Optimization models for electricity networks and renewable energy under uncertainty

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Optimization Models for Electricity Networks and Renewable Energy under Uncertainty

by

M. Mohsen Moarefdoost

Presented to the Graduate and Research Committee

of Lehigh University

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in

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Abstract

This work focuses on developing optimization models and algorithms to solve problems in electricity networks and renewable energy. The steady rise of electricity demand in the world, along with the deployment of volatile renewable energy resources in greater quantities, will require many researchers, policymakers, and other stakeholders in the field of power management to understand these challenges and use new methods, approaches and technologies to modernize the electric grid. We study reliable and efficient electricity dispatch with minimum costs in power networks and efficient and economic harvesting of ocean wave energy by optimizing wave farm configuration.

First, we focus on improving the use of energy storage units in electricity grids in the presence of unreliable generators. We present methods for optimizing generation and storage decisions in an electricity network with multiple unreliable generators, each co-located with one energy storage unit (e.g., battery), and multiple loads under power flow constraints. This problem cannot be optimized easily using stochastic programming and/or dynamic programming approaches. Therefore, in this study, we present several heuristic methods to find an approximate optimal solution for this system. Each heuristic involves decomposing the network into several single-generator, single-battery, multi-load systems and solving them optimally using dynamic programming, then obtaining a solution for the original problem by recombining. We discuss the computational performance of the proposed heuristics as well as insights gained from the models.
In addition, we introduce efficient and economic electricity dispatch policies in electricity grids under uncertainty arising from renewable penetration. The increased introduction of renewable energy sources (RES) such as wind into electricity networks has made the management and operation of these networks more difficult. The variability in the power output of renewable energy sources requires flexible generation units to ramp up and ramp down more frequently to maintain the power balance and reliability of the power network. Moreover, renewable energy sources are uncertain in nature, which adds another complexity to the system. In this work, we study generation ramping costs and constraints in power networks, and develop a chance constrained optimization model for generation and dispatch policies in order to manage the risk and maintain the reliability of the power system. We study necessary conditions under which the chance constrained model of the system can be reformulated as a suitable deterministic optimization problem for large-scale networks.

Finally, we propose novel and efficient optimization models and algorithms for designing and operating arrays of wave energy conversion devices in both deterministic and stochastic sea environments. We present models and heuristic algorithms for choosing optimal locations of wave energy conversion (WEC) devices within an array, or wave farm. The location problem can have a significant impact on the total power of the farm due to the constructive or destructive interactions among the incident ocean waves and the scattered and radiated waves produced by the WECs. In the deterministic models, our algorithm chooses WEC locations to maximize the performance of a wave farm as measured by a well known performance measure called the $q$-factor, under the point absorber approximation. In addition, we study the location problem under uncertainty, and propose modeling approaches for mitigating the effect of uncertainty assuming single component but stochastic sinusoidal waves, as well as for irregular waves with a spectral representation, under a modification of the $q$-factor. We formulate the problems, study the properties and theoretical characteristics of the proposed
models for a simple 2-WEC case, and develop a heuristic algorithm to choose WEC locations to maximize the performance of a wave farm.
Chapter 1

Introduction

Electrical energy is an essential part of almost every aspect of modern life and important for the growth of every economy. According to the U.S. Energy Information Administration (EIA), electricity consumption in the U.S. was nearly 3.856 trillion kWh in 2011, which was 13 times greater than electricity use in 1950. This steady increase in electricity demand requires affordable, reliable and sustainable renewable energy along with efficient and cost-effective management of the electrical power grid. Incorporating these renewable energy sources, which in some cases are highly variable, into the power grid and operating the power grid economically are the main challenges in the future. In this research, we aim to address these challenges from an optimization and economics point of view.

In the first topic (chapter 2), we analyze the effect of power storage capability in the power grid when we have unreliable sources of energy. Recently, new advances in electricity storage technology have made stored energy more efficient, cost effective and reliable. Energy storage systems (ESSs) could fill the gap between electricity generation and consumption, as well as helping to integrate renewable energy into the power grid. There are different types of energy storage technologies which can serve different applications in the power grid such as load leveling, frequency regulation, etc. Here,
we are not concerned with the energy storage technologies and their applications, but rather how to use this capability efficiently and economically. We provide methods and algorithms for optimizing generation and storage in an electricity network with multiple unreliable generators, each co-located with one storage unit, and multiple loads under power flow constraints.

In the second topic (chapter 3), we evaluate the effect of ramp capability on the power network by analyzing ramping costs for different power generators and their effects on generation and dispatch policies under uncertainty, which is due to the introduction of renewable energy resources into the power grid. The increased penetration of renewable generation requires more variability and flexibility on the part of controllable power generating units. Understanding the cost of having flexible power generation is an important part of managing the power system at minimal cost. We propose mathematical and optimization models to address these problems and challenges.

Finally, in chapter 4 we address the problem of optimally configuring arrays of wave energy converter (WEC) devices. Ocean wave energy is a source of renewable energy that is more consistent and predictable than other renewable resources such as wind and solar. There is great worldwide potential in utilizing ocean waves for green production. However, the high cost of wave energy production is the main barrier in realizing this potential. To overcome this barrier, not only do we need to design cost-effective, robust and efficient wave energy converter devices, but also we need to design and configure large-scale arrays of multiple WECs, known as wave farms, to optimize the performance of the system and benefit from economies of scale. Thus, we propose optimization methods and efficient algorithms to WEC location problems (WECLP) under uncertainty.
Chapter 2

Electricity Generation and Storage

Under Uncertainty

2.1 Introduction

In this chapter, we examine the effect of power storage capability in the power grid when we have unreliable energy resources. The steady rise of electricity demand in the United States, along with the deployment of variable renewable energy resources in greater quantities, will require many researchers, policymakers, and other stakeholders in the field of power management to understand these challenges and use new methods, approaches and technologies to modernize the electric grid. The demand for electricity has risen steadily world-wide; for example, U.S. electricity demand is expected to increase from 3.8 trillion kilowatt hours (kWh) in 2012 to nearly 5 trillion kWh in 2040 [102]. At the same time, volatile, non-hydropower renewable electricity generation will increase by more than 140% from 2012 to 2040 [102]. The combination of these factors will place more stress on electricity generation, transmission, and distribution infrastructure. This will increase the risk of disruptions within the electricity system—a risk that is already very costly, with interruptions costing U.S. electricity consumers
approximately $80 billion per year [55]. This risk can be mitigated through the deployment of emerging energy storage technologies, as well as quantitative methods to optimize the dispatch of both generation and storage resources. Other ways to mitigate the risk, e.g., investing in a more robust infrastructure, are also important but are not the focus of the dissertation.

Electricity ESSs have gained a lot of attention recently due to their potential applications and services within the power grid, which will result in the increased reliability and resiliency of the grid. ESSs serve different applications such as frequency regulation, load leveling, power quality and peak shaving in the power grid. However, from an optimization and algorithmic point of view, the effect of energy storage in complex electrical power networks under uncertainty has not been well studied.

In this work, we propose a mathematical framework to optimize generation and storage dispatch in an electricity transmission network over a finite horizon under capacity and network constraints. More specifically, our model determines the amount of energy produced by each generator and the amount of energy stored in each energy storage unit (or battery\(^1\)) in every time period in order to minimize energy generation, storage and shortage costs.\(^2\) Moreover, we assume that each generator faces stochastic Markovian supply disruptions, and each load node produces deterministic but time-varying demands. This problem cannot be optimized easily using stochastic programming and/or dynamic programming (DP) approaches. Therefore, we present several heuristic methods to find an approximate optimal solution for this system. Each heuristic involves decomposing the network into several single-generator, single-battery, multi-load systems and solving them optimally using dynamic programming, then obtaining a solution for the original problem by recombining.

\(^1\)We will use the terms “battery” and “energy storage system” interchangeably, though the models apply to other types of storage as well.
\(^2\)Without loss of generality, we assume each time period has a duration of one time unit; thus, the amount of power is mathematically equivalent to the amount of energy, and we use these terms interchangeably throughout this chapter.
2.2 Literature Review

Energy storage systems (ESSs) and their application have recently been studied extensively. There are a large number of presentations, reports and papers examining ESSs from different perspectives. We can categorize this body of literature in three main sub-categories. The first and main body of literature is concerned with electricity storage technologies. In this area, different storage technologies and their applications and performance are studied [11, 38, 79]. Interested readers may refer to Chen et al. [17], Masaud et al. [69], or Dunn et al. [25] for further and more comprehensive reviews.

The second sub-category considers the electricity storage market’s benefits, challenges and policies. Some works are focused on market barriers and the potential deployment of ESSs [6, 27, 75]. Others deal with market design and performance with ESSs in the presence of renewables [13, 52, 91, 98].

Finally, in the third sub-category, the utilization and operation of electricity storage in the power grid from an optimization and economics point of view has been studied [8, 43, 82, 108, 114]. Many different problems have been considered, ranging from the strategic level to the tactical and operational levels of the power grid with storage units and with or without renewables. For example, Shu and Jirutitijaroen [90] propose a stochastic dynamic programming framework to obtain the optimal policy for hourly operation of an ESS in a power grid connected to wind power. Taylor et al. [96] investigate the interaction between a storage unit and a variable power generator and argue that the optimal storage scheduling strategy is a base-stock policy, which is well studied in the area of inventory theory. This sub-category is most closely related to our work.

From an operational point of view, there is a need to study and evaluate the effect of energy storage systems on the electrical power grid, especially power flows. Economic generation and dispatch policies for electricity networks without storage capability, or the optimal power flow problem (OPF), has been studied extensively in the
area of power networks [23, 44, 110, 115]. Recently, due to new advances in energy storage systems, the OPF problem has taken on an additional complication. These developments in storing electrical energy enable us to manage electricity distribution more efficiently through flexible, continuous and smooth supply of power. The major difference between the classical OPF problem and the OPF problem with energy storage is the additional inter-temporal constraints which link successive time periods in the OPF problem with energy storage. Chandy et al. [15], Gayme and Topcu [37] and Baker et al. [4] study OPF models with energy storage under deterministic supply and demand. Chandy et al. [15] consider the linearized DC approximation of the OPF problem for “single-generator, single-load” (SGSL) and “multiple-generator, multiple-load” (MGML) cases, and characterize the optimal generation schedule using KKT conditions for the SGSL case. Gayme and Topcu [37] expand on this work in three ways. First, they relax the small-angle assumption in Chandy et al. [15] that requires the difference in voltage angles to be small for all pairs of buses. Second, they consider both active and reactive power. Finally, they set bounds on the charge/discharge rate of energy storage. Baker et al. [4] consider a model similar to Chandy et al. [15] and try to find conditions under which the “linear independence constraint qualification” (LICQ) holds. The KKT conditions are necessary when LICQ holds, and are sufficient when the problem is convex, so they argue that when LICQ holds the problem is solvable by the Newton-Raphson method.

Our work in this chapter has two major contributions: First, we introduce stochastic Markovian disruptions on each generator into the OPF model with storage. Second, we propose three heuristics to solve this problem, each using a different decomposition method. More specifically, we emphasize algorithmic methods while the current literature focuses on proving structural properties. Generator disruptions in power networks are analogous to supply disruptions in the supply chain management literature, as studied by Güllü et al. [40], Parlar et al. [81], Snyder et al. [94], Tomlin [100] and others.
and this analogy is exploited in our heuristics. For simplicity, we consider only real power and try to understand the impact of uncertainty on generation and storage. We also assume a deterministic and time-varying demand profile for every load node.

The structure of this chapter is as follows. We introduce the problem in Section 2.3. In Section 2.4, we discuss the proposed heuristics. We present an estimation of optimization error in Section 2.5, and computational analysis in Section 2.6. Conclusions and directions for future work are discussed in Section 2.7.

2.3 Model and Problem Formulation

In this section, we provide a model for the multiple-generator, multiple-load (MGML) power flow system with energy storage under Markovian disruptions. We consider a power network of $n$ buses (nodes) containing a set $G$ of generator–storage nodes and a set $D$ of demand nodes ($N = G \cup D$). An edge $(i,k)$ in the network represents a transmission line between node $i \in N$ and node $k \in N$. A complex admittance, $Y_{ik} = \xi_{ik} + \gamma_{ik} \sqrt{-1}$, is associated with the line $(i,k)$; the admittance provides a measure of how easily the line allows power to flow. ($Y_{ik} = 0$ if node $i$ and $k$ are not connected.) We assume a finite time horizon of length $T$ and a deterministic and time-varying demand profile. For every demand node $i \in D$, $d_i(t)$ is the demand of node $i$ for $t = 1, 2, \ldots, T$. We assume independent Markovian disruptions on every generator $i \in G$ in every time period $t$, as depicted in Fig. 2.1. Using Markov models to model disruptions is common in the literature. One may consider other failure processes, which would complicate the analysis (and is, we feel, outside the scope of this chapter). It is possible to incorporate different disruption and recovery probabilities for different generators in different time periods, just by adding a subscript $t$ to them, and this will not affect the solution methodology.

---

3 A “bus” in a power network is analogous to a “node” in graph theory terminology. We use these terms interchangeably throughout this chapter.
Figure 2.1: Markovian power supply process

Let $S_i(t)$ be the state of generator $i \in \mathcal{G}$ in period $t$, where $S_i(t) = 0$ if the generator is down (or disrupted) and 1 otherwise. The disruption and recovery probabilities at generator $i \in \mathcal{G}$ are given by $\pi_{i0}$ and $\pi_{01}$, respectively. For ease of exposition, we assume that all generators have the same disruption and recovery probabilities, though the models, algorithms, and analytical results below generalize easily to the case of heterogeneous probabilities. For each generator located in node $i \in \mathcal{G}$, we have state-dependent bounds on the power generation:

$$0 \leq g_i(t) \leq \begin{cases} 0, & S_i(t) = 0 \\ G_i, & S_i(t) = 1 \end{cases} \forall i \in \mathcal{G}, \quad (2.1)$$

where $G_i$ is the nominal capacity of the generator. We assume that each generator has its own dedicated, infinite-capacity, higher-cost source of backup energy, e.g., a diesel generator. (The difference in cost between generator $i$ and its backup source is analogous to a lost-sales cost in inventory management.) Thus, if generator $i$ is down at time $t$, the demand is satisfied either by energy stored in the battery or by the spare source. Let $g_{si}(t) (i \in \mathcal{G})$ denote the amount of power drawn from the spare source at $i$ in time period $t$.

In a typical power flow network, the net power export from node $i \in \mathcal{N}$ at time $t$
under the DC approximation
d on approximation is a widely used linearization of the exact AC power flow. A short summary of AC/DC power flow analysis is given in Appendix C.

\[ q_i(t) = \sum_{k \in N} V_i V_k \gamma_{ik} (\theta_i(t) - \theta_k(t)), \]

where \( V_i \) and \( \gamma_{ik} \) are parameters of the power system and represent, respectively, the voltage magnitude and the imaginary part of the admittance for edge \((i, k)\); and where \( \theta_i \) is the voltage angle of node \( i \), a decision variable. The flow over line \((i, k)\) is given by \( V_i V_k \gamma_{ik} (\theta_i(t) - \theta_k(t)) \), and the system operator can change the flow magnitude and its direction (from node \( i \) to node \( k \) or from node \( k \) to node \( i \)) over line \((i, k)\) by changing the values of the voltage angles of node \( i \) and node \( k \), i.e., \( \theta_i(t) \) and \( \theta_k(t) \). However, the difference in voltage angles between two connected buses must be relatively small in order for the DC approximation to be valid. We will enforce this by the following constraints:

\[ -\delta \leq \theta_i(t) - \theta_j(t) \leq \delta \quad \forall i \neq j \in N, \]

where \( \delta > 0 \) is sufficiently small. The net power import into demand node \( i \in D \) at time \( t \) is \(-d_i(t)\). Thus we have:

\[ q_i(t) = -d_i(t) \quad \forall i \in D. \]

Moreover, the net power imported into generation node \( i \in G \) is \( g_i(t) + g_{si}(t) + r^d_i(t) - r^c_i(t) \), where \( b_i(t) \) is the battery energy level, \( r^d_i(t) \geq 0 \) is the discharged energy and \( r^c_i(t) \geq 0 \) is the charged energy at time \( t \). We also have bounds on the battery energy level \( b_i(t) \):

\[ 0 \leq b_i(t) \leq B_i \quad \forall i \in G, \]

where \( B_i \) is the battery capacity (\( \forall i \in G \)). One may consider bounds for the battery
energy charge and discharge in each time period; however, for simplicity we assume that these quantities are unbounded. Conservation of flow for generation node \(i\) at time \(t\) is enforced by:

\[
q_i(t) = g_i(t) + g_{si}(t) + r^d_i(t) - r^r_i(t) \quad \forall i \in \mathcal{G} \tag{2.6}
\]

\[
b_i(t) = b_i(t - 1) + \nu_ir^r_i(t) - \frac{r^d_i(t)}{\tau_i} \quad \forall i \in \mathcal{G}, \tag{2.7}
\]

\[
r^r_i(t) \geq 0, r^d_i(t) \geq 0 \quad \forall i \in \mathcal{G} \tag{2.8}
\]

where, \(\nu_i\) is the charging efficiency and \(\tau_i\) is the discharging efficiency of battery \(i\). Finally, there is a capacity constraint on each transmission line:

\[
V_iV_j \gamma_{ij}(\theta_i(t) - \theta_j(t)) \leq \bar{q}_{ij} \quad \forall i \neq j \in \mathcal{N}, \tag{2.9}
\]

where \(\bar{q}_{ij}\) is the line capacity from node \(i\) to \(j\). In power engineering, constraints (2.2), (2.4), (2.6) and (2.9) form the power flow constraints:

\[
q_i(t) = \sum_{k \in \mathcal{N}} V_iV_k \gamma_{ik}(\theta_i(t) - \theta_k(t)) \quad \forall i \in \mathcal{N}, \forall t
\]

\[
q_i(t) + d_i(t) = 0 \quad \forall i \in \mathcal{D}, \forall t
\]

\[
q_i(t) = g_i(t) + g_{si}(t) + r^d_i(t) - r^r_i(t) \quad \forall i \in \mathcal{G}
\]

\[
b_i(t) = b_i(t - 1) + \nu_ir^r_i(t) - \frac{r^d_i(t)}{\tau_i} \quad \forall i \in \mathcal{G}
\]

\[
V_iV_j \gamma_{ij}(\theta_i(t) - \theta_j(t)) \leq \bar{q}_{ij} \quad \forall i \neq j \in \mathcal{N}, \forall t
\]

In this work, we consider a quadratic cost for generation, with \(\lambda_i(t), \forall i \in \mathcal{G}\), as the quadratic cost coefficient, similar to Chandy et al. [15]. Also, we consider linear holding (storage) and deficit costs with unit cost parameters \(h\) and \(p\), respectively. In this work, the holding cost is the cost of maintaining energy in the battery (representing
maintenance and lifetime costs, etc.) and increases as the battery level of charge increases [97]. This type of holding cost is different from the cost function considered by Chandy et al. [15], who consider a more general storage cost function and assume that the storage cost decreases as the charge level increases. However, it is easy to modify our algorithms to accommodate the type of holding cost considered by Chandy et al. [15].

The mathematical optimization model is:

\[
\min_{t=1}^{T} \sum_{j \in G} \left( \frac{1}{2} \lambda_j(t) g_j(t)^2 + h b_j(t) + p g_{s_j}(t) \right) + \sum_{j \in G} \left( h b_j(T+1) + p g_{s_j}(T+1) \right) \\
\text{over: } g_j(t), b_j(t), g_{s_j}(t), \theta_j(t) \\
\text{subject to: (2.2) - (2.9).}
\]

Note that \( h b(T + 1) + p g_s(T + 1) \) is the terminal cost on the final battery energy level. This cost is required for the stability and feasibility of the power system at the end of the horizon [15, 112].

### 2.4 Heuristics

Under stochastic disruptions on generators, constraint (2.1) is stochastic and is either \( g_i(t) = 0 \) or \( 0 \leq g_i(t) \leq G_i \), in every time period and for each generator. The number of scenarios is exponential in the number of generators and time periods, and therefore the problem of optimizing power generation and dispatch is difficult to solve using off-the-shelf solvers or direct implementation of DP for real systems with a large number of generators. For example, if there are 10 generators and batteries and 24 time periods, and if the battery levels are discretized to 100 discrete values, then there are \( 100^{10} \cdot 2^{10} \approx 10^{23} \) states that must be enumerated in each time period of a DP, or \( (2^{10})^{24} \approx 10^{72} \) scenarios that must be enumerated for a stochastic optimization problem. Therefore, we propose a heuristic approach for solving this problem.
The solution procedures proposed in this work decompose the electricity network into several single-generator, single-battery, multiple-load sub-systems. For each sub-system, our procedures find generation and storage quantities by temporarily ignoring the power flow constraints and consolidating the loads’ demands, then transforming the resulting problem into an inventory optimization problem, which we solve using dynamic programming. More specifically, by ignoring the power flow constraints, each sub-system contains one generator which faces the consolidated loads’ demands and one battery to buffer against uncertainties. This problem is similar to a single-echelon inventory problem in which the generator is analogous to the supplier and the battery is analogous to the inventory. We then obtain a solution for the original network by recombining the solutions of all sub-systems.

This approach is optimal for a given sub-system if the corresponding subset of power flow constraints (2.9) is non-binding for the solution found and is heuristic otherwise. Either way, the approach is heuristic for the overall system. Our proposed heuristics each consist of three basic steps:

1. Partition the network into several sub-systems, each with one generator, one storage unit, and multiple loads. In some cases, a given load may be contained in multiple sub-systems. We propose three different partitioning methods to decompose the power network into sub-systems: Eigenvalue-Based (EV), Deterministic Optimization-Based (DO), and Stochastic Load Assignment-Based (SLA) partitioning.

2. Ignore the power flow constraints and find optimal generation and storage quantities for each sub-system using dynamic programming.

3. Recombine the solutions of all sub-systems to get a solution for the entire network. In this step, we take the physical constraints of the power network into account and modify the sub-systems’ solutions to get a near-optimal solution for
the whole network.

We discuss each of these three steps in the following sections.

## 2.4.1 Partitioning

We propose three methods for partitioning the network. In the first method, we partition the network into several disjoint sub-systems, whereas in the latter two methods, a given load may be contained in multiple sub-systems. In the following, we describe these methods in detail.

