A numerical model for determining the three-dimensional deformations of a golf shaft during the swing

Alexia M. Brylawski

Lehigh University

Follow this and additional works at: https://preserve.lehigh.edu/etd

Part of the Mechanical Engineering Commons

Recommended Citation
Brylawski, Alexia M., "A numerical model for determining the three-dimensional deformations of a golf shaft during the swing" (1994). Theses and Dissertations. 312.
https://preserve.lehigh.edu/etd/312

This Thesis is brought to you for free and open access by Lehigh Preserve. It has been accepted for inclusion in Theses and Dissertations by an authorized administrator of Lehigh Preserve. For more information, please contact preserve@lehigh.edu.
AUTHOR: 
Brylawski, Alexia M.

TITLE: 
A Numerical Model for Determining the Three-Dimensional Deformations of a Golf Shaft During the Swing

DATE: October 9, 1994
A Numerical Model for Determining the Three-Dimensional Deformations of a Golf Shaft During the Swing

by
Alexia M. Brylawski

A Thesis
Presented to the Graduate Committee of Lehigh University in Candidacy for the Degree of Master of Science in Department of Mechanical Engineering and Mechanics

Lehigh University
Bethlehem, Pennsylvania
August, 1994
This thesis is accepted and approved in partial fulfillment of the requirement for the Master of Science.

Aug. 26, 1994

Date

Prof. Stanley H. Johnson, Ph.D.
Thesis Advisor

Chairman of Department
Acknowledgements

I would like to express my appreciation to the many people who supported me over the past few years. Special consideration goes to Professor Stanley H. Johnson for his advise, insight, and trust in my ability to solve the problem on hand. I am grateful to the U.S. Golf Association for their generous support and for extending me the opportunity to enter into the world of science and golf. Finally, a warm thank you to my family and to Jim for their friendship and good humor.
# Table of Contents

Abstract .................................................................................................................. 1

1. Introduction ......................................................................................................... 2
   1.1 Background .................................................................................................... 2
   1.2 Motivation ..................................................................................................... 3
   1.3 Outline of Thesis ............................................................................................ 4

2. Analytical Model .................................................................................................. 5
   2.1 Body-Axis Coordinate System ...................................................................... 5
   2.2 Application of Hamilton’s Principle .............................................................. 5
      2.2.1 Kinetic Energy Expression .................................................................... 8
      2.2.2 Potential Energy Expression .................................................................. 10
      2.2.3 Non-Conservative Virtual Work Expression .......................................... 12
   2.3 Hamilton’s Principle for a Golf Shaft ............................................................ 17
      2.3.1 Equations of Motion for \( z \) and \( y \) Coordinates ............................... 19
      2.3.2 Possible Boundary Conditions for \( z \) and \( y \) Coordinates .................. 19
      2.3.3 Equation of Motion for \( \theta \) Coordinate ............................................... 20
      2.3.4 Possible Boundary Conditions for \( \theta \) Coordinate ............................... 20
   2.4 Modifications Applied to the Equations of Motion and Boundary Conditions .................................................. 20
   2.5 Specifying the Boundary Conditions .............................................................. 21
   2.6 Simplified Equations of Motion for \( z \) and \( y \) Coordinates .......................... 22
   2.7 Simplified Boundary Conditions for \( z \) and \( y \) Coordinates ....................... 22
   2.8 Simplified Equation of Motion for \( \theta \) Coordinate ....................................... 23
   2.9 Simplified Boundary Conditions for \( \theta \) Coordinate ..................................... 23

3. Kinetic Analysis of Torques and Forces Acting on a Golf Shaft ....................... 24
   3.1 Experimental Equipment .............................................................................. 24
   3.2 Body-Axis Unit Vectors ................................................................................ 27
   3.3 Rotation Matrix, Angular Velocity and Acceleration of the Shaft ............... 29
List of Figures

2.1 Body-Axis Coordinate System ......................................................... 6
2.2 Change in Length $ds - dz$ ................................................................. 14
3.1 Position of Reflective Markers Relative to the Inertial Coordinate System 26
3.2 Body-Axis Unit Vectors in Inertial Coordinates .............................. 28
3.3 Club Head in Dynamic Equilibrium .................................................. 32
3.4 Reaction at End of Shaft ................................................................. 37
5.1 Outer and Inner Diameter vs. Shaft Length ...................................... 50
5.2 Position of Marker 1 vs. Time in Inertial Coordinates .................... 53
5.3 Position of Marker 2 vs. Time in Inertial Coordinates .................... 54
5.4 Position of Marker 3 vs. Time in Inertial Coordinates .................... 55
5.5 Position of Marker 4 vs. Time in Inertial Coordinates .................... 56
5.6 Damping of Gripped Steel Shaft ...................................................... 57
5.7 Force Acting on End of Shaft vs. Time in Body-Axis Coordinates ....... 59
5.8 Moment Acting on End of Shaft vs. Time in Body-Axis Coordinates .... 60
5.9 Deflection in the $x$ Direction vs. Time .......................................... 62
5.10 Deflection in the $y$ Direction vs. Time .......................................... 63
5.11 Deflection in the $\theta$ Direction vs. Time ...................................... 64
6.1 Velocity of Marker 4 vs. Time in Inertial Coordinates .................. 66
6.2 Acceleration of Marker 4 vs. Time in Inertial Coordinates ............... 67
Abstract

A model that calculates the deflections of a golf shaft during the swing was developed. Applying Lagrangian dynamics and Hamilton's Principle to a shaft that is flexible and has varying cross-sectional geometry, the equations of motion and boundary conditions that characterize its behavior over the duration of the swing were generated. Incorporated into the boundary conditions are the forces and torques acting at the end of the shaft. These loads were calculated using a kinematic analysis of the motion of the golf club during the swing. A motion analysis system by Motion Analysis Corporation was utilized to collect the necessary data.

Solutions to the equations of motion for the shaft were calculated numerically by the Numerical Method of Lines (NUMOL) procedure. The results are presented as deflections along the shaft as functions of time. Bending deflections are determined in the plane of the club face and normal to the plane of the club face, and torsional deflections about the center-line of the shaft are also calculated.
Chapter 1
Introduction

1.1 Background

Although the game of golf has been played for hundreds of years, only in the past twenty-five have attempts been made to model the golf swing mathematically. Interest in modeling began with the publication of The Search for the Perfect Swing by Cochran and Stobbs in 1968 in which a comprehensive analysis of the components that classify a 'perfect swing' were presented [1]. Since that time numerous studies have been published which describe models used to analyze the golf swing.

Most of the earlier mathematical models made the assumption that the golf swing occurs in a fixed plane, called the swing plane [2,3,4,5,6]. The golfer's arm and the shaft were modeled by a double pendulum with its motion confined to the swing plane. These models were primarily used to calculate the shoulder and wrist torques generated by the golfer. High speed stroboscopic photography was the most common method used for the collection of data from which the angular velocity and acceleration of the arm and shaft system were calculated.

As technological advances allowed for high speed film or video to be used in conjunction with computers, visual images could be digitized and defined by three-dimensional vectors. This permitted the three-dimensional forces and torques applied to the club to be calculated and the assumption that the golf swing was confined to a plane was no longer applicable [7,8].
The next step in the evolution of the model of the golf swing was to account for the flexibility of the shaft by designating the lower link (the shaft) of the double pendulum model as flexible [9]. Previous models had assumed the shaft was a rigid body [2 through 8] because their primary goal was to calculate the torques produced by the human body. Although the torques calculated in [9] contained corrections to account for the out-of-plane forces, the model once again confined the golf swing to the swing plane.

Recently the deflections in the shaft were measured experimentally using strain-gage technology [10,11]. Although the behavior of the shaft can be studied by analyzing the experimental data obtained by direct measurements, the question of what forces produce these deflections remains unanswered.

1.2 Motivation

The work presented in this thesis was carried out to enhance the current level of understanding of the dynamic behavior of a golf shaft during the swing. The aim is to develop a mathematical model that completely describes the deflections occurring in the golf shaft during the swing due to three-dimensional force and torque input that is calculated from experimentally obtained data. The analytical model will allow for detailed study of the affects that force and torque inputs and geometric and material parameters have on the deflection of the shaft. Once a thorough understanding of the shaft dynamics is acquired, the ultimate objective is for this information to be used to develop a quick and simple procedure of matching shaft performance with a golfer's particular needs.
1.3 Outline of Thesis

This thesis is comprised of three components: the analytical, the numerical, and the experimental sections. Chapter two follows the development of the equations of motion and boundary conditions using Lagrangian dynamics for a continuous body. These equations characterize the deflections of the shaft. In chapter three a kinetic analysis of the three-dimensional forces and torques acting on the shaft is outlined. Chapter four describes the Numerical Method of Lines (NUMOL) approach used to solve the differential equations of motion and in chapter five the solution obtained by analyzing the swing of a steel-shafted driver is presented. Lastly, a discussion of the findings of this project is presented in chapter 6, followed by relevant conclusions and suggestions for further research.
Chapter 2
Analytical Model

2.1 Body-Axis Coordinate System

Deformation in the shaft of a golf club may be fully described by a bending deflection perpendicular to the central axis of the shaft and a torsional deflection about the same axis. Both deflections are functions of time and position along the shaft. For this model it is assumed that the longitudinal deformation is negligible because elongation of the shaft is so small as to not change the orientation of the club-face with the ball at impact.

Figure 2.1 depicts a coordinate system fixed to the shaft with its origin located at the grip. This is called the body-axis coordinate system. The deflections are defined with respect to this system with the $z$ direction along the central axis of the shaft, the $x$ direction normal to the shaft and in the plane of the club face, and the $y$ direction completing the right-hand orthogonal system. Torsional deflection, $\theta$, is measured about the $z$-axis in a counter-clockwise direction relative to the $z$-$z$ plane. The total bending deflection of the shaft is described in terms of $x$ and $y$ components.

2.2 Application of Hamilton's Principle

The approach used to formulate the equations of motion is discussed in Analytical Methods in Vibrations by Leonard Meirovitch [12]. The extended
Figure 2.1: Body-Axis Coordinate System
Hamilton’s Principle is applied to develop the equations of motion and boundary conditions that specify the deflections in the $x$, $y$, and $\theta$ directions of the golf shaft. Expressed in Equation (2.1), the principle incorporates the variation of the kinetic energy and virtual work of the shaft at any instant in time [13].

$$
\int_{t_1}^{t_2} (\delta T + \delta W) \, dt = 0
$$

(2.1)

Hamilton’s Principle can be expanded by recognizing that the virtual work may be separated into conservative and non-conservative components, as shown in Equations (2.2) [14]. The conservative term is derived from the potential energy of the shaft while the non-conservative term is due to distributed and point forces applied to the shaft.

$$
\delta W = -\delta V + \delta W_{nc}
$$

(2.2)

Rewriting Equation (2.1) in terms of the kinetic energy, potential energy, and the work due to non-conservative forces acting on the system produces the following expression.

$$
\int_{t_1}^{t_2} \delta T(t) \, dt - \int_{t_1}^{t_2} \delta V(t) \, dt + \int_{t_1}^{t_2} \delta W_{nc}(t) \, dt = 0
$$

(2.3)

When formulating the energy expression for a golf shaft it is best to examine each term of Equation (2.3) separately. The same procedure is followed to obtain the final form of each expression: first generate an expression for the total kinetic energy, potential energy, or non-conservative work; then take the
variation of this term; lastly expand the resulting expression and simplify. For all cases the effects of rotary inertia about the transverse axis and transverse shear are neglected due to the small cross-sectional dimensions of the shaft relative to its length [15].

### 2.2.1 Kinetic Energy Expression

The total kinetic energy of the shaft is the sum the kinetic energy generated by motion of the shaft in the \(x, y, \) and \(\theta\) directions.

\[
T(t) = \frac{1}{2} \int_0^L m_z \left( \frac{\partial x(z,t)}{\partial t} \right)^2 dz + \frac{1}{2} \int_0^L m_y \left( \frac{\partial y(z,t)}{\partial t} \right)^2 dz + \frac{1}{2} \int_0^L I_m(z) \left( \frac{\partial \theta(z,t)}{\partial t} \right)^2 dz
\]  

(2.4)

Where,

- \(T\) - Total kinetic energy
- \(t\) - Time
- \(m_z\) - Mass per unit length
- \(I_m\) - Mass moment of inertia
- \(L\) - Length of shaft

Assuming that \(\delta\) and \(\frac{\partial}{\partial t}\) are commutative operations, the variation of the kinetic energy of the system is presented in Equation (2.5).

\[
\delta T(t) = \int_0^L m_z \frac{\partial x(z,t)}{\partial t} \frac{\partial}{\partial t} (\delta x(z,t)) \, dz + \int_0^L m_y \frac{\partial y(z,t)}{\partial t} \frac{\partial}{\partial t} (\delta y(z,t)) \, dz +
\]

\[
\int_0^L I_m(z) \frac{\partial \theta(z,t)}{\partial t} \frac{\partial}{\partial t} (\delta \theta(z,t)) \, dz
\]

(2.5)

Consequently, the first term of Equation (2.3) becomes
\[
\int_{t_1}^{t_2} \delta T(t) \, dt = \int_{t_1}^{t_2} \int_0^L \left\{ m_z \frac{\partial x(z,t)}{\partial t} \frac{\partial}{\partial t} (\delta x(z,t)) + m_z \frac{\partial y(z,t)}{\partial t} \frac{\partial}{\partial t} (\delta y(z,t)) + \right. \\
I_{m(z)} \frac{\partial}{\partial t} \frac{\partial}{\partial t} (\delta \theta(z,t)) \right\} \, dz \, dt
\]

(2.6)

This equation can be expanded and simplified utilizing integration by parts with respect to time.

\[
\int_{t_1}^{t_2} \delta T(t) \, dt = \int_{t_1}^{t_2} \int_0^L \left\{ m_z \frac{\partial x(z,t)}{\partial t} \delta x(z,t) \right\}_{t_1}^{t_2} + m_z \frac{\partial y(z,t)}{\partial t} \delta y(z,t) \right\}_{t_1}^{t_2} + \\
I_{m(z)} \frac{\partial}{\partial t} \frac{\partial}{\partial t} (\delta \theta(z,t)) \left[_{t_1}^{t_2} \right] - \int_{t_1}^{t_2} \left( \frac{\partial}{\partial t} \left( m_z \frac{\partial x(z,t)}{\partial t} \right) \right) \delta x(z,t) \, dt + \\
\frac{\partial}{\partial t} \left( m_z \frac{\partial y(z,t)}{\partial t} \right) \delta y(z,t) \, dt + \frac{\partial}{\partial t} \left( I_{m(z)} \frac{\partial \theta(z,t)}{\partial t} \right) \delta \theta(z,t) \right\} \, dz
\]

(2.7)

By definition the variations \( \delta x \), \( \delta y \), and \( \delta \theta \) have values of zero at times \( t_1 \) and \( t_2 \). This results in the first three terms of Equation (2.7) vanishing from the expression. The final form of the kinetic energy term of Hamilton’s Principle is stated below.

\[
\int_{t_1}^{t_2} \delta T(t) \, dt = - \int_{t_1}^{t_2} \int_0^L \left\{ m_z \frac{\partial^2 x(z,t)}{\partial t^2} \delta x(z,t) + m_z \frac{\partial^2 y(z,t)}{\partial t^2} \delta y(z,t) + \\
I_{m(z)} \frac{\partial^2 \theta(z,t)}{\partial t^2} \delta \theta(z,t) \right\} \, dz \, dt
\]

(2.8)
2.2.2 Potential Energy Expression

Next, the total potential energy of the system is formulated. Equation (2.9) is composed of the components of the potential energy terms describing the elastic strain energy of the shaft.

\[
V(t) = \frac{1}{2} \int_0^L EI_A(z) \left( \frac{\partial^2 x(z,t)}{\partial z^2} \right)^2 dz + \frac{1}{2} \int_0^L EI_A(z) \left( \frac{\partial^2 y(z,t)}{\partial z^2} \right)^2 dz + \frac{1}{2} \int_0^L GJ(z) \left( \frac{\partial \theta(z,t)}{\partial z} \right)^2 dz \tag{2.9}
\]

Where,

- \( E \) - Modulus of elasticity
- \( G \) - Shear modulus
- \( V \) - Total potential energy
- \( I_A \) - Area moment of inertia
- \( J^* \) - Polar moment of inertia

Again, assuming \( \delta \) and \( \frac{\partial}{\partial z} \) are interchangeable in order of application, the variation of the total potential energy becomes

\[
\delta V(t) = \int_0^L EI_A(z) \frac{\partial^2 x(z,t)}{\partial z^2} \frac{\partial}{\partial z} \left( \delta \left( \frac{\partial x(z,t)}{\partial z} \right) \right) dz + \int_0^L EI_A(z) \frac{\partial^2 y(z,t)}{\partial z^2} \frac{\partial}{\partial z} \left( \delta \left( \frac{\partial y(z,t)}{\partial z} \right) \right) dz + \int_0^L GJ(z) \frac{\partial \theta(z,t)}{\partial z} \frac{\partial}{\partial z} \left( \delta \theta(z,t) \right) dz \tag{2.10}
\]

Integrating the above expression with respect to time yields the second term of Equation (2.3).
\[
\int_{t_1}^{t_2} \delta V(t) \, dt = \int_{t_1}^{t_2} \int_0^L \left\{ EI_A(z) \frac{\partial^2 x(z,t)}{\partial z^2} \frac{\partial}{\partial z} \left( \delta \left( \frac{\partial x(z,t)}{\partial z} \right) \right) + 
\]

\[
EI_A(z,t) \frac{\partial^2 y(z,t)}{\partial z^2} \frac{\partial}{\partial z} \left( \delta \left( \frac{\partial y(z,t)}{\partial z} \right) \right) + 
\]

\[
GJ(z) \frac{\partial \theta(z,t)}{\partial z} \left( \frac{\partial \theta(z,t)}{\partial z} \right) \right\} \, dz \, dt \quad \cdots (2.11)
\]

Expanding Equation (2.11) by employing integration by parts with respect to \( z \) yields

\[
\int_{t_1}^{t_2} \delta V(t) \, dt = \int_{t_1}^{t_2} \int_0^L \left\{ EI_A(z) \frac{\partial^2 x(z,t)}{\partial z^2} \delta \left( \frac{\partial x(z,t)}{\partial z} \right) \right\}_0^L + EI_A(z) \frac{\partial^2 y(z,t)}{\partial z^2} \delta \left( \frac{\partial y(z,t)}{\partial z} \right) \right\}_0^L + 
\]

\[
GJ(z) \frac{\partial \theta(z,t)}{\partial z} \delta \theta(z,t) \left. \right|_0^L - \int_0^L \frac{\partial}{\partial z} \left[ EI_A(z) \frac{\partial^2 x(z,t)}{\partial z^2} \right] \frac{\partial}{\partial z} \left( \delta x(z,t) \right) \, dz + 
\]

\[
\int_0^L \frac{\partial}{\partial z} \left[ EI_A(z) \frac{\partial^2 y(z,t)}{\partial z^2} \right] \frac{\partial}{\partial z} \left( \delta y(z,t) \right) \, dz + 
\]

\[
\int_0^L \frac{\partial}{\partial z} \left[ GJ(z) \frac{\partial \theta(z,t)}{\partial z} \right] \delta \theta(z,t) \, dz \right\} \, dt \quad (2.12)
\]

To further expand the above equation, repeat the integration by parts for the forth and fifth terms. The final form of the potential energy term of Hamilton's Principle becomes
\[
\int_{t_1}^{t_2} \delta V(t) \, dt = \int_{t_1}^{t_2} \left\{ EI_A(z) \frac{\partial^2 x(z,t)}{\partial z^2} \delta \left( \frac{\partial x(z,t)}{\partial z} \right)_0^L + EI_A(z) \frac{\partial^2 y(z,t)}{\partial z^2} \delta \left( \frac{\partial y(z,t)}{\partial z} \right)_0^L + 
\right. \\
GJ(z) \frac{\partial \theta(z,t)}{\partial z} \delta \theta(z,t) \left. \right|_0^L - \frac{\partial}{\partial z} \left( EI_A(z) \frac{\partial^2 x(z,t)}{\partial z^2} \right) \delta x(z,t) \left|_0^L - \right. \\
\frac{\partial}{\partial z} \left( EI_A(z) \frac{\partial^2 y(z,t)}{\partial z^2} \right) \delta y(z,t) \left|_0^L + \int_0^L \frac{\partial}{\partial z} \left[ EI_A(z) \frac{\partial^2 x(z,t)}{\partial z^2} \right] \delta x(z,t) \, dz + 
\right. \\
\int_0^L \frac{\partial}{\partial z} \left[ EI_A(z) \frac{\partial^2 y(z,t)}{\partial z^2} \right] \delta y(z,t) \, dz - \\
\int_0^L \frac{\partial}{\partial z} \left[ GJ(z) \frac{\partial \theta(z,t)}{\partial z} \right] \delta \theta(z,t) \, dz \right\} \, dt 
\] 

(2.13)

2.2.3 Non-Conservative Virtual Work Expression

The non-conservative component of the virtual work is a function of forces and moments applied at the ends of the shaft and distributed forces applied along the body. Incorporating all the sources of non-conservative work yields Equation (2.14).

\[
W_{nc}(t) = \int_0^L \left( p_x(z,t) \, x(z,t) + p_y(z,t) \, y(z,t) - p_z(z,t)(ds - dz) \right) \, dz +
\]

\[ F_x(t) \, x(z,t) \left|_0^L + F_y(t) \, y(z,t) \left|_0^L - F_z(t)(ds - dz) \right|_0^L + 
\]

\[ M_x(t) \, \alpha(z,t) \left|_0^L + M_y(t) \, \beta(z,t) \left|_0^L + M_z(t) \, \theta(z,t) \left|_0^L 
\] 

(2.14)
Where,

- $W_{nc}$ - Non-conservative work
- $p_u$ - Force per unit length in $x$, $y$, or $z$ direction
- $F_u$ - Force applied at end of shaft in $x$, $y$, or $z$ direction
- $M_u$ - Moment applied at end of shaft about $x$, $y$, or $z$ axis
- $ds$ - Differential distance along shaft
- $\beta$ - Slope $\left( \frac{\partial y}{\partial z} \right)$
- $\alpha$ - Slope $\left( \frac{\partial x}{\partial z} \right)$

Note that the work done by forces acting in the $z$ direction is negative, this is due to the forces acting opposite to the change in distance $ds - dz$.

