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Forbidden subgraphs for Hamiltonian problems on 2-trees

by

Caitlin Owens

A Dissertation
Presented to the Graduate Committee
of Lehigh University
in Candidacy for the Degree of
Doctor of Philosophy
in
Mathematics

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Approved and recommended for acceptance as a dissertation in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Caitlin Owens

Forbidden subgraphs for Hamiltonian problems on 2-trees

Date

Garth Isaak, Dissertation Director, Chair

Accepted Date

Committee Members

Don Davis

Vince Coll

James Hammer

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Abstract

The Hamiltonian path problem is a well-known NP-complete graph theory problem which is to determine whether or not it is possible to find a spanning path in a graph. Some variations on this problem include the 1HP and 2HP problems, which are to determine whether or not it is possible to find a Hamiltonian path in a graph if one or two endpoints of the path are fixed, respectively. Both problems are also NP-complete for graphs in general, though like the Hamiltonian path problem, they are polynomially solvable on certain types of graphs. 2-trees are a specific type of graph for which the 1HP, 2HP, and traditional Hamiltonian path problems are polynomially solvable. It is known that 2-trees have a Hamiltonian cycle if and only if they are 1-tough. However, the analogous statement for Hamiltonian paths does not hold. We will structurally characterize 2HP on 2-trees, and then use these results to structurally characterize 1HP and HP on 2-trees. We will define a family of 2-trees such that any 2-tree has a Hamiltonian path if and only if it does not contain any graph from that family as an induced graph.

Chapter 1

Introduction

In this chapter, we will review basic information regarding 2-trees and Hamiltonian problems. In Section 1.1, we will review basic terminology and definitions and in Section 1.2 we will provide basic results. In Chapter 2 we will introduce the new definitions and techniques that will be used in this dissertation. Chapter 3 and 4 will have our main results regarding the Hamiltonian path problems on 2-trees, and Chapter 5 will conclude the dissertation with future work.

1.1 Problem Description

The Hamiltonian path problem (HP) is to determine whether or not a given graph has a Hamiltonian path, i.e., a spanning path in the graph. Two variations of this problem, 1HP and 2HP, determine whether a given graph has a Hamiltonian path fixing one or two given vertices, respectively, as endpoints.

The 1HP problem is known to be polynomially solvable on interval graphs, cographs, and biconvex graphs. It is known to be NP-complete on chordal and comparability graphs. The complexity of the 1HP problem is unknown on both permutation graphs and convex graphs. The complexity of the 2HP problem is unknown on interval graphs [2], though it is known to be polynomially solvable on cographs [3].

For k-trees, a subclass of chordal graphs, HP, 1HP, and 2HP problems are polynomially solvable. They fall into the class of partial k-trees, graphs with treewidth at most k. The Hamiltonian path problem is polynomially solvable on graphs with bounded treewidth [11], using FPT algorithms, or algorithms which are fixed parameter tractable. Since adding a pendant edge to a k-tree keeps the graph in the class of partial k-trees, we can solve the 1HP and 2HP problems by adding a pendant edge to a given path endpoint and running the Hamiltonian path algorithm for partial k-trees on the resulting graph.

In [23], Renjith and Sadagopan give a linear-time algorithm for Hamiltonian paths in 2-trees. They also discuss some structural qualities of 2-trees having a Hamiltonian path. Most of their results involve multiple algorithms and structures of a graph, which is not a 2-tree, formed from the algorithms. In Chapter 4, we will take a different approach, by using toughness properties and results from the 2HP problem to give a list of forbidden induced sub-2-trees for which a 2-tree will not have a Hamiltonian path.

In this dissertation, we will give structural conditions of 2-trees and forbidden induced subgraphs for Hamiltonian paths not to exist in 1HP, 2HP, and the traditional Hamiltonian path problem.

1.2 Basic Definitions and Results

Definition 1.2.1. *A graph, G, is **Hamiltonian** if G contains a Hamiltonian cycle.*

We will use the following notations for the 1HP and 2HP problems.

Definition 1.2.2. *Given a graph G and $x_1 \in V$, an **x_1-Hamiltonian Path** in G is a Hamiltonian path which either begins or ends with x_1.*

Definition 1.2.3. *Given a graph G and $x_1, x_2 \in V$, an **(x_1,x_2)-Hamiltonian Path** in G is a Hamiltonian path between x_1 and x_2.*

Our focus for these problems will be on 2-trees, which are k-trees for $k = 2$.

Definition 1.2.4. *[24] Define a **k-tree** as follows:*

- *K_k, the complete graph on k vertices, is a k-tree, and*
- *If G is a k-tree, then the graph formed by adding a vertex adjacent to all vertices in a k-clique in G is a k-tree.*

Subgraphs of k-trees are called ***partial k-trees****. A* ***simplicial vertex*** *is a vertex whose neighbors form a clique. Simplicial vertices in a k-tree have degree k.*

Notation 1.2.5. *We will use the notation from [7] where $S_1(G)$ is the set of simplicial vertices in G.*

Definition 1.2.6. *For any graph $G = (V, E)$, and $X \subset V$, $G[X]$, the graph induced by X, has vertex set X, and edge set $E' \subset E$ such that $uv \in E'$ iff $uv \in E$ and $u, v \in X$.*

Notation 1.2.7. *For any graph $G = (V, E)$ and $v \in V$, $G - v$ denotes $G[V - \{v\}]$. Likewise, for $S \subset V$, $G - S$ denotes $G[V - S]$.*

In [24], Rose also characterizes k-trees as connected graphs which contain a k-clique but no $k+2$-clique, and such that every minimal x, y separator of G is a k-clique. An x, y separator is a set $S \subset V$ such that x and y lie in different components of $G - S$.

Stemming from Chvátal's conjecture that there exists a t_0 such that every t_0-tough graph is hamiltonian [10], many known results, including those from [7], regarding Hamiltonian problems in k-trees involve toughness conditions.

Definition 1.2.8. *For a graph $G = (V, E)$ and $S \subset V$, let $c(G - S)$ denote the number of components in $G - S$. Then G is* ***t-tough*** *if $|S| \geq t(c(G - S))$ for all cut-sets, S, i.e., $S \subset V$ such that $c(G - S) > 1$. A set, S such that $|S| = t(c(G - S))$ is called a* ***tough set****.*

Definition 1.2.9. *A graph G is* ***1-path-tough*** *if $|S| \geq (c(G - S) - 1)$ for all $S \subseteq V$.*

The following theorem, originally stated by Chvátal in [10], is well known and can be found in many graph theory textbooks.

Theorem 1.2.10. *[10] If a graph G has a Hamiltonian cycle, then G is 1-tough.*

Theorem 1.2.11. *If G has a Hamiltonian path, then G is 1-path-tough.*

Path tough has also been used in [12] to describe a graph, G, such that for any nonempty $S \subset V$, $G - S$ can be covered by at most $|S|$ vertex disjoint paths.

Closely related to toughness, we will often use the scattering number of a graph when proving that Hamiltonian paths do not exist in a graph.

Definition 1.2.12. *[16] The* ***scattering number*** *of a graph G is*

$$s(G) = \max_{S \subseteq V,\ c(G-S) \neq 1} \{c(G-S) - |S|\}.$$

Hence, if G is 1-tough, then $s(G) \leq 0$ and if G is 1-path-tough, then $s(G) \leq 1$. Furthermore, if G is a graph for which $s(G) \geq 2$, then G does not have a Hamiltonian path.

Additionally, for graphs with scattering number at least one, the scattering number of a graph gives a well known lower bound for the path partition number of a graph.

Notation 1.2.13. *$PP(G)$ denotes the path partition number of a graph, G, the minimum number of vertex disjoint paths required to cover the vertices of G.*

The path partition number has also been referred to as the path cover number.

Lemma 1.2.14. *For any graph G,*

$$PP(G) \geq \max_{U \subseteq V} \{c(G-U) - |U|\}.$$

The related k-fixed endpoint path partition problem is to determine the minimum number of vertex disjoint paths required to cover the vertices of G given that each vertex in a set T of k vertices are each endpoints of a path. In [4], Baker gives the following lower bound for the k-fixed endpoint path partition number of a graph G with respect to $T \subset V(G)$. This will be helpful when we look at 2HP.

Notation 1.2.15. *$PP(G;T)$ denotes the k-fixed endpoint path partition number of a graph G with respect to $T \subset V(G)$.*

Lemma 1.2.16 (Baker, 2013)**.** *[4] For any graph G and a set $T \subset V(G)$,*

$$PP(G;T) \geq \max_{U \subseteq V} \{c(G-U) - |S|\},$$

for $S = U - T$.

We will begin looking at the Hamiltonian path problems on 2-trees by looking at the toughness conditions regarding Hamiltonian cycles in k-trees from [7].

Theorem 1.2.17 (Broersma, Xiong, Yoshimoto, 2005)**.** *[7] If $G \neq K_2$ is a $\frac{k+1}{3}$-tough k-tree ($k \geq 2$), then G is Hamiltonian.*

For $k = 2$, the above theorem proves that 1-toughness is also a sufficient condition for 2-trees to have a Hamiltonian cycle. In their proof, the Broersma, Xiong, and Yoshimoto also prove that there is a cycle which contains all of the edges, $e = uv$, for which $c(G - \{u, v\}) = 1$. For 1-tough 2-trees, this is the only Hamiltonian cycle in the graph. In Theorem 1.2.23, we will restate and prove Theorem 1.2.17 for the special case when $k = 2$.

Knowing that 1-toughness is a sufficient condition for a 2-tree to have a Hamiltonian cycle, it seemed natural to check if there was a similar 1-path-toughness condition for 2-trees having Hamiltonian paths. For a cocomparability graph, G, G has a Hamiltonian cycle iff it is 1-tough, and likewise, G has a Hamiltonian path iff it is 1-path-tough [13]. However, while 1-path-toughness is a necessary condition, it is not a sufficient condition for a 2-tree to have a Hamiltonian path. We build an infinite class of 1-path-tough 2-trees which do not contain a Hamiltonian path, as demonstrated in Figure 1.3. These 1-path-tough 2-trees will not be 1-tough, since clearly if a 1-path-tough 2-tree is also a 1-tough 2-tree, then it will have a Hamiltonian path by Theorem 1.2.17. So, first we will discuss a few structural conditions to identify 2-trees which are and are not 1-tough. In [19], Markenzon, Justel, and Paciornik refer to a 1-tough 2-tree as a simple-clique 2-tree or SC 2-tree, but we will refer to these 2-trees by their toughness condition.

Definition 1.2.18. *The open neighborhood, $N_G(v)$, of a vertex v, is the set of vertices adjacent to v in G. We will drop the G, when the graph in question is clear. The closed neighborhood of a vertex v is $N[v] = N(v) \cup \{v\}$.*

Definition 1.2.19. *We will say a vertex, v, is* ***adjacent to an edge****, uw, if v is adjacent to both u and w. Furthermore, for any edge, $e = uw$, the* ***closed neighborhood of* e***, $N[e]$, will be defined as $N[e] = N[u] \cap N[w]$, and the open neighborhood of* **e***, $N(e)$, will be defined as $N(e) = N(u) \cap N(w)$.*

Definition 1.2.20. *A* ***t-edge*** *is an edge, e such that $|N(e)| = t$.*

Remark 1.2.21. *A t-edge will be shared by t distinct induced K_3's, or triangles.*

The following lemma and its proof are similar to that found in [23] with new notation. We provide an additional proof here for clarity and completeness.

Lemma 1.2.22. *Suppose $G \neq K_2$ is a 2-tree. If $xy \in E(G)$ is a t-edge then $c(G - \{x, y\}) = t$.*

Proof. We proceed by induction on $|V(G)| = n$. If $n = 3$, then $G = K_3$. Furthermore, all edges are 1-edges, and the claim is true. Suppose the claim is true for graphs with $n - 1$ vertices. Now, consider G a 2-tree with $|V(G)| = n$. Then there exists a simplicial vertex v, such that $G' = G - v$ is a 2-tree with $|V(G')| = n - 1$. Suppose that v is adjacent to uw in G. If $xy \neq uw, uv, vw$, then $xy \in E(G')$, and by the induction hypothesis, if xy is a t-edge then $c(G' - \{x, y\}) = t$. Since v is adjacent to u and w, then v is in the same component as u if $x = w$ or $y = w$, and v is in the same component as w if $x = u$ or $y = u$. So, $c(G - \{x, y\}) = c(G' - \{x, y\}) = t$. If uw is a t-edge in G', then $c(G' - \{u, w\}) = t$, and so in G, since uw is also adjacent to v, then uw is a $(t + 1)$-edge. Additionally, $c(G - \{u, w\}) = t + 1$ as v is only adjacent to u and w. In G, both uv and vw are 1-edges. Furthermore, $c(G - \{u, v\}) = 1$ since $c(G - \{u, v\}) = c(G' - \{u\}) = 1$, as G' is a 2-tree and minimal separators of 2-trees are 2-cliques. Likewise, $c(G - \{v, w\}) = 1$. □

Using our new terminology, we can restate Theorem 1.2.17 with a structural condition, as Theorem 1.2.23 below.

Theorem 1.2.23. *If $G \neq K_2$ is a 2-tree, then the following are equivalent:*

1. *G is 1-tough,*

2. *G contains no t-edges for $t \geq 3$, and*

3. *G is Hamiltonian.*

Proof. (1) $\Longrightarrow$ (2)
We will prove the contrapositive. If G contains a t-edge, xy, for $t \geq 3$, then $c(G - \{x, y\}) = t \geq 3 > 2 = |\{x, y\}|$, then G is not 1-tough.
(2) $\Longrightarrow$ (3)
We will prove, by induction on $|V(G)| = n$, that if G contains no t-edges for $t \geq 3$, then G contains a Hamiltonian cycle containing all of the 1-edges of G, and hence is Hamiltonian. If $n = 3$, then $G = K_3$. G only contains t-edges where $t = 1$ and G is Hamiltonian with Hamiltonian cycle containing all 1-edges. Suppose that all 2-trees with $n - 1$ vertices and only t-edges for $t \leq 2$ are Hamiltonian with Hamiltonian cycle containing all 1-edges. Now consider G a 2-tree with $|V(G)| = n$ such that G contains only t-edges for $t \leq 2$. Let v be a simplicial vertex of G, adjacent to $uw \in E(G)$. Then by the induction hypothesis, $G' = G - v$ is Hamiltonian and hence contains a Hamiltonian cycle, C, containing all 1-edges. Furthermore, uw must be a 1-edge in

G', since G contains no 3-edges. Therefore, replacing, uw in C with (u, v, w) creates a Hamiltonian cycle C' in G, containing all 1-edges.

(3) $\implies$ (1) Theorem 1.2.10. □

From Theorem 1.2.23, if G is 2-tree which is not 1-tough, then G contains at least one t-edge, xy, for $t \geq 3$. If G is 2-tree which is 1-path-tough, then by the lemma below, G cannot contain a t-edge, xy, for $t \geq 4$. However, there are 2-trees which are not 1-path-tough which do not contain a t-edge, xy, for $t \geq 4$. Furthermore, there are 1-path-tough 2-trees which do not contain a Hamiltonian path. So, for Hamiltonian paths in 2-trees, we will not have a necessary and sufficient condition using t-edges as in Theorem 1.2.23. In Theorem 4.1.15, we will prove necessary and sufficient conditions for a 2-tree to have a Hamiltonian path, using induced subgraphs. We could also restate (2) in Theorem 1.2.23 using an induced subgraph condition instead. If G is a 2-tree which contains a t-edge for $t \geq 3$, then G contains an induced $K_2 \vee 3K_1$.

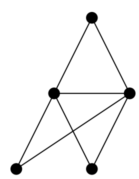


Figure 1.1: $K_2 \vee 3K_1$

Lemma 1.2.24. *If G is a 2-tree and contains a t-edge for $t \geq 4$, then G does not contain a Hamiltonian path.*

Proof. Let $xy \in E(G)$ be a t-edge for $t \geq 4$. Then, $c(G - \{x, y\}) - |\{x, y\}| = t - 2 \geq 2$, and G is not 1-path-tough. □

Lemma 1.2.25. *Suppose G is a 2-tree and contains a 3-edge, ab, such that ab is adjacent to two simplicial vertices, v_1 and v_2. Then G has a Hamiltonian path iff $G - v_1$ has a Hamiltonian path with either a or b as an endpoint of the path.*

Proof. $\impliedby$ Without loss of generality, assume $G - v_1$ has a Hamiltonian path, P, which begins with a as an endpoint. Then (v_1, P) is a Hamiltonian path in G.

$\implies$ Suppose G has a Hamiltonian path, P. Since $c(G - \{a, b\}) = 3$, then the endpoints of the Hamiltonian path must lie in two of the three components. Hence at least one of the simplicial vertices must be an endpoint of the path. Without loss of generality, let v_1 be an endpoint of the path. Since v_1 is only adjacent to a and b, then either a or b follows v_1 on the path. Since we cannot use v_1 again on the path, then the rest of the path must be in $G - v_1$, and hence $P - v_1$ is a Hamiltonian path in $G - v_1$ which has either a or b as an endpoint. □

From the above lemma, we can see that when a 2-tree contains a 3-edge, if there is a Hamiltonian path, there will be endpoint restrictions. Because of this, in Chapter 3, we begin our investigation looking at the 2HP problem on 2-trees, to extend these results to the the Hamiltonian path problem on 2-trees.

Definition 1.2.26. *A pair of vertices, u, v are called **false twins** if $N(u) = N(v)$. Vertices, u, v are called **twins** if $N[u] = N[v]$, i.e., the vertices are also adjacent.*

Definition 1.2.27. *Let P_n be a path with n vertices. Then $\mathbf{P_n^k}$, the k^{th} power of P_n, is a graph which has the same vertex set as P_n, but has edges between any vertices whose distance in P_n is at most k.*

Note that P_n^2 is a 2-tree. In particular, it is a special case of a 2-path graph. Originally introduced in [22] and further characterized in [19], a 2-tree with exactly two simplicial vertices is a **2-path graph**.

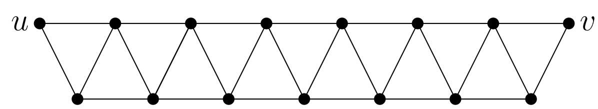


Figure 1.2: A specific example of a 2-path: P_{15}^2, with simplicial vertices u and v

Now consider $G = P_{17}^2$. We will form H from G by first adding a false twin of each of the simplicial vertices of G. Then we will add a pair of simplicial vertices adjacent to a 1-edge, ab, such that $(N[a] \cup N[b]) \cap (N[e] \cup N[f]) = \emptyset$, for any 3-edge, ef, as in Figure 1.3.

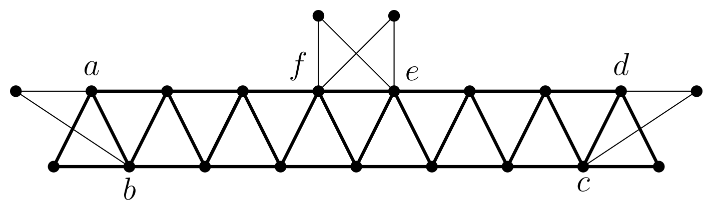


Figure 1.3: Example of a 1-path tough graph with no Hamiltonian path: H was constructed by adding simplicial vertices to $G = P_{17}^2$, shown in bold, such that $(N[a] \cup N[b]) \cap (N[e] \cup N[f]) = \emptyset$, for any 3-edges, ab, ef.

This is an example of a 1-path-tough graph which does not have a Hamiltonian path. The main idea, which will be formally proved in Chapter 4, for why there is no Hamiltonian path comes from Lemma 1.2.25 and that based on the construction of H, a Hamiltonian path in H would have three distinct endpoints. Furthermore, if we construct H from $G = P_n^2$ with larger n and add more pairs of simplicial vertices with the same properties as before, then we can create an infinite class of 1-path-tough graphs which do not contain a Hamiltonian path.

Definition 1.2.28. *A graph G is* ***Hamilton-connected*** *if there is a Hamiltonian path between all pairs of vertices of G.*

Theorem 1.2.29 (Kabela, preprint 2017)**.** *[17] Let $k \geq 3$. Every k-tree of toughness greater than $\frac{k}{3}$ is Hamilton-connected.*

The above theorem does not hold for $k = 2$. While 1-tough 2-trees are Hamiltonian, and contain a Hamiltonian path, they are not Hamilton-connected. Furthermore, even for $k = 3$, equality does not hold in the above theorem. In [17], Kabela gives examples of 1-tough planar 3-trees which do not contain a Hamiltonian path.

Since 1-tough 2-trees are not Hamilton-connected, in Chapter 3, we will discuss the 2HP problem on 1-tough 2-trees and which pairs of vertices will not be ends of a Hamiltonian path. We will then use these results to characterize the rest of the 2-trees with fixed endpoints which do not contain a Hamiltonian path. In Chapter 4 and 5 we will extend the results from 2HP on 2-trees to the traditional Hamiltonian path problem, and 1HP, respectively. In order to describe our results on 2HP, HP, and 1HP, we will begin the next chapter with a new toughness definition and special induced subgraphs of 2-trees which will help us define induced subgraphs which will not contain a Hamiltonian path. We will also discuss the types of induced subgraphs, which do not contain a Hamiltonian path, and which will prevent a general chordal graph from having a Hamiltonian path.

Chapter 2

New Approach

Since toughness and t-edges alone will not be enough to characterize 2HP, HP, and 1HP on 2-trees, we will take a new approach for a characterization by looking at induced subgraphs of 2-trees and by defining a new property regarding toughness.

In the next chapters, we will introduce families of 2-trees for which there do not exist Hamiltonian paths, or Hamiltonian paths with specified fixed endpoints. We will prove in Theorems 3.1.24, 3.2.10, 4.1.15, and 4.2.12, that if a 2-tree contains a graph from these families as an induced subgraph, the 2-tree will not have a Hamiltonian path. In this chapter, we will define special types of 2-trees, which will be useful in describing our families of graphs which do not contain Hamiltonian paths.

In general if a graph has an induced subgraph which is not Hamiltonian, we will not know whether or not our graph is Hamiltonian. In Section 2.1 we will define a special type of induced subgraph. If a graph has one of these induced subgraphs and is not Hamiltonian, then our graph will not be Hamiltonian as well. In Section 2.2, we will define specific 2-trees which will be the building blocks of our families of 2-trees in the later chapters. In Section 2.3, we will we define our new toughness property which will help us to prove graphs do not have Hamiltonian paths in later chapters.

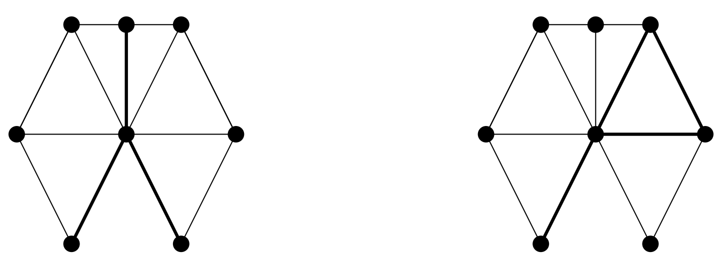
2.1 Induced Subgraphs

In [15], Goodman and Hedetniemi prove that 2-connected graphs which do not contain an induced $K_{1,3}$ or $N(1,0,0)$ are Hamiltonian. This is only a sufficient condition for a 2-connected graph to be Hamiltonian, whereas we will prove both necessary and sufficient conditions for Hamiltonian paths in 2-trees.

(a) $K_{1,3}$ (b) $N(1,0,0)$

Figure 2.1: A 2-connected graph not containing 2.1(a) and 2.1(b) as an induced subgraph is Hamiltonian.

Since 2-trees are 2-connected, 2-trees which do not contain an induced $K_{1,3}$ or $N(1,0,0)$ are Hamiltonian. Note that a 2-tree which contains a t-edge for $t \geq 3$ will contain an induced $K_{1,3}$, so if we are looking at 2-trees which do not contain an induced $K_{1,3}$, they will be 1-tough. However, we can also have 1-tough, and hence Hamiltonian, 2-trees which contain an induced $K_{1,3}$ and an induced $N(1,0,0)$, as in Figure 2.2.



(a) induced $K_{1,3}$ (b) induced $N(1,0,0)$

Figure 2.2: Example of a Hamiltonian 2-tree containing an induced $K_{1,3}$ and $N(1,0,0)$ where the induced $K_{1,3}$ and $N(1,0,0)$ are bolded

In general, a graph can be Hamiltonian even if it contains an induced subgraph which is not Hamiltonian. For example, a cycle is Hamiltonian, but no induced subgraph of a cycle is Hamiltonian. Even for the class of 2-trees, a Hamiltonian 2-tree can contain an induced subgraph which is not Hamiltonian, and so in this chapter we will present sufficiency conditions for which an induced subgraph which is not Hamiltonian will mean the chordal graph which contains it will not be Hamiltonian.

Consider the following 2-path, G, where all vertices are adjacent to a degree seven vertex.

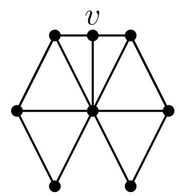


Figure 2.3: A 2-path, G, for which all vertices are adjacent to a degree seven vertex

Then $G[V(G) - v]$ below is an induced subgraph. Furthermore, $G[V(G) - v]$ is not Hamiltonian as it contains a cut-vertex, and hence is not 1-tough.

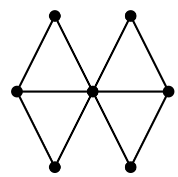


Figure 2.4: $G[V(G) - v]$ corresponding to G in Figure 2.3

Definition 2.1.1. *Let G be a k-tree. If H is an induced subgraph of G, which is also a k-tree, then H will be called an induced* ***sub-k-tree****.*

If we consider a 2-tree, G, which has an induced sub-2-tree, H, such that H does not contain a Hamiltonian path, then G also does not contain a Hamiltonian path. We will prove this below, though it is worthwhile to note that a parallel statement will not hold for chordal graphs in general.

Consider the class of chordal partial 3-trees, $\mathscr{C}$, which like 2-trees can be constructed recursively as follows:

1. K_2 is in $\mathscr{C}$, and
2. If G is in $\mathscr{C}$, then the graph formed by adding a vertex adjacent to all vertices in a 2-clique or a 3-clique in G is in $\mathscr{C}$.

Now consider G in Figure 2.5 below. We can see that $G \in \mathscr{C}$ by considering the edge labelled 87, the 'start' and adding vertices to the graph in decreasing consecutive order follows (2) in the recursive definition. Then G is Hamiltonian, with Hamiltonian cycle (6, 5, 1, 8, 2, 7, 4, 3, 6). However, we can find an induced subgraph of G which remains in the class of $\mathscr{C}$, which is not Hamiltonian.

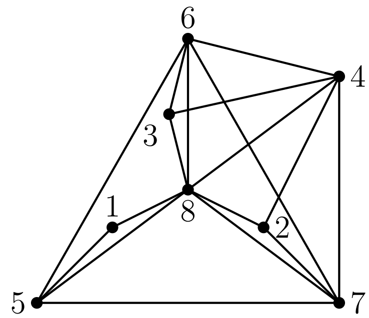


Figure 2.5: Hamiltonian $G \in \mathscr{C}$

The graph $G[V(G) - 4]$ is a well known 1-tough graph which is not Hamiltonian and furthermore, $G[V(G) - 4]$ is an induced subgraph of G which is also in $\mathscr{C}$.

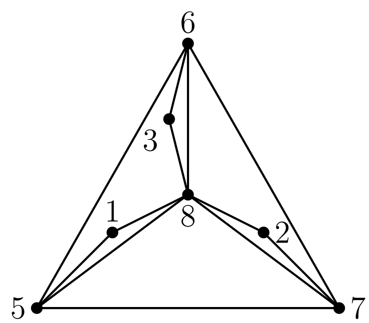


Figure 2.6: $G[V(G) - 4] \in \mathscr{C}$ corresponding to G in Figure 2.5 which is not Hamiltonian

A well known property of chordal graphs, which will help us distinguish between types of induced subgraphs, is that the vertices of a chordal graph can be labelled with a simplicial elimination ordering. A simplicial elimination ordering is also often called a perfect elimination ordering.

