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Space Frames with Biaxial Loading in Columns

ANALYSES OF BIAXially LOADED COLUMNS

by

Sakda Santathadaporn
Wai F. Chen

Fritz Engineering Laboratory Report No. 331.12
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American Iron and Steel Institute

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This report presents three different analyses of biaxially loaded columns. These are:

1. Limit analysis (Chapter 2);
2. Generalized Stress-Strain Relationships (Chapter 3); and
3. Stability analysis (Chapter 4).

In Chapter 2, limit theorems of perfect plasticity have been applied to obtain the yield surface equations. The lower bound solution is derived from considerations of equilibrium. Based on an assumed velocity field, the upper bound solution is obtained by equating the rate of external work to the rate of internal energy dissipation. The upper bound load is minimized and its value is found to be very close to the lower bound one.

The elastic-plastic solution of a biaxially loaded column is difficult to obtain because of nonlinearities involving material property and geometry. The problem of nonlinearity due to material property has been treated in Chapter 3. The stability problem including the geometric effect has been introduced in Chapter 4.

The methods developed in Chapters 3 and 4 are an incremental approach in which a linear relationship between the force increments and the deformation increments is introduced into the influence coefficient matrix. This matrix, representing the instantaneous stiffness or the tangent of the force-deformation
curve, is generally known as the tangent stiffness matrix.

All of the procedures have been programmed for digital computation and applied to different sample problems.

Factors considered in the study of the stability problem of biaxially loaded columns are:
1. Material, elastic or elastic-plastic;
2. Loading, proportional or nonproportional;
3. Residual stress;
4. End warping restraint;
5. End bending restraint;
6. Initial imperfection;
7. Externally applied twisting moment; and
8. Equations of equilibrium, with or without the non-linear terms.

The post-buckling behavior of centrally loaded columns and beam-columns (lateral-torsional buckling) has also been examined by using the concept of initial imperfection and the biaxial bending theory developed. This dissertation is limited to the problem of symmetrical loading with single curvature bending. The sway of columns and elastic unloading of yielded elements are not considered.
1. INTRODUCTION

1.1 Introduction

Of all the column problems, a biaxially loaded column is considered to be the most general case. In addition to the axial force, there are two bending moments, acting in the directions of the $x$ and $y$ axes of the reference coordinates (Fig. 1). The applied twisting moment may or may not exist at the ends of the column. These externally applied forces result from the space action between column and beam in a building frame.

In the past, space frames were treated and designed as planar structures. This idealization was necessary because of the lack of sufficient knowledge of the behavior and strength of biaxially loaded columns.

Solutions of biaxially loaded columns are difficult to obtain due to nonlinearities involving material property (stress-strain relationships) and geometry (deformed configuration of column). Although these nonlinearities are undesirable, they exist in many engineering problems. Neglect of either or both of them would yield an unrealistic solution to the problem. Fortunately, many complications of nonlinear analysis can be overcome by utilizing the incremental approach. The equations can be derived, provided that the current state of stress and strain is known. The approach yields a set of simultaneous linear equations with variable coefficients.
During the small increment of forces, it is assumed that these coefficients are constant. Thus, the method is an iterative process and requires successive corrections until equilibrium is satisfied. This method has recently gained in popularity in the analysis of nonlinear problems.

During the past decade, the problem of biaxially loaded columns has received considerable attention from many research workers in this country and abroad. The past development of the theory of biaxial bending can be divided into three stages as follows:

1. Elastic analysis;
2. Elastic-plastic analysis; and
3. Limit analysis.

The elastic behavior of biaxially loaded column has been extensively studied. Its solution only requires the consideration of the geometric nonlinearity and the maximum load of the column is reached when the deflections become excessively large.

A number of elastic-plastic solutions have also been obtained. These solutions are far more complicated than the elastic ones and generally require step-by-step calculations. Failure of columns is usually caused by the excessive yielding. Solutions are limited to an isolated column without externally applied twisting moments.

For a short or zero-length column, the failure of the column occurs when the whole cross-section becomes fully plastic. Limit
analysis, which is most suitable for this type of problem, can be applied to obtain interaction equations relating axial force and moments. Studies of interaction equations have been made by a number of investigators. All previous results are based on an equilibrium approach and, therefore, yield a lower bound solution to the problem.

The purpose of this report is to present three different types of analyses of biaxially loaded columns.

1. Limit analysis of cross-sections by applying limit theorems to obtain interaction equations (Chapter 2). Both upper and lower bound solutions are presented in order to compare the results.

2. Elastic-plastic analysis of cross-sections by utilizing the tangent stiffness method (Chapter 3).

3. Stability analysis of biaxially loaded columns by using the tangent stiffness method (Chapter 4).

There are many factors which affect the maximum strength of biaxially loaded columns. Some have already been studied, but many still need to be investigated. Factors considered in this study are:

1. Material, elastic or elastic-plastic;
2. Loading, proportional or nonproportional;
3. Residual stress;
4. End warping restraint;
5. End bending restraint;
6. Initial imperfection;
7. Externally applied twisting moment; and
8. Equilibrium equations, with or without the nonlinear terms.

The post-buckling behavior of centrally loaded columns and beam-columns (lateral-torsional buckling) has also been examined by using the concept of initial imperfection and the biaxial bending theory developed. This dissertation is limited to the problem of symmetrical loading with single curvature bending. The effects of the sway of columns and the elastic unloading of yielded elements are not taken into consideration. Although the wide-flange section is chosen for the numerical study, the methods are general and can be extended to other cross-sectional shapes.

1.2 Review of Previous Work

The elastic theory of columns and its solution can be found in several texts, as well as papers.\(^{(1-11)}\)* The contribution made by Wagner in 1929 for the torsional buckling of thin-walled open sections to the currently established solutions for torsional and flexural buckling gives an indication of the rapid progress in this area.\(^{(12)}\)

Analytical studies of elastic columns with a thin-walled open section, loaded biaxially with respect to the principal axes of the column cross section, are very extensive. Goodier,\(^{(4-6)}\) following the work on flexural-torsional buckling by Wagner,\(^{(12)}\)

*The numbers in parentheses refer to the list of references.*
Pretschner, and Kappus extended the governing differential equations to include columns under biaxial bending with symmetrical loading. Goodier's equations are simplified by the assumption that the twisting, as well as displacements, of any cross-sections of the column are small as compared to the eccentricities of the loading. Excellent discussion of the theory are given by Bleich, Timoshenko and Gere, and Kollbrunner and Meister.

Goodier's simplified equations have been solved exactly by Culver, and approximately by Thurlimann, Dabrowski, Prawel and Lee, and Trahair. Except for Trahair's solution, which includes the elastic restraint at the ends of the column, all other solutions are for simple support.

Harstead, Birnstiel and Leu have reported that Goodier's equations are not applicable for larger loads in the elastic range. This is due to the fact that as the rotation of the column cross section becomes larger, the error in Goodier's approximation becomes considerable.

Analytical studies of inelastic columns loaded eccentrically in a plane of symmetry have been investigated thoroughly and well understood since Von Karman's efforts in the early part of this century. References to the earlier work of the development are given by Bleich. An up-to-date summary of the research dealing with the inelastic behavior of wide-flange columns that are sufficiently braced in the lateral direction will appear in the revised Commentary on Plastic Design in Steel.
The inelastic behavior of beam-columns bent out of the plane of the applied moment and twist at the same time is generally too complicated for detailed solutions, even for the case of a doubly symmetrical section. Galambos defined failure of the column as the load at which the column begins to deflect laterally, accompanied by twisting. (26) The resulting solution is a conservative estimate of the ultimate load. Nevertheless, good correlation of this estimate with test results was observed.

The inelastic behavior of columns under biaxial loading has not been studied thoroughly, but a member of solutions are available. Pinadzhyan studied the ultimate load-carrying capacity of columns with H-shaped cross section. (27) Kloppel and Winkelmann have conducted experimental and analytical studies for isolated steel columns of channels and H sections. (28)

Birnstiel and Michalos, (29) following the related work of Johnston, (30) have presented a method for determining the ultimate load-carrying capacity of columns under biaxial loading. Warping strains due to nonuniform twist were considered. The procedure requires successive trials and corrections and needs considerable computational effort for a solution. Harstead, Birnstiel and Leu have succeeded in reducing the number of iterations to a few cycles by solving a system of simultaneous linear equations for the corrections at each station along the column. (22) Recently, Birnstiel and his co-workers have conducted experiments on isolated H-columns subjected to biaxial loading. (31) The effect of warping restraints at column ends on the ultimate load-carrying capacity of
the column and the effect of residual thermal strains on the behavior of the column were examined. The agreement between the theoretical and experimental results appears satisfactory.

Smith has considered the biaxially loaded columns in terms of the equations of three-dimensional elasticity, taking into account the nonlinear effects of the end tractions on a bilinearly-elastic column. (32) Finite-difference approximations to the governing partial differential equations are used, and a general procedure for the numerical step-by-step integration of these equations is presented. Several solutions were obtained for a solid rectangular column.

Ellis, Jury and Kirk have reported on experimental and theoretical studies of thin-walled box sections subjected to biaxial loading. (33) In their theory the plane section is assumed to remain plane after bending, and warping of the section is neglected. The failure criterion of overlapping shapes, which was used to find the maximum load-carrying capacity of columns in a single plane, was extended to the three-dimensional biaxial loading case. (34-36) The numerical results were found to agree with small scale tests.

El Darwish and Johnston have studied the particular shape composed of four corner angles with lacing on four sides. (37) The four angles were considered as point areas in their analysis. Torsion was neglected.
Experimental studies on small frameworks in which the columns were subjected to biaxial loading have been conducted at the University of Cambridge in England. The results of these tests and the effect of various end and loading conditions on the ultimate load-carrying capacity of the biaxially loaded columns were examined in the book by Baker, Horne and Heyman. (38)

Milner presented a theoretical and experimental study of biaxial columns under nonproportional loading. (39) In his procedure, the governing differential equations of equilibrium are first expressed in terms of finite differences, and a numerical integration procedure is used for the solution. The purpose was to investigate the effect of the irreversible nature of plastic strains and also the effect of the residual stress upon the elastic-plastic behavior of the H-section columns. Milner's results indicated that the effect of unloading after yielding had occurred in a biaxially loaded column was to strengthen the column rather than weaken it. The effect of the order of load application upon the failure load was observed to be significant. Also, the effect of residual stress was found to be less significant with increased eccentricity.

Another analytical procedure for determining the maximum load-carrying capacity of H-columns loaded biaxially was suggested by Ringo, (40) and later extended by McDonough. (41) The column is first assumed to be sufficiently braced in the lateral directions so that the entire section at the mid-height of the column can reach its plastic capacity. Successive reduction of the fully plastic stress distribution at mid-height is then introduced
because of the instability effect. An iterative method is then used until equilibrium based on the deformed configuration is satisfied.

A valuable extension of Jezek's approximations, \(^{(42-44)}\), which gave an analytical solution for eccentrically loaded steel columns based upon the perfectly plastic idealization for steel as well as assuming the sine curve shape of the deflected column axis, was given by Sharma and Gaylord.\(^{(45)}\) Jezek's theory proves useful in obtaining approximate results in a simple manner for columns subjected to biaxial loading.

All the methods previously reviewed are restricted for symmetrical loading. Only a few studies have been made on an un-symmetrical loading case. Syal and Sharma have proposed a numerical method for the study of the elastic behavior of biaxial columns with different loadings at each end.\(^{(46)}\) Razzaz,\(^{(47)}\) and Marshall and Ellis\(^{(48)}\) have conducted both experimental and theoretical studies on the elastic-plastic behavior of thin-walled box sections. The result of experimentation is in accordance with theoretical prediction.

A different approach to the problem of biaxially loaded columns is to extend the simple plastic hinge concept by taking into account the effect of axial force and biaxial moments on the fully plastic moment. Several investigators\(^{(41,49,50)}\) have constructed the interaction surfaces relating axial force and bending moments under the condition that the entire section will be plastic. The methods used thus far are based upon equilibrium (stress solution) and therefore yield a lower bound solution to the problem, according
to the limit theorems of perfect plasticity.\(^{(51,52)}\) An excellent review on the subject of interaction curves under different loading combinations is contained in the book by Hodge.\(^{(53)}\) Unfortunately, such an approach is invalid for compression members where geometrical change affects equilibrium. The present plastic analysis and design procedures do not consider biaxial loading of the columns, and hence have an unknown amount of inaccuracy when applied to space frames. The concept of the interaction surface approach, applicable to columns of zero length, can be considered as a first step in extending planar structural analysis to more realistic space frame analysis.\(^{(54,55,56)}\)

In Great Britain, the recommended practice for column design always considers the effect of biaxial loading.\(^{(57)}\) A recent review paper on the theory of biaxially loaded columns can now be found in the literature.\(^{(58)}\)
2. LIMIT ANALYSIS

2.1 Introduction

Figure 2 shows a wide-flange section subjected to an axial force $P$ and bending moments $M_x$ and $M_y$. The material is assumed to be elastic-perfectly plastic (Fig. 3); this property is the same for compression as well as tension. For convenience it is assumed that the cross section has one unit length.

As the externally applied forces are gradually increased, the section will undergo transformation from a completely elastic to a partially plastic stage. Eventually, the whole section will be yielded. At this stage the cross section can further deform without an increase in externally applied forces. The magnitudes of the force and the moments which cause the plastification of the cross section are referred to as limiting values.

Interaction equations which relate the limiting values of force and bending moments can be obtained directly from limit analysis. Studies of interaction relationships under biaxial bending have been made by a number of investigators. All previous results are based on an equilibrium approach and, therefore, yield a lower bound solution to the problem.

In this chapter, the lower bound solution will be derived by means of an integration technique. The upper bound solution will also be presented in order to compare the result with the lower
bound one. These solutions can be modified to include the twisting moment.\(^{(50)}\) The numerical study of the effect of torsion on the lower bound interaction curve will be given in the next chapter.

The sign convention used for the stress is positive when it is in tension. The axial force \(P\) and bending moments \(M_x\) and \(M_y\) are considered to be positive if they produce tensile stress in the first quadrant of the coordinate system. These positive vectors of force are shown in Fig. 2 and follow the right-hand rule. They will be utilized for this chapter only.

2.2 Lower Bound Solution

The lower bound theorem of limit analysis states that:

"The load computed on the basis of an assumed equilibrium state of stress distribution which does not violate the yield condition will be less than or at best equal to the true limit or ultimate load".

Figure 4 shows a fully plastified wide-flange section. The neutral axis \(y = f(x)\) or \(x = g(y)\) divides tensile from compressive stress. The equilibrium equations can be written as follows:

\[
P = - \int_{d/2 - t}^{d/2 + t} 2\sigma_y g(y) \, dy - \int_{d/2 - t}^{d/2 + t} 2\sigma_y g(y) \, dy - \int 2\sigma_y f(x) \, dx \tag{2.1a}
\]

\[
M_x = - \int_{d/2 - t}^{d/2 + t} 2\sigma_y y g(y) \, dy - \int_{d/2 - t}^{d/2 + t} 2\sigma_y y g(y) \, dy + \int \sigma_y \left[ \left( \frac{d}{2} - t \right)^2 - f^2(x) \right] \, dx \tag{2.1b}
\]
The first two integrals of Eqs. (2.1) are contributions due to the top and bottom flanges. The last integral is due to the web.

In order to derive a lower bound for the interaction equation, an arbitrary function of $f(x)$ or $g(y)$ is assumed and then substituted into the previous equations. It has been previously shown that, for a rectangular section, the best lower bound solution is obtained if the neutral axis is a straight line.\(^{(59)}\) This condition will be used for the wide-flange section without the proof. Thus, $f(x)$ or $g(y)$ is chosen as follows:

$$x = g(y) = - \lambda_3 y - \lambda_4$$

$$y = f(x) = - \frac{1}{\lambda_3} x - \frac{\lambda_4}{\lambda_3}$$

where $\lambda_3$ and $\lambda_4$ are arbitrary constants which define the location of a neutral axis.

