent with present worth, [2, p. 200] but it is rather long and complicated.

Cannaday, Colwell, and Paley [3] describe a technique that determines which, if any, of the many internal rates of return are relevant. Their method is used only for investment streams and is based on calculating the internal rates of return and adding one to each rate, creating x* values. Then they take the first derivative of the future worth function and plug in each value of x*. If the solution from the derivative is less than zero and the internal rate of return is greater than negative one, it is considered a relevant root. [3, pp. 21-27] Hajdasinski [4] extended the method of Cannaday, Colwell, and Paley, to the case of a borrowing stream. Unfortunately, these techniques cannot deal with cash flows with duplicate roots, multiple relevant roots, [3, p. 32] or no real zeros. [4, p. 351] They also cannot function when the interest rate varies with time and when the MARR falls outside the range of real internal rates of return. [4, pp. 351-352]

Hazen [5] recently published a method for determining whether to accept or reject a project with multiple internal rates of return, which is consistent with results from present worth. What is different about Hazen's method is that he can use any internal rate of return and each rate provides the same conclusion. To reach these conclusions the cash flow stream needs to be transformed into an investment stream. The present value needs to be recalculated and the investment stream needs to be classified as borrowing or investing. Finally, the minimum attractive rate of return and the internal rate of return need to be compared to determine whether the project is acceptable. [5, pp. 40-41] This is the only method that provides the same conclusion for all of the internal rates of return and provides the same solution as the present worth.

1.5 Thesis Outline

In this paper, we present a new method for determining project acceptability when multiple internal rates of return are present. The method identifies the relevant rate of return based on the MARR and the derivative of the slope. In order to do this, we first provide definitions and notation. Second, the step-by-step procedure of the new technique is presented. Third, several unique examples demonstrate the new method. Fourth, the results of the new approach and the present worth are compared. Finally, we establish what benefits this method produces over the other recognized techniques.

Chapter 2

Definitions And Notation

There are several definitions that we need to define before presenting the new technique. We first define a **cash flow stream** as $x_0, \ldots, x_n$, where n ranges from zero to the horizon, possibly infinity. The **present worth** of a project at an interest rate i, $PW(i)$ is defined as:

$$PW(i) = x_0 + \frac{x_1}{(1+i)} + \ldots + \frac{x_n}{(1+i)^n}$$

The **future worth** of a project at an interest rate i, $FW(i)$, is defined as:

$$FW(i) = x_0(1+i)^n + x_1(1+i)^{n-1} + \ldots + x_n$$

The **internal rate of return (IRR)** is "an interest rate that makes the [cash] flow's present worth equal to zero." [2, p. 180] It is represented by i^* and symbolically denoted as follows.

$$PW(i) = \sum_{t=0}^{n} \frac{x_t}{(1+i^*)^t} \equiv 0$$

A **conventional investment**, as originally defined by Bierman and Smidt [1] is an investment that contains "one or more negative cash outflows, followed by one or more positive cash inflows." [2, p. 188] A **non-conventional investment** is an investment, which "intersperses the positive and negative cash flows." [2, p. 188] The traditional definition of a **pure investment** is one in which the firm has money invested in the project during every period. [8, pp. 155-156] The project's unrecovered or investment balances are calculated at the project's internal rate of return, *i**, and are either zero or negative throughout the project's life *n*. The firm does not borrow from the project at anytime during its life and it exactly recovers its investment at the end of the project's life, earning interest at the IRR value, *i**, in the interim periods. [2, p. 189]

$$FW_t(i^*) = \sum_{j=0}^{t} x_j (1+i^*)^{t-j} \leq 0 \qquad t = (0,1,..,n-1)$$

And

$$FW_n(i^*) = \sum_{t=0}^{n} x_t (1+i^*)^{n-t} = 0$$

The traditional definition of a **mixed investment,** at rate *i*, is an investment in which the firm has money invested in the project during some periods and the firm owes the project money during others. [8, p. 156] A mixed investment does not qualify as a pure investment. Based on the definitions, a conventional investment can only be a pure investment, while a non-conventional investment can be either a pure or a mixed investment. [2, p. 189] We do not dispute these definitions; however, we provide a different perspective. We define the slope of the *PW*(*i*) function according to its first derivative as:

$$\frac{\partial PW(i)}{\partial i} = \frac{-x_1}{(1+i)^2} + \ldots + -\frac{x_n}{(1+i)^{(n+1)}}$$