Definition 2.1.2. *A labelling* $(v_1, ..., v_n)$ *is a* ***simplicial elimination ordering*** *of a graph* G*, if* v_i *is a simplicial vertex in* G_{i-1} *where* $G = G_0$ *and* $G_i = G_{i-1} - v_i$*.*

Definition 2.1.3. *Let* H *be an induced subgraph of* G*.* H *will be called an* ***SEO-induced subgraph*** *if there exists a simplicial elimination ordering,* $(v_1, ..., v_n)$*, of* G*, such that* $H = G[\{v_i, v_{i+1}, ..., v_n\}]$*.*

Lemma 2.1.4. *Let G be a chordal graph with simplicial vertex v.*

1. *If $G-v$ does not have a Hamiltonian path, then G does not have a Hamiltonian path.*

2. *If $G-v$ does not have a Hamiltonian cycle, then G does not have a Hamiltonian cycle.*

3. *If $G-v$ does not have an (x_1, x_2)-Hamiltonian path, then G does not have an (x_1, x_2)-Hamiltonian path.*

Proof. (a) Suppose that G has a Hamiltonian path, P. If v is an endpoint of the P, then $P-v$ is a Hamiltonian path in $G-v$. Now suppose that v is preceded by u and followed by w on P. Since v is simplicial, then u and w must be part of a clique, and hence are adjacent. Thus, replacing uv, vw on P with uw yields a Hamiltonian path in $G-v$. The proofs of (b) and (c) are similar. □

Corollary 2.1.5. *Let G be a chordal graph. If G contains an SEO-induced subgraph which does not contain a Hamiltonian path, then G does not contain a Hamiltonian path.*

Remark 2.1.6. *Note that $G[V(G)-4]$ in Figure 2.6 is not an SEO-induced subgraph of G in Figure 2.5 for any simplicial elimination ordering since the vertex labelled 4 is not simplicial in G.*

Proposition 2.1.7 (Proskurowski, 1980)**.** *[22] Given a k-tree Q and any k-clique B of Q, Q can be constructed from B by the iterative method of Definition 1.2.4.*

Remark 2.1.8. *Labelling the base subgraph $(n, ..., n-k+1)$, in Proposition 2.1.7 and successive simplicial vertices in the construction in decreasing consecutive order will yield a simplicial elimination ordering.*

Lemma 2.1.9. *Let G be a k-tree. If H is any induced sub-k-tree of G, then H is an SEO-induced subgraph of G.*

Proof. We will induct on $|V(H)| = m \leq |V(G)| = n$. If $m = k$, then H is a k-clique. Then from Proposition 2.1.7, the claim is true. Now suppose that for any induced sub-k-tree with $m-1$ vertices, that the claim is true. Now, suppose H is an induced sub-k-tree of G such that $|V(H)| = m$. Let w be a simplicial vertex in H. Then $H-w$ is an induced sub-k-tree of G such that $|V(H-w)| = m-1$. By the induction hypothesis, there exists a simplicial elimination ordering, $(v_1, ..., v_n)$, of G, such that $H - w = G[\{v_{n-m+2}, v_{n-m+3}, ..., v_n\}]$. If w is labelled v_{n-m+1}, then $H = G[\{v_{n-m+1}, v_{n-m+2}, v_{n-m+3}, ..., v_n\}]$ is an SEO-induced subgraph with the same labelling. If w is labelled $v_j \neq v_{n-m+1}$, then reduce by one all labels from v_{j+1} to v_{n-m+1} and relabel w as v_{n-m+1}. Note that w cannot be adjacent to any vertices with labels from v_{j+1} to v_{n-m+1} or $w = v_j$ would have degree more than k in G_{j-1}, contradicting that w is simplicial. So, the new labelling will still be a simplicial elimination ordering and $H = G[\{v_{n-m+1}, v_{n-m+2}, v_{n-m+3}, ..., v_n\}]$ is an SEO-induced subgraph under the new labelling. □

Corollary 2.1.10. *Let H be a k-tree which does not contain a Hamiltonian path. If H is an induced sub-k-tree of G, then G also does not contain a Hamiltonian path.*

Corollary 2.1.11. *Let H be a k-tree, with $x_1, x_2 \in V(G)$, which does not contain an (x_1, x_2)-Hamiltonian path. If H is an induced sub-k-tree of G, then G also does not contain an (x_1, x_2)-Hamiltonian path.*

Corollary 2.1.12. *If H is an induced sub-k-tree of G, then G can be constructed from H by the iterative method of Definition 1.2.4.*

2.2 Special Induced Sub-2-trees

In order to describe the sub-2-trees which we will later prove prevent 2-trees from having a Hamiltonian path, we will use graph amalgamation on disjoint graphs to create a connected graph as defined in the following definition.

Definition 2.2.1. *Given two disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, an amalgamation, G, of G_1 and G_2, will be constructed by identifying $x \in V_1$ and $y \in V_2$ such that if $G = (V, E)$,*
then $V = (V_1 - x) \cup (V_2 - y) \cup \{z\}$
and $E = \{ab : a, b \neq x, y \text{ and } ab \in E_1 \cup E_2\} \cup \{az : a \neq x, y \text{ and } ax \in E_1 \text{ or } ay \in E_2\}$.
This will be called the ***the amalgamation of x and y****, and z will be called* ***the (x, y)-amalgamated vertex****.*

Definition 2.2.2. *A* ***diamond graph*** *is a K_4 with one edge removed. The 2-edge of the diamond will be called the* ***central edge****.*

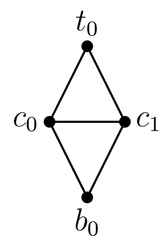


Figure 2.7: D_0, a diamond graph

Definition 2.2.3. *Let $D_0 = D_0(\emptyset)$, the 0-split diamond, be a diamond graph with the vertices on the central edge labelled c_0 and c_1, and the other two vertices with labels t_0 and b_0.*
*Given an $s \geq 1$ and $R \subseteq \{1, 2, ..., s\}$, such that $|R| = r$, the **s-split diamond** with respect to R is denoted $D_s(R)$ and is formed from $D_{s-1}(R-s)$ by adding c_{s+1} adjacent to*

(a) $t_{s-r}c_s$ and adding b_r adjacent to c_sc_{s+1} if $s \in R$, and

(b) b_rc_s and adding t_{s-r} adjacent to c_sc_{s+1} if $s \notin R$

*The vertices $\{c_0, c_1, ..., c_{s+1}\}$ will be called **central vertices**, c_0 and c_{s+1} will be called **exterior central vertices**, and the path the central vertices form will be called the **central path of the s-split diamond**. The vertices $\{t_0, t_1, ..., t_{s-r}\}$ will be called **top vertices** and $\{b_0, b_1, ..., b_r\}$ will be called **bottom vertices**.*

Remark 2.2.4. *We could create isomorphic graphs using different sets for R. For example, if $R' = \{s + 1 - i : i \in R\}$. Then $D_s(R)$ is isomorphic to $D_s(R')$.*

Remark 2.2.5. *The diamond graph is a 1-tough 2-tree, and since $D_s(R)$ and is formed from $D_{s-1}(R - s)$ by adding two simplicial vertices to $D_{s-1}(R - s)$, such that there are no t-edges for $t \geq 3$, then $D_s(R)$ is a 1-tough 2-tree, $\forall s, R$.*

We can see an example of this recursion as follows. Consider the 4-split diamond, $D_4(\{1, 3, 4\})$ below.

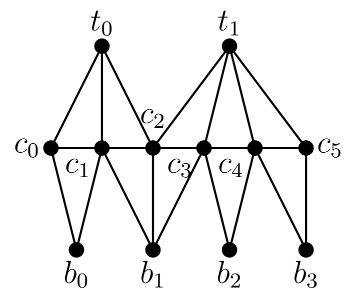


Figure 2.8: Example of an s-split diamond : $D_4(\{1, 3, 4\})$

From $D_4(\{1, 3, 4\})$ we can create two different 5-split diamonds, $D_5(\{1, 3, 4\})$, and $D_5(\{1, 3, 4, 5\})$, pictured below. In either case, we are adding two simplicial vertices to create an additional diamond which shares an edge with the 4-split diamond, and whose central edge extends the central path of the 4-split diamond.

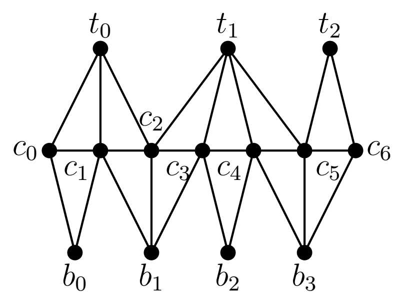


Figure 2.9: Example of an s-split diamond : $D_5(\{1, 3, 4\})$ constructed from $D_4(\{1, 3, 4\})$

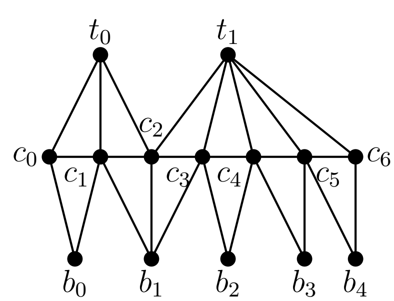


Figure 2.10: Example of an s-split diamond : $D_5(\{1, 3, 4, 5\})$ constructed from $D_4(\{1, 3, 4\})$

Definition 2.2.6. *Let $D^1_{s_1}(R_1), ...D^m_{s_m}(R_m)$ be disjoint $s_1, ..., s_m$-split diamonds respectively with $|R_i| = r_i$. Denote the central vertices of D^i, $\{c^i_0, c^i_1, .., c^i_{s_i+1}\}$, the top vertices of D^i, $\{t^i_0, t^i_1, .., t^i_{s_i-r_i}\}$, and the bottom vertices of D^i, $\{b^i_0, b^i_1, .., b^i_{r_i}\}$. Then an **ℓ-string of diamonds**, for $\ell = s_1 + s_2 + ... + s_m + m$ will be formed as follows:*

1. *Amalgamate $c^i_{s_i+1}$ with c^{i+1}_0, to form z_i and call z_i* ***the $(D^i, D^{i+1})-$amalgamated vertex**..*
2. *Then add exactly one of the following:*
   (a) *A path between $b^i_{r_i}$ and b^{i+1}_0 such that each vertex of the path is also adjacent to z_i, or*
   (b) *A path between $t^i_{s_i-r_i}$ and t^{i+1}_0 such that each vertex of the path is also adjacent to z_i*

An l-string of diamonds will be denoted

$$D^1_{s_1}; (x_1, \ell_1); D^2_{s_2}; (x_2, \ell_2); ....; D^{m-1}_{s_{m-1}}; (x_{m-1}, \ell_{m-1}); D^m_{s_m}$$

where $x_i = t$ if there is a path between $t^i_{s_i-r_i}$ and t^{i+1}_0, $x_i = b$ if there is a path between $b^i_{r_i}$ and b^{i+1}_0, and ℓ_i is the length of that path.
*The path formed from the central paths of the $s_1, ..., s_m$-split diamonds and the amalgamated vertices, $(c^1_0, ..., c^1_{s_1}, z_1, ..., z_{m-1}, c^m_1, ..., c^m_{s_m+1})$ will be called the **central path of the ℓ-string of diamonds**.*

Remark 2.2.7. *The paths in (2a) and (2b) above are added so that an ℓ-string of diamonds is a 2-tree.*

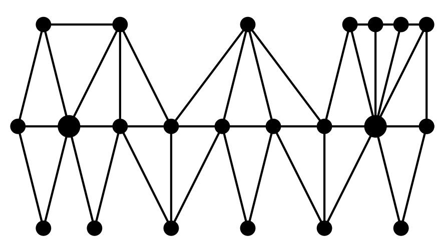


Figure 2.11: Example of an 8-string of diamonds, $D_0; (t, 1); D_5(\{1, 3, 4\}); (t, 3); D_0$, with amalgamated vertices shown as larger vertices

In this dissertation, we will be introducing families of forbidden induced sub-2-trees, with and without fixed endpoints, such that if G is a 2-tree, which contains an induced sub-2-tree in the family, then G will not have a Hamiltonian path. Note, however, that these will be families of forbidden induced sub-2-trees, as was the case in our example of a 1-path-tough graph which does not contain a Hamiltonian path (See Figure 1.3). In that example, our base graph was P_n^2, where we could create an infinite family of such graphs just by increasing n. Similarly in our lists, we will be able create infinite families of forbidden sub-2-trees, by increasing the number of vertices in a graph and the distance between an endpoint of a Hamiltonian path and a forbidden substructure. So, in order to create a primitive list of forbidden sub-2-trees, for which a 2-tree not having a Hamiltonian path must contain, then we will perform the following graph amalgamation.

Definition 2.2.8. *Suppose G is a 2-tree with $ab \in E(G)$. Let H be a 2-path with simplicial vertices x, y, such that x is adjacent to uv. Amalgamate G and $H - x$ by performing a vertex amalgamation of a and u and then b and v as in Definition 2.2.1. This process will be called an* ***amalgamation of a y-2-path with ab****.*

If, for a graph G, we have amalgamated a y-2-path with $ab \in E(G)$, it will be represented with a single curve between y and a and y and b, where $G[\{a, b, y\}]$ is some 2-path. An example of an amalgamation of a y-2-path with $t_0^2 c_1^2$ in $D_0; (t, 1); D_0$ is below.

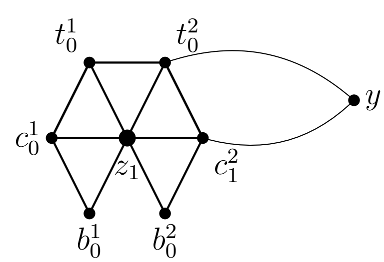


Figure 2.12: An amalgamation of a y–2-path with $t_0^2 c_1^2$ in $D_0; (t, 1); D_0$

Then Figures 2.13 and 2.14 below are both included in Figure 2.12. Also, if y is an endpoint fixed for a Hamiltonian path, then neither graphs in Figures 2.13 and 2.14 are induced subgraphs of one another.

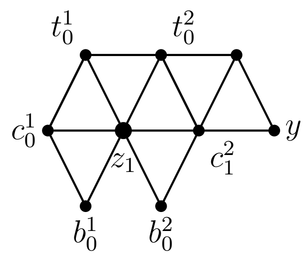


Figure 2.13: Specific example of a graph represented by Figure 2.12

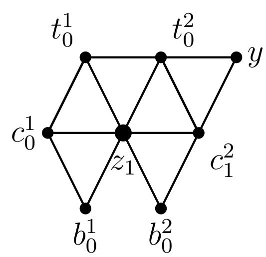


Figure 2.14: Specific example of a graph represented by Figure 2.12

2.3 A New Toughness Definition

We define a new toughness property which will be helpful in describing when there will not be a Hamiltonian path between two vertices. This definition will also relate to the ℓ-strings of diamonds defined in the previous section.

Definition 2.3.1. *A* ***tough path from v_1 to v_n*** *is a path $P = (v_1, v_2, ..., v_n)$ such that for all $i, j \in 1, ..., n$, with $i < j$ and $S_{v_i,v_j} = \{v_i, v_{i+1}, ...v_{j-1}, v_j\}$, $|S_{v_i,v_j}| = c(G - S_{v_i,v_j})$.*

Remark 2.3.2. *If G is a 1-tough graph then S_{v_i,v_j} is a tough set.*

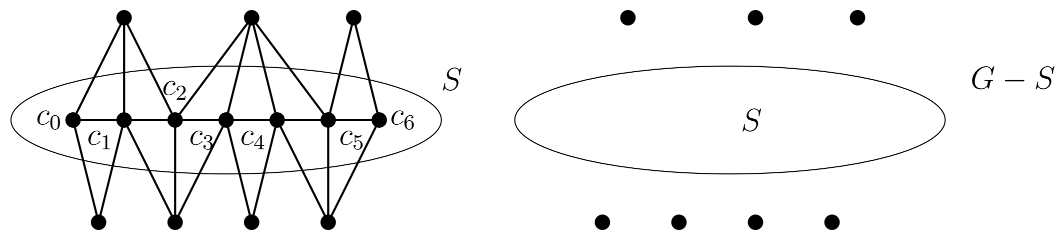


Figure 2.15: Example of a tough path: $(c_0, c_1, c_2, c_3, c_4, c_5, c_6)$

The following Lemma and proof are similar to a Lemma and proof for a toughness inequality in [7].

Lemma 2.3.3. *If v is a simplicial vertex in H and $G = H - v$, then $c(H - S) \geq c(G - S)$.*

Proof. If $c(H-S) < c(G-S)$, then v is adjacent to at least two components of $G-S$. But since v is simplicial, then $N(v)$ is a clique, and hence all neighbors not in S lie in the same component, a contradiction. □

Corollary 2.3.4. *Let H be a k-tree and $S \subset V(H)$ such that $c(H - S) = t$. If H is an induced sub-k-tree of a k-tree G, then $c(G - S) \geq t$.*

Proof. From Lemma 2.1.9, an induced sub-k-tree is an SEO-induced subgraph, so H can be formed by iteratively removing simplicial vertices. □

Lemma 2.3.5. *Let G be a 1-tough k-tree and H be an induced sub-k-tree of G. If P is a tough path in H, then P is a tough path in G.*

Proof. Since $P = (v_1, v_2, ..., v_n)$ is a tough path, then for all $i, j \in 1, ..., n$, with $i < j$, $|S_{v_i,v_j}| = c(H - S_{v_i,v_j})$. From Corollary 2.3.4, $c(G - S_{v_i,v_j}) \geq |S_{v_i,v_j}|$. But if $c(G - S_{v_i,v_j}) > |S_{v_i,v_j}|$, then G is not 1-tough, so we must have $|S_{v_i,v_j}| = c(G - S_{v_i,v_j})$, and hence P is a tough path in G. □

Lemma 2.3.6. *Let G be a 1-tough 2-tree. If there exists a tough set, U, such that $x_1, x_2 \in U$, then G does not have an (x_1, x_2)-Hamiltonian path.*

Proof. Since U is a tough set, then $|U| = c(G-U)$. Then $c(G-U) - |U - \{x_1, x_2\}| = c(G - U) - (|U| - 2) = c(G - U) - |U| + 2 = 2$. Hence, from Lemma 1.2.16, $PP(G; \{x_1, x_2\}) \geq 2$, and there cannot be a Hamiltonian path between x_1 and x_2. □

Lemma 2.3.7. *The central path of an s-split diamond is a tough path.*

Proof. We will proceed by induction on s. If $s = 0$, then we have the diamond graph D_0 with central path, (c_0, c_1), and $c(D_0 - c_0) = c(D_0 - c_1) = 1$. Furthermore, since t_0 and b_0 are both simplicial vertices and adjacent to c_0c_1, then $c(D_0 - \{c_0, c_1\} = 2$, and so the central path is a tough path. Suppose that the central path of an $(s-1)$-split diamond is a tough path. Now, consider $D_s(R)$. If $s \in R$ then b_r is simplicial, and $D_s(R) - \{b_r, c_{s+1}\} = D_{s-1}(R - s)$ is an $s-1$-split diamond, and hence $(c_0, c_1, ..., c_s)$ is a tough path in $D_{s-1}(R-s)$. From Lemma 2.3.5, $(c_0, c_1, ..., c_s)$ is also a tough path in $D_s(R)$. Furthermore, removing c_{s+1} from $D_s(R) - \{c_i, c_{i+1}, .., c_s\}$ for $0 \leq i \leq s$ will increase the number of components of the graph by one, as b_r and t_{s-r} will be in the same component of $D_s(R) - \{c_i, c_{i+1}, .., c_s\}$, as they are both adjacent to c_{s+1}, but different components of $D_s(R) - \{c_i, c_{i+1}, .., c_s, c_{s+1}\}$, since b_r is only adjacent c_sc_{s+1}. Hence the central path of $D_s(R)$ is a tough path. The proof is similar if $s \notin R$. □

Lemma 2.3.8. *The central path of an ℓ-string of diamonds is a tough path.*

Proof. We will proceed by induction on the number of amalgamated vertices, j. If $j = 0$, then the ℓ-string of diamonds is an $(l+1)$-split diamond, and by Lemma 2.3.7, the claim is true. Now suppose that when there are $j-1$ amalgamated vertices that the central path of an ℓ-string of diamonds is a tough path. Now, consider an ℓ-string of diamonds, $G = D^1_{s_1}; (x_1, \ell_1); D^2_{s_2}; (x_2, \ell_2); ....; (x_{j-1}, \ell_{j-1}); D^j_{s_j}; (x_j, \ell_j); D^{j+1}_{s_{j+1}}$. By the induction hypothesis, the central path, $P = (c^1_0, ..., c^j_{s_j+1})$, of $D^1_{s_1}; (x_1, \ell_1); D^2_{s_2}; (x_2, \ell_2); ....; (x_{j-1}, \ell_{j-1}); D^j_{s_j}$ is a tough path in $D^1_{s_1}; (x_1, \ell_1); D^2_{s_2}; (x_2, \ell_2); ....; (x_{j-1}, \ell_{j-1}); D^j_{s_j}$. Also, by Lemma 2.3.7, the central path, $P' = (c^{j+1}_0, ..., c^{j+1}_{s_j+1})$, of $D^{j+1}_{s_{j+1}}$, is a tough path in $D^{j+1}_{s_{j+1}}$. From Lemma 2.3.5, $(c^1_0, ..., c^j_{s_j+1} = z_j)$ and $(z_j = c^{j+1}_0, ..., c^{j+1}_{s_j+1})$ are also tough paths in G. Let $S_{w,c^j_{s_j}}$ be any consecutive subset of vertices from the tough path, P which ends with $c^j_{s_j}$. In $G - S_{w,c^j_{s_j}}$, $D^{j+1}_{s_{j+1}}$ is in one component, with some additional vertices. Hence, the combined path, $(P, P' - z_j)$, will be a tough path in G. □

Lemma 2.3.9. *Let $P = (v_1, v_2, ..., v_{n-1}, v_n)$ be a tough path in a 1-tough 2-tree, G. If v_{i-1} is adjacent to v_{i+1} in G, for some $2 \leq i \leq n-1$, then replacing (v_{i-1}, v_i, v_{i+1}) with (v_{i-1}, v_{i+1}) forms a tough path P'.*

Proof. Let $C_{v_{i+1}}$ be the component of $G - \{v_{i-1}, v_i\}$ which contains v_{i+1}. Since v_{i-1} is adjacent to v_{i+1}, then $v_{i-1} \in N(v_iv_{i+1})$. If $v_{i-1}v_{i+1}$ is a 1-edge, then v_{i-1} is an isolated vertex in $G[C_{v_{i+1}} \cup v_{i-1}v_i] - v_{i+1}$, and then $c(G - \{v_{i-1}, v_i, v_{i+1}\}) = 2$, contradicting that P is a tough path. So, $v_{i-1}v_{i+1}$ must be a 2-edge, and $c(G - \{v_{i-1}, v_{i+1}\}) = 2$. Furthermore, since $c(G - S_{v_{i+1},v_j}) = |S_{v_{i+1},v_j}|$ and $c(G - S_{v_k,v_{i-1}}) = |S_{v_k,v_{i-1}}|$, then $c(G - (S_{v_k,v_j} - v_i)) = |S_{v_k,v_j} - v_i|$. □

Definition 2.3.10. *A* ***short tough path*** *is a tough path $P = (v_1, v_2, ..., v_{n-1}, v_n)$, in a 2-tree, G, for which $v_{i-1}v_{i+1} \notin E(G)$ for any $2 \leq i \leq n-1$.*

Lemma 2.3.11. *If P is a short tough path in a 1-tough 2-tree, G, then P is the central path of an induced ℓ-string of diamonds in G.*

Proof. We will proceed by induction on the length, $\mathscr{L}$, of the tough path. If $\mathscr{L}$=1, then $P = (v_1, v_2)$, and since G is a 1-tough 2-tree, then v_1v_2 is a 2-edge in G. Hence, $G[N[v_1, v_2]]$ is a diamond graph, and a 1-string of diamonds, with central path (v_1, v_2). Suppose that the claim is true for tough paths of length $\mathscr{L} - 1$. Now, suppose G' is a 1-tough 2-tree with short tough path $P' = (v_1, v_2, ..., v_{\mathscr{L}}, v_{\mathscr{L}+1})$ of length $\mathscr{L}$. $P'' = (v_1, v_2, ..., v_{\mathscr{L}})$ is a short tough path of length of $\mathscr{L} - 1$, and by the induction hypothesis, P'' is the central path of an induced ℓ-string of diamonds in G'. Let $H = D^1_{s_1}; (x_1, \ell_1); D^2_{s_2}; (x_2, \ell_2); ....; D^{m-1}_{s_{m-1}}; (x_{m-1}, \ell_{m-1}); D^m_{s_m}$ be the induced ℓ-string of diamonds, and t', b' be the top and bottom vertices, respectively, adjacent to $v_{\mathscr{L}}$ in $D^m_{s_m}$. Since H is an induced sub-2-tree, then H is an SEO-induced subgraph of G', and hence there is a labelling of the vertices of G' such that G' can be constructed from H as in Definition 1.2.4 by iteratively adding vertices $\{y_1, y_2, ..., y_k\}$ in order of the labelling. Since P' is a short tough path, then $v_{\mathscr{L}-1}v_{\mathscr{L}+1} \notin E(G')$ and so $v_{\mathscr{L}+1} \in \{y_1, y_2, ..., y_k\}$. If $v_{\mathscr{L}+1} = y_i$ such that $i < j$ for any y_j a neighbor of $v_{\mathscr{L}}$, then $v_{\mathscr{L}+1}t' \in E(G')$ or $v_{\mathscr{L}+1}b' \in E(G')$. Furthermore, $v_{\mathscr{L}}, v_{\mathscr{L}+1}$ must be a 2-edge in G' and so there is another vertex, $x' \in \{y_1, y_2, ..., y_k\}$ adjacent to $v_{\mathscr{L}}, v_{\mathscr{L}+1}$, and this forms an $l + 1$-string of diamonds. Now suppose $v_{\mathscr{L}+1} = y_i$ such that $i > j$ for at least one y_j a neighbor of $v_{\mathscr{L}}$. Let $\{y_{j_1}, y_{j_2}, ..., y_{j_{k'}}\}$ be the vertices that are adjacent to $v_{\mathscr{L}}$ where $j_i < i$. Adding the vertices in $\{y_{j_1}, y_{j_2}, ..., y_{j_{k'}}\}$ followed by $v_{\mathscr{L}+1}$ as in Definition 1.2.4 forms a path $(t', y_{j_1}, y_{j_2}, ..., y_{j_{k'}}, v_{\mathscr{L}+1})$ or $(b', y_{j_1}, y_{j_2}, ..., y_{j_{k'}}, v_{\mathscr{L}+1})$ where all vertices on the path are adjacent to $v_{\mathscr{L}}$. Furthermore, $v_{\mathscr{L}}, v_{\mathscr{L}+1}$ must be a 2-edge in G' and so there is another vertex, $z' \in \{y_1, y_2, ..., y_k\}$ adjacent to $v_{\mathscr{L}}, v_{\mathscr{L}+1}$, and this forms an $l + 1$-string of diamonds which is an induced subgraph of G'. □

Lemma 2.3.12. *Let*

$G = D^1_0; (t, \ell_1); D^2_{s_2}(R_2); (x_2, \ell_2); ....; (x_{m-2}, \ell_{m-2}); D^{m-1}_{s_{m-1}}(R_{m-1}); (t, \ell_{m-1}); D^m_0.$

Then there is a tough path from t_0^1 to t_0^m, a tough path from c_0^1 to t_0^m, and a tough path from t_0^1 to c_1^m.