To simplify the result, the neutral axis passing through the web is assumed to be parallel to the x-axis, as shown in Fig. 5. This simplification of the stress distribution is not made for the top and bottom flanges. The following interaction equations are obtained from Eqs. (2.1) to (2.3):

$$P = 4 \sigma_y t \lambda_3 y_1 + 2 \sigma_y w y_1$$

(2.4a)
where

$$y_1 = \frac{\lambda_4}{\lambda_3}$$  (2.5)

If dimensionless quantities are defined by

$$p = \frac{p}{p_y}$$  (2.6a)

$$m_x = \frac{M_x}{M_{px}}$$  (2.6b)

$$m_y = \frac{M_y}{M_{py}}$$  (2.6c)

where

$$p_y = \sigma_y A = \sigma_y [2bt + w (d - 2t)]$$  (2.7a)

$$M_{px} = \sigma_y z_x = \sigma_y [bt (d - t) + w (\frac{d}{2} - t)^2]$$  (2.7b)

$$M_{py} = \sigma_y z_y = \sigma_y [\frac{tb^2}{2} + \frac{1}{4} w^2 (d - 2t)]$$  (2.7c)

Then the non-dimensional interaction equations may finally be written in the form

$$p = \frac{2}{A} [2t \lambda_3 y_1 + w y_1]$$  (2.8a)
The previous equations are derived for the case where the neutral axis passes through the top flange, the web and the bottom flange. These equations are, therefore, valid for

\[ 0 \leq y_1 \leq \frac{d}{2} - t \]  
(2.9)

and

\[ \frac{w}{y_1 + \frac{d}{2} - t} \leq \lambda_3 \leq \frac{b}{\frac{d}{2} + y_1} \]  
(2.10)

Once the values of \( p \) and \( m_x \) are given, the values of \( \lambda_3 \) and \( y_1 \), which define the location of the neutral axis, can be determined by Eqs. (2.8a) and (2.8b). The value of \( m_y \) is then computed from Eq. (2.8c). Equations (2.8a) and (2.8b) are nonlinear, so the Newton-Raphson method was employed to solve the equations for \( \lambda_3 \) and \( y_1 \). This method was programmed for computer solution.

Different locations of the neutral axis and their corresponding interaction equations are summarized in Table 1. The interaction curves for W12x31 and W14x426 are given in Figs. 6 and 7.

Pfrang and Toland, using a similar approach, obtained interaction curves for a few particular wide-flange sections. (49)
The analytical expressions are not available in their paper. Their results and the present solutions are compared in Fig. 8.

The same problem was also considered by Bruinette, McDonough, Morris and Fenves.\(^{(41,50,54)}\) Most of the solutions considered the effective depth of the web as the distance from the center of the top flange to the center of the bottom flange. This approximation was found to be in considerable error compared with the present solution, especially for large flange thickness and large axial force.\(^{(60)}\)

### 2.3 Upper Bound Solution

According to the theorems of limit analysis, there is no guarantee that the lower bound solution obtained in Art. 2.2 is a true ultimate load. The result must still be confirmed from the viewpoint of the upper bound theorem.

The upper bound theorem of limit analysis states that:

"The load computed on the basis of an assumed plastic velocity field by equating the external to internal rate of work for such a field will give an upper bound solution for the collapse or limit load".

Figure 9 shows the axial strain rate distribution on each of the flanges and the web. The assumption made is that the plane cross section for each plate element remains plane after deformation. Each are assumed to behave independently and to exert no restraint upon one another, except for the compatibility conditions at the junction of the flange and the web. Strain rate distribution across the section can be specified completely by six variables (generalized
strains): Three curvature rates ($K_t$, $K_w$, and $K_b$) and three strain rates at the centroid of each plate ($\dot{e}_t$, $\dot{e}_w$, and $\dot{e}_b$). Here, the subscripts $t$, $w$, and $b$ denote, respectively, the top flange, the web and the bottom flange. These six variables are not completely independent. They must satisfy the compatibility conditions (see Fig. 9)

\[ \dot{e}_t = \dot{e}_w + \frac{h}{2} \dot{K}_w \]  
\[ \dot{e}_b = \dot{e}_w - \frac{h}{2} \dot{K}_w \]  

where $h$ is the distance from the center of the top flange to the center of the bottom flange.

The rate of energy dissipation for the volume is computed from

\[ \dot{W}_1 = \int (\pm \sigma_y) \dot{e} dv \]  

where $\dot{e}$ is the rate of strain at any point on the section. Because the length is one unit, $dv$ can be replaced by $dA$.

Considering the web first, the strain rate $\dot{e}$ is given by

\[ \dot{e} = (y + y_1) \dot{K}_w \]  

where

\[ y_1 = \frac{\dot{e}_w}{K_w} \]  

which is the distance from the point of zero strain rate to the center of the web (see Fig. 9).
Similar expressions of the strain rate for the flanges can be derived. These expressions can be substituted in Eq. (2.13) and integrated. The following rate of internal energy dissipation is obtained:

$$
\dot{W}_i = \sigma_y t \left[ \frac{b^2}{4} K_t + \frac{h^2}{2} \left( \frac{\varepsilon_w}{W} \right)^2 K_t \right] \\
+ \sigma_y t \left[ \frac{b^2}{2} K_b + \left( \frac{\varepsilon_w}{W} \right)^2 \frac{K_w}{K_b} \right] \\
+ \sigma_y w \left( \frac{(h^2 - t^2)}{4} \frac{\varepsilon_w}{K_w} + \left( \frac{\varepsilon_w}{W} \right)^2 \frac{K_w}{K_w} \right) 
$$

(2.16)

The first and second terms are the contributions due to the top and bottom flanges; the third term is due to the web.

The rate of external work is given by

$$
\dot{W}_e = (P_t \dot{\varepsilon}_t + M_t \dot{K}_t) \\
+ (P_b \dot{\varepsilon}_b + M_b \dot{K}_b) \\
+ (P_w \dot{\varepsilon}_w + M_w \dot{K}_w) 
$$

(2.17)

where the generalized stresses $M_t$, $M_w$, $M_b$, $P_t$, $P_w$ and $P_b$ correspond to the generalized strains shown in Fig. 9.

The terms in the parenthesis in Eq. (2.17) are the contributions due, respectively, to the top flange, the bottom flange and the web. The axial forces $P_t$, $P_b$ and $P_w$ and bending moments $M_t$, $M_b$ and $M_w$
for each plate are related to the total stress resultants $P, M_x$ and $M_y$ as follows:

\begin{align}
\dot{P} &= \dot{P}_w + \dot{P}_t + \dot{P}_b \\
M_x &= \frac{M_w + M_t \frac{h}{2} - M_t \frac{h}{2} + M_b \frac{h}{2}}{2} \\
M_y &= M_t + M_b
\end{align}

(2.18a)  
(2.18b)  
(2.18c)

Using the above relationships and the compatibility conditions in Eqs. (2.11) and (2.12), Eq. (2.17) can be rewritten as:

\[ \dot{W}_e = \dot{W}_t \]

(2.19)

The upper bound solution is obtained by equating the total rate of work done by the external forces to the total rate of internal energy dissipation. That is

\[ \dot{W}_e = \dot{W}_i \]

(2.20)

or

\[ \dot{P} \dot{e}_w + M_x \dot{e}_w + M_t \dot{e}_t + M_b \dot{e}_b = \sigma_y \left[ \frac{b^2}{4} K_t + \frac{h}{2} \frac{\dot{e}_w}{K_t} \right] \]

\[ + \sigma_y \left[ \frac{b^2}{4} K_b + \frac{h}{2} \frac{\dot{e}_w}{K_b} \right] \]

\[ + \sigma_y \left[ \frac{(h-t)^2}{4} \frac{\dot{e}_w}{K_t} + \frac{\dot{e}_w}{K_t} \right] \]

(2.21)

In order to derive an upper bound solution for the interaction equation, arbitrary values of strain rates and curvature rates can be assumed. However, each different deformation pattern assumed may result in a different set of stress resultant over the wide-flange
section and, hence, will correspond to a different stress boundary value solution. For simplicity it is assumed that the curvature rates for the top and bottom flanges are the same and are equal to \( \dot{k}_f \); that is

\[
\dot{k}_t = \dot{k}_b = \dot{k}_f \tag{2.22}
\]

Thus, Eq. (2.21) reduces to

\[
P \dot{\varepsilon}_w + M_x \dot{k}_w + M_y \dot{k}_f = 2 \sigma_y \left[ \frac{b}{4} \frac{\dot{k}_w}{K_f} + \frac{h}{4} \frac{\dot{k}_f^2}{K_f} + \frac{\dot{\varepsilon}_w}{K_f} \right] + \sigma_y w \left[ \frac{(h - t)^2}{4} \frac{\dot{k}_w}{K_w} + \frac{\dot{\varepsilon}_w}{K_w} \right] \tag{2.23}
\]

If the following data are given or assumed, the upper bound value of \( P \) can be computed directly from Eq. (2.23):

1. Wide-flange section \((b, h, t, w)\)
2. Yield stress of the material \((\sigma_y)\)
3. Velocity field \((\dot{\varepsilon}_w, \dot{k}_w, \dot{k}_f)\) and
4. Bending moments \((M_x, M_y)\).

According to the upper bound theorem of limit analysis, the best choice of the deformation pattern (or velocity field) corresponds to the minimum value of the axial force \( P \). The best upper bound for the load \( P \) in Eq. (2.23) is found by minimizing \( P \) with respect to the variables \( \dot{\varepsilon}_w, \dot{k}_w \) and \( \dot{k}_f \).

Recalling that the right-hand side of Eq. (2.23) is the rate of dissipation \( \dot{W}_f \) (which is the function of \( \dot{\varepsilon}_w, \dot{k}_w \) and \( \dot{k}_f \)), the axial force \( P \) can be written in the form.
\[ P = \frac{1}{\dot{e}_w} (\dot{W}_i - M_x \dot{K}_w - M_y \dot{K}_f) \]  

(2.24)

The function \( P(\dot{e}_w, \dot{K}_w, \dot{K}_f) \) has a minimum value when its derivatives with respect to \( \dot{e}_w, \dot{K}_w \) and \( \dot{K}_f \) vanish. That is,

\[
\frac{\partial P}{\partial \dot{e}_w} = 0 = \frac{1}{\dot{e}_w} \left[ \frac{\partial \dot{W}_i}{\partial \dot{e}_w} - \frac{1}{\dot{e}_w} (\dot{W}_i - M_x \dot{K}_w - M_y \dot{K}_f) \right]
\]

\[ = \frac{1}{\dot{e}_w} \left[ \frac{\partial \dot{W}_i}{\partial \dot{e}_w} - P \right] \quad (2.25a)
\]

\[
\frac{\partial P}{\partial \dot{K}_w} = 0 = \frac{1}{\dot{e}_w} \left[ \frac{\partial \dot{W}_i}{\partial \dot{K}_w} - M_x \right] \quad (2.25b)
\]

\[
\frac{\partial P}{\partial \dot{K}_f} = 0 = \frac{1}{\dot{e}_w} \left[ \frac{\partial \dot{W}_i}{\partial \dot{K}_f} - M_y \right] \quad (2.25c)
\]

The value of \( \dot{e}_w \) is finite rather than infinite so the terms inside the bracket must be zero. Thus, the required conditions for the smallest upper bound of \( P \) are:

\[
P = \frac{\partial \dot{W}_i}{\partial \dot{e}_w} \quad (2.26a)
\]

\[
M_x = \frac{\partial \dot{W}_i}{\partial \dot{K}_w} \quad (2.26b)
\]

\[
M_y = \frac{\partial \dot{W}_i}{\partial \dot{K}_f} \quad (2.26c)
\]

These conditions also hold if, either \( M_x \) or \( M_y \) are selected as a function of \( \dot{e}_w, \dot{K}_w \) and \( \dot{K}_f \).
With dimensionless quantities defined in Eqs. (2.6) and \( \lambda_3 \) defined as the ratio of the curvature rate of the web to the curvature rate of the flange

\[
\lambda_3 = \frac{k_w}{k_f} \tag{2.27}
\]

the following interaction equations are obtained:

\[
p = \frac{2}{A} \left[ 2t \lambda_3 y_1 + w y_1 \right] \tag{2.28a}
\]

\[
m_x = \frac{1}{z_x} \left[ th^2 \lambda_3 + w \left[ \frac{(h - t)^2}{4} - y_1^2 \right] \right] \tag{2.28b}
\]

\[
m_y = \frac{2}{z_y} \left[ \frac{b}{4} - \frac{h}{4} \lambda_3^2 - \lambda_3^2 y_1^2 \right] \tag{2.28c}
\]

valid for

\[0 \leq y_1 \leq \frac{d}{2} - t \tag{2.29}\]

and

\[
\frac{w}{2} \leq \lambda_3 \leq \frac{b}{2} \frac{2}{y_1 + \frac{d}{2} - t} \leq \frac{b}{2} \frac{2}{\frac{d}{2} - \frac{t}{2} + y_1} \tag{2.30}
\]

The results of upper bound solutions for other cases corresponding to the lower bound ones are also given in Table 1. As can be seen from the table, the difference between the two solutions is the value of \( \lambda_3 \), which differs only by \( t^3/4 \). This difference is small if the thickness of the flange is small.
Tables 2 and 3 are the comparisons of the numerical results from both solutions for the W12x31 and W14x426 sections (for $p = 0.2$). Only slight differences were observed. Of interest is the values of the lower bound solutions that are slightly higher than those of the upper bound solutions. This occurs because of the assumption made in the upper bound solution that the section is thin and the strain rate was computed using the average value at the middle plane of the thickness. In other words, the neutral axis was assumed to pass perpendicularly to the flange plates as well as to the web, which is the main distinction between the two solutions.

2.4 Summary

Limit analysis is applied to obtain interaction equations for a wide-flange section under combined biaxial bending and axial force. Agreement between the present solution and the existing solution is observed. The difference between the lower bound and upper bound solutions is found to be small and it can be concluded that both are "exact".
3. GENERALIZED STRESS-STRAIN RELATIONSHIPS

3.1 Introduction

In structural engineering problems, the choice of generalized stresses for a given problem is not unique. For a biaxially loaded column, the appropriate set of generalized stresses are bending moments $M_x$ and $M_y$, and axial force $P$. The corresponding set of generalized strains are bending curvatures $\phi_x$ and $\phi_y$ and axial strain $\varepsilon_0$. These generalized stresses and strains are shown in Fig. 10 in the positive direction. For convenience, the following vectors of force and deformation are defined.

$$
\{f\} = \begin{bmatrix}
-M_x \\
M_y \\
-P
\end{bmatrix}
$$

$$
\{\chi\} = \begin{bmatrix}
\phi_x \\
\phi_y \\
\varepsilon_0
\end{bmatrix}
$$

This chapter formulates the relationship of generalized stresses $\{f\}$ and generalized strains $\{\chi\}$ based on the given stress-strain relationships. In the plastic range, the generalized strains cannot be uniquely determined by the generalized stresses but depend on the whole history of loading. This is because the plastic strain is an irreversible process.
However, it is possible to establish the linear relationship between the generalized stress increments \( \{ f \} \) and the generalized strain increments \( \{ \chi \} \), provided that the existing state of stress and strain is known. This relationship is introduced into the tangent stiffness matrix which is derived in the next section. Once this relationship is established, it is quite easy to find the corresponding path of generalized strains \( \{ \chi \} \) for a given path of generalized stresses \( \{ f \} \).

The problem of generalized stress-strain relationships is of fundamental importance for further development of stability theory. Without a thorough understanding of this basic relationship, the solution of the stability of biaxially loaded columns cannot be achieved.

In the present work, the elastic unloading of yielded fibers will be ignored. That is, the plastic strain is recoverable. The error from this assumption can be minimized if the load is monotonous increased. This discussion will be limited to the wide-flange section. Although the material is assumed to be elastic-perfectly plastic as shown in Fig. 3, the method developed herein can be applied to include strain-hardening without much change.

### 3.2 Mathematical Formulation

Consider the partially yielded cross-section in Fig. 11. Equilibrium is satisfied when the external forces are equal to the internal forces. This results in the following relationship between the generalized stresses and the stress \( \sigma \).
The sign for stress $\sigma$ is positive when it is tension.

The stress-strain relationship for an elastic-perfectly plastic material can be written in a mathematical form as:

$$
\sigma = \begin{cases} 
E\varepsilon & (|\varepsilon| < \varepsilon_y) \\
\sigma_y & (|\varepsilon| \geq \varepsilon_y)
\end{cases}
$$

(3.4)

where $\varepsilon_y$ is the yield strain. This expression does not take into consideration the irreversible nature of plastic strain.

It is assumed that the plane section for each of the thin-walled plates remains plane after deformation; however, the whole cross-section is allowed to warp. Thus, the strain $\varepsilon$ at a point in the cross-section can be expressed in a linear form as follows:

$$
\varepsilon = -y \dot{\phi}_x - x \dot{\phi}_y + \varepsilon_0 + \varepsilon_{wa} + \varepsilon_r
$$

(3.5)

Here, the tensile strain is assumed to be positive. All the normal strains, including the warping and residual strains $\varepsilon_{wa}$ and $\varepsilon_r$ are included. The negative signs in the first two terms of the equation indicate that the positive bending curvatures $\dot{\phi}_x$ and $\dot{\phi}_y$ produce compressive strains in the first quadrant of the $x$ and $y$ coordinate system.
If the values of $\dot{\phi}_x, \dot{\phi}_y, \epsilon_0, \epsilon_{wa},$ and $\epsilon_r$ are given, the strain at any point in the section can be evaluated. Consequently, the stress distribution and the generalized stresses are easily determined.