Before taking the variation of the non-conservative work expression an examination of the change in length $ds - dz$ is in order. This difference represents the change in projection in the $z$ direction of the shaft due to bending in the $x$ and $y$ directions. As illustrated in Figure 2.2, the length $ds$ is in the direction of the shaft and the length $dz$ is in the $z$ direction.

The distance $ds$ can be approximated by the following expression [16].

$$ ds = \left[ \left( \frac{\partial x}{\partial z} \right)^2 (dz)^2 + \left( \frac{\partial y}{\partial z} \right)^2 (dz)^2 + (dz)^2 \right]^{\frac{1}{2}} \quad (2.15) $$

Let

$$ (\frac{\partial x}{\partial z})^2 = (\frac{\partial x}{\partial z})^2 + (\frac{\partial y}{\partial z})^2 \quad (2.16) $$

Substitute expression (2.16) into Equation (2.15) and subtract the quantity $dz$ from both sides of the equation to obtain the change in projection.
Figure 2.2: Change in Length $ds - dz$
\[ ds - dz = \left[ \left( \frac{\partial r}{\partial z} \right)^2 (dz)^2 + (dz)^2 \right]^{\frac{1}{2}} - dz \]  \hspace{1cm} (2.17)

Apply the binomial series expansion to the right-hand-side of Equation (2.17) and truncate after the second term [17]. Equation (2.18) is an expression for the difference \( ds - dz \) with the value for \( (\frac{\partial r}{\partial z})^2 \) expressed in \( x \) and \( y \) components.

\[ ds - dz \approx \frac{1}{2} \left[ \left( \frac{\partial x}{\partial z} \right)^2 + \left( \frac{\partial y}{\partial z} \right)^2 \right] \]  \hspace{1cm} (2.18)

Place the above approximation into Equation (2.14) to obtain the total non-conservative work in terms of changes in the \( x \), \( y \), and \( \theta \) directions only.

\[
W_{nc}(t) = \int_0^L \left\{ p_x(z,t) \frac{\partial x(z,t)}{\partial z} + p_y(z,t) \frac{\partial y(z,t)}{\partial z} - \frac{p_x(z,t)}{2} \left[ \left( \frac{\partial x(z,t)}{\partial z} \right)^2 + \left( \frac{\partial y(z,t)}{\partial z} \right)^2 \right] \right\} dz + \\
F_x(t) \left. \left( \frac{\partial x(z,t)}{\partial z} \right) \right|_0^L + F_y(t) \left. \left( \frac{\partial y(z,t)}{\partial z} \right) \right|_0^L - \frac{F_z(t)}{2} \left| \left[ \left( \frac{\partial z(z,t)}{\partial z} \right)^2 + \left( \frac{\partial z(z,t)}{\partial z} \right)^2 \right] \right|_0^L + \\
M_x(t) \left. \left( \frac{\partial \theta(z,t)}{\partial z} \right) \right|_0^L + M_y(t) \left. \left( \frac{\partial \theta(z,t)}{\partial z} \right) \right|_0^L + M_x(t) \left. \theta(z,t) \right|_0^L  \hspace{1cm} (2.19)

The variation of the non-conservative component of the virtual work becomes...
\[ \delta W_{nc}(t) = \int_{0}^{L} \left\{ p_x(z,t) \delta x(z,t) + p_y(z,t) \delta y(z,t) - ight. \\
\left. p_x(z,t) \left[ \frac{\partial x(z,t)}{\partial z} \frac{\partial}{\partial z} \left( \delta x(z,t) \right) + \frac{\partial y(z,t)}{\partial z} \frac{\partial}{\partial z} \left( \delta y(z,t) \right) \right] \right\} dx + \\
F_x(t) \delta x(z,t) \left[ \frac{\partial x(z,t)}{\partial z} \frac{\partial}{\partial z} \left( \delta x(z,t) \right) + \frac{\partial y(z,t)}{\partial z} \frac{\partial}{\partial z} \left( \delta y(z,t) \right) \right]_0^L - \\
F_y(t) \delta y(z,t) \left[ \frac{\partial x(z,t)}{\partial z} \frac{\partial}{\partial z} \left( \delta x(z,t) \right) + \frac{\partial y(z,t)}{\partial z} \frac{\partial}{\partial z} \left( \delta y(z,t) \right) \right]_0^L + \\
M_x(t) \delta \left( \frac{\partial y(z,t)}{\partial z} \right) \left[ \frac{\partial x(z,t)}{\partial z} \frac{\partial}{\partial z} \left( \delta x(z,t) \right) + \frac{\partial y(z,t)}{\partial z} \frac{\partial}{\partial z} \left( \delta y(z,t) \right) \right]_0^L + M_y(t) \delta \theta(z,t) \left[ \frac{\partial x(z,t)}{\partial z} \frac{\partial}{\partial z} \left( \delta x(z,t) \right) + \frac{\partial y(z,t)}{\partial z} \frac{\partial}{\partial z} \left( \delta y(z,t) \right) \right]_0^L \\ 
(2.20) \]

The integral of Equation (2.20) with respect to time yields the third term of Hamilton's Principle (Equation 2.3).

\[
\int_{t_1}^{t_2} \delta W_{nc}(t) \, dt = \int_{t_1}^{t_2} \left\{ \int_{0}^{L} \left\{ p_x(z,t) \delta x(z,t) + p_y(z,t) \delta y(z,t) - ight. \\
\left. p_x(z,t) \left[ \frac{\partial x(z,t)}{\partial z} \frac{\partial}{\partial z} \left( \delta x(z,t) \right) + \frac{\partial y(z,t)}{\partial z} \frac{\partial}{\partial z} \left( \delta y(z,t) \right) \right] \right\} dx + \\
F_x(t) \delta x(z,t) \left[ \frac{\partial x(z,t)}{\partial z} \frac{\partial}{\partial z} \left( \delta x(z,t) \right) + \frac{\partial y(z,t)}{\partial z} \frac{\partial}{\partial z} \left( \delta y(z,t) \right) \right]_0^L - \\
F_y(t) \delta y(z,t) \left[ \frac{\partial x(z,t)}{\partial z} \frac{\partial}{\partial z} \left( \delta x(z,t) \right) + \frac{\partial y(z,t)}{\partial z} \frac{\partial}{\partial z} \left( \delta y(z,t) \right) \right]_0^L + \\
M_x(t) \delta \left( \frac{\partial y(z,t)}{\partial z} \right) \left[ \frac{\partial x(z,t)}{\partial z} \frac{\partial}{\partial z} \left( \delta x(z,t) \right) + \frac{\partial y(z,t)}{\partial z} \frac{\partial}{\partial z} \left( \delta y(z,t) \right) \right]_0^L + M_y(t) \delta \theta(z,t) \left[ \frac{\partial x(z,t)}{\partial z} \frac{\partial}{\partial z} \left( \delta x(z,t) \right) + \frac{\partial y(z,t)}{\partial z} \frac{\partial}{\partial z} \left( \delta y(z,t) \right) \right]_0^L \right\} dt \\
(2.21) 
\]

Integrate by parts with respect to \( z \) to expand the non-conservative term and obtain the final form as expressed below.
The three terms of Hamilton’s Principle are expressed as functions of their simplest components in Equations (2.8), (2.13), and (2.22). These expressions are substituted into Equation (2.3). Grouping similar terms, Hamilton’s Principle is restated below in a form that yields the equations of motion and boundary conditions of the shaft of a golf club.
\[
\int_{t_1}^{t_2} \delta T(t) \, dt - \int_{t_1}^{t_2} \delta V(t) \, dt + \int_{t_1}^{t_2} \delta \mathcal{W}_{\text{nc}}(t) \, dt =
\]
\[
\int_{t_1}^{t_2} \int_0^L -m \left( \frac{\partial^2 x(z,t)}{\partial t^2} + \frac{\partial^2}{\partial z^2} \left( EI_A(z) \frac{\partial^2 z(z,t)}{\partial z^2} \right) - p_x(z,t) - \frac{\partial}{\partial z} \left( p_x(z,t) \frac{\partial z(z,t)}{\partial z} \right) \right) \delta x(z,t) \, dz \, dt +
\]
\[
\int_{t_1}^{t_2} \int_0^L -m \left( \frac{\partial^2 y(z,t)}{\partial t^2} + \frac{\partial^2}{\partial z^2} \left( EI_A(z) \frac{\partial^2 y(z,t)}{\partial z^2} \right) - p_y(z,t) - \frac{\partial}{\partial z} \left( p_y(z,t) \frac{\partial y(z,t)}{\partial z} \right) \right) \delta y(z,t) \, dz \, dt +
\]
\[
\int_{t_1}^{t_2} \int_0^L \left\{ \left. \left[ \frac{\partial}{\partial z} \left( EI_A(z) \frac{\partial^2 x(z,t)}{\partial z^2} \right) - p_x(z,t) \frac{\partial x(z,t)}{\partial z} + F_x(t) \right] \right|_0^L \right\} \delta x(z,t) \, dt +
\]
\[
\int_{t_1}^{t_2} \int_0^L \left\{ \left. \left[ \frac{\partial}{\partial z} \left( EI_A(z) \frac{\partial^2 y(z,t)}{\partial z^2} \right) - p_y(z,t) \frac{\partial y(z,t)}{\partial z} + F_y(t) \right] \right|_0^L \right\} \delta y(z,t) \, dt +
\]
\[
\int_{t_1}^{t_2} \int_0^L \left\{ \left. \left[ -GJ(z) \frac{\partial \theta(z,t)}{\partial z} + M_z(t) \right] \right|_0^L \right\} \delta \theta(z,t) \, dt +
\]
\[
\int_{t_1}^{t_2} \int_0^L \left\{ \left. \left[ -EI_A(z) \frac{\partial^2 x(z,t)}{\partial z^2} - F_x(t) \frac{\partial x(z,t)}{\partial z} + M_x(t) \right] \right|_0^L \right\} \delta \left( \frac{\partial x(z,t)}{\partial z} \right) \, dt +
\]
\[
\int_{t_1}^{t_2} \int_0^L \left\{ \left. \left[ -EI_A(z) \frac{\partial^2 y(z,t)}{\partial z^2} - F_y(t) \frac{\partial y(z,t)}{\partial z} + M_y(t) \right] \right|_0^L \right\} \delta \left( \frac{\partial y(z,t)}{\partial z} \right) \, dt = 0 \quad (2.23)
\]

The variation terms \( \delta x, \delta y, \delta \theta, \delta \left( \frac{\partial x}{\partial z} \right), \) and \( \delta \left( \frac{\partial y}{\partial z} \right) \) are arbitrary for \( 0 \leq z \leq L \). Therefore to obtain a nontrivial solution to Hamilton's Principle the bracketed expressions in Equation (2.23) must equal zero. These expressions become the
equations of motion and possible boundary conditions of the system. Extracting the appropriate equations yields the following results.

2.3.1 Equations of Motion for $x$ and $y$ Coordinates

\[ m \frac{\partial^2 x(z,t)}{\partial t^2} + \frac{\partial^2}{\partial z^2} \left[ EI_A(z) \frac{\partial^2 x(z,t)}{\partial z^2} \right] - p_x(z,t) - \frac{\partial}{\partial z} \left[ p_x(z,t) \frac{\partial x(z,t)}{\partial z} \right] = 0 \]  

(2.24)

\[ m \frac{\partial^2 y(z,t)}{\partial t^2} + \frac{\partial^2}{\partial z^2} \left[ EI_A(z) \frac{\partial^2 y(z,t)}{\partial z^2} \right] - p_y(z,t) - \frac{\partial}{\partial z} \left[ p_y(z,t) \frac{\partial y(z,t)}{\partial z} \right] = 0 \]  

(2.25)

2.3.2 Possible Boundary Conditions for $x$ and $y$ Coordinates

\[ \delta x(z,t) \bigg|_0^L = 0 \quad \delta \left[ \frac{\partial x(z,t)}{\partial z} \right] \bigg|_0^L = 0 \]  

(2.26)(2.27)

\[ \left[ - EI_A(z) \frac{\partial^2 x(z,t)}{\partial z^2} - F_x(t) \frac{\partial x(z,t)}{\partial z} + M_y(t) \right] \bigg|_0^L = 0 \]  

(2.28)

\[ \left[ \frac{\partial}{\partial z} \left( EI_A(z) \frac{\partial^2 x(z,t)}{\partial z^2} \right) - p_x(z,t) \frac{\partial x(z,t)}{\partial z} + F_x(t) \right] \bigg|_0^L = 0 \]  

(2.29)

and

\[ \delta y(z,t) \bigg|_0^L = 0 \quad \delta \left[ \frac{\partial y(z,t)}{\partial z} \right] \bigg|_0^L = 0 \]  

(2.30)(2.31)

\[ \left[ - EI_A(z) \frac{\partial^2 y(z,t)}{\partial z^2} - F_y(t) \frac{\partial y(z,t)}{\partial z} + M_x(t) \right] \bigg|_0^L = 0 \]  

(2.32)

\[ \left[ \frac{\partial}{\partial z} \left( EI_A(z) \frac{\partial^2 y(z,t)}{\partial z^2} \right) - p_y(z,t) \frac{\partial y(z,t)}{\partial z} + F_y(t) \right] \bigg|_0^L = 0 \]  

(2.33)
2.3.3 Equation of Motion for $\theta$ Coordinate

$$-I_m(z) \frac{\partial^2 \theta(z,t)}{\partial t^2} + \frac{\partial}{\partial z} \left( GJ(z) \frac{\partial \theta(z,t)}{\partial z} \right) = 0$$  \hspace{1cm} (2.34)

2.3.4 Possible Boundary Conditions for $\theta$ Coordinate

$$\theta(z,t)|_0^L = 0 \hspace{1cm} \left[ -GJ(z) \frac{\partial \theta(z,t)}{\partial z} + M_z(t) \right]|_0^L = 0 \hspace{1cm} (2.35)(2.36)$$

2.4 Modification Applied to the Equations of Motion and Boundary Conditions

The equations of motion and boundary conditions provide a comprehensive representation of the deformations of the shaft. However, in order to obtain an initial solution to these equations some simplifying assumptions are made. The distributed forces applied to the shaft, designated by $P_x(z,t)$, $P_y(z,t)$, and $P_z(z,t)$ are neglected in the first solution of the model. This simplification is justified by the assumption that the distributed forces are smaller in magnitude than the forces acting at the ends of the shaft.

Additionally, a damping coefficient is added to the equation of motion to represent the natural damping that is provided by the golfer's hands. The general equation of motion for a free system may be described as

$$m\ddot{z} + c\dot{z} + kz = 0$$  \hspace{1cm} (2.37)

where $c$ is the damping coefficient of the system. Therefore, damping terms for
bending vibration are added to Equations (2.24) and (2.25) and a damping term for torsional vibration is added to Equation (2.34). The primary effect of these terms is to reduce the amplitude of the oscillations that are generated by the numerical solutions to the equations of motion. Were the damping terms not incorporated into the equations of motion the oscillations of the shaft would continue indefinitely.

2.5 Specifying the Boundary Conditions

To obtain a solution to the equations of motion the appropriate number of boundary conditions must be specified. The bending equations require four boundary conditions each to obtain a unique solution, while the torsional equation requires only two boundary conditions. Since the formulation of the problem using Hamilton's Principle produces more boundary conditions than are necessary for a unique solution, decisions are made regarding which boundary conditions apply to the situation of a golfer swinging a club. Boundary condition number one states that the butt end of the shaft must coincide with the origin of the body-axis coordinate system.

A second boundary condition must be specified at the origin for the bending equations. The assumption that the shaft does not undergo deflections at the grip end under the golfer's hands leads to the second boundary condition, similar to a cantilever beam. The cantilever boundary condition does not precisely model the physical boundary of a golfer gripping the shaft, nevertheless it is used to obtain an initial solution to the equations of motion.

At the club-head end of the shaft the boundary conditions are governed by
external moments and forces acting on the shaft. Equations (2.28), (2.29), (2.32), (2.33), and (2.36) evaluated at $z = L$ complete the description of the motion.

Accounting for the simplification of the equations of motion and the specification of the boundary conditions, Equations (2.38) through (2.50) become the equations that are solved to determine the deflection of a golf shaft during the swing. Solving these equations will yield the deflections of the shaft in the $x$, $y$, and $\theta$ directions as functions of time and distance from the grip end of the shaft.

2.6 Simplified Equations of Motion for $x$ and $y$ Coordinates

\[
m_x \frac{\partial^2 x(z,t)}{\partial t^2} + \frac{\partial^2}{\partial z^2} \left[ EI_A(z) \frac{\partial^2 x(z,t)}{\partial z^2} \right] + c \frac{\partial x(z,t)}{\partial t} = 0 \tag{2.38}
\]

\[
m_x \frac{\partial^2 y(z,t)}{\partial t^2} + \frac{\partial^2}{\partial z^2} \left[ EI_A(z) \frac{\partial^2 y(z,t)}{\partial z^2} \right] + c \frac{\partial y(z,t)}{\partial t} = 0 \tag{2.39}
\]

2.7 Simplified Boundary Conditions for $x$ and $y$ Coordinates

\[
x(0,t) = 0 \quad \frac{\partial x(0,t)}{\partial z} = 0 \tag{2.40} \tag{2.41}
\]

\[
\frac{\partial}{\partial z} \left( EI_A(L) \frac{\partial^2 x(L,t)}{\partial z^2} \right) + F_x(t) = 0 \tag{2.42}
\]

\[
- EI_A(L) \frac{\partial^2 x(L,t)}{\partial z^2} - F_y(t) \frac{\partial x(L,t)}{\partial z} + M_y(t) = 0 \tag{2.43}
\]

\[
y(0,t) = 0 \quad \frac{\partial y(0,t)}{\partial z} = 0 \tag{2.44} \tag{2.45}
\]
\[
\frac{\partial}{\partial z} \left( EI_A(L) \frac{\partial^2 y(L,t)}{\partial z^2} \right) + F_y(t) = 0
\]
\[
- EI_A(L) \frac{\partial^2 y(L,t)}{\partial z^2} - F(t) \frac{\partial y(L,t)}{\partial z} + M_z(t) = 0
\]

2.8 Simplified Equation of Motion for \( \theta \) Coordinate

\[
-I_m \frac{\partial^2 \theta(z,t)}{\partial t^2} + \frac{\partial}{\partial z} \left[ GJ(z) \frac{\partial \theta(z,t)}{\partial z} \right] + c_\theta \frac{\partial \theta(z,t)}{\partial t} = 0
\]

2.9 Simplified Boundary Conditions for \( \theta \) Coordinate

\[
\theta(0,t) = 0 \quad -GJ(z) \frac{\partial \theta(L,t)}{\partial z} + M_z(t) = 0
\]
Chapter 3
Kinetic Analysis of Torques and Forces Acting on a Golf Shaft

The boundary conditions developed in Chapter 2 contain terms which represent the forces and moments applied at the end of the shaft. Before the equations of motion for the system can be solved the magnitudes of these loads must be determined for a particular swing. This is accomplished by tracing the path of the center of mass of the club head and determining the forces and moments required to produce this motion. The linear acceleration, angular velocity, and angular acceleration of the club head, defined with respect to an inertial coordinate system, as well as the physical properties of the club head are needed to calculate these loads. Since the forces and moments that produce the motion of the club head are applied by the shaft, opposing forces and moments must act on the shaft. These are the loads specified by the boundary conditions of the system.