We define a project as a **pure investment** if for all *i* in $(-1,\infty)$:

$$\frac{\partial PW(i)}{\partial i} < 0$$

We define a project as **pure borrowing** if for all *i* in $(-1,\infty)$:

$$\frac{\partial PW(i)}{\partial i} > 0$$

A **mixed investment** is defined by a project having both positive and negative first derivatives over the interest rate interval (-1, ∞). A firm is considered **loaning** to a project (or project is borrowing from the firm) when the slope on the graph of the present value with respect to *i*, is decreasing. Thus, for a given interest rate *i*, a firm is loaning to a project if:

$$\frac{\partial PW(i)}{\partial i} = \frac{-x_1}{(1+i)^2} + \ldots + \frac{-x_n}{(1+i)^{(n+1)}} < 0$$

A firm is considered **borrowing** from a project (or a project is considered loaning to the firm) when the slope on the graph of the present value with respect to *i*, is increasing, or:

$$\frac{\partial PW(i)}{\partial i} = \frac{-x_1}{(1+i)^2} + \ldots + \frac{-x_n}{(1+i)^{(n+1)}} > 0$$

A **maximum or minimum point** of $PW(i)$ occurs at $\bar{i}$. At this point, a project changes from a lending to a borrowing opportunity or vice versa. The maximum or minimum occurs when:

$$\frac{\partial PW(i)}{\partial i} = \frac{-x_1}{(1+\bar{i})^2} + \ldots + \frac{-x_n}{(1+\bar{i})^{(n+1)}} = 0$$

Note that at the values of $\bar{i}$, the project is defined neither as loaning nor as borrowing. These terms will be used to explain the new method throughout the rest of this paper.

Chapter 3

Solution Procedure

The next step is to explain the new procedure for determining project acceptability. The following steps will be used to implement the new method.

1. Using the cash flow stream, define the present worth as a function of the interest rate *i*.

$$PW(i) = x_0 + \frac{x_1}{(1+i)} + \ldots + \frac{x_n}{(1+i)^n}$$

2. Solve for the internal rates of return. This can easily be done for real roots by graphing the function over the region $(-1, \infty)$ and finding the values where the function crosses the x-axis. However, graphing cannot locate the complex roots, the equation below must be

solved to determine the complex roots.

$$PW(i) = \sum_{t=0}^{n} \frac{x_t}{(1+i^*)^t} \equiv 0$$

3. Take the first derivative of the present worth function and set it equal to zero to identify all maximum and minimum points, $\bar{i}$. For *k* internal rates of return there will exist *(k-1)* optimal points.

$$\frac{\partial PV(i)}{\partial i} = \frac{-x_1}{(1+\bar{i})^2} + \ldots + \frac{-x_n}{(1+\bar{i})^{(n+1)}} = 0$$

4. Partition the graph at the optimal points, $\bar{i}$. For *k* internal rates of return there will exist *k* partitions and one IRR for each partition.

    a. The first partition is defined from $(-1, \bar{i}_1)$

    b. The last partition is defined from $(\bar{i}_{k-1}, \infty)$

    c. All other partitions are defined from $(\bar{i}_j, \bar{i}_{j+1})$ $\forall j \in (1,..,k-1)$

5. Determine the investment type of each partition:

Based on the definition of loaning, for all *i* in the partition the firm is loaning, if:

$$\frac{\partial PW(i)}{\partial i} = \frac{-x_1}{(1+i)^2} + \ldots + \frac{-x_n}{(1+i)^{(n+1)}} < 0$$

or borrowing if:

$$\frac{\partial PW(i)}{\partial i} = \frac{-x_1}{(1+i)^2} + \ldots + \frac{-x_n}{(1+i)^{(n+1)}} > 0$$

6. Locate the partition in which the minimum attractive rate of return resides. (If the MARR is equal to an $\bar{i}$, then choose a partition on either side of the MARR. Both partitions will reach the same conclusion.)

