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THE EFFECTS OF SIMPLE TRANSITIONS ON SUPERCRITICAL FLOW IN AN INCLINED OPEN CHANNEL

by

Gunnar Bagge

Fritz Engineering Laboratory Report No. 326.1
Open Channel Research

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Dr. L. S. Beedle is Acting Chairman of the Department of Civil Engineering and Director of Fritz Engineering Laboratory.
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LIST OF SYMBOLS

B = Initial width of the channel

\( d \) = Depth of flow, measured perpendicular to the bottom

\( F' \) = The Froude number, modified for inclined channel (\( = V^2 / g d \cos \beta \))

\( g \) = Acceleration of gravity

\( p \) = Pressure

\( u \) = Velocity in the x-direction, parallel to the undisturbed flow

\( v \) = Velocity in the y-direction, perpendicular to the undisturbed flow

\( V \) = Total velocity

\( w \) = Velocity in the z-direction, perpendicular to the bottom

\( \beta \) = Slope of the bottom

\( \theta \) = Change of angle in a straight flaring transition

\( \rho \) = Density of the liquid
This paper presents a theoretical analysis of the effects of disturbances on supercritical flows in an inclined channel, based on the Method of Characteristics.

Since the practical application of the method would be severely limited by the amount of manual computations involved, the adaptation of the results for computer treatment has been emphasized. A number of numerical examples concerning a straight flaring transition is presented.
1. INTRODUCTION

In many hydraulic construction projects, one is required to design transitions for water flows in open channels at high velocities, that is for supercritical flows. An accurate description of such a flow will be significantly more complex than that of a subcritical flow case, because of the formation of standing waves generated at the points of disturbance.

The theory of these flows was adapted from the analogous theory of supersonic flow of gases by von Karman\(^{(1)}\) and Freiwel\(^{(2)}\). The latter analysis was based on the Method of Characteristics and resulted in a fairly simple graphical solution for horizontal channels (explained in detail by Blaisdell\(^{(3)}\)). Most of the later work on the subject was similarly limited to horizontal or slightly inclined channels.\(^{(4,5,6,7)}\)

The present paper is an attempt to analyze supercritical flow in an inclined channel. The Method of Characteristics which is explained in detail by Owczarek\(^{(8)}\) is used. The method can be summarized as follows:

Using certain limiting assumptions (frictionless fluid, irrotational flow, hydrostatic pressure distribution) two differential equations can be derived, involving the partial derivatives of \(u\) and \(v\) with respect to \(y\) and \(x\), where \(u\) and \(v\) represent the velocities in the \(x\) and \(y\)
direction, respectively. From these equations two sets of characteristics can be found, expressed as:

\[
\frac{dy}{dx}_1 = f_1 (u, v, d) \quad \text{and} \quad \frac{dy}{dx}_2 = f_2 (u, v, d)
\]

where \(d\) indicates the depth of flow, measured perpendicular to the bottom. Along these lines \(u, v,\) and \(d\) vary in a prescribed manner, given in the compatibility relations:

\[
\frac{dv}{du}_1 = g_1 (u, v, d, \frac{dy}{du}) \quad \text{and} \quad \frac{dv}{du}_2 = g_2 (u, v, d, \frac{dy}{du})
\]

These equations, when written in finite difference form, make it possible to analyze the entire flow field, that is to determine values of \(u, v,\) and \(d\) at various points.
2. DERIVATION OF EQUATIONS

Using the Energy Equation and the Continuity Equation together with the condition for irrotational flow, two differential equations are derived, involving the partial derivations of $u$ and $v$ with respect to $x$ and $y$.

Assumptions

1. The flow is frictionless, without energy dissipation.

2. Acceleration perpendicular to the bottom is infinitely small compared to the acceleration of gravity (this results in a hydrostatic pressure distribution).