Because of the penetration of yielding and the moving of the elastic-plastic interface, the direct relationship between generalized stresses and generalized strains is nonlinear and difficult to obtain. An alternative method is to relate the increment of generalized stresses to the increment of generalized strains. The formulation of this type of problem requires a knowledge of the existing equilibrium state of stress and strain.

The problem can be stated as follows: Given the current generalized strain $\{\chi\}$ and generalized stress $\{f\}$, it is then desired to find the relationship between the rate of change of generalized strain $\{\dot{\chi}\}$ and the rate of change of generalized stress $\{\dot{f}\}$.

The advantage of considering the rate of change of the quantities is that, at any instantaneous time, the coefficients of the equations can be assumed constant. Thus, the resulting equations will be linear.

The rate equations of equilibrium are:

$$\dot{M}_x = \int \dot{\phi}_y \, dA \quad (3.6a)$$
$$\dot{M}_y = -\int \dot{\phi}_x \, dA \quad (3.6b)$$
$$\dot{P} = -\int \dot{\sigma} \, dA \quad (3.6c)$$
The rate of change of stress is given by

\[
\dot{\sigma} = \begin{cases} 
E \dot{\varepsilon} & (|\varepsilon| < \varepsilon_y) \\
0 & (|\varepsilon| \geq \varepsilon_y)
\end{cases} 
\tag{3.7}
\]

The strain rate equation is

\[
\dot{\varepsilon} = -y \dot{\phi}_x - x \dot{\phi}_y + \dot{\varepsilon}_o + \dot{\varepsilon}_{wa} + \dot{\varepsilon}_r
\tag{3.8}
\]

After the cooling process for a rolled or welded built-up section, the residual strain \(\varepsilon_r\) is independent of time; thus

\[
\dot{\varepsilon}_r = 0
\tag{3.9}
\]

Equation (3.8) reduces to

\[
\dot{\varepsilon} = -y \dot{\phi}_x - x \dot{\phi}_y + \dot{\varepsilon}_o + \dot{\varepsilon}_{wa}
\tag{3.10}
\]

Consider the rate of change of warping strain \(\dot{\varepsilon}_{wa}\). The stress resultants derived from this strain must be zero.

\[
\dot{M}_x: \quad \int \dot{\sigma}_{wa} y \, dA = \int E\dot{\varepsilon}_{wa} y \, dA = 0
\tag{3.11a}
\]

\[
\dot{M}_y: \quad \int \dot{\sigma}_{wa} x \, dA = \int E\dot{\varepsilon}_{wa} x \, dA = 0
\tag{3.11b}
\]

\[
\dot{P}: \quad \int \dot{\sigma}_{wa} \, dA = \int E\dot{\varepsilon}_{wa} \, dA = 0
\tag{3.11c}
\]

This is due to the fact that warping stresses are produced by self-equilibrating force.

Combination of Eqs. (3.6), (3.7) and (3.10) with Eqs. (3.11) gives a set of simultaneous linear equations which
can be written in matrix form as follows:

\[
\begin{bmatrix}
\dot{M}_x \\
\dot{M}_y \\
\dot{P}
\end{bmatrix} =
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} \\
Q_{21} & Q_{22} & Q_{23} \\
Q_{31} & Q_{32} & Q_{33}
\end{bmatrix}
\begin{bmatrix}
\dot{\phi}_x \\
\dot{\phi}_y \\
\dot{\epsilon}_o
\end{bmatrix}
\]  
(3.12)

where \( Q_{ij} \) is defined as

\[
Q_{11} = \int E y^2 \, dA 
\]  
(3.13a)

\[
Q_{22} = \int E x^2 \, dA 
\]  
(3.13b)

\[
Q_{33} = \int E \, dA 
\]  
(3.13c)

\[
Q_{12} = Q_{21} = \int E x y \, dA 
\]  
(3.13d)

\[
Q_{13} = Q_{31} = -\int E y \, dA 
\]  
(3.13e)

\[
Q_{23} = Q_{32} = -\int E x \, dA 
\]  
(3.13f)

Equation (3.12) can be rewritten as

\[
\{ \dot{\epsilon} \} = [Q] \{ \dot{\chi} \} .
\]  
(3.14)

The matrix \([Q]\), whose elements are given by Eqs. (3.13), is defined as the tangent stiffness matrix since it represents the tangent of the force-deformation curve as well as the stiffness of the cross-section.

If the section is elastic and \( x \) and \( y \) are the principal axes, the following equations are obtained:

\[
Q_{11} = EI_x 
\]  
(3.15a)

\[
Q_{22} = EI_y 
\]  
(3.15b)
\[ Q_{33} = EA \quad (3.15c) \]
\[ Q_{ij} = 0 \quad (i \neq j) \quad (3.15d) \]

Since the quantities \( Q_{ij} \) are constants throughout the elastic range, the dot can be deleted from Eq. (3.12) if the problem is elastic.

For the partially yielded section (Fig. 11), Eq. (3.15d) no longer holds. In general, none of the elements of the tangent stiffness matrix will be zero, except when the section is completely yielded. Unlike the elastic problems, the matrix \([Q]\) of the yielded section is a function of the current state of stress and strain as well as the properties of the material and the cross-section. For an elastic-perfectly plastic material, the value of \( E \) is zero in the yielded zone. Therefore, only the area of the elastic core will contribute to the integration in Eqs. (3.13). This implies that further increment of external forces is resisted by the remaining elastic area of the section.

If the material is other than elastic-perfectly plastic, the value of \( E \) in Eq. (3.13) must be replaced by the tangent modulus \( E_t \).

The convergent scheme of the method is illustrated graphically in Fig. 12. The curve OABC in the figure is the true force-deformation curve. Let \( \{f_A\} \) and \( \{\chi_A\} \) be any existing vectors in state "A" in which equilibrium is satisfied. It is now desired to compute the deformation in state "B" in which the prescribed force is \( \{f_B\} \). The increment of force at "A" is
The matrix $[Q_A]$ corresponding to $\{X_A\}$ can be determined from Eq. (3.13). This matrix is equivalent to the slope at point "A" on the curve. With the increment of external force $\{f_A\}$, the increment of deformations is obtained from

$$\{X_A\} = [Q_A]^{-1} \{f_A\}$$  \hspace{1cm} (3.17)

where $[Q_A]^{-1}$ is the inverse of the matrix $[Q_A]$.

This is an approximate solution, because $[Q_A]$ is calculated before the increment occurs. This matrix alters slightly as the elastic-plastic boundary moves during the increment of deformation. Nevertheless, the equation gives a good prediction of the increment of deformation, provided that the increment of external force is small. The first estimated deformation is given by the sum of $\{X_A\}$ and the incremental deformation predicted from Eq. (3.17)

$$\{X_1\} = \{X_A\} + \{\dot{X}_A\}$$  \hspace{1cm} (3.18)

This deformation gives rise to internal force $\{f_1\}$ which is not in equilibrium with the external force $\{f_B\}$. The first unbalanced force $\{f_1\}$ is computed from

$$\{f_1\} = \{f_B\} - \{f_1\}$$  \hspace{1cm} (3.19)

The next step is to find a correction vector $\{\dot{X}_1\}$ which will be added to $\{X_1\}$ in order to eliminate the unbalanced force. This correction vector is obtained from:

$$\{\dot{X}_1\} = [Q_1]^{-1} \{\dot{f}_1\}$$  \hspace{1cm} (3.20)
where $[Q_1]^{-1}$ is the inverse of the new tangent stiffness matrix $[Q_1]$ corresponding to the state $[\dot{f}_1]$ and $[\chi_1]$.

The process is repeated until the unbalanced force becomes zero or is within a tolerant limit.

If the increment of force is small, the first estimate of the increment of deformation from Eq. (3.17) is quite accurate, and the subsequent correction will be unnecessary. Even with a large incremental force, the solution will generally converge within just a few cycles of iteration. The unbalanced force resulting from each iterative cycle is always smaller than the previous one and they diminish rapidly.

With increasing external forces the plastic region spreads into the elastic core of the section. The values of the elements of the tangent stiffness matrix will also gradually decrease. This means that the section is being softened and results in a faster rate of increase in deformation. Finally, as the external force approaches the limiting value, the matrix $[Q]$ will be singular. At this point, the deformation is excessively large and the solution is difficult to obtain. The direct methods used to determine this limiting value have already been discussed in the limit analysis in Chapter 2.

The important development in this section is the establishment of the linear relationship between the increment of force and deformation in Eq. (3.12) making use of a strength of material method. The linear force-deformation relations can also be derived by applying Taylor's theorem. This method is given in Appendix 1.
3.3 Residual and Warping Strains

In previous derivations, both the residual and warping strains do not appear in the rate equations. However, their existence in Eq. (3.5) has an important influence on the tangent stiffness matrix.

When the section is free of external forces the resultant forces due to the sum of residual stresses over the whole cross-section must be zero. For a hot-rolled wide-flange section, the residual stress pattern can be closely idealized as shown in Fig. 13. With the symmetrical distribution of residual stresses, the equilibrium conditions for bending moments are automatically satisfied. The equilibrium of axial force requires that

$$\sigma_{rt} = \left[ \frac{bt}{bt + (d - 2t)w} \right] \sigma_{rc} \quad (3.21)$$

With x and y as the principal axes of the section, the residual stress at a point on the flanges or the web can be expressed as:

Flanges: $$\sigma = \sigma_{rt} - \left( \sigma_{rt} + \sigma_{rc} \right) \frac{2}{b} \left| x \right| \quad (3.22a)$$

Web: $$\sigma = \sigma_{rt} \quad (3.22b)$$

Thus, the residual strain is obtained by dividing the stress by Young's modulus E.

The warping strain of an elastic thin-walled section is given by:

$$\epsilon_{wa} = \omega_n \beta'' \quad (3.23)$$
where $\omega_n$ is the normalized unit warping and $\beta''$ is the second derivative of the twisting angle ($\beta$) with respect to $z$.

Equation (3.23) has been derived elsewhere and can be found in a number of references.\(^{10,11,61}\) The rate equation of warping strains is:

$$\dot{\epsilon}_{wa} = \omega_n \beta''$$

(3.24)

The normalized unit warping is sketched in Fig. 14 for a wide-flange shape. The warping strain for each of the thin-plates can be written as follows:

- Top flange: $\epsilon_{wa} = \frac{(d-t)}{2} \times \beta''$
  
  (3.25a)

- Bottom flange: $\epsilon_{wa} = -\frac{(d-t)}{2} \times \beta''$
  
  (3.25b)

- Web: $\epsilon_{wa} = 0$
  
  (3.25c)

Eqs. (3.25) are derived from the assumption that the section is completely elastic. It will also be assumed that the equations are valid in the inelastic range.

There is a similar feature between the residual and warping stresses. Both are induced from the self-equilibrating forces and therefore should not affect the limiting values of forces. However, the earlier yielding due to these stresses has an effect on the elastic-plastic behavior of the cross-section.
3.4 Inclusion of Twisting Moments

Up to the present, only the normal stresses have been taken into account, since the shear stresses are considered to be of second order effect. This may not be true if the column is also subjected to externally applied twisting moment. To include shear stresses makes the problem extremely difficult. However, neglecting shear stresses will generally result in an unsafe solution. This study considers only the shear stresses that result from the twisting moment.

The internal twisting moment of a thin-walled open section is composed of the St. Venant torsion, warping torsion and the torsion caused by the horizontal component of the inclined normal stresses.

\[
M_z = M^S_z + M^W_z + M^K_z
\]

\[
= Gk_T \beta' - EI_w \beta'' + \bar{K} \beta'
\]  

(3.26)

where \(Gk_T\) is the St. Venant torsional rigidity, \(EI_w\) is the warping torsional rigidity and \(\bar{K}\) is defined as

\[
\bar{K} = \int \sigma a^2 dA
\]  

(3.27)

where \(a\) is the distance from a point on the cross section to the shear center.

The shear flow due to the St. Venant torsion is illustrated in Fig. 15. Figure 16 shows the shear flow in the flange plate due to the warping torsion. The magnitude of the shear stress at a point on the cross-section has been derived and given elsewhere.\(^{(10,61)}\)
It is assumed that the shear distribution is not affected by the yielding. The yield point, under the combined shear and normal stresses, is governed by von Mises yield criterion which states that

$$\sigma^2 + 3 \tau^2 \leq \sigma_y^2$$  \hspace{1cm} (3.28)

This condition must be considered in Section 3.2.

In this study, the shear deformation due to torsion is neglected, since the method is restricted to small twisting moments. Although it is realized that such approach is not exact from the basis of continuum mechanics, it provides the first approximate solution to this type of problem.

3.5 Numerical Studies

Based upon the equations formulated, a computer program was developed to provide numerical results, since a closed form solution was not possible. The elements of the tangent stiffness matrix in Eqs. (3.13) must be evaluated numerically. This can be accomplished by dividing the cross-section into finite elements as shown in Fig. 17. Each of the flanges or the web contains m·n elements. The numbers have been selected for m = odd (m can be as small as one) and n = even. The strain, stress and the coordinates of each element are computed from the average value at its centroid.

The iterative procedure on which the program is based will now be described (see flow chart in Fig. 18):
1. After reading the input data, the cross-sectional properties and the coordinates of each elemental area are determined. If residual stresses or shear stresses due to torque or normal warping strains exist, they will be generated by the computer. Each initial variable of forces and deformations is set to zero.

2. Assuming an increment of external forces \( \{\mathbf{f}\} \), the corresponding increment of deformations \( \{\mathbf{\Delta x}\} \) is computed. The external forces \( \{\mathbf{f}\} \) are revised.

3. Revise the deformation vector \( \{\mathbf{x}\} \). If the computed unbalanced forces \( \{\mathbf{f}\} \) are sufficiently small, the program will be transferred to step (5). Otherwise, control proceeds to step (4).

4. Test the number of iterative cycles. The program will be terminated if it is too large. Otherwise the unbalanced forces \( \{\mathbf{f}\} \) and the correction vector \( \{\mathbf{\Delta x}\} \) will be computed. Control then returns to step (3).

5. Print the solution. Test to determine the determinant of the tangent stiffness matrix \( [Q] \). If it is too small, the program will be terminated. Control otherwise returns to step (2) for a new iteration.

The steps outlined above have led to a very rapid convergence. In most cases the correction for the unbalanced forces will be unnecessary since the solution converges within one cycle.
In computational step (2), the increment of axial force is applied first. The increment continues until the axial force reaches the required value. Then it is maintained constant and the bending moment about the x-axis is slowly increased until it attains the required magnitude. Finally, keeping the axial force and the bending moment about the x-axis constant, the bending moment about the y-axis is gradually applied until the cross-section is entirely plastic. It should be realized that the cross-section may have reached the plastic state prior to the application of $M_y$ or $M_x$.

The computational procedure described above has been programmed for the CDC 6400 computer. The cross-sectional problem tabulated in Table 4 can be divided into seven groups of studies from C1 to C7. The solutions which are shown graphically from Figs. 19 to 25 were all drawn by the computer. These curves are not smooth, but have a slight jagged appearance because the moving of the plotter is in succession of horizontal and vertical steps of 0.01 inches. All the values are nondimensionalized and the section used in this study is W8x31.

The moment-curvature curves for uniaxial bending case were studied in problems C1 and C2 (Figs. 19 and 20) for strong and weak axis bending respectively. The results are in agreement with those obtained by other method. (62)

Problem C3 (Fig. 21) was run for the biaxial bending case. The axial force was maintained constant at $0.3 P_{y}$. The figure shows the moment-curvature relationships for weak axis bending $(m_y - \phi_y)$ for various values of $m_x$.  


Studies of the influence of the residual stress and the normal warping stress were made in Problems C4 and C5 (Figs. 22 and 23). As can be seen, the presence of the residual stress and the normal warping stress has an effect on the elastic-plastic behavior of the moment-curvature curve. However, the limiting values are unaffected by these stresses.

The modified moment-curvature curves, including shear stresses due to the St. Venant and the warping torsions, were studied in problems C6 and C7 (Figs. 24 and 25). Evidently, the inclusion of twisting moment results in the change of the moment-curvature curves. It should be mentioned that, unlike axial force and bending moments, the values of twisting moments in these plots were nondimensionalized by dividing the twisting moment by the moment causing first yield. This study is limited to the case for \( m_s^z \leq 1 \) and \( m_w^z \leq 1 \). It is believed that large errors would be expected for the case \( m_s^z > 1 \) and \( m_w^z > 1 \).

Notice that all the curves plotted in Figs. 19 to 25 have only one characteristic; they asymptotically approach the limiting value as the curvatures tend to infinity.

Despite a number of solutions of the column obtained, the moment-curvature curves of biaxial bending problems for a wide-flange shape have not yet been presented in the literature. Therefore, the comparison is not available at the present moment. However, the accuracy of the solution can be checked by comparing the limiting
values with the "exact" interaction curve obtained in the previous chapter.