7. Compare the MARR to the IRR.

   a. If the partition is defined as loaning to a project, then:

      If the IRR > MARR Accept the project

      If the IRR = MARR The project is indifferent

      If the IRR < MARR Reject the project

   b. If the partition is defined as borrowing to a project, then:

      If the IRR < MARR Accept the project

      If the IRR = MARR The project is indifferent

      If the IRR > MARR Reject the project

Chapter 4

Illustration of Method

Now, several examples will be presented to demonstrate the new procedure. There are four pump problems; Hazen initially presented the first two. [5, p. 40] Each presents a different present worth function, which contains a unique scenario of the internal rate of return.

4.1 Pump Problem 1

The first pump problem is a cash flow stream with a present worth function that crosses the x-axis multiple times, representing multiple real roots.

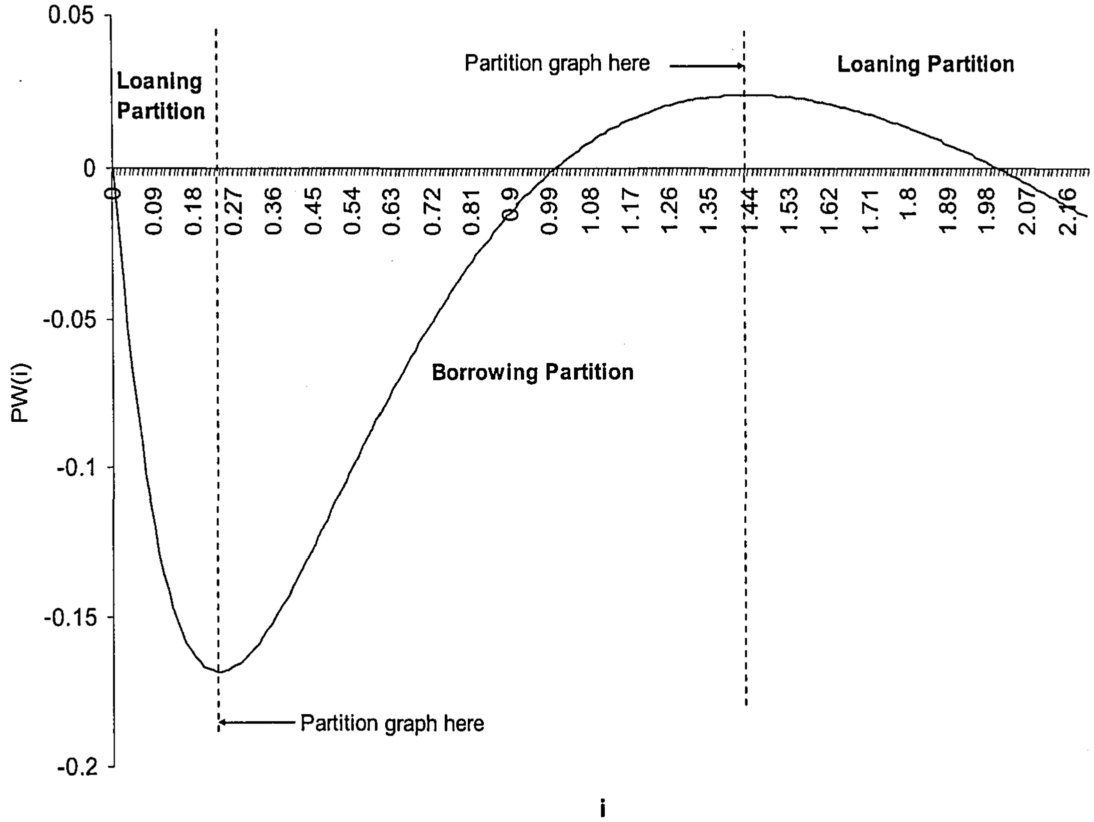

Figure 1. Present Worth Function of Pump One

The present worth function of the cash flow (-1, 6, -11, 6) is graphed in Figure 1 over $i \in (0,2.20)$. We now illustrate the decision procedure with a minimum attractive rate of return of 10%.