3.1 Experimental Equipment

To calculate the acceleration and angular velocity of the golf club an experiment is performed that records three-dimensional position data of several points on the shaft as functions of time. These position vectors are used to define the unit vectors of the body-axis coordinate system \((x, y, z)\) in terms of an inertial coordinate system \((X, Y, Z)\) fixed with respect to the earth.

The data is collected using the Motion Analysis Corporation's data
acquisition system installed at the United States Golf Association Technical Center in Far Hills, New Jersey. The function of the data acquisition system is to convert video images into digital data. A golf club is equipped with four reflective markers attached at different locations, as illustrated in Figure 3.1. The plane defined by markers 1, 2, and 3 is parallel to the plane of club face.

Four video cameras placed about the room record the motion of each reflective marker as the shaft is swung by a golfer. To insure that the swing is typical of one executed in a game situation a golf ball placed on a tee is the target. The frequency of sampling is 180 frames per second and the swing has a duration of approximately 2.5 seconds from the address position to the completion of the follow through. The process produces between four hundred and five hundred data points. The recorded video images are converted to digital data using the provided software. For the conversion to be successful at least three cameras need to record the path of a reflector simultaneously. In case it is not possible to obtain three images of a marker, due to the body of the golfer blocking the cameras view, a smoothing routine incorporated in the software package is used to generate the missing information. The final product of the data acquisition procedure is a three dimensional position vector for each reflective marker measured with respect to the origin of the inertial coordinate system. From these vectors the linear acceleration of the end of the shaft and a set of orthogonal unit vectors which define the body-axis coordinate system fixed to the shaft are calculated.
Figure 3.1: Position of Reflective Markers on Shaft Relative to the Inertial Coordinate System
### 3.2 Body-Axis Unit Vectors

Markers 1 and 2, illustrated in Figure 3.2, define the unit vector in the $\tilde{z}$ direction which is parallel to the central axis of the shaft. The defining equation is shown below.

$$\tilde{z} = \frac{(X_2 - X_1) \hat{i} + (Y_2 - Y_1) \hat{j} + (Z_2 - Z_1) \hat{k}}{\sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2}} = z_X \hat{i} + z_Y \hat{j} + z_Z \hat{k}$$ (3.1)

The $\tilde{z}$ direction unit vector cannot be calculated directly from the difference between two markers. Instead a vector from marker 1 to marker 3 is defined and separated into two components: one parallel to the shaft, the other normal to the shaft. The $x$ direction unit vector is in the direction of the normal component of the vector between markers 1 and 3. Figure 3.2 depicts the components described above.

The procedure used to calculate $\tilde{x}$ is as follows. First, define vector $\tilde{a}$ pointing from marker 1 to marker 3.

$$\tilde{a} = (X_3 - X_1) \hat{i} + (Y_3 - Y_1) \hat{j} + (Z_3 - Z_1) \hat{k} = a_X \hat{i} + a_Y \hat{j} + a_Z \hat{k}$$ (3.2)

Next, determine the projection of $\tilde{a}$ in the direction of the $\tilde{z}$ unit vector. Call this vector $\tilde{b}$.

$$\tilde{b} = (\tilde{a} \cdot \tilde{z}) \tilde{z} = (a_X z_X + a_Y z_Y + a_Z z_Z) \tilde{z} = b_X \hat{i} + b_Y \hat{j} + b_Z \hat{k}$$ (3.3)
Figure 3.2: Body-Axes Unit Vectors in Inertial Coordinates
To calculate the unit vector in the $\hat{z}$ direction subtract $\vec{a}$ from $\vec{b}$ and divide by the magnitude of the resulting vector.

$$
\vec{z} = \frac{(a_x - b_x) \hat{i} + (a_y - b_y) \hat{j} + (a_z - b_z) \hat{k}}{\sqrt{(a_x - b_x)^2 + (a_y - b_y)^2 + (a_z - b_z)^2}} = x_x \hat{i} + x_y \hat{j} + x_z \hat{k} \quad (3.4)
$$

The unit vector in the $\hat{y}$ direction is obtained from the cross product of vectors $\vec{z}$ and $\vec{z}$ and completes the right-hand triad. Unit vector $\vec{y}$, represented in Equation (3.5), is normal both to the shaft and the club face.

$$
\vec{y} = \vec{z} \times \vec{z} = \frac{(z_y z_x - z_x z_y) \hat{i} + (z_x z_y - z_y z_x) \hat{j} + (z_y z_x - z_x z_y) \hat{k}}{\sqrt{(z_y z_x - z_x z_y)^2 + (z_x z_y - z_y z_x)^2 + (z_y z_x - z_x z_y)^2}} \quad (3.5)
$$

### 3.3 Rotation Matrix, Angular Velocity and Acceleration of the Shaft

Two quantities, the rotation matrix and the angular velocity of the shaft, may be calculated directly from the unit vectors obtained in Section 3.2.

The rotation matrix is used to convert quantities defined in the inertial coordinate system into body-axis coordinates so as to conform with the coordinate system used to develop the equations of motion. The rotation matrix is expressed in Equation (3.6) [18].

$$
R = \begin{bmatrix}
\vec{z} \cdot \hat{i} & \vec{z} \cdot \hat{j} & \vec{z} \cdot \hat{k} \\
\vec{y} \cdot \hat{i} & \vec{y} \cdot \hat{j} & \vec{y} \cdot \hat{k} \\
\vec{z} \cdot \hat{k} & \vec{y} \cdot \hat{k} & \vec{z} \cdot \hat{k}
\end{bmatrix} = \begin{bmatrix}
x_x & y_x & z_x \\
x_y & y_y & z_y \\
x_z & y_z & z_z
\end{bmatrix} \quad (3.6)
$$
Additionally, the transpose of the rotation matrix, when used as described in Expression (3.7), will convert vector quantities from inertial components to body-axis components [19].

\[
\tilde{u}_{xyz} = \mathbf{R}^T \tilde{u}_{XYZ} \tag{3.7}
\]

This application is used to transform the linear acceleration, angular velocity, and angular acceleration of the club head from inertial components to body-axis components.

The angular velocity of the shaft, defined with respect to the inertial coordinate system, may be calculated according to the following formula [20].

\[
\begin{align*}
\omega_X &= \dot{z}_Z x_Y + \dot{y}_Z y_Z + \dot{z}_Z z_Y \\
\omega_Y &= \dot{z}_X x_Z + \dot{y}_X y_Z + \dot{z}_X z_Z \\
\omega_Z &= \dot{z}_Y x_Y + \dot{y}_Y y_Z + \dot{z}_Y z_X
\end{align*} \tag{3.8a, 3.8b, 3.8c}
\]

Where,

- \( \omega_V \) - Angular velocity in \( X, Y, \) or \( Z \) components
- \( \dot{u}_V \) - Unit vector \( x, y, z \) defined in \( X, Y, \) and \( Z \) components
- \( \ddot{u}_V \) - Time derivative of unit vector \( x, y, \) or \( z \) defined in \( X, Y, \) and \( Z \) components

The angular acceleration of the shaft and linear accelerations of marker 4 are used to calculate the acceleration of the center of mass and the angular momentum of the club head. The angular acceleration is defined as the derivative of the angular velocity with respect to time.

\[
\ddot{\omega} = \frac{d\omega}{dt} \tag{3.9}
\]
The linear acceleration of marker 4 (its location on the shaft is illustrated in Figure 3.1) is defined by the following equation.

\[ \ddot{r}_4 = \frac{d^2r_4}{dt^2} \]  

(3.10)

Where \( \ddot{r}_4 \) is the position vector of marker 4 with respect to the inertial origin.

### 3.4 Forces and Torques Acting on the Club Head

Figure 3.3 depicts the club head in dynamic equilibrium. To obtain the forces and moments acting at the hosel two equilibrium conditions are employed: (1) the sum of the forces applied to the club head are equal to zero and (2) the sum of the moments applied to the club head at the hosel are equal to zero [21]. Included are the forces and moments generated by the change in linear and angular momentum respectively. The equilibrium conditions are stated in Equation (3.11) and (3.12).

\[ \bar{F}_h - m\ddot{a}_g = 0 \]  

(3.11)

\[ \bar{M}_h - \bar{M}_g + \bar{r}_g \times ( - m\ddot{a}_g ) = 0 \]  

(3.12)

Where,

- \( \bar{F}_h \) - Force applied to the hosel of the club head.
- \( \ddot{a}_g \) - Linear acceleration of center of mass of the club head.
- \( m \) - Mass of the club head.

and

- \( \bar{M}_h \) - Moment applied to the club head at the hosel.
- \( \bar{M}_g \) - Moment due to change in angular momentum.
- \( \bar{r}_g \) - Vector from hosel to center of mass of club head.
Figure 3.3: Club Head in Dynamic Equilibrium
First determine the acceleration of the center of mass of the club head from the experimental data. Reflective marker 4, shown in Figure 3.1, is located near the hosel of the club head. Knowing the linear acceleration of marker 4 the acceleration of the center of mass of the club head is calculated using Equation (3.13). Keep in mind when calculating the acceleration in the \( Z \) direction the gravitational acceleration acting in the negative direction must be included.

\[
\ddot{\vec{a}}_g = \ddot{\vec{a}}_4 + \hat{\omega} \times (\hat{\omega} \times \hat{r}_4) + \ddot{\vec{r}}_4
\]  

(3.13)

Where,

- \( \ddot{\vec{a}}_g \) - Acceleration of center of mass.
- \( \ddot{\vec{a}}_4 \) - Acceleration of marker 4.
- \( \hat{\omega} \) - Angular velocity of club head.
- \( \ddot{\vec{r}}_4 \) - Vector from marker 4 to center of mass.

The acceleration of the center of mass of the club head is calculated in the inertial coordinate system and is converted to the body-axis coordinate system using the rotation matrix in accordance with Equation (3.7). Summing the forces applied to the club head and solving for the forces at the hosel result in the following expressions.

\[
F_{h,z} = m \ a_{g,z} \\
F_{h,y} = m \ a_{g,y} \\
F_{h,z} = m \ a_{g,z}
\]  

(3.14a)  
(3.14b)  
(3.14c)

To compute the moment applied at the hosel Equation (3.12) must be expanded. First examine the moment produced by the change in angular momentum.
where

\[ \ddot{\vec{H}} \] - Time derivative of angular momentum.
\[ \vec{\omega} \] - Angular velocity of club head.
\[ \vec{H} \] - Angular momentum of club head.

The angular momentum, \( \vec{H} \), is defined by the inertia matrix and angular velocity vector of the club head.

\[ \vec{H} = I \times \vec{\omega} \] \hspace{1cm} (3.16)

\[ H_x = I_{xx} \omega_x + I_{xy} \omega_y + I_{xz} \omega_z \] \hspace{1cm} (3.16a)
\[ H_y = I_{yx} \omega_x + I_{yy} \omega_y + I_{yz} \omega_z \] \hspace{1cm} (3.16b)
\[ H_z = I_{zx} \omega_x + I_{zy} \omega_y + I_{zz} \omega_z \] \hspace{1cm} (3.16c)

Where \( I \) is the inertia matrix measured with respect to the center of mass of the club head.

\[ I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \] \hspace{1cm} (3.17)

When defined with respect to the body-axis coordinate system, the inertia matrix remains constant with time, consequently the time derivative of the angular momentum becomes

\[ \dot{H}_x = I_{xx} \dot{\omega}_x + I_{xy} \dot{\omega}_y + I_{xz} \dot{\omega}_z \] \hspace{1cm} (3.18a)
\[ \dot{H}_y = I_{yx} \dot{\omega}_x + I_{yy} \dot{\omega}_y + I_{yz} \dot{\omega}_z \] \hspace{1cm} (3.18b)
\[ \dot{H}_z = I_{zx} \dot{\omega}_x + I_{zy} \dot{\omega}_y + I_{zz} \dot{\omega}_z \] \hspace{1cm} (3.18c)
The second term in Equation (3.15) can also be expanded into its components.

\[ \mathbf{\vec{\omega}} \times \mathbf{\vec{H}} = (\omega_y H_x - \omega_x H_y) \mathbf{j} + (\omega_z H_y - \omega_y H_z) \mathbf{k} \]  

Substitute the expressions for the angular momentum, Equation (3.16), into the above equation and add the result to Equation (3.18) to find the \( x \), \( y \), and \( z \) components of the moment applied to the club head resulting from the change in angular momentum. Note that the angular velocity and acceleration vectors must be transformed into the body-axis coordinate system before the moment due to angular momentum can be computed.

\[
M_{g,x} = I_{xx} \omega_x + I_{xy} \omega_y + I_{xz} \omega_z + \omega_y(I_{xx} \omega_x + I_{xy} \omega_y + I_{xz} \omega_z) - \\
\omega_x(I_{yx} \omega_x + I_{yy} \omega_y + I_{yz} \omega_z)
\]  

\[
M_{g,y} = I_{yx} \omega_x + I_{yy} \omega_y + I_{yz} \omega_z + \omega_x(I_{xx} \omega_x + I_{xy} \omega_y + I_{xz} \omega_z) - \\
\omega_y(I_{yx} \omega_x + I_{yy} \omega_y + I_{yz} \omega_z)
\]  

\[
M_{g,z} = I_{xz} \omega_x + I_{zy} \omega_y + I_{zz} \omega_z + \omega_x(I_{yx} \omega_x + I_{yy} \omega_y + I_{yz} \omega_z) - \\
\omega_y(I_{yx} \omega_x + I_{yy} \omega_y + I_{yz} \omega_z)
\]

The total moment acting at the hosel of the club head is comprised of the moment due to the angular momentum, defined by Equation (3.20), and the moment resulting from the force acting at the center of mass that is produced by the linear acceleration of the club head. Equation (3.21) yields the moment applied at the hosel of the club head about the \( x \), \( y \), and \( z \) axis.
\[ M_{h,x} = \bar{M}_{g,x} - (ma_{g,y} \Delta z + ma_{g,z} \Delta y) \]  
\[ M_{h,y} = \bar{M}_{g,y} - (ma_{g,x} \Delta z + ma_{g,z} \Delta x) \]  
\[ M_{h,z} = \bar{M}_{g,z} - (ma_{g,x} \Delta y + ma_{g,y} \Delta x) \]

where

\[ \Delta z \] - Component of vector \( \vec{r}_g \) in \( x \) direction
\[ \Delta y \] - Component of vector \( \vec{r}_g \) in \( y \) direction
\[ \Delta x \] - Component of vector \( \vec{r}_g \) in \( z \) direction

### 3.5 Forces and Torques Applied to the Shaft

The above forces and moments were calculated assuming the club head was a free-body moving in space. Since the club head is attached to the shaft at the hosel there must be equal but opposite forces and moments acting on the shaft. Refer to Figure 3.4 for a diagram illustrating the reactions acting at the end of the shaft. These forces and moments are of the opposite sign but equal in magnitude to the forces and moments applied to the club head at the hosel. Subsequently, the forces and moments defined by the boundary conditions of the equations of motion become

\[ \vec{F} = -\bar{F}_h \]  
\[ \vec{M} = -\bar{M}_h \]

Where, \( \vec{F} \) and \( \vec{M} \) are the force and moment acting on the end of the shaft, respectively.
Figure 3.4: Reactions at end of Shaft
3.6 Experimental Determination of the Damping Coefficient

To calculate the damping coefficient for bending vibration an experiment is performed that measures the response of the shaft to an impulse.

A golf shaft, without the club head attached, is instrumented with an accelerometer at the tapered end. A person holds the shaft with an appropriate grip while it is excited by an impulse that causes bending vibration. The movement of the shaft is record on an oscilloscope and a plot of the decay of the amplitude of vibration is obtained. To calculate the damping coefficient the natural frequency $\omega_n$ and the sequential amplitudes of the decaying wave form are recorded. The logarithmic decrement $\delta$ is calculated from this information [22].

$$\delta = \frac{1}{n} \ln \left( \frac{x_0}{x_n} \right)$$

(3.24)

where

- $\delta$ - Logarithmic decrement
- $x_0$ - Amplitude of first wave
- $x_n$ - Amplitude of $n^{th}$ wave
- $n$ - Number of elapsed cycles

From the logarithmic decrement, the damping factor $\zeta$ may be calculated by the following formula.

$$\zeta = \sqrt{\frac{\delta^2}{4\pi^2 + \delta^2}}$$

(3.25)

Finally, the damping coefficient $c$ is determined from

$$c = 2m\omega_n\zeta$$

(3.26)
Chapter 4
NUMOL Solution

In Chapters 2 and 3 the equations of motion and boundary conditions governing the behavior of the shaft were developed. All the information necessary to solve the equations of motion for the deflections in the $x$, $y$, and $\theta$ directions is known. The approach used to solve these problems is called the Numerical Methods of Lines (NUMOL) and it is discussed in detail in The Numerical Method of Lines by W. E. Schiesser [23]. Simply stated, this method involves transforming each partial differential equation (PDE) into a set of ordinary differential equations (ODEs) by replacing the continuous spatial derivatives with finite-difference approximations. The ODEs are then solved using a state-of-the-art numerical integrator.

4.1 Classifying the Equations of Motion

The deformations occurring in the shaft of a golf club can be categorized into two sorts: torsional motion, described by the coordinate $\theta$, and bending motion, described by the $x$ and $y$ coordinates. The main difference in the structure of these equations is the order of the spatial derivative. The torsional equation of motion has a second-order spatial derivative as the highest order spatial derivative and the bending equations of motion contain fourth-order spatial derivatives. Although the NUMOL procedure is similar for both types of equations, there exist sufficient differences to warrant an investigation into the
solution of each type.

The equation of motion and boundary conditions defining the torsional motion are stated in Equations (4.1) through (4.3). The variable \( \theta \) is replaced by \( u(z,t) \) to represent a general solution for this type of problem, where \( u(z,t) \) is a function of time and position along the shaft.

\[
\frac{\partial^2 u(z,t)}{\partial t^2} = \frac{1}{T_m(z)} \frac{\partial}{\partial z} \left[ GJ(z) \frac{\partial}{\partial z} \left( u(z,t) \right) \right] - \frac{c}{T_m(z)} \frac{\partial u(z,t)}{\partial t}
\]  
\( \text{(4.1)} \)

\[
u(0,t) = 0 \quad \text{(4.2)}
\]

\[
GJ(z) \frac{\partial}{\partial z}(u(L,t)) = M_z(t)
\]  
\( \text{(4.3)} \)

Since the bending equation of motion characterizes deflection in the \( x \) and \( y \) directions, the variable \( \nu(z,t) \) is chosen to represent either coordinate. The equation of motion is

\[
\frac{\partial^2 \nu(z,t)}{\partial t^2} = -\frac{1}{m_z} \frac{\partial^2}{\partial z^2} \left[ EI_A(z) \frac{\partial^2 \nu(z,t)}{\partial z^2} \right] - \frac{c}{m_z} \frac{\partial \nu(z,t)}{\partial t}
\]  
\( \text{(4.4)} \)

The following four boundary conditions accompany the bending equation of motion.

\[
u(0,t) = 0 \quad \frac{\partial \nu(0,t)}{\partial z} = 0
\]  
\( \text{(4.5)} \)  
\( \text{(4.6)} \)

\[
\frac{\partial^2 \nu(L,t)}{\partial z^2} = \frac{M_w(t)}{EI_A(L)} - \frac{F_z(t)}{EI_A(L)} \frac{\partial \nu(L,t)}{\partial z}
\]  
\( \text{(4.7)} \)

\[
\frac{\partial}{\partial z} \left[ EI_A(L) \frac{\partial^2 \nu(L,t)}{\partial z^2} \right] = F_{\nu}(t)
\]  
\( \text{(4.8)} \)
The NUMOL procedures for the two types of equations presented above are subsequently discussed.