The dark dots shown in Fig. 26 are the limiting values from the elastic-plastic analysis. The upper solid line is the "exact" yield curve. In addition, the theoretical line for the initial yielding which separates the elastic and the partially plastic zones is also plotted in this figure. The circles which are slightly off from the theoretical curve are the solutions from the numerical analysis. This discrepancy results from the numerical method which assumes that yielding is initiated at the centroid of the tipped element rather than at its outside corner. It is important to note that the area of the elastic-plastic zone is much larger than the elastic one. This indicates that there is a considerable amount of reserved column strength beyond the elastic limit.

The lower bound interaction curves for the W12x79 section which includes the St. Venant and the warping torsions are presented in Fig. 27 and 28. The axial force was held constant at 0.3 $P_y$ for all curves.

3.6 Summary

The elastic-plastic behavior of the cross-section has been examined in this chapter. It has been shown that formulating the equations from the viewpoint of the rate of change, this leads to the linear relationship between forces and deformations. The tangent stiffness matrix developed has been used to predict the incremental deformations and to estimate the correction vector for the unbalanced forces. This method is extremely powerful and
efficient for computer solution. The influence of residual stresses and normal warping stresses has also been investigated. The modified moment-curvature curve has been proposed to include shear stresses due to torsion.
4. STABILITY ANALYSIS

4.1 Introduction

The inelastic stability of biaxially loaded columns is extremely complex due to the nonlinearity of the material and geometry. Problems of this type are best solved by utilizing the incremental approach.

It has been the purpose of the present study to develop a general method of analysis so that a variety of problems can be covered. The method should have the capability of taking into account material yielding, residual stress, end warping restraint, end bending restraint, initial imperfection and externally applied twisting moment.

The important aspect of the development is to establish the influence coefficients which relate the increment of forces to the increment of deformations. The matrix of influence coefficients which is also called the tangent stiffness matrix, varies from time to time depending upon the penetration of yielding across the section, the magnitude of external loads and the displacements of the column. Due to the continuing change of the yielded pattern on the cross-section and geometric shape of the column, the influence coefficients of the matrix must be revised at each increment of load or during each cycle of iteration. Successive corrections are made until the equilibrium condition between the internal and external
forces is satisfied. Such an approach has also been used for other nonlinear problems of elastic and inelastic structures. (63-66)

The linear relationship between the increments of generalized stresses and generalized strains developed in the previous chapter is of great importance and makes the derivation in this chapter rather simple.

In the present investigation, the problems are limited to the case of symmetrical loading. The extension to the unsymmetrical case is straightforward and need only little modification.

4.2 Assumptions

Further assumptions, in addition to the ones previously made in Chapter 3, are

1. The column is a prismatic member.
2. All external forces are acting at the ends of the column.
3. There will be no local buckling failure.
4. Slopes of the column are sufficiently small so that the curvatures can be approximated by

\[ \frac{\dot{\phi}_x}{x} \approx \nabla'' \]
\[ \frac{\dot{\phi}_y}{y} \approx \nabla'' \]

(The prime is used to denote differentiation with respect to \( z \)).
5. The rotation of the column is small so that

\[ \cos \beta \approx 1 \]
\[ \sin \beta \approx \beta \]
6. The products of the quantities $U \beta$, $V \beta$, $U'' \beta$, $V'' \beta$, $V U''$, $U V''$, etc. are not negligible.

7. The initial imperfection is assumed to be a sine function

\[
U_i = U_{im} \sin \frac{n_i L}{L} \\
V_i = V_{im} \sin \frac{n_i L}{L} \\
\beta_i = \beta_{im} \sin \frac{n_i L}{L}
\]

8. Bending restraint can be provided at the column ends but twisting restraint will not be considered.

4.3 Mathematical Derivation

Consider the biaxially loaded column shown in Fig. 29. The reference axes $x$, $y$ and $z$ are a right-hand rectangular coordinate system and are stationary in space. The $z$-axis is directed along the column length and passes through the centroid of the cross-section at both ends of the column. The $\xi$, $\eta$ and $\zeta$ axes are a local coordinate system in which the $\zeta$ axis is tangential to the center line of the deflected column. The $\xi$ and $\eta$ axes are taken as the principal axes of the cross-section.

Figure 30 shows the cross-section of the wide-flange shape which is initially deflected by $U_i$ and $V_i$, and twisted by $\beta_i$. Under the application of the load, the shear center will move to a new location defined by $U_c$, $V_c$ and $\beta_c$ in which

\[
U_c = U_i + U
\]  

(4.1a)
\[ \begin{align*}
V_c &= V_i + V \\
\beta_c &= \beta_i + \beta
\end{align*} \]  

(4.1b) \hspace{1cm} (4.1c)

where \( U \) and \( V \) are the additional deflections and \( \beta \) is an additional twisting angle relative to the initial position. For a doubly symmetric section such as a wide-flange shape, the elastic shear center coincides with the centroid of the section. The shifting of the shear center due to yielding of the cross-section will not be considered in the present study.

Since the rotation and the slope of the column are assumed to be small, the vector of force can be transformed from one coordinate system to another by using the direction cosines which relate the two systems as follows:

\[
\begin{bmatrix}
x \\ y \\ z
\end{bmatrix} =
\begin{bmatrix}
1 & -\beta_c & U'_c \\
\beta_c & 1 & V'_c \\
-U'_c & -V'_c & 1
\end{bmatrix}
\begin{bmatrix}
\xi \\ \eta \\ \zeta
\end{bmatrix}
\]

(4.2)

4.3.1 External Forces

In consideration of the biaxially restrained column in Fig. 31, compatibility requires that the column ends and the attached springs, which furnish the bending restraint, will rotate through the same angle. Equilibrium at the joint requires that:

\[
\begin{align*}
M_{ox} &= M_{cx} + M_{rx} \\
M_{oy} &= M_{cy} + M_{ry} \\
M_{oz} &= M_{cz}
\end{align*} \]  

(4.3a) \hspace{1cm} (4.3b) \hspace{1cm} (4.3c)
where \( M_{ox}, M_{oy}, \) and \( M_{oz} \) are the externally applied moments, \( M_{cx}, M_{cy}, \) and \( M_{cz} \) are the moments acting at the column ends, and \( M_{rx} \) and \( M_{ry} \) are the restrained moment from rotational springs. No twisting restraint will be considered.

Because of symmetrical loading, the reactions \( R_x \) and \( R_y \) must be zero. The external bending and twisting moments with respect to the reference coordinates at any section of the column are given by:

\[
M_{x}(ext) = M_{cx} + P V_c \tag{4.4a}
\]

\[
M_{y}(ext) = M_{cy} - P U_c \tag{4.4b}
\]

\[
M_{z}(ext) = M_{cz} \tag{4.4c}
\]

These external moments can be transformed into local coordinates by using the transformation matrix from Eq. (4.2).

\[
M_{\xi}(ext) = M_{cx} + P V_c + (M_{cy} - P U_c) \beta_c - M_{cz} U'_c \tag{4.5a}
\]

\[
M_{\eta}(ext) = M_{cy} - P U_c - (M_{cx} + P V_c) \beta_c - M_{cz} V'_c \tag{4.5b}
\]

\[
M_{\zeta}(ext) = M_{cz} + (M_{cx} + P V_c) U'_c + (M_{cy} - P U_c) V'_c \tag{4.5c}
\]

Notice that the torque resulted from the component \( P U'_c \) and \( P V'_c \) about the shear center is zero since the shear center coincides with the centroid of the section.

In accordance with the assumption of small slope, the axial force in the direction of the \( \zeta \)-axis can be approximated by

\[
P_{\zeta}(ext) = P \tag{4.6}
\]
4.3.2 Internal Forces

A. Elastic Range

For the elastic case, the following equation possesses the most general form of the moment-thrust-curvature relationship, as derived in Chapter 3:

\[
\begin{align*}
\begin{pmatrix}
- \frac{M_{\xi}(\text{int})}{\eta''} \\
\frac{M_{\eta}(\text{int})}{\xi''} \\
- \frac{P_{\xi}(\text{int})}{\zeta''}
\end{pmatrix}
&= 
\begin{pmatrix}
Q_{11} & Q_{12} & Q_{13} \\
Q_{21} & Q_{22} & Q_{23} \\
Q_{31} & Q_{32} & Q_{33}
\end{pmatrix}
\begin{pmatrix}
\eta'' \\
\xi'' \\
\zeta''
\end{pmatrix} \\
&= 
\begin{pmatrix}
\eta'' \\
\xi'' \\
\zeta''
\end{pmatrix} \\
&= 
\begin{pmatrix}
Q_{11} \\
Q_{22} \\
Q_{33}
\end{pmatrix}
\end{align*}
\]  

(4.7)

where \(\eta''\) and \(\xi''\) are the curvatures referring to the local coordinates and \(Q_{ij}\) is redefined as:

\[
\begin{align*}
Q_{11} &= \int E \eta''^2 \, dA \\
Q_{22} &= \int E \xi''^2 \, dA \\
Q_{33} &= \int E \, dA \\
Q_{12} &= Q_{21} = \int E \xi'' \eta'' \, dA \\
Q_{13} &= Q_{31} = \int E \eta'' \, dA \\
Q_{23} &= Q_{32} = \int E \xi'' \, dA
\end{align*}
\]  

(4.8a, 4.8b, 4.8c, 4.8d, 4.8e, 4.8f)

Note that in the above equations \(\xi''\) and \(\eta''\) are not necessarily principal axes and \(\zeta''\) is not necessarily the centroidal axis.

Let the following vectors be defined:

\[
\begin{align*}
\{F\} \text{(int)} &= 
\begin{pmatrix}
- \frac{M_{\xi}(\text{int})}{\eta''} \\
\frac{M_{\eta}(\text{int})}{\xi''} \\
- \frac{P_{\xi}(\text{int})}{\zeta''}
\end{pmatrix}
\end{align*}
\]  

(4.9)
\[ \begin{align*}
\{ \delta \} &= \begin{bmatrix} \eta'' \\ \xi'' \\ \epsilon_o \end{bmatrix} \quad \text{(4.10)} \\
\{ \delta \} &= \begin{bmatrix} \nu'' \\ \psi'' \\ \epsilon_o \end{bmatrix} \quad \text{(4.11)}
\end{align*} \]

From Fig. 32 vectors \( \{ \delta \} \) and \( \{ \tilde{\delta} \} \) are related to each other by

\[ \{ \tilde{\delta} \} = [T] \{ \delta \} \quad \text{(4.12)} \]

where

\[ [T] = \begin{bmatrix} 1 & -\beta_c & 0 \\ \beta_c & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(4.13)} \]

which is a transformation matrix.

Equation (4.7) can be expressed as

\[ \{ \bar{F} \}_{\text{(int)}} = [Q] \{ \tilde{\delta} \} \]

\[ = [Q] [T] \{ \delta \} \quad \text{(4.14)} \]

\[ = [H] \{ \delta \} \]

in which

\[ [H] = [Q] [T] \quad \text{(4.15)} \]
The internal twisting moment is given by:

\[ M_{\xi(\text{int})} = (GK_{T} + \tilde{K}) \beta' - EI_{\omega} \beta'' \]  \hspace{1cm} (4.17)

By equating the external forces (Eqs. 4.5 and 4.6) to the internal forces (Eqs. 4.14 and 4.17), there results a set of four simultaneous, nonlinear, nonhomogeneous, differential equations with the four unknown variables \( U, V, \varepsilon, \) and \( \beta \). The exact solution for these equations was not attempted.

It is interesting to point out the difference between the present derivation and that obtained by Goodier. The present formulation has included the transformation of the bending curvatures (Eq. 4.12) and the terms involving the products of deformations. For an elastic column, Eq. (4.14) for the bending moments yields:

\[ M_{\xi(\text{int})} = -EI_{\xi} \eta'' = -EI_{\xi} (V'' - \beta_c U'') \]  \hspace{1cm} (4.18a)

\[ M_{\eta(\text{int})} = EI_{\eta} \xi'' = EI_{\eta} (U'' + \beta_c V'') \]  \hspace{1cm} (4.18b)
Goodier has simplified these equations by assuming that

\[ M_\xi(\text{int}) \approx -EI_\xi V'' \quad (4.19a) \]

\[ M_\eta(\text{int}) \approx EI_\eta U'' \quad (4.19b) \]

This simplification (Eq. 4.19a) is not justifiable for a large twisting angle. This is due to the fact that a wide-flange section is rather weak about the y-axis in comparison with the x-axis and generally \( V'' \ll U'' \). Therefore the assumption for \( V'' \gg \beta \frac{U''}{c} \) may not be true.

Goodier's approximate equations have been solved by many investigators.\(^{(16-21)}\) The exact solution was obtained by Culver.\(^{(16)}\) The solution which includes the nonlinear terms was first obtained in Ref. 22. Comparison of solutions from Goodier's theory and the present refined theory will be given in the numerical studies.

B. Elastic-Plastic Range

As was shown previously in Chapter 3, when yielding takes place, it is more convenient to work with the rate of change of the quantities. The generalized stress increment is related to the generalized strain increment by

\[ \{\dot{\tilde{F}}\}(\text{int}) = [Q] \{\dot{\tilde{\delta}}\} \quad (4.20) \]

After differentiation, Eq. (4.12) becomes

\[ \{\dot{\delta}\} = [T]\{\dot{\delta}\} + [\check{T}] \{\delta\} \quad (4.21) \]
Therefore
\[
\{\ddot{\mathbf{F}}\}_{\text{int}} = [\mathbf{Q}] [\mathbf{T}] \{\ddot{\delta}\} + [\mathbf{Q}] [\mathbf{T}] \{\dot{\delta}\}
\]
\[
= [\mathbf{H}] \{\ddot{\delta}\} + [\mathbf{H}] \{\dot{\delta}\} \quad (4.22)
\]
in which \([\mathbf{H}]\) is defined by Eq. (4.16) and
\[
[\mathbf{H}] = \beta \left[
\begin{array}{ccc}
Q_{12} & -Q_{11} & 0 \\
Q_{22} & -Q_{21} & 0 \\
Q_{32} & -Q_{31} & 0
\end{array}
\right] \quad (4.23)
\]

Notice that the following condition has been used.
\[
\dot{\beta}_c = \dot{\beta} \quad (4.24)
\]

This results from the fact that the initial twisting of the column is a time-independent function. Therefore
\[
\dot{\beta}_1 = 0 \quad (4.25)
\]

Equation (4.22) can also be obtained by differentiating Eq. (4.14). Equation (4.22) can be written in an expanded form as:
\[
\dot{M}_{\zeta}(\text{int}) = -H_{11} \ddot{V} - H_{12} \ddot{U} - H_{13} \dot{\epsilon}_o - (Q_{12} \dot{V}' - Q_{11} \dot{U}') \dot{\beta} \quad (4.26a)
\]
\[
\dot{M}_{\eta}(\text{int}) = -H_{21} \ddot{V} + H_{22} \ddot{U} + H_{23} \dot{\epsilon}_o + (Q_{22} \dot{V}' - Q_{21} \dot{U}') \dot{\beta} \quad (4.26b)
\]
\[
\dot{M}_{\zeta}(\text{int}) = -H_{31} \ddot{V} - H_{32} \ddot{U} - H_{33} \dot{\epsilon}_o - (Q_{32} \dot{V}' - Q_{31} \dot{U}') \dot{\beta} \quad (4.26c)
\]

From Eq. (4.17), the rate of change of internal twisting moment is:
\[
\dot{M}_{\zeta}(\text{int}) = (GK_T + K) \dot{\beta}'' - EI_w \dddot{\beta}''' \quad (4.26d)
\]
The St. Venant torsional rigidity $GK_u$ is unaltered by the yielding of the cross-section. The warping torsional rigidity $EI_{uw}$ is assumed to be directly proportional to the ratio of the remaining elastic area to the total area of the section.

4.3.3 Derivation of Tangent Stiffness Matrix

Knowing the rate of change of internal force in the preceding section, the next step is to determine the rate of change of external forces.

In order to achieve a workable solution to the problem, certain simplifications must be made. For convenience, the equations at the midspan of the column will be considered. Due to the symmetry of loads, the slopes at midspan are zero.

$$ \frac{U}{cm} = \frac{V}{cm} = 0 \quad (4.27) $$

The additional subscript "m" denotes the midspan of the column.