### Eigenvalue-Based Partitioning

The literature suggests several methods for partitioning power networks, e.g., [77, 111, 113]. Following the approach introduced by Muller and Quintana [77], our Eigenvalue-Based (EV) partitioning method uses the following steps:

1. Using the network topology and mutual admittances between every pair of buses in the network, we place all buses in a $K$-dimensional space by applying the method proposed by Kenneth [49]. In particular, we transform electrical “distances” between every two buses into Euclidean distances by solving the following optimization problem:

\[
\begin{align*}
\min_{X} & \quad z = \sum_{i=1}^{K} X_i^T \Phi X_i \\
\text{over:} & \quad X \\
\text{subject to:} & \quad X_i^T X_i = 1, \quad i = 1, 2, \ldots, K \\
& \quad X_i^T X_j = 0, \quad i \neq j = 1, 2, \ldots, K,
\end{align*}
\]

(2.11)

where $\Phi$ is the connectivity matrix. [77] suggest setting $\Phi$ equal to the negative of the inverse of the imaginary part of the bus-admittance matrix, $-\gamma_{bus}^{-1}$, where
\( \gamma_{bus} \) is defined by
\[
Y_{bus} = \xi_{bus} + \gamma_{bus} \sqrt{1},
\]
and
\[
Y_{bus}(i, j) = \begin{cases} 
-Y_{ij} & i \neq j \\
\sum_{j} Y_{ij} & i = j.
\end{cases}
\]

\( X_1, X_2, \ldots, X_K \) are \( K \) orthogonal vectors in an \( n \)-dimensional space (\( n \) is the number of nodes, and \( n \geq K \)). We partition the power network into several sub-systems and want them to be as separated as possible, in the sense that power injected by a generator in one sub-system should have a minimal effect on buses in another sub-system in the original network. Therefore, we set \( K = |G| \) since we want one sub-system per generator. The position of node \( i \) in the Euclidean space is
\[
a_i = (x_{i1}, x_{i2}, \ldots, x_{ik}) = (X_{1i}, X_{2i}, \ldots, X_{Ki})
\]
where \( X_{ki} \) is the \( i \)th coordinate of vector \( X_k \) (\( i = 1, 2, \ldots, n \) and \( k = 1, 2, \ldots, K \)).

2. After placing each node in the \( K \)-dimensional space, in order to partition the network, we place each node \( i \in D \) in the sub-system that contains its nearest generator with respect to the Euclidean coordinates, i.e., the sub-system that contains the generator \( j \in G \) that minimizes \( ||a_i - a_j|| \).

3. Finally, we modify our initial partition considering the supply and demand in each sub-system, to attempt to ensure that power generation capacity is greater than the total load in each period, plus some buffer. In the case of Markovian disruptions with \( \Pi = [\pi_{ij}] \) as the transition probability matrix, we have an ergodic Markov chain with states \( s \in \{0, 1\} \). The limiting distribution of the number of times that the chain is in state \( s \), \( S^{(T)}_s \), in the first \( T \) steps is asymptotically normal \([39, \text{pp. 463–464}] \), where \( E[S^{(T)}_s] = T\bar{\pi}_s \) and \( \text{Var}(S^{(T)}_s) = \sigma^2_s = T(2\bar{\pi}_s\bar{z}_{ss} - \bar{\pi}_s - \bar{\pi}^2_s) \).
Here, $\tilde{\pi}_s$ is the steady-state probability of the Markov chain and $\tilde{z}_{ss}$ is the $s$th diagonal element of the matrix $\tilde{Z} = (I - \Pi + \tilde{\pi})^{-1}$ [39]. Let $D_j$ be the set of load nodes in the sub-system of generator $j \in \mathcal{G}$ ($D_j \subseteq D$). Then we modify our partition so that for every sub-system $j \in \mathcal{G}$ we have:

$$TG_j\tilde{\pi}_1 - z^*_jG_j\tau_1 \geq \sum_{i \in D_j} \sum_{t=1}^T d_i(t).$$  \hspace{1cm} (2.12)

This rule ensures that the generation capacity of each partition satisfies the demand throughout the horizon with a certain confidence. (In our numerical tests, we set $z^*_\kappa = 3$ to ensure 99% confidence.) We try to enforce this condition by local swaps, iteratively selecting a sub-system for which (2.12) is violated and moving the demand node with the highest demand to a sub-system that has sufficient capacity. It is possible that for certain values of $z^*_\kappa$, there will be no feasible partition that satisfies (2.12). So, we set an iteration limit on the number of local swaps, and when this limit reached we reduce the value of $z^*_\kappa$ down to zero. Note that $1 - \kappa$ can be construed as the reliability level of the system’s capacity.

**Deterministic Optimization-Based Partitioning**

In the EV partitioning method, we assign every load to only one generator, and the demand of each load in a sub-system must be satisfied by the generation capacity of that generator. In the Deterministic Optimization-Based (DO) partitioning method, we instead partition the network in such a way that every load in the network can share its demand among the generation capacity of two or more generators. To do so, we first solve a deterministic version of the multi-period OPF model with storage (2.10), which is similar to the model by Chandy et al. [15], using an off-the-shelf convex optimization solver. We then solve a single-generator, single-load problem with stochastic disruptions for each generator, with the load served by each generator in each period set equal
to the generation quantity for that generator in the solution to the deterministic OPF problem. This single-generator, single-load problem is solved by DP, as described in Section 2.4.2.

**Stochastic Load Assignment-Based Partitioning**

Like the DO method, in the Stochastic Load Assignment-Based (SLA) method, we partition the network in such a way that the demand of each load is satisfied by the generation capacity of more than one generator in the power grid. In the SLA method, we ignore the physical power flow constraints and treat the problem like a network flow model, explicitly assigning each load’s demand to one or more generators. Stochastic disruptions are modeled approximately by requiring the total demand assigned to a given generator to be a few standard deviations below the mean power available from that generator, in a manner similar to that in (2.12). In particular, we solve the following optimization model:

\[
\min \sum_{t=1}^{T} \sum_{j \in \mathcal{G}} \left( \frac{1}{2} \lambda_j(t) g_j^2(t) + h b_j(t) + p g_{s,j}(t) \right) + \sum_{j \in \mathcal{G}} h b_j(T + 1) + p g_{s,j}(T + 1) \tag{2.13}
\]

over \(x_{ij}(t), g_j(t), g_{s,j}(t), b_j(t)\)

subject to:

\[
\sum_{t=1}^{T} \sum_{i \in \mathcal{D}} x_{ij}(t) \leq T \bar{\pi}_j G_j - z_a \sigma_1 G_j \quad \forall j \in \mathcal{G} \tag{2.14}
\]

\[
b_j(t + 1) - b_j(t) - g_j(t) + \sum_{i \in \mathcal{D}} x_{ij}(t) \geq 0 \quad \forall j \in \mathcal{G}, t = 1, 2, \ldots, T \tag{2.15}
\]

\[
g_{s,j}(t) + b_j(t) + g_j(t) - \sum_{i \in \mathcal{D}} x_{ij}(t) \geq 0 \quad \forall j \in \mathcal{G}, t = 1, 2, \ldots, T \tag{2.16}
\]

\[
\sum_{j \in \mathcal{G}} x_{ij} = 1 \quad \forall i \in \mathcal{D} \tag{2.17}
\]

\[
0 \leq x_{ij} \leq 1 \quad \forall j \in \mathcal{G}, \forall i \in \mathcal{D} \tag{2.18}
\]

\[
0 \leq b_j(t) \leq B_j \quad \forall j \in \mathcal{G}, t = 1, 2, \ldots, T \tag{2.19}
\]
\[ 0 \leq g_{ij}(t) \quad \forall j \in \mathcal{G}, t = 1, 2, \ldots, T \] (2.20)

\[ 0 \leq g_j(t) \leq G_j \quad \forall j \in \mathcal{G}, t = 1, 2, \ldots, T \] (2.21)

In this model, \( x_{ij} \) is the fraction of the load for bus \( i \in \mathcal{D} \) that is satisfied by generator \( j \in \mathcal{G} \), and \( z_\kappa \) is the \( \kappa \)-quantile of the standard normal distribution. Constraint (2.14) ensures that the total demand assigned to a given generator is \( z_\kappa \) standard deviations below the mean power available from that generator. Constraints (2.15) and (2.16) are dynamic constraints for balancing the demand, power generation, and battery level in each period for every generator. Constraints (2.17) guarantee that the demand of each load bus is satisfied completely. Finally, the lower bounds and upper bounds on the decision variables are defined by constraints (2.18) to (2.21). We solve this quadratic program using an off-the-shelf convex optimization solver.

### 2.4.2 Dynamic Programming

After partitioning the overall network into several sub-systems with one generator and multiple loads (with or without overlapping), we solve a DP for each sub-system to obtain the optimal levels of power generation and storage ignoring the rest of the network. For the dynamic programming model, the sequence of events in each period \( t \) is as follows:

1. The energy level stored in the battery, \( b \in [0, B] \), is inherited from the end of the previous period.

2. The state of the system (\( S \in \{0, 1\} \)) is observed.

3. The “generate-up-to” level \( y \) is chosen; if the system is up, \( y - \tau b \) units of power are generated.

4. Demand \( D(t) \) occurs and is satisfied as much as possible.
5. \( \max\{0, \nu(y - D(t))\} \) units of energy are stored in the battery.

6. Holding and stock-out costs (generation costs of the spare sources) are assessed.

It is worth mentioning that for different partitioning methods, we have different definitions of \( D(t) \). In particular, in the EV method, we have \( D(t) = \sum_{i \in \mathcal{D}} d_i(t) \) for any \( j \in \mathcal{G} \); for the SLA method, \( D(t) = \sum_{i \in \mathcal{D}} x_{ij}^* d_i(t) \) for any \( j \in \mathcal{G} \); and for the DO method, \( D(t) = q_j^*(t) \) for any \( j \in \mathcal{G} \).

The problem of finding \( y \) (and therefore \( g \) and \( b \)) for each sub-system, ignoring power flow constraints, is equivalent to a finite-horizon inventory optimization problem with deterministic demand and stochastic disruptions, and can therefore be solved via the following DP, see [93, Section 4.4.3]:

\[
\eta_t^1(b) = \min_{\tau b \leq y \leq \tau b + G_i} \left\{ \frac{1}{2} \lambda_i(t)(y - \tau b)^2 + L(y - D(t)) + \sum_{j=0}^{1} \pi_{ij} \eta_{t+1}^j (\nu[y - D(t)]^+) \right\} \tag{2.22}
\]

where

- \( b \) is the initial battery level at the beginning of period \( t, 0 \leq b \leq B_i \).
- \( L(b) = hb^+ + pb^- \) is the holding and stock-out cost function.
- \( \eta_t^1(b) \) is the optimal expected cost to operate the system in periods \( t, t + 1, \ldots, T \) when we start period \( t \) with \( b \) as the battery level and in up state.
- \( \eta_t^0(b) = L(\tau b - D(t)) + \sum_{j=0}^{1} \pi_{0j} \eta_{t+1}^j (\nu[b - D(t)]^+) \) is the expected cost to operate the system in periods \( t, t + 1, \ldots, T \) when we start period \( t \) with \( b \) as the battery level and in down state.
- The terminal value function is \( \eta_{T+1}(b) = hb^+ + pb^-, \forall 0 \leq b \leq B_i \).

After solving the DP above, we obtain \( b_i^*(t) \) and \( g_i^*(t) \) and \( g_{s,t}^*(t) \) (\( g_{s,t}^*(t) = [D_i(t) - y_i^*(t)]^+ \)) which minimize the generation and storage cost of sub-system \( i \). In the next section, we use these quantities to solve a power flow problem for the overall network.
2.4.3 Recombining

The solution \((b^*_i(t), g^*_i(t), g_{si}^*(t))\) obtained by solving the dynamic programming recursion (2.22) satisfies constraints (2.4)–(2.8). Therefore, given \(b^*_i(t)\), we can easily calculate \(r^e_i(t)\) and \(r^h_i(t)\) from (2.7). We can then calculate \(q^*_i(t)\) from (2.6) \(\forall i \in G\) and from (2.4) \(\forall i \in D\). It remains only to solve for \(\theta^*_i(t)\). To do so, we need to solve a system of linear equalities/inequalities consisting of (2.2) and (2.9), i.e, the power flow constraints. By defining the parameter \(u_i\) as

\[
u_i := \begin{cases} 
1, & i \in G \\
0, & i \in D
\end{cases}
\]

we can rewrite the net export power of each node as

\[
q_i(t) = (u_i - 1)d_i(t) + u_i \left( r^e_i(t) - r^h_i(t) + g^*_i(t) + g_{si}^*(t) \right), \quad (2.23)
\]

which replaces (2.6) and (2.4). Thus, we need to solve the following system of linear (in)equalities for \(\theta\):

\[
\begin{align*}
V_i V_j \gamma_{ij}(\theta_i(t) - \theta_j(t)) &\leq \bar{q}_{ij} \quad \forall i \neq j \in \mathcal{N} \quad (2.24a) \\
\sum_{j \in \mathcal{N}} V_i V_j \gamma_{ij}(\theta_i(t) - \theta_j(t)) &= q_i(t) \quad \forall i \in \mathcal{N} \quad (2.24b)
\end{align*}
\]

If the underlying graph of the power network is connected, i.e., there is a path between every pair of nodes, the system (2.24) is feasible in \(\theta\) and has infinite solution if and only if \(\sum_{i \in \mathcal{N}} q_i(t) = 0\) for \(t = 1, 2, \ldots, T\) [7].

As is common in the literature (e.g., Cain et al. [14]), it is convenient to work with the difference in voltage angles between buses \(i\) and \(j\), \(\delta_{ij}(t) = \theta_i(t) - \theta_j(t)\), rather than with the angles themselves. For notational convenience, we also define a new parameter
\[ \omega_{ij} = V_i V_j \gamma_{ij}. \]

**Lemma 2.1.** The system of linear inequalities (2.24) can be written as

\[
\begin{align*}
\Omega_1 \delta(t) &\leq \bar{q} \\ -\Omega_1 \delta(t) &\leq \bar{q} \\ \Omega_2 \delta(t) &= q(t)
\end{align*}
\]  

where \( \delta(t) = (\delta_{12}(t), \delta_{23}(t), \ldots, \delta_{n-1,n}(t))^T \) is the vector of the angle differences and the remaining quantities are constants:

1. \( \Omega_1 \) is an \( \frac{n(n-1)}{2} \times (n-1) \)-matrix made up of parameters \( \omega_{ij} \).

2. \( \bar{q} = (\bar{q}_{ij})_{j > i \in N} \) is an \( \frac{n(n-1)}{2} \)-vector made up of transmission line capacities.

3. \( q(t) = (q_i(t))_{i \in N} \) is an \( n \)-vector of net power export from each node.

4. \( \Omega_2 \) is an \( n \times (n-1) \)-matrix where:

\[
[\Omega_2]_{ij} = \begin{cases} 
-\sum_{k=1}^{j} \omega_{ki} & j < i \\
\sum_{k=j+1}^{n} \omega_{ik} & j \geq i
\end{cases}
\]

All proofs are given in the Appendix. The explicit expressions for the new vectors and matrices are given in the proof of Lemma 2.1. The last row of matrix \( \Omega_2 \) is the negative of the sum of all the other rows. Therefore, we exclude the row of \( \Omega_2 \) and \( q(t) \) corresponding to the slack bus;\(^5\) let \( \Omega'_2 \) and \( q'(t) \) be the resulting quantities. Then equalities (2.25c) reduce to the following system of \( n - 1 \) equations and \( n - 1 \) variables:

\[
\Omega'_2 \delta(t) = q'(t).
\]

\(^5\)The slack (or swing or reference) bus is a special generator bus that balances the real and reactive power in a power flow system. Its voltage angle and magnitude are assumed to be fixed, and the slack bus is normally numbered as bus 1, i.e., the first row.
The solution to this system is \( \delta^*(t) = \Omega_2^{-1} q'(t) \), and by substituting it into (2.25), we have
\[
\hat{\Omega}q'(t) \leq \bar{q}
\]
(2.27)
\[-\hat{\Omega}q'(t) \leq \bar{q},
\]
where \( \hat{\Omega} = \Omega_1 \Omega_2^{-1} \). (\( \hat{\Omega} \) is equivalent to the matrix of generation shift factors (GSFs) from power systems engineering, treating demand as negative generation.) Constraints (2.27) represent the line capacity constraints, so if \( \hat{\Omega} \) satisfies (2.27), then we have found feasible voltage angles for the generation and battery charging/discharging quantities produced by the individual DPs, and therefore \( \delta^*(t) \) is optimal for (2.25). On the other hand, if \( \hat{\Omega} \) does not satisfy (2.27), then the generation and battery quantities from the DP are not feasible for the network as a whole, in the sense that there are no feasible voltage angles that produce the required power flows, \( q^*(t) \). In that case, we attempt to find flows that are close to \( q^*(t) \) that do permit feasible voltage angles, and then modify the generation and charging/discharging quantities in order to obtain those flows. In particular, we try to find feasible flows by adding an error vector \( \epsilon(t) \) to \( q'(t) \) and minimizing the amount of error by solving the following quadratic problem:
\[
\min_{\epsilon} \sum_{t=1}^{T} \epsilon(t)^T \epsilon(t)
\]
(2.28)
over: \( \epsilon \)
subject to:
\[
\hat{\Omega} \epsilon(t) \leq \bar{q} - \hat{\Omega} q'(t) \quad \forall t = 1, 2, \ldots, T \quad (2.29)
\]
\[
\hat{\Omega} \epsilon(t) \geq \bar{q} - \hat{\Omega} q'(t) \quad \forall t = 1, 2, \ldots, T \quad (2.30)
\]
\[
\epsilon(t)^T \mathbf{1} = 0 \quad \forall t = 1, 2, \ldots, T \quad (2.31)
\]
\[
Lb \leq \epsilon(t) \leq Ub \quad \forall t = 1, 2, \ldots, T \quad (2.32)
\]
Constraints (2.29) and (2.30) guarantee that the added error vector provides a feasible
power flow. Constraints (2.31) require the sum of changes in the net power export of all nodes in the network to equal zero in order to have a balanced power supply and demand. Constraints (2.32) limit the magnitude of the error terms by lower and upper bounds defined by

\[
Lb_i = -u_i(\bar{\delta} + [\Omega_2^{\prime -1} q'(t)]_i)
\]

\[
Ub_i = \min\{u_i(B_i + G_i - g_i^*(t) - g_{s_i}^*(t) - r_i^d(t) + r_i^e(t)), u_i(\bar{\delta} - [\Omega_2^{\prime -1} q'(t)]_i)\},
\]

where \([\Omega_2^{\prime -1} q'(t)]_i\) is the \(i\)th element of vector \(\Omega_2^{\prime -1} q'(t)\). These bounds guarantee that the deviation in power injection of each node does not exceed the capacity of that node as well as ensuring that \(\delta_{ij}(t) = \theta_i(t) - \theta_j(t)\) are small enough such that the DC approximation is still valid—more specifically, that \(-\bar{\delta} \leq \delta_{ij}(t) \leq \bar{\delta}\) for small \(\bar{\delta} > 0\).

After solving the quadratic program (2.28)–(2.32), we allocate \(\epsilon^*_i(t)\) over \(g_i^*(t)\) and \(g_{s,i}^*(t)\) based on the following procedure:

- When \(\epsilon^*_i(t) < 0\), we need to reduce the net power export from node \(i \in \mathcal{N}\). To do so, we have three options: 1) charge the battery, 2) reduce \(g_i^*(t)\), or 3) reduce \(g_{s,i}^*(t)\) by \(|\epsilon^*_i(t)|\). Generally, reducing \(g_{s,i}^*(t)\) is preferred over reducing \(g_i^*(t)\) due to its expensive generation cost. Moreover, changing the battery level will have consequences on the future solution. Therefore, we first reduce \(g_{s,i}^*(t)\) by \(|\epsilon^*_i(t)|\), and then reduce \(g_i^*(t)\) by \(\max\{0, |\epsilon^*_i(t)| - g_{s,i}^*(t)\}\).

- When \(\epsilon^*_i(t) \geq 0\), we need to increase the net power export from node \(i \in \mathcal{N}\). To do so, we have three options: 1) discharge the battery, 2) increase \(g_i^*(t)\), or 3) increase \(g_{s,i}^*(t)\). We increase \(g_i^*(t)\) by \(\min\{|G_i, |\epsilon^*_i(t)|\}\), and \(g_{s,i}^*(t)\) by \(\max\{|\epsilon^*_i(t)| - G_i, 0\}\).
2.5 Estimation of Optimization Error

In this section, we try to gain insights into the performance of the proposed heuristics analytically by considering a simplified model of the power system. The real model is complicated and intractable, with many interacting parameters. Understanding the role and effect of these parameters on the performance of the heuristics is important. Identifying the key parameters and their effects will help a decision maker in algorithm selection. For example, we show below that when the penalty cost is not too large, the optimality gap is also not large large and our decomposition methods are a good choice of heuristic. We quantify the gap in Theorem 2.1 for a simplified system and expect this to be an approximation of the true system.

The heuristics try to optimize the OPF problem with storage under Markovian disruptions on the generators by decomposing the power network into several sub-systems. In order to analyze the effect of the decomposition on the optimal solution, we consider a simplified model of the system by making the following assumptions:

1. The time horizon is infinite: $T \to \infty$.

2. The generation and battery capacities are unlimited: $G_i, B_i \to \infty \forall i \in G$.

3. The demand is constant over time: $d_i(t) = d_i \forall i \in D$.

4. The disruption and recovery probabilities are homogeneous across the generators: $\pi_{i0}^i = \alpha$ and $\pi_{01}^i = \beta \forall i \in G$.

5. There are no physical power flow constraints, i.e., (2.2) and (2.9) are relaxed: $\bar{q}_{ij} \to \infty$.

For the simplified system, since we have infinite capacity for power generation, storage and transmission links, the power supply is disrupted when all generators are down and it is up (not disrupted) if at least one generator is up. The supply process is still Markovian, but with different disruption and recovery probabilities.
Lemma 2.2. The simplified system has recovery probability

\[ \beta' = 1 - (1 - \beta)^m, \]

(2.33)

and disruption probability

\[ \alpha' = \sum_{x=1}^{m} \binom{m}{x} \alpha^x (1 - \beta)^{m-x} \left( \frac{(1 - \pi)\pi^{m-x}}{1 - \pi^m} \right) = (1 - (1 - \beta)^m) \frac{\pi^m}{1 - \pi^m}, \]

(2.34)

where \( \pi = \frac{\alpha}{\beta + \alpha}, m = |G| \).

Lemma 2.3. For any \( 0 \leq \alpha \leq 1, 0 \leq \beta \leq 1 \) and for all integers \( m \geq 1 \), we have \( \beta \leq \beta' \) and \( \alpha' \leq \alpha \).