Proof. $G - \{c_0^1, t_0^1, b_0^1\}$ is an $(l-1)$-string of diamonds, and hence, by Lemma 2.3.8, the central path, P, is a (z_1, c_1^m)-tough path. Furthermore, by Lemma 2.3.5, it is a tough path in G. Let $S_{z_1,w}$ be any consecutive subset of vertices from the tough path which begins with z_1, the (D^1, D^2)-amalgamated vertex. Then, c_0^1, t_0^1, b_0^1, and t_0^2 are in the same component of $G - S_{z_1,w}$, but since c_0^1 is only adjacent to t_0^1, b_0^1, and z_1, then removing t_0^1 from $G - S_{z_1,w}$ will add a component. Hence (t_0^1, P) is a (t_0^1, c_1^m)-tough path. Similarly, there is a (c_0^1, t_0^m)-tough path which uses the central path from c_0^1 to z_{m-1}, the (D^{m-1}, D^m)-amalgamated vertex. Hence, there is a (z_1, t_0^m)-tough path, P' which uses the central path from z_1 to z_{m-1}. Let $S'_{z_1,w}$ be any consecutive subset of vertices from the tough path P' which begins with z_1. Then, c_0^1, t_0^1, b_0^1, and t_0^2 are in the same component of $G - S'_{z_1,w}$, but since c_0^1 is only adjacent to t_0^1, b_0^1, and z_1, then removing t_0^1 from $G - S'_{z_1,w}$ will add a component. Hence (t_0^1, P') is a (t_0^1, t_0^m)-tough path. □

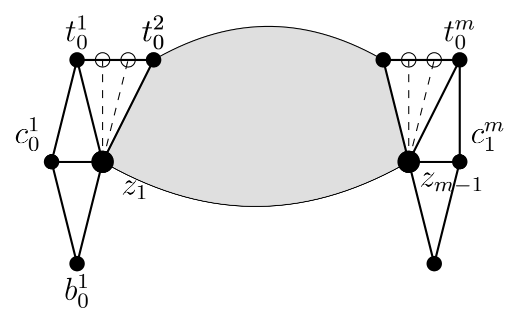


Figure 2.16: A general example of
$G = D_0^1; (t, \ell_1); D_{s_2}^2(R_2); (x_2, \ell_2); ....; D_{s_{m-1}}^{m-1}(R_{m-1}); (b, \ell_{m-1}); D_0^m$ in Lemma 2.3.12
where $D_{s_2}^2(R_2); (x_2, \ell_2); ....; (x_{m-2}, \ell_{m-2}); D_{s_{m-1}}^{m-1}(R_{m-1})$ with $z_1 = c_0^2$ and $z_{m-1} = c_{s_{m-1}+1}^{m-1}$, is shown in gray to preserve generality

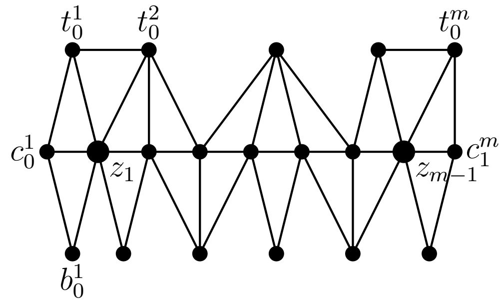


Figure 2.17: Specific example of G in Lemma 2.3.12: $D_0; (t, 1); D_5(\{1, 3, 4\}); (t, 1); D_0$

Corollary 2.3.13. *Let* $G = D_0^1; (t, \ell_1); D_{s_2}^2(R_2); (x_2, \ell_2); ....; (x_{m-2}, \ell_{m-2}); D_{s_{m-1}}^{m-1}(R_{m-1}); (b, \ell_{m-1}); D_0^m$. *Then there is a tough path from t_0^1 to b_0^m, a tough path from c_0^1 to b_0^m, and a tough path from t_0^1 to c_1^m.*

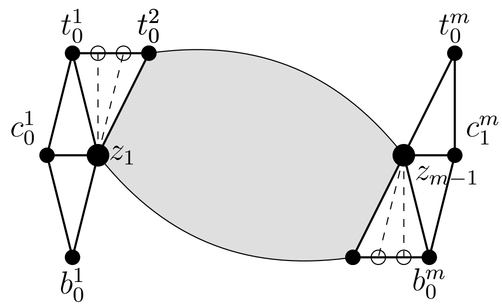


Figure 2.18: A general example of $G = D_0^1; (t, \ell_1); D_{s_2}^2(R_2); (x_2, \ell_2); ....; D_{s_{m-1}}^{m-1}(R_{m-1}); (b, \ell_{m-1}); D_0^m$ in Corollary 2.3.13 where $D_{s_2}^2(R_2); (x_2, \ell_2); ....; (x_{m-2}, \ell_{m-2}); D_{s_{m-1}}^{m-1}(R_{m-1})$ with $z_1 = c_0^2$ and $z_{m-1} = c_{s_{m-1}+1}^{m-1}$, is shown in gray to preserve generality

Corollary 2.3.14. *Let* $G = D_0^1; (t, \ell_1); D_{s_2}^2(R_2); (x_2, \ell_2); ....; (x_{m-2}, \ell_{m-2}); D_{s_{m-1}}^{m-1}(R_{m-1}); (t, \ell_{m-1}); D_0^m$. *Then G does not have a (t_0^1, t_0^m), (c_0^1, t_0^m), or (t_0^1, c_1^m)-Hamiltonian path.*

Corollary 2.3.15. *Let* $G = D_0^1; (t, \ell_1); D_{s_2}^2(R_2); (x_2, \ell_2); ....; (x_{m-2}, \ell_{m-2}); D_{s_{m-1}}^{m-1}(R_{m-1}); (b, \ell_{m-1}); D_0^m$. *Then G does not have a (t_0^1, b_0^m), (c_0^1, b_0^m), or (t_0^1, c_1^m)-Hamiltonian path.*

In addition to tough paths, some 2-trees will not have a Hamiltonian path because there exists a t-edge, ab, $t \geq 2$, and a component C of $G - \{a, b\}$, such that $G[C \cup \{a, b\}]$ does not have a Hamiltonian path. The following lemmas describe these cases.

Lemma 2.3.16. *Let G be a 2-tree, and $x_1, x_2 \in V(G)$. If there exists $ab \in E(G)$ such that:*

1. *x_1 and x_2 lie in different components, C_{x_1}, C_{x_2}, respectively, of $G - \{a, b\}$ and*
2. *In $G[V(C_{x_1}) \cup \{a, b\}]$ there is no (x_1, a)-Hamiltonian path and no (x_1, b)-Hamiltonian path,*

then G does not have an (x_1, x_2)-Hamiltonian path.

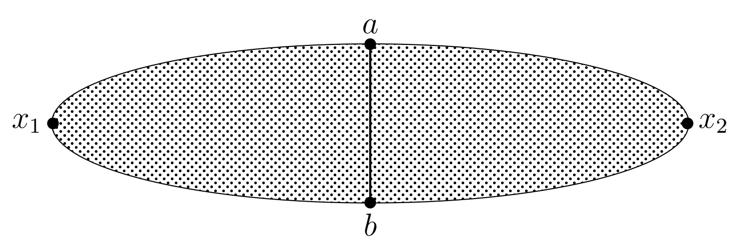


Figure 2.19: Graph G corresponding to Lemma 2.3.16 where the dotted section of the graph represents any 2-tree to preserve generality

Proof. Suppose that G has an (x_1, x_2)-Hamiltonian path, P, but in $G[V(C_{x_1}) \cup \{a, b\}]$ there is no (x_1, a)-Hamiltonian path and no (x_1, b)-Hamiltonian path. Then, P must alternate between vertices from $V(C_{x_1})$ and $V(C_{x_2})$ using $\{a, b\}$, beginning with x_1 and ending with x_2. However, P cannot switch from vertices in $V(C_{x_1})$ to $V(C_{x_2})$ and then back to $V(C_{x_1})$, as then a and b would be used in the path already, and there would be no path back to C_{x_2}. So either:

(a) There is an x_1-Hamiltonian path, P_1, in C_{x_1} and an x_2-Hamiltonian path, P_2, in C_{x_2}, and $P = (P_1, a, b, P_2)$ or $P = (P_1, b, a, P_2)$, or

(b) There is an x_1-Hamiltonian path, P_1, in C_{x_1} and two paths, P_{21} and P_{22} in C_{x_2}, and $P = (P_1, a, P_{21}, b, P_{22})$ or $P = (P_1, b, P_{21}, a, P_{22})$, or

(c) There are two paths, P_{11} and P_{12} in C_{x_1}, and an x_2- Hamiltonian path, P_2, in C_{x_2}, and $P = (P_{11}, a, P_{12}, b, P_2)$ or $P = (P_{11}, b, P_{12}, a, P_2)$.

In (a) and (c), P begins with an (x_1, a) or (x_1, b)-Hamiltonian path in $G[V(C_{x_1}) \cup \{a, b\}]$, a contradiction. In (b), there is an x_1-Hamiltonian path, P_1 in C_{x_1} which connects to either a or b. Since a is adjacent to b, then (P_1, a, b) and (P_1, b, a) are (x_1, b) and (x_1, a)-Hamiltonian paths, respectively, in $G[V(C_{x_1}) \cup \{a, b\}]$, a contradiction. □

Corollary 2.3.17. *Let G be a 2-tree, and $x_1, x_2 \in V(G)$. If there exists $ab \in E(G)$ such that:*

1. *x_1 and x_2 lie in different components, C_{x_1}, C_{x_2}, respectively, of $G - \{a, b\}$, and*
2. *In $G[V(C_{x_1}) \cup \{a, b\}]$ there is a tough path from x_1 to a and a tough path from x_1 to b,*

then G does not have an (x_1, x_2)-Hamiltonian path.

Lemma 2.3.18. *Let G be a 2-tree, and $x_1, x_2 \in V(G)$. If there exist $ab, cd \in E(G)$ such that:*

1. *x_1 and x_2 lie in different components of $G - \{a, b\}$ and $G - \{c, d\}$, and*
2. *In $G - \{a, b, c, d\}$, x_1, x_2 lie in C_{x_1}, C_{x_2}, respectively, such that in $G[V(G) - V(C_{x_1}) - V(C_{x_2})]$ there are no (a, c), (a, d), (b, c), or (b, d)-Hamiltonian paths, then G does not have an (x_1, x_2)-Hamiltonian path.*

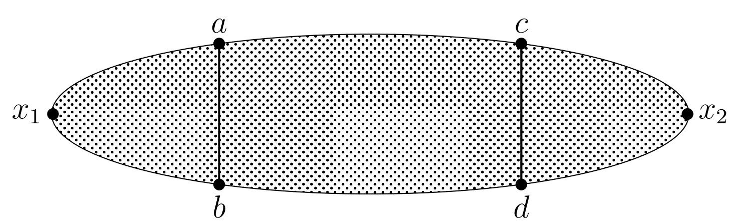


Figure 2.20: Graph G corresponding to Lemma 2.3.18, where the dotted section of the graph represents any 2-tree to preserve generality

Proof. From [24], $G[V(G) - V(C_{x_1})]$ is a 2-tree. By Lemma 2.3.16, there is no Hamiltonian path between a and x_2 in $G[V(G) - V(C_{x_1})]$ and there is no Hamiltonian path between b and x_2 in $G[V(G) - V(C_{x_1})]$. Hence, Lemma 2.3.16, there is no Hamiltonian path between x_1 and x_2. □

Corollary 2.3.19. *Let G be a 2-tree, and $x_1, x_2 \in V(G)$. If there exist $ab, cd \in E(G)$ such that:*

1. *x_1 and x_2 lie in different components of $G - \{a, b\}$ and $G - \{c, d\}$, and*
2. *In $G - \{a, b, c, d\}$, x_1, x_2 lie in C_{x_1}, C_{x_2}, respectively, such that in $G[V(G) - V(C_{x_1}) - V(C_{x_2})]$ there are tough paths from a to c, a to d, b to c, and b to d, then G does not have an (x_1, x_2)-Hamiltonian path.*

Chapter 3

2HP

Since the Hamiltonian path problem on 2-trees can be reduced to the 2HP problem on 2-trees, we will first prove results for 2HP on 2-trees to later extend to the Hamiltonian path problem on 2-trees. Since we know that 1-tough 2-trees contain a Hamiltonian path, we begin our investigation of 2HP with 1-tough 2-trees in Section 3.1. We will begin this section by defining a family, $\mathscr{F}^1$, of 1-tough 2-trees, with specified vertices, x_1 and x_2, which we will later prove contain no (x_1, x_2)-Hamiltonian path. In Theorem 3.1.24, we will also prove that a 1-tough 2-tree which does not contain, as an induced sub-2-tree, one of the graphs, with specified vertices, x_1 and x_2, in $\mathscr{F}^1$, will have an (x_1, x_2)-Hamiltonian path.

We will then extend the results from Section 3.1 to the 2HP problem on 2-trees with scattering number at most one, in Section 3.2. We will begin this section by defining a family, $\mathscr{F}^2$, of graphs, with specified vertices, x_1 and x_2, which contains $\mathscr{F}^1$, and for which we will later prove contain no (x_1, x_2)-Hamiltonian path. In Theorem 3.2.10, we will also prove that a 2-tree with scattering number at most one, which does not contain, as an induced sub-2-tree, one of the graphs, with specified vertices, x_1 and x_2, in $\mathscr{F}^2$, will have an (x_1, x_2)-Hamiltonian path.

3.1 2HP on 1-tough 2-trees

Definition 3.1.1. *Define* $\mathscr{F}^1 = \{F_a^1, F_b^1, F_c^1, F_d^1, F_e^1, F_f^1\}$ *where:*

(a) F_a^1 *is constructed from* D_0 *by:*

(i) Adding a simplicial vertex adjacent to c_0t_0*, and*

(ii) Amalgamating an x_2-2-path with t_0c_1.

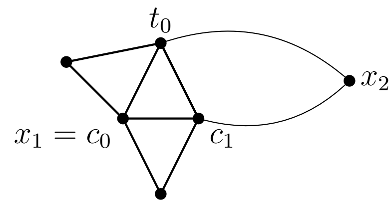


Figure 3.1: An example of F_a^1

(b) F_b^1 is an ℓ-string of diamonds,
$D_{s_1}^1; (x_1, \ell_1); D_{s_2}^2; (x_2, \ell_2); ....; D_{s_{m-1}}^{m-1}; (x_{m-1}, \ell_{m-1}); D_{s_m}^m$, *with* $x_1 = c_0^1$ *and* $x_2 = c_{s_m+1}^m$.

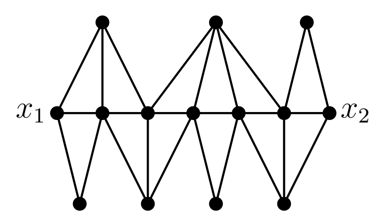


Figure 3.2: An example of F_b^1: $D_5(\{1, 3, 4, 5\})$ with $x_1 = c_0^1$ and $x_2 = c_6^1$

(c) F_c^1 is constructed from
$D_{s_1}^1(R_1); (x_1, \ell_1); ....; (x_{m-2}, \ell_{m-2}); D_{s_{m-1}}^{m-1}(R_{m-1}); (t, \ell_{m-1}); D_0^m$, $m \geq 2$, *where* $x_1 = c_0^1$, *by amalgamating an x_2-2-path with $t_0^m c_1^m$.*

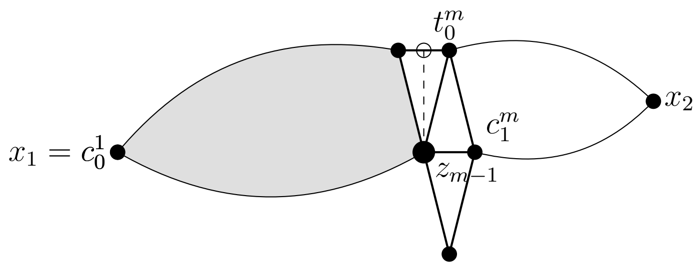


Figure 3.3: A general example of F_c^1:
$D_{s_1}^1(R_1); (x_1, \ell_1); ....; (x_{m-2}, \ell_{m-2}); D_{s_{m-1}}^{m-1}(R_{m-1}); (t, \ell_{m-1}); D_0^m$
$m \geq 2$, with an amalgamated x_2-2-path and such that $x_1 = c_0^1$
where $D_{s_1}^1(R_1); (x_1, \ell_1); ....; (x_{m-2}, \ell_{m-2}); D_{s_{m-1}}^{m-1}(R_{m-1})$, with $x_1 = c_0^1$
and $z_{m-1} = c_{s_{m-1}+1}^{m-1}$, is shown in gray to preserve generality

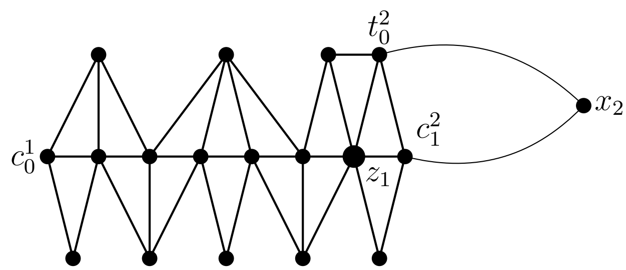


Figure 3.4: An example of F_c^1: $D_5(\{1, 3, 4, 5\}); (t, 1); D_0$ with an amalgamated x_2-2-path and such that $x_1 = c_0^1$

(d) F_d^1 *is constructed from*
$D_0^1; (t, \ell_1); D_{s_2}^2(R_2); (x_2, \ell_2); ....; (x_{m-2}, \ell_{m-2}); D_{s_{m-1}}^{m-1}(R_{m-1}); (t, \ell_{m-1}); D_0^m$, $m \geq 3$, *by:*

(i) Amalgamating an x_1*-2-path with* $t_0^1 c_0^1$*, and*

(ii) Amalgamating an x_2*-2-path with* $t_0^m c_1^m$*.*

OR

F_d^1 *is constructed from*
$D_0^1; (t, \ell_1); D_{s_2}^2(R_2); (x_2, \ell_2); ....; (x_{m-2}, \ell_{m-2}); D_{s_{m-1}}^{m-1}(R_{m-1}); (b, \ell_{m-1}); D_0^m$,
$m \geq 3$, *by:*

(i) Amalgamating an x_1-2-path with $t_0^1c_0^1$, and

(ii) Amalgamating an x_2-2-path with $b_0^mc_1^m$.

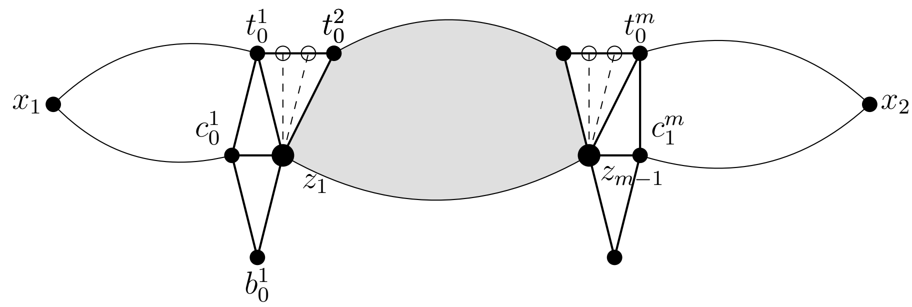


Figure 3.5: A general example of F_d^1:
$D_0^1; (t, \ell_1); D_{s_2}^2(R_2); (x_2, \ell_2); ....; (x_{m-2}, \ell_{m-2}); D_{s_{m-1}}^{m-1}(R_{m-1}); (t, \ell_{m-1}); D_0^m$
$m \geq 3$, with amalgamated x_1 and x_2-2-paths
where $D_{s_2}^2(R_2); (x_2, \ell_2); ....; (x_{m-2}, \ell_{m-2}); D_{s_{m-1}}^{m-1}(R_{m-1})$ with $z_1 = c_0^2$
and $z_{m-1} = c_{s_{m-1}+1}^{m-1}$, is shown in gray to preserve generality

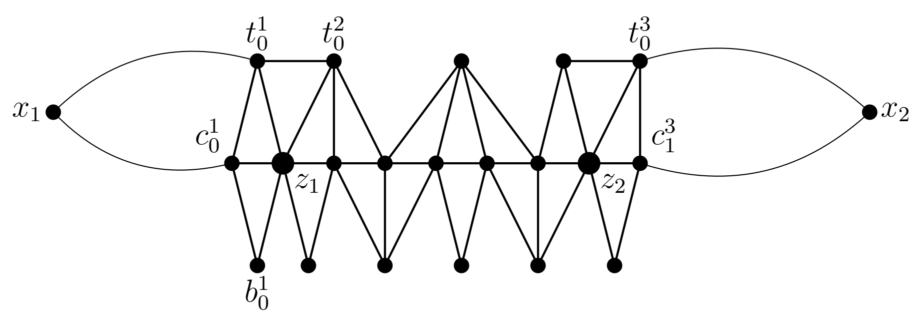


Figure 3.6: An example of F_d^1: $D_0; (t, 1); D_5(\{1, 3, 4, 5\}); (t, 1); D_0$ with amalgamated x_1 and x_2-2-paths

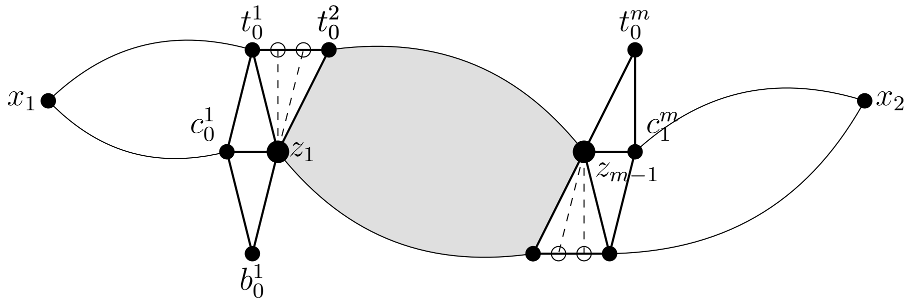


Figure 3.7: A general example of F_d^1:
$D_0^1; (t, \ell_1); D_{s_2}^2(R_2); (x_2, \ell_2); ....; (x_{m-2}, \ell_{m-2}); D_{s_{m-1}}^{m-1}(R_{m-1}); (b, \ell_{m-1}); D_0^m$
$m \geq 3$, with amalgamated x_1 and x_2-2-paths
where $D_{s_2}^2(R_2); (x_2, \ell_2); ....; (x_{m-2}, \ell_{m-2}); D_{s_{m-1}}^{m-1}(R_{m-1})$ with $z_1 = c_0^2$
and $z_{m-1} = c_{s_{m-1}+1}^{m-1}$, is shown in gray to preserve generality

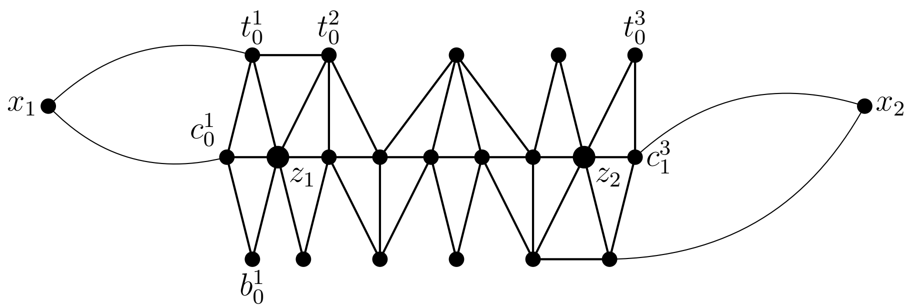


Figure 3.8: An example of F_d^1: $D_0; (t, 1); D_5(\{1, 3, 4, 5\}); (b, 1); D_0$ with amalgamated x_1 and x_2-2-paths

(e) *F_e^1 is constructed from $G = D_0^1; (t, 1); D_0^2$ by amalgamating an x_1-2-path with $t_0^1 c_0^1$, amalgamating an x_2-2-path with $t_0^2 c_1^2$, and by adding a simplicial vertex adjacent to t_0^1, t_0^2.*

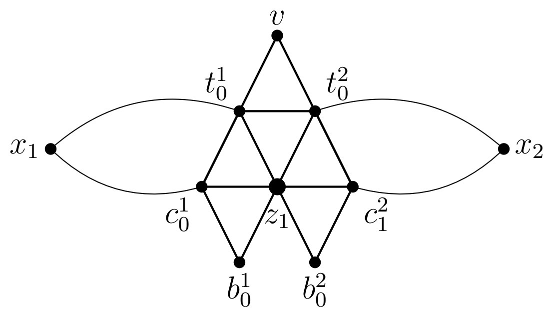


Figure 3.9: Example of F_e^1, $D_0^1; (t, 1); D_0^2$ with amalgamated x_1 and x_2-2-paths and a simplicial vertex adjacent to t_0^1, t_0^2

(f) F_f^1 is constructed from $G = D_0^1; (t, \ell); D_0^2$, for $l \geq 2$ by amalgamating an x_1-2-path with $t_0^1 c_0^1$ and amalgamating an x_2-2-path with $t_0^2 c_1^2$.

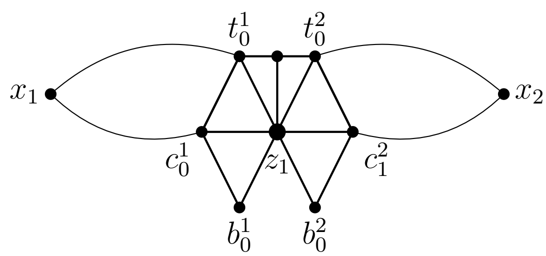


Figure 3.10: An example of F_f^1, $D_0^1; (t, 2); D_0^2$ with amalgamated x_1 and x_2-2-paths

Lemma 3.1.2. *The graph F_a^1 does not have an (x_1, x_2)-Hamiltonian path.*

Proof. The paths $(x_1 = c_0, t_0)$ and $(x_1 = c_0, c_1)$ are tough paths. Furthermore, x_1 and x_2 are in different components of $F_a^1 - \{t_0, c_1\}$, and hence, by Lemma 2.3.16, there is no (x_1, x_2)-Hamiltonian path. □

Lemma 3.1.3. *The graph F_b^1 does not have an (x_1, x_2)-Hamiltonian path.*

Proof. From Lemma 2.3.8, there is a tough path from $x_1 = c_0^1$ to $x_2 = c_{s_m+1}^m$. Hence, from Lemma 2.3.6, there is no (x_1, x_2)-Hamiltonian path. □

Lemma 3.1.4. *The graph F_c^1 does not have an (x_1, x_2)-Hamiltonian path.*

Proof. From the proof of Lemma 2.3.12, there is a tough path from c_0^1 to t_0^m. From Lemma 2.3.8, there is a tough path from c_0^1 to c_1^m. Furthermore, c_0^1 and x_2 are in different components of $F_c^1 - \{t_0^m, c_1^m\}$ and hence, by Lemma 2.3.16, there is no $(x_1 = c_0^1, x_2)$-Hamiltonian path. □

Lemma 3.1.5. *The graph F_d^1 does not have an (x_1, x_2)-Hamiltonian path.*

Proof. If F_d^1 is constructed from
$D_0^1; (t, \ell_1); D_{s_1}^2(R_1); (x_1, \ell_2); ....; (x_{m-1}, \ell_{m-2}); D_{s_{m-2}}^{m-1}(R_{m-2}); (t, \ell_{m-1}); D_0^m$,
then from Lemma 2.3.12, there are tough paths from t_0^1 to t_0^m, from t_0^1 to c_1^m, and from t_0^m to c_0^1. Likewise, if F_d^1 is constructed from
$D_0^1; (t, \ell_1); D_{s_1}^2(R_1); (x_1, \ell_2); ....; (x_{m-1}, \ell_{m-2}); D_{s_{m-2}}^{m-1}(R_{m-2}); (b, \ell_{m-1}); D_0^m$,
then from Corollary 2.3.13 there are tough paths from t_0^1 to b_0^m, from t_0^1 to c_1^m, and from c_0^1 to b_0^m. From Lemma 2.3.8, there is a tough path from c_0^1 to c_1^m. Furthermore, if F_d^1 is constructed from
$D_0^1; (t, \ell_1); D_{s_1}^2(R_1); (x_1, \ell_2); ....; (x_{m-1}, \ell_{m-2}); D_{s_{m-2}}^{m-1}(R_{m-2}); (t, \ell_{m-1}); D_0^m$,
then x_1 and x_2 are in different components of $F_d^1 - \{t_0^1, c_0^1\}$ and $F_d^1 - \{t_0^m, c_1^m\}$, and hence, by Lemma 2.3.18, there is no (x_1, x_2)-Hamiltonian path. Likewise, if F_d^1 is constructed from
$D_0^1; (t, \ell_1); D_{s_1}^2(R_1); (x_1, \ell_2); ....; (x_{m-1}, \ell_{m-2}); D_{s_{m-2}}^{m-1}(R_{m-2}); (b, \ell_{m-1}); D_0^m$,
then x_1 and x_2 are in different components of $F_d^1 - \{t_0^1, c_0^1\}$ and $F_d^1 - \{b_0^m, c_1^m\}$, and hence, by Lemma 2.3.18, there is no (x_1, x_2)-Hamiltonian path. □

Lemma 3.1.6. *The graph F_e^1 does not have an (x_1, x_2)-Hamiltonian path.*

Proof. The paths (t_0^1, z_1, t_0^2), (c_0^1, z_1, c_1^2), (t_0^1, z_1, c_1^2), and (c_0^1, z_1, t_0^2) are tough paths. Furthermore, x_1 and x_2 are in different components of $F_e^1 - \{t_0^1, c_0^1\}$ and $F_e^1 - \{t_0^2, c_1^2\}$, and hence, by Lemma 2.3.18, there is no (x_1, x_2)-Hamiltonian path. □

Lemma 3.1.7. *The graph F_f^1 does not have an (x_1, x_2)-Hamiltonian path.*

Proof. The paths (t_0^1, z_1, t_0^2), (c_0^1, z_1, c_1^2), (t_0^1, z_1, c_1^2), and (c_0^1, z_1, t_0^2) are tough paths. Furthermore, x_1 and x_2 are in different components of $F_f^1 - \{t_0^1, c_0^1\}$ and $F_f^1 - \{t_0^2, c_1^2\}$, and hence, by Lemma 2.3.18, there is no (x_1, x_2)-Hamiltonian path. □

3.1.1 Paths in ℓ-strings of diamonds

Definition 3.1.8. *A **forced edge,** $e = uv$, is an edge that must be used in the Hamiltonian Path (if one exists). Incident forced edges form a **forced path**.*

Since G is a 2-tree, simplicial vertices have degree 2, and hence lie on a forced path if they are not endpoints of the path. To simplify the graphs we are considering, we use a reduction process on our graphs which contracts sections of the graph where there is a forced path. This process is similar to that used when reducing a series-parallel network of resistors. These series-parallel networks are partial 2-trees and this reduction method has been used to find the resistance in the network. It has also been used to find the probability that a communication network will work. In both cases the edges are labelled with weights: resistance, and probabilities, respectively [1].