4.2 Discretization of Spatial Derivatives

The first step in obtaining a NUMOL solution for any type of equation is to discretize the spatial variable. For the case of the equations of motion of a golf club, \( z \) is the variable to be discretized. This is accomplished by dividing the range of \( z \) (the length of the shaft) into \( N-1 \) number of segments. Thus \( z \) changes from a continuous variable to a function of \( N \) points identified by the integer \( i \). Specifically, the boundary at \( z = 0 \) corresponds to \( i = 1 \) and at \( z = L \) corresponds to \( i = N \).

The process of discretizing the spatial variable generates \( N \) differential equations, one for each point. Stating the equations of motion in terms of \( z_i \) converts it from one equation into \( N \) equations. Expression (4.9) is the discretized form of the equation of motion for the torsional coordinate and Equations (4.10) and (4.11) are the related boundary conditions.

\[
\frac{d^2 u_i(t)}{dt^2} = \frac{1}{I_{m,i}} \frac{d}{dz} \left( GJ_i \frac{du_i(t)}{dz} \right) - \frac{c}{I_{m,i}} \frac{\partial u_i(t)}{\partial t} \\
(4.9)
\]

\[
u_1(t) = 0 \quad GJ_N \frac{d u_N(t)}{dz} = M_i(t) \quad (4.10) (4.11)
\]

By the same procedure, discretizing the bending equation of motion results in Expression (4.12).
\[
\frac{d^2 v_i(t)}{dt^2} = -\frac{1}{m_z} \frac{d^2}{dz^2} \left( EI_{A,i} \frac{\partial^2 v_i(t)}{\partial z^2} \right) - \frac{c}{m_z} \frac{\partial v_i(t)}{\partial t} \tag{4.12}
\]

The associated boundary conditions become

\[
v_i(t) = 0 \quad \quad \frac{\partial v_i(t)}{\partial z} = 0 \tag{4.13} \tag{4.14}
\]

\[
EI_{A,N} \frac{\partial^2 v_N(t)}{\partial z^2} = M_w(t) - F_x(t) \frac{\partial v_N(t)}{\partial z} \tag{4.15}
\]

\[
\frac{\partial}{\partial z} \left( EI_{A,N} \frac{\partial^2 v_N(t)}{\partial z^2} \right) = F_w(t) \tag{4.16}
\]

4.3 Evaluating the Right-Hand Side of the Equations of Motion

The next step in the NUMOL procedure is to evaluate the right-hand side of Equations (4.9) and (4.12) utilizing approximations for the spatial derivative and incorporating the boundary conditions. These approximations are often developed by finite differences and the applied formulas are presented in Appendix A of this document.

The right-hand side of Equation (4.9) is evaluated first. If the term GJ were constant the second-order derivative could be computed directly. However this is not the case for a golf club because the shaft is tapered. To overcome this problem the right-hand side of Equation (4.9) is computed in two steps. First the derivative \( \frac{du_i(t)}{dz} \) is evaluated using a three point difference formula that incorporates boundary condition (4.10). This value is multiplied by the coefficient GJ to obtain a new intermediate value called \( T_i \).
Boundary condition (4.11) is applied to the above equation for \( i = N \) and the derivative \( \frac{dT_i(t)}{dz} \) is determined using the same three-point approximation for the first derivative. The right-hand side of the equation of motion for the torsional coordinate becomes the following.

\[
\frac{d^2 u_i(t)}{dt^2} = \frac{1}{I_{m,i}} \frac{dT_i(t)}{dz} - \frac{c}{I_{m,i}} \frac{\partial u_i(t)}{\partial t} \tag{4.18}
\]

The same logic is followed when calculating the right-hand side of Equation (4.12) since the value \( EI_i \) is also variable for a golf shaft. However, in this case the second derivative \( \frac{d^2 v_i(t)}{dz^2} \) is calculated directly instead of applying the first derivative approximation twice. This requires the use of two boundary conditions, namely (4.13) and (4.14). Boundary condition (4.14) cannot be applied as it is presented above because the formula for the second-order derivative is defined in terms of the spatial variable \( v_i \) only. Instead this boundary condition must be manipulated so that the value for point 2 \( (v_2) \) can be expressed in terms of the solution at other points on the spatial grid. This is accomplished by replacing boundary condition (4.14) with a 5-point finite-difference spatial approximation for the first derivative, as shown below.

\[
\frac{dv_1}{dz} \approx \frac{1}{60} \frac{1}{\Delta z} \left[ -147v_1 + 360v_2 - 450v_3 + 400v_4 - 225v_5 + 72v_6 - 10v_7 \right] = 0 \tag{4.19}
\]

Solve for \( v_2 \).
\[ v_2 = \frac{147v_1 + 450v_3 - 400v_4 + 225v_5 - 72v_6 + 10v_7}{60} \quad (4.20) \]

The second order derivative \( \frac{d^2v_i(t)}{dz^2} \) may now be calculated incorporating both boundary conditions at \( i = 1 \). The result is multiplied by the factor \( EI_i \) to obtain a temporary variable called \( S_i(t) \).

\[
S_i(t) \triangleq EI_i \frac{d^2v_i}{dz^2} \quad (4.21)
\]

Before the second derivative with respect to \( z \) of variable \( S_i \) is calculated boundary conditions (4.15) and (4.16) must be modified. Boundary condition (4.15) contains the term \( \frac{\partial v_N}{\partial z} \) which is expanded using a five point spatial derivative approximation. The resulting expression is shown in Equation (4.23).

\[
S_N = M_w(t) - F_z(t) \frac{\partial v_N}{\partial z} \quad (4.22)
\]

Expanded,

\[
S_N \simeq M_w(t) - F_z(t) \frac{147v_N - 360v_{N-1} + 450v_{N-2} - 400v_{N-3} + 225v_{N-4} - 72v_{N-5} + 10v_{N-6}}{60\Delta z} \quad (4.23)
\]

To incorporate the second boundary condition, Equation (4.16) is be manipulated to obtain an expression for the point \( S_{N-1} \) using the same finite-difference approximation as in Equation (4.23). Expanded in terms of \( S_i \), boundary condition (4.16) becomes

\[
\frac{\partial S_N}{\partial z} \simeq \frac{147 S_N - 360 S_{N-1} + 450 S_{N-2} - 400 S_{N-3} + 225 S_{N-4} - 72 S_{N-5} + 10 S_{N-6}}{60 \Delta z} = F_v(t) \quad (4.24)
\]
Solve the above equation for $S_{N-1}$.

$$S_{N-1} = \frac{147 S_N + 450 S_{N-2} - 400 S_{N-3} + 225 S_{N-4} - 72 S_{N-5} + 10 S_{N-6} + 60 \Delta x F_v(t)}{360} \quad (4.25)$$

Employ these two boundary conditions and calculate the second derivative of $S_i$.

The right-hand side of Equation (4.12) is expressed below.

$$\frac{d^2 v_i}{dt^2} = \frac{1}{m_z} \frac{d^2 S_i}{dz^2} - \frac{\epsilon}{m_z} \frac{\partial v_i}{\partial t} \quad (4.26)$$

### 4.4 Initial Conditions

With the right-hand side of both types of equations of motion known, the final step in obtaining a solution is to integrate each side with respect to time. A problem arises in that the integrator used to solve the equations is only applicable for first-order ordinary differential equations. Subsequently, it is necessary to separate the second-order equation in time into two first-order equations. This procedure applies to both categories of equations of motion.

Define the following variables:

$$u_1 = u \quad u_2 = \frac{d}{dt}(u_1) \quad (4.27)$$

As a consequence of the previous statement,

$$\frac{d(u_2)}{dt} = \frac{d^2(u_1)}{dt^2} \quad (4.28)$$
Substitute the above expressions into Equations (4.18) and (4.26) thereby separating each one into two equations. The sets of first order ODEs representing the torsional and bending motions are expressed in Equation (4.29), (4.30) and (4.31), (4.32) respectively.

\[
\frac{du_{1_i}}{dt} = u_{2_i} \\
\frac{du_{2_i}}{dt} = \frac{1}{I_{m,i}} \frac{dT_i}{dz} - \frac{c}{I_{m,i}} v_{2_i} \tag{4.29} \tag{4.30}
\]

\[
\frac{dv_{1_i}}{dt} = v_{2_i} \\
\frac{dv_{2_i}}{dt} = -\frac{1}{m_i} \frac{d^2S_i}{dz^2} - \frac{c}{m_i} v_{2_i} \tag{4.31} \tag{4.32}
\]

The equations of motion are ready to be solved once the initial conditions of the problem have been specified. The initial conditions are needed to provide the first evaluation of the right-hand sides of the equations of motion. Since the golf club is stationary when the ball is addressed it is assumed that all the deflections in the shaft are zero initially. Incorporated into the initial conditions is the assumption that the deflection due to the gravitational force is negligible. The velocity of the shaft is also zero at this time. Stated symbolically, the initial conditions of the problem for all values of \( z_i \) are

\[
\begin{align*}
    u_{1_i}(0) &= u_{2_i}(0) = v_{1_i}(0) = v_{2_i}(0) = 0 
\end{align*}
\]

4.5 Solution

Integrating Equations (4.29) through (4.32) is accomplished with the use of the numerical integrator LSODE (Livermore Solver for Ordinary Differential Equation) [24]. LSODE integrates the equations of motion over the period of
interest, in this case the duration of the swing. The user must supply a time step, $\Delta t$, at which the solution to the problem is desired. The time increment $\Delta t$ must be small enough so that the integration can be completed in five hundred or fewer internally specified time steps. The step size is dependent on the stiffness of the problem; a large value for $\Delta t$ will suffice for a non-stiff problem while a small value for $\Delta t$ is required to solve a stiff problem.

When satisfactory time steps have been found for both the bending and torsional equations of motion, the final solutions are calculated and presented as deflections in the $x$, $y$, and $\theta$ directions at $N$ points along the shaft at intervals of $\Delta t$ for the duration of the golf swing.
Chapter 5
Results for a Trial Golf Swing

5.1 Equipment

To test the procedure developed in Chapters two though four, relevant data was collected from a golf swing using a steel shafted driver. This experiment was performed with a True Temper Dynamic Gold parallel tip shaft with a stiffness rating of R300. Attached to the shaft was a H&G number one metal wood club head. A golfer swung the club while the motion analysis system recorded the data needed to calculate the motion of the swing. The physical parameters of the shaft were measured directly.

5.2 Material Properties and Physical Parameters of the Golf Shaft

The equations of motion for the shaft contain terms which represent the material properties of the shaft. Specifically the modulus of elasticity (E), shear modulus (G), and mass density (ρ). The shaft material is assumed to be steel with the following material properties [25,26].

\[
E = 20 \times 10^{10} \frac{N}{m^2} \quad G = 7.7 \times 10^{10} \frac{N}{m^2} \quad \rho = 7900 \frac{kg}{m^3}
\]

Additionally, physical parameters of the shaft such as length, mass per unit length, area moment of inertia, polar moment of inertia, and mass moment of
inertia are needed to obtain a solution for the deflection of the shaft. The length and the mass of the shaft are measured directly and the moments of inertia at a particular cross section are calculated from the inner and outer diameters of the shaft. The outer diameters and inner diameters were measured for each step of the tapered shaft and are constant for a given step of the shaft, as illustrated in Figure 5.1. A table of the shaft dimensions is found in Appendix B.

Knowing the outer and inner diameter at each step, the area, polar and mass moments of inertia are calculated using Equations (5.1) through (5.3) respectively.

\[ I_A(z) = \frac{\pi}{64} [d_o(z)^4 - d_i(z)^4] \]  
\[ J(z) = 2 \cdot I_A(z) \]  
\[ I_M(z) = \rho \cdot I_A(z) \]

Where,

- \( I_A \) - Area moment of inertia for a cross section.
- \( J \) - Polar moment of inertia for a cross section.
- \( I_M \) - Mass moment of inertia per unit length.
- \( d_o \) - Outer diameter of shaft.
- \( d_i \) - Inner diameter of shaft.

The mass per unit length of the shaft was assumed to be constant based on knowledge of the manufacturing process (the shaft is initially a plain tube and is squeezed into its tapered form). This assumption was confirmed by finding the center of mass of the shaft to be in the middle of the shaft, as would be expected for a shaft with a constant mass per unit length. Therefore the mass per unit length is calculated by measuring the total mass of the shaft and dividing by the length. Note that the length includes the portion of the shaft that is fitted into the club-head hosel.
Figure 5.1: Outer and Inner Diameters vs. Shaft Length
\[ m_s = 0.1216 \text{ kg} \quad L_s = 1.059 \text{ m} \]
\[ m_z = 0.115 \frac{\text{kg}}{\text{m}} \quad L = 1.024 \text{ m} \]

Where,

- \( m_s \) - Mass of shaft.
- \( L_s \) - Total length of shaft.
- \( m_z \) - Mass per unit length of shaft.
- \( L \) - Length of shaft from butt to top of hosel.

### 5.3 Physical Parameters of the Club Head

The physical parameters of the club head required to obtain a solution include the inertia matrix, the total mass, and the location of the center of mass relative to the top of the hosel. These were measured by Dr. S.H. Johnson at Lehigh University using an apparatus that measures the inertia matrix of the club-head assembly about its center of mass [27].

\[
m = 0.2012 \text{ kg}
\]

\[
\vec{r}_g = \begin{bmatrix}
0.0279 \\
0.0163 \\
0.0756
\end{bmatrix} \text{ meters}
\]

\[
I_g = \begin{bmatrix}
1.618 & -0.117 & -0.535 \\
-0.117 & 1.971 & -0.106 \\
-0.535 & -0.106 & 1.738
\end{bmatrix} \times 10^{-4} \text{ kg} \cdot \text{m}^2
\]

where

- \( m \) - Mass of club head
- \( \vec{r}_g \) - Vector from hosel to center of mass
- \( I_g \) - Inertia matrix of club head

Additionally, to calculate the acceleration of the center of mass the distance
from marker 4 to the center of mass is required. This value is specified by vector $\vec{r}_4$ for this example.

$$\vec{r}_4 = \begin{bmatrix} 0.0279 \\ 0.0163 \\ 0.0979 \end{bmatrix} \text{ meters}$$

5.4 Position Data

Position vectors measured relative to the inertial origin were generated with the motion analysis package described in Section 3.1. Vectors consisting of $X, Y,$ and $Z$ components are developed for each of the four markers described in Figure 3.1. The resulting position data for markers one through four are illustrated in Figures 5.2 through 5.5 respectively.

5.5 Damping Coefficient

The damping coefficient for bending vibration was calculated in accordance with the procedure outlined in Section 3.6 for the response curve shown in Figure 5.6. This figure is a plot of the screen display of the Ono Sokki Dynamic Analyzer used to measure the magnitudes and frequency of the response curve. The damping coefficient for the True Temper shaft is calculated from the later portion of the response curve, which has a measured frequency of 40 Hz and a logarithmic decrement ($\delta$) of 0.55. This yields a damping factor ($\zeta$) of 0.09 and from that value the damping coefficient divided by the mass per unit length ($\frac{\zeta}{\rho}$)
Figure 5.2: Position of Marker 1 vs. Time in Inertial Coordinates
Figure 5.3: Position of Marker 2 vs. Time in Inertial Coordinates
Figure 5.4: Position of Marker 3 vs. Time in Inertial Coordinates
Figure 5.5: Position of Marker 4 vs. Time in Inertial Coordinates
Figure 5.6: Damping of Gripped Steel Shaft
is calculated to be 45. This value is used for the term \( \frac{c}{m_i} \) of Equation (4.32).

The damping coefficient for the torsional vibration equations of motion has a value of 7, which was arrived at experimentally. The value chosen for the torsional damping coefficient decreased the effect of a high frequency oscillation that occurred in the initial part of the solution without changing the magnitude of the resulting deflection significantly [28]. This value of the damping coefficient allowed for a larger time steps to be taken and therefore less effort was required by the integrator to obtain a solution.

5.6 Forces and Torques Applied to the Shaft

A FORTRAN program was written that imports the position data and calculates the forces and moments applied at the end of the shaft. The program and related subroutines is found in Appendix C. Figure 5.7 depicts the forces \( F_x \), \( F_y \), and \( F_z \) defined in body-axis coordinates and Figure 5.8 shows the applied moments about the \( x \), \( y \), and \( z \) axes. The vertical line indicates the time of impact with the ball.

5.7 Grid Size and Time Step Required for a Solution

Three FORTRAN programs were developed to solve the equations of motion for the shaft; one for each deflection coordinate. The programs import the appropriate forces and moments, as specified by the boundary conditions, and return the deflections along the shaft as functions of time. Each program consists of a "main" program and three subroutines that specify the initial
Figure 5.7: Force Acting on End of Shaft vs. Time in Body-Axis Coordinates
Figure 5.8: Moment Acting on End of Shaft vs. Time in Body-Axis Coordinates
conditions, boundary conditions, and the print format. The programs are printed in Appendix D.

For this example the shaft was segmented into forty parts, therefore forty-one grid points were used to obtain the solution. The value of the time step is unique for each problem because it is dependent on the stability of the problem. For this example:

\[
\Delta t_\theta = 5.569 \times 10^{-5} \text{ sec} \quad \Delta t_x = 2.882 \times 10^{-4} \text{ sec} \quad \Delta t_y = 2.882 \times 10^{-4} \text{ sec}
\]

Once the step size has been determined, a spline routine available in the mathematical software Matlab was used to generate the missing data points for the input forces and moments [29].

5.8 Resulting Deflections

The solutions for the \( x, y, \) and \( \theta \) equations of motion are presented for six evenly spaced points along the shaft in Figures 5.9 through 5.11. The maximum deflection occurs at the hosel while the grip end of the shaft is stationary as specified by the boundary conditions.
Figure 5.9: Deflection in the x Direction vs. Time
Figure 5.10: Deflection in the y Direction vs. Time

Impact

$\Delta L = 0.205$ meters
Figure 5.11: Deflection in the $\theta$ Direction vs. Time

$\Delta L = 0.205$ meters

Deflection - Degrees

Time - Seconds

Impact
6.1 Experimental Procedure

The process of gathering the experimental data required to generate a solution to the numerical model yielded two topics worthy of discussion. The first pertains to the difficulty encountered when determining the stiffness characteristics of the shaft. The stiffness of the shaft can be measured from either influence coefficients or the area moment of inertia, which is a function of the outer and inner diameters of the shaft. Since the influence coefficients can in theory be determined through non-destructive testing, an attempt was made to measure shaft stiffness using this method. This proved to be unsuccessful due to the large number of precision measurements required to obtain an accurate result. The alternative method for calculating shaft stiffness necessitated the destruction of the shaft in order to measure the inner diameter. Although the destructive method produced accurate results, it could have been avoided if the shaft dimensions were provided by the manufacturer.

The second topic of discussion concerns the noise observed in the force and torque data supplied to the model through the boundary conditions. This data is a function of the derivatives of the position data recorded with the motion analysis system. The observed noise is generated when taking the first and second derivatives with respect to time of the position data for marker four (Figure 5.5). Plots of the velocity and acceleration of marker four, Figures 6.1
Figure 6.1: Velocity of Marker 4 vs. Time in Inertial Coordinates
Figure 6.2: Acceleration of Marker 4 vs. Time in Inertial Coordinates
and 6.2 respectively, depict the source and propagation of the noise.