Thus, Eq. (4.5) reduces to

$$ M_{e}(ext) = M_{cx} + P \frac{V}{cm} + (M_{cy} - P U_{cm}) \beta_{cm} \quad (4.28a) $$

$$ M_{\eta}(ext) = M_{cy} - P U_{cm} - (M_{cx} + P \frac{V}{cm}) \beta_{cm} \quad (4.28b) $$

$$ M_{\zeta}(ext) = M_{cz} \quad (4.28c) $$

The column under study is supposed to be twisted by $M_{cz}$ prior to the application of the axial force and the bending moment. Equation (4.28c) has then already been satisfied and it is necessary to replace this equation by the rate of change of $M_{\zeta}(ext)$ (with respect to $z$) at the mid-length.

$$ M'_{\zeta}(ext) = (M_{cx} + P \frac{V}{cm}) \frac{U''}{cm} + (M_{cy} - P U_{cm}) \frac{V''}{cm} \quad (4.29) $$
The rate equations of the external forces are now obtained by differentiating Eqs. (4.28), (4.6) and (4.29).

\[
\dot{M}_{\xi}(\text{ext}) = M_{cx} + P V_{cm} + P \dot{V}_{m} + (M_{cy} - PU_{cm}) \beta_{m} \\
+ (M_{cy} - PU_{cm} - PU_{m}) \beta_{cm}
\]  

(4.30a)

\[
\dot{M}_{\eta}(\text{ext}) = M_{cy} - PU_{cm} - PU_{m} - (M_{cx} + PV_{cm}) \beta_{m} \\
- (M_{cx} + PV_{cm} + PV_{m}) \beta_{cm}
\]  

(4.30b)

\[
\dot{P}_{\zeta}(\text{ext}) = \dot{P}
\]  

(4.30c)

\[
\ddot{M}_{\xi}(\text{ext}) = (M_{cx} + PV_{cm} + PV_{m}) U'' + (M_{cy} + PV_{cm}) \ddot{V}_{m} \\
+ (M_{cy} - PU_{cm} - PU_{m}) V'' + (M_{cy} - PU_{cm}) \ddot{V}_{m}
\]  

(4.30d)

Again the following conditions have been used

\[
\dot{U}_{cm} = \dot{U}_{m}
\]  

(4.31a)

\[
\dot{V}_{cm} = \dot{V}_{m}
\]  

(4.31b)

\[
\dot{\beta}_{cm} = \dot{\beta}_{m}
\]  

(4.31c)

because the initial deflections are independent of time.

It is now necessary to express \( U_{m}, \dot{V}_{m}, \ddot{V}_{m}, \dot{\beta}_{m} \), and \( \dddot{\beta}_{m} \) in terms of \( \dot{U}_{m}, \dot{V}_{m} \) and \( \dot{\beta}_{m} \) so that the resulting equations will be linear. This can be done by assuming the deflected shape of the column as the following functions:

\[
V = A_1 \sin \frac{\pi z}{L} + 4A_2 \frac{z}{L} (1 - \frac{z}{L})
\]  

(4.32a)

\[
U = B_1 \sin \frac{\pi z}{L} + 4B_2 \frac{z}{L} (1 - \frac{z}{L})
\]  

(4.32b)

and

\[
\beta = \beta_{m} \sin \frac{\pi z}{L} \quad \text{(Warping permitted)}
\]  

(4.32c)

or

\[
\beta = \frac{\beta_{m}}{2} (1 - \cos \frac{2\pi z}{L}) \quad \text{(Warping restrained)}
\]  

(4.32d)
It has been shown in Appendix 2 that at midspan

\[ \ddot{V}_m = -\frac{L^2}{\gamma_1} \dot{V}_m \]  
(4.33a)

\[ \ddot{U}_m = -\frac{L^2}{\gamma_2} \dot{U}_m \]  
(4.33b)

\[ \ddot{\beta}_m = -\frac{1}{\gamma_3} \left( \frac{L}{n} \right)^2 \dddot{\beta}_m \]  
(4.33c)

\[ \dddot{\beta}_m = -\gamma_3 \left( \frac{L}{n} \right)^2 \ddot{\beta}_m \]  
(4.33d)

where \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) are factors that depend on the boundary conditions of the column.

\[ \gamma_1 = \begin{cases} 
\frac{n^2}{8} & (M_{cx} = 0) \\
\frac{n^2}{8} & (M_{cx} \neq 0) 
\end{cases} \]  
(4.34a)

\[ \gamma_2 = \begin{cases} 
\frac{n^2}{8} & (M_{cy} = 0) \\
\frac{n^2}{8} & (M_{cy} \neq 0) 
\end{cases} \]  
(4.34b)

\[ \gamma_3 = \begin{cases} 
1 & (Warping Free) \\
2 & (Warping Fully Restrained) 
\end{cases} \]  
(4.34c)

By equating the rate of change of external forces in Eqs. (4.30) to the rate of change of internal forces in Eqs. (4.26) and by using the relationships from Eqs. (4.33), the final equations, after some manipulation, can be expressed in matrix form as:

\[ \{ \dot{W} \} = [R] \{ \dot{\lambda} \]  
(4.35)

in which \( \{ \dot{W} \} \) is the rate of change of external force vector, \( \{ \dot{\lambda} \} \) is
the rate of change of deformation vector and \([R]\) is the tangent stiffness matrix of size 4 \(\times\) 4 which includes the instability effect.

\[
\begin{align*}
\{\dot{\omega}\} &= \begin{bmatrix} -\dot{M}_{ex} - \dot{P}V_{cm} - (\dot{M}_{cy} - \dot{P}U_{cm}) \beta_{cm} \\ \dot{M}_{cy} - \dot{P}U_{cm} - (\dot{M}_{ex} + \dot{P}V_{cm}) \beta_{cm} \\ -\dot{P} \\ \frac{1}{\gamma_3} \left(\frac{L}{\eta}\right)^2 \left[ (\dot{M}_{ex} + \dot{P}V_{cm}) U'''_{cm} + (\dot{M}_{cy} - \dot{P}U_{cm}) V'''_{cm} \right] \end{bmatrix} \\
\{\dot{\Delta}\} &= \begin{bmatrix} \dot{V}'''_m \\ \dot{U}'''_m \\ \dot{\epsilon}_m \\ \dot{\beta}'''_m \end{bmatrix}
\end{align*}
\] (4.36)

\[
[R] =
\begin{bmatrix}
H_{11} - P \left(\frac{L}{\eta}\right)^2 \gamma_1 & H_{12} + P \left(\frac{L}{\eta}\right)^2 \beta_{cm} & H_{13} & \frac{1}{\gamma_3} \left(\frac{L}{\eta}\right)^2 \left[ (Q_{12} V'''_m - Q_{11} U'''_m) \\ + (M_{cy} - P U_{cm}) \right] \\
H_{21} - P \left(\frac{L}{\eta}\right)^2 \beta_{cm} & H_{22} - P \left(\frac{L}{\eta}\right)^2 \gamma_2 & H_{23} & \frac{1}{\gamma_3} \left(\frac{L}{\eta}\right)^2 \left[ (Q_{22} V'''_m - Q_{21} U'''_m) \\ + (M_{ex} + P V_{cm}) \right] \\
0 & 0 & H_{33} & \frac{1}{\gamma_3} \left(\frac{L}{\eta}\right)^2 \left[ (Q_{32} V'''_m - Q_{31} U'''_m) \\ + (GK_T + K)(\frac{L}{\eta}) + \gamma EI_3 \right] \\
- P \left(\frac{L}{\eta}\right)^2 U'''_{cm} & + P \left(\frac{L}{\eta}\right)^2 V'''_{cm} & 0 & 0
\end{bmatrix}
\] (4.38)
Equation (4.35) is similar to Eq. (3.14) in that both
give the relationships between the rate of change of the force
vector and the rate of change of the deformation vector. The stiff-
ess matrix \([R]\), accounting for the yielding and the instability
effect, can be interpreted as the tangent of the load-deformation
curve of the column. As can be seen from Eq. (4.38), the tangent
stiffness matrix \([R]\) is the function of the following quantities:

1. Properties of the material and cross-section, \(E, G, K, \bar{K}, I_0, \) and \(Q_{ij} \) (\(\sigma_x, \sigma_y, M_z, \sigma_y \) are included
in \(Q_{ij} \));
2. Current external forces, \(P, M_{cx}, \) and \(M_{cy} \);
3. Current displacements and deformations, \(U_{cm}, V_{cm},
\beta_{cm}, e_m, u_m''\) and \(v_m''\);
4. Boundary conditions, \(\gamma_1, \gamma_2, \) and \(\gamma_3 \); and
5. Length of the column \(L\).

Notice that \(H_{ij} \) is merely a function of \(Q_{ij} \) and \(\beta_{cm} \) as defined
in Eq. (4.16).

The procedure of solving stability problem is similar to
the method suggested in Chapter 3. For a given increment of forces,
the corresponding increment of deformations is calculated by applying
Eq. (4.35). Both of the increments are then superimposed with the
previous values before the increments occur. Based on the new de-
f ormations, the internal forces are evaluated and compared to the external
ones. If the differences are too large, a new tangent stiffness
matrix \([R]\) will be determined from the latest deformations and
external forces. Replacing the incremental forces in Eq. (4.36)
by the unbalanced forces and substituting the revised matrix \([R]\) into the equation, the correction vector \([\Delta]\), which is defined in the same way as the incremental deformation vector, can be evaluated. This correction vector is then added to the last predicted deformations and again the unbalanced forces are determined. The iteration continues until the unbalanced forces diminish or are small within a tolerable limit. In many cases, the solution converges within a few cycles. This method is very suitable for the computer solution.

Usually it is convenient to start the computation from the point where the column is free of the external forces. At the beginning, the column remains elastic, and therefore the slope of the load-deformation curve will be steepest at this point. Figure 33 shows the elastic and elastic-plastic behavior of columns. If no yielding takes place, the column will follow the dashed curve OED which is asymptotic to the elastic buckling load. The buckling load of elastic column can be defined as the load at which the slope of the force deformation curves is zero, that is, when the increment of deformations does not give rise to the increment of forces. This can happen if the deformations are infinite. Such a problem is of only academic interest and seldom exists in the real situation except for columns with extremely long lengths.

The true behavior of the column actually involves yielding of the material. At point \(E\), yielding begins in the column section. Further increments of the load result in a rapid increase of deformations. The elastic-plastic column behavior (curve EA) will
deviate from the elastic column curve (ED). As the load continues to increase and reaches point A, then a sudden collapse may occur if the column is subjected to the dead load. This phenomenon can be shown by the straight line AB in the figure. At A, there will be no increment of external loads, that is

\[
\{\dot{W}\} = \{0\} \tag{4.39}
\]

It should be noted that this condition also holds for the elastic column when point D approaches infinity.

Hence, Eq. (4.35) becomes homogeneous. In order that nontrivial solutions exist, the determinant of the matrix \([R]\) must vanish.

\[
|R| = 0 \tag{4.40}
\]

The buckling criterion defined above is valid for inelastic as well as elastic columns. Unfortunately, unlike the elastic column, the buckling load of the inelastic column cannot be computed directly from this condition because the coefficients of the matrix are also a function of the external loads. However, the inelastic buckling load can be obtained from a plot of its load-deformation curve.

There is, as yet, another load-deformation curve of the column shown in Fig. 33 as indicated by OEAC. The curve, consisting of the ascending and descending portions, is usually observed for columns loaded by a testing machine. The ascending portion is the same as that obtained from the column under the dead load. The descending or unloading portion occurs due to excessive yielding of the column and simultaneous drop of the load in the machine during the test. Therefore, at a certain load level, there always exists
two equilibrium positions, stable and unstable. At the apex of the curve, the equilibrium is neutral. This later type of load-deformation curve is the primary objective of this study.

4.4 Numerical Studies

In general, the external loads acting on the column can be separated into two types; proportional and nonproportional. The formulation of the equations in the last section is applicable for both cases. Two different programs have been written for the CDC 6400 computer to demonstrate the application of the proposed theory.

4.4.1 Proportional Loading

For the case of proportional loading, the externally applied moments can be expressed as function of the axial force as follows:

\[
\begin{align*}
\dot{M}_{ox} &= -P e_y \\
\dot{M}_{oy} &= P e_x
\end{align*} \tag{4.41a}
\]

where \( e_x \) and \( e_y \) are the eccentricities of the load \( P \) in the direction of the \( x \) and \( y \) axes respectively.

The increment of bending moments acting on the column ends are

\[
\begin{align*}
\dot{M}_{cx} &= \dot{M}_{ox} - \dot{M}_{rx} \\
\dot{M}_{cy} &= \dot{M}_{oy} - \dot{M}_{ry}
\end{align*} \tag{4.42a}
\]

During the first cycle of the iteration, \( \dot{M}_{rx} \) and \( \dot{M}_{ry} \) are the unknown quantities and, therefore, are assumed to be zero. Thus Eq. (4.36) becomes
\[ \{ \dot{W} \} = P \begin{bmatrix} e_y - V_{cm} - (e_x - U_{cm}) \beta_{cm} \\ e_x - U_{cm} + (e_y - V_{cm}) \beta_{cm} \\ -1 \\
\frac{1}{\nu^3} \frac{L}{h^2} \left[ (-e_y + V_{cm}) U''_{cm} + (e_x - U_{cm}) V''_{cm} \right] \end{bmatrix} \] (4.43)

which is a function of the increment of axial force \( \dot{P} \). By assuming \( \dot{V}_m'' \), it is then easy to compute the corresponding values of \( \dot{P} \), \( \dot{U}_m'' \), \( \dot{\varepsilon}_m \) and \( \dot{\beta}_m'' \) from Eq. (4.35).

The iteration procedure to be followed in the analysis of the biaxially loaded column consists of the following steps (see flow chart in Fig. 34):

1. Read input data and generate all the initial values which are necessary for the computation.

2. Assign an increment of curvature \( \dot{V}_m'' \) and compute the corresponding increments of axial force, curvatures and axial strain; \( \{ P, \dot{U}_m'', \dot{\varepsilon}_m, \dot{\beta}_m'' \} \). From the computed values of \( \dot{P} \), the total axial force \( P \) is revised.

3. Revise the deformation vector \( \{ \dot{V}_m'', \dot{U}_m'', \dot{\varepsilon}_m, \dot{\beta}_m'' \} \) and compute the corresponding external and internal forces. If the differences between the external and internal forces are sufficiently small, control will be transferred to step (6). Otherwise, proceed to step (4).
4. If the number of iteration cycle are too large, program proceeds to step (5). Otherwise, compute the unbalanced forces and the correction vector, then program will be transferred back to step (3).

5. Make appropriate adjustment by either decreasing the increment of curvature \(\gamma_{m}^{n}\) or forcing column to undergo transformation from loading (stable equilibrium) to unloading (unstable equilibrium) portion of the load-displacement curve. Then, control returns to step (2).

6. Test the converged displacements. If they are smaller than the previous ones, program then returns to step (5). Otherwise, print the solution. If the column has not been weakened yet, return to step (2) for a new iteration. Otherwise program will be terminated.

The adjustment in step (5) was first used in Ref. 22. This step is necessary when the load displacement curve almost reaches its maximum value. At this point, slope is very small and the determinant of the tangent stiffness matrix is nearly zero. The system of the equations becomes ill conditioned and requires a larger number of iteration cycles for the solution.

4.4.2 Solutions of Proportional Loading Case

Biaxially loaded columns under study for the case of proportional loading are divided into 14 groups from P1 to P14. Table 5 lists all the problems and information needed for the input data for the computer program.
Problem P1 has been solved twice using first the refined equations derived in the AIAA 4,3 and secondly the approximate equations of Goodier which neglect all the nonlinear terms. The solutions are compared in Fig. 35. The column is completely elastic and is permitted to warp freely at both ends. Culver's solutions, using Goodier's approximate equations, are shown by the dots in the figure. (16) Harstead, Birnstiel and Leu have also solved the same problem. Their solutions, shown by the small circles, included the nonlinear terms. (22) As can be seen from Fig. 35, the present solution using approximate equations agree with those obtained by Culver. The difference of the solutions between the refined and Goodier's equations are clearly seen in the figure. This difference became larger as the load goes higher. Goodier's equations are not justified since the displacement $V_m$ is underestimated.

Problem P2 is similar to problem P1 except that column ends were prevented from warping. The results are summarized in Fig. 36. Again, large discrepancies are observed between the refined and Goodier's equations at high load level.

Studies of the effect of residual stress and end warping on the strength of the inelastic columns are examined in Problems P3 to P6. Solutions are shown in Figs. 37 to 40. The results indicate that the presence of residual stress reduces the maximum load of the column and the prevention of the column ends from warping strengthens the column. In all cases, the maximum loads computed from the present theory are slightly lower than those obtained
in Ref. 22. This is probably due to the fact that the present formulation neglects elastic unloading of plastic elements, whereas Ref. 22 has taken this effect into account.

The purpose of problems P7 and P8 is to compare the solutions from the present theory with the tested data obtained by Birnstiel and his associates. (31) The theoretical curves, in Figs. 41 and 42, are shown by the solid line and the small circles are the experimental results. General agreement is observed from these two figures. The twisting angle \( \beta_m \) in Fig. 41 shows a large discrepancy at loads higher than 60 kips. This disagreement probably resulted from the twist of the end fixture of the column during the test. (22, 31)

The initial imperfection of the column has been investigated in Problems P9 and P10 and the results are shown in Figs. 43 and 44. The column remains elastic throughout the loading. Again, note that there is a large difference between the displacement \( V_m \) calculated from the approximate equations suggested by Goodier and the results from this study. Solutions obtained by Culver are shown by the small circles in the figures.