Notation 3.1.9. *Let (G, u, v) denote a 2-tree, G, with $u, v \in V(G)$, and let $S_1^*(G, u, v)$ denote the set of simplicial vertices in $G - \{u, v\}$.*

Definition 3.1.10. *Given a 1-tough 2-tree, G, such that $G \neq K_3$, then the **reduced graph** of (G, u, v), is formed using the following algorithm:*

1. *Let $w \in S_1^*(G, u, v)$, and x, y the neighbors of w. If $G - w \neq K_2$, remove w and turn the edge xy into a forced edge.*
2. *Repeat (1) for all $w \in S_1^*(G, u, v)$. Define the resulting graph to be G_1^*.*
3. *For $i \geq 2$, let $S_i^* = S_1^*(G_{i-1}^*, u, v)$ where $G_{i-1}^* \neq K_3$ is the graph formed by repeating (1) for $G = G_{i-1}^*$ and for all $w \in S_1^*(G_{i-1}^*, u, v)$.*

Repeat (3) for all $i = 2, 3, ..., j$ for j such that $S_j^* = \emptyset$ or $G_j^* = K_3$. This is the reduced graph of (G, u, v).
For F the set of forced edges, let (H, u, v, F) denote the reduced graph of (G, u, v), for G a 1-tough 2-tree.

Since simplicial vertices in 2-trees are not adjacent [7], when we remove the vertices in each S_i^*, regardless of order, we will end up with the same graph, unless removing all vertices in S_i^* results in K_2. In this case, if we change the order of removal of vertices, we will end up with different, but isomorphic graphs.

Furthermore, the reduction process removes all simplicial vertices other than the two given endpoints, and hence the resulting graph is a 2-path.

In order to describe a Hamiltonian path in a 2-path graph, we will use a specific simpliical elimination ordering to create a labelling for our vertices.

Definition 3.1.11. ***Algorithm for labelling a 2-path:***
Let H be a 2-path with simplicial vertices, $\{u, v\}$, with $|V(H)| = n$.

1. *Label $\{u, v\}$ with 1 and n. Remove vertex labelled 1.*
2. *Label the new simplicial vertex (not the one labelled n), consecutively and remove.*
3. *Repeat (2) until all that remains is a K_3.*
4. *Label the last K_3 by starting with the original 2-path (labels intact) and removing the vertex labelled n. Label the new simplicial vertex (which is not labelled) with $n - 1$. Label the remaining vertex $n - 2$.*

Remark 3.1.12. *Since simplicial vertices in $G - S_1(G)$ are adjacent to vertices in $S_1(G)$ [7], vertices that are consecutively labelled will be adjacent. Hence, following the ordering in the labelling algorithm consecutively will yield a Hamiltonian path.*

Definition 3.1.13. *[22] A **k−caterpillar**, P, is a k-tree in which deletions of all simplicial vertices results in a k-path.*

Definition 3.1.14. *Let (H, x_1, x_2, F) be the reduced graph of (G, x_1, x_2). The **caterpillar representation**, (H', x_1, x_2), of a graph (G, x_1, x_2) is created by adding $|F|$ simplicial vertices to (H, x_1, x_2, F), making each vertex adjacent to exactly one forced edge, and changing all forced edges back to regular edges.*

Remark 3.1.15. *(H', x_1, x_2) could also have been constructed by removing one less simplicial vertex from each of the forced edges in the reduced graph algorithm, though it would be more difficult to define. Furthermore, since H is a 2-tree, then H' is also a 2-tree and since the forced edges were changed back to regular edges, (H', x_1, x_2) is an induced sub-2-tree of (G, x_1, x_2).*

Remark 3.1.16. *Since x_1 and x_2 are simplicial in (H, x_1, x_2, F), then they are incident to at most two forced edges, and hence in (H', x_1, x_2), they have degree at most four.*

Lemma 3.1.17. *Let G be a 1-tough 2-tree with $x_1, x_2 \in V(G)$. Let (H, x_1, x_2, F) be the reduced graph of (G, x_1, x_2) and (H', x_1, x_2) the caterpillar representation of (G, x_1, x_2). Then the following are equivalent:*

1. *G has an (x_1, x_2)-Hamiltonian path,*
2. *(H', x_1, x_2) has an (x_1, x_2)-Hamiltonian path, and*
3. *(H, x_1, x_2, F) has an (x_1, x_2)-Hamiltonian path which uses all of the edges in F.*

Proof. (1)$\Longrightarrow$ (2)
Suppose (H', x_1, x_2) does not have an (x_1, x_2)-Hamiltonian path. Since (H', x_1, x_2) is an induced sub-2-tree of (G, x_1, x_2), then by Corollary 2.1.10, G does not have an (x_1, x_2)-Hamiltonian path.

(2) $\Longrightarrow$ (3)
Suppose (H', x_1, x_2) has an (x_1, x_2)-Hamiltonian path, P. Let $v \neq x_1, x_2$ be a simplicial vertex with neighbors u and w. Then $P = (x_1, ..., u, v, w, ..., x_2)$ or $P = (x_1, ..., w, v, u, ..., x_2)$. Furthermore, because H' is a 2-tree, then $uw \in E(H')$, and from the reduction algorithm $uw \in F$. Replacing (u, v, w) or (w, v, u) by (u, w) in P, then P is a Hamiltonian path using exactly one forced edge. Repeating this process for all $S_1^*(H', x_1, x_2)$, then P will be a Hamiltonian path in (H, x_1, x_2, F).

(3) $\Longrightarrow$ (1) Suppose (H, x_1, x_2, F) has an (x_1, x_2)-Hamiltonian path, P, which uses all of the edges in F. Consider $xy \in F$. In (G, x_1, x_2), xy is incident to at least one vertex, v, which is not in (H, x_1, x_2, F) so that $c(G - \{x, y\}) = 2$. Let C_v be the component of $G - \{x, y\}$ which contains v. From [24], $G[C_v \cup xy]$ is a 2-tree, and from Lemma 1.2.23, it is also 1-tough and so it contains a Hamiltonian cycle C. In $G[C_v \cup xy]$, xy is a 1-edge and hence lies on C. Thus, there is a Hamiltonian path, P', in $G[C_v \cup xy]$ from x to y, and we can replace xy in P with P'. Repeating this process for all $f \in F$ will yield a Hamiltonian path in (G, x_1, x_2). □

Lemma 3.1.18. *Let (H', x_1, x_2) the caterpillar representation of (G, x_1, x_2), where G is a 1-tough 2-tree. If (H', x_1, x_2) is a 2-path, then (H', x_1, x_2) has an (x_1, x_2)-Hamiltonian path.*

Proof. If (H', x_1, x_2) is a 2-path, then the caterpillar representation is the same as the reduced graph, and hence the reduced graph does not have any forced edges. Thus, taking the specified simplicial ordering from Definition 3.1.11 in consecutive order will yield a Hamiltonian path. □

Lemma 3.1.19. *Let (H', x_1, x_2) the caterpillar representation of (G, x_1, x_2), where G is a 1-tough 2-tree. If (H', x_1, x_2) has an (x_1, x_2)-Hamiltonian path, and x_1x_2 is not a 1-edge in (H', x_1, x_2), then x_1, x_2 have degree at most three in (H', x_1, x_2).*

Proof. Suppose that in (H', x_1, x_2), x_1 has degree four. If x_1x_2 is a 2-edge in (H', x_1, x_2), then (x_1, x_2) is a trivial tough path, and hence (H', x_1, x_2) does not have an (x_1, x_2)-Hamiltonian path by Lemma 2.3.6. So, suppose x_1 is not adjacent to x_2 in (H', x_1, x_2). Then (H', x_1, x_2) contains F_a^1 as an induced sub-2-tree, and hence does not have an (x_1, x_2)-Hamiltonian path by Corollary 2.1.10. Similarly if x_2 has degree four. □

In Theorem 3.1.24, we will use the caterpillar representation of a 2-tree and the paths through s-split diamonds and ℓ-strings of diamonds in the lemmas below to construct a path through any 2-tree which does not contain $F_x^1 \in \mathscr{F}^1$.

Lemma 3.1.20. *Let $D_s(R)$ be an s-split diamond. Then there is a unique (c_0, b_r)-Hamiltonian path and a unique (c_0, t_{s-r})-Hamiltonian path.*

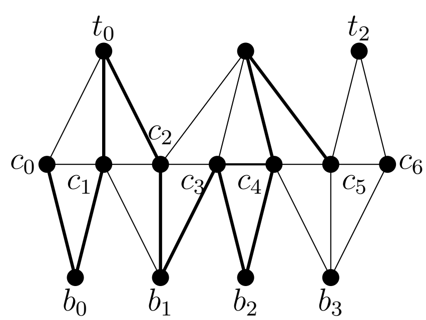


Figure 3.11: An example of a Hamiltonian path in an s-split diamond, $D_5(\{1, 3, 4\})$

Proof. If $s = 0$ then both b_0 and t_0 are adjacent to c_0 on the unique Hamiltonian cycle, C. So, using the edges in C, there are unique (c_0, b_0) and (c_0, t_0)-Hamiltonian paths. Now, suppose that the claim is true for an $(s-1)$-split diamond. Consider $D_s(R)$ an s-split diamond. Then c_0 is adjacent to a simplicial vertex, either t_0 or b_0. Without loss of generality, assume b_0 is simplicial. Then (c_0, b_0, c_1) is a forced path and $D_s(R) - \{c_0, b_0\}$ is an $(s-1)$-split diamond. By the induction hypothesis, there is a (c_1, b_r) and (c_1, t_{s-r})-Hamiltonian path, P and P' respectively. Hence (c_0, b_0, P) and (c_0, b_0, P') are unique (c_0, b_r) and $(c_0, t_{s-r})-$Hamiltonian paths, respectively. $\square$

Remark 3.1.21. *The path created uses all of the edges, other than the central path, except for* $c_{i-1}t_j$ *if* $i \in R$, $c_i b_k$ *if* $i \notin R$ *and* $i+1 \notin R$, *in addition to avoiding* $c_s b_r$ *in a* (c_0, b_r)*-Hamiltonian path, and* $c_s t_{s-r}$ *in a* (c_0, t_{s-r})*-Hamiltonian path.*

Lemma 3.1.22. *Let*
$G = D^1_{s_1}(R_1); (x_1, \ell_1); D^2_{s_2}(R_2); (x_2, \ell_2); ....; (x_m, \ell_m); D^{m+1}_{s_{m+1}}(R_{m+1})$ *be an* ℓ*-string of diamonds with* z_i *the* (D^i, D^{i+1})*-amalgamated vertex for all* i*. If* y *is the simplicial vertex in* $\{b^{m+1}_{r_{m+1}}, t^{m+1}_{s_{m+1}-r_{m+1}}\}$ *and neither* $t^i_0 \neq y$ *nor* $b^i_0 \neq y$ *is simplicial for* $i > 1$*, then there is a unique* (c_0, y)*-Hamiltonian path.*

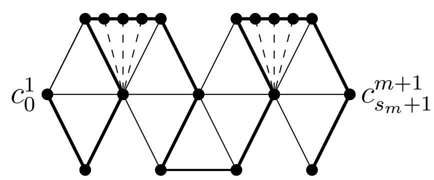


Figure 3.12: An example of a Hamiltonian path in an ℓ-string of diamonds: $D_0; (t, \ell_1); D_0; (b, 1); D_0; (t, \ell_3); D_0$

Proof. We proceed by induction on the number of amalgamated vertices, m. If $m = 0$, then G is an s-split diamond, and by Lemma 3.1.20, the claim is true. Now, suppose that for an ℓ-string of diamonds with $m-1$ amalgamated vertices, that the claim is true. Now, let G' be an ℓ-string of diamonds with m amalgamated vertices. Without loss of generality, let $x_1 = t$. From Lemma 3.1.20, there is a $(c^1_0, t^1_{s_1-r_1})$-path, P, which covers all of the vertices in $D^1_{s_1}$. Furthermore, in $G' - (D^1_{s_1} - t^1_{s_1-r_1})$, the path, P', from $t^1_{s_1-r_1}$ to t^2_0 is forced since the only other vertex adjacent to vertices on this path is z_1, which is not in $G' - (D^1_{s_1} - t^1_{s_1-r_1})$. Also, $G' - (D^1_{s_1} - z_1) - (P' - t^2_0)$ is a string of diamonds with $m-1$ amalgamated vertices where t^2_0 is simplicial but no other $t^i_0 \neq y$ nor $b^i_0 \neq y$ is simplicial for $i \geq 1$. By the induction hypothesis, there is a unique $(z_1 = c^2_0, y)$-Hamiltonian path, P''. Furthermore, since t^2_0 is simplicial in $G' - (D^1_{s_1} - z_1) - (P' - t^2_0)$, the path must begin with (c^2_0, t^2_0). Replacing (c^2_0, t^2_0) with (c^2_0, P') in P'' and preceding this path with P, yields a unique (c^1_0, y)-Hamiltonian path.

$\square$

Remark 3.1.23. *In addition to the unused edges from Lemma 3.1.20, this path will also avoid the edges* $c_0^i b_0^i$ *and* $c_0^i t_0^i$*.*

Theorem 3.1.24. *If* G *is a 1-tough 2-tree with* $x_1, x_2 \in V(G)$*, then the following are equivalent:*

1. *G contains* $F^1 \in \mathscr{F}^1$ *as an induced sub-2-tree,*
2. *One of the following tough conditions hold:*
   - (a) *There exists a tough path from* x_1 *to* x_2*,*
   - (b) *There exists* $ab \in E(G)$ *such that* x_1 *and* x_2 *lie in different components,* C_{x_1}*,* C_{x_2}*, respectively of* $G - \{a, b\}$ *and such that in* $G[V(C_{x_1}) \cup \{a, b\}]$ *there is a tough path from* x_1 *to* a *and a tough path from* x_1 *to* b*, or*
   - (c) *There exists* $ab, cd \in E(G)$ *such that* x_1 *and* x_2 *lie in different components of* $G - \{a, b\}$ *and* $G - \{c, d\}$ *and such that if* x_1 *and* x_2 *lie in components,* C_{x_1}*,* C_{x_2}*, respectively of* $G - \{a, b, c, d\}$ *where in* $G[V - V(C_{x_1}) - V(C_{x_2})]$ *there are tough paths from* a *to* c*,* a *to* d*,* b *to* c*, and* b *to* d*.*
3. *G does not have an* (x_1, x_2) *Hamiltonian path.*

Proof. (1)$\Longrightarrow$(2)

(A) If G contains F_a^1, $x_1 t_0$ and $x_1 c_1$ are tough paths and x_1 and x_2 are in different components of $G - \{t_0 c_1\}$.

(B) If G contains F_b^1, the central path is a tough path from $x_1 = c_0^1$ to $x_2 = c_{s_m+1}^m$.

(C) If G contains F_c^1, there is a tough path from $x_1 = c_0^1$ to c_1^m and to t_0^m and $x_1 = c_0^1$ and x_2 are in different components of $G - \{t_0^m, c_1^m\}$.

(D) If G contains F_d^1, there is a tough path from c_0^1 to c_1^m and to t_0^m and a tough path from t_0^1 to c_1^m and to t_0^m and x_1 and x_2 are in different components of $G - \{t_0^1, c_0^1\}$ and of $G - \{t_0^m, c_1^m\}$ OR there is a tough path from c_0^1 to c_1^m and to b_0^m and a tough path from t_0^1 to c_1^m and to b_0^m and x_1 and x_2 are in different components of $G - \{t_0^1, c_0^1\}$ and of $G - \{b_0^m, c_1^m\}$.

(E) If G contains F_e^1, there is a tough path from c_0^1 to c_1^2 and to t_0^2 and a tough path from t_0^1 to c_1^2 and to t_0^2 and x_1 and x_2 are in different components of $G - \{t_0^1, c_0^1\}$ and of $G - \{t_0^2, c_1^2\}$.

(F) If G contains F_f^1, there is a tough path from c_0^1 to c_1^2 and to t_0^2 and a tough path from t_0^1 to c_1^2 and to t_0^2 and x_1 and x_2 are in different components of $G - \{t_0^1, c_0^1\}$ and of $G - \{t_0^2, c_1^2\}$.

(2)$\Longrightarrow$(3)

(a) Lemma 2.3.6

(b) Corollary 2.3.17

(c) Corollary 2.3.19

(3)$\Longrightarrow$(1)

Suppose G does not contain any $F_x^1 \in \mathscr{F}^1$ as an induced sub-2-tree. Let (H', x_1, x_2) be the caterpillar representation of (G, x_1, x_2). Then (H', x_1, x_2) does not contain any of $F_x^1 \in \mathscr{F}^1$ as an induced sub-2-tree. If (H', x_1, x_2) is a 2-path, then (H', x_1, x_2) will have an (x_1, x_2)-Hamiltonian path by Lemma 3.1.18, so we will assume that (H', x_1, x_2) is not a 2-path. Also since (H', x_1, x_2) does not contain any $F_x^1 \in \mathscr{F}^1$ as an induced sub-2-tree, then x_1 and x_2 have degree two or three in (H', x_1, x_2). In the following cases we will construct paths in (H', x_1, x_2).

Case (A) Suppose x_1 is a simplicial vertex in (H', x_1, x_2). Since (H', x_1, x_2) is not a 2-path, there is at least one simplicial vertex, other than x_1 and x_2. Let v_1 be the vertex with the smallest label from Definition 3.1.11, which is adjacent to a simplicial vertex, $s_1 \neq x_1, x_2$. Using that same labelling, in consecutive order, there is a path, P'_A, from x_1 to y_1, a vertex which is labelled one less than v_1. Since v_1 is adjacent to a simplicial vertex, then there is a tough path which starts at v_1. Let P_A be a maximal short tough path beginning at x_1. By Lemma 2.3.11, P_A is the central path of an ℓ-string of diamonds,

$D^1_{s_1}(R_1); (w_1, \ell_1); D^2_{s_2}(R_2); (w_2, \ell_2); ....; (w_{m-1}, \ell_{m-1}); D^m_{s_m}(R_m)$. Without loss of generality, suppose $w_1 = t$. Since (H', x_1, x_2) does not contain any $F_x^1 \in \mathscr{F}^1$ as an induced sub-2-tree, then $x_2 \neq c^m_{s_m+1}$ and neither $t_0^i \neq x_2$ nor $b_0^i \neq x_2$ is simplicial for $i > 2$.

(I) Suppose t_0^1 is adjacent to a simplicial vertex as well. Then, $t_0^2 \neq x_2$ and $b_0^2 \neq x_2$ are not simplicial and continuing from P'_A, we can take the path to $v_1 = c_0^1$ and continue the path as in Case B with $x_1 = v_1 = c_0^1$.

(II) Suppose t_0^1 is not adjacent to a simplicial vertex. Then, continuing from P'_A, we can take the path $(t_0^1, v_1 = c_0^1, s_1 = b_0^1, c_1^1)$.

(a) If $c_1^1 = z_1$ is an amalgamated vertex, and $\ell_1 = 1$, then removing all of the visited vertices other than $c_1^1 = z_1$, we will have an $(l-1)$-string of diamonds. If $\ell_1 > 1$, then b_0^2 is not simplicial and we will have an $(l-1)$-string of diamonds with additional vertices, left from the path of length ℓ_1 between t_0^1 and t_0^2. In either case, we can then extend P'_A by using the path given in Lemma 3.1.22 beginning at the amalgamated vertex, $c_1^1 = z_1$. We can continue the construction of the path as in Case (B)(II).

(b) If $c_1^1 = z_1$ is not an amalgamated vertex, then $R_1 \neq \emptyset$, so let $R_1 = \{q_1, q_2, ..., q_{r_1}\}$. Let q_i be the first value such that $q_{i-1} \neq q_i - 1$.

(i) If no such value exists, then $c_{s_1+1}^1 = z_1$ is an amalgamated vertex and b_j^1 is simplicial for all $j \in \{1, ..., s_1\}$. So there is a forced path from c_1^1 to $c_{s_1+1}^1 = z_1$ which uses all edges $c_k b_k$ and $c_{k+1} b_k$ for $1 \leq k \leq s_1 + 1$. By assumption, t_0^1 is not adjacent to a simplicial vertex, so this path uses all possible edges which could have a simplicial vertex adjacent in (H', x_1, x_2). Hence, replacing any edges of the path which are adjacent to simplicial vertices in (H', x_1, x_2), with the path through the simplicial vertex, we have a path in (H', x_1, x_2). Furthermore, removing the visited vertices, other than $c_{s_1+1}^1 = z_1$, we will have an $(l - (s_1 + 1))$-string of diamonds if $\ell_1 = 1$ and if $\ell_1 > 1$, then we will have an $(l-(s_1+1))$-string of diamonds with additional vertices, left from the path of length ℓ_1 between t_0^1 and t_0^2. In either case, we can then extend P'_A by using the path given in Lemma 3.1.22 beginning at the amalgamated vertex, $c_{s_1+1}^1 = z_1$. We can continue the construction of the path as in Case (B)(II).

(ii) If a q_i exists, then b_j^1 is simplicial for $0 \leq q_{i-1} - 1 = q_{i-2}$, and hence there is a forced path from c_1^1 to $c_{q_{i-1}}^1$, which uses all edges $c_k b_k$ and $c_{k+1} b_k$ for $1 \leq k \leq q_{i-2}$. By assumption,

t_0^1 is not adjacent to a simplicial vertex, so this path uses all possible edges which could have a simplicial vertex adjacent in (H', x_1, x_2). Hence, replacing any edges of the path which are adjacent to simplicial vertices in (H', x_1, x_2), with the path through the simplicial vertex, we have a path in (H', x_1, x_2). Furthermore, removing the visited vertices, other than $c^1_{q_{i-1}}$, and removing the visited vertices, other than $c^1_{q_{i-1}}$, we will have a 1-tough 2-tree with $c^1_{q_{i-1}}$ simplicial, and we can continue this path by repeating Case (A) with $x_1 = c^1_{q_{i-1}}$.

Case (B) Suppose x_1 is not a simplicial vertex in (H', x_1, x_2). Then x_1 has degree three and hence is adjacent to a simplicial vertex. Hence, x_1 lies on a tough path. Let P_B be a maximal short tough path beginning at x_1. By Lemma 2.3.11, P_B is the central path of an ℓ-string of diamonds, $D^1_{s_1}(R_1); (w_1, \ell_1); D^2_{s_2}(R_2); (w_2, \ell_2); ....; (w_{m-1}, \ell_{m-1}); D^m_{s_m}(R_m)$. Without loss of generality, suppose $w_1 = t$. Since (H', x_1, x_2), ad (H', x_1, x_2) does not contain any $F^1_x \in \mathscr{F}^1$ as an induced sub-2-tree, then $x_2 \neq c^m_{s_m+1}$ and neither $t^i_0 \neq x_2$ nor $b^i_0 \neq x_2$ is simplicial for $i > 1$.

(I) If the ℓ-string of diamonds contains no amalgamated vertices, then we have an $(l-1)$-split diamond, $D_{s_1}(R_1)$. From Lemma 3.1.20, there are (x_1, b_{r_m}) and $(x_1, t_{s_m - r_m})$ paths, P'_{B1} and P'_{B2}, respectively, which cover all of the vertices in $D_{s_1}(R_1)$. Note that since the $(l-1)$-split diamond is an induced sub-2-tree, it is possible for 1-edges of the $(l-1)$-split diamond to be adjacent to simplicial vertices in (H', x_1, x_2). Such edges would correspond to edges that would need to be used in a path through the $(l-1)$-split diamond. However, attaching simplicial vertices to any unused edges in the path from Lemma 3.1.20 would form an induced sub-2-tree in $\mathscr{F}^1$. And so, for any edges of the $(l-1)$-split diamond which are adjacent to a simplicial vertex in (H', x_1, x_2) we can replace the edge on P'_{B1} or P'_{B2} with the path through the simplicial vertex to form a path, P''_{B1} and P''_{B2}, respectively, in (H', x_1, x_2). If $b_{r_m} = x_2$, then P''_{B1} is an (x_1, x_2)-Hamiltonian path in (H', x_1, x_2). If $t_{s_m - r_m} = x_2$, then P''_{B2} is an (x_1, x_2)-Hamiltonian path in (H', x_1, x_2). So now suppose that $b_{r_m}, t_{s_m-r_m} \neq x_2$. Let $y = b_{r_m}$ if $(H', x_1, x_2) - \{b_{r_m}c_{s_1+1}\}$ leaves x_1 and x_2 in different components, and $y = t_{s_m-r_m}$ if $(H', x_1, x_2) - \{t_{s_m-r_m}c_{s_1+1}\}$ leaves x_1 and x_2 in different components. Let $P''_y = P''_{B1}$ if $y = b_{r_m}$ and $P''_y = P''_{B2}$ if $y = t_{s_m-r_m}$. $P''_y - y$ is an (x_1, c_{s_1+1})-path and furthermore $(H', x_1, x_2) - (P''_y - \{y, c_{s_1+1}\})$ is a 1-tough 2-tree. Additionally, c_{s_1+1} is simplicial since if it weren't, then P_B would not be maximal. So we can finish constructing the Hamiltonian path by finding an (c_{s_1+1}, x_2)-Hamiltonian path in $(H', x_1, x_2) - (P''_y - \{y, c_{s_1+1}\})$ using Case (A).