Energy Equation

![Figure 1](image)
At \( x = x_0 \) the flow is assumed to be uniform, with constant velocity \( V_o \) (\( = u_o \)). Applying the energy equation from point 0 to I gives:

\[
p + \frac{1}{2} \rho V^2 + \rho g (z \cos \beta - (x - x_0) \sin \beta) = p_o + \rho g z_o \cos \beta + \frac{1}{2} \rho V_o^2 \quad \text{or}
\]

\[
V^2 - V_o^2 = \frac{2}{\rho} (p - p_o) + 2g \left((z_o - z) \cos \beta + (x - x_0) \sin \beta\right) \quad (1)
\]

With static pressure distribution (Assumption 2)

\[
p = \rho g (d - z) \cos \beta \quad (2)
\]

Differentiation of equation (1) gives:

\[
\frac{\partial V}{\partial x} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + g \sin \beta \quad (3)
\]

\[
\frac{\partial V}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial y} \quad (4)
\]

\[
\frac{\partial V}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial z} - g \cos \beta \quad (5)
\]

while differentiation of (2) results in:

\[
\frac{\partial p}{\partial x} = \rho g \frac{\partial d}{\partial x} \cos \beta \quad (6)
\]

\[
\frac{\partial p}{\partial y} = \rho g \frac{\partial d}{\partial y} \cos \beta \quad (7)
\]

\[
\frac{\partial p}{\partial z} = - \rho g \cos \beta \quad (8)
\]

Equations (6) and (7) state that the accelerations in the \( x \) and \( y \) directions are independent of \( z \). Therefore, \( u \) and \( v \) must also be independent of \( z \), since this was the case during the initial (uniform) flow.
Equations (5) and (8) combined give that \( \frac{\partial V}{\partial z} = 0 \) or that \( V \) is independent of \( z \). Now \( w = \sqrt{v^2 - u^2 - v^2} \) must similarly be independent of \( z \), and since it equals 0 at the bottom one may conclude that \( w < < u \) or \( v \), or that \( V^2 = u^2 + v^2 \).

Combining equations (3), (6), and (9) results in:

\[
\frac{\partial d}{\partial x} = \tan\beta - \frac{v}{g \cos\beta} \frac{\partial V}{\partial x} - \frac{u}{g \cos\beta} \frac{\partial u}{\partial x}
\]

Similarly equations (4), (7), and (9) give:

\[
\frac{\partial d}{\partial y} = - \frac{v}{g \cos\beta} \frac{\partial V}{\partial y} - \frac{u}{g \cos\beta} \frac{\partial u}{\partial y}
\]

Continuity Equation

As the flow can be considered incompressible, the continuity equation becomes:

\[
\delta y \left[ \left( \frac{\partial (d) (u)}{\partial y} + \frac{1}{2} \frac{\partial \left[ \left( d \right) (u) \right]}{\partial \delta y} \delta y \right) - \left[ (d) (u) + \frac{\partial \left[ \left( d \right) (u) \right]}{\partial \delta y} \delta y + \frac{1}{2} \frac{\partial \left[ \left( d \right) (u) \right]}{\partial \delta x} \delta x \right] + \right] + \delta x \left[ \left( \frac{\partial (d) (v)}{\partial x} + \frac{1}{2} \frac{\partial \left[ \left( d \right) (v) \right]}{\partial \delta x} \delta x \right) - \left[ (d) (v) + \frac{\partial \left[ \left( d \right) (v) \right]}{\partial \delta y} \delta y + \frac{1}{2} \frac{\partial \left[ \left( d \right) (v) \right]}{\partial \delta x} \delta x \right] \right] = 0
\]

or

\[
\frac{\partial \left[ \left( d \right) (u) \right]}{\partial \delta x} + \frac{\partial \left[ \left( d \right) (v) \right]}{\partial \delta y} = 0
\]
which can be written as:

\[ d \frac{\partial u}{\partial x} + u \frac{\partial d}{\partial x} + d \frac{\partial v}{\partial y} + v \frac{\partial d}{\partial y} = 0 \]  

(12)

**Final Differential Equations**

Introducing the expressions for \( \frac{\partial d}{\partial x} \) and \( \frac{\partial d}{\partial y} \) from equations (10), (11) into the continuity equation (12) one gets:

\[ \frac{\partial u}{\partial x} \left( 1 - \frac{u^2}{gd \cos \beta} \right) + \frac{\partial v}{\partial y} \left( 1 - \frac{v^2}{gd \cos \beta} \right) - \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \frac{uv}{gd \cos \beta} + \frac{u}{d} \tan \beta = 0 \]  