There have been a number of studies of the effect of initial imperfection on elastic as well as inelastic columns which are axially loaded. For the inelastic case, all of the earlier studies permitted the column to deflect in one direction. There is no known solution which allows the column to deflect in two perpendicular directions and twist at the same time. Problem
Problem P11 examined this behavior. The column is a W12x161 section and is initially twisted \((\beta_{im} = 0.05\) radians). The initial displacement in the x and y directions are equal and arbitrarily assumed to vary from \(-0.055\) to \(-0.880\) inches. Results shown in Fig. 45 indicate a substantial decrease in the maximum load due to the initial imperfections. This particular cross-section and length of column was selected because test results were available. The tangent modulus load \(P_T\) computed based on the measured residual stress is 1010 kips. The maximum loads from the test results varied from 900 to 1170 kips. Since the initial twist of the column was not measured, comparisons between the test data and the theoretical solutions are not possible. The technique of determining the "true" maximum strength of a centrally loaded column is illustrated in Fig. 46. The maximum strength values and the corresponding initial imperfections read from Fig. 45 are plotted into this figure. The value of the "true" maximum strength is the load at which the initial displacement \(\delta_0\) is zero.

Problem P12 considered the effect of bending restraint on the biaxially loaded column. For the space subassemblage sketched in Fig. 47, it is assumed that the beams connected to the column ends can only resist bending moment about their strong axis; no bending moment about the weak axis or twisting moment in the beam will be considered. Furthermore, the effect of column shortening on the beam bending will also be ignored. For the problem study, all the beams are made of S4x7.7 and have a length of 200 inches. These beams which are simply supported at one end behave elastically until plastic hinge forms at the joints. The rotational
stiffness $K$ and the plastic moment $M_p$ are calculated respectively from $3EI/\ell$ (= 2700 kip-in/rad,) and $\sigma_y Z$ (= 126 kip-in). Figure 48 shows the moment-rotation curve for the beam at the connected end. The column solution, shown in Fig. 49, is compared with the nonrestrained column from problem P3. The maximum strength for the restrained column is relatively higher than the nonrestrained one.

The solution which includes the externally applied twisting moment $M_{oz}$ is given in problem P13. The column is first twisted. The St. Venant and warping torsions at the midspan of the column can be determined from the known angle of twist which is derived in Appendix 3. These twisting moments must be included in the moment-thrust-curvature curve by the method proposed in Chapter 3. The value of external twisting moment $M_{oz}$ of 35 kip-in has been arbitrarily chosen. The solutions for both cases, warping permit and warping restraint, are plotted in Fig. 50. The result indicates that the column has a greater strength if the end warping is prevented. Comparing with problems P3 and P4, the inclusion of the twisting moment obviously causes the reduction of the strength of the column.

The problem of inelastic lateral-torsional buckling has been extensively investigated in the past, but the post-buckling behavior of this type of column has not been obtained yet. The purpose herein is to determine the failure load and the behavior of this column by introducing the concept of initial imperfection. In problem P14 the initial displacement $U_{im}$ is assumed to be -0.02 inches and the values of $V_{im}$ and $\beta_{im}$ are both zero. The column is
eccentrically loaded about the strong axis \((e_y = 3.0 \text{ in.})\). The load acts slightly eccentrically about the weak axis \((e_x = 0.01 \text{ in.})\). The curves are shown for two cases in Fig. 51. The solid curve indicates the case of warping fully restrained, and the dashed curve was computed for the case of warping free. It can be observed that the lateral displacement \(U_m\) and the rotation \(\beta_m\) are small until the axial force approaches a certain critical value which corresponds to the inception of lateral-torsional buckling. Again, the results obviously indicate a higher strength of the column for the case with warping restraint.

4.4.3 Nonproportional Loading

This study will be restricted to only one specific loading path similar to that used in the generalized stress-strain problem in Chapter 3. The axial force \(P\) and the bending moment \(M_{ox}\) are applied respectively and maintained constant. Then the column is bent about the weak axis until failure. The increment of external forces defined in Eq. (4.36) will differ for each increment of the loading path. For instance, during the increment of axial force, the increment of the strong and weak axis bending moments are zero, that is \(M_{ox} = M_{oy} = 0\). Thus,

\[
P: [\dot{W}] = \dot{P} = \\
\begin{bmatrix}
-V_{cm} + U_{cm} \beta_{cm} \\
-U_{cm} - V_{cm} \beta_{cm} \\
-1 \\
-\frac{1}{3} \left( \frac{L}{m} \right)^{\frac{2}{3}} \left( V_{cm} V_{cm}'' - U_{cm} V_{cm}'' \right)
\end{bmatrix} \tag{4.44}
\]
Similar vectors due to the increment of moments \( M_{ox} \) and \( M_{oy} \) will be

\[
M_{ox}: \{ \dot{W} \} = M_{ox} \begin{bmatrix}
-1 \\
-\beta_{cm} \\
0 \\
\frac{1}{Y_3} \left( \frac{L}{L'} \right)^2 \ddot{U}_{cm}
\end{bmatrix}
\]

and

\[
M_{oy}: \{ \dot{W} \} = M_{oy} \begin{bmatrix}
-\beta_{cm} \\
1 \\
0 \\
\frac{1}{Y_3} \left( \frac{L}{L'} \right)^2 \ddot{V}_{cm}
\end{bmatrix}
\]

The computational steps outlined in Section 4.4.1 are still valid for the nonproportional loading case. However, modification in step (2) is needed to account for the different loading sequence. All other steps will remain the same. Figure 52 shows the brief flow chart for the nonproportional loading case.

4.4.4 Solutions of Nonproportional Loading Case

Table 6 summarizes the five different column studies which have been selected for the case of nonproportional loading.

Solutions of Problems N1 and N3 are plotted in Fig. 53. Both problems are identical, except that Problem N1 is elastic while Problem N3 has included the yielding effect of the material. The values of axial force \( P \) and strong axis bending \( M_{ox} \) are
arbitrarily assumed and held constant at 135 kips and -270 kip-in respectively. The negative sign of \( M_{ox} \) is used so that the displacement \( V_m \) will move in the negative direction of the y-axis. The ends of the columns are free to warp. The figure shows the plot between the weak axis bending moment \( M_{oy} \) and the displacement \( U_m \).

Figure 54 shows the solutions of Problems N2 and N4. The column ends are fully restrained against warping. The results indicates a slight increase of the maximum strength due to this restraint effect.

Problem N5 investigates the influence of strong axis moment on the weak axis bending. The axial force \( P \) remains constant at 135 kips. Two difference values of strong axis moments, -270 and -675 kip-in, are chosen for this study. The solutions, shown in Fig. 55, differ significantly from one another.

It is now possible to develop the maximum strength interaction curves for biaxially loaded columns. These interaction curves are shown in Fig. 56 for the W8x31 section. The ratios of \( P/P_y \) and \( \sigma_{rc}/\sigma_y \) are equal to 0.3. The slenderness ratios \( L/r_x \) of the column are 0, 20, 40 and 60 respectively. For \( L/r_x = 97 \), the column buckles elastically and, therefore, no bending moment can be applied to it. For the case of \( L/r_x = 0 \), the solution is obtained from the limit analysis presented in Chapter 2. The solid curves represent the strength of the column when its ends are free to warp, and the curves for the case of warping restraint are shown.
dashed in the figure. Note that the warping restraint does not affect the column strength for small value of $M_{ox}/M_{px}$. However, its effect appears to be significant as the value of $M_{ox}/M_{px}$ is high. The greatest difference occurs at $M_{oy}/M_{py} = 0$. This phenomenon can be explained as follows.

Consider an extreme case when $M_{oy}/M_{py} = 0$. The column is bent in the plane of greatest flexural rigidity and therefore will buckle laterally at a certain critical value of the load. Because of the prevention of warping at the column ends, this will result in a larger stiffness of the column and hence increase its maximum strength. For the case of $M_{ox}/M_{px} = 0$, the column is subjected to bending about the weak axis and there will be no lateral-torsional buckling. The failure of the column always occurs due to excessive bending in the plane of the applied moment. Thus the strength of the column is unaffected by the end warping.

It is also observed from Fig. 56 that the effect of warping of the column end becomes less significant as the slenderness ratio decreases. For the limiting case of $L/r_x = 0$, no influence results from the warping restraint of the column ends at all.

4.5 Summary

In this chapter a number of problems on the stability of biaxially loaded columns have been solved utilizing the tangent stiffness method. The discussion has covered two different types of loading, proportional and nonproportional. The influences of material yielding, residual stress, end warping restraint, end
bending restraint, initial imperfection and externally applied twisting moment have been included. Comparison between solutions using Goodier's approximate equations and the proposed refined equations indicates that the Goodier equations result in an erroneous solution when the applied load and the twist of the column are large. The problems of centrally loaded column and lateral-torsional buckling of beam-columns have also been examined using the biaxial bending theory and the concept of initial imperfection. Good correlation exists between the present and the existing solutions. The agreement between the predicted values and the results reported from the test appears to be satisfactory.
5. SUMMARY AND CONCLUSIONS

This report presents three different analyses of biaxially loaded columns. These are:

1. Limit analysis (Chapter 2);
2. Generalized stress-strain relationships (Chapter 3); and
3. Stability analysis (Chapter 4).

In Chapter 2, the upper and lower bound theorems of limit analysis have been applied to obtain the yield surface equations.

Chapter 3 is devoted to the problem of the generalized stress-strain relationships. The method is a stepwise linearization procedure, in which a linear relationship between the force increments and the deformation increments is introduced into the influence coefficient matrix. This matrix, representing the instantaneous stiffness of the cross-section or the tangent of the force-deformation curve, is generally known as the tangent stiffness matrix.

The stability of biaxially loaded columns is treated in Chapter 4. The method is also based on an incremental approach. The tangent stiffness matrix derived in this chapter is an extension of Chapter 3 to include the geometric effects. Solutions also cover the centrally loaded columns and the post-buckling behavior of lateral-torsional buckling of beam-columns.
All of the procedures have been programmed for digital computation and applied to different sample problems.

The following conclusions can be drawn from the results of this study:

1. For sections subjected to axial force and biaxial bending, the best lower bound solution of limit analysis is obtained by assuming the neutral axis as a straight line. This result has also been confirmed by the upper bound solution.

2. The tangent stiffness method developed is suitable for use in elastic-plastic analysis of the cross-section. It has been shown that simple relationships exist between the generalized stress increment and the generalized strain increment.

3. The formulation of the stability problem, also based on the tangent stiffness method, is similar to the matrix displacement method of structural analysis. The maximum load carrying capacity of the column is reached when the determinant of the stiffness matrix is zero. This condition is true for both elastic and inelastic cases.

4. It has been clearly demonstrated that the neglect of nonlinear terms (product of deformations) results in an erroneous solution when the applied load and the twist of the column are large. This error is most severe for the deflection in the strong axis direction.
5. Failure of a column is dominated by excessive deflection in the weak axis direction.

6. In general, the prevention of end warping increases the stiffness and strength of the column. However, this restraint does not affect the column strength when it is bent about the weak axis alone. Furthermore, the influence of warping restraint decreases as the length of the column decreases.

7. Presence of residual stress and the inclusion of externally applied twisting moments reduce the maximum strength of biaxially loaded columns. The amount of reduction can not be specified since it depends on the boundary conditions and the length of the individual column.

8. The maximum strength of a centrally loaded column is very much affected by the initial imperfection of the column.

9. The provision of bending restraint at the column ends raises the load carrying capacity of the column.

10. The post-buckling behavior of lateral-torsional buckling problem can be conveniently determined by utilizing the concept of initial imperfection and the biaxial bending theory.

11. Present solutions indicate good agreement with test results and a number of existing solutions.
The present work deals with symmetrically loaded columns bent in single curvature. The extension to the unsymmetrical case is straightforward. The method is capable to include columns free to sway, strain-hardening of the material and other cross-sectional shape.
6. APPENDICES
APPENDIX 1

Derivation of Tangent Stiffness Matrix by Use of Taylor's Theorem

The analysis of the problem involving the nonlinearity of material properties is very complex, even in the simplest form of the stress-strain relationship, such as the elastic-perfectly plastic material. The attempt has been to transform the problem into a set of simultaneous linear equations by considering the rate of change of quantities as presented in Chapter 3. Based upon the same idea, Gurfinkel obtained a set of equations by applying Taylor's theorem to the nonlinear problem of biaxially loaded footings.\(^7\) The intent here is to point out the relationship between the elements of the tangent stiffness matrix \([Q]\) and the coefficients from Taylor expansion series.

If the plastic strain is reversible, it is then possible to express generalized stresses in terms of generalized strains as follows:

\[
\begin{align*}
M_x &= M_x(\dot{\phi}_x, \dot{\phi}_y, \varepsilon_o) \quad (A1.1a) \\
M_y &= M_y(\dot{\phi}_x, \dot{\phi}_y, \varepsilon_o) \quad (A1.1b) \\
P &= P(\dot{\phi}_x, \dot{\phi}_y, \varepsilon_o) \quad (A1.1c)
\end{align*}
\]

These generalized strains are different from those used by Gurfinkel. At the existing state, \(M_x(A), M_y(A), P(A)\) are known. The problem is then to find the solution for:

\[
\begin{align*}
M_x(B) &= M_x(A) + \delta M_x \quad (A1.2a) \\
M_y(B) &= M_y(A) + \delta M_y \quad (A1.2b) \\
P(B) &= P(A) + \delta P \quad (A1.2c)
\end{align*}
\]
Applying Taylor's theorem and neglecting all terms containing derivatives higher than the first order, the resulting equations are

\[
M_x(B) = M_x(A) + \frac{\partial M_x}{\partial \phi_x} \frac{\partial \phi_x}{\partial \phi_o} \frac{\partial \phi_x}{\partial \phi_o} \frac{\partial M_x}{\partial \phi_o} \frac{\partial \phi_x}{\partial \phi_o} \frac{\partial \phi_x}{\partial \phi_o} \frac{\partial \phi_x}{\partial \phi_o} (A)
\]

\[
M_y(B) = M_y(A) + \frac{\partial M_y}{\partial \phi_x} \frac{\partial \phi_x}{\partial \phi_o} \frac{\partial \phi_x}{\partial \phi_o} \frac{\partial M_y}{\partial \phi_o} \frac{\partial \phi_x}{\partial \phi_o} \frac{\partial \phi_x}{\partial \phi_o} \frac{\partial \phi_x}{\partial \phi_o} (A)
\]

\[
P(B) = P(A) + \frac{\partial P}{\partial \phi_x} \frac{\partial \phi_x}{\partial \phi_o} \frac{\partial \phi_x}{\partial \phi_o} \frac{\partial P}{\partial \phi_o} \frac{\partial \phi_x}{\partial \phi_o} \frac{\partial \phi_x}{\partial \phi_o} \frac{\partial \phi_x}{\partial \phi_o} (A)
\]

Combination of Eqs. (A1.2) and (A1.3) gives

\[
\begin{pmatrix}
\delta M_x \\
\delta M_y \\
\delta P
\end{pmatrix} =
\begin{pmatrix}
\frac{\partial M_x}{\partial \phi_x} & \frac{\partial M_x}{\partial \phi_y} & \frac{\partial M_x}{\partial \phi_o} \\
\frac{\partial M_y}{\partial \phi_x} & \frac{\partial M_y}{\partial \phi_y} & \frac{\partial M_y}{\partial \phi_o} \\
\frac{\partial P}{\partial \phi_x} & \frac{\partial P}{\partial \phi_y} & \frac{\partial P}{\partial \phi_o}
\end{pmatrix}
\begin{pmatrix}
\delta \phi_x \\
\delta \phi_y \\
\delta \phi_o
\end{pmatrix}
\]

(A1.4)

The coefficient of the stiffness matrix can be numerically evaluated.

For instance, by assuming an increment \(\Delta \phi_x\) (\(\Delta \phi_y = \Delta \phi_o = 0\)), the first column of the stiffness matrix is obtained by dividing the corresponding increment of forces \(\Delta M_x\), \(\Delta M_y\) and \(\Delta P\) by \(\Delta \phi_x\). These ratios will be close to the values of the derivative if \(\Delta \phi_x\) approaches zero, that is

\[
\lim_{\Delta \phi_x \to 0} \frac{\Delta M_x}{\Delta \phi_x} = \frac{\partial M_x}{\partial \phi_x} \quad \text{(A1.5a)}
\]

\[
\lim_{\Delta \phi_x \to 0} \frac{\Delta M_y}{\Delta \phi_x} = \frac{\partial M_y}{\partial \phi_x} \quad \text{(A1.5b)}
\]
\[ \lim_{\Delta \Phi_x \to 0} \frac{\Delta P}{\Delta \Phi_x} = \frac{\partial P}{\partial \Phi_x} \]  

(A1.5c)

This technique has been utilized in Ref. 70.