(II) Suppose the ℓ-string of diamonds contains at least one amalgamated vertex. Using the path in Lemma 3.1.22, we have a path, P'_B from x_1 to y, where $y = b^m_{r_m}$ if $y_{m-1} = t$ in
$D^1_{s_1}(R_1); (w_1, \ell_1); D^2_{s_2}(R_2); (w_2, \ell_2); ....; (w_{m-1}, \ell_{m-1}); D^m_{s_m}(R_m)$, and
$y = t^m_{s_m-r_m}$ if $y_{m-1} = b$ in
$D^1_{s_1}(R_1); (w_1, \ell_1); D^2_{s_2}(R_2); (w_2, \ell_2); ....; (w_{m-1}, \ell_{m-1}); D^m_{s_m}(R_m)$, such that all vertices in the ℓ-string of diamonds are covered. Note that since the ℓ-string of diamonds is an induced sub-2-tree, it is possible for 1-edges of the ℓ-string of diamonds to be adjacent to simplicial vertices in (H', x_1, x_2), which would correspond to edges that would need to be used in a path through the ℓ-string of diamonds. However, attaching simplicial vertices to any unused edges in the path from Lemmas 3.1.20 and 3.1.22 would form an induced sub-2-tree in $\mathscr{F}^1$. And so, for any edges of the ℓ-string of diamonds which are adjacent to a simplicial vertex in (H', x_1, x_2) we can replace the edge on P'_B with the path through the simplicial vertex to form a path, P''_B in (H', x_1, x_2). If $y = x_2$, then P''_B is an (x_1, x_2)-Hamiltonian path in (H', x_1, x_2). If $y \neq x_2$, then $P''_B - y$ is an $(x_1, c^m_{s_m+1})$-path. Furthermore, $(H', x_1, x_2) - (P''_B - \{y, c^m_{s_m+1}\})$ is a 1-tough 2-tree and $c^m_{s_m+1}$ is simplicial since if it weren't, then P_B would not be maximal. So we can finish constructing the Hamiltonian path by finding an $(c^m_{s_m+1}, x_2)$-Hamiltonian path in $(H', x_1, x_2) - (P'' - \{y, c^m_{s_m+1}\})$ using Case (A). $\square$

Remark 3.1.25. *In the cases when G is a 1-tough 2-tree, which do not contain an (x_1, x_2)-Hamiltonian path, we can partition G into two vertex disjoint paths with x_1*

the end of one path and x_2 the end of the other. We can do this by breaking the Hamiltonian cycle in G into two paths.

3.2 2HP on 2-trees

Definition 3.2.1. *Define* $\mathscr{F}^2 = \{\mathscr{F}^1, F_a^2, F_b^2, F_c^2, F_d^2, F_e^2\}$ *where:*

(a) F_a^2 *is a 2-tree with vertices* x_1, x_2*, and a 3-edge,* ef*, such that either:*

(i) x_1 *and* x_2 *are in the same component of* $G - \{e, f\}$*, or*

(ii) $e \in \{x_1, x_2\}$.

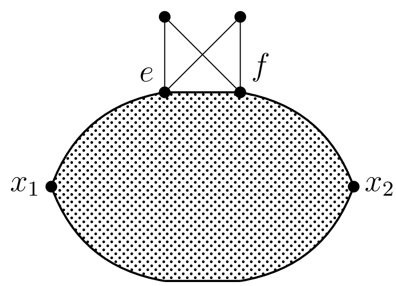


Figure 3.13: General example of F_a^2 such that x_1 and x_2 are in the same component of $G - \{e, f\}$, and to preserve generality, the dotted section of the graph represents any 2-tree with scattering number at most one

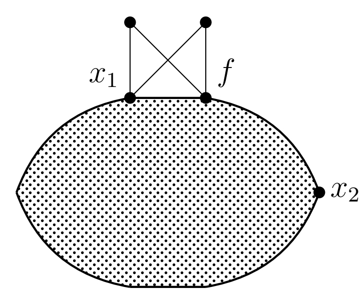


Figure 3.14: General example of F_a^2 such that $e \in \{x_1, x_2\}$ and to preserve generality, the dotted section of the graph represents any 2-tree with scattering number at most one

(b) F_b^2 *is a 2-tree with vertices* x_1, x_2*, which contains a 3-edge,* ab*, such that:*

(i) x_1 *and* x_2 *are in different components of* $F_b^2 - \{a, b\}$*,*

(ii) $N(a) - \{x_1, x_2\}$ *contains two simplicial vertices,*

(iii) $N(b) - \{x_1, x_2\}$ *contains two simplicial vertices, and*

(iv) In $F_b^2 - \{a, b\}$ *two of the simplicial vertices lie in the same component.*

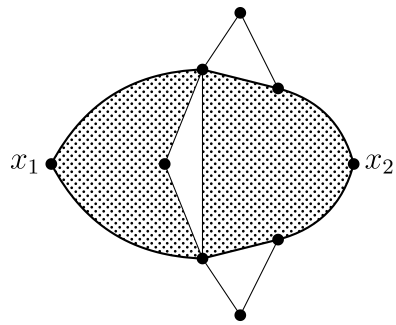


Figure 3.15: General example of F_b^2 and to preserve generality, the dotted section of the graph represents any 2-tree with scattering number at most one

(c) F_c^2 *is a 2-tree with vertices* x_1, x_2*, which contains a 3-edge, ab, such that* x_1 *and* x_2 *are in different components of* $F_c^2 - \{a, b\}$ *and* $N(a) - \{x_1, x_2\}$ *contains three simplicial vertices.*

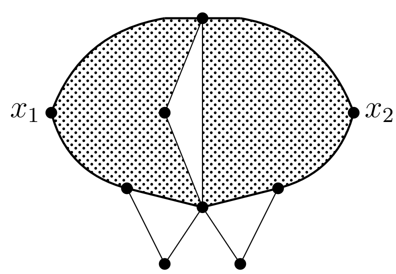


Figure 3.16: General example of F_c^2 and to preserve generality, the dotted section of the graph represents any 2-tree with scattering number at most one

(d) F_d^2 *is constructed from* $D_{s_1}^1(R_1); (x_1, \ell_1); ....; D_{s_{m-1}}^{m-1}(R_{m-1}); (t, \ell_{m-1}); D_0^m$*,* $m \geq 2$*, by:*

(i) Amalgamating an x_2*-2-path with* $t_0^m c_1^m$*, and*

(ii) Amalgamating an x_1*-2-path with* $c_0^1 c_1^1$*.*

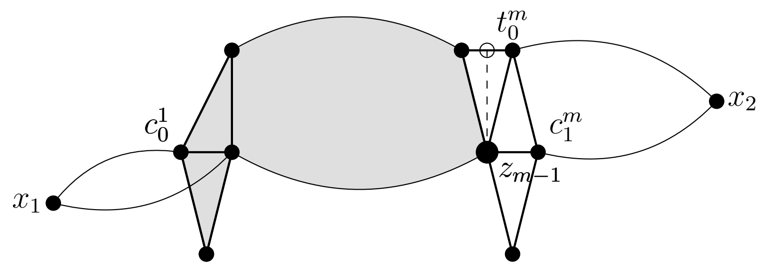


Figure 3.17: A general example of F_d^2:
$D_{s_1}^1(R_1); (x_1, \ell_1); D_{s_2}^2(R_2); (x_2, \ell_2); ....; D_{s_{m-1}}^{m-1}(R_{m-1}); (t, \ell_{m-1}); D_0^m$
$m \geq 2$, with an x_2-2-path amalgamated with $t_0^m c_1^m$, and x_1-2-path amalgamated with $c_0^1 c_1^1$
where $D_{s_1}^1(R_1); (x_1, \ell_1); D_{s_2}^2(R_2); (x_2, \ell_2); ....; (x_{m-2}, \ell_{m-2}); D_{s_{m-1}}^{m-1}(R_{m-1})$ is shown in gray to preserve generality

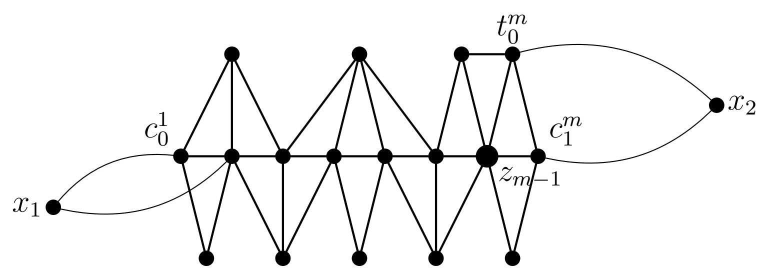


Figure 3.18: Specific example of F_d^2: $D_5(\{1, 3, 4, 5\}); (t, 1); D_0$ with x_1-2-path amalgamated with $c_0^1 c_1^1$ and x_2-2-path amalgamated with $t_0^2 c_1^2$

(e) F_e^2 *is constructed from an* ℓ*-string of diamonds, with* $x_1 = c_0^1$*, by amalgamating an* x_2*-2-path with* $c_{s_m}^m c_{s_m+1}^m$*.*

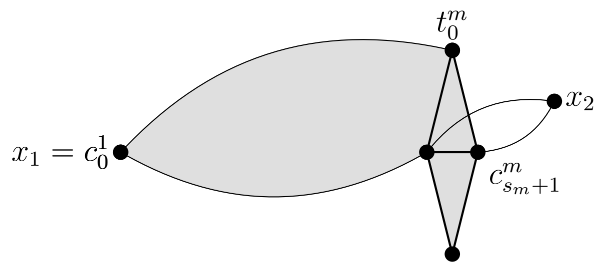


Figure 3.19: A general example of F_e^2: $D_{s_1}^1(R_1); (x_1, \ell_1); ....; (x_{m-1}, \ell_{m-1}); D_{s_m}^m(R_m)$ with an amalgamated x_2-2-path and such that $x_1 = c_0^1$ where $D_{s_1}^1(R_1); (x_1, \ell_1); ....; (x_{m-1}, \ell_{m-1}); D_{s_m}^m(R_m)$ is shown in gray to preserve generality

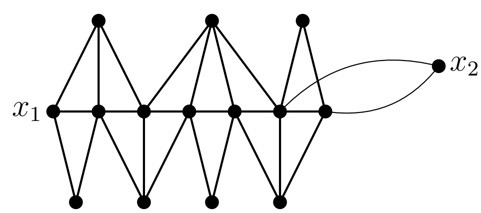


Figure 3.20: Specific example of F_e^2: $D_5(\{1, 3, 4, 5\})$ with $x_1 = c_0$ and x_2-2-path amalgamated with c_5c_6

Lemma 3.2.2. *The graph F_a^2 does not have an (x_1, x_2)-Hamiltonian path.*

Proof. Since ef is a 3-edge, then $c(G - \{e, f\}) = 3$ and hence if G has a Hamiltonian path, then the ends of the path must lie in two of the three components of $G - \{e, f\}$. So, if x_1 and x_2 are in the same component of $G - \{e, f\}$, then G does not have an (x_1, x_2)-Hamiltonian path. Similarly, if $e \in \{x_1, x_2\}$, then x_1 or x_2 is not in one of the components of $G - \{e, f\}$ and G does not have an (x_1, x_2)-Hamiltonian path. □

Lemma 3.2.3. *The graph F_b^2 does not have an (x_1, x_2)-Hamiltonian path.*

Proof. Let u be the simplicial vertex in $N(ab) - \{x_1, x_2\}$, v the simplicial vertex in $N(a) - \{x_1, x_2, u\}$, and w the simplicial vertex in $N(b) - \{x_1, x_2, u\}$. Suppose that G contains an (x_1, x_2)-Hamiltonian path, P. Since u, v and w are simplicial and not endpoints of P, then P must contain (v, a, u, b, w). But since v and w are in the same component of $G - \{a, b\}$, then x_1 and x_2 need to be in the same component of $G - \{a, b\}$, a contradiction. □

Lemma 3.2.4. *The graph F_c^2 does not have an (x_1, x_2)-Hamiltonian path.*

Proof. Let u, v and w be the simplicial vertices in $N(a) - \{x_1, x_2\}$, and suppose G has an (x_1, x_2)-Hamiltonian path. Since u, v and w are not endpoints to the path, then u, v and w must all be either preceded or followed by a. But that means that a must be used at least twice on the Hamiltonian path, a contradiction. □

Lemma 3.2.5. *The graph F_d^2 does not have an (x_1, x_2)-Hamiltonian path.*

Proof. Since $c(H - c_0^1 c_1^1) = 3$, then if H has a Hamiltonian path, there must be a Hamiltonian path in each of the components, and c_0^1 and c_1^1 must connect the paths. Furthermore, if H has an (x_1, x_2)-Hamiltonian path, then the path must start in the component of $H - c_0^1 c_1^1$ which contains x_1 and end in the component which contains x_2. But that would mean that H has a (c_0^1, x_2) or (c_1^1, x_2)-Hamiltonian path, a contradiction to Lemma 3.1.4. □

Lemma 3.2.6. *The graph F_e^2 does not have an (x_1, x_2)-Hamiltonian path.*

Proof. Since $c(H - c_{s_m}^m c_{s_m+1}^m) = 3$, then if H has a Hamiltonian path, there must be a Hamiltonian path in each of the components, and $c_{s_m}^m$ and $c_{s_m+1}^m$ must connect the paths. Furthermore, if H has an (c_0^1, x_2)-Hamiltonian path, then the path must start in the component of $H - c_{s_m}^m c_{s_m+1}^m$ which contains c_0^1 and end in the component which contains x_2. But that would mean that H has a $(c_0^1, c_{s_m}^m)$ or $(c_0^1, c_{s_m+1}^m)$-Hamiltonian path, a contradiction to Lemma 3.1.3. □

Similar to the reduced graph of a 1-tough 2-tree with fixed endpoints, we will create a reduced graph of a 2-tree with scattering number one and fixed points, in order to more easily describe the paths in the 2-trees, as follows.

Definition 3.2.7. *Given a 2-tree, G with $s(G) = 1$, then the* ***reduced graph of*** *(G, u, v), is formed using the following algorithm:*

1. *For every 3-edge ab with components of $G - \{a, b\}$, $C_{ab}^1, C_{ab}^2, C_{ab}^3$, if $G[C_{ab}^i \cup \{a, b\}]$ is 1-tough and does not contain u or v, then replace C_{ab}^i with a simplicial vertex adjacent to ab.*

2. *Let $w \in S_1^*(G, u, v)$, and x, y the neighbors of w. If xy is not a 3-edge, remove w and turn the edge xy into a forced edge.*

3. *Repeat (2) for all $w \in S_1^*(G, u, v)$. Define the resulting graph to be G_1^*.*

4. *For $i \geq 2$, let $S_i^* = S_1^*(G_{i-1}^*, u, v)$ where G_{i-1}^* is the graph formed by repeating (2) for $G = G_{i-1}^*$ and for all $w \in S_1^*(G_{i-1}^*, u, v)$.*

Repeat (4) for all $i = 2, 3, ..., j$ for j such that $S_j^* = \emptyset$ or for all $s \in S_j^*$, $N(s)$ is a 3-edge. This is the reduced graph of (G, u, v).
For F the set of forced edges, let (H, u, v, F) denote the reduced graph of (G, u, v), for G a 2-tree containing at least one 3-edge.

Since simplicial vertices in 2-trees are not adjacent [7], when we remove the vertices in each S_i, regardless of order, we will end up with the same graph.

Remark 3.2.8. *When creating the reduced graph of a 2-tree with scattering number one with no fixed endpoints, $S_1^*(G, u, v)$ will be replaced by $S_1(G)$.*

We will form the corresponding caterpillar representation of (G, u, v) as in Chapter 2.

Lemma 3.2.9. *Let G be a 2-tree with $x_1, x_2 \in V(G)$ and $s(G) = 1$. Let (H, x_1, x_2, F) be the reduced graph of (G, x_1, x_2), and (H', x_1, x_2) the caterpillar representation of (G, x_1, x_2). Then the following are equivalent:*

1. *G has an (x_1, x_2)-Hamiltonian path,*

2. *(H', x_1, x_2) has an (x_1, x_2)-Hamiltonian path, and*

3. *(H, x_1, x_2, F) has an (x_1, x_2)-Hamiltonian path which uses all of the edges in F.*

Proof. (1)$\Longrightarrow$ (2)
Suppose (H', x_1, x_2) does not have an (x_1, x_2)-Hamiltonian path. Since (H', x_1, x_2) is an induced sub-2-tree of (G, x_1, x_2), then by Corollary 2.1.10, G does not have an (x_1, x_2)-Hamiltonian path.

(2) $\Longrightarrow$ (3)
Suppose (H', x_1, x_2) has an (x_1, x_2)-Hamiltonian path, P. Let $v \neq x_1, x_2$ be a simplicial vertex with neighbors u and w. Then $P = (x_1, ..., u, v, w, ..., x_2)$ or $P = (x_1, ..., w, v, u, ..., x_2)$. Furthermore, because H' is a 2-tree, then $uw \in E(H')$, and from the reduction algorithm $uw \in F$. Replacing (u, v, w) or (w, v, u) by (u, w) in P, then P is a Hamiltonian path using exactly one forced edge. Repeating this process for all $S_1^*(H', x_1, x_2)$, then P will be a Hamiltonian path in (H, x_1, x_2, F).

(3) $\implies$ (1) Suppose (H, x_1, x_2, F) has an (x_1, x_2)-Hamiltonian path, P, which uses all of the edges in F. Let ab be a 3-edge in G with components of $G - \{a, b\}$, $C^1_{ab}, C^2_{ab}, C^3_{ab}$, where $G[C^i_{ab} \cup \{a, b\}]$ and is 1-tough and does not contain x_1 or x_2. In (H, x_1, x_2, F), C^i_{ab} has been replaced by the simplicial vertex, v^i_{ab}. Since $G[C^i_{ab} \cup \{a, b\}]$ is 1-tough, then $G[C^i_{ab} \cup \{a, b\}]$ has a Hamiltonian cycle, C using all 1-edges in $G[C^i_{ab} \cup \{a, b\}]$. Hence, since ab is a 1-edge in $G[C^i_{ab} \cup \{a, b\}]$, then there is an (a, b)-Hamiltonian path P' in $G[C^i_{ab} \cup \{a, b\}]$. Since v^i_{ab} is on the interior of P in (H, x_1, x_2, F), then we can replace (a, v^i_{ab}, b) on P with P'. Now, consider $xy \in F$. In G, xy is incident to at least one vertex, v, which is not in (H, x_1, x_2, F) so that $c(G - \{x, y\}) = 2$. Let C_v be the component of $G - \{x, y\}$ which contains v. From [24], $G[C_v \cup xy]$ is a 2-tree, and from the reduction algorithm, $G[C_v \cup xy]$ must be 1-tough and so it contains a Hamiltonian cycle C. In $G[C_v \cup xy]$, xy is a 1-edge and hence lies on C. Thus, there is a Hamiltonian path, P'', in $G[C_v \cup xy]$ from x to y, and we can replace xy in P with P'. Repeating these processes for all $f \in F$ and all 3-edges, cd and all C^i_{cd} such that $G[C^i_{cd} \cup \{c, d\}]$ is 1-tough, will yield an (x_1, x_2)-Hamiltonian path in G. □

Theorem 3.2.10. *If G is a 2-tree with $x, y \in V(G)$, then G has an (x, y)-Hamiltonian path iff $s(G) \leq 1$ and (G, x, y) does not contain any $F^2 \in \mathscr{F}^2$.*

Proof. $\implies$ If $s(G) \geq 2$, then G is not 1-path-tough, and G does not contain a Hamiltonian path.

1. If $(G, x, y) = F^2_a$, then (G, x, y) does not have an (x, y)-Hamiltonian path by Lemma 3.2.2. If (G, x, y) contains F^2_a as an induced sub-2-tree, then (G, x, y) does not have an (x, y)-Hamiltonian path by Corollary 2.1.11.

2. If $(G, x, y) = F^2_b$, then (G, x, y) does not have an (x, y)-Hamiltonian path by Lemma 3.2.3. If (G, x, y) contains F^2_b as an induced sub-2-tree, then (G, x, y) does not have an (x, y)-Hamiltonian path by Corollary 2.1.11.

3. If $(G, x, y) = F^2_c$, then (G, x, y) does not have an (x, y)-Hamiltonian path by Lemma 3.2.4. If (G, x, y) contains F^2_c as an induced sub-2-tree, then (G, x, y) does not have an (x, y)-Hamiltonian path by Corollary 2.1.11.

4. If (G, x, y) contains an $F^1_x \in \mathscr{F}^1 \subset \mathscr{F}^2$, then G does not have an (x, y)-Hamiltonian path by Theorem 3.1.24 and Corollary 2.1.11.

$\Longleftarrow$

Suppose G does not have an (x, y)-Hamiltonian path, but that $s(G) \leq 1$. Since $s(G) \leq 1$, then G contains no t-edges for $t \geq 4$. We will proceed by induction on the number of 3-edges, m. If $m = 0$, then by Theorem 3.1.24, (G, x, y) contains an $F_x^1 \in \mathscr{F}^1 \subset \mathscr{F}^2$. Suppose the claim is true for all graphs with $(m-1)$ 3-edges. Now consider G a 2-tree with $s(G) \leq 1$ such that G does not have an (x, y)-Hamiltonian path with m 3-edges. Let (H', x, y) be the caterpillar representation of G. Then $s(H') \leq 1$ and H' does not have an (x, y)-Hamiltonian path. Suppose H' does not contain F_a^2. Denote the 3-edges in H', $S_i = s_i s_i'$ for all $1 \leq i \leq m$. Then the 3-edges in H' can be ordered $S_1, S_2, ...S_m$ so that for all i, x and y are in different components of $H' - S_i$, in $H' - S_1$, x is in a different component than s_i and s_i' for all i, in $H' - S_m$, y is in a different component than s_i and s_i' for all i, and such that for all $i \in \{1, 2, ..., m-2\}$, s_i and s_{i+2} are in different components of $H' - S_{i+1}$. Let C_1 be the component of $H' - S_1$ which contains x. Let C_2 be the component of $H' - S_1$ which contains y. Let H_1' be the graph constructed from $G[C_1 \cup S_1]$ by adding a simplicial vertex, v_1, adjacent to S_1. Let H_2' be the graph constructed from $G[C_2 \cup S_1]$ by adding a simplicial vertex, v_2, adjacent to S_1. Let $S_1 = ab$. If H_1' has an (x, a)-Hamiltonian path, P, then because v_1 is simplicial, then P ends with (b, v_1, a). Likewise, if H_2' has a (b, y)-Hamiltonian path, P', then P' begins with (b, v_2, a). So, in H', $(P - \{a, v_1, b\}, P')$ is an (x, y)-Hamiltonian path. Similarly if H_1' has an (x, b)-Hamiltonian path and H_2' has an (a, y)-Hamiltonian path. So, since H' does not have an (x, y)-Hamiltonian path, either (1) H_1' has neither an (x, a)-Hamiltonian path nor an (x, b)-Hamiltonian path, or (2) H_2' has neither an (a, y)-Hamiltonian path nor an (b, y)-Hamiltonian path, or (3) H_1' only has an (x, a)-Hamiltonian path while H_2' only has an (a, y)-Hamiltonian path, or (4) H_1' only has an (x, b)-Hamiltonian path while H_2' only has an (b, y)-Hamiltonian path.

1. If H_1' does not have an (x, a)-Hamiltonian path, then by Theorem 3.1.24, then (H_1', x, a) contains an $F_x^1 \in \mathscr{F}^1 \subset \mathscr{F}^2$. Likewise, if H_1' does not have an (x, b)-Hamiltonian path, then by Theorem 3.1.24, then (H_1', x, b) contains an $F_x^1 \in \mathscr{F}^1 \subset \mathscr{F}^2$. Note first that if there is an ef such that x and a lie in different components of $H_1' - \{e, f\}$, then since a and b are adjacent, then either b is in the same component as a in $H_1' - \{e, f\}$, or $b \in \{e, f\}$. Also, if in (H_1', x, a), and similarly for (H_1', x, b), there is an ef such that x and a lies in different components, C_x, C_a, respectively of $H_1' - \{e, f\}$ and such that in

$G[V(C_x) \cup \{e, f\}]$ there is a tough path from x to e and a tough path from x to f, then we have F_a^1 or F_c^1, with $x = c_0^1$. In either case, y will be in the same component as a, and hence (H', x, y) also contains F_a^1 or F_c^1. Similarly, if (H_1', x, a) and/or (H_1', x, b) contains F_d^1, F_e^1, or F_f^1, then (H', x, y) also contains F_d^1, F_e^1, or F_f^1. Now, suppose that (H_1', x, a) and (H_1', x, b) contain F_b^1, so in H_1' there is a tough path from x to a and a tough path from x to b. If the short tough path from x to a contains the short tough path from x to b, then in (H', x, y) we have F_e^2. Otherwise, we have x and y in different components of $H' - \{a, b\}$ and so we have F_a^1 or F_c^1 in (H', x, y), with $x = c_0^1$. If (H_1', x, a) contains F_a^1 with $x_1 = a$ and (H_1', x, b) contains F_a^1 with $x_1 = b$, then (H', x, y) contains F_b^2. Now, suppose (H_1', x, a) contains F_c^1 or F_a^1 with $x_1 = a$ and (H_1', x, b) contains F_c^1 with $x_1 = b$. The case when (H_1', x, b) contains F_c^1 or F_a^1 with $x_1 = b$ and (H_1', x, a) contains F_c^1 with $x_1 = a$ is similar. If the tough paths starting at a and b do not intersect, then (H', x, y) contains F_b^2. If ab is an edge of one of the tough paths, then (H', x, y) contains F_d^2. If the tough paths starting at a and b intersect, but ab is not an edge of one of the tough paths, then (H', x, y) contains F_f^1 or F_d^1.

2. If H_2' does not have an (a, y)-Hamiltonian path, then by the induction hypothesis, (H_2', y, a) contains an $F_x^2 \in \mathscr{F}^2$. Likewise, if H_2' does not have an (b, y)-Hamiltonian path, then by the induction hypothesis, (H_2', y, b) contains an $F_x^2 \in \mathscr{F}^2$. As above, if (H_2', y, a) or (H_2', y, b) contains an $F_x^1 \in \mathscr{F}^1 \subset \mathscr{F}^2$, then (H', x, y) contains an $F_x^2 \in \mathscr{F}^2$. If (H_2', y, a) and/or (H_2', y, b) contains F_b^2 or F_c^2, then since x will be in the same component as a and/or b, respectively, then (H', x, y) will also contain F_b^2 or F_c^2. Similarly, if (H_2', y, a) and/or (H_2', y, b) contains F_d^2 or F_e^2, (H', x, y) will also contain F_d^2 or F_e^2, respectively.

3. Without loss of generality, assume H_1' only has an (x, b)-Hamiltonian path while H_2' only has a (b, y)-Hamiltonian path. Then (H_1', x, a) and (H_2', y, a) contain an $F_x^2 \in \mathscr{F}^2$. But since H_1' has an (x, b)-Hamiltonian path while H_2' has a (b, y)-Hamiltonian path, then (H_1', x, b) and (H_2', y, b) cannot contain an $F_x^2 \in \mathscr{F}^2$. Then, (H_1', x, a) must contain F_a^1, F_b^1, F_c^1 and (H_2', y, a) must contain F_a^1, F_b^1, F_c^1, or F_e^2. If (H_1', x, a) and (H_2', y, a) both contain F_a^1, then (H', x, y) contains F_c^2. If (H_1', x, a) and (H_2', y, a) both contain F_c^1, then (H', x, y) contains F_d^1. If (H_1', x, a) and (H_2', y, a) both contain F_b^1, then there is a tough path from x to y and hence (H', x, y) also contains F_b^1. If one contains F_b^1 and the other

contains F_a^1 or F_c^1, then we have F_c^1 in (H', x, y). If one contains F_a^1 and the other contains F_c^1, then we have F_d^1 in (H', x, y). If (H_2', y, a) contains F_e^2 and the other contains F_a^1 or F_c^1, then we have F_d^2 in (H', x, y). Lastly if (H_2', y, a) contains F_e^2 and the other contains F_b^1, then we have F_e^2 in (H', x, y). $\square$

Chapter 4

Using 2HP to Characterize HP and 1HP

As mentioned earlier in this dissertations, the Hamiltonian path problem on 2-trees is closely related to 2HP on 2-trees, and will use the results from the previous chapter on 2HP to prove necessary and sufficient conditions for which a 2-tree will not have a Hamiltonian path in Theorem 4.1.15 in section 4.1. We will begin as in the previous chapter by defining a family, $\mathscr{H}$, of 2-trees which will not have a Hamiltonian path. In Theorem 4.1.15, we will prove that any 2-tree with scattering number at most one, which does not contain one of the graphs in $\mathscr{H}$ as an induced sub-2-tree, will have a Hamiltonian path. In section 4.2, we will use the results from 2HP on 2-trees to prove necessary and sufficient conditions for which a 2-tree with a specified vertex, x_2, will not have an x_2-Hamiltonian path in Theorem 4.2.12. We will begin as in the previous chapters by defining a family, $\mathscr{I}$, of 2-trees, with a specified vertex, x_2, which will not have an x_2-Hamiltonian path. In Theorem 4.2.12, we will prove that any 2-tree with scattering number at most one, which does not contain one of the graphs in $\mathscr{I}$ as an induced sub-2-tree, will have an x_2-Hamiltonian path.