(13)

Since the flow is frictionless and initially irrotational it will remain so. Therefore

\[ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \]  

(14)

These equations are suitable for analysis by the Method of Characteristics as shown in the following section.
3. SOLUTION BY METHOD OF CHARACTERISTICS

From the preceding equations expressions for the slope of the characteristics are derived with compatibility relations controlling the variation of \( u, v \) and \( d \) along these characteristics. Equations (13) and (14) can be written in the form:

\[
\begin{align*}
A_1 \frac{\partial u}{\partial x} + A_2 \frac{\partial u}{\partial y} + A_3 \frac{\partial v}{\partial x} + A_4 \frac{\partial v}{\partial y} &= F_1 \\
B_1 \frac{\partial u}{\partial x} + B_2 \frac{\partial u}{\partial y} + B_3 \frac{\partial v}{\partial x} + B_4 \frac{\partial v}{\partial y} &= F_2
\end{align*}
\]

where:

\[
\begin{align*}
A_1 &= 1 - \frac{u^2}{gd \cos \beta}, \quad A_2 = A_3 = \frac{-uv}{gd \cos \beta}, \quad A_4 = 1 - \frac{v^2}{gd \cos \beta}, \quad F_1 = \frac{u}{d} \tan \beta \\
B_1 &= 0, \quad B_2 = -1, \quad B_3 = +1, \quad B_4 = 0, \quad F_2 = 0
\end{align*}
\]

The physical characteristics can now be determined from the following equation *):

\[
(A_1 B_3 - A_2 B_1) \left( \frac{dy}{dx} \right)^2 - (A_1 B_4 + A_2 B_3 - A_4 B_1 - A_3 B_2) \frac{dy}{dx} + (A_3 B_4 - A_4 B_2) = 0 
\]

(15)

Introducing the Froude number (modified for inclined channel):

\[
F' = \frac{u^2 + v^2}{gd \cos \beta}
\]

(16)

* Further details can be found in reference (8), pp. 278-290.
the solutions of equation (15) become:

\[
\frac{dy}{dx} = \frac{uv}{gd \cos \beta} - \frac{\sqrt{F'} - 1}{1 - \frac{u^2}{gd \cos \beta}}
\]  
(17a)

and

\[
\frac{dy}{dx} = \frac{uv}{gd \cos \beta} + \frac{\sqrt{F'} - 1}{1 - \frac{u^2}{gd \cos \beta}}
\]  
(17b)

Solutions corresponding to equation (17a) will be called the \(C_+\) characteristics while the ones resulting from equation (17b) will be called the \(C_-\) characteristics. The method is obviously applicable for supercritical flow only (where \(F' > 1\)).

Along the physical characteristics certain compatibility relations must be satisfied. These can be expressed in the following equation *):

\[
\frac{dv}{du} = \frac{A_3 B_2 - A_2 B_3}{A_4 B_3 - A_3 B_4} - \frac{A_4 B_2 - A_2 B_4}{(A_4 B_3 - A_3 B_4) \frac{dy}{dx}} - \frac{(F_2 B_4 - A_4 F_2) \frac{dy}{du}}{(A_4 B_3 - A_3 B_4) \frac{dy}{dx}} + \\
+ \frac{(F B_3 - A_3 F_2) \frac{dy}{dx}}{A_4 B_3 - A_3 B_4}
\]

Inserting the expression for \(\frac{dy}{dx}\) given in equation (17a) this reduces to:

\[
\frac{dv}{du} = \frac{uv}{gd \cos \beta} + \frac{\sqrt{F'} - 1}{1 - \frac{u^2}{gd \cos \beta}} \frac{\frac{dy}{du}}{\frac{dy}{dx}}
\]  
(18a)

*) See footnote on preceding page.
Similarly, for the $C_-$ characteristics (equation (17b)):

$$
\frac{dv}{du} = \frac{uv}{gd \cos \beta} - \sqrt{\frac{f'}{1 - \frac{u}{d} \tan \beta \left(\frac{dv}{du}\right)}} - 1 - \frac{v^2}{gd \cos \beta}
$$

Equations (18a) and (18b) must be satisfied along the $C_+$ and the $C_-$ characteristics, respectively.