(1) Define the present worth function.

(2) Solve for the internal rates of return. Based on the definition of IRR, i^* = 0, 1, and 2.

(3) Determine where to partition the function based on the optimal points. From the definition of an optimal point, find $\bar{i}$ = 0.232408 and 1.4342582.

(4) From this information, establish the following partitions: (-1, 0.232408), (0.232408, 1.4342582), and (1.4342582, ∞).

(5) Based on the definition of loaning to a project, partition one and three are defined as loaning, and partition two is defined as borrowing from a project.

(6) Determine where the MARR is located. Based on our partitions the MARR of 10% is found in partition one, which is a loaning partition.

(7) In partition one, i^* = 0, and the IRR < MARR, thus we reject the project.

Note the decisions for all relevant minimum attractive rates of return in Table 1, as seen in Figure1, are consistent with present worth.

Table 1. Decision Table for Pump One

IRR	MARR	Decision
$-1 \leq i^* \leq 0.232$	IRR > MARR	Accept
$-1 \leq i^* \leq 0.232$	IRR < MARR	Reject
$-1 \leq i^* \leq 0.232$	IRR = MARR	Indifferent
$0.232 \leq i^* \leq 1.43$	IRR < MARR	Accept
$0.232 \leq i^* \leq 1.43$	IRR > MARR	Reject
$0.232 \leq i^* \leq 1.43$	IRR = MARR	Indifferent
$i^* \geq 1.43$	IRR > MARR	Accept
$i^* \geq 1.43$	IRR < MARR	Reject
$i^* \geq 1.43$	IRR = MARR	Indifferent

4.2 Pump Problem 2

In our second example, we will face the scenario when the optimal point $\bar{i}$ equals the internal rate of return i^*.