By comparison between Eqs. (A1.4) and (3.12) it is apparent that

\[
\frac{\partial M_x}{\partial \Phi_x} = -Q_{11} = -\int E y^2 \, dA \tag{A1.6a}
\]

\[
\frac{\partial M_y}{\partial \Phi_y} = Q_{22} = \int E x^2 \, dA \tag{A1.6b}
\]

\[
\frac{\partial P}{\partial e_o} = -Q_{33} = -\int E \, dA \tag{A1.6c}
\]

\[
\frac{\partial M_x}{\partial \Phi_y} = -\frac{\partial M_y}{\partial \Phi_x} = -Q_{12} = -\int E x y \, dA \tag{A1.6d}
\]

\[
\frac{\partial M_x}{\partial e_o} = \frac{\partial P}{\partial \Phi_x} = -Q_{13} = \int E y \, dA \tag{A1.6e}
\]

\[
\frac{\partial M_y}{\partial e_o} = -\frac{\partial P}{\partial \Phi_y} = Q_{23} = -\int E x \, dA \tag{A1.6f}
\]
APPENDIX 2

Determination of the Relationships of the Rate of Change
Of Deflections and Curvatures

1. Displacement $V$ and $U$

The displacement $V$ of the column in the direction of the $y$-axis can be closely approximated by the function

$$ V = A_1 \sin \frac{\pi z}{L} + 4 A_2 \frac{z}{L} \left(1 - \frac{z}{L}\right) \tag{A2.1} $$

this satisfies the boundary condition that displacements at both
column ends are zero. After differentiating with respect to $z$ twice,
it becomes

$$ V'' = -A_1 \left(\frac{\pi}{L}\right)^2 \sin \frac{\pi z}{L} - A_2 \left(\frac{8}{L^2}\right) \tag{A2.2} $$

$A_2$ is determined from the boundary condition of the curvature
at the column end ($z = L$) which gives

$$ V'' \bigg|_{z=L} = -A_2 \left(\frac{8}{L^2}\right) = -\mu_x \frac{M_{cx}}{EI_x} \tag{A2.3} $$
or

$$ A_2 = \mu_x \frac{M_{cx} L^2}{8EI_x} \tag{A2.4} $$

The multiplication factor $\mu_x$, accounting for the yielding and the
biaxial bending effects on the curvature, can be determined from the
method presented in Chapter 3. This factor is of primary importance
for short columns in which yielding spreads from the midspan towards
the ends. Most of the columns under this investigation are of inter-
mediate or long length and, therefore, their ends will remain elastic.
The value of $\mu_x$ equal to 1.0 will be used throughout this entire study.

From Eq. (A2.1), the deflection at midspan ($z = \frac{L}{2}$) is

$$V_m = A_1 + A_2$$

(A2.5)

the curvature at the same section is

$$V''_m = -\left(\frac{\pi}{L}\right)^2 A_1 - \left(\frac{8}{L^2}\right) A_2$$

(A2.6)

Elimination of either $A_1$ or $A_2$ from these two equations gives

$$V''_m = -\left(\frac{\pi}{L}\right)^2 V_m + \left(\frac{2}{L^2}\right) A_2$$

(A2.7)

or

$$V''_m = -\left(\frac{8}{L^2}\right) V_m - \left(\frac{\pi^2 - 8}{L^2}\right) A_1$$

(A2.8)

Similarly, the rate equation takes the form

$$\dot{V''}_m = -\left(\frac{\pi}{L}\right)^2 \dot{V}_m + \left(\frac{2}{L^2}\right) \dot{A}_2$$

(A2.9)

or

$$\dot{V''}_m = -\left(\frac{8}{L^2}\right) \dot{V}_m - \left(\frac{\pi^2 - 8}{L^2}\right) \dot{A}_1$$

(A2.10)

If $M_{cx} = 0$, from Eq. (A2.4)

$$A_2 = \dot{A}_2 = 0$$

(A2.11)

and hence

$$\dot{V''}_m = -\left(\frac{\pi}{L}\right)^2 \dot{V}_m$$

(A2.12)
If $M_{cx} \neq 0$, then $A_2 \neq 0$, and

$$|V_m| > |A_2| > |A_1|$$  \hspace{1cm} \text{(A2.13)}

this is generally true for the case of single curvature bending. It is expected that the condition also holds for

$$|\dot{V}_m| > |\dot{A}_2| > |\dot{A}_1|$$  \hspace{1cm} \text{(A2.14)}

Equation (A2.10), after neglecting the term containing $A_1$, becomes

$$\dot{V}''_m = -\left(\frac{8}{L^2}\right) \frac{V_m}{L^2}$$  \hspace{1cm} \text{(A2.15)}

In general, one can write

$$\dot{V}_m = -\left(\frac{L^2}{\gamma_1}\right) \frac{V''_m}{L^2}$$  \hspace{1cm} \text{(A2.16)}

where

$$\gamma_1 = \begin{cases} \pi^2 & (M_{cx} = 0) \\ 8 & (M_{cx} \neq 0) \end{cases}$$  \hspace{1cm} \text{(A2.17)}

In fact the value of $\gamma_1$ for the case $M_{cx} \neq 0$ should lie somewhere between 8 and $\pi^2$ depending upon the magnitude of $M_{cx}$. However it has been found that the maximum strength of the column is insensitive to this factor, the value of $\gamma_1$ given in Eq. (A2.17) will be adopted in the study. This similar finding has also been observed in the uniaxial bending of beam-columns. \textsuperscript{(71,72)}

Similarly, by assuming the displacement $U$ as

$$U = B_1 \sin \frac{\pi z}{L} + 4 B_2 \frac{z}{L} (1 - \frac{z}{L})$$  \hspace{1cm} \text{(A2.18)}
The same results are obtained

\[ \ddot{U}_m = -\frac{L}{\gamma_2} \dot{U}_m'' \]  \hspace{1cm} (A2.19)

where

\[ \gamma_2 = \begin{cases} \pi^2 & (\gamma_{cy} = 0) \\ 8 & (\gamma_{cy} \neq 0) \end{cases} \]  \hspace{1cm} (A2.20)

2. Twisting Angle $\beta$

**Free warping.** When the ends of the column is free to warp, the following assumed function

\[ \beta = \beta_m \sin \frac{\pi z}{L} \]  \hspace{1cm} (A2.21)

satisfies the boundary condition.

Differentiating twice and substituting $z = \frac{L}{2}$, the resulting equation is

\[ \beta_m'' = -\left(\frac{\pi}{L}\right)^2 \beta_m \]  \hspace{1cm} (A2.22)

The rate equation is

\[ \dot{\beta}_m'' = -\left(\frac{\pi}{L}\right)^2 \dot{\beta}_m \]  \hspace{1cm} (A2.23)

Again differentiating Eq. (A2.23) twice, it gives

\[ \beta_m'''' = -\left(\frac{\pi}{L}\right)^2 \beta_m'' \]  \hspace{1cm} (A2.24)

Warping Fully Restrained. In order to satisfy the boundary condition, the following function is assumed

\[ \beta = \frac{\beta_m}{2} \left(1 - \cos \frac{2\pi z}{L}\right) \]  \hspace{1cm} (A2.25)
from which
\[ \dot{\beta}_m'' = -2 \left( \frac{\pi}{L} \right)^2 \ddot{\beta}_m \]  
(A2.26)

and
\[ \dot{\beta}_m'''' = -2 \left( \frac{\pi}{L} \right)^2 \beta''_m \]  
(A2.27)

In summary, the following general form can be given for both cases of free and restrained warping,

\[ \dot{\beta}_m = -\frac{1}{\gamma_3} \left( \frac{\pi}{L} \right)^2 \ddot{\beta}_m'' \]  
(A2.28)

and
\[ \dot{\beta}_m'''' = -\gamma_3 \left( \frac{\pi}{L} \right)^2 \beta''_m \]  
(A2.29)

where
\[ \gamma_3 = \begin{cases} 1 & \text{(Warping Permitted)} \\ 2 & \text{(Warping Fully Restrained)} \end{cases} \]  
(A2.30)

which is the warping restraint factor.
APPENDIX 3

Determination of the Twisting Angle of Column

Under Pure Twisting Moment

The equilibrium equation of twisting moment for a thin-walled open section subjected to the externally applied torque is given by:

\[ EI_\omega \beta''' - GK_T \beta' = - M_{cz} \quad (A3.1) \]

Denoting

\[ \lambda^2 = \frac{GK_T}{EI_\omega} \quad (A3.2) \]

the solution of the differential equation is

\[ \beta = C_1 + C_2 \cosh \lambda z + C_3 \sinh \lambda z + \frac{M_{cz} z}{GK_T} \quad (A3.3) \]

where \( C_1, C_2 \), and \( C_3 \) are integration constants and can be determined from the boundary conditions of the column.

For convenience, during the application of twisting moment \( M_{cz} \), the cross-section of the column at the midspan will be assumed to be stationary. Thus

\[ \beta \bigg|_{z = \frac{L}{2}} = 0 \]

(A3.4)

If the column ends are permitted to warp, the second derivative of twisting angle \( \beta \) must be zero, that is

\[ \beta''_{z=0} = \beta''_{z=L} = 0 \quad (A3.5) \]
From the three boundary conditions in Eqs. (A3.4) and (A3.5), the final solution is

$$\beta = \frac{Mc}{GK_T} (z - L)$$  \hspace{1cm} (A3.6)

For the case of end warping fully restrained, the first derivative of the twisting angle at the ends must be zero.

$$\beta'_{z=0} = \beta'_{z=L} = 0$$  \hspace{1cm} (A3.7)

Thus, combining the boundary conditions in Eqs. (A3.4) and (A3.7), the final solution is

$$\beta = \frac{Mc}{\lambda Gk_T} \left[ \tanh \frac{\lambda L}{2} \cosh \lambda z - \sinh \lambda z + \lambda (z - \frac{L}{2}) \right]$$  \hspace{1cm} (A3.8)

Now, either St. Venant or warping torsion can be determined from the known expression of the twisting angle $\beta$. 
7. **NOTATIONS**

In the following symbols, the subscripts (ext), (int) and m which stand for external, internal and midspan will not be shown.

- $A$ = total cross sectional area;
- $a$ = distance from point on cross-section to shear center;
- $b$ = width of wide-flange section;
- $d$ = depth of wide-flange section;
- $E$ = modulus of elasticity;
- $e_x, e_y$ = eccentricity of load in x and y direction;
- $[F]$ = vector of moment about local coordinate defined by Eq. (4.9);
- $\dot{[F]}$ = rate of $[F]$;
- $[f]$ = vector of force defined by Eqs. (3.1);
- $\dot{[f]}$ = rate of $[f]$;
- $G$ = shear modulus;
- $[H]$ = matrix defined by Eq. (4.16);
- $\dot{[H]}$ = rate of $[H]$;
- $h$ = distance from center of top flange to center of bottom flange;
- $I_x, I_y$ = moment of inertia about x and y axis;
- $I_\xi, I_\eta$ = moment of inertia about $\xi$ and $\eta$ axis;
- $I_w$ = warping moment of inertia;
- $K$ = cross sectional constant defined by Eq. (3.27);
- $K_b, K_t, K_w$ = rate of curvature of bottom flange, top flange and web respectively;
\( K_f \) = rate of curvature of flange (when \( K_b = K_t = K_f \));

\( K_T \) = St. Venant torsion constant;

\( K_x, K_y \) = Stiffness factor of beam about x and y axis;

\( L \) = length of column;

\( l \) = length of beam;

\( M_{b}, M_{t}, M_{w} \) = moment acting on bottom flange, top flange and web respectively;

\( M_{x}, M_{y} \) = moment about x and y axis;

\( M_{px} \) = fully plastic moment about x axis when no axial force or moment about y axis is acting;

\( M_{py} \) = fully plastic moment about y axis when no axial force or moment about x axis is acting;

\( M_{bx}, M_{by} \) = end moment acting at bottom of column;

\( M_{tx}, M_{ty} \) = end moment acting at top of column;

\( M_{ox}, M_{oy}, M_{oz} \) = end moment about x, y and z axis;

\( M_{cx}, M_{cy}, M_{cz} \) = moment acting at ends of column;

\( M_{rx}, M_{ry} \) = restrained moment about x and y axis;

\( M_{\xi}, M_{\eta}, M_{\zeta} \) = moment about \( \xi, \eta \) and \( \zeta \) axis;

\( M_{ox}, M_{oy}, M_{oz} \) = rate of end moment about x, y and z axis;

\( M_{cx}, M_{cy}, M_{cz} \) = rate of moment acting at ends of column;

\( M_{rx}, M_{ry} \) = rate of restrained moment about x and y axis;

\( M_{\xi}, M_{\eta}, M_{\zeta} \) = rate of moment about \( \xi, \eta \) and \( \zeta \) axis;

\( M_{pl}, M_{p2} \) = plastic moment of beam about x and y axis respectively;

\( M_z \) = torsion;

\( M_z^s \) = St. Venant torsion;

\( M_z^w \) = warping torsion;
\( \tau \) = torsion caused by horizontal component of inclined normal stresses;
\( M_z \) = \( M_x / M_{px} \);
\( m_{uz} \) = \( M_y / M_{py} \);
\( s \) = nondimensional St. Venant torsion;
\( \tilde{w} \) = nondimensional warping torsion;
\( m, n \) = number of grid defined in Fig. 17;
\( P \) = axial force;
\( P_b, P_t, P_w \) = axial force acting on bottom flange, top flange and web respectively;
\( P_{max} \) = maximum strength of column;
\( P_T \) = tangent modulus load;
\( P_y \) = yield load \( (P_y = \sigma_y A) \);
\( P_z \) = axial force along \( \zeta \) axis;
\( \dot{P} \) = rate of axial force;
\( p \) = \( P/P_y \);
\([Q]\) = tangent stiffness matrix of cross section defined by Eq. (3.13) or (4.8);
\([R]\) = tangent stiffness matrix of beam-column defined by Eq. (4.38);
\( R_{bx}, R_{by} \) = reaction at bottom of column;
\( R_{tx}, R_{ty} \) = reaction at top of column;
\( R_x, R_y \) = reaction in \( x \) and \( y \) direction;
\( r_x \) = radius of gyration about \( x \) axis;
\([T]\) = transformation matrix defined by Eq. (4.13);
\( t \) = thickness of flange;
\( U \) = deflection of shear center in \( x \) direction;
\( \dot{U} \) = rate of \( U \);
\( \dot{U}_c \) = rate of \( U_c \);
\( \dot{U}_i \) = rate of \( U_i \);
\( V \) = deflection of shear center in \( y \) direction;
\( V_c \) = deflection of shear center in \( y \) direction measured from origin of \( y \) axis;
\( V_i \) = initial deflection of shear center in \( y \) direction;
\( \dot{V} \) = rate of \( V \);
\( \dot{V}_c \) = rate of \( V_c \);
\( \dot{V}_i \) = rate of \( V_i \);
\{ \dot{W} \} = rate of vector of external force defined by Eq. (4.36);
\( \dot{W}_e \) = total rate of external work;
\( \dot{W}_i \) = total rate of internal energy dissipation;
\( w \) = thickness of web;
\( x, y, z \) = reference coordinate system;
\( y_1 \) = parameter defined by Eqs. (2.5) and (2.15);
\( Z_x \) = plastic modulus \( Z_x = M_p x / \sigma_y \);
\( Z_y \) = plastic modulus \( Z_y = M_p y / \sigma_y \);
\( \beta \) = angle of twist resulted from applied load;
\( \beta_c \) = total angle of twist;
\( \beta_i \) = initial twisting angle;
\( \dot{\beta} \) = rate of \( \beta \);
\( \dot{\beta}_c \) = rate of \( \beta_c \);
\( \dot{\beta}_i \) = rate of \( \beta_i \);
\( \gamma_1, \gamma_2, \gamma_3 \) = factor depending on boundary condition defined by Eq. (4.34);

\( \dot{\delta} \) = rate of deformation vector defined by Eq. (4.37);

\( \delta \) = deformation vector defined by Eq. (4.11);

\( \ddot{\delta} \) = deformation vector defined by Eq. (4.10);

\( \dot{\delta} \) = rate of \( \delta \);

\( \ddot{\delta} \) = rate of \( \ddot{\delta} \);

\( \delta_0 \) = initial deflection at midspan of column \( (U_{im} = V_{im} = \delta_0) \);

\( e \) = strain;

\( e_o \) = strain at origin of coordinate system;

\( e_r \) = residual strain;

\( e_{wa} \) = warping strain;

\( e_y \) = yield strain;

\( \dot{e} \) = strain rate;

\( \dot{e}_o \) = strain rate at origin of coordinate system;

\( \dot{e}_{wa} \) = warping strain rate;