Using the above equations, it is possible to obtain numerical solutions for specific transition problems, as indicated in the following section.
4. ADAPTATION OF SOLUTION FOR COMPUTER TREATMENT

The equations of the characteristics and the compatibility equations will be applied to solve a flow problem involving a straight flaring transition. The method can be summarized as follows:

The solution \((u, v \text{ and } d)\) is assumed known at \(P_1\) and \(P_2\). Inserting the values of \(u, v\) and \(d\) from \(P_1\) into equation (17a) the approximate slope of the \(C_+\) characteristic passing through \(P_1\) can be determined. Similarly, using equation (17b) the slope of the \(C_-\) characteristic passing through \(P_2\) can be found, after which the approximate location of point \(Q\) is determined.

Now equation (18a) can be applied from \(P_1\) to \(Q\), using finite differences and inserting the values of \(u, v, \text{ and } d\) from \(P_1\) while \(\Delta y\) is known as \(y_Q = y_{P_1}\). Similarly, equation (18b) is used from \(P_2\) to \(Q\) and the two equations yield values of \(u\) and \(v\) for point \(Q\). Using the energy equation the corresponding value of \(d\) at \(Q\) is determined.

This procedure is subsequently repeated, using the average values of \(u, v\) and \(d\) between \(P_1\) and \(Q\) and \(P_2\) and \(Q\), respectively, until sufficient accuracy is achieved.

For points on the boundaries the method must be modified, as shown in the following detailed description.
Simple (straight) Flaring Transition

The channel is initially divided into a number of increments, depending on the desired accuracy. Although 5 is chosen here, the approach can easily be modified for a different number of points across the channel.

Four types of node must be considered, that is the lines connecting points 0-5 (I), points 6, 12, 18 etc. (II), points 11, 17, 23 etc. (III) plus the region inside the boundaries, points 7-10, 13-16 etc. (IV).

I. Points 0-5

Figure 3

Figure 4
In this region \( v = 0 \), which reduces equations (18b) to:

\[
\frac{dv}{du} = 0 = \sqrt{F'} - 1 - \frac{u}{d} \tan \beta \left( \frac{dv}{du} \right)
\]

which can be rewritten, using finite differences, as:

\[
\Delta u = - \frac{u \tan \beta \left( \Delta y \right)}{d \sqrt{F'} - 1}
\]

(19)

in which form it is suitable for iteration. A fixed value of \( \Delta y \) is assumed for each point* and in the first approximation \( u_{A-1} \) and \( d_{A-1} \) are used to determine \( \Delta u \). After that

\[
u_A = u_{A-1} + \Delta u
\]

and

\[
d_A = \frac{u_{A-1}}{u_A} \cdot d_{A-1} \quad \text{(from continuity)}
\]

In successive approximations \( u = \frac{1}{2} (u_A + u_{A-1}) \) and \( d = \frac{1}{2} (d_A + d_{A-1}) \) are used, until sufficient accuracy is achieved.

Since \( v = 0 \), equation (17b) can be rewritten as:

\[
\frac{\Delta y}{\Delta x} = \frac{\sqrt{F'} - 1}{1 - F'}, \quad \text{or}
\]

\[
\Delta x = - \Delta y \sqrt{F'} - 1
\]

(20)

from which equation \( \Delta x \) now can be determined.

* The most accurate results are obtained if the first \( \Delta y \) increment selected (between point 0 and 1) is very small.
II. Points 6, 12, 18 etc.