This system is equivalent to an infinite-horizon newsvendor system with disruptions and deterministic demand, for which the optimal base stock level is [93]:

\[ S^* = D + DF'^{-1}\left( \frac{p}{p + h} \right), \]

(2.35)

where

\[ F'(k) = 1 - \frac{\alpha'}{\alpha' + \beta'}(1 - \beta')^k \]

(2.36)

and \( F'^{-1}(y) \) is taken to mean the smallest \( k \) such that \( F'(k) \geq y \). (Similar notation is used for other discrete cdfs later in our analysis.) Also, its optimal cost is [93]:

\[ g(S^*) = \sum_{x=0}^{\infty} \psi_x [h(S^* - xD - D)^+ + p(xD + D - S^*)^-], \]

(2.37)

where

\[
\psi_x = \begin{cases} \frac{\alpha'}{\alpha' + \beta'}, & x = 0 \\ \frac{\alpha' \beta'}{\alpha' + \beta'}(1 - \beta')^{x-1}, & x \geq 1 \end{cases}
\]

(2.38)

is the pmf of the disruption process, i.e., the probability that the current period is the \( x \)th period of a disruption (or not disrupted at all, if \( x = 0 \)). As a result of the simplifying
assumptions in this network, it does not matter how the $S$ units are allocated among the generators/batteries, and therefore we can talk about a single quantity $S$ rather than a vector of $S$ values (one per generator).

Now, we decompose this simplified system (with any desired method) into $m = |G|$ sub-systems, each of which contains a generator and faces a deterministic demand $\tilde{d}_i$, $\forall i \in G$, which is equal to the sum of the loads in that sub-system, with $\sum_{i \in G} \tilde{d}_i = D$. Each sub-system is also analogous to an infinite-horizon newsvendor system with disruptions and deterministic demand. Considering $\alpha$ and $\beta$ as the disruption and recovery probabilities, respectively, for sub-system $i \in G$, the optimal base stock level of sub-system $i \in G$ is

$$S^*_i = \tilde{d}_i + \tilde{d}_i F^{-1}\left(\frac{p}{p + h}\right),$$

where

$$F(k) = 1 - \frac{\alpha}{\alpha + \beta} (1 - \beta)^k.$$  \hspace{1cm} (2.40)

**Lemma 2.4.** For the cumulative distribution functions defined in (2.36) and (2.40), we have:

$$F'(k) \geq F(k), \quad \forall k = 0, 1, 2, \ldots$$ \hspace{1cm} (2.41)

**Lemma 2.5.** The sum of the optimal base stock levels of all sub-systems is greater than the base stock level of the whole system, i.e.,

$$\sum_{i \in G} S^*_i = \sum_{i \in G} \tilde{d}_i \left[1 + F^{-1}\left(\frac{p}{p + h}\right)\right] \geq D + DF^{-1}\left(\frac{p}{p + h}\right) = S^*.$$ \hspace{1cm} (2.42)

Notice that when $k^* = 1 + F^{-1}\left(\frac{p}{p + h}\right)$ and $k' = 1 + F'^{-1}\left(\frac{p}{p + h}\right)$, then by Lemma 2.4 and Lemma 2.5, we have $k' \leq k^*$ and $S^* \leq \sum_{i \in G} S^*_i$. The following theorem quantifies the deviation from the optimal cost when we decompose the system into $m$ sub-systems.
Theorem 2.1. Let $\Delta S = \sum_{i \in G} S_i^* - S_i^*$, $\Delta k = k^* - k'$. (Note that $\Delta S = \Delta kD$.) Then,

$$
g(S^* + \Delta S) - g(S^*) = D(h + p) \left( \epsilon \Delta k + \sum_{x=k'}^{k'+\Delta k-1} \psi_x(k' + \Delta k - x - 1) \right)$$

where $\epsilon = F'(k' - 1) - \frac{p}{p+h}$. Note that $S^* + \Delta S$ is the base-stock level in the recombined system.

Corollary 2.1. The percent increase in cost due to partitioning the network is independent of the total demand of the network.

Note that the gap provided in Theorem 2.1 is neither a lower nor an upper bound. It is simply an estimate of the actual optimality gap, based on the simpler network. In this theorem, we see that the gap is related to the storage and penalty costs as well as the difference between $k^*$ and $k'$. The difference between $k^*$ and $k'$ is a function of the number of generators and of the disruption and recovery probabilities. The gap will increase as the number of generators increases and as the disruption probability increases. It will decrease as the recovery probability increases. Thus, we expect our decomposition methods to be more effective when the recovery probability is high and/or the disruption probability is low. The actual and estimated gaps are compared numerically in Section 2.6.1.

2.6 Numerical Experiments

We ran a numerical experiment to evaluate the performance of our heuristics versus each other and versus the optimal solution (when possible), as well as the behavior of the solutions as the parameters change. We conducted two sets of experiments; one uses small instances based on a 6-bus system from Wood and Wollenberg [109] for which we can obtain optimal solutions using DP, while the other uses the IEEE14, IEEE30,
and IEEE57 benchmark instances. Detailed information related to each benchmark instance is provided in Table A.7. All optimization problems were solved using CPLEX 12.5.0.0.

For all $i \in D$, we set $d_i(t) = \left( iH_1 |\sin(\frac{i \pi}{T}) | + H_2 \right) \sum_{i \in G} G_i$, where $H_1$ is a parameter chosen between 0 and 1, $H_2$ is chosen systematically from $\{0.1, 0.2, \cdots, 1.9, 2.0\}$ to be a proxy for demand magnitude, and $T = 24$ is a 24-hour planning horizon. We define $d_i(t)$ in this way to model typical peak/off-peak patterns approximately. We set the battery capacity to 10% of the generation capacity, $B_i = 0.1G_i$, and consider their charge and discharge efficiency as $\nu_i = 1$ and $\tau_i = 1$. For all $i, j \in N$, we set line capacities $\bar{q}_{ij}$ in a way that each line can transmit 10% of total maximum capacity, we set $\lambda_i(t) = \lambda_i$, which are different for different types of power plants and are provided in Table A.7, and $h = 0.1\lambda_i$. Finally, we set $\frac{h}{p^{en}} \in \{0.90, 0.95, 0.99\}$, $\pi_{10} \in \{0.02, 0.06, 0.10\}$ and $\pi_{01} \in \{0.85, 0.90, 0.95\}$.

### 2.6.1 Accuracy of Heuristics

Since obtaining the optimal solution for large networks is difficult, and impossible in many cases, we designed small test problems for the 6-bus system with two generators, one swing bus and three loads to evaluate the performance of our heuristics with respect to the optimal solution. To obtain optimal solutions, we use a modified version of our dynamic programming algorithm in which the state consists of the vectors of battery levels and generator states. (We omit the details of this DP due to space considerations.) We evaluate the performance of the heuristics under the various values of the system parameters discussed above.

Figures 2.2–2.10 provide results on the costs attained by the three heuristics (labeled EV, DO, and SLA in the figures) versus the optimal cost attained by the modified DP.

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6The test instances can be found at http://www.ee.washington.edu/research/pstca.
Figure 2.2: 6-bus system with $\pi_{01} = 0.85$ and $\frac{p}{p+h} = 0.90$

Figure 2.3: 6-bus system with $\pi_{01} = 0.85$ and $\frac{p}{p+h} = 0.95$
Figure 2.4: 6-bus system with $\pi_{01} = 0.85$ and $\frac{p}{p+h} = 0.99$

Figure 2.5: 6-bus system with $\pi_{01} = 0.90$ and $\frac{p}{p+h} = 0.90$
Figure 2.6: 6-bus system with \( \pi_{01} = 0.90 \) and \( \frac{p}{p+h} = 0.95 \)

Figure 2.7: 6-bus system with \( \pi_{01} = 0.90 \) and \( \frac{p}{p+h} = 0.99 \)
Figure 2.8: 6-bus system with $\pi_{01} = 0.95$ and $\frac{p}{p+h} = 0.90$

Figure 2.9: 6-bus system with $\pi_{01} = 0.95$ and $\frac{p}{p+h} = 0.95$
In these tiny problems, the gap between the modified DP and the heuristics decreases as the demand increases. This implies that the heuristics perform better when the overall capacity is tighter. Since many real power networks have tightly constrained transmission lines, batteries and/or generators, the proposed heuristics seem to be practical and effective. Moreover, the optimality gap is smaller for smaller (and more realistic) values of the generators’ disruption probability ($\pi_{10}$), and for larger (and more realistic) values of the newsvendor fractile ($\frac{p}{p + h}$). Tables A.1–A.3 provide the average optimality gaps attained by the three heuristics, as well as the average gap estimate discussed in Section 2.5 (in the column labeled Analytical Gap). Tables A.4–A.6 list the average CPU times in seconds. The first three columns in both tables list the instance parameters. CPU times for all three heuristics are comparable (and less than 20 seconds on average), but the DP is unacceptably slow, even for these small instances (with an average of approximately 2200 seconds).

2.6.2 Behavior of Heuristics

In this section, we report on the results of an experiment on the IEEE14, IEEE30, and IEEE57 benchmark problems to evaluate the performance of our heuristics versus each
other, as well as the behavior of the solutions as the parameters change. In Figures 2.11–2.19, we plot the expected costs of the solutions returned by the three proposed heuristics (EV, DO, and SLA). As expected, when the demand (that is, its proxy $H_2$) and the disruption probability increase, the expected cost of the solutions obtained from all heuristics increases as well. The expected costs decrease as the recovery probability increases, but the cost is less sensitive to changes in the recovery probability than the disruption probability. For all instances, as the demand increases, the EV heuristic starts to perform worse than the SLA and DO heuristics. This is because in the DO and SLA heuristics we assign loads to the generators based on the distribution of loads and generation capacities throughout the grid, so when the bounds are tight (demand is high), the DO and SLA heuristics are more effective. In contrast, the EV heuristic performs worse since the EV partitioning method is based only on physical admittances and not on how loads and capacities are distributed in the network. In the IEEE14 system, the DO heuristic performs slightly better than SLA; however, as the size of the system increases, the SLA heuristic shows better results. It is also worth mentioning that the quality of the results obtained by both DO and SLA is affected by the optimization problems they solve for the load assignment. That is, the SLA heuristic outperforms the DO heuristic for large systems when it can solve the optimization problem (2.13)–(2.21) optimally. However, in our implementation, we set an iteration limit for the inner optimization problems in SLA and DO, and the SLA algorithm reaches this limit for some of the more difficult problems, such as the IEEE57 instance and for large values of the demand.

Table A.8 shows the average solution time of the heuristics for each instance. All heuristics solved all instances in less than two minute on average.
Figure 2.11: IEEE14 with $\pi_{01} = 0.85$

Figure 2.12: IEEE14 with $\pi_{01} = 0.90$
Figure 2.13: IEEE14 with $\pi_{01} = 0.95$

Figure 2.14: IEEE30 with $\pi_{01} = 0.85$
Figure 2.15: IEEE30 with $\pi_{01} = 0.90$

Figure 2.16: IEEE30 with $\pi_{01} = 0.95$
Figure 2.17: IEEE57 with $\pi_{01} = 0.85$

Figure 2.18: IEEE57 with $\pi_{01} = 0.90$
2.7 Conclusion

In this chapter, we introduce three heuristics to approximate the optimal generation and storage dispatch in electricity networks with unreliable generators. By comparing the solutions from our heuristics with the optimal solution for small instances, we find our heuristics to be effective on practical problems where there is limited capacity and small disruption probabilities. On average the CPU time of the heuristics in the worst case is less than 20 seconds, while the CPU time to obtain the optimal solution using DP is 2200 seconds. We analyze the gap between the heuristics and the optimal solution in a simplified version of the problem in order to provide an estimate on the actual gap, and observe that the gap is an increasing function of the disruption probability and the newsvendor fractile and a decreasing function of the recovery probability. This theoretical result is supported by the numerical study. Moreover, we evaluate the performance of the proposed heuristics on three benchmark instances and find that the DO and SLA heuristics perform better than the EV heuristic. Also, in large scale problems, the SLA heuristic shows better results. This result highlights the importance of balanced load sharing among generators and the expected available capacity (2.12) throughout the
planning horizon in the partitioned power network. The DO and SLA heuristics take advantage of balanced load among the generators in partitioning the network, while the EV heuristic simply uses the electrical distances to partition the network. Moreover, between the SLA and DO heuristics, the SLA considers the expected available capacity.

Future research on this problem can proceed in at least three areas: modeling, solution method, and application. First, in the area of modeling, this work can be extended either by considering stochastic demand, AC power flow constraints, and/or other parameters and constraints in the systems. Second, there is still room for improvement of the heuristics. For example, the SLA heuristic could be modified in the way that it assigns loads to the generator dynamically throughout the horizon. Third, the proposed solution method should be tested in practice to evaluate the robustness of these methods when faced with the complexities of real-world problems.

Finally, it would be worthwhile to explore ways to apply these solution methods to networks other than power networks. For example, consider a supply chain network in which there exist bounds on the links connecting demand and supply nodes. By considering \( \omega_{ij} = V_i V_j \gamma_{ij} \) in (2.24) as the cost of the link connecting nodes \( i \) and \( j \), \( \bar{q}_{ij} \) in (2.27) as its capacity limit and \( \bar{q}' \) in (2.27) as the supply-demand vector of all nodes, a feasible solution to the system (2.27) is a feasible flow for the network. Therefore, this system can be added to a network flow optimization problem.
Chapter 3

Ramp Capability in Electricity Networks under Uncertainty

3.1 Introduction

In this chapter, we study generation ramping costs and constraints in power networks under uncertainty, and develop a chance constrained optimization model for generation and dispatch policies in order to manage the risk and maintain the reliability of the power system. We apply two common solution approaches for the chance constrained optimization model and study the challenges and properties of the problem.

Introducing renewable power resources into the power grid brings about new challenges and problems related to system operations and reliability. The increased penetration of uncertain and variable renewable generation adds a new dimension to the problem of balancing power (matching power supply and power demand), and requires more flexibility on the part of controllable and dispatchable power generating units. These units must ramp down and ramp up, known as ramping, or stop and start, known as cycling, more frequently to maintain the power balance and provide reliable power. Ramping and cycling result in additional wear-and-tear costs and emissions from fossil-
fuel power plants [54, 59]. Ramping and cycling costs range from $150 million to $450 million at 30% renewable penetration, and these costs can reduce the value of the renewable energy by up to 2.4% [46]. Therefore, effective mathematical and robust models are required to reduce these costs and increase the profitability of renewable resources. This problem is important due to increased use of renewable power resources in electricity grids. Here, we answer the question of how to dispatch reliable power into electricity grid efficiently and economically, when there exist uncertain and volatile renewable resources.

In this work, we study generation ramping costs and constraints and examine the effect of ramping on generation and dispatch policies from an optimization point of view in an electricity network under uncertainty. We develop a chance constrained optimization model to analyze the effect of ramping on generation and dispatch policies in a network where there exist uncertain renewable power generators. We use two approaches to convert the stochastic chance constrained model to a tractable and deterministic model, and compare their properties and challenges.

### 3.2 Literature Review

The majority of studies in the ramping literature are evolving along two main streams. The first stream of the literature focuses on the wear and tear cost of ramping for fossil-fuel power plants [57, 58, 59, 60, 61, 89]. These works quantify and estimate the cost of ramping on different types of conventional power plants. Kumar et al. [54] obtain estimates for ramping cost based on generation type and size by applying statistical and engineering accounting methods. In a similar study, Jordan and Venkataraman [46] evaluate the increased cost of ramping due to the integration of wind and solar resources in the Western grid. Navid and Rosenwald [78] provide an optimization model for managing ramping with high penetration of renewable resources.
In the second stream, researchers and practitioners study the effect of ramping cost and constraints on the optimal power dispatch, called dynamic economic dispatch. Here, research has been done to optimally dispatch electricity while considering costs and constraints on generators’ ramp rate [45, 67, 86, 107]. Kumano [53] studies a dynamic economic load dispatch problem with constraints related to unit output and output ramp rate. Sherestha et al. [88] study the strategic use of ramping rates in power systems with price and demand volatility. They develop a set of ramping processes for ramping costs and rates for an economic dispatch problem, and investigate the impact of the ramping process. Tanaka [95] propose a quadratic cost function for ramping and develop an optimal control model to obtain the optimal pricing policy in the case of a steep change in electricity load. Similarly, Attaviriyanupap et al. [3] propose a heuristic algorithm based on SQP for solving a dynamic economic dispatch problem with non-smooth ramping cost function. These works consider ramping cost and constraints in a deterministic setting for dynamic economic dispatch or the dynamic optimal power flow problem. However, uncertainty is an important part of any realistic system. In the current study, we solve a stochastic dynamic economic dispatch problem with power flow and ramping constraints. In this work, we consider penetration of volatile and uncertain renewable resources and propose an optimization framework to manage the power system under high penetration of wind power.

The structure of this chapter is as follows. In Section 3.3, we study the effect of ramping cost on the dispatch policies of renewable generators. In Section 3.4, we develop mathematical optimization models to address the effect of ramping in power networks when there is uncertain power output from wind power plants. In Section 3.6, we provide numerical analysis. Finally, in Section 3.7, we conclude our work.
3.3 Ramping Cost and Renewable Dispatch Policies

From the perspective of damage to the system’s components, frequent ramping up and/or ramping down of fossil-fueled power plants causes thermal and pressure stresses which are the main reasons for thermal creep, fatigue and creep-fatigue interactions [58, 59, 89]. Creep is a time dependent and permanent deformation of materials when they are subjected to stress at high temperatures, and fatigue is a failure occurring under cyclic stress. Moreover, when a component faces cyclic stresses under high temperature, this component also experiences creep-fatigue interaction, which amplifies the damaging effect of creep and fatigue. These types of damage could result in reducing the life of components and thus increase capital and maintenance costs. In addition to the capital and maintenance cost, ramping up and/or ramping down increase fuel inefficiency and thus fuel consumption and cost increase. As mentioned by Kumar et al. [54], faster ramp rates mean increased costs, and the relationship is not linear. According to Viswanathan [104], during ramping up, the system faces creep and creep-fatigue interaction damage, and it faces fatigue damage in a ramping up and ramping down cycle. Therefore, a legitimate ramping cost function is nonlinear in ramp rate and asymmetric around zero, i.e., the ramping cost is greater for positive ramp rates than negative ones. However, for the sake of analysis, we relax the asymmetric condition and consider a quadratic cost function for ramping cost throughout this chapter.

Ramping cost has a significant effect on renewable dispatch policies due to the fact that renewable generators have variable power output and this variability requires frequent ramping by the conventional generators, and the cost of ramping could degrade the value of renewable sources. We could have two renewable dispatch policies, full and partial. In the full dispatch policy, we inject all observed renewable power to the grid, while in the partial dispatch policy, we inject a fraction of the observed renewable power to the grid in order to reduce ramping on the conventional generators. However,
the choice between these policies is not clear. At first glance, the full renewable dispatch policy seems to dominate the partial renewable dispatch policy due to its low marginal generation cost. It would be expected that if the system operator has resources with low marginal cost, it would try to dispatch as much as possible. However, the output of renewable resources is variable, and this variability requires ramping by the conventional generators. Thus it might be less costly to dispatch renewable power partially.

To show the effect of ramping cost on renewable dispatch, we consider a simplified system where there are no power flow and capacity constraints, we have only one conventional generator and one uncontrollable generator. Given the dispatch policy in period $t - 1$, and the observed renewable power in time periods $t - 1$ and $t$, we want to find the dispatch policy in period $t$ in order to minimize the total cost. Let $D$ be the constant demand, thus we have the following optimization problem:

$$\min C_p(p^t) + C_r(r^t) \quad (3.1)$$

s.t.

\begin{align*}
    p^t + y^t &= D \quad \forall t \quad (3.2) \\
    p^t - p^{t-1} - r^t &= 0 \quad \forall t \quad (3.3) \\
    0 \leq y^t \leq w^t \quad \forall t \quad (3.4)
\end{align*}

In this model, $p^t$ and $r^t$ are decision variables respectively representing the power generated (MWh) by the conventional generator and the amount of ramping up/down (MWh) in time period $t$. The decision variable $y^t$ is the amount of power out of the observed renewable power injected to the system at time $t$. In addition, $p^{t-1} = D - y^{t-1}$ represents the dispatch policy of period $t - 1$ and is given. By considering a linear generation cost and quadratic ramping cost, one can write the model as:

$$\min c_p D - c_p y^t + \frac{1}{2} c_r (y^t - y')^2 \quad (3.5)$$
Given $y^{t-1}$, the optimal value for $y^t$ is:

$$y^*_t = \begin{cases} \frac{c_p}{c_r} + y^{t-1}, & \text{if } w^t \geq \frac{c_p}{c_r} + y^{t-1} \\ w^t, & \text{otherwise} \end{cases}$$ (3.7)

Define $w^t - y^{t-1}$ as the maximum available ramp capacity in period $t$. The system prefers full renewable dispatch if the marginal ramping cost at the maximum available ramp capacity is less than the marginal generation cost, i.e., $c_p \geq c_r (w^t - y^{t-1})$, Figure 3.1(a). Moreover, the system prefers partial renewable dispatch if the marginal ramping cost at the maximum available ramp capacity is greater than the marginal generation cost, i.e., $c_p \leq c_r (w^t - y^{t-1})$, Figure 3.1(b).

![Figure 3.1: Renewable dispatch policy](image)

In the numerical simulation in Section 3.6, we observe that these results hold for realistic problems, too.
3.4 Deterministic Formulation

We observe that the ramping cost is important and has significant effects on the generation and dispatch policies in the power network. These effects need to be quantified and addressed properly to have a reliable dispatch with minimum cost. In this study, first, we develop a mathematical DC approximation of the optimal power flow model with ramping cost and constraints in a deterministic setting. Then, we extend it to a chance constrained optimization model to optimize the effect of ramping cost on the electricity grid and provide reliable service when there exist variable and uncertain renewable sources.