The most probable explanation for this problem is that the motion analysis system does not have the resolution or sampling frequency required to produce smooth position data. An attempt to remedy the situation was made by generating a polynomial curve fit, however a satisfactory solution was not obtained.

Historically, obtaining accurate force and torque data has been a problem commonly faced by researchers studying the golf swing. A variety of smoothing techniques have been used to overcome this problem. Budney and Bellow used a fourth-order polynomial curve fit to model the angular position data [5], Milburn smoothed the angular displacement data before taking the derivatives [6], and Vaughan performed time differentiation on the position vectors by using a cubic spline technique [7]. Additionally, Neal and Wilson used a Butterworth second-order digital filter to smooth their data which resulted in the second derivative at the end points being forced to zero [8]. Due to time limitations, alternate methods of obtaining position, velocity, or acceleration data were not pursued.

6.2 Forces and Torques

The input forces and torques are calculated despite the noise generated by the differentiation process. Examining Figure 5.7, one notices that the axial force, $F_z$, has a peak magnitude approximately three times either of the transverse forces. The axial force produces tension in the shaft, which results in additional bending stiffness of varying magnitude. This temporal bending stiffness reaches a maximum just prior to impact, corresponding to the
maximum magnitude of the axial force.

Prior to impact a rapid change in magnitude and direction are noted for the transverse forces, $F_x$ and $F_y$. This indicates that the maximum acceleration of the club head was achieved before impact with the ball for this particular swing, and that the club head is actually decelerating at impact.

Unlike the forces, the moments acting at the end of the shaft reach peak magnitudes at impact. The moments about the $x$ and $y$ axes have a maximum magnitude of five times the moment about the $z$ axis, which is the only factor contributing to the torsional deflection.

### 6.3 Deflection

The deflections of the shaft observed in Figures 5.9 through 5.11 were computed without smoothing the input forces or moments. The results are presented at six points along the shaft spaced 0.205 meters apart. Deflections at the grip end are zero for all cases, while the largest deflections occur the hosel. Viewing Figure 5.9 one observes that the deflection in the $x$ direction reaches a maximum prior to impact, however at impact there is virtually no deflection in the shaft. The deflection in the $y$ direction, observed in Figure 5.10, also shows minimal deflection at impact. For both cases a high amplitude oscillation appears in the solution after impact. The rapid change in direction of the transverse forces prior to impact may produce the oscillations in the solution. Similar behavior is observed at time equal to 0.4 seconds, which corresponds to the time of the noise in the acceleration of marker four.

The solution for the torsional deflection, $\theta$, does not contain the same
oscillations observed for the bending deflections; this deflection is directly proportional to the moment about the $z$ axis. This leads to the conclusion that the oscillations observed for the bending deflection may be due to a numerical stability problem since the bending equations of motion contain fourth-order spatial derivatives and the torsional equations of motion have only second-order spatial derivatives.

6.4 Future Work

Before the model can be applied to shaft design it must be validated. The first step towards this goal is to acquire smoother position data. When confidence in the precision of the input forces and torques has been established further analysis of the deflections can be performed. One method for validating the model is to compare the numerical solution with experimentally obtained results, such as those measured by Butler and Winfield [11].

It is also recommended that further study into the validity of the cantilever boundary condition be pursued. In this model the cantilever boundary condition was assumed in the interest of obtaining an initial solution.

Finally, it is important to develop a model that incorporates the equations of motion for non-homogeneous materials. Shafts are often manufactured from composite materials where the strength is dependent upon the direction of the fibers.
References


13. Meirovitch 131

14. Meirovitch 130

15. Meirovitch 135

16. Meirovitch 441


19. Craig 23

20. Craig 163


24. Hindmarsh, A. C., ACM SIGNUM Newsl. 15, No. 4, 10, 1980


APPENDIX A
Spatial Derivatives

First-Order Formulas:

\[
\frac{\partial u_i}{\partial x} = \frac{1}{720} \frac{1}{\Delta x} \left[ -12 u_{i-3} + 108 u_{i-2} - 540 u_{i-1} + 540 u_{i+1} - 108 u_{i+2} + 12 u_{i+3} \right]
\]

\[
\frac{\partial u_i}{\partial x} = \frac{1}{720} \frac{1}{\Delta x} \left[ -1764 u_1 + 4320 u_2 - 5400 u_3 + 4800 u_4 - 2700 u_5 + 864 u_6 - 120 u_7 \right]
\]

\[
\frac{\partial u_i}{\partial x} = \frac{1}{720} \frac{1}{\Delta x} \left[ -120 u_1 - 924 u_2 + 1800 u_3 - 1200 u_4 + 600 u_5 - 180 u_6 + 24 u_7 \right]
\]

\[
\frac{\partial u_l}{\partial x} = \frac{1}{720} \frac{1}{\Delta x} \left[ 24 u_1 - 288 u_2 - 420 u_3 + 960 u_4 - 360 u_5 + 96 u_6 - 12 u_7 \right]
\]

\[
\frac{\partial u_{n-2}}{\partial x} = \frac{1}{720} \frac{1}{\Delta x} \left[ 12 u_{n-6} - 96 u_{n-5} + 360 u_{n-4} - 960 u_{n-3} + 420 u_{n-2} + 288 u_{n-1} - 24 u_n \right]
\]

\[
\frac{\partial u_{n-1}}{\partial x} = \frac{1}{720} \frac{1}{\Delta x} \left[ -24 u_{n-6} + 180 u_{n-5} - 600 u_{n-4} + 1200 u_{n-3} - 1800 u_{n-2} + 924 u_{n-1} + 120 u_n \right]
\]

\[
\frac{\partial u_n}{\partial x} = \frac{1}{720} \frac{1}{\Delta x} \left[ 120 u_{n-6} - 864 u_{n-5} + 2700 u_{n-4} - 4800 u_{n-3} + 5400 u_{n-2} - 4320 u_{n-1} + 1764 u_n \right]
\]
Second-Order Formulas:

\[ \frac{\partial^2 u_1}{\partial x^2} = \frac{1}{180 \Delta x^2} [2u_{1,3} - 27u_{1,2} + 270u_{1,1} - 490u_1 + 270u_{1,1} - 27u_{1,2} + 2u_{1,3}] \]

\[ \frac{\partial^2 u_2}{\partial x^2} = \frac{1}{180 \Delta x^2} [812u_1 - 3132u_2 + 5265u_3 - 5080u_4 + 2970u_5 - 972u_6 + 137u_7] \]

\[ \frac{\partial^2 u_3}{\partial x^2} = \frac{1}{180 \Delta x^2} [137u_1 - 147u_2 - 255u_3 + 470u_4 - 285u_5 + 93u_6 - 13u_7] \]

\[ \frac{\partial^2 u_4}{\partial x^2} = \frac{1}{180 \Delta x^2} [-13u_1 + 228u_2 - 420u_3 - 200u_4 + 15u_5 - 12u_6 + 2u_7] \]

\[ \frac{\partial^2 u_{n-2}}{\partial x^2} = \frac{1}{180 \Delta x^2} [-13u_n + 288u_{n-1} - 420u_{n-2} + 200u_{n-3} + 15u_{n-4} - 12u_{n-5} + 2u_{n-6}] \]

\[ \frac{\partial^2 u_{n-1}}{\partial x^2} = \frac{1}{180 \Delta x^2} [137u_{n-1} - 147u_{n-2} - 255u_{n-3} + 470u_{n-4} - 285u_{n-5} + 93u_{n-6} - 13u_{n-7}] \]

\[ \frac{\partial^2 u_6}{\partial x^2} = \frac{1}{180 \Delta x^2} [812u_n - 3232u_{n-1} + 5265u_{n-2} - 5080u_{n-3} + 2970u_{n-4} - 972u_{n-5} + 137u_{n-6}] \]
Third-Order Formulas:

\[
\frac{\partial^3 u_4}{\partial x^3} = -\frac{1}{8\Delta x^3}[-u_{i-3} + 8u_{i+2} - 13u_{i-1} + 13u_{i+1} - 8u_{i+3} + u_{i+5}]
\]

\[
\frac{\partial^3 u_1}{\partial x^3} = -\frac{1}{8\Delta x^3}[49u_1 - 232u_2 + 461u_3 - 496u_4 + 307u_5 - 104u_6 + 15u_7]
\]

\[
\frac{\partial^3 u_2}{\partial x^3} = -\frac{1}{8\Delta x^3}[-15u_1 + 56u_2 - 83u_3 + 64u_4 - 29u_5 + 8u_6 - u_7]
\]

\[
\frac{\partial^3 u_3}{\partial x^3} = -\frac{1}{8\Delta x^3}[u_1 + 8u_2 - 35u_3 + 48u_4 - 29u_5 + 8u_6 - u_7]
\]

\[
\frac{\partial^3 u_{n-2}}{\partial x^3} = -\frac{1}{8\Delta x^3}[u_n + 8u_{n-1} - 35u_{n-3} + 48u_{n-4} - 29u_{n-5} + 8u_{n-6} - u_{n-6}]
\]

\[
\frac{\partial^3 u_{n-1}}{\partial x^3} = -\frac{1}{8\Delta x^3}[-15u_n + 56u_{n-1} - 83u_{n-3} + 64u_{n-3} - 29u_{n-4} + 8u_{n-5} - u_{n-6}]
\]

\[
\frac{\partial^3 u_n}{\partial x^3} = -\frac{1}{8\Delta x^3}[49u_n - 232u_{n-2} + 461u_{n-2} - 496u_{n-3} + 307u_{n-4} - 104u_{n-5} + 15u_{n-6}]
\]
Fourth-Order Formulas:

\[
\frac{\partial^4 u_1}{\partial x^4} = -\frac{1}{3! \Delta x^4} \left[ u_{i-3} - 12 u_{i-2} + 39 u_{i-1} - 56 u_i + 39 u_{i+1} - 12 u_{i+2} + u_{i+3} \right]
\]

\[
\frac{\partial^4 u_2}{\partial x^4} = \frac{1}{3! \Delta x^4} \left[ 35 u_1 - 186 u_2 + 411 u_3 - 484 u_4 + 321 u_5 - 114 u_6 + 17 u_7 \right]
\]

\[
\frac{\partial^4 u_3}{\partial x^4} = \frac{1}{3! \Delta x^4} \left[ 17 u_1 - 84 u_2 + 171 u_3 - 184 u_4 + 111 u_5 - 36 u_6 + 5 u_7 \right]
\]

\[
\frac{\partial^4 u_4}{\partial x^4} = \frac{1}{3! \Delta x^4} \left[ 5 u_1 - 18 u_2 + 21 u_3 - 4 u_4 - 9 u_5 + 6 u_6 - u_7 \right]
\]

\[
\frac{\partial^4 u_{n-2}}{\partial x^4} = \frac{1}{3! \Delta x^4} \left[ 5 u_{n-1} - 18 u_{n-2} + 21 u_{n-3} - 4 u_{n-4} - 9 u_{n-5} + 6 u_{n-6} - u_{n-7} \right]
\]

\[
\frac{\partial^4 u_{n-1}}{\partial x^4} = \frac{1}{3! \Delta x^4} \left[ 17 u_{n-1} - 84 u_{n-2} + 171 u_{n-3} - 184 u_{n-4} + 111 u_{n-5} - 36 u_{n-6} + 5 u_{n-7} \right]
\]

\[
\frac{\partial^4 u_n}{\partial x^4} = \frac{1}{3! \Delta x^4} \left[ 35 u_n - 186 u_{n-1} + 411 u_{n-2} - 484 u_{n-3} + 321 u_{n-4} - 114 u_{n-5} + 17 u_{n-6} \right]
\]

\[
\frac{\partial^4 u_5}{\partial x^4} = -\frac{1}{3! \Delta x^4} \left[ 112 u_1 - 174 u_2 + 39 u_3 + 84 u_4 - 96 u_5 + 42 u_6 - 7 u_7 + 60 \frac{\partial u_1}{\partial x} \Delta x \right]
\]

\[
\frac{\partial^4 u_6}{\partial x^4} = \frac{1}{3! \Delta x^4} \left[ 182 u_1 - 546 u_2 + 861 u_3 - 884 u_4 + 546 u_5 - 186 u_6 + 27 u_7 - 60 \frac{\partial u_1}{\partial x} \Delta x \right]
\]
Shaft Dimensions

<table>
<thead>
<tr>
<th>Step Number</th>
<th>Outer Diameter Meters</th>
<th>Inner Diameter Meters</th>
<th>Length of Step Meters</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.52654e-02</td>
<td>1.45796e-02</td>
<td>0.0429</td>
</tr>
<tr>
<td>2</td>
<td>1.48844e-02</td>
<td>1.42240e-02</td>
<td>0.0762</td>
</tr>
<tr>
<td>3</td>
<td>1.45288e-02</td>
<td>1.37668e-02</td>
<td>0.0762</td>
</tr>
<tr>
<td>4</td>
<td>1.39954e-02</td>
<td>1.32588e-02</td>
<td>0.0476</td>
</tr>
<tr>
<td>5</td>
<td>1.35890e-02</td>
<td>1.27762e-02</td>
<td>0.0476</td>
</tr>
<tr>
<td>6</td>
<td>1.30556e-02</td>
<td>1.22682e-02</td>
<td>0.0476</td>
</tr>
<tr>
<td>7</td>
<td>1.25222e-02</td>
<td>1.16840e-02</td>
<td>0.0476</td>
</tr>
<tr>
<td>8</td>
<td>1.20650e-02</td>
<td>1.11760e-02</td>
<td>0.0476</td>
</tr>
<tr>
<td>9</td>
<td>1.14808e-02</td>
<td>1.05918e-02</td>
<td>0.0476</td>
</tr>
<tr>
<td>10</td>
<td>1.10236e-02</td>
<td>1.00838e-02</td>
<td>0.0476</td>
</tr>
<tr>
<td>11</td>
<td>1.04902e-02</td>
<td>9.55040e-03</td>
<td>0.0476</td>
</tr>
<tr>
<td>12</td>
<td>1.00584e-02</td>
<td>9.09320e-03</td>
<td>0.0476</td>
</tr>
<tr>
<td>13</td>
<td>9.70280e-03</td>
<td>8.76300e-03</td>
<td>0.0476</td>
</tr>
<tr>
<td>14</td>
<td>9.37260e-03</td>
<td>8.38200e-03</td>
<td>0.0476</td>
</tr>
<tr>
<td>15</td>
<td>8.91540e-03</td>
<td>7.87400e-03</td>
<td>0.0476</td>
</tr>
<tr>
<td>16</td>
<td>8.53440e-03</td>
<td>7.46760e-03</td>
<td>0.2572</td>
</tr>
</tbody>
</table>
APPENDIX C
PROGRAM FORCES
IMPLICIT DOUBLE PRECISION (A-H,O-Z)

PROGRAM FORCES:
1) CALLS SUBROUTINE UNIT WHICH CALCULATES THE UNIT VECTOR THAT
   DEFINE THE BODY-AXIS COORDINATE SYSTEM.
2) CALLS SUBROUTINE ANGLE WHICH CALCULATES THE ANGULAR VELOCITY C
   AND ACCELERATION OF THE BODY-AXIS COORDINATE SYSTEM RELATIVE
   TO THE INERTIAL COORDINATE SYSTEM.
3) READS IN THE POSITION DATA FOR THE HOSEL.
4) DEFINES THE ROTATION MATRIX FOR THE SYSTEM.
5) CALCULATES THE ACCELERATION OF THE HOSEL USING DSS044.
6) CALCULATES THE FORCES APPLIED TO THE CENTER OF MASS IN THE
   INERTIAL COORDINATE SYSTEM.
7) FINDS THE MOMENT GENERATED BY MOVING THE APPLIED FORCES FROM
   THE CENTER OF MASS TO THE HOSEL.
8) TRANSFORMS THE PREVIOUSLY DETERMINED FORCES, MOMENTS, ANGULAR
   VELOCITY AND ANGULAR ACCELERATION INTO BODY-AXIS COORDINATES.
9) DETERMINES THE MOMENT DUE TO ANGULAR MOMENTUM.
10) ADDS BOTH MOMENT COMPONENTS AND PRINTS SOLUTIONS.

******************************************************************************

VARIABLE LIST

INERTIAL COORDINATES
- PX, PY, PZ - POSITION OF MOVING ORIGIN (HOSEL) (m)
- VX, VY, VZ - VELOCITY OF MOVING ORIGIN (HOSEL) (m/s)
- AX, AY, AZ - ACCELERATION OF MOVING ORIGIN (HOSEL) (m/s^2)
- R(9,4,15) - DIRECTION COSINE MATRIX (ROTATION MATRIX)
- ACM(X,Y,Z) - ACCELERATION OF CENTER OF MASS (m/s^2)
- F(X,Y,Z) - FORCES ACTING AT CENTER OF MASS (N)
- Z(X,Y,Z) - UNIT VECTOR IN z DIRECTION (m)
- X(X,Y,Z) - UNIT VECTOR IN x DIRECTION (m)
- Y(X,Y,Z) - UNIT VECTOR IN y DIRECTION (m)
- W(X,Y,Z) - ANGULAR VELOCITY (rad/s)
- A(X,Y,Z) - ANGULAR ACCELERATION (rad/s^2)
- PMC(X,Y,Z) - POSITION FROM MARKER TO CENTER OF MASS (m)

BODY-AXIS COORDINATES
- CI(6) - MOMENTS OF INERTIA OF CLUB HEAD (kg/m^2)
- P(3) - POSITION OF CENTER OF MASS OF CLUB HEAD
   RELATIVE TO TOP OF HOSEL (m)
- FM - DISTANCE FROM MARKER TO CENTER OF MASS, IN z DIRECTION.
- CM - MASS OF CLUB HEAD (kg)
- FL(X,Y,Z) - FORCES AT ORIGIN (N)
- WL(X,Y,Z) - ANGULAR VELOCITY (rad/s)
- AL(X,Y,Z) - ANGULAR ACCELERATION (rad/s^2)
- CMM(X,Y,Z) - MOMENT DUE TO ANGULAR MOMENTUM (N*m)
- TOM(X,Y,Z) - TOTAL MOMENT (N*m)
- HM(X,Y,Z) - MOMENT DUE TO MOVEMENT OF FORCE (N*m)

OTHER PARAMETERS
- ZM - MASS/LENGTH (kg/m)
- G - GRAVITY (m/s^2)
- TL - INITIAL TIME (s)
- TU - FINAL TIME (s)
C *********************************************************************
C DIMENSION STATEMENT
COMMON/A/ X(3,415), Y(3,415), Z(3,415)
COMMON/B/ W(3,415), A(3,415)
DIMENSION CI(6), P(3), R(9,415)
DIMENSION PX(415), PY(415), PZ(415)
DIMENSION VX(415), VY(415), VZ(415)
DIMENSION AX(415), AY(415), AZ(415)
DIMENSION ACM(3,415), F(3,415)
DIMENSION HM(3,415), FL(3,415), WL(3,415)
DIMENSION AL(3,415), CMM(3,415)
DIMENSION TOM(3,415), PMC(3,415)
C *********************************************************************
C OPEN FILES
C OPEN(UNIT=6,FILE='spt4.dat')
OPEN(UNIT=7,FILE='force.end')
OPEN(UNIT=8,FILE='moment.end')
C DEFINE CONSTANTS
C NUMBER OF POINTS
K = 415
C INERTIA MATRIX FOR NUMBER ONE WOOD WITH RESPECT TO CENTER OF MASS.
C CI(1) = Ixx CI(4) = Ixy = Iyx
CI(2) = Iyy CI(5) = Iyz = Izy
CI(3) = Izz CI(6) = Ixz = Izx
CI(1) = 1.619D-04
CI(2) = 1.972D-04
CI(3) = 1.738D-04
CI(4) = -1.166D-05
CI(5) = -1.058D-05
CI(6) = -5.356D-05
C P(1) = 2.794D-02
P(2) = 1.623D-02
P(3) = 7.559D-02
PM = 9.782D-02
C CM = 2.012D-01
G = 9.81D0
TL = 0.0D0
TU = 415.D0/180.D0
C CALL SUBROUTINE UNIT TO GET UNIT VECTORS IN INERTIAL COORDINATES
CALL UNIT
C CALL SUBROUTINE ANGLE TO GET ANGULAR VELOCITY AND ACCELERATION
CALL ANGLE
READ IN DATA FOR POINT FOUR AND CONVERT TO METERS
DO 10 J = 1, 1415
   READ(6, 100) PX(J), PY(J), PZ(J)
   PX(J) = PX(J)/1000.0
   PY(J) = PY(J)/1000.0
   PZ(J) = PZ(J)/1000.0
10 CONTINUE