Using initial values of $u = u_{A-5}$, $v = v_{A-5}$ and $d = d_{A-5}$, $\frac{dy}{dx}$ can be determined from equation (17a). The (x,y) - coordinates of A are then found as the intersection between the two lines:

0 - (A): $y_A = \frac{1}{2} B + t x_A$

(A-5) - (A): $y_A = y_{A-5} + \frac{dy}{dx} (x_A - x_{A-5})$

which give:

$$x_A = \frac{\frac{1}{2} B - y_{A-5} + \frac{dy}{dx} x_{A-5}}{\frac{dy}{dx} - t} \tag{21}$$

and

$$y_A = \frac{1}{2} B + t x_A \tag{22}$$

Now:

$$\Delta x = x_A - x_{A-5} \tag{23}$$

$$\Delta y = y_A - y_{A-5} \tag{24}$$

Equation (18a) can be written as:

$$\frac{\Delta v}{\Delta u} = G_1 + \frac{G_3}{\Delta u} \tag{25}$$

where

$$G_1 = \frac{uv}{gd \cos \beta} + \frac{\sqrt{F_1 - 1}}{1 - \frac{v^2}{gd \cos \beta}} \tag{26}$$
and 

\[ G_3 = \frac{\frac{u}{d} \tan \beta \, (\Delta y)}{1 - \frac{v^2}{gd \cos \beta}} \]  

(\Delta y \text{ is known from equation (24)})

The boundary conditions for \( v_A \) and \( u_A \) can be written as:

\[ v_A = t \, u_A \]
\[ v_{A-5} + \Delta u = t \, (u_{A-5} + \Delta u) \quad \text{or} \quad \Delta v = t \, \Delta u = tu_{A-5} - v_{A-5} \]

Equation (23) and (26) can now be solved for \( \Delta v \) and \( \Delta u \) resulting in:

\[ \Delta u = \frac{G_3 - tu_{A-5} + v_{A-5}}{t - G_1} \]  

and 

\[ \Delta v = G_1 \Delta u + G_3 \]

At this point, the (approximate) values of \( u_A \) and \( v_A \) can be determined:

\[ u_A = u_{A-5} + \Delta u \]
\[ v_A = v_{A-5} + \Delta v \]

The depth at point A is found using the energy equation:

\[ d_{A-5} \cos \beta + \Delta x \sin \beta + \frac{1}{2g} \, v_{A-5}^2 = d_A \cos \beta + \frac{1}{2g} \, v_A^2 \]

\[ d_A = d_{A-5} + \Delta x \tan \beta + \frac{(u_{A-5}^2 + v_{A-5}^2 - u_A^2 - v_A^2)}{2g \cos \beta} \]  

(31)
The iteration process can now be repeated, using

\[ u = \frac{1}{2} (u_{A-1} + u_A) \]

\[ v = \frac{1}{2} (v_{A-1} + v_A) \]

and

\[ d = \frac{1}{2} (d_{A-1} + d_A) \]

until sufficient accuracy is achieved.

III. Points 11, 17, 23 etc.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure6}
\caption{Figure 6}
\end{figure}

Using \( u = u_{A-1} \), \( v = v_{A-1} \) and \( d = d_{A-1} \), \( \frac{dy}{dx} \) can be determined from equation (17b). The \((x,y)\) coordinates of point \( A \) will then become:

\[ x_A = x_{A-1} - \frac{y_{A-1}}{\frac{dy}{dx}} \]  \hspace{1cm} (32)

and

\[ y_A = 0 \]  \hspace{1cm} (33)

\[ \Delta x = x_A - x_{A-1} \]  \hspace{1cm} (34)

\[ \Delta y = y_A - y_{A-1} \]  \hspace{1cm} (35)
With known values of $u$, $v$, $d$ and $\Delta y$, equation (18b) can now be rewritten as:

$$\frac{\Delta v}{\Delta u} = G_2 + \frac{G_3}{\Delta u}$$  \hspace{1cm} (36)$$

where

$$G_2 = \frac{uv}{gd \cos \beta} - \sqrt{\frac{F}{1 - \frac{v^2}{gd \cos \beta}}}$$ \hspace{1cm} (37)$$

and $G_3$ is expressed in equation (27).