We consider a network of \( n \) buses containing a set \( G \) of generator buses and a set \( D \) of demand (load) buses (\( N = G \cup D \)). These sets are connected by transmission branches defined in set \( \mathcal{L} \). \( Y \in \mathbb{C}^{n \times n} \) is the admittance matrix, and \( B \) is the corresponding susceptance matrix. Let \( \theta_i^t \) and \( V_i \) be the voltage phase angle and voltage amplitude of bus \( i \in N \) at time \( t \). Then the DC power flow equations are:

\[
V_i V_j B_{i j} (\theta_i^t - \theta_j^t) \leq \bar{f}_{i j} \quad \forall i \neq j \in N
\]

\[
\sum_{j \in N} V_i V_j B_{i j} (\theta_i^t - \theta_j^t) = q_i^t \quad \forall i \in N
\]  

(3.8)

where \( q_i^t \) is the real power injection of node \( i \in N \) and \( \bar{f}_{i j} \) is the thermal capacity limit on branch connecting nodes \((i, j) \in N \). Note that the net power injection for a load bus is negative, i.e.,

\[
\begin{cases}
q_i^t \geq 0 & i \in G \\
q_i^t \leq 0 & i \in D.
\end{cases}
\]  

(3.9)

Under the connectedness assumption for the underlying graph, the system (3.8) is feasible in \( \theta^t \) if and only if \( \sum_{i \in N} q_i^t = 0 \) for \( t = 1, 2, ..., T \) [7]. Defining new variables \( \delta_{i j}^t = \theta_i^t - \theta_j^t \) to be the voltage angle difference from bus \( i \) to bus \( j \) and new parameters

\[1\text{This is a standard way to model the angle variables in OPF problems. See Cain et al. [14]}

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\[ \omega_{ij} = V_i V_j B_{ij}, \] one can rewrite system (3.8) as the following system of linear inequalities (The details of the derivation can be found in appendix B):

\[
\begin{align*}
\hat{B} \hat{q}' &\leq \bar{f} \\
-\hat{B} \hat{q}' &\leq \bar{f},
\end{align*}
\] (3.10)

where \( \hat{B} \) is a special \(|\mathcal{L}| \times (n - 1)\)-matrix, \( \bar{f} \) is an \(|\mathcal{L}|\)-vector of line capacity limits, and \( \hat{q}' \) is an \((n - 1)\)-vector of net power export from each node \((n = |\mathcal{N}|)\). Note that \( \hat{q}' \) is the vector \( q' \) excluding the reference bus.

Let \( p'_i \) be the power generated by generator \( i \in \mathcal{G} \) and \( r'_i \) be its power ramp up/down rate (when \( r'_i \geq 0 \), generator \( i \) is ramping up, and ramping down otherwise) at time \( t = 1, 2, \ldots, T \), and \( \hat{q}'_i = -d'_i \), \( \forall i \in \mathcal{D} \), and \( \hat{q}'_i = p'_i \), \( \forall i \in \mathcal{G} \), where \( d'_i \) is the load of bus \( i \) at time \( t \). Without loss of generality, assume that rows of \( \hat{q}' \) and columns of \( \hat{B} \) are arranged in such a way that the first \(|\mathcal{D}|\) rows of \( \hat{q}' \) correspond to the load buses and the remaining rows correspond to the generators. Thus, we can partition \( \hat{B} \) into two matrices \( \hat{B}^d \) and \( \hat{B}^g \), where \( \hat{B}^d \) contains the columns of matrix \( \hat{B} \) corresponding to the load buses (or negative generation) and \( \hat{B}^g \) contains the columns of matrix \( \hat{B} \) corresponding to the generator buses. Finally, for \( t = 1, 2, \ldots, T \), \( \forall i \in \mathcal{G} \) and \( \forall k \in \mathcal{L} \) we can write the mathematical model as:

\[
\begin{align*}
\min & \sum_{i \in \mathcal{G}} \sum_{t=1}^{T} C_p(p'_i) + C_r(r'_i) \\
\text{s. t.} & \\
\sum_{i \in \mathcal{D}} d'_i - \sum_{i \in \mathcal{G}} p'_i = 0 & \quad t = 1, 2, \ldots, T \quad (3.12) \\
\sum_{i \in \mathcal{G}} \hat{B}^d_{ki} p'_i &\leq \bar{f}_k + \sum_{i \in \mathcal{D}} \hat{B}^d_{ki} d'_i & \forall k \in \mathcal{L}, t = 1, 2, \ldots, T \quad (3.13) \\
-\sum_{i \in \mathcal{G}} \hat{B}^g_{ki} p'_i &\leq \bar{f}_k - \sum_{i \in \mathcal{D}} \hat{B}^g_{ki} d'_i & \forall k \in \mathcal{L}, t = 1, 2, \ldots, T \quad (3.14)
\end{align*}
\]
\[ p_i^t = p_i^{t-1} + r_i^t \tau \quad \forall i \in \mathcal{G}, \ t = 1, 2, \cdots, T \tag{3.15} \]

\[ P_i^{\text{min}} \leq p_i^t \leq P_i^{\text{max}} \quad \forall i \in \mathcal{G}, \ t = 1, 2, \cdots, T \tag{3.16} \]

\[ r_i^{\text{min}} \leq r_i^t \tau \leq r_i^{\text{max}} \quad \forall i \in \mathcal{G}, \ t = 1, 2, \cdots, T \tag{3.17} \]

In this model, \( p_i^t \) and \( r_i^t \) are decision variables for \( \forall i \in \mathcal{G} \) and \( \tau \) is the elapsed time between time periods \( t \) and \( t+1 \) for \( t = 1, 2, \cdots, T \). The objective cost function (3.11) consists of the power generation cost, \( C_p(\cdot) \) and the ramping cost, \( C_r(\cdot) \). In the mathematical model, we consider \( C_p(\cdot) \) to be a linear function of the power generation level and the ramping cost, \( C_r(\cdot) \), to be a quadratic function of the ramp rate. Constraints (3.13) and (3.14) are the DC power flow constraints on each branch \( k \in \mathcal{L} \). In constraints (3.12), \( d_i(t) \) is the demand of bus \( i \in \mathcal{D} \) in time period \( t \). These constraint are the power balance equations, ensuring that the total power demand equals the total power supply in each time period. Constraints (3.15) are the ramping inter-temporal constraints for the active power. Finally, the lower bounds and upper bounds on the decision variables are defined by constraints (3.16) and (3.17). Notice that we have initial conditions on \( p_i^0 = p_{0i} \) for \( i \in \mathcal{G} \).

### 3.5 Stochastic Formulation

Now, assume that the set of generator buses, \( \mathcal{G} \), contains a set of controllable and conventional generators, \( \mathcal{G} \), and a set of uncontrollable generators, i.e., wind generators, \( \tilde{\mathcal{G}} \). Let \( \tilde{p}_j^t \) be a random variable representing the output of uncontrollable generator \( j \in \tilde{\mathcal{G}} \) in time period \( t \) and \( x_j^t \in [0, 1] \) be the fraction of its power dispatched at time \( t \) by the system operator. We will rewrite the equality constraints (3.12) as:

\[ \sum_{i \in \mathcal{D}} d_i^t - \sum_{j \in \mathcal{\tilde{G}}} x_j^t \tilde{p}_j^t - \sum_{i \in \mathcal{G}} p_i^t = 0 \quad t = 1, 2, \cdots, T \tag{3.18} \]
The output of uncontrollable generator $j \in \hat{G}$ is random ($\bar{p}_j$), and we need power balance equality constraints (3.18) to hold for any possible uncertain realizations of $\bar{p}_j$, $\forall j \in \hat{G}$. As stated in Bienstock et al. [7], the controllable power output needs to be adjusted in real time in response to the uncertain output of uncontrollable generators and this requires a secondary control, which they call affine control, to reset the output of conventional generators within ramping period $\tau$. If the power output of uncontrollable generator $j \in \hat{G}$ has a form like

$$\bar{p}_j = \mu_j + \epsilon_j,$$

(3.19)

then the affine control that resets the output of conventional generator $i \in \hat{G}$ will be:

$$p_i = \bar{p}_i - \frac{1}{|\hat{G}|} \sum_{j \in \hat{G}} x_j \epsilon_j,$$

(3.20)

By these definitions, the equality constraints (3.18) hold for any possible uncertain realization of $\bar{p}_j$, $\forall j \in \hat{G}$, if

$$\sum_{i \in D} d_i - \sum_{j \in \hat{G}} x_j \mu_j - \sum_{i \in \hat{G}} \bar{p}_i = 0, \quad \forall t.$$

(3.21)

Finally, as a consequence of affine control we have:

$$r'_i = \bar{r}'_i - \frac{1}{|\hat{G}|} \sum_{j \in \hat{G}} x_j \epsilon_j' + \frac{1}{|\hat{G}|} \sum_{j \in \hat{G}} x_j^{-1} \epsilon_j'^{-1},$$

(3.22)

thus, the equality constraints (3.15) hold if

$$\bar{p}_i = \bar{p}^{-1}_i + r_i \tau, \quad \forall i \in G, \forall t.$$

(3.23)

Moreover, in the stochastic model, constraints (3.13), (3.14), (3.16), and (3.17) are random. We want to make sure that these random constraints are satisfied with a certain
probability. This leads to a \textit{chance constrained optimization} model for the stochastic case. Let $\hat{B}^\delta$ be the matrix of columns of $\hat{B}^r$ corresponding to the controllable generators, and $\hat{B}^\delta$ be the matrix of columns of $\hat{B}^u$ corresponding to the uncontrollable generators. The chance constraints corresponding to (3.13) and (3.14) are respectively written as:

\begin{equation}
\operatorname{Prob}\left\{-\sum_{i \in \bar{G}} \hat{B}_{ki}^\delta p^i_k - \sum_{j \in G} \hat{B}_{kj}^\delta x^j_k \bar{p}^j_k \geq -\bar{f}_k - \sum_{i \in \bar{D}} \hat{B}_{ki}^d d^i_k \right\} \geq 1 - \gamma_r, \quad \forall k \in L, t = 1, 2 \cdots, T
\end{equation}

(3.24)

\begin{equation}
\operatorname{Prob}\left\{\sum_{i \in \bar{G}} \hat{B}_{ki}^\delta p^i_k + \sum_{j \in G} \hat{B}_{kj}^\delta x^j_k \bar{p}^j_k \geq -\bar{f}_k + \sum_{i \in \bar{D}} \hat{B}_{ki}^d d^i_k \right\} \geq 1 - \gamma_l, \quad \forall k \in L, t = 1, 2 \cdots, T
\end{equation}

(3.25)

where $\bar{p}^j_k$ and $p^i_k$ are defined as (3.19) and (3.20), respectively. Similarly, the corresponding chance constraints of (3.16) are:

\begin{equation}
\operatorname{Prob}\left\{-\bar{p}^i_k + \frac{1}{|\bar{G}|} \sum_{j \in \bar{G}} x^j_k e^j_k \geq -P^\text{max}_i \right\} \geq 1 - \gamma_r, \quad \forall i \in \bar{G}, t = 1, 2 \cdots, T.
\end{equation}

(3.26)

\begin{equation}
\operatorname{Prob}\left\{\bar{p}^i_k - \frac{1}{|\bar{G}|} \sum_{j \in \bar{G}} x^j_k e^j_k \geq P^\text{min}_i \right\} \geq 1 - \gamma_l, \quad \forall i \in \bar{G}, t = 1, 2 \cdots, T.
\end{equation}

(3.27)

finally, the corresponding chance constraints of (3.17) are:

\begin{equation}
\operatorname{Prob}\left\{-\bar{r}^i_k + \frac{1}{|\bar{G}|} \sum_{j \in \bar{G}} x^j_k e^j_k - \frac{1}{|\bar{G}|} \sum_{j \in \bar{G}} x^j_k e^j_k \geq -r^\text{max}_i \right\} \geq 1 - \gamma_r, \quad \forall i \in \bar{G}, t = 1, 2 \cdots, T,
\end{equation}

(3.28)

\begin{equation}
\operatorname{Prob}\left\{\bar{r}^i_k - \frac{1}{|\bar{G}|} \sum_{j \in \bar{G}} x^j_k e^j_k + \frac{1}{|\bar{G}|} \sum_{j \in \bar{G}} x^j_k e^j_k \geq r^\text{min}_i \right\} \geq 1 - \gamma_l, \quad \forall i \in \bar{G}, t = 1, 2 \cdots, T.
\end{equation}

(3.29)
We need to convert the probabilistic constraints (3.24), (3.25), (3.26), (3.27), (3.28) and (3.29) into their deterministic equivalent in order to have a tractable optimization problem. Generally, to handle probabilistic constraints of the form

\[ Pr\{G(x, \xi) \geq R\} \geq 1 - \gamma, \]

we need to compute \( Pr\{G(x, \xi) \geq R\} \) [2]. To do so, we can use two approaches. First, by sampling a number of realizations (scenarios) for \( \xi \), we can approximate \( Pr\{G(x, \xi) \geq R\} \) and write the equivalent optimization problem as a mixed-integer program [65, 66].\(^2\)

Second, if we have probability distribution of \( G(x, \xi) \), we can compute \( Pr\{G(x, \xi) \geq R\} \) accurately [9, 16, 31]. The result would be a deterministic equivalent which can either be convex or non-convex. Assume that the probability distribution for the source of uncertainty is known (in this study it is wind power), and consider the following probabilistic constraint:

\[ \text{Prob}\{\xi^T x \geq R\} \geq 1 - \gamma, \] \hspace{1cm} (3.30)

where \( x \) is the vector of decision variables and \( \xi \) is a random vector with mean \( \mu \) and variance-covariance matrix \( \Sigma \). The deterministic equivalent of (3.30) is the following nonlinear constraint [9]:

\[ \mu^T x + F^{-1}(\gamma)x^T \Sigma x \geq R, \] \hspace{1cm} (3.31)

where \( F(\cdot)(\psi) \) is the cumulative probability distribution of the following random variable:

\[ \psi = \frac{\mu^T x - \xi^T x}{\sqrt{x^T \Sigma x}}, \]

and \( F^{-1}(\gamma) \) is its inverse. Bonami and Lejeune [9] show that when \( 0 < \gamma \leq 0.5 \) and the probability distribution of \( \xi^T x \) is symmetric or positively skewed, the deterministic equivalent (3.31) of the probabilistic constraint (3.30) is a second-order cone constraint.\(^2\)

Appendix D provides the details of the MIP formulation of the problem.
3.5.1 Second-Order Cone Formulation

As mentioned earlier, it is possible to write the deterministic equivalent of a chance constraint using the probability distribution of the source of uncertainty. Now, assume that we have the probability distribution of the random variables \( \varepsilon_j^t \) for \( t = 1, 2, \ldots, T \) and \( \forall j \in \tilde{G} \) and their affine combinations. Thus, one might compute the probability constraints (3.24), (3.25), (3.26), (3.27), (3.28) and (3.29) accurately. Assume that we are given the probability distribution of \( \sum_{j \in \tilde{G}} a_j \varepsilon_j^t \), \( \forall t \) and \( F_a(\cdot) \) is its cumulative probability distribution (\( F_{a}^{-1}(\cdot) \) is its inverse). Also, \( \forall i, j \in \tilde{G} \) and \( \forall t, E(\varepsilon_j^t) = 0, \text{cov}(\varepsilon_j^t, \varepsilon_j^{t-1}) = \sigma_{ij} \), and \( \text{cov}(\varepsilon_j^t, \varepsilon_j^{t-1}) = 0. \) Thus, we can use the fractile formulation to obtain the deterministic form of the probabilistic constraints (3.24), (3.25), (3.26), (3.27), (3.28) and (3.29).

For that purpose, define the following parameters:

\[
D_k^t = \sum_{i \in D} \tilde{B}_{ki}^t d_i, \quad \forall k \in \mathcal{L}, \forall t \\
\tilde{p}_k = \frac{1}{|\tilde{G}|} \sum_{i \in \tilde{G}} \tilde{B}_{ki}, \quad \forall k \in \mathcal{L} \\
\varepsilon^t_{jk} = \mu_j \tilde{B}_{kj}, \quad \forall j \in \tilde{G}, \forall k \in \mathcal{L}, \forall t \\
\sigma^k_{i,j} = (\tilde{B}_{ki} - \tilde{p}_k)(\tilde{B}_{kj} - \tilde{p}_k)\sigma_{ij}, \quad \forall i, j \in \tilde{G}, \forall k \in \mathcal{L}
\]

thus, the deterministic equivalent of the chance constrained optimization model is:

\[
\min \sum_{i \in \tilde{G}} \sum_{t=1}^{T} E_w[C_p(\tilde{p}_i - \frac{1}{|G|} \sum_{j \in G} x_j \varepsilon_j^t) + C_r \left( \tilde{r}_i - \frac{1}{|G|} \sum_{j \in G} x_j \varepsilon_j^t - x_j^{t-1} \varepsilon_j^{t-1} \right)] \tag{3.32}
\]

s.t.

\[
\sum_{i \in G} d_i^t - \sum_{j \in G} \mu_j - \sum_{i \in G} \tilde{p}_i = 0 \quad \forall t \tag{3.33}
\]

\[
- \sum_{i \in G} \tilde{B}_{ki}^t \xi_k^t - \sum_{j \in G} \xi_j^t x_j^t + D_k^t + f_i + F_{a}^{-1}(\gamma_r) \sqrt{\sum_{i \in G} \sum_{j \in G} \sigma^k_{i,j} x_j^t x_j^t} \geq 0 \quad \forall t, k \in \mathcal{L} \tag{3.34}
\]
\[
\sum_{i \in \bar{G}} \hat{B}_{ki}^g \hat{p}_t + \sum_{j \in \bar{G}} \xi_{kj}^i x_j^t - D_k^t + \hat{f}_k + F^{-1}_{(\alpha)}(\gamma_t) \sqrt{\sum_{i \in \bar{G}} \sum_{j \in \bar{G}} \sigma_{ij} x_i^t x_j^t} \geq 0 \quad \forall t, k \in \mathcal{L} 
\]

(3.35)

\[
\bar{r}_k^t |\bar{G}| - r_k^{\text{min}} |\bar{G}| + F^{-1}_{(\alpha)}(\gamma_t) \sqrt{\sum_{i \in \bar{G}} \sum_{j \in \bar{G}} \sigma_{ij} (x_i^t x_j^t + x_i^{t-1} x_j^{t-1})} \geq 0 \quad \forall t, k \in \bar{G} 
\]

(3.36)

\[
-\bar{r}_k^t |\bar{G}| + r_k^{\text{max}} |\bar{G}| + F^{-1}_{(\alpha)}(\gamma_t) \sqrt{\sum_{i \in \bar{G}} \sum_{j \in \bar{G}} \sigma_{ij} (x_i^t x_j^t + x_i^{t-1} x_j^{t-1})} \geq 0 \quad \forall t, k \in \bar{G} 
\]

(3.37)

\[
\bar{p}_k^t |\bar{G}| - p_k^{\text{min}} |\bar{G}| + F^{-1}_{(\alpha)}(\gamma_t) \sqrt{\sum_{i \in \bar{G}} \sum_{j \in \bar{G}} \sigma_{ij} x_i^t x_j^t} \geq 0 \quad \forall t, k \in \bar{G} 
\]

(3.38)

\[
-\bar{p}_k^t |\bar{G}| + p_k^{\text{max}} |\bar{G}| + F^{-1}_{(\alpha)}(\gamma_t) \sqrt{\sum_{i \in \bar{G}} \sum_{j \in \bar{G}} \sigma_{ij} x_i^t x_j^t} \geq 0 \quad \forall t, k \in \bar{G} 
\]

(3.39)

\[
\bar{p}_k^t = \bar{p}_k^{t-1} + \bar{r}_i^t \quad \forall t, k \in \bar{G} 
\]

(3.40)

\[
0 \leq x_j^t \leq 1 \quad \forall t, j \in \bar{G} 
\]

(3.41)

The objective function (3.32) for the case of linear generation cost and quadratic ramping cost has the following form:

\[
\sum_{i=1}^{T} \sum_{i \in \bar{G}} c_i^i \bar{p}_i^t + \frac{1}{2} c_{ij}^i (\bar{p}_j^t)^2 + \frac{1}{|\bar{G}| r^2} \sum_{i=1}^{T} \sum_{i,j \in \bar{G}} \sum_{i,j \in \bar{G}} c_{ij}^i \sigma_{ij} (x_i^t x_j^t + x_i^{t-1} x_j^{t-1}) 
\]

Note that to have a convex optimization model, \( F^{-1}_{(\alpha)}(\cdot) \) should be independent of the decision variables, \( x_i, \forall i \in \bar{G} \), and be non-positive, \( F^{-1}_{(\alpha)}(\cdot) \leq 0 \). Therefore, first, we need to find conditions under which \( F^{-1}_{(\alpha)}(\cdot) \leq 0 \). Second, we need to obtain a good approximation for \( F^{-1}_{(\alpha)}(\cdot) \) to be independent of \( x_i, \forall i \in \bar{G} \).
Here, the source of uncertainty is the wind power output of a wind farm. Thus, to have a convex nonlinear optimization problem, we need to explore its probability distribution and properties. Despite numerous theoretical and empirical studies on wind speed probability distribution [5, 24, 35, 41, 47, 76, 87, 101], there are few works that address specifically wind power probability distributions [42, 64, 116]. Previous studies in the literature [70, 76, 101] show that the wind speed profile at a given location can be fitted closely by a Weibull or a Rayleigh distribution over time.

A wind farm consists of multiple wind turbines. The probability distribution of the wind farm’s power output is the same as the probability distribution of the power output of a wind turbine with different mean and variance, assuming that all wind turbines in the farm are identical and experience almost the same wind speed. For an idealized wind turbine, the wind power as function of a given wind speed is [64]:

\[
P_w(v) = \begin{cases} 
0 & v < v_i \\
\frac{v^3 - v_i^3}{v_r^3 - v_i^3}P_{\text{max}} & v_i \leq v \leq v_r \\
P_{\text{max}} & v_r \leq v \leq v_o \\
0 & v > v_o,
\end{cases}
\]

(3.42)

where \(P_w\) is the wind power output (kW or MW), \(P_{\text{max}}\) is the rated wind power (kW or MW), \(v_i\) is the cut-in wind speed (miles per hour), \(v_r\) is the rated wind speed (miles per hour) and \(v_o\) is the cut-out wind speed (miles per hour).

It is clear that the wind power has a mixed discrete-continuous probability distribution. Let \(f_v, F_v\) be the pdf and cdf of the wind speed random variable, respectively, and let \(f_{P_w}\) and \(F_{P_w}\) be the pdf and cdf of the wind power random variable, respectively.
Then we have:

\[ f_{p_\omega}(p) = \begin{cases} 
\alpha_0 = F_V(v_i) + 1 - F_V(v_o), & p = 0 \\
\frac{dF_V(v(p))}{dv} \cdot \frac{dx(p)}{dp}, & 0 < p < P_{\text{max}} \\
\alpha_{\text{max}} = F_V(v_o) - F_V(v_r), & p = P_{\text{max}} 
\end{cases} \]  

(3.43)

where \( v(p) = \left( \frac{p(v_i^3 - v_r^3) + P_{\text{max}}^3}{v_i^3} \right)^\frac{1}{3} \).