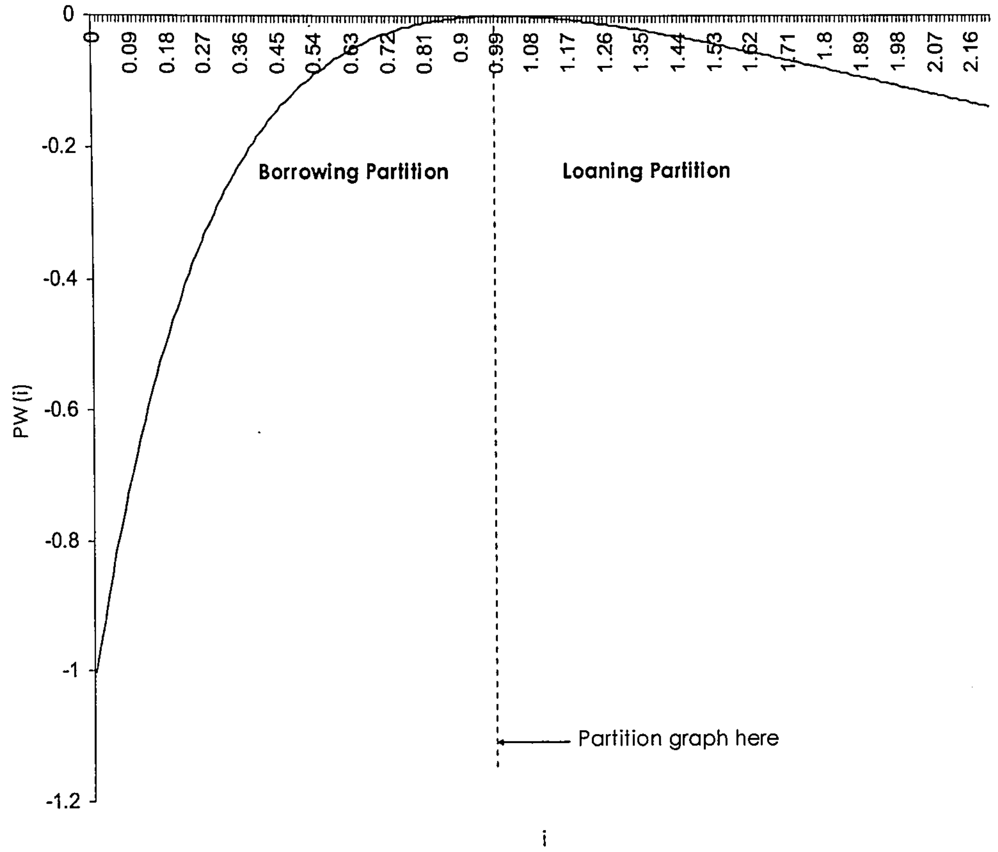

Figure 2. Present Worth Function of Pump Two

The present worth function of the cash flow (-1, 4, -4) is illustrated in Figure 2 over $i \in (0,2.4)$. The MARR remains 10%.

(1) Determine the present worth function.

(2) Find that $i^* = 1$.

(3) Find that the maximum point for this particular function is $\bar{i} = 1$.

(4) Based on $\bar{i}$ create the following partitions: (-1, 1) and (1, ∞).

(5) Define partition one as borrowing and partition two as loaning.

(6) Based on our partitions, the MARR is located in partition one, which is the borrowing partition.

(7) In partition one, $i^* = 1$ and the IRR > MARR. Since it is a borrowing partition, the project will be rejected.

Note our results in Table 2 are consistent with present worth.

Table 2. Decision Table for Pump Two

IRR	MARR	Decision
$-1 \leq i^* \leq 1$	IRR < MARR	Accept
$-1 \leq i^* \leq 1$	IRR > MARR	Reject
$-1 \leq i^* \leq 1$	IRR = MARR	Indifferent
$i^* \geq 1$	IRR > MARR	Accept
$i^* \geq 1$	IRR < MARR	Reject
$i^* \geq 1$	IRR = MARR	Indifferent

4.3 Pump Problem 3

The third pump problem deals with the issue of complex roots. This function contains both real and complex roots. There are two complex roots, followed by one real root. When this scenario occurs, we ignore the complex roots and analyze using only the real root.