DEFINE ROTATION MATRIX
DO 15 J = 1, K
   R(1, J) = X(1, J)
   R(2, J) = Y(1, J)
   R(3, J) = Z(1, J)
   R(4, J) = X(2, J)
   R(5, J) = Y(2, J)
   R(6, J) = Z(2, J)
   R(7, J) = X(3, J)
   R(8, J) = Y(3, J)
   R(9, J) = Z(3, J)
15 CONTINUE

CALL DSS044(TL, TU, K, PX, VX, AX, 1, 1)
CALL DSS044(TL, TU, K, PY, VY, AY, 1, 1)
CALL DSS044(TL, TU, K, PZ, VZ, AZ, 1, 1)

STEP ONE: ACCELERATION OF HOSEL USING DSS044

STEP TWO: CALCULATE THE SEPARATE COMPONENTS OF ACM
FORCES ACTING AT HOSEL OF CLUB HEAD IN INERTIAL SYSTEM

\[
\begin{align*}
F(1,J) &= CM*ACM(1,J) \\
F(2,J) &= CM*ACM(2,J) \\
F(3,J) &= CM*ACM(3,J)
\end{align*}
\]

DEFINE FORCES, MOMENTS, OMEGA, AND ALPHA IN MOVING COORDINATES AT HOSEL

DEFINE LOCAL ACCELERATION IN MOVING COORDINATES

\[
\begin{align*}
F(xyz) &= \text{ROTATION(Transpose)} \ast F(XYZ) \\
M(xyz) &= \text{ROTATION(Transpose)} \ast M0(XYZ) \\
A(xyz) &= \text{ROTATION(Transpose)} \ast A(XYZ)
\end{align*}
\]

DO 23 L = 1, 3

\[
\begin{align*}
FL(L,J) &= R(L,J) \ast F(1,J) + R(L+3,J) \ast F(2,J) + R(L+6,J) \ast F(3,J) \\
WL(L,J) &= R(L,J) \ast W(1,J) + R(L+3,J) \ast W(2,J) + R(L+6,J) \ast W(3,J) \\
AL(L,J) &= R(L,J) \ast A(1,J) + R(L+3,J) \ast A(2,J) + R(L+6,J) \ast A(3,J)
\end{align*}
\]

CONTINUE 23

CALCULATE MOMENT ABOUT HOSEL GENERATED BY EFFECTIVE FORCES

\[
\begin{align*}
HM(1,J) &= P(3) \ast FL(2,J) - P(2) \ast FL(3,J) \\
HM(2,J) &= P(1) \ast FL(3,J) - P(3) \ast FL(1,J) \\
HM(3,J) &= P(2) \ast FL(1,J) - P(1) \ast FL(2,J)
\end{align*}
\]

MOMENT AT CENTER OF MASS IN LOCAL COORDINATES DUE TO ANGULAR MOMENTUM

\[
\begin{align*}
CMM(CM) &= \text{ALPHA} \ast I + \text{OMEGA} \times (\text{OMEGA} \times I) \\
CMM(1,J) &= CI(1) \ast AL(1,J) + CI(4) \ast AL(2,J) + CI(6) \ast AL(3,J) + \\
&\quad WL(2,J) \ast (CI(6) \ast WL(1,J) + CI(5) \ast WL(2,J) + CI(3) \ast WL(3,J)) - \\
&\quad WL(3,J) \ast (CI(4) \ast WL(2,J) + CI(2) \ast WL(2,J) + CI(5) \ast WL(3,J)) \\
CMM(2,J) &= CI(4) \ast AL(1,J) + CI(2) \ast AL(2,J) + CI(5) \ast AL(3,J) + \\
&\quad WL(3,J) \ast (CI(1) \ast WL(1,J) + CI(4) \ast WL(2,J) + CI(3) \ast WL(3,J)) - \\
&\quad WL(1,J) \ast (CI(6) \ast WL(1,J) + CI(5) \ast WL(2,J) + CI(3) \ast WL(3,J)) \\
CMM(3,J) &= CI(6) \ast AL(1,J) + CI(5) \ast AL(2,J) + CI(3) \ast AL(3,J) + \\
&\quad WL(1,J) \ast (CI(4) \ast WL(1,J) + CI(2) \ast WL(2,J) + CI(5) \ast WL(3,J)) - \\
&\quad WL(2,J) \ast (CI(1) \ast WL(1,J) + CI(4) \ast WL(2,J) + CI(6) \ast WL(3,J))
\end{align*}
\]

TOTAL MOMENT: \(TOM = HM + CMM\)

DO 24 L = 1, 3

\[
\begin{align*}
TOM(L,J) &= (HM(L,J) + CMM(L,J))
\end{align*}
\]

CONTINUE 24

WRITE MOMENTS AND FORCES TO APPLIED TO SHAFT OUTPUT FILES:

\[
\begin{align*}
\text{WRITE(7,101)} &\quad -FL(1,j), -FL(2,j), -FL(3,j) \\
\text{WRITE(8,101)} &\quad -TOM(1,j), -TOM(2,j), -TOM(3,j)
\end{align*}
\]

CONTINUE 20
C ********************************************************************************
C FORMAT STATEMENTS
C
101 FORMAT(3F20.10)
C
END
SUBROUTINE UNIT
IMPLICIT DOUBLE PRECISION (A-H,O-Z)

1. IS CALLED BY MAIN PROGRAM FORCES TO CALCULATE THE UNIT VECTORS
IN THE X,Y,Z DIRECTIONS.
2. IMPORT POSITION DATA FROM REFLECTIVE MARKERS 1, 2, AND 3
3. CALCULATES ORTHOGONAL UNIT VECTOR OF MOVING COORDINATE SYSTEM
AND RETURNS THE RESULTS TO THE PROGRAM FORCES

VARIABLE INDEX

PT1(X,Y,Z) - POSITION OF MARKER 1 ON SHAFT (m)
PT2(X,Y,Z) - POSITION OF MARKER 2 ON SHAFT (m)
PT3(X,Y,Z) - POSITION OF MARKER 3 ON SHAFT (m)
Z(X,Y,Z) - UNIT VECTOR IN Z DIRECTION (m)
X(X,Y,Z) - UNIT VECTOR IN X DIRECTION (m)
Y(X,Y,Z) - UNIT VECTOR IN Y DIRECTION (m)
A(X,Y,Z) - VECTOR FROM POINT 1 TO POINT 3
AP(X,Y,Z) - PROJECTION OF A IN Z DIRECTION
AN(X,Y,Z) - COMPONENT OF A NORMAL TO Z DIRECTION, IN X
W(X,Y,Z) - ANGULAR VELOCITY (rad/s)
ALF(X,Y,Z) - ANGULAR ACCELERATION (rad/s^2)

COMMON/A/ X(3,415), Y(3,415), Z(3,415)
COMMON/B/ W(3,415), ALF(3,415)

DIMENSION PT1(3,415), PT2(3,415), PT3(3,415)
DIMENSION A(3,415), AP(3,415), AN(3,415)
DIMENSION EL(4,415), C(415)

OPEN FILES
OPEN(UNIT= 9, FILE='spt1.dat')
OPEN(UNIT=10, FILE='spt2.dat')
OPEN(UNIT=11, FILE='spt3.dat')

ASSIGN CONSTANTS

N = 415

READ IN DATA POINTS

DO 10 I=1,N
READ(9,100) PT1(1,I), PT1(2,I), PT1(3,I)
READ(10,100) PT2(1,I), PT2(2,I), PT2(3,I)
READ(11,100) PT3(1,I), PT3(2,I), PT3(3,I)
10 CONTINUE

READ POSITION DATA TO METERS FROM MM

DO 20 I = 1,N
DO 30 J=1,3
PT1(J,I) = PT1(J,I)/1000.D0
PT2(J,I) = PT2(J,I)/1000.D0
PT3(J,I) = PT3(J,I)/1000.D0
30 CONTINUE
20 CONTINUE

z DIRECTION UNIT VECTOR
C DO 40 I = 1,N
C      EL(1,I) = DSQRT((PT2(1,I) - PT1(1,I))**2 +
1   (PT2(2,I) - PT1(2,I))**2 +
2   (PT2(3,I) - PT1(3,I))**2)
C DO 50 J = 1,3
3     Z(J,I) = (PT2(J,I) - PT1(J,I)) / EL(1,I)
50 CONTINUE
C **********************************************
C CALCULATE x-DIRECTION UNIT VECTOR BY
C      A = Ap + An
C      Ap IS THE PROJECTION OF A ONTO z
C      An IS A VECTOR IN THE x DIRECTION
C      Ap = (A "dot" z)*z = C*z
C DO 60 J = 1,3
6.  A(J,I) = PT3(J,I) - PT1(J,I)
. 60 CONTINUE
C C(I) = A(1,I)*Z(1,I) + A(2,I)*Z(2,I) + A(3,I)*Z(3,I)
C DO 70 J = 1,3
7.  AP(J,I) = C(I)*Z(J,I)
   AN(J,I) = A(J,I) - AP(J,I)
70 CONTINUE
C EL(3,I) = DSQRT(AN(1,I)**2+AN(2,I)**2+AN(3,I)**2)
C DO 80 J = 1,3
8.  X(J,I) = AN(J,I)/EL(3,I)
80 CONTINUE
C **********************************************
C y DIRECTION UNIT VECTOR:  y = z 'cross' x : RIGHT HAND RULE
C C EL(4,I) = DSQRT((Z(2,I)*X(3,I) - Z(3,I)*X(2,I))**2 +
1   (Z(3,I)*X(1,I) - Z(1,I)*X(3,I))**2 +
2   (Z(1,I)*X(2,I) - Z(2,I)*X(1,I))**2)
C      Y(1,I) = (Z(2,I)*X(3,I) - Z(3,I)*X(2,I)) / EL(4,I)
      Y(2,I) = (Z(3,I)*X(1,I) - Z(1,I)*X(3,I)) / EL(4,I)
      Y(3,I) = (Z(1,I)*X(2,I) - Z(2,I)*X(1,I)) / EL(4,I)
C **********************************************
C FORMAT STATEMENTS
100 FORMAT(3F11.4)
C **********************************************
40 CONTINUE
RETURN
END
SUBROUTINE ANGLE
IMPLICIT DOUBLE PRECISION (A-H,O-Z)

SUBROUTINE ANGLE:
1. IS CALLED BY THE MAIN PROGRAM FORCES TO CALCULATE THE ANGULAR
VELOCITY AND ACCELERATION OF THE GOLF SHAFT.
2. UNIT VECTORS ARE PASSED TO SUBROUTINE THROUGH COMMON BLOCK C
/A/.
3. ANGULAR VELOCITY AND ACCELERATION ARE RETURNED TO MAIN PROGRAM
THROUGH COMMON BLOCK /B/.

VARIABLE INDEX
Z(X,Y,Z) - UNIT VECTOR IN z DIRECTION (m)
X(X,Y,Z) - UNIT VECTOR IN x DIRECTION (m)
Y(X,Y,Z) - UNIT VECTOR IN y DIRECTION (m)
W(X,Y,Z) - ANGULAR VELOCITY (rad/s)
ALF(X,Y,Z) - ANGULAR ACCELERATION (rad/s^2)
TU - INITIAL TIME
TL - FINAL TIME
XX, XY, XZ - COMPONENTS OF X(X,Y,Z)
YY, YY, YZ - COMPONENTS OF Y(X,Y,Z)
ZX, ZY, ZZ - COMPONENTS OF Z(X,Y,Z)
XXT, XYT, XZT - COMPONENTS OF THE TIME DERIVATIVE OF X(X,Y,Z)
YXT, YYT, YZT - COMPONENTS OF THE TIME DERIVATIVE OF Y(X,Y,Z)
ZXT, ZYT, ZZT - COMPONENTS OF THE TIME DERIVATIVE OF Z(X,Y,Z)
WX, WY, WZ - COMPONENTS OF THE ANGULAR VELOCITY W(X,Y,Z)
AX, AY, AZ - COMPONENTS OF THE ANGULAR ACCELERATION A(X,Y,Z)

COMMON/A/ X(3,415), Y(3,415), Z(3,415)
COMMON/B/ W(3,415), ALF(3,415)

DIMENSION XX(415), XY(415), XZ(415)
DIMENSION YX(415), YY(415), YZ(415)
DIMENSION ZX(415), ZY(415), ZZ(415)
DIMENSION XXT(415), XYT(415), XZT(415)
DIMENSION YXT(415), YYT(415), YZT(415)
DIMENSION ZXT(415), ZYT(415), ZZT(415)
DIMENSION WX(415), WY(415), WZ(415)
DIMENSION AX(415), AY(415), AZ(415)

DEFINE CONSTANTS
N = 415
TL = 0.D0
TU = 2.30555556D0

SEPARATE UNIT VECTOR MATRICES INTO COMPONENTS
DO 10 I = 1,N
   XX(I) = X(1,1)
   XY(I) = X(2,1)
   XZ(I) = X(3,1)
   YX(I) = Y(1,1)
   YY(I) = Y(2,1)
   YZ(I) = Y(3,1)
   ZX(I) = Z(1,1)
   ZY(I) = Z(2,1)
   ZZ(I) = Z(3,1)
10   CONTINUE

CALL DSS004 TO COMPUTE TIME DERIVATIVE OF THE UNIT VECTORS

- 89 -
CALL DSS004(TL, TU, N, XX, XXT)
CALL DSS004(TL, TU, N, XY, XYT)
CALL DSS004(TL, TU, N, XZ, XZT)
CALL DSS004(TL, TU, N, YY, YYT)
CALL DSS004(TL, TU, N, YZ, YZT)
CALL DSS004(TL, TU, N, ZX, ZXT)
CALL DSS004(TL, TU, N, ZZ, ZYT)
CALL DSS004(TL, TU, N, ZZ, ZZZT)

C **************************************************************************
C CALCULATE ANGULAR VELOCITY: CRAIG (P. 163) EQ 5.40
C **************************************************************************
DO 20 I = 1, N
   WX(I) = XZT(I)*XY(I)+YZT(I)*YY(I)+ZZT(I)*ZY(I)
   WY(I) = XXT(I)*XZ(I)+YXT(I)*YZ(I)+ZXT(I)*ZZ(I)
   WZ(I) = XYT(I)*XX(I)+YYT(I)*YX(I)+ZYT(I)*ZX(I)
20 CONTINUE

C **************************************************************************
C CALL DSS004 TO CALCULATE ANGULAR ACCELERATION
C **************************************************************************
CALL DSS004(TL, TU, N, WX, AX)
CALL DSS004(TL, TU, N, WY, AY)
CALL DSS004(TL, TU, N, WZ, AZ)

C **************************************************************************
C COMBINE COMPONENTS OF OMEGA AND ALPHA TO FORM MATRICES W AND ALF
C **************************************************************************
DO 30 I = 1, N
   W(1, I) = WX(I)
   W(2, I) = WY(I)
   W(3, I) = WZ(I)
   ALF(1, I) = AX(I)
   ALF(2, I) = AY(I)
   ALF(3, I) = AZ(I)
30 CONTINUE

C **************************************************************************
C RETURN
C **************************************************************************
RETURN
END
APPENDIX D
PROGRAM SHAFTX.MAIN

PROGRAM MAINX

PROGRAM MAINX CALLS: (1) SUBROUTINE INITX TO DEFINE THE ODE INITIAL CONDITIONS, (2) SUBROUTINE DLSODE TO INTEGRATE THE ODES AND (3) SUBROUTINE PRINTX TO PRINT THE SOLUTION

DECLARE DOUBLE PRECISION
IMPLIED DOUBLE PRECISION (A-H,O-Z)

COMMON STATEMENTS
COMMON/T/ T, NSTOP, NORUN
COMMON/Y/ Y(450)
COMMON/F/ F(450)
COMMON/IO/ NI, NO
COMMON/L/ LCOUNT

DIMENSIONING ARRAYS REQUIRED FOR DLSODE
DIMENSION YV(450), RWORK(7484), IWORK(102)

DIMENSION ARRAYS FOR JMAPP
DIMENSION A(82,82), YOLD(82), FOLD(82)

EXTERNAL THE DERIVATIVE ROUTINE CALLED BY DLSODE
EXTERNAL FCN

ARRAY FOR THE TITLE, CHARACTERS END OF RUNS
CHARACTER TITLE(20)*4, ENDRUN(3)*4

DEFINE THE CHARACTERS END OF RUNS
DATA ENDRUN/'END ','OF R', 'UNS '/

DEFINE THE INPUT/OUTPUT UNIT NUMBERS
NI=5
NO=6

OPEN INPUT AND OUTPUT FILES
OPEN(NI,FILE='shaftx.dat')
OPEN(NO,FILE='shaftx.out')

INITIALIZE THE RUN COUNTER
NORUN=0

BEGIN A RUN
NORUN = NORUN + 1

INITIALIZE THE RUN TERMINATION VARIABLE
NSTOP = 0

INITIALIZE LCOUNT
LCOUNT = 1

READ THE FIRST LINE OF DATA
READ(NI,1000,END=999) (TITLE(I),I=1,20)

TEST FOR END OF RUNS IN THE DATA
DO 2 I=1,3
   IF (TITLE(I).NE.ENDRUN(I)) GO TO 3

CONTINUE

-92-
AN END OF RUNS HAS BEEN READ, SO TERMINATE EXECUTION
STOP

READ THE SECOND LINE OF DATA
READ(NI,1001,END=999)TO,TF,TP

READ THE THIRD LINE OF DATA
READ(NI,1002,END=999)NEQN,ERROR

PRINT A DATA SUMMARY
WRITE(NO,1003)NORUN,(TITLE(I),I=1,20),
TO,TF,TP,
NEQN,ERROR

INITIALIZE TIME
T=TO

SET THE INITIAL CONDITIONS
CALL INITX

SET THE INITIAL DERIVATIVES (FOR POSSIBLE PRINTING)
CALL DERVX

PRINT THE INITIAL CONDITIONS FOR SUBROUTINE DLSODE
CALL PRINTX

SET THE INITIAL CONDITIONS FOR SUBROUTINE DLSODE
TV=TO
DO 5 I=1,NEQN
  YV(I)=Y(I)
CONTINUE

SET THE PARAMETERS FOR SUBROUTINE DLSODE
ITOL = 1
RTOL = ERROR
ATOL = ERROR
LRW = 7484
LIW = 102
IOPT = 0
ITASK = 1
ISTATE = 1
TOUT = TP
MF = 10

CALL LSODE(FCN,NEQN,YV,TV,TOUT,ITOL,RTOL,ATOL,ITASK,ISTATE,
  IOPT,RWORK,LRW,IWORK,LIW,JAC,MF)