In the SOCP formulation of (3.32)–(3.41), \( F^{-1}_V(\gamma) \) is non-positive if the wind power probability distribution is positively skewed for \( 0 < \gamma \leq 0.5 \), or \( F^{-1}_p(\gamma) \leq 0 \) in a certain rage of \( \gamma \).

**Lemma 3.1.** The wind probability distribution defined in (3.43) is positively skewed if

\[ F_V^{-1}(\alpha) \leq \left( \alpha_{\text{max}}v_r^3 + \alpha_0v_i^3 + \min\{v_i^3f_V(v_i), v_r^3f_V(v_r)\} \right)^\frac{1}{3}, \]  

(3.44)

where \( \alpha = 0.5 + F_V(v_i) - \alpha_0 \).

**Proof of Lemma 3.1.** A probability distribution of a random variable is positively skewed if its median is strictly smaller than its mean. The mean for the wind power probability distribution (3.43) is:

\[
\mu_{p_\omega} = E(P) = P_{\text{max}}\alpha_{\text{max}} + \int_{v_i}^{v_r} \left( \frac{v_i^3 - v_r^3}{v_i^3} P_{\text{max}} \right) f_V(v)dv \\
= P_{\text{max}} \left( \alpha_{\text{max}} - F_V(v_r) \frac{v_i^3}{v_i^3 - v_r^3} + F_V(v_i) \frac{v_r^3}{v_i^3 - v_r^3} + \int_{v_i}^{v_r} \frac{v_i^3}{v_i^3 - v_r^3} f_V(v)dv \right). 
\]

(3.45)

We can show that

\[
\mu_{p_\omega} \geq P_{\text{max}} \left( \alpha_{\text{max}} - F_V(v_r) \frac{v_i^3}{v_i^3 - v_r^3} + F_V(v_i) \frac{v_r^3}{v_i^3 - v_r^3} + \min\left\{ \frac{v_i^3}{v_i^3 - v_r^3} f_V(v_i), \frac{v_r^3}{v_i^3 - v_r^3} f_V(v_r) \right\} \right). 
\]

(3.46)

Let \( P_{\text{m}} \) be the median of the wind power distribution. Then,

\[
Pr[p \leq P_{\text{m}}] = 0.5 \iff Pr[v \leq \bar{V}] + \alpha_0 = 0.5 \iff F_V(\bar{V}) = 0.5 + F_V(v_i) - \alpha_0 \]  

(3.47)

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Let $\tilde{\alpha} = 0.5 + F_V(v_i) - \alpha_0$. Then, $\tilde{V} = F_V^{-1}(\tilde{\alpha})$ and $P_m = \frac{\bar{v}^3 - v^3_i}{v^3_r - v^3_i} P_{max}$. The wind power distribution has positive skewness if $P_m \leq \mu_{P_m}$, and this is always true if

$$F_V^{-1}(\tilde{\alpha}) \leq \left( \alpha_{max}(v^3_r - v^3_i) - F_V(v_i)v^3_i + F_V(v_i)v^3_i + \min\{v^3_i f_V(v_i), v^3_i f_V(v_i)\} + v^3_i \right)^{\frac{1}{2}} \leq \left( \alpha_{max}v^3_r + \alpha_0 v^3_i + \min\{v^3_i f_V(v_i), v^3_i f_V(v_i)\} \right)^{\frac{1}{2}}.$$ 

Therefore, given (3.44), for a single wind farm, $F_V^{-1}(\gamma) \leq 0$ when $0 < \gamma \leq 0.5$. However, condition (3.44) is satisfied when the wind speed mean is close to the cut-in speed, $v_i$, which is not very common. To resolve this issue, one might relax the positive skewness condition by bounding the reliability probability, $\gamma$, in such a way that $F_V^{-1}(\gamma) \leq 0$.

**Theorem 3.1.** For a single wind farm, the deterministic equivalent of (3.30) for the case of uncertain wind power is a convex second-order cone if there exists $\gamma_c \in (0, 1)$ such that $0 < \gamma \leq \gamma_c$ and

$$\gamma_c \leq \alpha_0 + F_V(v_i) + F_V \left( (\alpha_0 v^3_i + \alpha_{max}v^3_r)^{\frac{1}{2}} \right). \tag{3.48}$$

**Proof of Theorem 3.1** Assume that $\exists \gamma_c$ such that $F_V^{-1}(\gamma) \leq 0$ for $0 < \gamma \leq \gamma_c$. Then, $\exists p_c$ such that $F_{p_c}(p_c) = \gamma_c$ and $p_c = P_w(v_c)$. Thus, we have $\gamma_c = \alpha_0 + F_V(v_c) - F_V(v_i)$ and

$$v_c = F_V^{-1}(\gamma_c - \alpha_0 + F_V(v_i)).$$

In order to have $F_V^{-1}(\gamma) \leq 0$ for $0 < \gamma \leq \gamma_c$, we need to have $p_c \leq \mu_{P_m}$, which implies

$$\frac{v^3_c - v^3_i}{v^3_r - v^3_i} P_{max} \leq P_{max} \left( \alpha_{max} - F_V(v_c) \right) \frac{v^3_i}{v^3_r - v^3_i} + F_V(v_i) \frac{v^3_i}{v^3_r - v^3_i} + \int_{v_i}^{v_r} \frac{v^3}{v^3_r - v^3_i} f_V(v)dv,$$
and with further algebraic simplification, we get

\[ v_c \leq \left( \alpha_0 v_i^3 + \alpha_{\text{max}} v_r^3 + \int_{v_i}^{v_r} v^3 f_v(v) dv \right)^{\frac{1}{3}}. \]

This inequality holds if

\[ v_c \leq \left( \alpha_0 v_i^3 + \alpha_{\text{max}} v_r^3 \right)^{\frac{1}{3}}, \]

where \( v_c = F_v^{-1}(\gamma_c - \alpha_0 + F_v(v_i)) \). Therefore,

\[ \gamma_c \leq \alpha_0 + F_v(v_i) + F_v \left( (\alpha_0 v_i^3 + \alpha_{\text{max}} v_r^3)^{\frac{1}{3}} \right) \]

\[ \square \]

Theorem 3.1 provides conditions under which \( F_{(s)}^{-1}(\gamma) \leq 0 \) for a single wind farm. However, in the real problem, a power system contains more than one wind farm and we are interested in the following random variable:

\[ \bar{P}_{(s)} = \sum_{i=1}^{N} x_i P_{w}^i, \]

where \( N \) is the number of wind farms. If \( N \) is sufficiently large and farms are distant from each other, \( \bar{P}_{(s)} \) will be approximately normal according to the law of large numbers and \( F_{(s)}^{-1}(\gamma) \) can be easily obtained from a normal table. However, when \( N \) is small, we need to approximate \( F_{(s)}^{-1}(\gamma) \). For small enough \( \gamma \), we have

\[ N \max\{F_{P_w}^{-1}(\gamma) : i\} \leq F_{(s)}^{-1}(\gamma) \leq \max\{F_{P_w}^{-1}(\gamma) : i\}, \quad (3.49) \]

thus, if these bounds are tight, we can use the lower bound as a proper approximation for \( F_{(s)}^{-1}(\gamma) \). Given the wind power distribution, obtaining \( N \max\{F_{P_w}^{-1}(\gamma) : i\} \) is not a difficult task. Moreover, with this approximation, the SOCP optimization problem is
convex. However, using the lower bound in the SOCP would result in a more conserva-
tive solution.

To evaluate the accuracy of the approximation, we run a numerical simulation con-
sidering the Rayleigh probability distribution for the underlying wind speed. We con-
sider wind farms with 10 wind turbines. We assume all turbines in a farm are almost
identical in terms of maximum rated power and technical characteristics. Figure 3.2(a),
Figure 3.2(b), Figure 3.2(c) and Figure 3.2(d) show the values of $\max\{F_{P_{iw}}^{-1}(\gamma) : i\}$ and
$N \max\{F_{P_{iw}}^{-1}(\gamma) : i\}$ for a power system with $N = 2, 3, 4$ and $N = 5$ wind farms. The
farms have different power capacities and chosen randomly between 9 MW and 300
MW, as the minimum and maximum installed capacities of existing wind farms in PJM
in 2010.\footnote{PJM’s EIA-411 report.} These figures illustrate that for realistic situations and for the small number
of farms, approximation (3.49) is reasonable. The lower bound is not a tight bound for
cases where the number of farms is big, however, when in this case we can use Normal
approximation due to the central limit theorem.

\section*{3.6 Numerical Analysis}

In Section 3.4, we develop methodological optimization models to minimize cost and
improve power system reliability. Here, we run a numerical experiment to study the
performance of the proposed methodologies in detail. We consider IEEE14 system with
2 wind farms and 3 conventional generators, a IEEE30 system with 2 wind farms and 4
conventional generators, and a IEEE118 system with 4 wind farms and 50 conventional
generators . We run the experiment for $T = 24$ hours with different levels for the
variability in the mean of wind power output: low, medium and high. The variability is
defined by the change in the mean of wind power between two consecutive periods. To
measure the variability in the power output of wind generator $j \in \mathcal{G}$, we consider the
The wind output has no variability when $\eta = 0$, and it increases as $\eta$ increases. The value of $\eta$ for high, medium and low variability is 0.7, 0.3 and 0.1, respectively. Since our focus in this work is on the effect of wind variability and uncertainty on generation and ramping costs, we consider almost constant and flat demand profile in order to isolate ramping due to load following from ramping due to the wind integration. For ramping, we consider two capability levels, 10% and 30%. The capability level
is defined as the percentage of total capacity of a conventional generator that can be dedicated to ramping. Finally, we run the experiment for $1 - \gamma = 0.95$ and $1 - \gamma = 0.98$ reliability levels.

We solve the MIP formulation with 200 randomly generated scenarios, and the SOCP formulation using Gurobi 5.6 on a desktop PC with 2.10GHz x86 64-bit Intel(R) Core(TM)2 Duo CPU and 4.00 GB RAM. It turns out that the MIP formulation of IEEE118 system cannot be solved in reasonable time (less than 10000 seconds) even if we reduce the number of scenarios. Thus, we report only the solution of the SOCP formulation for IEEE118 system. All numerical results are in Tables 3.1-3.12. In addition, it might be of interest to see the dispatch policy for different wind variability and different solution approaches. Figures 3.3- 3.8 show the changes in power dispatch of conventional generators and wind generators for 98% reliability and 10% ramp capability for IEEE14 and IEEE30 systems.

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Table 3.1: 14-bus system with 10% ramp capability and 98% reliability
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Table 3.2: 14-bus system with 10% ramp capability and 95% reliability

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Table 3.3: 14-bus system with 30% ramp capability and 98% reliability
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Table 3.4: 14-bus system with 30% ramp capability and 95% reliability

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Table 3.5: 30-bus system with 10% ramp capability and 98% reliability
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Table 3.6: 30-bus system with 10% ramp capability and 95% reliability

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Table 3.7: 30-bus system with 30% ramp capability and 98% reliability
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Table 3.8: 30-bus system with 30% ramp capability and 95% reliability

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Table 3.9: 118-bus system with 10% ramp capability and 98% reliability
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Table 3.10: 118-bus system with 10% ramp capability and 95% reliability

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Table 3.11: 118-bus system with 30% ramp capability and 98% reliability
### Table 3.12: 118-bus system with 30% ramp capability and 95% reliability

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Figure 3.3: IEEE14 system under low wind variability with 10% ramp capability and 98% reliability
Figure 3.4: IEEE14 system under medium wind variability with 10% ramp capability and 98% reliability

Figure 3.5: IEEE14 system under high wind variability with 10% ramp capability and 98% reliability
Figure 3.6: IEEE30 system under low wind variability with 10% ramp capability and 98% reliability

Figure 3.7: IEEE30 system under medium wind variability with 10% ramp capability and 98% reliability

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These numerical results indicate that the SOCP methodology and formulation provide more conservative solutions than the MIP, both in total ramping and total wind power penetration. For example, in the 14-bus problem with low wind variability, Figure 3.3, the total power output of conventional generators in the MIP solution fluctuates more between 225 MW and 250 MW over the horizon, as oppose to the total power output of conventional generators in the SOCP solution (between 250 MW and 260 MW). In this test problem, as reported in Table 3.1, under low wind variability, the total wind power dispatch is 518 MW for the MIP solution and is 152 MW for the SOCP solution. Also, the total ramping cost is 16700$ for the MIP solution and is 5300$ for the SOCP solution. We observe almost the same trend in the other test problems, however, as the size of the system increases, the MIP methodology and formulation become ineffective in terms of computational performance. In addition, the results show that as the wind variability increases, the SOCP and MIP methodologies dispatch slightly less
wind power to reduce the cost and damaging effects of ramping.

3.7 Conclusion

In this chapter, we study ramping costs and constraints along with their effects on generation and dispatch policies. We develop and propose a stochastic chance constrained optimization framework to analyze the ramping effects in electricity networks when there exist uncertain and volatile renewable energy resources. We demonstrate that ramping can have a significant effect on the renewable dispatch policy and that is necessary to minimize the operational costs of the system while maintaining a certain reliability. To solve the stochastic chance constrained optimization model, we use mixed integer programming (MIP) by sampling scenarios and second order conic programming (SOCP), which are common in the literature, to transform the stochastic model into its equivalent deterministic model. The SOCP formulation requires certain conditions to be satisfied to have a convex optimization problem. In this work, we prove conditions for wind power and wind farms under which the SOCP formulation is convex.

The numerical experiment shows the impracticality of the MIP methodology compared to the SOCP formulation. The numerical results indicate that the SOCP methodology provides more conservative solutions than the MIP, both in total ramping and total wind power dispatch. However, as the size of the system increases, the MIP formulation becomes ineffective in terms of computational performance. In addition, the results show that as the wind variability increases, the SOCP and MIP methodologies dispatch slightly less wind power to reduce the cost and damaging effects of ramping.

For future research, it is necessary to obtain and estimate more accurate ramping costs for different conventional power plants. Moreover, considering other renewable sources such as wave and solar and their complications in the modeling and optimization is interesting and should be studied.
Chapter 4

Wave Energy Converter Location

Problems

4.1 Introduction

In this chapter, we study optimization algorithms and methods for the problem of finding an \textit{optimal configuration for an array of wave energy converter (WEC) devices}. Renewable energy, such as wave energy, plays a significant role in sustainable energy development. Ocean wave energy represents a large untapped source of energy in the world and potentially offers a vast source of sustainable energy. According to the Electric Power Research Institute [26], the total potential wave energy resource along the U.S. continental shelf edge is estimated to be 1,170 TWh per year, which is almost one third of the annual electricity consumption in the U.S. This energy resource has the advantage of being in close proximity to the coastal load centers in the U.S., which makes the transmission of the generated energy more efficient than, say, wind farms located in the geographical center of the country. Moreover, wave energy is more predictable and stable than wind and solar energy. For a summary of emerging WEC technologies, see, e.g., Karimirad [48], Kishore et al. [50], or López and Iglesias [63].
However, uncertainties are still present and need to be investigated and mitigated in order to have optimized power from wave energy systems. There is still a need to design cost-effective, robust and standardized WEC devices to operate in realistic ocean environments and capture the maximum power from ocean waves. In addition to the technological issues of designing efficient and standardized WEC devices, which is out of the scope of this research, the problem of optimal and cost effective configuration of multiple devices, known as a wave farm, needs more research in order to make ocean wave energy economically competitive and attractive.

In the present work, we study the problem of determining the optimal layout of multiple WECs in a wave farm. A wave farm’s configuration or layout can have a significant effect on the power output of the farm, depending on the nature of the interference (constructive or destructive) among the incident ocean waves and the scattered and radiated waves produced by the WECs. In the interactions between waves and WECs or other rigid bodies, there are three types of waves [73]: Incident waves represent disturbances due to natural forces and occur whether or not the body is present; in other words, they are the incoming ocean waves. Scattered waves correspond to the collision and dispersion of incident waves with a fixed body; they represent the way in which incoming waves are deformed by the body. Finally, the motion of the body itself produces radiated waves, similar to the circular waves produced by a stone dropped into a lake. (The same three types of waves are relevant for electromagnetic waves.)

When the incident wave hits a WEC, therefore, it produces scattered and radiated waves that interact with each other and with the incident wave. Downstream WECs experience the combined wave and, in turn, produce their own scattered and radiated waves. We would like all of the WECs to be located at points of constructive interference so that they all experience waves whose amplitudes are as large as possible, thus maximizing the power produced. Unfortunately, the layout problem is complex, since the mathematical model of the hydrodynamic interactions is nonconvex and computa-
tionally expensive, and since the objective function is non-separable, i.e., the power at a given WEC depends on the locations of all the other WECs.

WECs absorb mechanical power from waves and convert it to electrical power. The conversion process is a function of how the WECs are engineered (and is often proprietary), and therefore there is no readily available function relating the mechanical power absorbed with the electrical power produced. Following the literature, therefore, we treat the absorbed mechanical power as a proxy for the electrical power produced; when we say “power,” we mean mechanical power. The hydrodynamics of simple wave farms are relatively well understood, as is the absorbed power (see Section 4.3), but computing the exact absorbed power requires a boundary element code such as WAMIT [106]. To evaluate the absorbed power for a given layout can take seconds or even minutes (for a large number of WECs and/or complex WEC geometries), and except for very special cases, a closed form analytical expression is out of the question. Therefore, exact calculations are computationally prohibitive within an optimization context. Instead, we make use of a well established approximation in our formulation, called the **point-absorber approximation**. In the point-absorber approximation, the devices are assumed to be small enough with respect to the wavelength of incident waves that the scattered waves can be neglected.

It is common in the literature to focus on the *q*-factor, which is the ratio between the total power absorbed by *N* WECs in a wave farm to the power that would be absorbed by *N* WECs acting in isolation:

\[
q = \frac{\sum_{n=1}^{N} P_n}{NP_0},
\]

where *P*<sub>*n*</sub> is the power absorbed by the *n*th device in the array and *P*<sub>0</sub> is the power absorbed by a single device acting in isolation (a constant). We follow this convention and use the *q*-factor as our objective function.

In the next section, we study the past research in this domain, then in Section 4.3
briefly describe the theoretical background for arrays of multiple wave energy converters. In Section 4.4, we formulate the deterministic optimization problem and explore its theoretical and structural properties. We present a solution algorithm for the deterministic model in Section 4.5. Next, we extend the analysis to the real ocean environment in Section 4.6. Finally, we discuss our conclusions in Section 4.8.

4.2 Literature Review

Extracting energy from ocean waves has a history of more than two centuries, when inventors proposed many different devices to harvest ocean wave power [30]. The first attempt to optimize wave farm layouts is by Thomas and Evans [99], who consider devices with simple geometry (e.g., spheres) laid out in simple arrangements (e.g., rows) and conclude that the spacing among devices has a larger impact on $q$ than the device geometry. Fitzgerald and Thomas [32] appear to be the first to consider general configurations of WECs in the plane. By employing the point-absorber approximation, they solve the 5-WEC layout optimization problem using a sequential quadratic programming (SQP) solver with multiple manually chosen starting points but do not propose a general optimization method. Our heuristic also contains a SQP-like step, but, unlike Fitzgerald and Thomas [32], our heuristic chooses starting points systematically, based on insights gained from our analytical solution to the 2-WEC problem. Indeed, our study shows that the choice of starting point for local optimization is important. We also allow the number of devices to be general, whereas Fitzgerald and Thomas [32] focus exclusively on the 5-WEC case.

Child [18] and Child and Venugopal [19, 20] consider the layout problem for WECs with simplified geometries and develop mathematical equations for calculating the exact $q$-factor. They argue it is advantageous for each device to be located at the intersection points of certain parabolas centered at the other devices. They use this to develop a
heuristic they call the parabolic intersection (PI) method, which they find is less accurate but faster than a genetic algorithm (GA) that they also introduce. The current work differs from theirs in the sense that we consider the point-absorber approximation rather than the exact calculation, in light of the computational difficulties discussed above. In addition, they consider both real and reactive tuning as the control mechanism for each individual device in the array, while we assume optimal tuning as is common in the literature [e.g., 12, 28, 32, 71, 72, 99].

Similar to our work, Mao [68] and Snyder and Moarefdoost [92] study the wave farm layout problem under the point-absorber approximation, and propose a “tuned” GA and a greedy search heuristic (respectively) to optimize the wave farm layout. However, their proposed optimization algorithms do not exploit the properties of the layout problem or the structure of near-optimal solutions.

The current body of literature lacks fast and efficient optimization algorithms for large-scale arrays, along with analysis under stochastic and realistic ocean environments. There are many time domain models in the literature for design and control of a single WEC [21, 34, 51]. On the other hand, there are few works that address wave farm design under an irregular wave regime [18, 22, 33, 36, 68]; however, these works do not provide any solution approach for the layout optimization problem. In this research, we provide models and algorithms for choosing optimal locations of WEC devices within an array in regular and irregular ocean environments.

4.3 Theory

4.3.1 Hydrodynamic Background

The total mean mechanical power absorbed by an array of $N$ identical WECs oscillating in one mode of motion, such as heave, under the standard assumptions of linear wave
theory is given by [12, 99]:

\[ P = \frac{1}{4} (U'X + X'U) - \frac{1}{2} U'BU, \] (4.2)

where \( U \) is a column vector of complex velocity amplitudes (determined from the equations of motion); \( X \) is a column vector of complex exciting forces (i.e., forces acting on a floating system due to the waves) of both the incident and scattered waves; \( B \) is a matrix of real damping coefficients (i.e., parameters that quantify the reduction of oscillations in an oscillatory system); and an asterisk denotes complex conjugate transpose.

Throughout, we assume that the incident waves consist of a single sinusoid, characterized by the wave angle \( \beta \) and wavenumber \( k = \frac{2\pi}{\lambda} \), where \( \lambda \) is the wavelength. (The wavenumber is similar to frequency: Whereas frequency measures waves per unit time, wavenumber measures waves per unit distance.) Device \( m \in \{1, 2, \ldots, N\} \) is located at point \((\alpha_m, d_m)\) in the polar coordinate system (Figure 4.1).