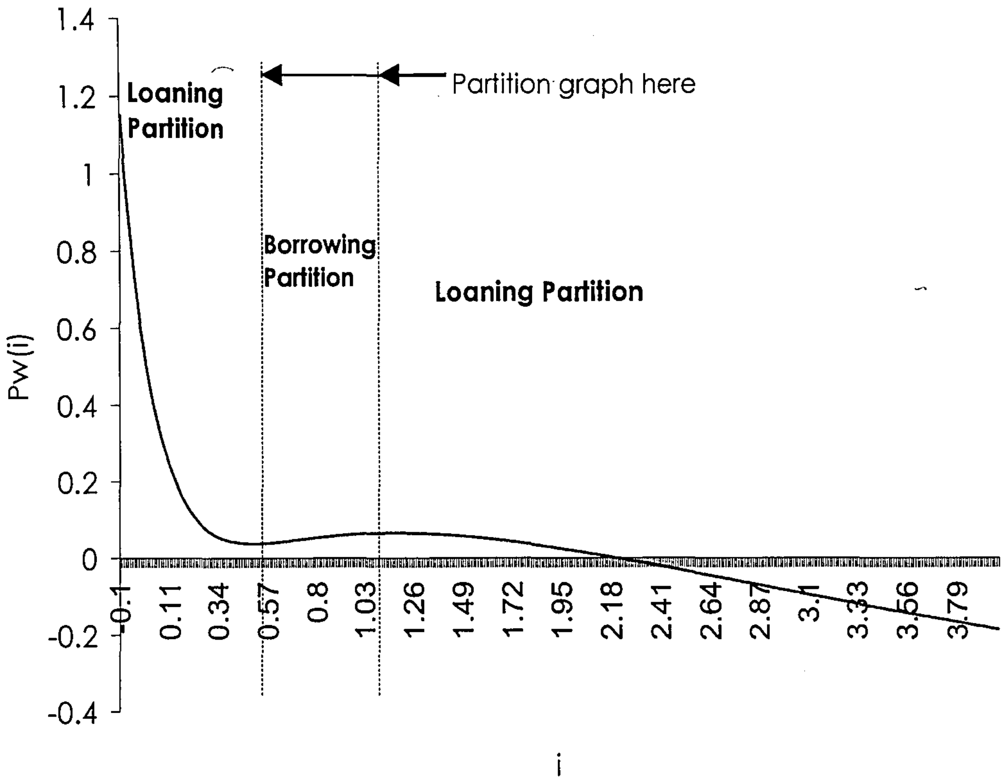

Figure 3. Present Worth Function of Pump Three

Illustrated in Figure 3 over $i \in (-0.12, 3.98)$ is the present worth function of the cash flow (-1, 6, -11, 6.5). The MARR remains at 10%.

(1) Determine the present worth function.

(2) Find that i^* = 2.19 and 4.0426 ± 0.2544i. In this case two of the three internal rates of return are complex.

When this occurs we repartition the graph to only contain partitions with real internal rates of return. Only real IRRs are used to determine project acceptability.

(3) Find that the minimum and maximum points for this particular function are $\bar{i} = 0.5$ and 1.167.

(4) Based on $\bar{i}$ create the following partitions: (-1, 0.5), (0.5, 1.167), and (1.167, ∞). Since the complex roots are found in partitions one and two, we repartition partitions one and two. Since we have only one real IRR our new partition spans the entire graph, (-1, ∞).

(5) Since our real partition was defined as loaning, our new partition (-1, ∞) is defined as loaning.

(6) The MARR will be compared to the only real IRR.

(7) $i^* = 2.19$ therefore, the IRR > MARR. Since it is a loaning partition we will accept the project.

Our results are found in Table 3 and they are consistent with present worth.

Table 3. Decision Table for Pump Three

IRR	MARR	Decision
$-1 \geq i^* \geq \infty$	IRR > MARR	Accept
$-1 \geq i^* \geq \infty$	IRR < MARR	Reject
$-1 \geq i^* \geq \infty$	IRR = MARR	Indifferent

4.4 Pump Problem 4

Like the third pump problem, the fourth example deals with complex and real roots. However in this case, there is a real root, followed by two complex roots, and then a real root. Figure 4 illustrates over $i \in (-0.3, 1.78)$, the present worth function of the cash flow (0.25, -40, 65, -1, -25, -49.5, 40). The MARR remains 10%.

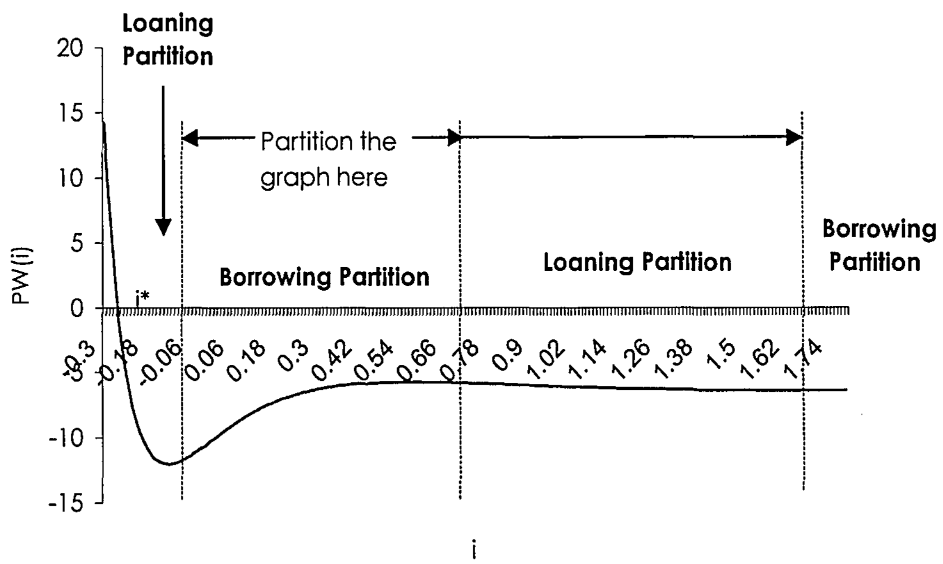

Figure 4. Present Worth Function of Pump Four

(1) Determine the present worth function.