In this case the boundary conditions become:

$$v_A = 0$$ \hspace{1cm} (38)$$

or

$$\Delta v = - v_{A-1}$$ \hspace{1cm} (39)$$

Combining equations (36) and (39) gives:

$$\Delta u = - \frac{v_{A-1} + G_3}{G_2}$$ \hspace{1cm} (40)$$

$$u_A = u_{A-1} + \Delta u$$

The depth $d_A$ may now be found using an expression analogous to the one in equation (31) after which the iteration can continue using average values of $u$, $v$, and $d$ between point $(A)$ and point $(A-1)$.

IV. Points 7-10, 13-16, etc.
\( \frac{dy}{dx} \) can be determined from equation (17a) using values of \( u, v \) and \( d \) (as \( u_p, v_p \) and \( d_p \)) corresponding to point (A-5). Similarly, \( \frac{dy}{dx} \) is determined from equation (17b) using values \((u_n, v_n \) and \( d_n \)) from point (A-1).

Now, the \((x,y)\) coordinates of A can be found as the intersection between the two lines:

\[
y - y_{A-5} = \left( \frac{dy}{dx} \right)_p (x - x_{A-5})
\]

and

\[
y - y_{A-1} = \left( \frac{dy}{dx} \right)_n (x - x_{A-1})
\]

This gives:

\[
x_A = \frac{y_{A-1} - y_{A-5} + \left( \frac{dy}{dx} \right)_p x_{A-5} - \left( \frac{dy}{dx} \right)_n x_{A-1}}{\left( \frac{dy}{dx} \right)_p - \left( \frac{dy}{dx} \right)_n}
\]

(41)

and

\[
y_A = y_{A-5} + \left( \frac{dy}{dx} \right)_p (x_A - x_{A-5})
\]

(42)

\[
(\Delta x)_p = x_A - x_{A-5}
\]

\[
(\Delta y)_p = y_A - y_{A-5}
\]

\[
(\Delta x)_n = x_A - x_{A-1}
\]

\[
(\Delta y)_n = y_A - y_{A-1}
\]

Introducing:

\[
G_4 = \frac{u_p}{gd} \frac{v_p}{p} + \frac{\sqrt{F_1}}{p} - \frac{T_1}{p} - 1
\]

\[
\frac{v_p}{gd_p \cos\beta}
\]

2
\[
G_5 = \frac{\frac{u_p}{d} \tan \beta (\Delta y)_p}{2} \frac{v}{1 - \frac{v^2}{g d_p \cos \beta}}
\]
\[
G_6 = \frac{\frac{u_n}{d} \frac{v}{g d_n \cos \beta} - \sqrt{F_n - 1}}{2} \frac{v}{1 - \frac{v^2}{g d_n \cos \beta}}
\]
\[
G_7 = -\frac{\frac{u_n}{d} \tan \beta (\Delta y)_n}{2} \frac{v}{1 - \frac{v^2}{g d_n \cos \beta}}
\]
equations (18a) and (18b) can be written as:

\[
\frac{(\Delta v)_p}{(\Delta u)_p} = G_4 + \frac{G_5}{(\Delta u)_p} \quad (43)
\]
and

\[
\frac{(\Delta v)_n}{(\Delta u)_n} = G_6 + \frac{G_7}{(\Delta u)_n} \quad (44)
\]

From geometry, the following additional relations are achieved:

\[
(\Delta u)_p - (\Delta u)_n = u_{A-1} - u_{A-5} = G_8 \quad (45)
\]
\[
(\Delta v)_p - (\Delta v)_n = v_{A-1} - v_{A-5} = G_9 \quad (46)
\]

Equations (43) through (46) can be solved giving:

\[
(\Delta v)_p = \frac{G_8 G_4 - \frac{G_7 G_4}{G_6} - \frac{G_9 G_4}{G_6} + G_5}{1 - \frac{G_4}{G_6}} \quad (47)
\]
and \( (\Delta u)_p = \frac{(\Delta v)_p - G_5}{G_4} \) \( (48) \)

Now \( u_A \) and \( v_A \) can be expressed as:

\[ u_A = u_{A-5} + (\Delta u)_p \]

and \( v_A = v_{A-5} + (\Delta v)_p \)

For \( d_A \) the expression in equation (31) can be used. The iteration can now continue, using the average values of \( u, v \) and \( d \) between \( A-5 \) and \( A \) for the \( C_+ \) characteristics (subscript \( p \)) or between \( A-1 \) and \( A \) for the \( C_- \) characteristics (subscript \( n \)).
5. Examples

The variables in a flow problem involving a straight flaring transition can be expressed in dimensionless form as:

- the slope, $\beta$
- the direction change, $\theta$
- the initial (modified) Froude No., $F'_0$
- and the initial depth to width ratio, $d_0/\beta$

Since this paper is primarily concerned with the effects of the slope, constant values of the three remaining variables will be chosen for a series of examples.