![Figure 4.1: Location of device m](image)

In (4.2), \( U \) is a control variable that represents the amplitude of the devices in the direction of motion. These amplitudes are realized using a braking or damping mechanism within the WEC. Not all control strategies \( U \) are attainable, due to engineering constraints on the devices, but, like most of the literature, we ignore this consideration.
and assume an optimal control strategy. In particular, if the WEC locations are fixed, then (4.2) may be maximized over the control variables $U$, and the maximum power is [99]

$$P_{\text{max}} = \frac{1}{8} X^* B^{-1} X,$$

(4.3)

which is attained when $U = \frac{1}{2} B^{-1} X$. The expression for the power in (4.3) is optimized with respect to the control variables $U$ but for fixed WEC locations. We wish to optimize the locations in order to maximize (4.3). Unfortunately, to calculate $P_{\text{max}}$ in (4.3) requires calculating the so-called hydrodynamic coefficients $B$ and $X$. Since these coefficients depend on the shape, the geometry and—most relevant to our problem—the locations of the WEC devices, they must be computed numerically. As noted above, classical numerical methods are too slow to be practical within an optimization context; therefore, we employ the point-absorber approximation.

Under the point-absorber approximation, the $q$-factor in (4.1) has an analytical expression [99]:

$$q = \frac{1}{N} L^* J^{-1} L,$$

(4.4)

where $L$ is an $N$-dimensional column vector with

$$L_m = e^{ikd_m \cos(\beta - \alpha_m)}$$

(4.5)

and $J$ is an $N \times N$ matrix with

$$J_{mn} = J_0(kd_{mn});$$

(4.6)

$J_0$ is the Bessel function of the first kind with order 0 and $d_{mn}$ is the distance between device $m$ and device $n$. The advantage of (4.4) is that it does not involve the hydrodynamic coefficients and can thus be evaluated efficiently. Note that, although $L$ is complex, $q$ is guaranteed to be real since $(L^* J^{-1} L)^* = L^* J^{-1} L$ (since $J$ is symmetric and real).
4.4 Deterministic Optimization Model

We want to choose the locations of the WECs (in terms of $\alpha$ and $d$) in order to maximize the $q$-factor. The mathematical formulation of the Wave Energy Converter Location Problem (WECLP) is:

$$\max_{d, \alpha} q(d, \alpha; \beta, k) = \frac{1}{N} L^* J^{-1} L$$

(4.7)

subject to

$$d_{mn} \geq d_0 \lambda \quad \forall m, n = 1, 2, ..., N; m \neq n$$

(4.8)

$$(d_n, \alpha_n) \in R \quad \forall n = 1, 2, ..., N$$

(4.9)

where $d_0 > 0$ is a constant, $\lambda$ is the incident wavelength, and $R$ is the region for locating WECs. In (4.7), the notation $q(d, \alpha; \beta, k)$ indicates the $q$-factor for solution $(d, \alpha)$ under wave angle $\beta$ and wavenumber $k$. We will often shorten this notation to simply $q$.

The objective function depends on the decision variables $d$ and $\alpha$ through $L$ and $J$, as discussed above. Constraints (4.8) ensure a minimum level of separation between the devices, which reflect physical constraints and are also necessary for the point-absorber approximation to remain valid.

**Proposition 4.1.** Let $(d, \alpha)$ be a solution to the WECLP and let $\beta$ and $k$ be a wave angle and wavenumber, respectively. Then for any wave angle $\beta'$ and any wavenumber $k'$, there exists a solution $(d', \alpha')$ such that

$$q(d, \alpha; \beta, k) = q(d', \alpha'; \beta', k').$$

**Proof of Proposition 4.1.** First suppose the wavenumber $k$ changes to $k'$. Let $d_n' = \ldots$
\[ k \frac{d_n}{k'} d_n, \forall n \in \{1, 2, \ldots, N\}. \] Since

\[ d'_{mn} = \sqrt{d_n^2 + d_m^2 - 2d_n d_m \cos(\alpha_m - \alpha_n)} = \sqrt{\frac{k^2}{k'}^2 (d_n^2 + d_m^2 - 2d_n d_m \cos(\alpha_m - \alpha_n))} = \frac{k}{k'} d_{mn}, \]

for all \( m, n \in \{1, 2, \ldots, N\} \), we have \( J_0(k' d_{mn}) = J_0(k d_{mn}) \), and hence matrix \( J \) in (4.4) remains unchanged. Moreover, for all \( m \in \{1, 2, \ldots, N\} \), we have

\[ L'_m = e^{ik' d_m \cos(\beta - \alpha_m)} = e^{ik' d_m \cos(\beta - \alpha_m)} = L_m, \]

which means that vector \( L \) in (4.4) remains unchanged, as well. Therefore, we have

\[ q(d, \alpha; \beta, k) = q \left( \frac{k}{k'}, d, \alpha; \beta, k' \right), \]

as desired. Similarly, one can show that

\[ q(d, \alpha; \beta, k) = q(d, \alpha + (\beta' - \beta) \mathbf{1}; \beta', k), \]

where \( \mathbf{1} \) is a vector of all ones. \( \square \)

**Proposition 4.2.** If we translate the layout along either or both axes, \( q \) remains unchanged.

**Proof of Proposition 4.2**  For all \( n \in \{1, 2, \ldots, N\} \), we know that

\[ x_n = d_n \cos(\alpha_n) \]
\[ y_n = d_n \sin(\alpha_n). \]

Thus we can write (4.5) as:

\[ L_n = e^{ik(x_n \cos(\beta) + y_n \sin(\beta))}, \]
and for all \( n, m \in \{1, 2, \ldots, N\} \), we can write \( d_{mn} \) in (4.6) as:

\[
d_{mn} = \sqrt{(x_n - x_m)^2 + (y_n - y_m)^2}.
\]

In addition, let \( J_{mn}^{-1} \) be the element in row \( m \) and column \( n \) of matrix \( J^{-1} \). Then we can expand (4.4) as:

\[
q = \frac{1}{N} \sum_{n=1}^{N} J_{nn}^{-1} + \frac{1}{N} \sum_{n=1}^{N-1} \sum_{m=n+1}^{N} 2J_{nm}^{-1} \cos(z_{nm}),
\]

(4.10)

where \( z_{nm} = k[(x_n - x_m) \cos(\beta) + (y_n - y_m) \sin(\beta)] \). Now, consider a translation along the 2-dimensional vector \((u, v)\), i.e., \( \forall n \in \{1, 2, \ldots, N\} \),

\[
x'_n = x_n + u
\]

\[
y'_n = y_n + v.
\]

Under the Cartesian definition of \( d_{mn} \), it is clear that \( d_{mn} \) remains unchanged, as does \( J_{nm} \) in (4.6), for all \( n, m \in \{1, 2, \ldots, N\} \). Moreover, since \( z_{nm} \) in (4.10) remains unchanged, the \( q \)-factor does not change if we translate the layout along vector \((u, v)\). □

Proposition 4.1 demonstrates that the WECLP is isomorphic with respect to \( \beta \) and \( k \). Therefore, an instance of the WECLP is completely specified by \( N \), the number of devices.

Lemma 4.1. Matrix \( J \) in (4.4) is positive definite if it is invertible.

Proof of Lemma 4.1 We know that \( q \geq 0 \), thus by (4.4) we can conclude that \( J^{-1} \geq 0 \), which means that \( \det(J^{-1}) \geq 0 \). Since matrix \( J \) is invertible, \( \det(J) \neq 0 \) and also \( \det(J^{-1}) \neq 0 \). So, it is clear that \( \det(J^{-1}) > 0 \), which means \( J^{-1} \) is positive definite. □

The following proposition provides bounds on the value of the \( q \)-factor in (4.4).
**Proposition 4.3.** Let $0 < \lambda_{min} = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{N-1} \leq \lambda_N = \lambda_{max}$ be the $N$ eigenvalues of the matrix $J$. Then

\[
\frac{1}{\lambda_N} \leq q \leq \frac{1}{\lambda_1}.
\] (4.11)

**Proof of Proposition 4.3** Since matrix $J^{-1}$ is symmetric, $\frac{L^* J^{-1} L}{L^* L}$ is a Rayleigh quotient, so by the min–max theorem,

\[
\mu_{min} \leq \frac{L^* J^{-1} L}{L^* L} \leq \mu_{max},
\]

for $\|L\| \neq 0$, where $\mu_{min}$ and $\mu_{max}$ are the minimum and maximum eigenvalues of $J^{-1}$, respectively [84]. Thus,

\[
\mu_{min} \leq \frac{Nq}{L^* L} \leq \mu_{max}.
\]

The proof follows from the fact that $L^* L = N$ and $\mu_{min} = \frac{1}{\lambda_N}$ and $\mu_{max} = \frac{1}{\lambda_1}$. □

These bounds are illustrated in Figure 4.2 and Figure 4.3 for the 5-WEC configuration in Fitzgerald and Thomas [32] as the location of the first WEC is changing along the $x$-axis and $y$-axis.

![Figure 4.2: Bounds and $q$-factor vs. location of the first WEC along $x$-axis](image-url)
Although the lower bound is not as tight as desired, these bounds not only are faster to compute, but also they are robust with respect to the incident wave direction, $\beta$, since these bounds depend only on the $J$ matrix, which does not depend on $\beta$. So, by maximizing the lower bound, we can obtain a solution that is robust with respect to $\beta$.

Moreover, the maximum of the upper bound and the maximum of the $q$-factor often coincide, and they coincide exactly for the special case of $N = 2$. According to Falnes [28] and Snyder and Moarefdoost [92], the maximum $q$-factor for the 2-WEC case is:

$$q^* = \frac{1}{1 - |J_0(kd_{12}^*)|},$$

(4.12)

where $d_{12}^*$ is the optimal distance between two devices.

**Proposition 4.4.** Let $0 < \lambda_{\min} = \lambda_1 \leq \lambda_2 = \lambda_{\max}$ be two eigenvalues of the matrix $J$. Then

$$q^* = \frac{1}{\lambda_1}.$$

(4.13)
**Proof of Proposition 4.4** For the case of $N = 2$, the matrix $J$ is:

$$J = \begin{bmatrix}
1 & J_0(kd_{12}) \\
J_0(kd_{12}) & 1
\end{bmatrix},$$

and its eigenvalues are:

$$\lambda = 1 \pm J_0(kd_{12}).$$

Thus, the minimum eigenvalue is $\lambda_1 = 1 - |J_0(kd_{12})|$. Therefore, at optimality,

$$\frac{1}{\lambda_1} = \frac{1}{1 - |J_0(kd_{12})^*|} = q^*.$$

\[\square\]

### 4.5 Heuristic Algorithm

The optimization model developed in Section 4.4 is a nonconvex function of the WEC locations with many locally optimal points. See Figure 4.4, which plots the $q$-factor vs. the location of device 1 in the 5-device layout S5A given by [32] (blue circles in Figure 4.5) assuming the incident wave angle $\beta = 0$; the location is plotted in Cartesian coordinates scaled by the wavenumber $k$. Hence, in the following, we develop an efficient heuristic algorithm based the structural properties of the model.
4.5.1 Structure of Near-Optimal Solutions

In order to motivate our heuristic, we first explore the structure of optimal and near-optimal solutions. In what follows, if $d$ is a distance, then we refer to $kd$ as the non-dimensional distance. In the non-dimensional distance, the distance is normalized so that the unit of distance is one angular wavelength $\lambda$ ($\lambda = \lambda/2\pi$). For example, if $d = 1000\text{m}$ and $k = 0.1$ (so that $\lambda = 62.8\text{m}$ and $\lambda = 10$), then $kd = 100\text{m} = 10$ angular
wavelengths. We will also use the term *non-dimensional layout* to refer to a layout whose Cartesian coordinates have been scaled by $k$.

For the problem with $N = 2$, Snyder and Moarefdoost [92] show that the optimal non-dimensional distance between the two devices occurs at the smallest local optimizer (min or max) of the Bessel function $J_0(\cdot)$ that is greater than the separation limit, or equivalently it occurs at the smallest root of Bessel function $J_1(\cdot)$ that is greater than the separation limit.\footnote{Note that due to Bessel function properties, we have $J_1(x) = \frac{dJ_0(x)}{dx}$.} For larger $N$, a similar feature seems to hold approximately: Figure 4.6 shows the best known non-dimensional layout (from Fitzgerald and Thomas [32]) for the case of $N = 5$, with $q^* = 2.777$, assuming $\beta = 0$. The circles are centered at the WECs and have radius equal to $j_n$ (the $n$th Bessel optimizer) for particular integer values of $n$. Note that the pairwise non-dimensional distances among the WECs are very nearly equal to the Bessel optimizers. In particular, the non-dimensional distance from WEC 1 (located at the origin) to all other WECs equals $j_7$. The non-dimensional distances between the remaining pairs of WECs are all very close to values in $\{j_4, j_7, j_8, j_{10}, j_{12}\}$. Our preliminary results suggest that a similar property holds for near-optimal solutions for other values of $N$, as well. We use this intuition to develop a heuristic algorithm for wave farms with a general number $N$ of devices.

### 4.5.2 Phase 1: Finding Master Layouts

The proposed heuristic has two main phases. In the first, we search for a *master layout* in which the non-dimensional distance between each pair of devices equals one of the local optimizers of the Bessel function $J_0(\cdot)$ or the roots of the Bessel function $J_1(\cdot)$. In the second phase, we attempt to improve the solution using a continuous local optimization procedure.

To generate master layouts, we randomly generate a matrix of pairwise non-dimensional distances that equal Bessel optimizers, and then attempt to find WEC coordinates that
have this distance matrix. In particular, for each pair $m, n (m, n = 1, 2, \ldots, N)$, we choose the non-dimensional distance $kd_{mn}$ between $m$ and $n$ uniformly from $\{j_1, j_2, \ldots, j_M\}$ for $m, n = 1, 2, \ldots, N$ and a sufficiently large parameter $M$. The parameter $M$ plays an important role in the heuristic’s performance, in terms of both time and accuracy. If we choose $M$ too small, the search space is reduced and it will be unlikely to find good master layouts. For example, in the best known 5-WEC layout, the maximum non-dimensional distance among the pairs of WECs approximately equals the 12th Bessel optimum, so if we set $M$ equal to, say, 5, we may not find the best layout. On the other hand, if we set $M$ to be large, the algorithm will execute more slowly. Our preliminary numerical experiments suggest that $M = 2N + 3$ is a good choice. Given $M$, we define the discrete search set, $S(M)$, as the set of the first $M$ local optimizers of the Bessel function $J_0(\cdot)$ or as the set of the first $M$ roots of the Bessel function $J_1(\cdot)$.

The output of this process is an $N \times N$ distance matrix, $D$, whose elements are the
The next step is to find a master layout that corresponds to the distance matrix $D$, if one exists. First, we check whether $D$ is a Euclidean distance matrix (EDM), i.e., whether there exists a set of $N$ distinct points in the 2-dimensional Euclidean space whose pairwise distances equal the entries of the matrix $D$. It is easy to test whether $D$ is an EDM using the following theorem by Schoenberg [85]:

**Theorem 4.1** (Schoenberg [85]). A symmetric matrix $D$ with zero diagonal and non-negative off-diagonal elements is an EDM if and only if

$$-\left( I - \frac{1}{N}11^T \right) D \left( I - \frac{1}{N}11^T \right) \succeq 0.$$  

If $D$ is not an EDM, we simply discard it. If it is an EDM, we can find the corresponding Cartesian coordinates using methods developed in the area of Euclidean distance geometry and multidimensional scaling [10, 62]. Consider an $N \times 2$ matrix $X$ given by

$$X = \begin{pmatrix} X_1^T \\ X_2^T \\ \vdots \\ X_N^T \end{pmatrix},$$

where $X_m^T = (x_m, y_m)$ are the coordinates for device $m$. If $X$ is the coordinate matrix, then we can compute the distance matrix $D$ as

$$D^2(X) = \text{diag}(XX^T) 1^T + 1 \text{diag}(XX^T)^T - 2XX^T.$$ (4.14)

Here, $\text{diag}(XX^T)$ is an $N$-by-1 vector of the diagonal elements of $XX^T$, and $D^2$ is the element-wise square of matrix $D$. It can be proven that matrix $D$ is invariant to translation, rotation and reflection of $X$ [80]; thus, given $D$, there are infinitely many...
coordinate matrices \( X \) that generate \( D \). There are several algorithms from Euclidean
distance geometry and multidimensional scaling (MDS) that can be used to recover
the coordinate matrix \( X \) from a given distance matrix \( D \). We use an algorithm called
Classic Scaling to recover the corresponding coordinates [e.g., 10]; see Algorithm 1.

\begin{algorithm}
\textbf{Algorithm 1} Classic Scaling Algorithm

1. Given \( D \), compute \( D^2 \) by squaring elements of \( D \).
2. Compute the eigenvalues and eigenvectors of
   \[
   \Gamma = \frac{1}{2} \left( I - \frac{1}{N} 11^T \right) D^2 \left( I - \frac{1}{N} 11^T \right).
   \]
3. Choose the two largest eigenvalues and their corresponding eigenvectors. Note
   that the number of nonzero eigenvalues is the dimension of the solution; if there
   are fewer than two nonzero eigenvalues, then the layout is linear and thus easy to
   recover.
4. The coordinate matrix is given by:
   \[
   X = Q_2 \Lambda_2^{1/2},
   \]
   \begin{equation}
   (4.15)
   \end{equation}
   where \( Q \) is the square \( N \)-by-\( N \) matrix whose \( n \)th column is the \( n \)th eigenvector of
   \( \Gamma \) and \( \Lambda \) is the diagonal matrix whose diagonal elements are the corresponding
   eigenvalues of \( \Gamma \).
\end{algorithm}

4.5.3 Phase 2: Local Improvement

In the second phase of the heuristic, we take the resulting coordinates as the WEC loca-
tions in the master layout and use a continuous, convex optimization solver to attempt
to improve the solution to maximize the \( q \)-factor. (Any convex optimization solver may
be used; we used MATLAB’s \texttt{fminunc} and \texttt{fmincon} functions. See additional details
below.) The motivation for this phase is that the master layout found in the first phase
may fall within one of the “humps” in Figure 4.4 but not at its peak; that is, it may fall
near, but not at, a local maximum. The continuous-optimization step ensures that we
reach a local maximum.

Algorithm 2 formally states the steps for our two-phase heuristic.

**Algorithm 2** Two-Phase Heuristic

0. Given: CPU time limit $T$, parameter $M$. Set $q_{max} := 0$.

1. Randomly generate an $N \times N$ symmetric matrix whose elements equal one of the elements of the discrete search set, $S(M)$, and store it in $D$.

2. Check whether $D$ is an EDM using Theorem 4.1. If $D$ is an EDM, go to 4, otherwise go to 1.

3. Find $x$, $y$ coordinates that correspond to the distance matrix $D$ using Algorithm 1.

4. Using the layout defined by $(x, y)$ as the initial solution, optimize all locations locally using a continuous optimization algorithm. Let $q :=$ the $q$-factor of the resulting solution.

5. Let $q_{max} := \max\{q, q_{max}\}$.

6. If the elapsed CPU time $\geq T$, STOP, otherwise go to 1.

4.5.4 Symmetry

The best known solution for $N = 5$ from Fitzgerald and Thomas [32] is symmetric with respect to a line parallel to the wave direction $\beta$. In fact, we conjecture that this property is always true of optimal solutions:

**Conjecture 4.1.** For $N \geq 2$, layouts that are symmetric with respect to the wave direction $\beta$ dominate asymmetric layouts.

If true, Conjecture 4.1 would imply that we can simplify the optimization problem.
Thus, we can enforce symmetry using the constraints

\[
\forall n = 1, \ldots, N/2, \quad \begin{cases}
\alpha_n - \beta = \beta - \alpha_{2n} \\
 d_n = d_{2n}
\end{cases}
\] (4.16)

if \(N\) is even or

\[
\forall n = 1, \ldots, (N + 1)/2, \quad \begin{cases}
\alpha_n - \beta = \beta - \alpha_{2n-1} \\
 d_n = d_{2n-1}
\end{cases}
\] (4.17)

if \(N\) is odd. We add these constraints to the problem solved by the solver in step 4 of Algorithm 2. (We used MATLAB’s \texttt{fmincon} and \texttt{fminunc} functions for the problems with and without symmetry constraints, respectively.) In the computational results below, we refer to the problems with and without symmetry constraints as WSymC and WOSymC, respectively.

### 4.5.5 Computational Results

We tested our two-phase heuristic for \(N = 3, \ldots, 15\). (Recall that an instance is completely specified by \(N\) since the problem is isomorphic with respect to \(\beta\) and \(k\), by Proposition 4.1.) We are aware of very few other computational studies for the WE-CLP to use as benchmarks. Folley and Whittaker [33] and Fitzgerald and Thomas [32] consider \(N = 5\) only and report a solution with \(q = 2.777\), which we believe is optimal; and Mao [68] provides results of his “tuned” genetic algorithm (GA) for \(N \leq 15\). The results of Child [18] and Child and Venugopal [19, 20] are not comparable to ours because they use a different method to calculate the \(q\)-factor and make different assumptions regarding the control mechanisms used by the WECs. Therefore, we compare our heuristic (with and without symmetry constraints) to Mao’s (2013) tuned GA and with two general-purpose global optimization solvers, NOMAD [56] and PSWARM [103].

We conducted two sets of experiments, one using a 1-hour time limit and the other using a 24-hour time limit, for all algorithms. All algorithms are implemented and tested...
in MATLAB (R2014a) on a PC desktop with 2.10GHz x86 64-bit Intel(R) Core(TM)2 Duo CPU and 4.00 GB RAM.

Tables 4.1 and 4.2 and Figures 4.7(a) and 4.7(b) present the results of the 1- and 24-hour experiments, respectively. They list the best reported $q$-factor for $N = 3, \ldots, 15$ for our heuristic and the other three algorithms. In the tables, the maximum $q$-factor found by any approach for a given $N$ is indicated in boldface.

Our heuristic finds the best known solution from the literature for $N = 5$ [32, 33] and consistently dominates NOMAD and PSWARM for both 1- and 24-hour time limits. It outperforms the tuned GA by Mao [68] for all instances except $N = 8$ and $N = 11$ in the 1-hour experiment and except for $N = 12$ in the 24-hour experiment. The two-phase heuristic with symmetry constraints tends to outperform the heuristic without, especially for the longer run time. The gap between our heuristics and the others tends to increase as $N$ increases. These results highlight the importance of exploiting properties of the solution structure in the solution approach to the WECLP, as our heuristic, and to a lesser extent that of Mao [68], do.

It is worth noting that the results in the “$q$ [WOSymC]” column are not necessarily asymmetric, but rather that the local optimization phase does not enforce the symmetry constraint explicitly. Also, the instances for which the WOSymC heuristic finds a better solution than WSymC (e.g., $N = 9, 10, \text{and } 15$ in Table 4.1) do not necessarily disprove Conjecture 4.1, since WSymC may not have found the optimal solution.
Figure 4.7: Best q-factor obtained by heuristics

![Graphs showing the best q-factor obtained by heuristics for 1-hour and 24-hour run times.]