(2) Find that $i^* = -0.2616, 0.1319 \pm 0.5393i$, and 157.36. In this case, two of the four internal rates of return are complex. Once again, we will use only the real internal rates of return to determine project acceptability.

(3) Find that the minimum and maximum points for this particular function are $\bar{i}$ = -0.1222, 0.5624, and 1.6592.

(4) Based on $\bar{i}$ create the following partitions: (-1, -0.1222), (-0.1222, 0.5624), (0.5624, 1.6592) and (1.6592, ∞). Since the complex roots are found in partitions two and three, we repartition, creating the following partitions: (-1, 0.5624) and (0.5624, ∞). In partition one, we use the first real IRR for analysis and in partition two; we use the second real IRR for analysis.

(5) Since the partition containing the first real IRR was originally defined as loaning, the new partition containing that IRR is also loaning. Therefore, the new partition one is a loaning partition. Since, the second internal rate of return was originally found in a borrowing partition, it will remain in a borrowing partition. Therefore, the new partition two will be defined as borrowing.

(6) The MARR is located in partition one.

(7) i^* = -0.2616 in partition one and the IRR < MARR. Since it is a loaning partition we will reject the project.

The following results in Table 4 are consistent with present worth.

Table 4. Decision Table for Pump Four

IRR	MARR	Decision
$-1 \leq i^* \leq 0.5624$	IRR > MARR	Accept
$-1 \leq i^* \leq 0.5624$	IRR < MARR	Reject
$-1 \leq i^* \leq 0.5624$	IRR = MARR	Indifferent
$0.5624 \leq i^* \leq \infty$	IRR < MARR	Accept
$0.5624 \leq i^* \leq \infty$	IRR > MARR	Reject
$0.5624 \leq i^* \leq \infty$	IRR = MARR	Indifferent

In the case where no real internal rates of return exist, we cannot use this method of analysis to determine project acceptability because there does not exist an IRR to compare to the MARR.

Chapter 5

Equivalence to Present Worth

5.1 Real Roots

Examining Figure 1, Table 1, Figure 2, and Table 2, it is clear that our method provides consistent results with present worth. We formalize this with the following Lemma and Theorem.

Lemma 1: There is at most one real root per partition

Proof: By definition, the slope between any two consecutive optima is monotonic. Since the sign of the slope of a function must remain constant between two successive optimal points, the function can contain at most one root.

Theorem 1: The internal rate of return is consistent with the present worth decision for any given minimum attractive rate of return.

According to the present worth decision criterion, if the $PW(MARR) > 0$, then the project should be accepted. Thus to prove consistency, if the $PW(MARR) > 0$, either $i^* < MARR$ and $\frac{\partial PW(i)}{\partial i} > 0$ or $i^* > MARR$ and $\frac{\partial PW(i)}{\partial i} < 0$ must be true.

Note that by our method, a partition is defined by either a positive or negative slope and by Lemma 1, a partition has at most one real root. By definition $PW(i^*) = 0$.

Assume $\frac{\partial PW(i)}{\partial i} > 0$. For any $i^* < i$ the present worth remains positive. Thus, if the $i^* < MARR$, the $PW(MARR) > 0$. Assume $\frac{\partial PW(i)}{\partial i} < 0$. For any $i^* > i$ the present worth remains positive. Thus, if the $i^* > MARR$, the $PW(MARR) > 0$.

According to the present worth decision criterion, if the $PW(MARR) < 0$, then the project should be rejected. Thus to prove consistency, if the $PW(MARR) > 0$, either $i^* > MARR$ and $\frac{\partial PW(i)}{\partial i} > 0$ or $i^* < MARR$ and $\frac{\partial PW(i)}{\partial i} < 0$ must be true. By definition $PW(i^*) = 0$.