4.1 Hamiltonian Path Problem

Definition 4.1.1. *Define* $\mathscr{H} = \{H_a, H_b, H_c, H_d, H_e, H_f, H_g\}$ *where:*

(a) H_a *is a 2-tree which contains three 3-edges,* ab, cd, *and* ef, *none of which are incident, such that:*

(i) cd *and* ef *are in the same component of* $G - \{a, b\}$

(ii) ab and cd are in the same component of $G - \{e, f\}$

(iii) ab and ef are in the same component of $G - \{c, d\}$

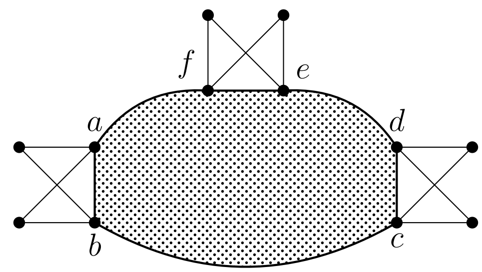


Figure 4.1: A general example of H_a where the dotted section of the graph represents any 2-tree with scattering number at most one to preserve generality

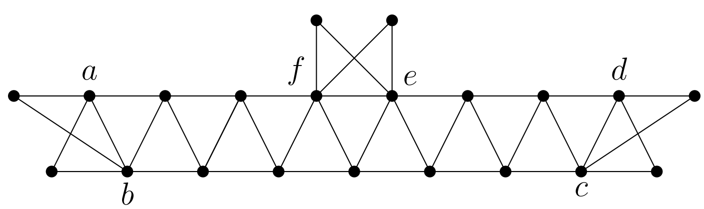


Figure 4.2: A specific example of H_a: P_{15}^2 with three pairs of simplicial vertices added

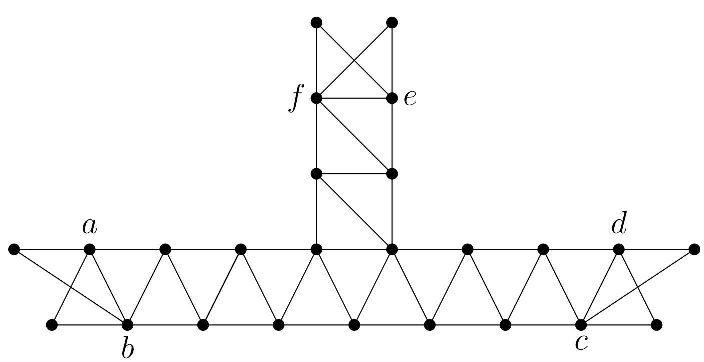


Figure 4.3: A specific example of H_a

(b) H_b *is a 2-tree which contains exactly two 3-edges, ab and cd, such that:*

(i) ab is not incident to cd,

(ii) ab and cd are each adjacent to two simplicial vertices, and

(iii) N(ab) contains two simplicial vertices.

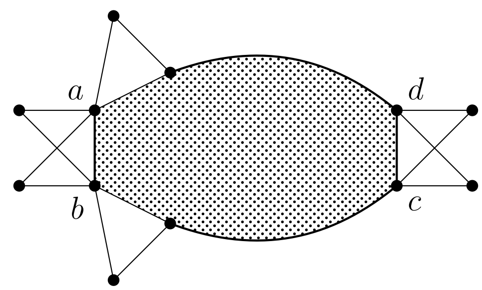


Figure 4.4: A general example of H_b. To preserve generality, the dotted section of the graph represents any 2-tree with scattering number at most one.

(c) H_c *is a 2-tree which contains three 3-edges such that for one of the 3-edges, ef,:*

(i) Two of the three components of $G - \{e, f\}$ *contain a 3-edge, and*

(ii) e is adjacent to three simplicial vertices which are all in different components of $G - \{e, f\}$.

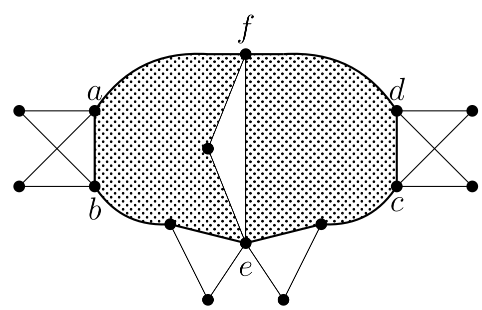


Figure 4.5: A general example of H_c where the dotted section of the graph represents any 2-tree with scattering number at most one to preserve generality

(d) H_d *is constructed from*

$G = D^1_{s_1}(R_1); (x_1, \ell_1); ....; (x_{m-2}, \ell_{m-2}); D^{m-1}_{s_{m-1}}(R_{m-1}); (t, \ell_{m-1}); D^m_0$, $m \geq 2$, *by :*

(i) Amalgamating an x_2*-2-path with* $t^m_0 c^m_1$,

(ii) Adding a false twin, x'_2, *of* x_2, *and*

(iii) Adding a simplicial vertex adjacent to $c_0^1 c_1^1$.

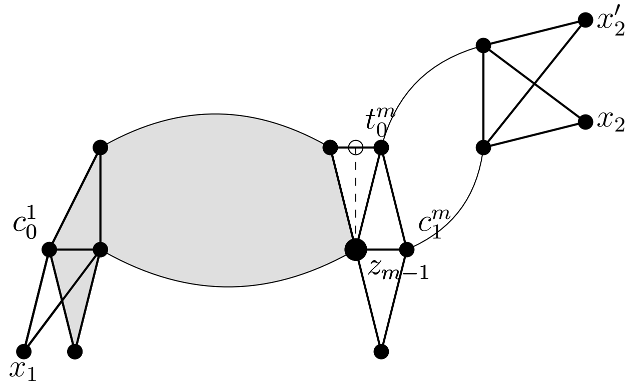


Figure 4.6: A general example of H_d:
$D^1_{s_1}(R_1); (x_1, \ell_1); D^2_{s_2}(R_2); (x_2, \ell_2); ....; D^{m-1}_{s_{m-1}}(R_{m-1}); (t, \ell_{m-1}); D^m_0$
$m \geq 2$, with an x_2-2-path amalgamated with $t_0^2 c_1^2$, and a simplicial vertex added to $c_0^1 c_1^1$
where $D^2_{s_2}(R_2); (x_2, \ell_2); ....; (x_{m-2}, \ell_{m-2}); D^{m-1}_{s_{m-1}}(R_{m-1})$ is shown in gray to preserve generality

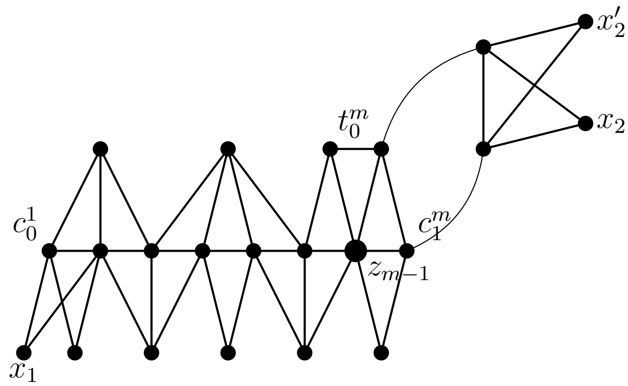


Figure 4.7: Specific example of H_d: $D_5(\{1, 3, 4\}); (t, 1); D_0$ with an x_2-2-path amalgamated with $t_0^2 c_1^2$, and a simplicial vertex added to $c_0^1 c_1^1$

(e) H_e is constructed from $G = D_0^1; (t, \ell); D_0^2$, for $l \geq 2$, by:

(i) Amalgamating an x_1-2-path with $t_0^1 c_0^1$,

(ii) Amalgamating an x_2-2-path with $t_0^2 c_1^2$,

(iii) Adding a false twin, x_1'*, of* x_1*, and*

(iv) Adding a false twin, x_2'*, of* x_2*.*

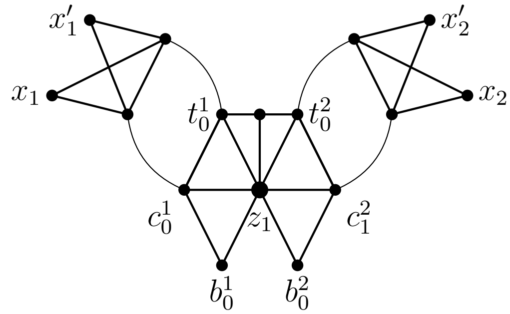


Figure 4.8: Specific example of H_e with $\ell = 2$

(f) H_f *is constructed from* $G = D_0^1; (t, 1); D_0^2$*, by:*

(i) Amalgamating an x_1*-2-path with* $t_0^1 c_0^1$*,*

(ii) Amalgamating an x_2*-2-path with* $t_0^2 c_1^2$*,*

(iii) Adding a false twin, x_1'*, of* x_1*,*

(iv) Adding a false twin, x_2'*, of* x_2*, and*

(v) Adding a simplicial vertex adjacent to $t_0^1 t_0^2$*.*

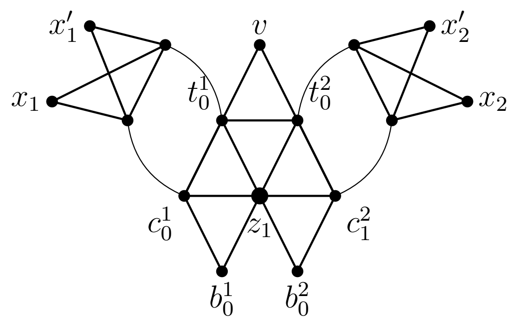


Figure 4.9: Example of H_f

(g) H_g *is constructed from*
$D_0^1; (t, \ell_1); D_{s_2}^2(R_2); (x_2, \ell_2); ....; (x_{m-2}, \ell_{m-2}); D_{s_{m-1}}^{m-1}(R_{m-1}); (t, \ell_{m-1}); D_0^m,\ m \geq 3,$
by:

(i) Amalgamating an x_1*-2-path with* $t_0^1 c_0^1$*,*

(ii) Amalgamating an x_2*-2-path with* $t_0^m c_1^m$*,*

(iii) Adding a false twin, x_1'*, of* x_1*, and*

(iv) Adding a false twin, x_2'*, of* x_2*.*

OR

H_g *is constructed from*
$D_0^1; (t, \ell_1); D_{s_2}^2(R_2); (x_2, \ell_2); ....; (x_{m-2}, \ell_{m-2}); D_{s_{m-1}}^{m-1}(R_{m-1}); (b, \ell_{m-1}); D_0^m,\ m \geq 3,$
by:

(i) Amalgamating an x_1*-2-path with* $t_0^1 c_0^1$*,*

(ii) Amalgamating an x_2*-2-path with* $b_0^m c_1^m$*,*

(iii) Adding a false twin, x_1'*, of* x_1*, and*

(iv) Adding a false twin, x_2'*, of* x_2*.*

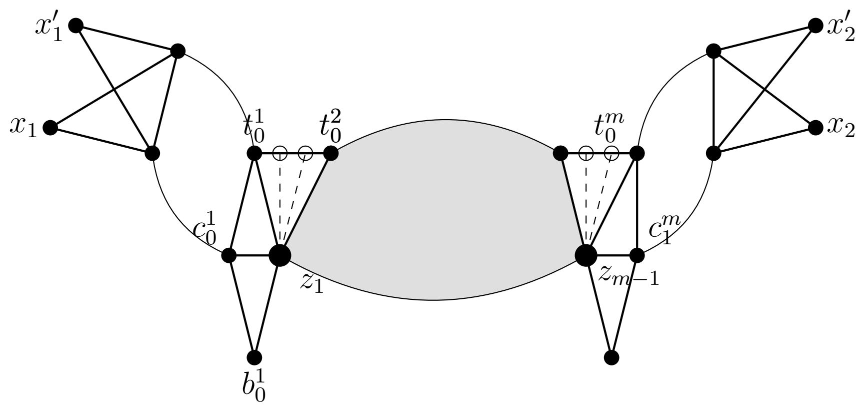


Figure 4.10: A general example of H_g:
$D_0^1; (t, \ell_1); D_{s_2}^2(R_2); (x_2, \ell_2); ....; D_{s_{m-1}}^{m-1}(R_{m-1}); (t, \ell_{m-1}); D_0^m$
$m \geq 3$, with amalgamated x_1 and x_2-2-paths
where $D_{s_2}^2(R_2); (x_2, \ell_2); ....; (x_{m-2}, \ell_{m-2}); D_{s_{m-1}}^{m-1}(R_{m-1})$ with $z_1 = c_0^2$
and $z_{m-1} = c_{s_{m-1}+1}^{m-1}$, is shown in gray to preserve generality

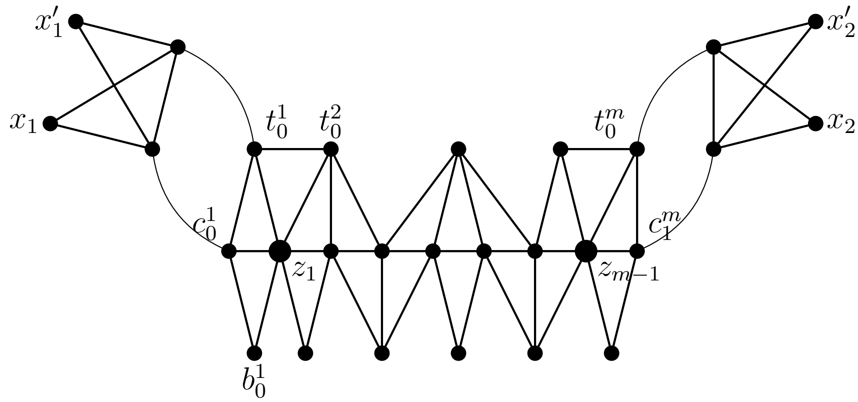


Figure 4.11: Specific example of H_g: $D_0; (t, 1); D_5(\{1, 3, 4\}); (t, 1); D_0$ by amalgamating an x_1-2-path with $t_0^1 c_0^1$, amalgamating an x_2-2-path with $t_0^m c_1^m$, and adding false twins, x'_1, x'_2 of x_1, x_2

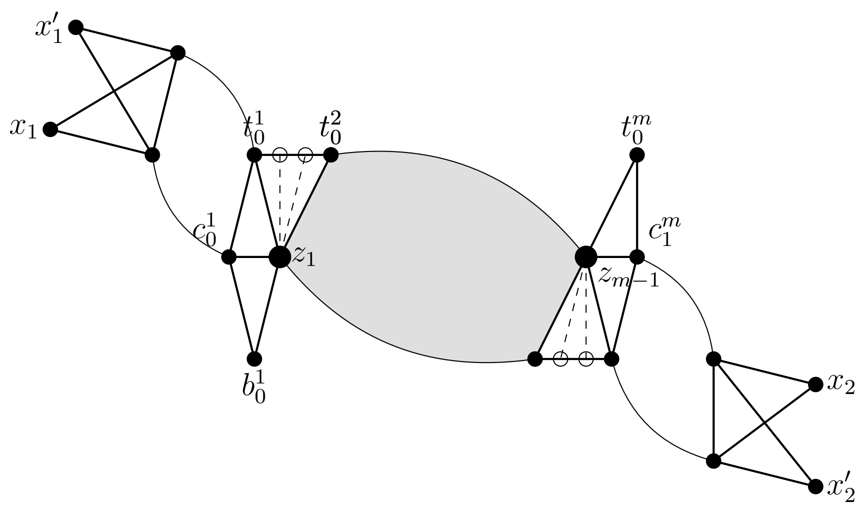


Figure 4.12: A general example of H_g:
$D_0^1; (t, \ell_1); D_{s_2}^2(R_2); (x_2, \ell_2); ....; D_{s_{m-1}}^{m-1}(R_{m-1}); (b, \ell_{m-1}); D_0^m$
$m \geq 3$, with amalgamated x_1 and x_2-2-paths.
where $D_{s_2}^2(R_2); (x_2, \ell_2); ....; (x_{m-2}, \ell_{m-2}); D_{s_{m-1}}^{m-1}(R_{m-1})$ with $z_1 = c_0^2$ and $z_{m-1} = c_{s_{m-1}+1}^{m-1}$, is shown in gray to preserve generality

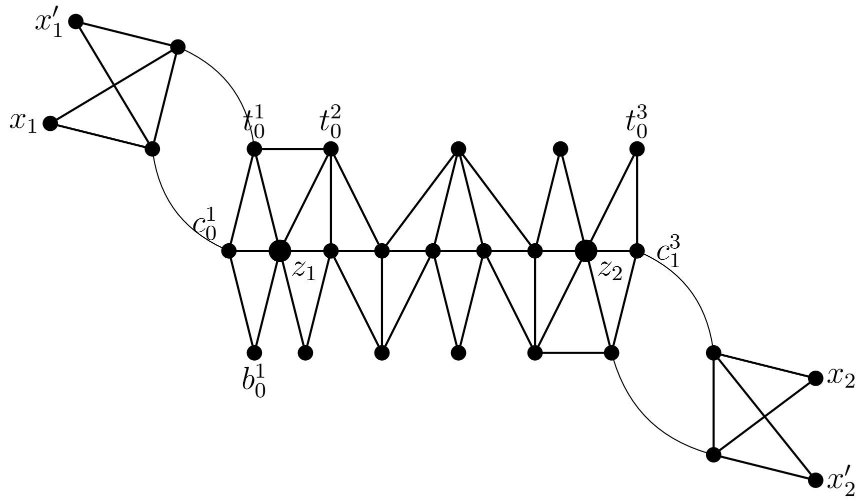


Figure 4.13: An example of H_g: $D_0; (t, 1); D_5(\{1, 3, 4, 5\}); (b, 1); D_0$ with amalgamated x_1 and x_2-2-paths

Note that in $\mathscr{H}$, all graphs have at least two 3-edges. In this section we will be discussing 2-trees which contain at least two 3-edges, but no t-edge for $t \geq 4$. From Lemma 1.2.24, if G is a 2-tree which contains a t-edge for $t \geq 4$, then G does not contain a Hamiltonian path. Furthermore, from Lemma 1.2.23 if G is a 2-tree which only contains t-edges for $t \leq 2$, then G is 1-tough and hence contains a Hamiltonian path. 2-trees with exactly one 3-edge and no t-edges for $t \geq 4$ have a Hamiltonian path, by Lemma 4.1.2 below.

Lemma 4.1.2. *If G is a 2-tree which contains exactly one 3-edge and no t-edges for $t \geq 4$, then G has a Hamiltonian path.*

Proof. Let ab be the 3-edge in G. Let C_1, C_2, C_3 be the components of $G - \{a, b\}$. From [24], $G[C_1 \cup \{a, b\}]$ is a 2-tree, and since G contains no other 3-edges and no t-edges for $t \geq 4$, then it is also 1-tough. Hence, $G[C_1 \cup \{a, b\}]$ contains a Hamiltonian cycle C which contains all 1-edges in $G[C_1 \cup \{a, b\}]$. Since ab is a 1-edge in $G[C_1 \cup \{a, b\}]$, then ab lies on C, so $G[C_1 \cup \{a, b\}]$ has an (a, b)-Hamiltonian path, P. $G[C_1 \cup C_2 \cup \{a, b\}]$ is also a 1-tough 2-tree, so there is a b-Hamiltonian path, P', in $G[C_1 \cup C_2 \cup \{a, b\}]$. Taking $P - a$ followed by P' yields a Hamiltonian path in G. □

Lemma 4.1.3. *Let G be a 1-tough 2-tree with tough path $P = (v_1, v_2, ..., v_{n-1}, v_n)$. If H is constructed by adding a simplicial vertex adjacent to $v_i v_{i+1}$ and a simplicial vertex adjacent to $v_j v_{j+1}$, $i < j$, then H does not contain a Hamiltonian path.*

Proof. Let $S_{v_i,v_{j+1}} = \{v_i, v_{i+1}, ..., v_j, v_{j+1}$. Since P is a tough path, $G - S_{v_i,v_{j+1}} = |S_{v_i,v_{j+1}}|$. Then $c(H - S_{v_i,v_{j+1}}) = |S_{v_i,v_{j+1}}| + 2$ and hence $s(H) \geq 2$ and so H does not have a Hamiltonian path. □

Since in Theorem 4.1.15, we assume scattering number at most one, we do not include in $\mathscr{H}$, $\mathscr{F}^2$, or $\mathscr{I}$, graphs which have the properties of Lemma 4.1.3. However, in the cases of H_d, H_e, H_f, and H_g, if the x_2-2-path, and likewise x_1-2-path, that is amalgamated to our graphs is a diamond with simplicial vertex x_2, then the graph produced will have scattering number at least two. In the future, we would like to characterize the 2-trees which have scattering number two or more, such that we could prove characterization theorems for HP, 1HP, and 2HP on 2-trees which rely only on a forbidden family and do not include scattering number conditions.

Corollary 4.1.4. *If H is constructed from an ℓ-string of diamonds, by adding two simplicial vertices, each adjacent to a different edge on the central path, then H does not contain a Hamiltonian path.*

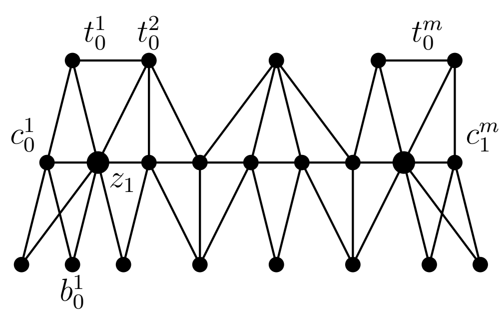


Figure 4.14: An example of Lemma 4.1.3: $D_0; (t, 1); D_5(\{1, 3, 4, 5\}); (t, 1); D_0$ with a simplicial vertex added to $c_0^1 z_1$ and a simplicial vertex added to $z_2 c_1^m$

Lemma 4.1.5. *The graph H_a does not have a Hamiltonian path.*

Proof. Suppose G has a Hamiltonian path, P. By assumption, $c(G - \{a, b\}) = 3$, and cd and ef are in the same component of $G - \{a, b\}$. Hence, at least one endpoint, x_1, of P must lie in a different component of $G - \{a, b\}$ than cd and ef. Likewise at least one endpoint, x_2, of P must lie in a different component of $G - \{c, d\}$ than ab and ef, and at least one endpoint, x_3 of P must lie in a different component of $G - \{e, f\}$ than ab and cd. But since x_1 is in a different component of $G - \{a, b\}$ than cd and

ef, then x_1 is in the same component as ab in $G - \{c, d\}$ and in $G - \{e, f\}$, and so $x_1 \neq x_2, x_3$. Similarly, $x_2 \neq x_3$, and P must have three distinct endpoints. Hence G does not have a Hamiltonian path. □

Lemma 4.1.6. *The graph H_b does not have a Hamiltonian path.*

Proof. Let ab and cd be the only two 3-edges in G. Let s^1_{ab} be a simplicial vertex adjacent to ab and s^1_{cd} be a simplicial vertex adjacent to cd. Then $G - \{s^1_{ab}, s^1_{cd}\}$ is a 1-tough 2-tree. Furthermore, since ab was adjacent to four simplicial vertices and ab is not incident to cd, then in $G - \{s^1_{ab}, s^1_{cd}\}$, a and b are each adjacent to two simplicial vertices, none of which can be c or d. Hence, $(G - \{s^1_{ab}, s^1_{cd}\}, a, c)$, $(G - \{s^1_{ab}, s^1_{cd}\}, a, d)$, $(G - \{s^1_{ab}, s^1_{cd}\}, b, c)$, and $(G - \{s^1_{ab}, s^1_{cd}\}, b, d)$ all contain an induced forbidden sub-2-tree $F^1_a \in \mathscr{F}^1$ from Chapter 2. Thus, $G - \{s^1_{ab}, s^1_{cd}\}$ does not have an (a, c), (a, d), (b, c), or (b, d)-Hamiltonian path. Hence, from Lemma 1.2.25, G does not contain a Hamiltonian path. □

Lemma 4.1.7. *The graph H_c does not have a Hamiltonian path.*

Proof. Let ab and cd be 3-edges which lie in different components of $G - \{e, f\}$, and suppose that G contains a Hamiltonian path, P. Then $c(G - \{a, b\}) = 3$, and cd and ef are in the same component of $G - \{a, b\}$. Hence, at least one endpoint of P must lie in a different component of $G - \{a, b\}$ than cd and ef. Likewise at least one endpoint of P must lie in a different component of $G - \{c, d\}$ than ab and ef. Let u, v, w be the simplicial vertices adjacent to e. None of u, v, w can be an endpoint of P as they will either be in the same component of $G - \{a, b\}$ and $G - \{c, d\}$ as ef or they will be one of $\{a, b, c, d\}$. Thus, on P, u, v, w must all be preceded or followed by e. But then e must appear on P at least twice, and hence G does not have a Hamiltonian path. □

Lemma 4.1.8. *The graph H_d does not have a Hamiltonian path.*

Proof. Suppose x_2, x'_2 are adjacent to cd and x'_1 the simplicial vertex which was added to G which was made adjacent $c^1_0 c^1_1$. Suppose $cd = t^m_0 c^m_1$, and S_G is the set of all vertices on the central path of G. Then, $c(H - S_G) = |S_G| + 1$ and since x_2 and x'_2 are adjacent to $t^m_0 c^m_0$, then, $c(H - (S_G \cup \{t^m_0\})) = |S_G \cup \{t^m_0\}| + 2$, and H has scattering number at least two and does not have a Hamiltonian path. Otherwise, by Lemma 1.2.25, H has a Hamiltonian path iff $H - \{x'_1, x'_2\}$ has a (c^1_0, c), (c^1_0, d), (c^1_1, c), or (c^1_1, d)-Hamiltonian path. But no such path exists in $H - \{x'_1, x'_2\}$ by Theorem 3.1.24, since

$(H-\{x_1', x_2'\}, c_0^1, c)$, $(H-\{x_1', x_2'\}, c_0^1, d)$, $(H-\{x_1', x_2'\}, c_1^1, c)$, and $(H-\{x_1', x_2'\}, c_0^1, d)$ have induced subtrees from $F_a^1 \in \mathscr{F}^1$ or $F_c^1 \in \mathscr{F}^1$. So H does not have a Hamiltonian path. □

Lemma 4.1.9. *The graph H_e does not have a Hamiltonian path.*

Proof. Suppose x_1, x_1' are adjacent to ab and x_2, x_2' are adjacent to cd. If $ab = t_0^1 c_0^1$, $cd = t_0^2 c_1^2$, and $S = \{t_0^1, c_0^1, t_0^2, c_1^2, z_1\}$, then $c(H-S) = 7 = |S| + 2$ and hence H has scattering number at least two and does not have a Hamiltonian path. Otherwise, by Lemma 1.2.25, H has a Hamiltonian path iff $H - \{x_1', x_2'\}$ has a (a, c), (a, d), (b, c), or (b, d)-Hamiltonian path. But no such path exists in $H - \{x_1', x_2'\}$ by Theorem 3.1.24, since $(H-\{x_1', x_2'\}, a, c)$, $(H-\{x_1', x_2'\}, a, d)$, $(H-\{x_1', x_2'\}, b, c)$, and $(H-\{x_1', x_2'\}, b, d)$ have induced subtrees from $F_f^1 \in \mathscr{F}^1$. So H does not have a Hamiltonian path.