PRINT THE SOLUTION AT THE NEXT PRINT POINT
T=TV
DO 6 I=1,NEQN
  Y(I)=YV(I)
CONTINUE

LCOUNT = LCOUNT + 1

CALL DERVX
CALL PRINTX

TEST FOR AN ERROR CONDITION
IF(ISTATE.NE.2)THEN
PRINT A MESSAGE INDICATING ERROR CONDITION  
WRITE(NO,1004)ISTATE  
GO ON TO NEXT RUN  
GO TO 1  
END IF  
CHECK FOR RUN TERMINATION  
IF(NSTOP.NE.0)GO TO 1  
CHECK FOR THE END OF THE RUN  
TOUT=TV+TP  
IF(TV.LT.(TF-0.5D0*TP))GO TO 4  
THE CURRENT RUN IS COMPLETE, SO PRINT COMPUTATIONAL STATISTICS  
WRITE(NO,1005)RWORK(11),IWORK(14),IWORK(11),IWORK(11),IWORK(12),IWORK(13)  
GO TO 1  
*---------------------------------------------------------------------* 
* 1000 FORMAT(20A4)  
* 1001 FORMAT(3D10.0)  
* 1002 FORMAT(I5,2X,D10.0)  
* 1003 FORMAT(1H1,  
* 1 ' RUN NO. - ',I3,2X,20A4,//,  
* 2 ' INITIAL T - ',D10.3,//,  
* 3 ' FINAL T - ',D10.3,//,  
* 4 ' PRINT T - ',D10.3,//,  
* 5 ' NUMBER OF DIFFERENTIAL EQUATIONS - ',I3,//,  
* 6 ' MAXIMUM INTEGRATION ERROR - ',D10.3,//,  
* 7 1H1)  
* 1004 FORMAT(1H, //,' ISTATE = ',I3,//,  
* 1 ' INDICATING AN INTEGRATION ERROR, SO THE CURRENT RUN' //,  
* 2 ' IS TERMINATED. PLEASE REFER TO THE DOCUMENTATION FOR' //,  
* 3 ' SUBROUTINE', //,'DLSODE' //,  
* 4 ' FOR AN EXPLANATION OF THESE ERROR INDICATORS' )  
* 1005 FORMAT(1H, //,' COMPUTATIONAL STATISTICS' //,  
* 1 ' LAST STEP SIZE ' //,D10.3,//,  
* 2 ' LAST ORDER OF THE METHOD ' //,I10,//,  
* 3 ' TOTAL NUMBER OF STEPS TAKEN ' //,I10,//,  
* 4 ' NUMBER OF FUNCTION EVALUATIONS ' //,I10,//,  
* 5 ' NUMBER OF JACOBIAN EVALUATIONS ' //,I10,//)  
END  
*---------------------------------------------------------------------* 
SUBROUTINE FCN(NEQN,TV,YV,YDOT)  
SUBROUTINE FCN IS AN INTERFACE ROUTINE BETWEEN SUBROUTINES RKF45 AND DERV  
DECLARE DOUBLE PRECISION  
IMPLICIT DOUBLE PRECISION (A-H, O-Z)  
COMMON STATEMENTS  
COMMON/T/ T, NSTOP, NORUN
COMMON /IO/ NI, NO

TRANSFER INDEPENDENT VARIABLE, DEPENDENT VARIABLE VECTOR
FOR USE IN SUBROUTINE DERV
T=TV
DO 1 I=1,NEQN
   Y(I)=YV(I)
1 CONTINUE

EVALUATE THE DERIVATIVE
CALL DERVX

TRANSFER THE DERIVATIVE VECTOR FOR USE BY SUBROUTINE DLSODE
DO 2 I=1,NEQN
   YDOT(I)=F(I)
2 CONTINUE
RETURN
END
NUMOL SOLUTION FOR GOLF SHAFT: X EQUATION
0. 2.3055556 .000288230
82 .000000001
END OF RUNS
SUBROUTINE INITX
 IMPLICIT DOUBLE PRECISION (A-H,O-Z)

C SUBROUTINE INITX IS CALLED BY THE PROGRAM SHAFTX.MAIN TO:
C 1) READ IN EXTERNAL DATA FILES FX.IN AND DIAM.IN
C 2) DEFINE THE CONSTANTS OF THE PROBLEM
C 3) ASSIGN THE INITIAL VALUES OF X

C VARIABLE INDEX FOR SUBROUTINES INITAL, DERV, AND PRINT:
C COMMON VARIABLES:
C COMMON VARIABLES:
C TIME - TIME (sec)
C NSTOP - RUN STOP INDICATOR
C NORUN - RUN NUMBER
C U1 - REPRESENTS X (m)
C U2 - THE TIME DERIVATIVE OF U1 (m/s)
C U1T - THE TIME DERIVATIVE OF U1 (m/s)
C U2T - THE TIME DERIVATIVE OF U2 (m/s^2)
C NI - THE NUMBER OF INPUT FILE SHAFTX.DAT
C NO - THE NUMBER OF OUTPUT FILE SHAFTX.OUT
C DZ - THE SPATIAL DIFFERENCE IN Z DIRECTION (m)
C TM - THE MASS PER UNIT LENGTH (kg/m)
C ZL - THE LENGTH OF THE SHAFT (m)
C N - NUMBER OF ODES
C FX - FORCE APPLIED AT END OF SHAFT IN X DIRECTION (N)
C FZ - FORCE APPLIED AT END OF SHAFT IN Y DIRECTION (N)
C YM - MOMENT APPLIED AT END OF SHAFT ABOUT Y AXIS (N*m)
C C - TERM REPRESENTING TiE (N*m^2)
C LCOUNT - TIME COUNT

C VARIABLES IN SUBROUTINE INITAL ONLY:
C DO - OUTER DIAMETER OF SHAFT (m)
C DI - INNER DIAMETER OF SHAFT (m)
C TI - AREA MOMENT OF INERTIA (m^4)
C E - MODULUS OF ELASTICITY (N/m^2)

C VARIABLES IN SUBROUTINE DERV ONLY:
C U1Z1 - FIRST DERIVATIVE OF U1 WITH RESPECT TO Z
C U1Z2 - SECOND DERIVATIVE OF U1 WITH RESPECT TO Z
C TZ1 - FIRST DERIVATIVE OF TEMP WITH RESPECT TO Z
C TZ2 - SECOND DERIVATIVE OF TEMP WITH RESPECT TO Z
C TEMP - TEMPORARY TERM REPRESENTING C*U1Z2

C COMMON AND DIMENSION STATEMENTS:
C COMMON /T/ TIME, NSTOP, NORUN
C COMMON /Y/ U1(41), U2(41)
C COMMON /F/ U1T(41), U2T(41)
C COMMON /IO/ NI, NO
C COMMON /C/ DZ, TM, ZL
C COMMON /R/ FX(8000), FZ(8000), YM(8000), C(41)
C COMMON /L/ LCOUNT, ITIME
C DIMENSION DO(41), DI(41), TI(41)

C OPEN AND READ EXTERNAL FILES:
C OPEN(UNIT= 9,FILE='diam.in')
OPEN(UNIT=10,FILE='fx.in')

DO 1 I = 1,41
  READ(9,100) DO(I), DI(I)
  CONTINUE

DO 2 I = 1,8000
  READ(10,101) FX(I), FZ(I), YM(I)
  CONTINUE

ASSIGN PROBLEM CONSTANTS:
ITIME = 0
N = 41
TM = 0.115D0
E = 20.0D+10
  ZL = 1.024D0
  DZ = ZL/DFLOAT(N-1)

INITIAL CONDITION:
  DO 3 I = 1,N
  U1(I) = 0.D0
  U2(I) = 0.D0
  CONTINUE

DEFINE VARIABLES CHANGING WITH LENGTH:
  TI(I) = DACOS(-1.DO)/64.DO*(DO(I)**4 - DI(I)**4)
  C(I) = E*TI(I)
  CONTINUE

FORMAT STATEMENTS:
100 FORMAT (2D16.8)
101 FORMAT(3D16.8)
RETURN
END

SUBROUTINE DERVX
IMPLICIT DOUBLE PRECISION (A-H,O-Z)

SUBROUTINE DERV IS CALLED BY INTEGRATOR LSODE TO DEFINE THE
ODES AND BOUNDARY CONDITIONS FOR THE NUMOL SOLUTION.

COMMON AND DIMENSION STATEMENTS:
  COMMON/T/ TIME, NSTOP, NORUN
  COMMON/I/ NI, NO
  COMMON/C/ DZ, TM, ZL
  COMMON/R/ FX(8000), FZ(8000), YM(8000), C(41)
  COMMON/L/ LCOUNT, ITIME
  DIMENSION U1Z1(41), U1Z2(41), TZ1(41), TZ2(41), TEMP(41)
  N = 41

BOUNDARY CONDITIONS AND DIFFERENTIATION:
BOUNDARY CONDITION NUMBER 1 AND 2:

\[ Y(0) = 0 \quad \frac{DY(0)}{DZ} = 0.0 \]

BC 1 BECOMES: \[ U1(1) = 0.0 \]

SOLVE FOR \( U1 \) AT POINT 2 USING THE 7 POINT APPROXIMATION OF THE FIRST DERIVATIVE. BC 2 BECOMES:

\[
U1(2) = \frac{(147.0D*U1(1)+450.0D*U1(3)-400.0D*U1(4)+225.0D*U1(5)-72.0D*U1(6)+10.0D*U1(7))/360.0D}{(147.0D*U1(1)+450.0D*U1(3)-400.0D*U1(4)+225.0D*U1(5)-72.0D*U1(6)+10.0D*U1(7))/360.0D}
\]

CALCULATE \( D^2Y/DZ^2 \) USING A FIVE POINT APPROXIMATION OF THE SECOND DERIVATIVE, THEN APPLY BOUNDARY CONDITIONS 3 AND 4.

BC 3 BECOMES:

\[ \text{T} \text{EM}(N) = E*TI*U1Z2(N) = YM(LCOUNT) - FZ(LCOUNT)*U1Z1(N) \]

BC 4 BECOMES:

\[ \frac{D(E*TI*U1Z2(N))}{DZ} = -FX(LCOUNT) \]

BOUNDARY CONDITIONS 1 AND 2:

\[
U1(1) = 0.0
\]

\[
U1(2) = \frac{(147.0D*U1(1)+450.0D*U1(3)-400.0D*U1(4)+225.0D*U1(5)-72.0D*U1(6)+10.0D*U1(7))/360.0D}{(147.0D*U1(1)+450.0D*U1(3)-400.0D*U1(4)+225.0D*U1(5)-72.0D*U1(6)+10.0D*U1(7))/360.0D}
\]

SECOND DERIVATIVE U1Z2:

CALL DSS044(0.0D,ZL,N,U1,U1Z1,U1Z2,1,1)

MULTIPLY U1Z2 BY C TO GET VARIABLE TEMP:

\[
\text{DO } 10 \text{ I = } 1, N-2
\]

\[
\text{TEMP}(I) = C(I)*U1Z2(I)
\]

\[
\text{CONTINUE}
\]

BOUNDARY CONDITIONS 3 AND 4:

\[
\text{TEMP}(N) = YM(LCOUNT) - FZ(LCOUNT)*(10.0D*U1(N-6)
\]

\[
- 72.0D*U1(N-5) + 225.0D*U1(N-4)
\]

\[
- 400.0D*U1(N-3) + 450.0D*U1(N-2)
\]

\[
- 360.0D*U1(N-1) + 147.0D*U1(N))/60.0D*DZ)
\]

\[
\text{TEMP}(N-1) = (147.0D*TEMP(N) + 450.0D*TEMP(N-2)
\]

\[
- 72.0D*TEMP(N-5) + 225.0D*TEMP(N-4)
\]

\[
- 400.0D*TEMP(N-3) + 450.0D*TEMP(N-2)
\]

\[
+ 60.0D*FX(LCOUNT)*DZ)/360.0D
\]

SECOND DERIVATIVE OF TEMP:

CALL DSS044(0.0D,ZL,N,TEMP,TZ1,TZ2,1,1)

DEFINE EQUATION OF MOTION:

\[
\text{DO } 100 \text{ I = } 1, N
\]

\[
U1T(I) = U2(I)
\]

\[
U2T(I) = -TZ2(I)/TM - 45.0D*U2(I)
\]

100
CONTINUE
RETURN
END

*******************************************************************************
*******************************************************************************

SUBROUTINE PRINTX
IMPLICIT DOUBLE PRECISION (A-H,O-Z)
SUBROUTINE PRINT IS CALLED BY MAIN PROGRAM SHAFTX.MAIN TO PRINT
THE NUMOL SOLUTION OF THE EQUATION OF MOTION IN THE X DIRECTION.

COMMON STATEMENTS:

COMMON/T/ TIME, NSTOP, NORUN
1 /Y/ U1(41), U2(41)
2 /F/ U1T(41), U2T(41)
COMMON/I0/ NI, NO
COMMON/C/ DZ, TM, ZL
COMMON/R/ FX(8000), FZ(8000), YM(8000), C(41)
COMMON/L/ LCOUNT, ITIME

OPEN OUTPUT FILES:
OPEN(UNIT=15,FILE='x.out')

PRINT EVERY 10TH NUMOL SOLUTIONS:
ITIME = ITIME + 1
IF(ITIME .GT. 9) THEN
   ITIME = 0
   WRITE(15,5) (U1(I), I=1,41,8)
ENDIF

FORMAT STATEMENTS:
5 FORMAT(6F9.5)
RETURN
END
PROGRAM SHAFTY_MAIN

PROGRAM MAINY
C
C PROGRAM MAIN CALLS: (1) SUBROUTINE INITY TO DEFINE THE ODE
C INITIAL CONDITIONS, (2) SUBROUTINE DLSODE TO INTEGRATE THE ODES
C AND (3) SUBROUTINE PRINTY TO PRINT THE SOLUTION
C
C DECLARE DOUBLE PRECISION
C IMPLICIT DOUBLE PRECISION (A-H,O-Z)
C
C COMMON STATEMENTS
C COMMON/T/ T, NSTOP, NORUN
1 /Y/ Y(450)
2 /F/ F(450)
C COMMON/IO/ NI, NO
C COMMON/L/ LCOUNT
C
C DIMENSIONING ARRAYS REQUIRED FOR DLSODE
C DIMENSION YV(450), RWORK(7484), IWORK(102)
C
C DIMENSION ARRAYS FOR JMAP
C DIMENSION A(82,82), YOLD(82), FOLD(82)
C
C EXTERNAL THE DERIVATIVE ROUTINE CALLED BY DLSODE
C EXTERNAL FCN
C
C ARRAY FOR THE TITLE, CHARACTERS END OF RUNS
C CHARACTER TITLE(20)*4, ENDRUN(3)*4
C
C DEFINE THE CHARACTERS END OF RUNS
C DATA ENDRUN/'END ','OF RUNS '/
C
C DEFINE THE INPUT/OUTPUT UNIT NUMBERS
C NI=5
C NO=6
C
C OPEN INPUT AND OUTPUT FILES
C OPEN(NI,FILE='shafty.dat')
C OPEN(NO,FILE='shafty.out')
C
C INITIALIZE THE RUN COUNTER
C NORUN=0
C
C BEGIN A RUN
C NORUN = NORUN + 1
C
C INITIALIZE THE RUN TERMINATION VARIABLE
C NSTOP = 0
C
C INITIALIZE LCOUNT
C LCOUNT = 1
C
C READ THE FIRST LINE OF DATA
C READ(NI,1000,END=999) (TITLE(I), I=1,20)
C
C TEST FOR END OF RUNS IN THE DATA
C DO 2 I=1,3
C IF (TITLE(I).NE. ENDRUN(I)) GO TO 3
C CONTINUE
AN END OF RUNS HAS BEEN READ, SO TERMINATE EXECUTION
STOP

READ THE SECOND LINE OF DATA
READ(NI,1001,END=999)T0,TF,TP

READ THE THIRD LINE OF DATA
READ(NI,1002,END=999)NEQN,ERROR

PRINT A DATA SUMMARY
WRITE(NO,1003)NORU~,(TITLE(I),I=1,20), 1 T0,TF,TP,
 2 NEQN,ERROR

INITIALIZE TIME
T=T0

SET THE INITIAL CONDITIONS
CALL INITY

SET THE INITIAL DERIVATIVES (FOR POSSIBLE PRINTING)
CALL DERVY

PRINT THE INITIAL CONDITIONS FOR SUBROUTINE DLSODE
CALL PRINTY

SET THE INITIAL CONDITIONS FOR SUBROUTINE DLSODE
TV=T0
DO 5 I=1,NEQN
  YV(I)=Y(I)
  CONTINUE

SET THE PARAMETERS FOR SUBROUTINE DLSODE
ITOL = 1
RTOL = ERROR
ATOL = ERROR
LRW = 7484
LIW = 102
IOPT = 0
ITASK = 1
ISTATE = 1
TOUT = TP
MF = 10

CALL LSODE(FCN,NEQN,YV,TV,TOUT,ITOL,RTOL,ATOL,ITASK,ISTATE,
  IOPT,RWORK,LRW,IWORK,LIW,JAC,MF)

PRINT THE SOLUTION AT THE NEXT PRINT POINT
T=TV
DO 6 I=1,NEQN
  Y(I)=YV(I)
  CONTINUE

LCOUNT = LCOUNT + 1

CALL DERVY
CALL PRINTY

TEST FOR AN ERROR CONDITION
IF(ISTATE.NE.2)THEN
PRINT A MESSAGE INDICATING ERROR CONDITION
WRITE(NO,1004)ISTATE
GO ON TO NEXT RUN
GO TO 1
END IF

CHECK FOR RUN TERMINATION
IF(NSTOP.NE.0)GO TO 1

CHECK FOR THE END OF THE RUN
TOUT=TV+TP
IF(TV.LT. (TF-0.5D0*TP))GO TO 4

THE CURRENT RUN IS COMPLETE, SO PRINT COMPUTATIONAL STATISTICS
WRITE(NO,1005)RWORK(11),IWORK(14),IWORK(11),IWORK(12),IWORK(13)
GO TO 1

********************************************************************
********************************************************************
* FORMATS
********************************************************************

1000 FORMAT(20A4)
1001 FORMAT(3D10.0)
1002 FORMAT(I5,20X,D10.0)
1003 FORMAT(1H1,
  1 ' RUN NO. - ',I3,2X,20A4,///,
  2 ' INITIAL T - ',D10.3,///,
  3 ' FINAL T - ',D10.3,///,
  4 ' PRINT T - ',.D10.3,///,
  5 ' NUMBER OF DIFFERENTIAL EQUATIONS - ',I3,///,
  6 ' MAXIMUM INTEGRATION ERROR - ',D10.3,///,
  7 1H1)
1004 FORMAT(1H1,' ISTATE = ',I3,///,
  1 ' INDICATING AN INTEGRATION ERROR, SO THE CURRENT RUN' ,///,
  2 ' IS TERMINATED. PLEASE REFER TO THE DOCUMENTATION FOR' ,///,
  3 ' SUBROUTINE',//,25X,'DLSODE',//,
  4 ' FOR AN EXPLANATION OF THESE ERROR INDICATORS')
1005 FORMAT(1H1,' COMPUTATIONAL STATISTICS',///,
  1 ' LAST STEP SIZE',D10.3,///,
  2 ' LAST ORDER OF THE METHOD',I10,///,
  3 ' TOTAL NUMBER OF STEPS TAKEN',I10,///,
  4 ' NUMBER OF FUNCTION EVALUATIONS',I10,///,
  5 ' NUMBER OF JACOBIAN EVALUATIONS',I10,///)

END

******************************************************************************
******************************************************************************
SUBROUTINE FCN(NEQN,TV,YV,YDOT)
SUBROUTINE FCN IS AN INTERFACE ROUTINE BETWEEN SUBROUTINES RKF45 AND DERV
DECLARE DOUBLE PRECISION
IMPLICIT DOUBLE PRECISION (A-H, O-Z)
COMMON STATEMENTS

- 103 -
COMMON/T/ T, NSTOP, NORUN
1 /Y/ Y(I)
2 /F/ F(I)
COMMON/IO/ NI, NO

C ABSOLUTE DIMENSION THE DEPENDENT VARIABLE, DERIVATIVE VECTORS
DOUBLE PRECISION YV(450), YDOT(450)

C TRANSFER INDEPENDENT VARIABLE, DEPENDENT VARIABLE VECTOR
FOR USE IN SUBROUTINE DERVY

T=TV
DO 1 I=1,NEQN
   Y(I)=YV(I)
   CONTINUE

C EVALUATE THE DERIVATIVE
CALL DERVY

C TRANSFER THE DERIVATIVE VECTOR FOR USE BY SUBROUTINE DLSODE
DO 2 I=1,NEQN
   YDOT(I)=F(I)
   CONTINUE
RETURN
END
NUMOL SOLUTION FOR GOLF SHAFT: Y EQUATION

0.  2.3055556  .000288230
82  .000000001

END OF RUNS
* PROGRAM SHAFTY.DP *

SUBROUTINE INITY
IMPLICIT DOUBLE PRECISION (A-H,O-Z)

SUBROUTINE INITY IS CALLED BY THE PROGRAM SHAFTY.MAIN TO:
1) READ IN EXTERNAL DATA FILES FY.IN AND DIAM.IN
2) DEFINE THE CONSTANTS OF THE PROBLEM
3) ASSIGN THE INITIAL VALUES OF Y