Assume $\frac{\partial PW(i)}{\partial i} > 0$. For any $i^* > i$ the present worth remains negative. Thus, if the $i^* > MARR$, the $PW(MARR) < 0$. Assume $\frac{\partial PW(i)}{\partial i} < 0$.

For any $i^* < i$ the present worth remains negative. Thus, if the $i^* < MARR$, the $PW(MARR) < 0$.

When looking at a graph with a single IRR, it is understandable that the single, unique IRR is a measure of efficiency. But when multiple rates occur it can be difficult to see that all of the rates determine the efficiency of a project. In pump two, the project is initially borrowing, followed by loaning. Consider an investor looking to invest in this project, with a MARR of 20%. Partition one is a borrowing partition. Since the IRR for that partition or the rate at which capital would be invested in this project is 100%, the investor could earn more by investing in the project rather than 20%. So the investor would accept the project. If the investor has a MARR of 160% and wishes to borrow from this project, this project would be worthwhile because the capital borrowed would grow at 100% rather than at the MARR of 160%, which is a better borrowing rate. Therefore, partitioning the graph and focusing discussion on the relevant partition leads to similar intuition found with a single internal rate of return.

5.2 Complex Roots

Recall Figure 4, Table 4, Figure 5, and Table 5. When a partition contains complex roots, the new partition is consistent with the present worth decision, as it has one real root and we assume the slope does not change sign (strictly loaning or borrowing). Thus, Lemma 1 and Theorem 1 follow directly.

It is difficult to interpret complex roots. In out partitioning scheme, they are removed from analysis and we assume that a project's status (loaning or borrowing) does not change. We can say this because the presence of complex roots means the slope has changed from a change in cash flow signs (as noted by Lorie and Savage), but not significantly enough to produce a real root (cross the x-axis). Thus, complex roots seem to signify minor changes in cash flow streams, but they are not large enough to change the "status" of an investment. As with local minimum in optimization, these roots can be ignored to find the optimal solution.

Chapter 6

Conclusion

In this paper we present a new method for determining project acceptability when multiple internal rates of return are present. Like most other methods, this technique is consistent with present worth. Unlike the other methods, this method considers the minimum attractive rate of return when determining which IRR is to be used to determine project suitability. We presented four examples, each a different scenario with multiple internal rates of return. The first example presents two real internal rates of return, which is consistent with present worth. The second demonstrates when the optimal point is equal to the IRR; the solution is consistent with present worth. The third and fourth examples contain both real and complex roots. When this occurs, only the real roots are used to determine project acceptability, but the results are also consistent with present worth. Therefore, no matter what the cash flow, except when there are no real roots, this method determines project acceptability and is consistent with present worth.

In the future we hope to find a better understanding of complex roots, including a method of using complex roots in IRR analysis. We also would like to have a deeper understanding of the meaning of multiple internal rates of return. Finally, we would like to find a formal tie with the technique created by Teichroew, Robichek, and Montalbano or the method created by Hazen. Perhaps the link may lie in investment balances. These three items would be beneficial in understanding multiple internal rates of return.

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Vita

Ingrid Christine Schafrick was born on February 20, 1980 in Cleveland, Ohio to Maria and Edwin Schafrick. She grew up and attended high school in Lakewood, Ohio before moving to Bethlehem, Pennsylvania in 1998 to attend Lehigh University. She received her Bachelor of Science in Industrial Engineering from Lehigh University in May of 2002 with a minor in German. She was a member of the gymnastics club, the running club, and the cycling team during her college career. Her academic honors include Alpha Pi Mu and Tau Beta Pi. She was also on the Dean's List and a Presidential Scholar. During the fall of 2000 and the summer of 2001, she worked for IBM in Endicott, New York as an Industrial Engineering Intern. During the summer of 2002 she worked for Lutron Electronics Company Inc. in Coopersburg, Pennsylvania as an Operations Intern. After graduation she plans to move to New York City and work for Standard and Poors as a Corporate Value Consultant.

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