□

Lemma 4.1.10. *The graph H_f does not have a Hamiltonian path.*

Proof. Suppose x_1, x_1' are adjacent to ab and x_2, x_2' are adjacent to cd. If $ab = t_0^1 c_0^1$, $cd = t_0^2 c_1^2$, and $S = \{t_0^1, c_0^1, t_0^2, c_1^2, z_1\}$, then $c(H-S) = 7 = |S| + 2$ and hence H has scattering number at least two and does not have a Hamiltonian path. Otherwise, by Lemma 1.2.25, H has a Hamiltonian path iff $H - \{x_1', x_2'\}$ has a (a, c), (a, d), (b, c), or (b, d)-Hamiltonian path. But no such path exists in $H - \{x_1', x_2'\}$ by Theorem 3.1.24, since $(H-\{x_1', x_2'\}, a, c)$, $(H-\{x_1', x_2'\}, a, d)$, $(H-\{x_1', x_2'\}, b, c)$, and $(H-\{x_1', x_2'\}, b, d)$ have induced subtrees from $F_e^1 \in \mathscr{F}^1$. So H does not have a Hamiltonian path.

□

Lemma 4.1.11. *The graph H_g does not have a Hamiltonian path.*

Proof. Suppose x_1, x_1' are adjacent to ab and x_2, x_2' are adjacent to cd. Suppose $ab = t_0^1 c_0^1$ and $cd = t_0^m c_1^m$, and S_G is the set of all vertices on the central path of G. Then, $c(H - S_G) = |S_G|$ and since x_1 and x_1' are adjacent to $t_0^1 c_0^1$, x_2 and x_2' are adjacent to $t_0^m c_0^m$, then, $c(H - (S_G \cup \{t_0^1, t_0^m\})) = |S_G \cup \{t_0^1, t_0^m\}| + 2$, and H has scattering number at least two and does not have a Hamiltonian path. Otherwise, by Lemma 1.2.25, H has a Hamiltonian path iff $H - \{x_1', x_2'\}$ has a (a, c), (a, d), (b, c), or (b, d)-Hamiltonian path. But no such path exists in $H - \{x_1', x_2'\}$ by Theorem 3.1.24, since $(H-\{x_1', x_2'\}, a, c)$, $(H-\{x_1', x_2'\}, a, d)$, $(H-\{x_1', x_2'\}, b, c)$, and $(H-\{x_1', x_2'\}, b, d)$ have induced subtrees from $F_c^1 \in \mathscr{F}^1$. So H does not have a Hamiltonian path. □

Similar to the reduced graph of a 2-tree with scattering number one with fixed endpoints, we will create a reduced graph of a 2-tree with scattering number one, without fixed endpoints, as follows.

Definition 4.1.12. *Given a 2-tree, G with $s(G) = 1$, then the* ***reduced graph of*** *G, is formed using the following algorithm:*

1. *For every 3-edge ab with components of $G - \{a, b\}$, $C^1_{ab}, C^2_{ab}, C^3_{ab}$, if $G[C^i_{ab} \cup \{a, b\}]$ is 1-tough then replace C^i_{ab} with a simplicial vertex adjacent to ab.*
2. *Let $w \in S_1(G)$, and x, y the neighbors of w. If xy is not a 3-edge, remove w and turn the edge xy into a forced edge.*
3. *Repeat (2) for all $w \in S_1(G)$. Define the resulting graph to be G_1.*
4. *For $i \geq 2$, let $S_i = S_1(G_{i-1})$ where G_{i-1} is the graph formed by repeating (2) for $G = G_{i-1}$ and for all $w \in S_1(G_{i-1})$.*

Repeat (4) for all $i = 2, 3, ..., j$ for j such that $S_j = \emptyset$ or for all $s \in S_j$, $N(s)$ is a 3-edge. This is the reduced graph of G.
For F the set of forced edges, let (H, F) denote the reduced graph of G, for G a 2-tree containing at least one 3-edge.

Since simplicial vertices in 2-trees are not adjacent [7], when we remove the vertices in each S_i, regardless of order, we will end up with the same graph.

We will form the corresponding caterpillar representation of G as in Chapter 2.

Definition 4.1.13. *Let G be a 2-tree with $s(G) = 1$ and (H, F) be the reduced graph of G. The* ***caterpillar representation****, H', of G is created by adding $|F|$ simplicial vertices to (H, F), making each vertex adjacent to exactly one forced edge, and changing all forced edges back to regular edges.*

Lemma 4.1.14. *Let G be a 2-tree with $s(G) = 1$, (H, F) be the reduced graph of G, and H' the caterpillar representation of G. Then the following are equivalent:*

1. *G has a Hamiltonian path,*
2. *H' has a Hamiltonian path, and*
3. *(H, F) has a Hamiltonian path which uses all of the edges in F.*

Proof. (1)$\Longrightarrow$ (2)

Suppose H' does not have a Hamiltonian path. Since H' is an induced sub-2-tree of G, then by Corollary 2.1.10, G does not have a Hamiltonian path.

(2) $\Longrightarrow$ (3)

Suppose H' has a Hamiltonian path, P. Let v be a simplicial vertex with neighbors u and w, such that uw is not a 3-edge. Then $P = (x_1, ..., u, v, w, ..., x_2)$ or $P = (x_1, ..., w, v, u, ..., x_2)$. Furthermore, because H' is a 2-tree, then $uw \in E(H')$, and from the reduction algorithm $uw \in F$. Replacing (u, v, w) or (w, v, u) by (u, w) in P, then P is a Hamiltonian path using exactly one forced edge. Repeating this process for all $s \in S_1\ell_(H')$, such that s is not adjacent to a 3-edge, then P will be a Hamiltonian path in (H, F).

(3) $\Longrightarrow$ (1) Suppose (H, F) has a Hamiltonian path, P, which uses all of the edges in F. Let ab be a 3-edge in G with components of $G - \{a, b\}$, $C^1_{ab}, C^2_{ab}, C^3_{ab}$, where $G[C^i_{ab} \cup \{a, b\}]$ is 1-tough. In (H, F), C^i_{ab} has been replaced by the simplicial vertex, v^i_{ab}. Since $G[C^i_{ab} \cup \{a, b\}]$ is 1-tough, then $G[C^i_{ab} \cup \{a, b\}]$ has a Hamiltonian cycle, C using all 1-edges in $G[C^i_{ab} \cup \{a, b\}]$. Hence, since ab is a 1-edge in $G[C^i_{ab} \cup \{a, b\}]$, then there is an (a, b)-Hamiltonian path P' in $G[C^i_{ab} \cup \{a, b\}]$. So, if v^i_{ab} is on the interior of P in (H, F), then we can replace (a, v^i_{ab}, b) on P with P'. If v^i_{ab} is an endpoint of P in (H, F), then we can replace (v^i_{ab}, b) or (v^i_{ab}, a) on P with $P' - a$ or $P' - b$, respectively. Now, consider $xy \in F$. In G, xy is incident to at least one vertex, v, which is not in (H, F) so that $c(G - \{x, y\}) = 2$. Let C_v be the component of $G - \{x, y\}$ which contains v. From [24], $G[C_v \cup xy]$ is a 2-tree, and from the reduction algorithm, $G[C_v \cup xy]$ must be 1-tough and so it contains a Hamiltonian cycle C. In $G[C_v \cup xy]$, xy is a 1-edge and hence lies on C. Thus, there is a Hamiltonian path, P'', in $G[C_v \cup xy]$ from x to y, and we can replace xy in P with P'. Repeating these processes for all $f \in F$ and all 3-edges, cd and all C^i_{cd} such that $G[C^i_{cd} \cup \{c, d\}]$ is 1-tough, will yield a Hamiltonian path in G. □

Theorem 4.1.15. *If G is a 2-tree, then G has a Hamiltonian path iff $s(G) \leq 1$ and G does not contain any $H \in \mathscr{H}$ as an induced sub-2-tree.*

Proof. $\Longrightarrow$

If $s(G) \geq 2$, then G is not 1-path-tough, and G does not contain a Hamiltonian path.

1. If $G = H_a$, then G does not have a Hamiltonian path by Lemma 4.1.5. If G contains H_a as an induced sub-2-tree, then G does not have a Hamiltonian path by Corollary 2.1.10.

2. If $G = H_b$, then G does not have a Hamiltonian path by Lemma 4.1.6. If G contains H_b as an induced sub-2-tree, then G does not have a Hamiltonian path by Corollary 2.1.10.

3. If $G = H_c$, then G does not have a Hamiltonian path by Lemma 4.1.7. If G contains H_c as an induced sub-2-tree, then G does not have a Hamiltonian path by Corollary 2.1.10.

4. If $G = H_d$, then G does not have a Hamiltonian path by Lemma 4.1.8. If G contains H_d as an induced sub-2-tree, then G does not have a Hamiltonian path by Corollary 2.1.10.

5. If $G = H_e$, then G does not have a Hamiltonian path by Lemma 4.1.9. If G contains H_e as an induced sub-2-tree, then G does not have a Hamiltonian path by Corollary 2.1.10.

6. If $G = H_f$, then G does not have a Hamiltonian path by Lemma 4.1.10. If G contains H_f as an induced sub-2-tree, then G does not have a Hamiltonian path by Corollary 2.1.10.

7. If $G = H_g$, then G does not have a Hamiltonian path by Lemma 4.1.11. If G contains H_g as an induced sub-2-tree, then G does not have a Hamiltonian path by Corollary 2.1.10.

$\Longleftarrow$

Suppose G does not have a Hamiltonian path, but that $s(G) \leq 1$. Since $s(G) \leq 1$, then G contains no t-edges for $t \geq 4$. If G contains m 3-edges for $m \leq 1$, then G has a Hamiltonian path. So G has m 3-edges for $m \geq 2$. Let H' be the caterpillar representation of G. Then $s(H') \leq 1$ and H' does not have a Hamiltonian path by Lemma 4.1.14. Suppose that H' does not contain H_a. Denote the 3-edges in H', $S_i = s_i s_i'$ for all $1 \leq i \leq m$. Then the 3-edges in H' can be ordered $S_1, S_2, ...S_m$ so that in $H' - S_1$, all $s_i, s_i' \neq s_1, s_1'$ are in the same component, in $H' - S_m$, all $s_i, s_i' \neq s_m, s_m'$ are in the same component, and such that for all $i \in \{1, 2, ..., m-2\}$, s_i and s_{i+2} are in different components of $H' - S_{i+1}$. From the reduction algorithm, S_1 and S_m are each adjacent to two simplicial vertices. Furthermore, since $c(G - S_i) = 3$, then, if H' has a Hamiltonian path, one of the simplicial vertices adjacent to S_1 must be an endpoint of the path, and likewise, one of the simplicial vertices adjacent to S_m must be an endpoint of the path. Without loss of generality, label one of the simplicial vertices adjacent to S_1, x_1, and one of the simplicial vertices adjacent to S_m, x_2. So since H' does not have a Hamiltonian path, then H' does not have an (x_1, x_2)-Hamiltonian path. So, by Theorem 3.2.10, (H', x_1, x_2) must contain an $F^2 \in \mathscr{F}^2$ as an induced sub-2-tree. Also, since x_1 and x_2 are simplicial, then (H', x_1, x_2) must contain F_a^2, F_b^2, F_c^2, F_d^2, F_f^1, F_e^1, or F_d^1. Adding a false twin of x_1 and x_2 and removing the labels, we will get the forbidden induced sub-2-trees for H' without fixed endpoints. Using this process on F_f^1 forms H_e, on F_e^1 forms H_f, on F_d^1 forms H_g, on F_a^2 forms H_a, and on F_c^2 forms H_c^2. For F_d^2 and F_b^2, we can leave x_1 and just remove the label, as the x_1 is amalgamated with a 3-edge and hence already forcing x_1 or the other simplicial vertex as an end. Using this process on F_d^2 forms H_d^2, and on F_b^2 forms H_b . □

4.2 1HP

Definition 4.2.1. *Define* $\mathscr{I} = \{I_a, I_b, I_c, I_d, I_e, I_f, I_g, I_h, I_i, I_j\}$ *where:*

(a) I_a *is a 2-tree with vertex* x_2*, which contains two 3-edges, ab and cd, which are not incident, such that:*

(i) cd *and* x_2 *are in the same component of* $G - \{a, b\}$*, and*

(ii) ab *and* x_2 *are in the same component of* $G - \{c, d\}$*, or*

(iii) $x_2 \in \{a, b, c, d\}$.

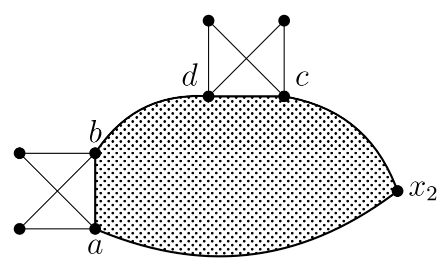


Figure 4.15: A general example of I_a where the dotted section of the graph represents any 2-tree with scattering number at most one to preserve generality

(b) I_b is a 2-tree which contains exactly one 3-edge, ab, such that:

(i) $N(ab) - x_2$ contains two simplicial vertices,

(ii) $N(a) - x_2$ contains two simplicial vertices, and

(iii) $N(b) - x_2$ contains two simplicial vertices.

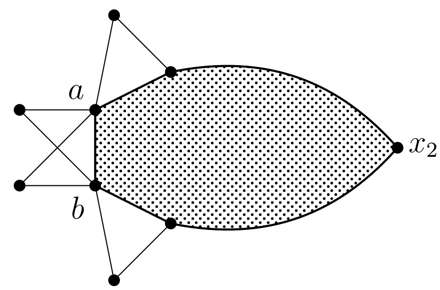


Figure 4.16: General example of I_b where the dotted section of the graph represents any 2-tree with scattering number at most one to preserve generality

(c) I_c is a 2-tree which contains at least two 3-edges such that for one of the 3-edges, ef:

(i) One component of $G - \{e, f\}$ contains a 3-edge, which is in a different component of $G - \{e, f\}$ than x_2, and

(ii) e is adjacent to three simplicial vertices in $G - x_2$.

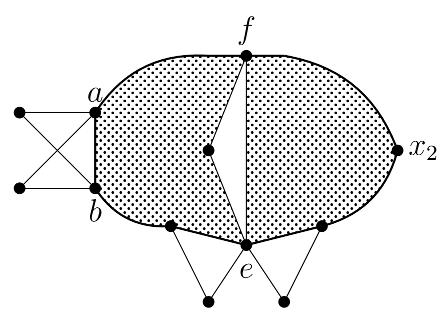


Figure 4.17: General example of I_c where the dotted section of the graph represents any 2-tree with scattering number at most one to preserve generality

(d) *I_d is constructed from an ℓ string of diamonds by adding a simplicial vertex adjacent to $c_0^1 c_1^1$ and where $x_2 = c_{s_m+1}^m$.*

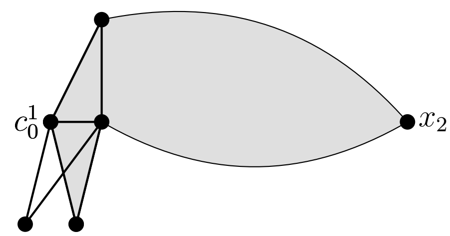


Figure 4.18: A general example of I_d:
$D_{s_1}^1(R_1); (x_1, \ell_1); ....; (x_{m-2}, \ell_{m-2}); D_{s_{m-1}}^{m-1}(R_{m-1}); (x_{m-1}, \ell_{m-1}); D_{s_m}^m(R_m)$
with added simplicial vertex adjacent to $c_0^1 c_1^1$, where $x_2 = c_{s_m+1}^m$, and
where $D_{s_1}^1(R_1); (x_1, \ell_1); ....; (x_{m-2}, \ell_{m-2}); D_{s_{m-1}}^{m-1}(R_{m-1}); (x_{m-1}, \ell_{m-1}); D_{s_m}^m(R_m)$,
is shown in gray to preserve generality

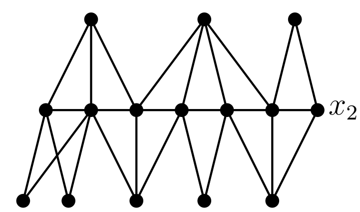


Figure 4.19: Specific example of I_d: $D_5(\{1, 3, 4, 5\})$ with $x_2 = c_6^1$

(e) *I_e is constructed from $D_{s_1}^1(R_1); (x_1, \ell_1); ....; (x_{m-2}, \ell_{m-2}); D_{s_{m-1}}^{m-1}(R_{m-1}); (t, \ell_{m-1}); D_0^m$, $m \geq 2$, by amalgamating an x_2-2-path with $t_0^m c_1^m$ and adding a simplicial vertex adjacent to $c_0^1 c_1^1$.*

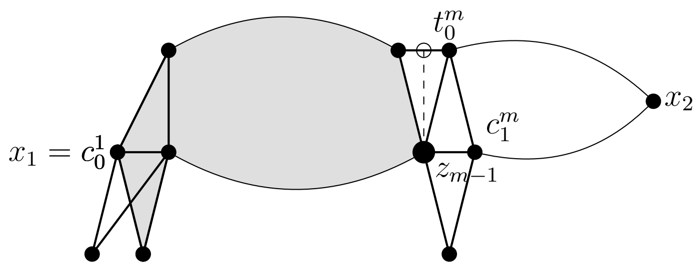


Figure 4.20: A general example of I_e:
$D^1_{s_1}(R_1); (x_1, \ell_1); ....; (x_{m-2}, \ell_{m-2}); D^{m-1}_{s_{m-1}}(R_{m-1}); (t, \ell_{m-1}); D^m_0$
$m \geq 2$, with an amalgamated x_2-2-path and an added simplicial vertex adjacent to $c^1_0c^1_1$
where $D^1_{s_1}(R_1); (x_1, \ell_1); ....; (x_{m-2}, \ell_{m-2}); D^{m-1}_{s_{m-1}}(R_{m-1})$, with $x_1 = c^1_0$ and $z_{m-1} = c^{m-1}_{s_{m-1}+1}$, is shown in gray to preserve generality

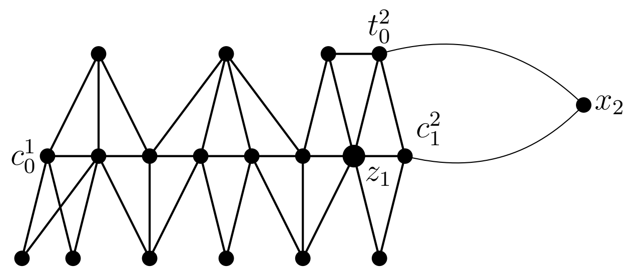


Figure 4.21: Specific example of I_e: $D_5(\{1, 3, 4, 5\}); (t, 1); D_0$ with an added simplicial vertex adjacent to $c^1_0c^1_1$ and an x_2-2-path amalgamated with $t^2_0c^2_1$

(f) I_f *is constructed from*
$D^1_{s_1}(R_1); (x_1, \ell_1); ....; (x_{m-2}, \ell_{m-2}); D^{m-1}_{s_{m-1}}(R_{m-1}); (t, \ell_{m-1}); D^m_0$, $m \geq 2$, *with* $x_1 = c^1_0$, *by amalgamating an* x_2*-2-path with* $t^m_0c^m_1$ *and adding a false twin* x'_2 *of* x_2*.*

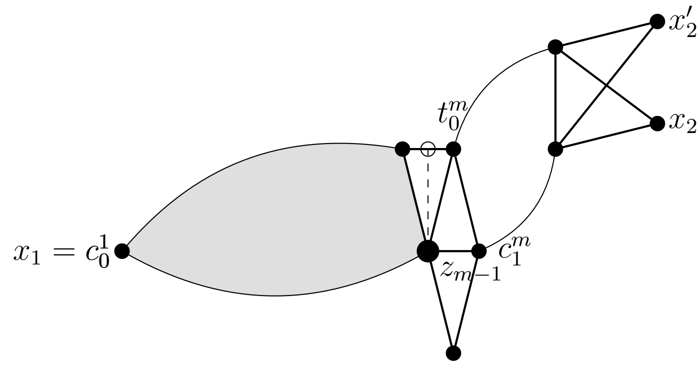


Figure 4.22: A general example of
I_f: $D^1_{s_1}(R_1); (x_1, \ell_1); ....; (x_{m-2}, \ell_{m-2}); D^{m-1}_{s_{m-1}}(R_{m-1}); (t, \ell_{m-1}); D^m_0$
$m \geq 2$, with an amalgamated x_2-2-path, such that $x_1 = c^1_0$, and where $D^1_{s_1}(R_1); (x_1, \ell_1); ....; (x_{m-2}, \ell_{m-2}); D^{m-1}_{s_{m-1}}(R_{m-1})$, with $x_1 = c^1_0$ and $z_{m-1} = c^{m-1}_{s_{m-1}+1}$, is shown in gray to preserve generality

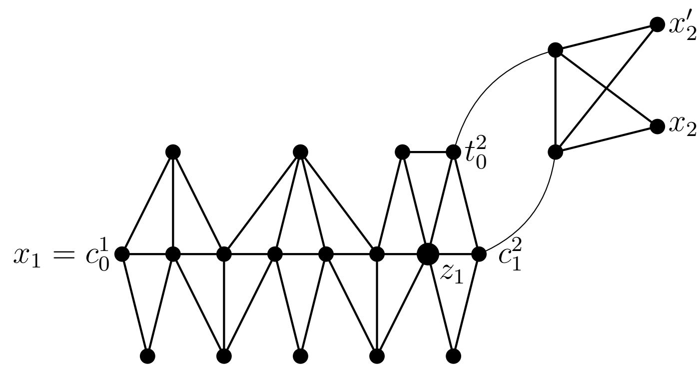


Figure 4.23: Specific example of I_f: $D_5(\{1, 3, 4, 5\}); (t, 1); D_0$ with $x_1 = c^1_0$ and an x_2-2-path amalgamated with $t^2_0 c^2_1$

(g) I_g *is constructed from* $D^1_0; (t, \ell); D^2_0$*, for* $l \geq 2$ *by:*

(i) Amalgamating an x_1*-2-path with* $t^1_0 c^1_0$*,*

(ii) Amalgamating an x_2*-2-path with* $t^2_0 c^2_1$*, and*

(iii) Adding a false twin x'_1 *of* x_1*.*

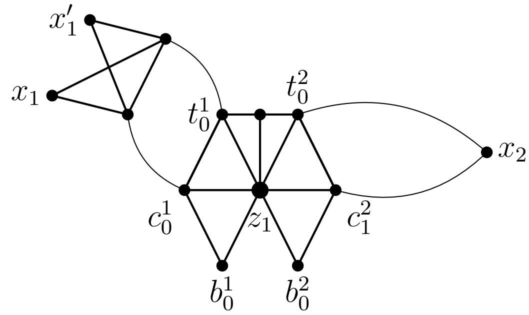


Figure 4.24: Example of I_g

(h) I_h *is constructed from* $D_0^1; (t, 1); D_0^2$ *by:*

(i) Amalgamating an x_1*-2-path with* $t_0^1 c_0^1$*,*

(ii) Amalgamating an x_2*-2-path with* $t_0^2 c_1^2$*,*

(iii) Adding a false twin x_1' *of* x_1*, and*

(iv) Adding a simplicial vertex adjacent to $t_0^1 t_0^2$*.*

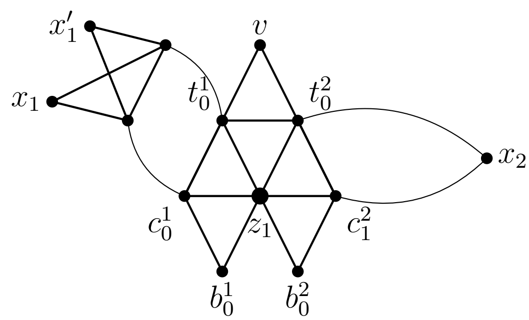


Figure 4.25: Example of I_h.

(i) I_i *is constructed from*
$D_0^1; (t, \ell_1); D_{s_1}^2(R_1); (x_1, \ell_2); ....; (x_{m-1}, \ell_{m-2}); D_{s_{m-2}}^{m-1}(R_{m-2}); (t, \ell_{m-1}); D_0^m,\ m \geq 3,$
by:

(i) Amalgamating an x_1*-2-path with* $t_0^1 c_0^1$*,*

(ii) Amalgamating an x_2*-2-path with* $t_0^m c_1^m$*, and*

(iii) Adding a false twin x_1' of x_1.

OR

I_i is constructed from
$D_0^1; (t, \ell_1); D_{s_1}^2(R_1); (x_1, \ell_2); ....; (x_{m-1}, \ell_{m-2}); D_{s_{m-2}}^{m-1}(R_{m-2}); (b, \ell_{m-1}); D_0^m,\ m \geq 3,$
by:

(i) Amalgamating an x_1-2-path with $t_0^1 c_0^1$,

(ii) Amalgamating an x_2-2-path with $b_0^m c_1^m$, and

(iii) Adding a false twin x_1' of x_1.