VARIABLE INDEX FOR SUBROUTINES INITY, DERVY, AND PRINTY:

COMMON VARIABLES:
- TIME - TIME (sec)
- NSTOP - RUN STOP INDICATOR
- NORUN - RUN NUMBER
- U1 - REPRESENTS Y (m)
- U2 - THE TIME DERIVATIVE OF U1 (m/s)
- U1T - THE TIME DERIVATIVE OF U1 (m/s)
- U2T - THE TIME DERIVATIVE OF U2 (m/s^2)
- NI - THE NUMBER OF INPUT FILE SHAFTY.DAT
- NO - THE NUMBER OF OUTPUT FILE SHAFTY.OUT
- DZ - THE SPATIAL DIFFERENCE IN Z DIRECTION (m)
- TM - THE MASS PER UNIT LENGTH (kg/m)
- ZL - THE LENGTH OF THE SHAFT (m)
- N - NUMBER OF ODES
- FY - FORCE APPLIED AT END OF SHAFT IN Y DIRECTION (N)
- FZ - FORCE APPLIED AT END OF SHAFT IN Z DIRECTION (N)
- XM - MOMENT APPLIED AT END OF SHAFT ABOUT X AXIS (N*m)
- C - TERM REPRESENTING T*I*E (N*m^2)
- LCOUNT - TIME COUNT

VARIABLES IN SUBROUTINE INITY ONLY:
- DO - OUTER DIAMETER OF SHAFT (m)
- DI - INNER DIAMETER OF SHAFT (m)
- TI - AREA MOMENT OF INERTIA (m^4)
- E - MODULUS OF ELASTICITY (N/m^2)

VARIABLES IN SUBROUTINE DERVY ONLY:
- U1ZI - FIRST DERIVATIVE OF U1 WITH RESPECT TO Z
- U1Z2 - SECOND DERIVATIVE OF U1 WITH RESPECT TO Z
- T1Z1 - FIRST DERIVATIVE OF TEMP WITH RESPECT TO Z
- T1Z2 - SECOND DERIVATIVE OF TEMP WITH RESPECT TO Z
- TEMP - TEMPORARY TERM REPRESENTING C*U1Z2

COMMON AND DIMENSION STATEMENTS:

COMMON/T/ TIME, NSTOP, NORUN
1 /Y/ U1(41), U2(41)
2 /F/ U1T(41), U2T(41)
COMMON/IO/ NI, NO
COMMON/C/ DZ, TM, ZL
COMMON/R/ FY(8000), FZ(8000), XM(8000), C(41)
COMMON/L/ LCOUNT, ITIME
DIMENSION DO(41), DI(41), TI(41)

OPEN AND READ EXTERNAL FILES:
OPEN(UNIT= 9,FILE='diam.in')
OPEN(UNIT=10,FILE='fy.in')

1 DO 1 I = 1,41
   READ(9,100) DO(I), DI(I)
   CONTINUE

2 DO 2 I = 1,8000
   READ(10,101) FY(I), FZ(I), XM(I)
   CONTINUE

ASSIGN PROBLEM CONSTANTS:
   ITIME = 0
   N = 41
   TM = 0.115 DO
   E = 20.0 DO+10
   ZL = 1.024 DO
   DZ = ZL/DFLOAT(N-1)

INITIAL CONDITION:
   DO 3 I = 1,N
   U1(I) = 0. DO
   U2(I) = 0. DO

DEFINE VARIABLES CHANGING WITH LENGTH:
   TI(I) = DACOS(-1.D0)/64.DO*(DO(I)**4 - DI(I)**4)
   C(I) = E*TI(I)

FORMAT STATEMENTS:
   100 FORMAT(2D16.8)
   101 FORMAT(3D16.8)

RETURN
END

SUBROUTINE DERVY
IMPLICIT DOUBLE PRECISION (A-H,O-Z)

COMMON AND DIMENSION STATEMENTS:
   COMMON/T/ TIME, NSTOP, NORUN
   COMMON/Y/ U1(41), U2(41)
   COMMON/F/ U1T(41), U2T(41)
   COMMON/C/ DZ, TM, ZL
   COMMON/R/ FY(8000), FZ(8000), XM(8000), C(41)
   COMMON/L/ LCOUNT, ITIME
   DIMENSION U1Z1(41), U1Z2(41), TZ1(41), TZ2(41), TEMP(41)
   N = 41

BOUNDARY CONDITIONS AND DIFFERENTIATION:
BOUNDARY CONDITION NUMBER 1 AND 2:
Y(0) = 0  DY(0)/DZ = 0.DO

BC 1 BECOMES:  U1(1) = 0.DO

SOLVE FOR U1 AT POINT 2 USING THE 7 POINT APPROXIMATION
OF THE FIRST DERIVATIVE.  BC 2 BECOMES:

    U1(2) = (147.DO*U1(1)+450.DO*U1(3)-400.DO*U1(4)+225.DO*U1(5)
             - 72.DO*U1(6)+ 10.DO*U1(7))/360.DO

CALCULATE D^2Y/DZ^2 USING A FIVE POINT APPROXIMATION OF THE
SECOND DERIVATIVE, THEN APPLY BOUNDARY CONDITIONS 3 AND 4.

BC 3 BECOMES:
TEMP(N) = E*TI*U1Z2(N) = XM(LCOUNT) - FZ(LCOUNT)*U1Z1(N)

BC 4 BECOMES:
D[E*TI*U1Z2(N)]/DZ = - FY(LCOUNT)

BOUNDARY CONDITIONS 1 AND 2:
U1(1) = 0.DO
U1(2) = (147.DO*U1(1)+450.DO*U1(3)-400.DO*U1(4)+225.DO*U1(5)
             - 72.DO*U1(6)+ 10.DO*U1(7))/360.DO

SECOND DERIVATIVE U1Z2:
CALL DSS044(0.DO,2L,N,U1,U1Z1,U1Z2,1,1)

CONVERT TO VARIABLE TEMP:
DO 10 I = 1,N-2
    TEMP(I) = C(I)*U1Z2(I)
10 CONTINUE

BOUNDARY CONDITIONS 3 AND 4:
TEMP(N) = XM(LCOUNT) - FZ(LCOUNT)*(10.DO*U1(N-6)
    - 72.DO*U1(N-5) + 225.DO*U1(N-4)
    - 400.DO*U1(N-3) + 450.DO*U1(N-2)
    - 360.DO*U1(N-1) + 147.DO*U1(N))/60.DO*DY

TEMP(N-1) = ( 147.DO*TEMP(N-2) + 450.DO*TEMP(N-1)
       - 400.DO*TEMP(N-3) + 225.DO*TEMP(N-4)
       - 72.DO*TEMP(N-5) + 10.DO*TEMP(N-6)
       +  60.DO*FY(LCOUNT)*DZ)/360.DO

SECOND DERIVATIVE OF TEMP:
CALL DSS044(0.DO,2L,N,TEMP,TZ1,TZ2,1,1)

DEFINE EQUATION OF MOTION:
DO 100 I=1,N
    U1T(I) = U2(I)
    U2T(I) = - TZ2(I)/TM - 45.DO*U2(I)
100 CONTINUE
RETURN
END

****************************************************************
****************************************************************
SUBROUTINE PRINTY
IMPLICIT DOUBLE PRECISION (A-H,O-Z)
(*)
SUBROUTINE PRINTY IS CALLED BY MAIN PROGRAM SHAFTY_MAIN TO PRINT THE NUMOL SOLUTION OF THE EQUATION OF MOTION IN THE Y DIRECTION.

COMMON STATEMENTS:
COMMON/T/ TIME, NSTOP, NORUN
1 /Y/ U1(41), U2(41)
2 /F/ U1T(41), U2T(41)
COMMON/IO/ NI, NO
COMMON/C/ DZ, TM, ZL
COMMON/R/ FY(8000), FZ(8000), XM(8000), C(41)
COMMON/L/ LCOUNT, ITIME

OPEN OUTPUT FILES:
OPEN(UNIT=15, FILE='y.out')

PRINT NUMOL SOLUTIONS:
ITIME = ITIME + 1
IF(ITIME .GT. 9) THEN
  WRITE(15,5) (U1(I), I=1,41,8)
ENDIF

FORMAT STATEMENTS:
5 FORMAT(6F9.5)
RETURN
END
PROGRAM MAIN

C
C PROGRAM MAIN CALLS: (1) SUBROUTINE INITAL TO DEFINE THE ODE
C INITIAL CONDITIONS, (2) SUBROUTINE DLSODE TO INTEGRATE THE ODES
C AND (3) SUBROUTINE PRINT TO PRINT THE SOLUTION
C
C DECLARE DOUBLE PRECISION
IMPLICIT DOUBLE PRECISION (A-H, O-Z)

NORUN.

EXTERNAL THE DERIVATIVE ROUTINE CALLED BY DLSODE
EXTERNAL FCN

DIMENSIONING ARRAYS REQUIRED FOR DLSODE
DIMENSION YV(450), RWORK(7484), IWORK(102)

DIMENSIONING ARRAYS REQUIRED FOR JMAP
DIMENSION A(82, 82), YOLD(82), FOLD(82)

ARRAY FOR THE TITLE, CHARACTERS END OF RUNS
CHARACTER TITLE(20)*4, ENDRUN(3)*4

DEFINE THE CHARACTERS END OF RUNS
DATA ENDRUN/'END ', 'OF R', 'UNS '/

DEFINE THE INPUT/OUTPUT UNIT NUMBERS
NI=5
NO=6
NE=7

OPEN INPUT AND OUTPUT FILES
OPEN(NI, FILE='theta.dat')
OPEN(NO, FILE='theta.out')

INITIALIZE THE RUN COUNTER
NORUN=0

BEGIN A RUN
NORUN = NORUN + 1

INITIALIZE THE RUN TERMINATION VARIABLE
NSTOP = 0.

READ THE FIRST LINE OF DATA
READ(NI, 1000, END=999) (TITLE(I), I=1, 20)

TEST FOR END OF RUNS IN THE DATA
DO 2 I=1, 3
   IF (TITLE(I).NE. ENDRUN(I)) GO TO 3
2 CONTINUE
AN END OF RUNS HAS BEEN READ, SO TERMINATE EXECUTION
STOP

READ THE SECOND LINE OF DATA
READ(NI,1001,END=999)T0,TF,TP

READ THE THIRD LINE OF DATA
READ(NI,1002,END=999)NEQN,ERROR

PRINT A DATA SUMMARY
WRITE(NO,1003)NORUN,(TITLE(I),I=1,20),
1 T0,TF,TP,
2 NEQN,ERROR

INITIALIZE TIME
T=T0

INITIALIZE LCOUNT
LCOUNT=1

SET THE INITIAL CONDITIONS
CALL INITIAL

SET THE INITIAL DERIVATIVES (FOR POSSIBLE PRINTING)
CALL Derv

PRINT THE INITIAL CONDITIONS FOR SUBROUTINE DLSODE
CALL PRINT(NI,NO)

SET THE INITIAL CONDITIONS FOR SUBROUTINE DLSODE
TV=T0
DO 5 I=1,NEQN
YV(I)=Y(I)
5 CONTINUE

SET THE PARAMETERS FOR SUBROUTINE DLSODE
ITOL = 1
RTOL = ERROR
ATOL = ERROR
LRW = 7484
LIW = 102
IOPT = 0
ITASK = 1
ISTATE = 1

CHECK INITIAL TIME STEP
TOUT=TP
MF = 10

CALL LSODE(FCN,NEQN,YV,TV,TOUT,ITOL,RTOL,ATOL,ITASK,ISTATE,
1 IOPT,RWORK,LRW,IWORK,LIW,JAC,MF)

PRINT THE SOLUTION AT THE NEXT PRINT POINT
T=TV
DO 6 I=1,NEQN
Y(I)=YV(I)
6 CONTINUE

LCOUNT=LCOUNT+1
CALL DERV
CALL PRINT(NI,NO)

C
C TEST FOR AN ERROR CONDITION
IF(ISTATE.NE.2)THEN
C
C PRINT A MESSAGE INDICATING ERROR CONDITION
WRITE(NO,1004) ISTATE
C
C GO ON TO NEXT RUN
GO TO 1
END IF
C
C CHECK FOR RUN TERMINATION
IF(NSTOP.NE.0)GO TO 1
C
TOUT=TV+TP
IF(TV.LT.(TF-0.5D0*TP))GO TO 4
C
CALL JMAP
CALL JMAP(NEQN,A,Y,YOLD,F,FOLD,T)
C
THE CURRENT RUN IS COMPLETE, SO PRINT COMPUTATIONAL STATISTICS
WRITE(NO,1005)RWORK(11),IWORK(14),IWORK(11),IWORK(12),IWORK(13)
C
GO TO 1
C
********************************************************************
C
 SUBROUTINE FCN (NEQN, TV, YV, YDOT)
 C
 SUBROUTINE FCN IS AN INTERFACE ROUTINE BETWEEN SUBROUTINES RKF45

---112---
C AND DERV
C IMPLICIT DOUBLE PRECISION (A-H, O-Z)
C COMMON STATEMENTS
COMMON/T/, T, NSTOP, NORUN
1 /Y/ Y(I)
2 /F/ F(I)
C ABSOLUTE DIMENSION THE DEPENDENT VARIABLE, DERIVATIVE VECTORS
DOUBLE PRECISION YV(450), YDOT(450)
C TRANSFER INDEPENDENT VARIABLE, DEPENDENT VARIABLE VECTOR
FOR USE IN SUBROUTINE DERV
T=TV
DO 1 I=1,NEQN
Y(I)=YV(I)
1 CONTINUE
C EVALUATE THE DERIVATIVE
CALL DERV
C TRANSFER THE DERIVATIVE VECTOR FOR USE BY SUBROUTINE DLSODE
DO 2 I=1,NEQN
YDOT(I)=F(I)
2 CONTINUE
RETURN
END
<table>
<thead>
<tr>
<th>NUMOL SOLUTION OF SHAFT: TORSIONAL EQUATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.   2.3055556  .000055690  82  1.0D-10</td>
</tr>
<tr>
<td>END OF RUNS</td>
</tr>
</tbody>
</table>
SUBROUTINE INITAL
IMPLICIT DOUBLE PRECISION (A-H,O-Z)

SUBROUTINE INITAL IS CALLED BY THE MAIN PROGRAM THETA.MAIN TO:
1) READ IN EXTERNAL DATA SUCH AS THE MOMENT IN THE Z DIRECTION
   AND THE OD AND ID OF THE SHAFT.
2) DEFINE THE CONSTANTS OF THE PROBLEM.
3) ASSIGN THE INITIAL VALUES OF THETA.

VARIABLE INDEX FOR SUBROUTINES INITAL, DERV, AND PRINT:

- TIME - TIME (sec)
- U1 - REPRESENTS THETA (rad)
- U2 - THE TIME DERIVATIVE OF U1 (rad/s)
- U1T - THE TIME DERIVATIVE OF U1 (rad/s)
- U2T - THE TIME DERIVATIVE OF U2 (rad/s^2)
- U1Z - THE SPATIAL DERIVATIVE OF U1 (rad/m)
- T - THE TEMPORARY VARIABLE G*TJ*U1Z
- TZ - THE SPATIAL DERIVATIVE OF T
- G - SHEAR MODULUS
- TM - THE MASS MOMENT OF INERTIA PER UNIT LENGTH
- TJ - THE POLAR MOMENT OF INERTIA
- ZL - THE LENGTH OF THE SHAFT (m)
- DZ - THE SPATIAL DIFFERENCE IN Z DIRECTION
- ZM - THE MOMENT APPLIED TO THE END OF THE SHAFT ABOUT Z AXIS
- N - NUMBER OF ODES
- LCOUNT - TIME COUNT
- U1ZZ - THE SECOND SPATIAL DERIVATIVE OF U1 (rad/m^2)
- DO - OUTER DIAMETER OF SHAFT (m)
- DI - INNER DIAMETER OF SHAFT (m)

COMMON AND DIMENSION STATEMENTS:

COMMON/T/ TIME
  1 /Y/ U1(41), U2(41)
  2 /F/ U1T(41), U2T(41)
  3 /R/ U1Z(41), T(41), TZ(41),
  4 G, TM(41), TJ(41), ZL, DZ
  5 /B/ ZM(41401)
  6 /I/ N

COMMON/C/ LCOUNT

DIMENSION DO(41), DI(41)

OPEN AND READ EXTERNAL FILES:
OPEN(UNIT= 9,FILE='diam.in')
OPEN(UNIT=10,FILE='mz.in')

DO 1 I = 1, 41401
  READ(10,100) ZM(I)
1 CONTINUE

DO 2 I = 1,41
  READ(9,101) DO(I), DI(I)
2 CONTINUE

- 115 -
ASSIGN CONSTANTS:
   \( G = 7.70 \times 10^{10} \)
   \( ZL = 1.024 \times 10^0 \)
   \( DZ = ZL/DFLOAT(N-1) \)
   \( N = 41 \)

INITIAL CONDITION:
   DO 10 I = 1,N
     U1(I) = 0.0
     U2(I) = 0.0
   CONTINUE

CALCULATE MASS AND POLAR MOMENTS OF INERTIA:
   \( TJ(I) = \frac{\text{DACOS}(-1.0)}{32.0} \times (DO(I)^4 - DI(I)^4) \)
   \( TM(I) = 7900.0 \times TJ(I) \)
   CONTINUE

FORMAT STATEMENTS:
100  FORMAT(D16.8)
101  FORMAT(2D16.8)
RETURN
END

******************************************************************
******************************************************************
------------------------------------------------------------------
SUBROUTINE DERV
IMPLICIT DOUBLE PRECISION (A-J,O-Z)

BOUNDARY CONDITION 2: \( G \times TJ \times \frac{D[\text{THETA(L)}]}{DZ} = 2M \) BECOMES
   \( G \times TJ \times U1Z(N) = 2M \)
   \( U1(1) = 0.0 \)

CALCULATE THE DERIVATIVE, U1Z:
   CALL DSS002(0.0, ZL, N, U1, U1Z)
DEFINE TEMPORARY VARIABLE, T:
DO 11 I = 1,N-1
   T(I) = G*TJ(I)*U1Z(I)
11 CONTINUE
T(N) = 2M(LCOUNT)

CALCULATE THE DERIVATIVE, TZ:
CALL DSS002 (0.DO,ZL,N,T,TZ)

DEFINE THE EQUATIONS OF MOTION:
DO 2 I = 1,N
   U1T(I) = U2(I)
2 CONTINUE
ADD DAMPING TO EQUATION OF MOTION TO REDUCE IMPACT OF VIBRATION
   U2T(I) = - 7.DO*U2(I) + TZ(I)/TM(I)
RETURN

END

********************************************************************

********************************************************************

SUBROUTINE PRINT(NI,NO)
IMPLICIT DOUBLE PRECISION (A-J,O-Z)
------------------------------------------------------------------
SUBROUTINE PRINT IS CALLED BY PROGRAM THETA.MAIN TO PRINT THE
NUMOL SOLUTION.
------------------------------------------------------------------
COMMON AND DIMENSION STATEMENTS:
COMMON/T/ TIME, NSTOP, NORUN
1 /Y/ U1(41), U2(41)
2 /F/ U1T(41), U2T(41)
3 /R/ U1Z(41), T(41), TZ(41),
   G, TJ(41), TM(41), ZL, DZ
6 /B/ ZM(41401)
7 /I/ N
COMMON/C/ LCOUNT
------------------------------------------------------------------
OPEN OUTPUT FILES
OPEN(UNIT=9,FILE='theta.end')
PRINT NUMOL SOLUTIONS
WRITE(9,3) (U1(I),I=1,N,8)
3 FORMAT (6F15.7)
RETURN
END
Alexia Brylawski, a native of Boise, Idaho has Bachelor and Master of Science degrees in Mechanical Engineering from Lehigh University in 1992 and 1994, respectively. Her honors include: election to Tau Beta Pi and Pi Tau Sigma, Engineering Honor Societies. She served as president of the latter during 1991-1992. Her work experience compromises: NSF's Research Experience for Undergraduates at Lehigh University, 1990 and at the University of Maine, 1991. She interned at General Chemical Corporation, Claymont, DE in 1992.
END OF TITLE