(b) 24-hour run time

Table 4.1: Results of two-phase heuristic vs. GA, NOMAD and PSWARM for $T = 1$ (hr)

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Table 4.1: Results of two-phase heuristic vs. GA, NOMAD and PSWARM for $T = 1$ (hr)
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Table 4.2: Results of two-phase heuristic vs. GA, NOMAD and PSWARM for \(T = 24\) (hr)

4.6 **WECLP under Uncertainty**

The methods and analysis provided in Section 4.4 and Section 4.5 are based on deterministic ocean states, i.e., waves are regular with deterministic wavenumber, frequency, and direction. However, waves in the ocean are irregular and stochastic, and these can have a substantial degrading effect on the power of the wave farm [92]. Thus, we need optimization models that design layouts that perform well for realistic sea states. In addition, real ocean waves are irregular, i.e., have multiple sinusoidal components rather
than the single component assumed here, and there is a need to develop models to accommodate a more realistic model of the ocean waves and the resulting hydrodynamics.

### 4.6.1 The Effect of Uncertainty

The uncertainties in the ocean environments can have a substantial degrading effect on the power of the wave farm. In fact, many authors (e.g., [12, 20, 32, 33, 68, 74, 92]) have lamented the fact that a wave farm optimized for a particular wave environment (wave heading angle or wavenumber) performs quite poorly when the environment changes just a little. For example, the best-known 5-device layout in Fitzgerald and Thomas [32] performs quite well if the incident waves arrive at an angle of $\beta = 0$, but the performance degrades almost immediately as $\beta$ changes; see the solid blue curve in Figure 4.8. We consider two models for mitigating the effect of uncertainty on the total power. The first model maximizes the expected value of the $q$-factor when the wave direction, $\beta$, is stochastic with known distribution:

$$\max_{(d,\alpha)} E_{\beta}[q(\beta)] = E \left[ \frac{1}{N} L^* J^{-1} L \right]$$

subject to

$$d_{mn} \geq d_0 \lambda \quad \forall m, n = 1, 2, ..., N; \ m \neq n$$

$$(d_n, \alpha_n) \in R \quad \forall n = 1, 2, ..., N$$

where $q(\beta)$ is written so as to stress that $\beta$ is changing. This is an example of a *stochastic optimization* model. The second model maximizes the worst-case solution over a range of $\beta$ values and is an example of *robust optimization*:

$$\max_{(d,\alpha)} \min_{\beta_0 \leq \beta \leq \beta_0} \{q(\beta)\} = \min_{\beta} \left( \frac{1}{N} L^* J^{-1} L \right)$$

subject to

$$97$$
\[ d_{mn} \geq d_{0} \lambda \quad \forall m, n = 1, 2, ..., N; m \neq n \]

\[(d_n, \alpha_n) \in R \quad \forall n = 1, 2, ..., N\]

where \( \beta_l \) and \( \beta_u \) are lower and upper bounds for the range of wave direction, \( \beta \). Figure 4.8 plots \( q \) vs. \( \beta/\pi \) for 5-WEC solutions found by optimizing these two objectives using a genetic algorithm [68]. The red curve plots the stochastic solution, which maximizes \( E[q(\beta)] \), while the green curve plots the robust solution, which maximizes \( \min\{q(\beta)\} \). Both solutions are significantly more robust than the deterministic solution, in the sense that they perform at \( q > 1 \) for a much broader range of \( \beta \) values. Of course, this comes at some expense, since \( q(\beta = 0) \) is smaller for the stochastic and robust solutions than for the deterministic solution, as is typical for optimization under uncertainty. The stochastic and robust solutions are also worse in the tails, but this is of less concern since the tails represent unrealistic wave angles such as waves headed out to sea from shore. In the following subsections, we treat and analyze models (4.18) and (4.19), separately.

Figure 4.8: \( q \)-factor vs. \( \beta/\pi \), for 5-WEC layout that • Maximize \( q \) assuming \( \beta = 0 \), • Maximize \( \min_{\beta}\{q\} \) assuming \( \beta \in [-\pi/8, \pi/8] \), and • Maximize \( E_{\beta}\{q\} \) assuming \( \beta \sim N(0, (\frac{\pi}{8})^2) \).
4.6.2 Max-Min Optimization Model

In the max-min (robust) model, we reduce the effect of uncertainty in the wave heading, $\beta$, by maximizing the worst case output. For convenience, we work with the expanded form of the $q$-factor. One can show that

$$q(\beta) = \frac{1}{N} \left( \sum_{n=1}^{N} J_{nn}^{-1} + \sum_{n=1}^{N-1} \sum_{m=n+1}^{N} 2J_{nm}^{-1} \cos \left[ k d_n \cos (\beta - \alpha_n) - k d_m \cos (\beta - \alpha_m) \right] \right). \quad (4.20)$$

In the max-min optimization problem, we have:

$$\max_{d, \alpha} \left\{ \min_{\beta \leq \beta \leq \beta_u} \frac{1}{N} \left( \sum_{n=1}^{N} J_{nn}^{-1} + \min_{\beta} \sum_{n=1}^{N-1} \sum_{m=n+1}^{N} 2J_{nm}^{-1} \cos \left[ k d_n \cos (\beta - \alpha_n) - k d_m \cos (\beta - \alpha_m) \right] \right) \right\}. \quad (4.21)$$

Here, the minimum value of $2J_{nm}^{-1} \cos (kd_n \cos (\beta - \alpha_n) - kd_m \cos (\beta - \alpha_m))$ is $-2J_{nm}^{-1}$ when $J_{nm} \geq 0$, and is $2J_{nm}^{-1}$ otherwise, for $n, m = 1, 2, \cdots, N$ when $\beta_u - \beta_l \geq \pi$. Thus, we have:

$$\min_{\beta_l \leq \beta \leq \beta_u} q(\beta) = \frac{1}{N} \left( \sum_{n=1}^{N} J_{nn}^{-1} - \sum_{n=1}^{N-1} \sum_{m=n+1}^{N} 2|J_{nm}| \right),$$

and the min-max optimization problem simplifies to:

$$\max_{d} \frac{1}{N} \left( \sum_{n=1}^{N} J_{nn}^{-1} - \sum_{n=1}^{N-1} \sum_{m=n+1}^{N} 2|J_{nm}| \right). \quad (4.21)$$

The condition $\beta_u - \beta_l \geq \pi$ means that the wave angle can vary by more than $\pi$, which is unrealistic. However, without this assumption, the optimization model is much more difficult to solve, at least analytically. Therefore, for the sake of analysis, we assume that $\beta_u - \beta_l \geq \pi$ holds and consider a 2-WEC array. For $N = 2$, without loss of generality, we put the first device at the origin and the second one at the point $(d, \alpha)$ in polar coordinates; then we can write the max-min problem as:

$$\max_{d} \frac{1 - |J_0(kd)|}{1 - J_0(kd)^2}.$$
In this problem, if $J_0(kd) \geq 0$, then

$$\max_d \frac{1 - |J_0(kd)|}{1 - J_0(kd)^2} = \frac{1 - J_0(kd)}{1 - J_0(kd)^2} = \frac{1}{1 + J_0(kd)}.$$ 

We get the maximum value for the above function if we set $J_0(kd) = 0$, and its maximum is 1. Now, assume that $J_0(kd) < 0$; then

$$\max_d \frac{1 - |J_0(kd)|}{1 - J_0(kd)^2} = \frac{1 + J_0(kd)}{1 - J_0(kd)^2} = \frac{1}{1 - J_0(kd)};$$

since $J_0(kd) < 0$, the maximum value for this function is 1 and is obtained if we set $J_0(kd) = 0$. Therefore, the optimal value of the objective function is 1 and the optimal non-dimensional distance between the two devices, $kd^*$, occurs at the roots of Bessel function $J_0(\cdot)$. Figure 4.9(a) shows the objective function value, $\frac{1 - |J_0(kd)|}{1 - J_0(kd)^2}$, versus the value of the non-dimensional distance $kd$. In this figure, the blue line is the objective function, $\frac{1 - |J_0(kd)|}{1 - J_0(kd)^2}$, and the dashed red line is the Bessel function, $J_0(kd)$. The fact that $q \leq 1$ implies that the two WECs together can never perform better than if they were separated; that is, there is no synergy between the devices. This is not a desirable property of wave farms. However, for more realistic situations with $\beta_u - \beta_l < \frac{\pi}{2}$, the optimal max-min objective will be greater than one. The optimal value of max-min $q$-factor, calculated by discretization of the search space and enumeration, as a function of $\beta_u - \beta_l$ is depicted in Figure 4.9(b).
Figure 4.9: Max-Min solution for 2-WEC problem.

4.6.3 Maximum Expected Value Problem

As we observed, the solution of the max-min optimization problem can be conservative. Thus, we may use the other optimization model, the maximum expected value model.
However, this problem is analytically challenging to study and in some cases obtaining a closed form solution is impossible. For the expected value optimization problem, we have:

$$\max_{d, \alpha} \left\{ E_\beta [q(\beta)] = \frac{1}{N} \sum_{n=1}^{N} J_{n}^{-1} + \frac{1}{N} \sum_{m=n+1}^{N} 2J_{nm}^{-1} E_\beta [\cos (kd_n \cos(\beta - \alpha_n) - kd_m \cos(\beta - \alpha_m))] \right\}.$$  

In this problem, computing $E_\beta [\cos (kd_n \cos(\beta - \alpha_n) - kd_m \cos(\beta - \alpha_m))]$ analytically is important but difficult. To get some insight into this problem, we start by analyzing the expected value problem for arrays with $N = 2$ WECs. In this case, the expected value optimization problem is:

$$\max_{d, \alpha} \left\{ E_\beta [q(\beta)] = \frac{1}{1 - J_0(kd)} E_\beta [\cos(kd \cos(\beta - \alpha))] \right\}. \quad (4.22)$$

**Proposition 4.5.** Suppose $\beta$ is uniformly distributed between $\beta_0$ and $\beta_0 + \pi$, where $\beta_0$ is a constant. Then for any choice of locations for the 2-WEC problem, $E_\beta[q(\beta)] = 1$. Therefore, any solution is optimal.

**Proof of Proposition 4.5.** According to the Bessel function properties [1], we have

$$\cos(kd \cos(\beta - \alpha)) = J_0(kd) + 2 \sum_{r=1}^{\infty} (-1)^r J_{2r}(kd) \cos(2r\beta - 2r\alpha),$$

and

$$E_\beta [\cos(kd \cos(\beta - \alpha))] = J_0(kd) + 2 \sum_{r=1}^{\infty} (-1)^r J_{2r}(kd) E_\beta [\cos(2r\beta - 2r\alpha)]. \quad (4.23)$$

When $\beta$ is uniformly distributed between $\beta_0$ and $\beta_0 + \pi$, $E_\beta [\cos(2r\beta - 2r\alpha)] = 0$, for $r = 1, 2, \cdots$. Therefore,
\[ E_\beta [\cos(kd \cos(\beta - \alpha))] = J_0(kd), \]

and

\[ E_\beta [q(\beta)] = \frac{1 - J_0(kd)^2}{1 - J_0(kd)^2} = 1. \]

**Proposition 4.6.** Suppose \( \beta \) is normally distributed with a mean of 0 and a variance of \( \sigma^2 \). Then if the first WEC is located at (0, 0) and the second at \((d, \alpha)\), we have

\[ E_\beta [q(\beta)] = 1 - \frac{2J_0(kd) \sum_{r=1}^{\infty} (-1)^r J_{2r}(kd)e^{-2r^2\sigma^2} \cos(2r\alpha)}{1 - J_0(kd)^2}. \] (4.24)

**Proof of Proposition 4.6.** Based on equation (4.23), in order to evaluate \( E_\beta [q(\beta)] \), we need to compute \( E_\beta [\cos(2r\beta - 2r\alpha)] \), for \( r = 1, 2, \cdots \). According to Euler’s formula for complex numbers, we have

\[ E_\beta [\cos(2r\beta - 2r\alpha)] + iE_\beta [\sin(2r\beta - 2r\alpha)] = E_\beta \left[ e^{i(2r\beta - 2r\alpha)} \right] = E_\beta \left[ e^{2ri\beta} \right] e^{i(-2r\alpha)}. \]

\( E_\beta \left[ e^{2ri\beta} \right] \) can be easily computed knowing the moment generating function of the random variable \( \beta \). When \( \beta \sim \mathcal{N}(0, \sigma^2) \), the value of \( E_\beta \left[ e^{2ri\beta} \right] \) is equal to \( e^{\frac{1}{2}\sigma^2(2r)^2} \). Hence, we have

\[ E_\beta [\cos(2r\beta - 2r\alpha)] + iE_\beta [\sin(2r\beta - 2r\alpha)] = e^{-2r^2\sigma^2} \cos(2r\alpha) - ie^{-2r^2\sigma^2} \sin(2r\alpha), \]

and this implies that \( E_\beta [\cos(2r\beta - 2r\alpha)] = e^{-2r^2\sigma^2} \cos(2r\alpha) \). By plugging the computed value of \( E_\beta [\cos(2r\beta - 2r\alpha)] \) back into equations (4.23) and (4.22), we obtain (4.24). □
From Proposition 4.6, we observe that as the variance of $\beta$ increases, the expected value of the $q$-factor converges to one, which is the optimal value for the max-min model for large ranges of $\beta$. This indicates that the mathematical models tend to provide conservative solutions as the uncertainty increases.

As mentioned in Section 4.5.1, the optimal non-dimensional distance, $kd^*$, between two WECs in a deterministic 2-WEC problem occurs at the roots of Bessel function $J_1(\cdot)$. We observe that when the uncertainty is large, both in the max-min problem and the maximum expected value problem, the optimal non-dimensional distance, $k\tilde{d}^*$, is at the roots of $J_0(\cdot)$. This means that $\tilde{d}^* \leq d^*$, i.e., as the uncertainty increases, it is better to locate the devices closer to each other.

### 4.6.4 Approximate $q$-factor for Irregular Waves

The analysis in the previous sections does not capture the real ocean environment effectively. Real ocean waves are of stochastic nature and can be considered as a superposition of a number of regular waves, each with its own phase, frequency, amplitude and direction of propagation. In particular, each component, indexed by $j$, has a wave amplitude, $A^j$, a wavenumber, $k^j$, a wave direction, $\beta^j$, an angular frequency, $\omega^j$ ($\omega^j = \sqrt{gk^j \tanh(k^j h)}$), and a phase $\gamma^j$ (usually considered random). Then, the sea surface elevation at point $(x, y)$ at time $t$ is:

$$\eta(x, y, t) = \sum_j A^j \cos(k^j x \cos(\beta^j) + k^j y \sin(\beta^j) - \omega^j t + \gamma^j). \quad (4.25)$$

Consider $\Phi = \Phi(x, y, x, t)$ as the velocity potential of an irrotational and inviscid fluid at point $(x, y, z)$ at time $t$. Then, the following hydro-dynamic equations must hold [29]:

$$\nabla^2 \Phi = 0 \quad \text{Throughout the fluid} \quad (4.26)$$

$$\frac{\partial \Phi}{\partial \nu} = \vec{v} \cdot \nabla \Phi \quad \text{On the surface of a rigid body} \quad (4.27)$$

104
\[
\frac{\partial \Phi}{\partial \nu} = 0 \quad \text{On the sea bed} \quad (4.28)
\]
\[
\eta(x, y, 0, t) = -\frac{1}{g} \frac{\partial \Phi}{\partial t} \quad \text{On the mean free surface} \quad (4.29)
\]
\[
\frac{\partial \Phi(x, y, z, t)}{\partial t} + \frac{g}{\partial z} \frac{\partial \Phi(x, y, z, t)}{\partial z} \bigg|_{z=0} = 0 \quad \text{On the mean free surface} \quad (4.30)
\]
\[
P = \mathcal{P}(x, y, z, t) \approx -\rho \frac{\partial \Phi(x, y, z, t)}{\partial t} \quad (4.31)
\]

Here, \( \rho \) is the fluid density, \( g \) is the acceleration due to gravity, \( \nu \) is the outward unit normal direction to the surface of the body, \( \frac{\partial}{\partial \nu} \) is the normal derivative in the direction \( \nu \), and \( \mathcal{P} \) is the hydro-dynamic pressure corresponding to \( \Phi \).

Now, let \( \Phi^r_m \) be the radiated velocity potential due to the oscillation of device \( m \) and \( \Phi^i \) be the velocity potential due to incident waves. Thus, under the point absorber approximation and linear wave theory assumptions, the total velocity potential of an array of \( N \) devices in the ocean is:

\[
\Phi = \Phi^0 + \sum_{m=1}^{N} \Phi^r_m. \quad (4.32)
\]

If device \( m \) oscillates harmonically with frequency \( \omega^r_m \), its radiated velocity potential is of the form [12]:

\[
\Phi^r_m = \text{Re}\{\phi_m e^{i(\omega m t + \phi)}\}. \quad (4.33)
\]

Hence, the complex radiated velocity potential of device \( m \) oscillating with complex velocity amplitude \( U_m \) is [28, 72]:

\[
\hat{\Phi}^r_m = \hat{\phi}_m U_m, \quad (4.34)
\]

where \( \hat{\phi}_m = \phi_m e^{i\phi} \). As a result, the complex radiated hydro-dynamic pressure is:

\[
\mathcal{P}_m \approx -\rho \frac{\partial \hat{\phi}_m}{\partial t} U_m. \quad (4.35)
\]

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Note that in the case of simple harmonic incident waves, the optimum radiated velocity potential is when all devices oscillate with the same frequency as the incident wave frequency \( \omega_r \). Here, we assume devices oscillate with equal frequency, i.e., \( \omega'_m = \omega_r \) for \( m = 1, 2, ..., N \), and treat \( \omega_r \) as a decision variable. It is worth mentioning that \( \omega_r = \sqrt{gk\tanh(kr)} \).

The absorbed power of a wave farm with \( N \) interacting WECs is [28]:

\[
P = \frac{1}{4} \sum_{m=1}^{N} (X_m U_m^* + X_m^* U_m) - \frac{1}{2} \sum_{m=1}^{N} \sum_{n=1}^{N} B_{mn} U_m^* U_n,
\]

(4.36)

where \( B_{nm} \) is the radiated damping coefficient between device \( n \) and \( m \), \( X_m \) is the excitation force acting upon device \( m \) due to the incident waves and \( U_m \) is the complex velocity amplitude of device \( m \). The radiated damping coefficient and excitation force will be computed as:

\[
B_{nm}(\omega_r) = \int_{\Gamma_n} \int_{\Gamma_m} \mathcal{P}_m \frac{\partial \hat{\phi}_n^*}{\partial \nu} dS
\]

(4.37)

\[
X_m = -\int_{\Gamma_m} \int_{\Gamma_n} \mathcal{P}_0 \cdot \vec{v} dS.
\]

(4.38)

When linear wave theory holds, the velocity potential due to the incident wave is [18]:

\[
\hat{\Phi}^0 = \sum_j \hat{\Phi}_j^0,
\]

(4.39)

where \( \hat{\Phi}_j^0 \) is the complex incident velocity potential due to wave component \( j \). As a result, the complex incident hydrodynamic pressure is:

\[
\mathcal{P}_0 = \sum_j \mathcal{P}_0^j.
\]

(4.40)
Hence, the excitation force (4.38) will be written as:

$$X_m = - \int \int \Gamma_m \mathcal{P}_0 \cdot \dot{v} dS = - \int \int \Gamma_m \sum_j \mathcal{P}_j \dot{v} dS = \sum_j X^i_m, \quad (4.41)$$

where $X^i_m$ is the excitation force on device $m$ due to wave component $j$. Under the point absorber approximation, there exit closed form solutions for $X^i_m$ and $B_{nm}$. In deep water, they are [28]:

$$B_{nm}(\omega^r) = \frac{\rho k^r \sqrt{g k^r} \Lambda_n \Lambda_m J_0(k^r d_{nm})}{2} \quad (4.42)$$

$$X^i_m = \rho g \Lambda_m A^j e^{-i k^r d_m \cos(\beta^j_j - \alpha^m_m)}. \quad (4.43)$$

Here, $\Lambda_m$ is a constant related to the geometry of device $m$. Consider vector $\hat{X}$ and matrix $\hat{B}$, where

$$[\hat{B}]_{nm} = B_{nm}(\omega^r) \quad (4.44)$$

$$[\hat{X}]_m = \sum_j X^i_m. \quad (4.45)$$

Thus, the approximate absorbed power of the wave farm with $N$ interacting WECs in irregular waves is:

$$\hat{P} = \frac{1}{4}(U^* \hat{X} + \hat{X}^* U) - \frac{1}{2} U^* \hat{B}^{-1} U, \quad (4.46)$$

where $U$ is the column vector of complex velocity amplitudes. The sub-optimal maximum power is:

$$\hat{P}_{\text{max}} = \frac{1}{8} \hat{X}^* \hat{B}^{-1} \hat{X}. \quad (4.47)$$

Finally, the approximate $q$-factor for the case of irregular waves is:

$$\tilde{q} = \frac{\hat{P}_{\text{max}}}{N \hat{P}_0}. \quad (4.48)$$
where $\tilde{P}_0$ is the absorbed power of a single WEC in isolation.

In the real ocean, there are enormous historical data and measurements for the sea surface elevation. These statistical measurements would help in estimating the wave spectrum where

$$\overline{\eta^2(x, y, t)} = \int S(\omega, \beta) d\beta d\omega,$$

and by definition (Falnes [29] equation 4.188):

$$\frac{1}{2} |A(\omega, \beta)|^2 \Delta \beta \Delta \omega = S(\omega, \beta);$$

here $S(\omega, \beta)$ is called the wave energy spectrum. Therefore, the excitation forces in equation (4.43) can be written as

$$X^j_m = \rho g \Lambda_m \sqrt{\frac{2S(\omega^j, \beta^j)}{\Delta \beta \Delta \omega}} e^{-ik^j d_n \cos(\beta^j - \alpha_m)}$$

assuming that the components of the irregular waves are separated at regular intervals in wave frequency (or wavenumber) by $\Delta \omega$ and wave direction by $\Delta \beta$. Finally, the WECLP optimization problem for real random ocean waves is:

$$\max_{(d, \alpha)} \bar{q} = \frac{1}{8} \bar{X}^T \bar{B}^{-1} \bar{X}$$

subject to

$$d_{mn} \geq d_0 \lambda \quad \forall m, n = 1, 2, \ldots, N; m \neq n$$

$$(d_n, \alpha_n) \in R \quad \forall n = 1, 2, \ldots, N$$

where $\bar{X}_m = \sum_j \rho g \Lambda_m \sqrt{\frac{2S(\omega^j, \beta^j)}{\Delta \beta \Delta \omega}} e^{-ik^j d_n \cos(\beta^j - \alpha_m)}$ for $m = 1, 2, \ldots, N$. 