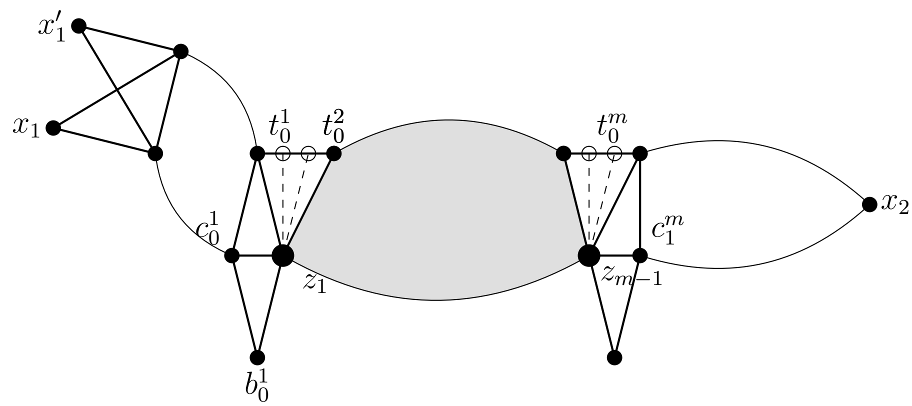


Figure 4.26: A general example of I_i:
$D_0^1; (t, \ell_1); D_{s_2}^2(R_2); (x_2, \ell_2); ....; (x_{m-2}, \ell_{m-2}); D_{s_{m-1}}^{m-1}(R_{m-1}); (t, \ell_{m-1}); D_0^m$
$m \geq 3$, with amalgamated x_1 and x_2-2-paths, and
where $D_{s_2}^2(R_2); (x_2, \ell_2); ....; (x_{m-2}, \ell_{m-2}); D_{s_{m-1}}^{m-1}(R_{m-1})$ with $z_1 = c_0^2$
and $z_{m-1} = c_{s_{m-1}+1}^{m-1}$, is shown in gray to preserve generality

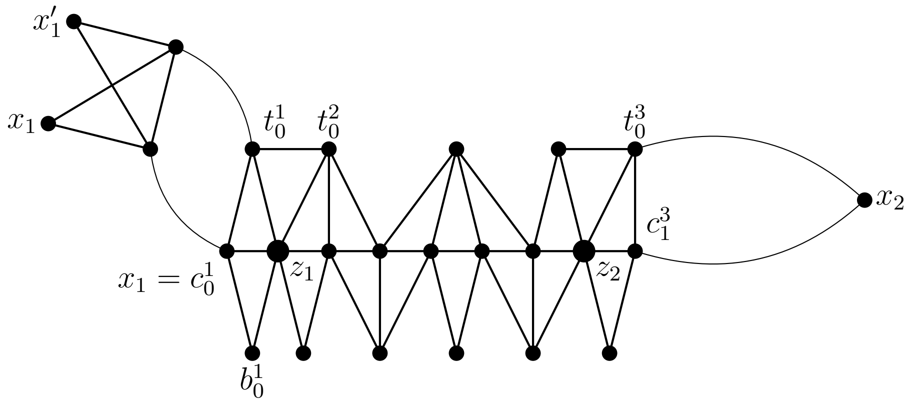


Figure 4.27: Specific example of I_i: $D_0; (t, 1); D_5(\{1, 3, 4, 5\}); (t, 1); D_0$ with $x_1 = c_0^1$ and an x_2-2-path amalgamated with $t_0^2 c_1^2$

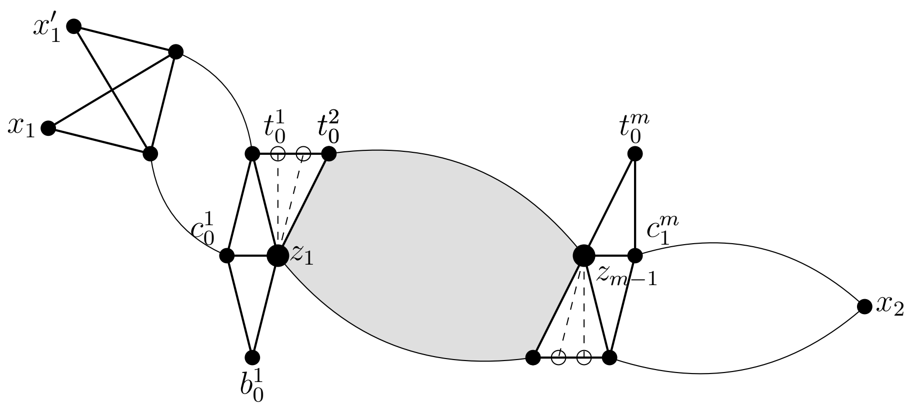


Figure 4.28: A general example of I_i:
$D_0^1; (t, \ell_1); D_{s_2}^2(R_2); (x_2, \ell_2); ....; (x_{m-2}, \ell_{m-2}); D_{s_{m-1}}^{m-1}(R_{m-1}); (b, \ell_{m-1}); D_0^m$
$m \geq 3$, with amalgamated x_1 and x_2-2-paths, and
where $D_{s_2}^2(R_2); (x_2, \ell_2); ....; (x_{m-2}, \ell_{m-2}); D_{s_{m-1}}^{m-1}(R_{m-1})$ with $z_1 = c_0^2$
and $z_{m-1} = c_{s_{m-1}+1}^{m-1}$, is shown in gray to preserve generality

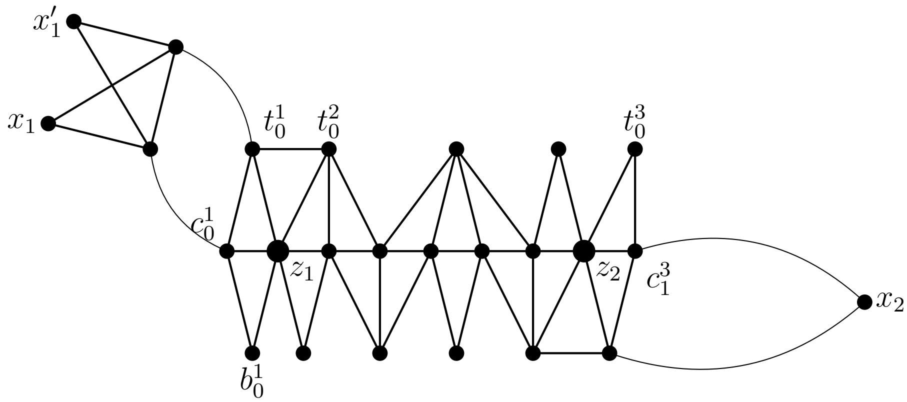


Figure 4.29: An example of I_i: $D_0; (t,1); D_5(\{1,3,4,5\}); (b,1); D_0$ with amalgamated x_1 and x_2-2-paths

(j) I_j is constructed from D_0 by:

(i) Adding a simplicial vertex adjacent to c_0t_0,

(ii) Amalgamating an x_1-2-path with t_0c_1, and

(iii) Adding a false twin, x_1', or x_1.

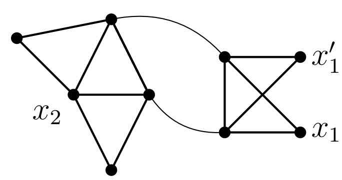


Figure 4.30: Example of I_j

Lemma 4.2.2. *The graph I_a does not have an x_2-Hamiltonian path.*

Proof. Suppose G has an x_2-Hamiltonian path, P. Since $c(G - \{a,b\}) = 3$, then the one endpoint of P, x_1, must be in one of the components of $G - \{a,b\}$ that does not contain cd and x_2. Likewise, one endpoint of P, x_3, must be in one of the components of $G - \{c,d\}$ that does not contain ab and x_2. Clearly, $x_1, x_3 \neq x_2$. Additionally, since x_1 is in a different component of $G - \{a,b\}$ than cd, then it will be in the same component as ab in $G - \{c,d\}$, and hence $x_1 \neq x_3$. Thus, P has three distinct endpoints, a contradiction. □

Lemma 4.2.3. *The graph I_b does not have an x_2-Hamiltonian path.*

Proof. Let v_1, v_2 be the simplicial vertices adjacent to ab. By Lemma 1.2.25, G has an x_2-Hamiltonian path iff $G - v_1$ has an (a, x_2)-Hamiltonian path or $G - v_1$ has a (b, x_2)-Hamiltonian path. However, $(G - v_1, a, x_2)$ and $(G - v_1, b, x_2)$ both contain $F_a^1 \in \mathscr{F}^1$ as an induced sub-2-tree, and hence by Theorem 3.1.24, $G - v_1$ does not have an (a, x_2)-Hamiltonian path or a (b, x_2)-Hamiltonian path. Hence G does not have an x_2-Hamiltonian path. □

Lemma 4.2.4. *The graph I_c does not have an x_2-Hamiltonian path.*

Proof. Let ab be the 3-edge which is in a different component of $G - \{e, f\}$ than x_2, and suppose that G has an x_2-Hamiltonian path, P. Since $c(G - \{a, b\}) = 3$, then the one endpoint of P, x_1, must be in one of the components of $G - \{a, b\}$ that does not contain ef and x_2. Let $u, v, w \neq x_2$ be the simplicial vertices adjacent to e. None of u, v, w can be an endpoint of P as they will either be in the same component of $G - \{a, b\}$ as ef or they will be one of $\{a, b\}$. Thus, on P, u, v, w must all be preceded or followed by e. But then e must appear on P at least twice, and hence G does not have an x_2-Hamiltonian path. □

Lemma 4.2.5. *The graph I_d does not have an x_2-Hamiltonian path.*

Proof. Let v be the simplicial vertex made adjacent to $c_0^1 c_1^1$. Since c_0^1, c_1^1, and $c_{s_m+1}^m$ are all vertices on the central path, then in $H - v$, there are $(c_0^1, c_{s_m+1}^m)$ and $(c_1^1, c_{s_m+1}^m)$-tough paths. Hence in $H - v$, there does not exist a $(c_0^1, c_{s_m+1}^m)$ or $(c_1^1, c_{s_m+1}^m)$-Hamiltonian path. Thus, by Lemma 1.2.25, there is no $x_2 = c_{s_m+1}^m$-Hamiltonian path in H. □

Lemma 4.2.6. *The graph I_e does not have an x_2-Hamiltonian path.*

Proof. Let v_1, v_1' the simplicial vertices adjacent to $c_0^1 c_1^1$. $(H - v_1', c_0^1, x_2)$ and $(H - v_1', c_1^1, x_2)$ have $F_c^1 \in \mathscr{F}^1$ as an induced sub-2-tree, and hence by Theorem 3.1.24, $H - v_1'$ does not have a (c_0^1, x_2)-Hamiltonian path or a (c_1^1, x_2)-Hamiltonian path. Thus, by Lemma 1.2.25, H does not have a x_2-Hamiltonian path. □

Lemma 4.2.7. *The graph I_f does not have an x_2-Hamiltonian path.*

Proof. Suppose x_2, x_2' is adjacent to ab. $(H - x_2', c_0^1, a)$ and $(H - x_2', c_0^1, b)$ have $F_b^1 \in \mathscr{F}^1$ or $F_c^1 \in \mathscr{F}^1$ as an induced sub-2-tree, and hence by Theorem 3.1.24, $H - x_2'$ does not have a (c_0^1, a)-Hamiltonian path or a (c_0^1, b)-Hamiltonian path. Thus, by Lemma 1.2.25, H does not have an $x_2 = c_0^1$-Hamiltonian path. □

Lemma 4.2.8. *The graph I_g does not have an x_2-Hamiltonian path.*

Proof. Suppose x_1, x_1' is adjacent to ab. $(H - x_1', x_2, a)$ and $(H - x_1', x_2, b)$ have $F_b^1 \in \mathscr{F}^1$ or $F_f^1 \in \mathscr{F}^1$ as an induced sub-2-tree, and hence by Theorem 3.1.24, $H - x_1'$ does not have an (x_2, a)-Hamiltonian path or an (x_2, b)-Hamiltonian path. Thus, by Lemma 1.2.25, H does not have an x_2-Hamiltonian path. □

Lemma 4.2.9. *The graph I_h does not have an x_2-Hamiltonian path.*

Proof. Suppose x_1, x_1' is adjacent to ab. $(H - x_1', x_2, a)$ and $(H - x_1', x_2, b)$ have $F_b^1 \in \mathscr{F}^1$ or $F_e^1 \in \mathscr{F}^1$ as an induced sub-2-tree, and hence by Theorem 3.1.24, $H - x_1'$ does not have an (x_2, a)-Hamiltonian path or an (x_2, b)-Hamiltonian path. Thus, by Lemma 1.2.25, H does not have an x_2-Hamiltonian path. □

Lemma 4.2.10. *The graph I_i does not have an x_2-Hamiltonian path.*

Proof. Suppose x_1, x_1' is adjacent to ab. $(H - x_1', x_2, a)$ and $(H - x_1', x_2, b)$ have $F_c^1 \in \mathscr{F}^1$ or $F_d^1 \in \mathscr{F}^1$ as an induced sub-2-tree, and hence by Theorem 3.1.24, $H - x_1'$ does not have an (x_2, a)-Hamiltonian path or an (x_2, b)-Hamiltonian path. Thus, by Lemma 1.2.25, H does not have an x_2-Hamiltonian path. □

Lemma 4.2.11. *The graph I_j does not have an x_2-Hamiltonian path.*

Proof. Suppose x_1, x_1' is adjacent to ab. $(H - x_1', x_2, a)$ and $(H - x_1', x_2, b)$ have $F_a^1 \in \mathscr{F}^1$ or $F_b^1 \in \mathscr{F}^1$ as an induced sub-2-tree, and hence by Theorem 3.1.24, $H - x_1'$ does not have an (x_2, a)-Hamiltonian path or an (x_2, b)-Hamiltonian path. Thus, by Lemma 1.2.25, H does not have an x_2-Hamiltonian path. □

Theorem 4.2.12. *If G is a 2-tree with $x_2 \in V(G)$, then (G, x_2) has an x_2-Hamiltonian path iff $s(G) \leq 1$ and G does not contain any $I \in \mathscr{I}$ as an induced sub-2-tree.*

Proof. $\Longrightarrow$ If $s(G) \geq 2$, then G is not 1-path-tough, and G does not contain a Hamiltonian path.

1. If $(G, x_2) = I_a$, then (G, x_2) does not have an x_2-Hamiltonian path by Lemma 4.2.2. If (G, x_2) contains I_a as an induced sub-2-tree, then (G, x_2) does not have an x_2-Hamiltonian path by Corollary 2.1.11.

2. If $(G, x_2) = I_b$, then (G, x_2) does not have an x_2-Hamiltonian path by Lemma 4.2.3. If (G, x_2) contains I_b as an induced sub-2-tree, then (G, x_2) does not have an x_2-Hamiltonian path by Corollary 2.1.11.

3. If $(G, x_2) = I_c$, then (G, x_2) does not have an x_2-Hamiltonian path by Lemma 4.2.4. If (G, x_2) contains I_c as an induced sub-2-tree, then (G, x_2) does not have an x_2-Hamiltonian path by Corollary 2.1.11.

4. If $(G, x_2) = I_d$, then (G, x_2) does not have an x_2-Hamiltonian path by Lemma 4.2.5. If (G, x_2) contains I_d as an induced sub-2-tree, then (G, x_2) does not have an x_2-Hamiltonian path by Corollary 2.1.11.

5. If $(G, x_2) = I_e$, then (G, x_2) does not have an x_2-Hamiltonian path by Lemma 4.2.6. If (G, x_2) contains I_e as an induced sub-2-tree, then (G, x_2) does not have an x_2-Hamiltonian path by Corollary 2.1.11.

6. If $(G, x_2) = I_f$, then (G, x_2) does not have an x_2-Hamiltonian path by Lemma 4.2.7. If (G, x_2) contains I_f as an induced sub-2-tree, then (G, x_2) does not have an x_2-Hamiltonian path by Corollary 2.1.11.

7. If $(G, x_2) = I_g$, then (G, x_2) does not have an x_2-Hamiltonian path by Lemma 4.2.8. If (G, x_2) contains I_g as an induced sub-2-tree, then (G, x_2) does not have an x_2-Hamiltonian path by Corollary 2.1.11.

8. If $(G, x_2) = I_h$, then (G, x_2) does not have an x_2-Hamiltonian path by Lemma 4.2.9. If (G, x_2) contains I_h as an induced sub-2-tree, then (G, x_2) does not have an x_2-Hamiltonian path by Corollary 2.1.11.

9. If $(G, x_2) = I_i$, then (G, x_2) does not have an x_2-Hamiltonian path by Lemma 4.2.10. If (G, x_2) contains I_i as an induced sub-2-tree, then (G, x_2) does not have an x_2-Hamiltonian path by Corollary 2.1.11.

10. If $(G, x_2) = I_j$, then (G, x_2) does not have an x_2-Hamiltonian path by Lemma 4.2.11. If (G, x_2) contains I_j as an induced sub-2-tree, then (G, x_2) does not have an x_2-Hamiltonian path by Corollary 2.1.11.

$\Longleftarrow$

Suppose G does not have an x_2-Hamiltonian path, but that $s(G) \leq 1$. Since $s(G) \leq 1$, then G contains no t-edges for $t \geq 4$. If G contains m 3-edges for $m = 0$, then G has an x_2-Hamiltonian path. So G has m 3-edges for $m \geq 1$. Let H' be the caterpillar representation of G. Then $s(H') \leq 1$ and H' does not have a x_2-Hamiltonian path by Lemma 4.1.14. Suppose that (H', x_2) does not contain I_a.

Denote the 3-edges in H', $S_i = s_i s_i'$ for all $1 \leq i \leq m$. Then the 3-edges in H' can be ordered $S_1, S_2, ...S_m$ so that in $H' - S_m$, x_2 is in a different component than s_i and s_i' for all i, in $H' - S_1$, all $s_i, s_i' \neq s_1, s_1'$ are in the same component, in $H' - S_m$, all $s_i, s_i' \neq s_m, s_m'$ are in the same component, and such that for all $i \in \{1, 2, ..., m-2\}$, s_i and s_{i+2} are in different components of $H' - S_{i+1}$. From the reduction algorithm, S_1 is adjacent to two simplicial vertices. Furthermore, since $c(G - S_1) = 3$, then, if H' has an (x_2)-Hamiltonian path, one of the simplicial vertices adjacent to S_1 must be an endpoint of the path. Without loss of generality, label one of the simplicial vertices adjacent to S_1, x_1. So since H' does not have an (x_2)-Hamiltonian path, then H' does not have an (x_1, x_2)-Hamiltonian path. So, by Theorem 3.2.10, (H', x_1, x_2) must contain an $F_x^2 \in \mathscr{F}^2$ as an induced sub-2-tree. Also, since x_2 is simplicial, then (H', x_1, x_2) must contain F_a^2, F_b^2, F_c^2, F_d^2, F_e^2, F_f^1, F_e^1, F_d^1, F_c^1, or F_a^1. Adding a false twin of x_1 and removing the label, we will get the forbidden induced sub-2-trees for H' with one fixed endpoint. Using this process on F_f^1 forms I_f, on F_e^1 forms I_g, on F_d^1 forms I_h, on F_c^1 forms I_e, on F_a^1 forms I_i, on F_a^2 forms I_a, on F_c^2 forms I_c^2, on F_e^2 forms I_e^2. For F_d^2 and F_b^2, we can leave x_1 and just remove the label, as the x_1 is amalgamated with a 3-edge and hence already forcing x_1 or the other simplicial vertex as an end. Using this process on F_d^2 forms I_d, and on F_b^2 forms I_b . □

Chapter 5

Conclusion

In Chapter 2, we introduced a new toughness condition and introduced a new approach for characterizing Hamiltonian problems on 2-trees by describing a forbidden list of induced sub-2-trees for which 2-trees will not have Hamiltonian paths. While the approach of defining a forbidden list of induced subgraphs will not work for graphs in general, this approach will work for induced k-trees in a k-tree, as proved in Chapter 2. In Chapter 3, we characterized 2HP on 1-tough 2-trees by giving necessary and sufficient conditions for a 1-tough 2-tree with fixed vertices, x_1, x_2, to have an (x_1, x_2)-Hamiltonian using both toughness conditions and defining a family, $\mathscr{F}^1$ of 2-trees for which a 1-tough 2-tree containing a graph in $\mathscr{F}^1$ as an induced subgraph will not have a Hamiltonian path. Additionally, in Chapter 3, we used the results for 2HP on 1-tough 2-trees to similarly characterize the 2-trees which are not 1-tough as containing a 2-tree in a family, $\mathscr{F}^2$, as an induced subgraph. Furthermore, we used the results in Chapters 2 and 3 to characterize the Hamilonian path problem on 2-trees in Chapter 4 and 1HP on 2-trees in Chapter 5, by defining forbidden families of 2-trees, $\mathscr{H}$ and $\mathscr{I}$, respectively.

In the future, it is possible that we could extend these methods on 2-trees to other generalizations of the Hamiltonian path problem, like the Path Partition problem or the k-Fixed Endpoint Path Partition problem. It is also possible that we could try to extend these results to 3-trees or k-trees. Since adding a vertex adjacent to all vertices in a 2-tree would form a 3-tree, our forbidden lists would be a starting point for investigating these problems on 3-trees.

Bibliography

[1] Arnborg, S. (1985). Efficient algorithms for combinatorial problems on graphs with bounded decomposability-a survey. *BIT Numerical Mathematics*, 25(1):1–23.

[2] Asdre, K. and Nikolopoulos, S. D. (2010a). The 1-fixed-endpoint path cover problem is polynomial on interval graphs. *Algorithmica*, 58(3):679–710.

[3] Asdre, K. and Nikolopoulos, S. D. (2010b). A polynomial solution to the k-fixed-endpoint path cover problem on proper interval graphs. *Theoretical Computer Science*, 411(7-9):967–975.

[4] Baker, B. A. (2013). The k-fixed-endpoint path partition problem.

[5] Bodlaender, H. L. (1988). Nc-algorithms for graphs with small treewidth. In *International Workshop on Graph-Theoretic Concepts in Computer Science*, pages 1–10. Springer.

[6] Bodlaender, H. L. (1996). *A partial k-arboretum of graphs with bounded treewidth*, volume 1996. Utrecht University: Information and Computing Sciences.

[7] Broersma, H., Xiong, L., and Yoshimoto, K. (2007). Toughness and hamiltonicity in k-trees. *Discrete mathematics*, 307(7):832–838.

[8] Chartrand, G., Lesniak, L., and Zhang, P. (2011). *Graphs & digraphs*. CRC press.

[9] Chen, G., Jacobson, M. S., Kézdy, A. E., and Lehel, J. (1998). Tough enough chordal graphs are hamiltonian. *Networks: An International Journal*, 31(1):29–38.

[10] Chvátal, V. (1973). Tough graphs and hamiltonian circuits. *Discrete Mathematics*, 5(3):215–228.

[11] Cygan, M., Nederlof, J., Pilipczuk, M., Pilipczuk, M., van Rooij, J. M., and Wojtaszczyk, J. O. (2011). Solving connectivity problems parameterized by treewidth

in single exponential time. In *Foundations of computer science (focs), 2011 ieee 52nd annual symposium on*, pages 150–159. IEEE.

[12] Dankelmann, P., Niessen, T., and Schiermeyer, I. (1994). On path-tough graphs. *SIAM Journal on Discrete Mathematics*, 7(4):571–584.

[13] Deogun, J. S., Kratsch, D., and Steiner, G. (1997). 1-tough cocomparability graphs are hamiltonian. *Discrete Mathematics*, 170(1-3):99–106.

[14] Faudree, R. J. and Gould, R. J. (1997). Characterizing forbidden pairs for hamiltonian properties. *Discrete Mathematics*, 173(1-3):45–60.

[15] Goodman, S. and Hedetniemi, S. (1974). Sufficient conditions for a graph to be hamiltonian. *Journal of Combinatorial Theory, Series B*, 16(2):175–180.

[16] Jung, H. A. (1978). On a class of posets and the corresponding comparability graphs. *Journal of Combinatorial Theory, Series B*, 24(2):125–133.

[17] Kabela, A. (2017). Long paths and toughness of k-trees and chordal planar graphs. *arXiv preprint arXiv:1707.08026.*

[18] Kabela, A. and Kaiser, T. (2015). 10-tough chordal graphs are hamiltonian. *Electronic Notes in Discrete Mathematics*, 49:331–336.

[19] Markenzon, L., Justel, C. M., and Paciornik, N. (2005). Subclasses of k-trees: characterization and recognition. *Discrete Applied Mathematics*, 154(5):818–825.

[20] Nešetřil, J. (1979). Amalgamation of graphs and its applications. *Annals of the New York Academy of Sciences*, 319(1):415–428.

[21] Patil, H. (1986). On the structure of k-trees. *Journal of Combinatorics, Information and System Sciences*, 11(2-4):57–64.

[22] Proskurowski, A. (1984). Separating subgraphs in k-trees: cables and caterpillars. *Discrete Mathematics*, 49(3):275–285.

[23] Renjith, P. and Sadagopan, N. (2016). 2-trees: Structural insights and the study of hamiltonian paths. *arXiv preprint arXiv:1511.02038.*

[24] Rose, D. J. (1974). On simple characterizations of k-trees. *Discrete mathematics*, 7(3-4):317–322.

[25] Shook, J. M. and Wei, B. (2010). Some properties of k-trees. *Discrete Mathematics*, 310(17-18):2415–2425.

[26] Winkler, P. M. (1981). Graphic characterization of k-trees. In *Proceedings of the 12th Southeastern Conference on Combinatorics, Graph Theory and Computing*, volume 33, pages 349–357.

Curriculum vitae

Caitlin Owens
Christmas-Saucon Hall, Lehigh University
14 E Packer Ave
Bethlehem, PA, 18015
cmo312@lehigh.edu

EDUCATION

Lehigh University	Bethlehem, PA 18015
Ph.D. candidate in Mathematics	August 2012-Present
Advisor: Garth Isaak	(degree exp. January 2019)
GPA: 3.76	
M.S. in Mathematics	August 2014
Kutztown University	Kutztown, PA 19530
Post-Baccalaureate Certification - Secondary Ed Math	May 2011 - May 2012
Graduate curriculum GPA: 4.00	
Undergraduate curriculum GPA: 3.84	
PA Instructional I Certification: Mathematics 7-12	July 2012
Ithaca College	Ithaca, NY 14850
B.A. in Mathematics	August 2005 - May 2008
Mathematics GPA: 4.00	
Minors: Economics, Honors Humanities and Sciences	
Overall GPA: 3.89	

TEACHING EXPERIENCE

Adjunct Faculty Instructor, Calculus II — Summer 2018
Muhlenberg College — Allentown, PA 18104

Instructor, Calculus with Business Applications — Spring 2018
Lehigh University — Bethlehem, PA 18015

Adjunct Faculty Instructor, Calculus II — Summer 2017
Muhlenberg College — Allentown, PA 18104

Instructor, PreCalculus, Calculus I — Summer 2016, 2017, 2018
Lehigh University Summer Scholars Institute (LUSSI)
Lehigh University — Bethlehem, PA 18015

Instructor, PreCalculus — Fall 2016
Lehigh University — Bethlehem, PA 18015

Adjunct Faculty Instructor, Calculus II — Summer 2016
Moravian College — Bethlehem, PA 18018

Teaching Assistant — August 2014-Present
Lehigh University — Bethlehem, PA 18015
Recitations Taught: Calculus I, II, III, and
Calculus with Business Applications

Graduate Assistant (Tutor) — August 2012-May 2014
Lehigh University — Bethlehem, PA 18015
Group tutor for: Calculus II and III

Student Teacher — March 2012-May 2012
Lehighton High School — Lehighton, PA 18235
Courses Taught: Algebra I, Algebra II, inclusion class

Student Teacher January 2012-March 2012
Palmerton Jr. High School Palmerton, PA 18071
Courses Taught: Pre-Algebra, Algebra I, inclusion class

SERVICE

Proctor/Grader for Lehigh University High School Math Contest (annual)
Lehigh University 2013-2018

Tutor in Lehigh University Math Help Center August 2014-May 2018
Lehigh University Mathematics Department

Graduate Student Mentor May 2015-May 2018
Lehigh University Mathematics Department

Mathematics Graduate Liaison Committee Member
Lehigh University Mathematics Department September 2016-May 2018

Building Committee Member March 2017-May 2018
Lehigh University

Graduate Student Senate Senator August 2015-May 2018
Lehigh University

Graduate Student Intercollegiate Mathematics Seminar Co-President
Lehigh University Mathematics Department April 2015-April 2017

Graduate Student Senate Treasurer April 2015-May 2016
Lehigh University

Dean's Graduate Student Advisory Council Member
Lehigh University College of Arts and Sciences September 2015-May 2016

OTHER WORK EXPERIENCE

Operations Analyst/ Cost Accountant, Hatfield Quality Meats, Hatfield, PA 19440

HONORS

- American Mathematical Society (AMS) Travel Grant Recipient
- High Scorer on the Praxis II Content Area Mathematics
- Dean's List all semesters, Ithaca College
- Oracle Honor Society, Ithaca College
- Phi Kappa Phi National Honor Society, Ithaca College
- Pi Mu Epsilon National Mathematics Honor Society, Ithaca College
- Omicron Delta Epsilon International Economics Honor Society, Ithaca College
- Senior Mathematics Award, Ithaca College
- Ramanujan Mathematics Award, Ithaca College
- First Year Mathematics Award, Ithaca College
- Graduated Magna cum Laude, Ithaca College

INVITED TALKS

- Pi Mu Epsilon Induction Ceremony, Ithaca College, March 2018
- Epsilon Series, Moravian College, November 2017
- Cedar Crest College, November 2017
- Penn State University-Hazleton, October 2017
- Mathematics Seminar, University of Scranton, October 2017
- Mathematics/Computer Science Colloquium Series, Muhlenberg College, April 2017

- Graduate Student Intercollegiate Mathematics Seminar, Lehigh University, February 2017

- Mathematics and Computer Science Lecture Series, DeSales University, November 2016

- Graduate Student Intercollegiate Mathematics Seminar, Lehigh University, September 2015

OTHER TALKS

- Joint Mathematics Meetings, San Diego, January 2018

- Hudson River Undergraduate Mathematics Conference, Sienna College, 2007

- Ithaca College James J. Whalen Academic Symposium, Ithaca College, 2007

SKILLS

- Proficient in: Microsoft Office; Activstudio Professional/ Promethean Board; SMARTBoard; Geometer's Sketchpad; TI Connect; LaTeX; Coursesite; MyLab; Canvas

- Working Knowledge of: Mathematica, MATLAB, EViews, SPSS