\( \dot{e}_b, \dot{e}_t, \dot{e}_w \) = strain rate at center of bottom flange, top flange and web respectively;

\( \xi, \eta, \zeta \) = local coordinate system;

\( \theta \) = angle or slope;

\( \lambda \) = \( \sqrt{GK_T/\bar{E}I_w} \);

\( \lambda_3, \lambda_4 \) = parameter defined location of neutral axis;

\( \mu_x \) = multiplication factor;

\( \sigma \) = stress;

\( \sigma_{rc} \) = residual stress in compression;

\( \sigma_{rt} \) = residual stress in tension;

\( \sigma_y \) = yield stress;

\( \dot{\sigma} \) = stress rate;
\( \dot{\sigma}_{wa} \) = rate of warping stress;
\( \tau \) = shear stress;
\( \dot{\psi}_x, \dot{\psi}_y \) = curvature about x and y axis;
\( \dot{\psi}_x, \dot{\psi}_y \) = rate of curvature about x and y axis;
\( \phi_x, \phi_y \) = nondimensional curvature;
\( \{\chi\} \) = deformation vector defined in Eq. (3.2);
\( \{\dot{\chi}\} \) = rate of \( \{\chi\} \); and
\( \omega_n \) = normalized unit warping;
8. TABLES
<table>
<thead>
<tr>
<th>Case</th>
<th>Location of N.A.</th>
<th>Equation</th>
<th>Valid for</th>
<th>Lower Bound</th>
<th>Upper Bound</th>
</tr>
</thead>
</table>
| 1    | ![Diagram 1](image1.png) | \[ p = \frac{2L}{A} [2t \lambda_3 y_1 + w y_1] \] \[ m_x = \frac{1}{Z_x} \left\{ \frac{4}{3} n_3 \lambda_3 + w \left[ (\frac{d-t}{2})^2 - y_1^2 \right] \right\} \] \[ m_y = \frac{2}{Z_y} \left[ \frac{1}{4} n_3 \lambda_3^2 - t \lambda_3^2 y_1^2 \right] \] | \[ 0 \leq y_1 \leq \frac{d}{2} - t \] | \[ 0 \leq y_1 \leq \frac{d}{2} - t \] |}
| 2    | ![Diagram 2](image2.png) | \[ p = \frac{2L}{A} [2t \lambda_3 y_1 + w (\frac{d-t}{2})] \] \[ m_x = \frac{4}{3} n_3 \lambda_3 \] \[ m_y = \frac{2}{Z_y} \left[ \frac{1}{4} n_3 \lambda_3^2 - t \lambda_3^2 y_1^2 \right] \] | \[ \frac{d}{2} - t \leq y_1 \] | \[ \frac{d}{2} - t \leq y_1 \] |}
<p>| 3    | <img src="image3.png" alt="Diagram 3" /> | [ p = \frac{L}{A} [b t - n_2 \lambda_3 + 2t \lambda_3 y_1 + w y_1] ] [ m_x = \frac{L}{Z_x} \left{ \frac{b}{2} (d-t) + \frac{2}{3} n_3 \lambda_3 - n_2 \lambda_3 y_1 + w \left[ (\frac{d-t}{2})^2 - y_1^2 \right] \right} ] [ m_y = \frac{1}{Z_y} \left[ \frac{1}{4} n_3 \lambda_3^2 + n_2 \lambda_3^2 y_1 - t \lambda_3 y_1^2 \right] ] | [ 0 \leq y_1 \leq \frac{d}{2} - t ] | [ 0 \leq y_1 \leq \frac{d}{2} - t ] |</p>
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<td>4</td>
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<td>[ p = \frac{1}{A} [bt - n_2 \lambda_3 + 2t \lambda_3 y_1 + 2w \left( \frac{d}{2} - t \right)] ]</td>
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<td>[ m_x = \frac{1}{Z_x} \left[ -\frac{bt}{2} (d-t) + \frac{2}{3} n_3 \lambda_3 - n_2 \lambda_3 y_1 \right] ]</td>
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<td>[ \frac{\omega (d-2t)}{A} \leq p \leq 1 ]</td>
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<td>[ m_x = \frac{A}{2Z_x} \left[ d(1-p) - \frac{A}{2b} (1-p)^2 \right] ]</td>
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<td>5b</td>
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## TABLE 1 (cont'd)

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<td>$m_y = \frac{A}{2z} \left[ b \left( 1-p \right) - \frac{A}{4t} (1-p)^2 \right]$</td>
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<td>$m_y = 1 - \frac{A^2}{4dz} p^2$</td>
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**Remarks:**

\[
\lambda_3 = \tan \theta \\
y_1 = \frac{\lambda_4}{\lambda_3} \\
x = -\lambda_3 y - \lambda_4 \\
A = 2bt + w (d-2t) \\
Z_x = bt (d-t) + w \left( \frac{d}{2} - t \right)^2 \\
Z_y = \frac{1}{2} tb^2 + \frac{1}{4} w^2 (d-2t) \\
n_2 = \left( \frac{d}{2} \right)^2 - \left( \frac{d}{2} - t \right)^2 \\
= t (d-t) \\
\text{Lower Bound} \\
n_3 = \left( \frac{d}{2} \right)^3 - \left( \frac{d}{2} - t \right)^3 \\
= \frac{3}{4} t (d-t)^2 + \frac{1}{4} t^3 \\
\text{Upper Bound} \\
n_3 = \frac{3}{4} t (d-t)^2 \\
\]
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<th>Upper Bound</th>
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**TABLE 2**
Comparison of Lower and Upper Bound Solutions
(W12x31)
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TABLE 3
Comparison of Lower and Upper Bound Solutions
($W_{14}x426$)
# Table 4: Studies of Cross-Sectional Problems

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<th>Problem</th>
<th>Axial Force $P_{(max)}$</th>
<th>Bending Moment $m_x(\max)$</th>
<th>Bending Moment $m_y(\max)$</th>
<th>Residual Stress $\sigma_{rc}$</th>
<th>Normal Warping Strain $s^N \frac{\sigma_y}{\infty}$</th>
<th>St. Venant Torsion $m_z$</th>
<th>Warping Torsion $\dot{w}$</th>
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All sections are W8x31
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$\sigma_y = 36$ ksi
$m = 9$ $n = 20$

The values in the parenthesis are obtained from the computer output.
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### TABLE 6  STUDIES OF NON-PROPORTIONAL LOADING CASE

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9. FIGURES
Fig. 1 A Biaxially Loaded Column
Fig. 2 Forces Acting on the Wide-Flange Section
Fig. 3 Stress-Strain Relationships for Elastic-Perfectly Plastic Material
Fig. 4 A Fully Plastified Wide-Flange Section
Fig. 5 Simplified Stress Distribution

\[ \lambda_3 = \tan \theta \]
Fig. 6 Interaction Curves for Biaxial Bending
Fig. 7  Interaction Curves for Biaxial Bending
Fig. 8  Comparison with Results from Reference 49
Fig. 9 Strain Rate Distribution
Fig. 10 Positive Vectors of Forces and Deformations
Fig. 11 A Partially-Yielded Section
Fig. 12 Convergent Scheme for Force-Deformation Curve
Fig. 13 Residual Stress Distribution
Fig. 14 Normalized Unit Warping ($w_n$)
Fig. 15 Shear Flow due to St. Venant Torsion
Fig. 16 Shear Flow due to Warping Torsion
Fig. 17 A Wide-Flange Section Divided into Finite Elements
COLUMN IS LOADED IN THE FOLLOWING SEQUENCE:

1. ASSUME $P$
   - COMPUTE $\dot{X} = \{\ddot{x}, \ddot{y}, \ddot{e}\}$
   - REVISE $P$
   - REPEAT TILL $P = P_{\text{max}}$

2. KEEP $P$ CONSTANT
   - ASSUME $M_x$
   - COMPUTE $\dot{X}$
   - REVISE $M_x$
   - REPEAT TILL $M_x = M_x(\text{max})$

3. KEEP $P$ AND $M_x$ CONSTANT
   - ASSUME $M_y$
   - COMPUTE $\dot{X}$
   - REVISE $M_y$
   - REPEAT TILL SECTION IS FULLY PLASTIC

READ INPUT DATA

COMPUTE:
- SECTIONAL PROPERTIES
- COORDINATES OF EACH ELEMENT AREA
- RESIDUAL STRESSES (IF ANY)
- SHEARING STRESSES DUE TO TORQUE (IF ANY)
- NORMAL WARping STRAINS (IF ANY)

SET INITIAL VALUES OF FORCES, CURVATURES, AXIAL STRAIN, AND INCREMENT OF EXTERNAL FORCES

ITERATION BEGINS

COMPUTE:
- UNBALANCED FORCES AND CORRECTION VECTOR
  $$\dot{X} = \{\ddot{x}, \ddot{y}, \ddot{e}\}$$

REVISE DEFORMATION
$$X = X + \dot{X}$$

COMPUTE INTERNAL FORCES

ARE DIFFERENCES BETWEEN EXT. AND INT. FORCES ACCEPTABLE?

IS NO. OF CYCLES TOO LARGE?

HAS COLUMN BEEN WEAKENED?

PRINT SOLUTIONS

NO

YES

STOP.

Fig. 18 Brief Flow Chart for Cross-Sectional Analysis
Fig. 20 Uniaxial Bending About Weak Axis (Problem C2)
Fig. 21 Moment-Curvature Curves for Biaxial Bending (Problem C3)
Fig. 22 Effect of Residual Stress (Problem C4)

- Without Residual Stress
- With Residual Stress ($\sigma_{rc} = 0.3 \sigma_y$)

$W8 \times 31$
$p = 0.3$
$m_x = 0.4$
Fig. 23 Effect of Normal Warping Stress (Problem C5)

- $\beta'' = 0$
- $\beta'' = 0.00002$

$W8\times31$
- $p = 0.3$
- $m_x = 0.4$
Fig. 24 Effect of St. Venant Torsion (Problem C6)

- $m_z = 0$
- $W 8 \times 31$
- $p = 0.3$
- $m_x = 0.4$
Fig. 25 Effect of Warping Torsion (Shear Stress) (Problem C7)
Fig. 26 Elastic and Elastic-Plastic Interaction Curves for Biaxial Bending
Fig. 27 Effect of St. Venant Torsion on Interaction Curve
Fig. 28 Effect of Warping Torsion on Interaction Curve
Fig. 29 A Biaxially-Loaded Column Under Symmetrical Loading
Fig. 30 Displacement of the Cross-Section
Fig. 31 Forces and Deformations on a Restrained Column Biaxially Loaded
Fig. 32 Transformation of Curvatures

\[ \eta'' = \nu'' - \beta_c \ U'' \]
\[ \xi'' = \beta_c \ V'' + U'' \]
Fig. 33 Load-Deformation Curves of Beam-Columns
READ INPUT DATA

COMPUTE:
SECTIONAL AND COLUMN PROPERTIES
COORDINATES OF EACH ELEMENT AREA
RESIDUAL STRESSES (IF ANY)
SHEARING STRESSES DUE TO TORQUE (IF ANY)

SET INITIAL VALUES OF AXIAL FORCE,
CURVATURES, AXIAL STRAIN, END SLOPES,
DISPLACEMENTS, AND TRANSFORMATION MATRIX

ITERATION BEGINS

ASSUME \( \dot{V}_m \)
COMPUTE CORRESPONDING
\( \{ \frac{\dot{P}}{, \dot{U}_m, \dot{\epsilon}_m, \dot{\delta}_m \} \}
REVISE P

COMPUTE:
UNBALANCED FORCES
AND CORRECTION
VECTOR
\( \Delta = [ \dot{V}_m, \dot{U}_m, \dot{\epsilon}_m, \dot{\delta}_m ] \)

REVISE DEFORMATION
\( \Delta = \Delta + \Delta \)
COMPUTE EXT. AND INT.
FORCES

ARE DIFFERENCES
BETWEEN EXT. AND INT.
FORCES ACCEPTABLE?

ARE CONVERGED VALUES
OF DISPLACEMENTS
LARGER THAN PREVIOUS
ONES?

MAKE APPROPRIATE ADJUSTMENT,
EITHER DECREASE INCREMENT
OF CURVATURE OR FORCE
COLUMN TO JUMP FROM
LOADING TO UNLOADING PORTION
OF LOAD-DISPLACEMENT CURVE
(THIS ADJUSTMENT IS
NECESSARY WHEN LOAD IS
CLOSE TO ITS MAX, AND
TANGENT STIFFNESS MATRIX
APPROACHES SINGULARITY)

PRINT SOLUTIONS

HAS COLUMN BEEN
WEAKENED?

Fig. 34 Brief Flow Chart for Proportional Loading Case
Fig. 35 Load-Displacement Curves (Problem Pl)
Fig. 36 Load-Displacement Curves (Problem P2)
Fig. 37 Load-Displacement Curves (Problem P3)
Fig. 38 Load-Displacement Curves (Problem P4)
Fig. 39 Load-Displacement Curves (Problem P5)
Fig. 40 Load-Displacement Curves (Problem P6)
Fig. 41 Load-Displacement Curves (Problem P7)
Fig. 42 Load-Displacement Curves (Problem P8)
Fig. 43 Load-Displacement Curves (Problem P9)
Fig. 44 Load-Displacement Curves (Problem P10)
$P_y = 1290$

$P_{max} = 1170$

$P_T = 1010$

0.110

0.055 $\delta_0$ (in.)

-0.220

-0.440

-0.680

$\beta_m = 0.05$

$L = 160$ in.

$\sigma_y = 27.5$ ksi

$\sigma_{rc} = 0.4 \sigma_y$

Fig. 45 Load-Displacement Curves (Problem P11)
Fig. 46 Relationship Between the Maximum Strength and the Initial Imperfection of a Centrally Loaded Column
Fig. 47 A Space Subassemblage Under Proportional Loading
Fig. 48 Moment-Rotation Relationship of Beam
Fig. 49 Load-Displacement Curves (Problem P12)

\[ P (\text{kips}) \]

<table>
<thead>
<tr>
<th>( U_m ) (in)</th>
<th>( V_m ) (in)</th>
<th>( \beta_m ) (rad.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nonrestrained</td>
<td>Restrainted</td>
<td></td>
</tr>
</tbody>
</table>

\[ K_x = K_y = 2700 \text{ kip-in/\text{rad.}} \]
\[ M_{p1} = M_{p2} = 126 \text{ kip-in} \]
Fig. 50 Load-Displacement Curves (Problem P13)
Fig. 51 Load-Displacement Curves (Problem P14)
COLUMN IS LOADED IN THE FOLLOWING SEQUENCE

1. ASSUME $\hat{p}$
   COMPUTE $\Delta$

2. KEEP $p$ CONSTANT
   ASSUME $\hat{m}$
   COMPUTE $\Delta$
   REPEAT THIS STEP UNTIL $m_{dx}$ REACHES GIVEN VALUE

3. KEEP $p$ AND $m_{dx}$ CONSTANT
   ASSUME $\hat{u}''$
   COMPUTE $\{\hat{v}''', \hat{m}_{oy}, \hat{e}_m, \hat{\hat{u}}''\}$
   REVISE $m_{oy}$

REVISE DEFORMATION
$\Delta = \Delta + \tilde{\Delta}$
COMPUTE EXT. AND INT. FORCES

ARE DIFFERENCES BETWEEN EXT. AND INT. FORCES ACCEPTABLE?

ARE CONVERGED VALUES OF DISPLACEMENTS LARGER THAN PREVIOUS ONES?

MAKE APPROPRIATE ADJUSTMENT, EITHER DECREASE INCREMENT OF CURVATURE OR FORCE COLUMM TO JUMP FROM LOADING TO UNLOADING PORTION OF LOAD-DISPLACEMENT CURVE (THIS ADJUSTMENT IS NECESSARY WHEN LOAD IS CLOSE TO ITS MAX. AND TANGENT STIFFNESS MATRIX APPROACHES SINGULARITY)

PRINT SOLUTIONS

HAS COLUMN BEEN WEAKENED?

STOP

Fig. 52 Brief Flow Chart for Nonproportional Loading Case
Fig. 53 Load-Displacement Curves (Problem N1 and N3)
Fig. 54 Load-Displacement Curves (Problems N2 and N4)
W 14x43
L = 220 in.
\( \sigma_y = 36 \text{ ksi} \)
\( \sigma_{rc} = 0.3 \sigma_y \)
\( P = 135 \text{ kips} \)

\( M_{ox} = -270 \text{ kip-in.} \)
\( M_{oy} = -675 \text{ kip-in.} \)

Fig. 55 Load-Displacement Curves (Problem N5)
Fig. 56 Maximum-Strength Interaction Curves

- Warping Permit
- Warping Restraint

\[ \frac{L}{r_x} = 0 \]

\[ \frac{L}{r_x} = 97 \]

Euler Buckling

\[ \frac{P}{P_y} = 0.3 \]

\[ \sigma_{rc} = 0.3 \sigma_y \]
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