![Figure 8](image)

The most significant changes in the flow regime occur along the (expansion) wave generated at point O and its reflections. In the regions between
the waves (regions, I, II and III) the flow variables remain approximately constant at any cross-section.

Figures 9 through 13 show that the effects of change in slope can be summarized as follows:

(1) With increasing slope the system of waves and reflections will be moved further downstream from the transition.

(2) The increase of the Froude Number and the decrease in depth of the flow in the transition will be more pronounced with increasing slope.
Figure 9 Examples, plots of transition
Figure 10  Depth at wall versus distance
Figure 11 Depth at centerline versus distance
Figure 12  Froude No. (modified) at wall versus distance
CONCLUSIONS

It is obvious from the preceding examples that even a moderate change in slope will have a significant effect on the supercritical flow in an open channel transition. Although the analysis presented in this paper is rather complicated for direct practical use it would be feasible to design a number of computer programs for various types of transitions, which could then be applied to actual flow problems.

The method is limited by the initial assumptions of frictionless and irrotational flow with hydrostatic pressure distribution. Future experimental studies would be needed to investigate the importance of these limitations.
APPENDIX

Computer Program
<table>
<thead>
<tr>
<th>SEQ</th>
<th>LBL</th>
<th>TYP</th>
<th>STATEMENT</th>
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<tr>
<td>001</td>
<td>10</td>
<td>D</td>
<td>X(50), Y(50), U(50), V(50), D(50)</td>
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<tr>
<td>002</td>
<td>20</td>
<td>ENDE</td>
<td></td>
</tr>
<tr>
<td>003</td>
<td>25</td>
<td>PL</td>
<td></td>
</tr>
<tr>
<td>004</td>
<td>26</td>
<td>PVLSLOPE, THETA, B, F, D(0)</td>
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<tr>
<td>005</td>
<td>30</td>
<td>T(SIN, [THETA<em>3, 14159/180]), C(SIN, [THETA</em>3, 14159/180])</td>
<td></td>
</tr>
<tr>
<td>006</td>
<td>31</td>
<td>180</td>
<td></td>
</tr>
<tr>
<td>007</td>
<td>40</td>
<td>BETA=SLPLOT3, 14159/180</td>
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</tr>
<tr>
<td>008</td>
<td>50</td>
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<td></td>
</tr>
<tr>
<td>009</td>
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<td>PL</td>
<td></td>
</tr>
<tr>
<td>010</td>
<td>60</td>
<td>PL</td>
<td>X/B, Y/B, U/C</td>
</tr>
<tr>
<td>011</td>
<td>61</td>
<td>V</td>
<td>D/B, F/C</td>
</tr>
<tr>
<td>012</td>
<td>62</td>
<td>A</td>
<td></td>
</tr>
<tr>
<td>013</td>
<td>70</td>
<td>PL</td>
<td></td>
</tr>
<tr>
<td>014</td>
<td>80</td>
<td>A+5</td>
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<tr>
<td>015</td>
<td>90</td>
<td>X(0)=0, Y(0)=W/2, V(0)=0</td>
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<tr>
<td>016</td>
<td>94</td>
<td>U(0)=SORT, [32, 2*DI0]+COS, [BETA]*F</td>
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</tr>
<tr>
<td>017</td>
<td>95</td>
<td>PHI N</td>
<td></td>
</tr>
<tr>
<td>018</td>
<td>100</td>
<td>PV</td>
<td>X(0)+Y(0)+U(0)+2/[32, 2*DI0]+COS, [BETA])</td>
</tr>
<tr>
<td>019</td>
<td>105</td>
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</tr>
<tr>
<td>020</td>
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<tr>
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<td>222</td>
<td>A+41</td>
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<td>224</td>
<td>A+47</td>
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<td>037</td>
<td>230</td>
<td>A+50 (ENDE)</td>
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</tr>
<tr>
<td>038</td>
<td>240</td>
<td>U=U(A-5), V=V(A-5), D=DI(A-5)</td>
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<tr>
<td>039</td>
<td>250</td>
<td>U=U(A-1), V=V(A-1), D=DI(A-1)</td>
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</tr>
<tr>
<td>040</td>
<td>260</td>
<td>GP=32, 2*DI0+COS, [BETA]</td>
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<tr>
<td>041</td>
<td>280</td>
<td>C=GP*ON/DP</td>
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<tr>
<td>042</td>
<td>290</td>
<td>FF=([UP+LP+VP+VP]/GP)</td>
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</tr>
<tr>
<td>043</td>
<td>300</td>
<td>FN=([UN+UN+VN+VN]/GN)</td>
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<tr>
<td>044</td>
<td>310</td>
<td>DYDXP=([UP+VP]/GP+SORT, [FP+1]/[1-UP+UP/GP])</td>
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</tr>
<tr>
<td>045</td>
<td>320</td>
<td>DIYDXN=([UN+VN+GN+SORT, [FN+1]/[1-UN+UN/GN])</td>
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<tr>
<td>046</td>
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<td>X(A)=T(Y(A-1)+Y(A-5)+Y(A-9)+DYDXP+Y(A-1)+DYDXN)</td>
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<tr>
<td>047</td>
<td>340</td>
<td>Y(A)=T(Y(A-1)+Y(A-5)+DYDXP+Y(A-1)+DYDXN)</td>
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<tr>
<td>048</td>
<td>341</td>
<td>U=U+VP+GP+SORT, [FP+1][1-VP+VP/GP]</td>
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<tr>
<td>049</td>
<td>350</td>
<td>DYP=U+VP+DP+SORT, [FP+1][1-VP+VP/GP]</td>
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<tr>
<td>050</td>
<td>360</td>
<td>DYNN=Y(A)+Y(A-1)</td>
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<tr>
<td>051</td>
<td>375</td>
<td>C=([UP+VP]/GP+SORT, [FP+1][1-VP+VP/GP]</td>
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<tr>
<td>052</td>
<td>380</td>
<td>C=[[UN+VN+GN]+SORT, [FN+1][1-VN+VN/GN]</td>
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<tr>
<td>#</td>
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<td>LABEL</td>
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<tr>
<td>108.</td>
<td>781</td>
<td>*2+V[A-1]*2-U[A-1]<em>2+64,4</em>COS(BETA)]</td>
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<td>DD=D[A-1]+DD/2</td>
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<td>N=ABS([S-L]/L)+0.005</td>
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<td>ABS((D-W1)/W)+0.005</td>
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<td>ABS([F-X/3]*0.005</td>
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<td>ABS([S-L]/L)+0.005</td>
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<td>AR L=U[A-1], N=DD/2</td>
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<td>MH S=L, R=H, DU=B/R</td>
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<td>FA=U/0.005</td>
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<tr>
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<td>856</td>
<td>DU=D/A, DU/0.005</td>
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<td>119.</td>
<td>860</td>
<td>DU=D/A, DU/0.005</td>
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<td>DD=D[A-1]+DD/2</td>
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<td>DU=U[A-1]-D[A-1]</td>
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<td>L=U[A-1]-DL/2</td>
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<tr>
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<td>N=ABS([S-L]+DD/2</td>
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<td>128.</td>
<td>928</td>
<td>ABS([F-X/3]*0.005</td>
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<tr>
<td>129.</td>
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<td>ABS([S-L]/L)+0.005</td>
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<td>DX=D/Y+SQRT([F]*1)</td>
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<td>131.</td>
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<td>X[A]-X[A-1]+DX</td>
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<td>Y[A]-Y[A-1]+DY</td>
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<td>134.</td>
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<td>ENDS END OF PROGRAM</td>
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