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4.7 Solution Algorithm

Solving the optimization problem defined in (4.52) analytically is difficult. Thus, we develop a numerical optimization algorithm to solve this problem. As before, first, we examine the optimal solution of the 2-WEC problem. We consider a multi-directional spectrum describing the irregular sea state as:

\[ S(\omega, \beta) = S(\omega; H_s, \omega_p)D(\beta; \beta_0), \]  

(4.53)

where \( S(\omega; H_s, \omega_p) \) is the power spectrum with significant wave height \( H_s \) and peak frequency \( \omega_p \), and \( D(\beta; \beta_0) \) is:

\[ D(\beta) = \begin{cases} 
\frac{2}{\pi} \cos^2(\beta - \beta_0) & |\beta - \beta_0| < \frac{\pi}{2} \\
0 & |\beta - \beta_0| \geq \frac{\pi}{2}, 
\end{cases} \]

where \( \beta_0 \) is the predominant angle of the incident waves. The range \( 0.5 \leq \omega \leq 2.5 \) rad/s is considered for the frequency, and the the range of \( -\frac{\pi}{16} \leq \beta \leq \frac{\pi}{16} \) is considered for the angle of wave direction when \( \beta_0 = 0 \). In the experiment, we place one WEC at the origin and the second one at point \((\alpha, d)\) in polar coordinates. Table 4.3 shows the result of this numerical study. We obtain the optimal values of \( \alpha \) and \( d \) by enumerating the discretized space.

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<th>( \omega_p )</th>
<th>( \omega_r )</th>
<th>( \bar{q}^* )</th>
<th>( \alpha^* )</th>
<th>( d^* )</th>
<th>( k_d d^\ast )</th>
<th>( k_r d^\ast )</th>
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<td>( \frac{\pi}{2} )</td>
<td>9.50</td>
<td>3.81</td>
<td>3.81</td>
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Table 4.3: Numerical experiment for a 2-WEC farm in irregular waves
We observe that the optimal non-dimensional distance, $k_d^*$, between two WECs for different cases approximately occurs at one of the roots of the Bessel function $J_1(\cdot)$. This fact indicates that the radiated wavenumber due to radiated frequency $\omega_r$ is a key factor in designing a wave farm’s layout. Table 4.4 shows the first eight roots of the Bessel functions $J_0(\cdot)$ and $J_1(\cdot)$.

<table>
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<tr>
<th>$i$ :</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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<tbody>
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<td>$J_0(x_i) = 0$</td>
<td>2.40</td>
<td>5.52</td>
<td>8.65</td>
<td>11.79</td>
<td>14.93</td>
<td>18.07</td>
<td>21.21</td>
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</tr>
<tr>
<td>$J_1(x_i) = 0$</td>
<td>0.00</td>
<td>3.83</td>
<td>7.02</td>
<td>10.17</td>
<td>13.32</td>
<td>16.47</td>
<td>19.62</td>
<td>22.76</td>
</tr>
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</table>

Table 4.4: First eight roots of Bessel functions $J_0(\cdot)$ and $J_1(\cdot)$

Therefore, we can simply modify the two-phase Algorithm 2 to use for the uncertain cases and irregular sea states. We replace the $q$-factor in the optimization with the objective function $g$, where $g$ is either (4.52), (4.18) or (4.19). Moreover, we modify the discrete set $S(M)$ to be the set of both the first $M$ roots of the Bessel function $J_0(\cdot)$, and the first $M$ roots of the Bessel function $J_1(\cdot)$. Thus, to generate master layouts, we randomly generate a matrix of pairwise non-dimensional distances that equal roots of the Bessel function $J_0(\cdot)$ or the Bessel function $J_1(\cdot)$. Algorithm 3 formally states the modified steps of our two-phase heuristic.
Algorithm 3 Modified Two-Phase Heuristic

0. Given: CPU time limit \( T \), parameter \( M \), and objective \( g \). Set \( g_{\text{max}} := 0 \).

1. Randomly generate an \( N \times N \) symmetric matrix whose elements equal one of the elements of the modified discrete search set \( S(M) \), and store it in \( D \).

2. Check whether \( D \) is an EDM using Theorem 4.1. If \( D \) is an EDM, go to 4, otherwise go to 1.

3. Find \( x, y \) coordinates that correspond to the distance matrix \( D \) using Algorithm 1.

4. Using the layout defined by \((x, y)\) as the initial solution, optimize all locations locally using a continuous optimization algorithm. Let \( q := \) the \( q \)-factor of the resulting solution.

5. Let \( g_{\text{max}} := \max\{g, g_{\text{max}}\} \).

6. If the elapsed CPU time \( \geq T \), STOP, otherwise go to 1.

We test this algorithm for the max-min and maximum expected value problems, and for the approximate spectral \( q \)-factor defined in (4.52). For the approximate \( q \)-factor problem, we consider the multi-directional spectrum in (4.53) with significant wave height \( H_s = 0.8 \) m, peak frequency \( \omega_p = 2 \) rad/s, and predominant angle of incident \( \beta_0 = 0 \). The range \( 0.5 \leq \omega \leq 2.5 \) rad/s is considered for the frequency, \( \omega_r = 2 \) rad/s is considered as the radiated frequency, and the range \(-\pi/4 \leq \beta \leq \pi/4\) is considered for the wave direction. For the expected value problem, we consider \( \beta \) to be normally distributed with mean \( \beta_0 \) and variance \( \sigma^2 \). We run the experiment with \( \sigma \in \{\frac{\pi}{4}, \frac{\pi}{8}\} \) and report the results in Tables 4.5–4.9.

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<table>
<thead>
<tr>
<th>Number of WECs</th>
<th>$\min_{\beta}[q^*]$</th>
<th>Optimal Layout</th>
</tr>
</thead>
</table>
| $N = 3$        | 1.120           | $kx^* = (0.0; 0.0; -1.8)$  
                      |                 | $ky^* = (-3.0; 3.0; 0.0)$ |
| $N = 4$        | 1.190           | $kx^* = (0.5; 0.5; -1.4; -1.4)$  
                      |                 | $ky^* = (1.6; -1.6; 4.9; -4.9)$ |
| $N = 5$        | 1.216           | $kx^* = (0.3; 0.3; 0.0; 0.0; 0.2)$  
                      |                 | $ky^* = (-3.3; 3.3; 9.9; -9.9; 0.0)$ |
| $N = 6$        | 1.241           | $kx^* = (-0.4; -0.4; 0.3; 0.3; -0.6; -0.6)$  
                      |                 | $ky^* = (-11.6; 11.6; 4.9; -4.9; 1.6; -1.6)$ |

Table 4.5: Max-Min problem with $-\frac{\pi}{4} \leq \beta \leq \frac{\pi}{4}$

<table>
<thead>
<tr>
<th>Number of WECs</th>
<th>$\min_{\beta}[q^*]$</th>
<th>Optimal Layout</th>
</tr>
</thead>
</table>
| $N = 3$        | 1.481           | $kx^* = (0.0; 0.0; 0.0)$  
                      |                 | $ky^* = (3.5; -3.5; 0.0)$ |
| $N = 4$        | 1.568           | $kx^* = (0.0; 0.0; 0.0; 0.0)$  
                      |                 | $ky^* = (-1.6; 1.6; 5.6; -5.6)$ |
| $N = 5$        | 1.647           | $kx^* = (0.0; 0.0; 0.0; 0.0; 0.0)$  
                      |                 | $ky^* = (3.7; -3.7; -7.6; 7.6; 0.0)$ |
| $N = 6$        | 1.692           | $kx^* = (0.0; 0.0; 0.0; 0.0; 0.0; 0.0)$  
                      |                 | $ky^* = (2.15; -2.15; 5.7; -5.7; 9.8; -9.8)$ |

Table 4.6: Max-Min problem with $-\frac{\pi}{8} \leq \beta \leq \frac{\pi}{8}$
<table>
<thead>
<tr>
<th>Number of WECs</th>
<th>$E_{\theta}[q]^*$</th>
<th>Optimal Layout</th>
</tr>
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</table>
| $N = 3$       | 1.160            | $kx^* = (0.0; 0.0; 0.0)$  
                    |                  | $ky^* = (-3.5; 3.5; 0.0)$  |
| $N = 4$       | 1.179            | $kx^* = (0.0; 0.0; 0.0; 0.0)$  
                    |                  | $ky^* = (-5.3; 5.3; -1.7; 1.7)$  |
| $N = 5$       | 1.190            | $kx^* = (0.0; 0.0; 0.0; 0.0; 0.0)$  
                    |                  | $ky^* = (-3.5; 3.5; 7.1; -7.1; 0.0)$  |
| $N = 6$       | 1.199            | $kx^* = (0.0; 0.0; 0.0; 0.0; 0.0; 0.0)$  
                    |                  | $ky^* = (-5.4; 5.4; -9.0; 9.0; -1.8; 1.8)$  |

Table 4.7: Maximum expected value problem with $\sigma = \frac{\pi}{4}$

<table>
<thead>
<tr>
<th>Number of WECs</th>
<th>$E_{\theta}[q]^*$</th>
<th>Optimal Layout</th>
</tr>
</thead>
</table>
| $N = 3$       | 1.460            | $kx^* = (0.0; 0.0; 0.0)$  
                    |                  | $ky^* = (3.7; -3.7; 0.0)$  |
| $N = 4$       | 1.512            | $kx^* = (0.0; 0.0; 0.0; 0.0)$  
                    |                  | $ky^* = (5.8; -5.8; 1.9; -1.9)$  |
| $N = 5$       | 1.552            | $kx^* = (0.0; 0.0; 0.0; 0.0; 0.0; 0.0)$  
                    |                  | $ky^* = (7.8; -7.8; 3.9; -3.9; 0.0)$  |
| $N = 6$       | 1.577            | $kx^* = (0.0; 0.0; 0.0; 0.0; 0.0; 0.0; 0.0; 0.0)$  
                    |                  | $ky^* = (2.0; -2.0; 9.8; -9.8; 5.9; -5.9)$  |

Table 4.8: Maximum expected value problem with $\sigma = \frac{\pi}{8}$
From these results, we observe that the good layouts for all proposed models are symmetric with respect to the predominant direction of incident $\beta_0$. In addition, as the number of WECs increases, the optimal objective values of all models increase slightly. As anticipated, in the expected value problem, an increase in the wave direction’s standard deviation will degrade the wave farms’ performances. This is also true for the max-min problem where an increase in the wave direction’s range will decrease the wave farms’ performances.

### 4.8 Conclusion

In this chapter, we study optimization models and algorithms for the optimal configuration of wave farms. We develop theoretical properties for the $q$-factor under the point-absorber approximation. Moreover, we develop two optimization models, called the max-min model and the maximum expected value model, for the WECLP under uncertainty considering simple regular waves with random wave heading direction, $\beta$. We prove structural properties of the max-min model and the maximum expected value
model for a 2-WEC layout and observe that the optimal distance between the two devices decreases as the uncertainty increases. Finally, we provide an approximate performance measure for the design of wave energy farms in irregular ocean waves.

Based on the analytical and numerical solutions for the 2-WEC problem, we gain insights into the relative spacing among the WECs in near-optimal solutions in both deterministic and stochastic environments. Based on these insights, we propose a heuristic optimization algorithm for choosing layouts to maximize the $q$-factor in a deterministic setting, and modify it to be applicable for the uncertain and spectral ocean environments.

Our results demonstrate that the heuristic algorithm, Algorithm 2, generally outperforms the WECLP-specific genetic algorithm by Mao [68] and the general-purpose global solvers PSWARM and NOMAD. This heuristic does not exploit our result that the WECLP problem is isomorphic. One possible method for improving a given solution would be to take the solution, find the $\beta$ and/or $k$ for which the solution gives the greatest $q$-factor, and then rotate and/or scale the resulting layout to reconfigure it for the original $\beta$ and $k$. The local optimization step could be added, as well. Developing a more efficient algorithm for finding good master layouts is another possible avenue for future research.

Moreover, we provide models and a solution algorithm for ocean environments where the wave heading angle $\beta$ is stochastic. To our knowledge, there are no such results in the literature for comparison; however, testing accuracy of the proposed models is an important study for the future research. Also, there is a need for a closed-form objective function for the max-min and expected value optimization models, and obtaining such a function should be pursued.

In addition, we propose an optimization model for irregular ocean waves and develop a modified algorithm to solve it. However, the accuracy of the model is still an open question. One might perform a time domain analysis to obtain a better model for
irregular wave regimes.
Bibliography


[34] F. Fusco, J. Gilloteaux, and J. Ringwood. A study on prediction requirements in time-domain control of wave energy converters. *Proceeding of Control Applications in Marine Systems*. 120


## Appendix A

### Tables for Chapter 2

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Table A.1: Average Gap: heuristics vs. analytical estimate (1/3)
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Table A.2: Average Gap: heuristics vs. analytical estimate (2/3)

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Table A.3: Average Gap: heuristics vs. analytical estimate (3/3)
<table>
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<tr>
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<th>$\frac{p}{p+h}$</th>
<th>EV</th>
<th>DO</th>
<th>SLA</th>
<th>Modified DP</th>
</tr>
</thead>
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Table A.4: Average CPU time (sec.): heuristics vs. optimal (1/3)

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Table A.5: Average CPU time (sec.): heuristics vs. optimal (2/3)
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<th>DO</th>
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<th>Modified DP</th>
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Table A.6: Average CPU time (sec.): heuristics vs. optimal (3/3)

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<th>IEEE30</th>
<th>IEEE57</th>
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</thead>
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<td>1 2 3 4</td>
<td>1 2 3 4</td>
<td>1 2 3 4 5</td>
</tr>
<tr>
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<td>14 10 10</td>
<td>14 10 10</td>
<td>10 14 10 55 10 41</td>
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Table A.7: Test Problems

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Table A.8: Average CPU time (sec.)
Appendix B

Proofs for Chapter 2

Proof of Lemma 2.1  The following properties hold for the new quantities:

1. \( \omega_{ij} = V_i V_j \gamma_{ij} = V_j V_i \gamma_{ji} = \omega_{ji} \), since \( \gamma_{ij} = \gamma_{ji} \).

2. \( \delta_{ii}(t) = \theta_i(t) - \theta_i(t) = 0 \) \( \forall i \in \mathcal{N} \).

3. \( \delta_{ij}(t) = \theta_i(t) - \theta_j(t) = -(\theta_j(t) - \theta_i(t)) = -\delta_{ji}(t) \) \( \forall i, j \in \mathcal{N} \).

4. \( \delta_{ij}(t) + \delta_{jk}(t) = \theta_i(t) - \theta_j(t) + \theta_j(t) - \theta_k(t) = \theta_i(t) - \theta_k(t) = \delta_{ik}(t) \) \( \forall i, j, k \in \mathcal{N} \).

5. \( \delta_{ij}(t) = \sum_{k=i}^{j-1} \delta_{k,k+1} \) by property 4, where \( \sum_{k=i}^{j-1} \delta_{k,k+1} = 0 \) by convention.

Property 5 implies that the information from all of the \( n^2 - n \) new decision variables \( \delta_{ij}(t) \) is captured by the \( n - 1 \) variables \( \delta_{12}(t), \delta_{23}(t), \delta_{34}(t), \ldots, \delta_{n-1,n}(t) \).

Plugging in \( \omega_{ij} = V_i V_j \gamma_{ij} \) and \( \delta_{ij}(t) = \theta_i(t) - \theta_j(t) \) to (2.24a) yields \( \omega_{ij} \delta_{ij}(t) \leq \bar{q}_{ij} \) \( \forall i, j \in \mathcal{N} \). The physics of the power system dictates that \( \bar{q}_{ij} = \bar{q}_{ji} \); then we also have \( \omega_{ji} \delta_{ji}(t) \leq \bar{q}_{ij} \), \( \forall i, j \in \mathcal{N} \). Thus, by properties 1 and 3, for all \( i \leq j \in \mathcal{N} \) we have:

\[
\omega_{ij} \delta_{ij}(t) \leq \bar{q}_{ij} \tag{B.1}
\]

\[
-\omega_{ij} \delta_{ij}(t) \leq \bar{q}_{ij}.
\]
Using properties 2 and 5, we can write (B.1) as:

\[ \omega_{ij} \sum_{k=i}^{j-1} \delta_{k,k+1} \leq \bar{q}_{ij}, \quad \forall i < j \in \mathcal{N} \]

(B.2)

\[ -\omega_{ij} \sum_{k=i}^{j-1} \delta_{k,k+1} \leq \bar{q}_{ij}, \quad \forall i < j \in \mathcal{N}. \]

Letting $|\mathcal{N}| = n$, the matrix form of the system of linear inequalities (B.2) is

\[
\mathbf{\Omega}_1 \mathbf{\delta}(t) \leq \bar{q} \\
-\mathbf{\Omega}_1 \mathbf{\delta}(t) \leq \bar{q},
\]

(B.3)

where

1. $\mathbf{\Omega}_1$ is an $\frac{n(n-1)}{2} \times (n-1)$-matrix with the following structure:

\[
\begin{pmatrix}
\omega_{12} & 0 & 0 & 0 & \ldots & 0 \\
\omega_{13} & \omega_{13} & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\omega_{1n} & \omega_{1n} & \omega_{1n} & \omega_{1n} & \ldots & \omega_{1n} \\
0 & \omega_{23} & 0 & 0 & \ldots & 0 \\
0 & \omega_{24} & \omega_{24} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \omega_{2n} & \omega_{2n} & \omega_{2n} & \ldots & \omega_{2n} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \omega_{n-2,n-1} & 0 \\
0 & 0 & 0 & \ldots & \omega_{n-2,n-1} & 0 \\
0 & 0 & 0 & \ldots & \omega_{n-2,n} & \omega_{n-2,n} \\
0 & 0 & 0 & \ldots & 0 & \omega_{n-1,n}
\end{pmatrix}
\]

2. $\mathbf{\delta}(t) = (\delta_{12}(t), \delta_{23}(t), \ldots, \delta_{n-1,n}(t))^T$ is an $(n-1)$-vector of decision variables.

135
3. \( \bar{q} = (\bar{q}_{ij})_{j \in \mathcal{N}} \) is an \( \frac{n(n-1)}{2} \)-vector.

Next, we plug \( \omega_{ij} = V_i V_j \gamma_{ij} \) and \( \delta_{ij}(t) = \theta_i(t) - \theta_j(t) \) into (2.24b) to get

\[
\sum_{j \in \mathcal{N}} \omega_{ij} \delta_{ij}(t) = q_i(t), \quad \forall i \in \mathcal{N}.
\]  

(B.4)

The left-hand side of (B.4) is equal to

\[
\begin{align*}
\sum_{j=1}^{i-1} \omega_{ij} \delta_{ij}(t) + \sum_{j=i}^{n} \omega_{ij} \delta_{ij}(t) \\
= \sum_{j=1}^{i-1} -\omega_{ij} \delta_{ji}(t) + \sum_{j=i}^{n} \omega_{ij} \delta_{ji}(t) \quad \text{(by properties 1 and 3)} \\
= \sum_{j=1}^{i-1} \delta_{j,k+1}(t) \left( \sum_{k=i}^{j-1} \omega_{ij} \right) + \sum_{j=i}^{n} \omega_{ij} \left( \sum_{k=i}^{j-1} \delta_{j,k+1}(t) \right) \quad \text{(by property 5)} \\
= \sum_{k=1}^{i-1} \delta_{k,k+1}(t) \left( \sum_{j=1}^{k} -\omega_{ij} \right) + \sum_{k=i}^{n-1} \delta_{k,k+1}(t) \left( \sum_{j=k+1}^{n} \omega_{ij} \right) = q_i(t) \quad \text{(changing the order of the sums)}.
\end{align*}
\]

In matrix form, we have

\[
\begin{align*}
\Omega_2 \delta(t) &= q(t), \\
\quad (B.5)
\end{align*}
\]

where

1. \( q(t) = (q_i(t))_{i=1,2,...,n} \) is an \( n \)-vector.

2. \( \Omega_2 \) is an \( n \times (n-1) \)-matrix whose elements are

\[
[\Omega_2]_{ik} = \begin{cases} 
-\sum_{j=1}^{i-1} \omega_{ij}, & k < i \\
\sum_{j=k+1}^{n} \omega_{ij}, & k \geq i.
\end{cases}
\]

\( \square \)

**Proof of Lemma 2.2**  
Let \( S_t \) be the state of the system of \( m \) generators at time \( t \), where each generator follows an independent Markovian disruption process with disruption
and recovery probabilities $\alpha$ and $\beta$, respectively. Thus,

$$\beta' = Pr\{S_{r+1} = \text{Up} | S_r = \text{Down}\}$$
$$= 1 - Pr\{S_{r+1} = \text{Down} | S_r = \text{Down}\}$$
$$= 1 - (1 - \beta)^m.$$ 

Also,

$$\alpha' = Pr\{S_{r+1} = \text{Down} | S_r = \text{Up}\}$$
$$= \sum_{x=1}^{m} Pr\{S_{r+1} = \text{Down} | S_r = \text{Up} \& x \text{ generators are Up}\} Pr\{x \text{ generators are Up} | S_r = \text{Up}\}$$
$$= \sum_{x=1}^{m} \alpha'^x (1 - \beta)^{m-x} Pr\{x \text{ generators are Up} | S_r = \text{Up}\}.$$ 

Using Bayes’ rule, we have:

$$Pr\{x \text{ generators are Up} | S_r = \text{Up}\} = \frac{Pr\{S_r = \text{Up} | x \text{ generators are Up}\} Pr\{x \text{ generators are Up}\}}{Pr\{S_r = \text{Up}\}}.$$ 

Clearly, $Pr\{S_r = \text{Up} | x \text{ generators are Up}\} = 1$ for $x \geq 1$. Also,

$$Pr\{S_r = \text{Up}\} = 1 - Pr\{S_r = \text{Down}\},$$

and $Pr\{S_r = \text{Down}\}$ is equivalent to the probability that all generators are down, thus

$$Pr\{S_r = \text{Down}\} = \pi^m,$$

where $\pi$ is the steady state probability of a generator being down, i.e., $\pi = \frac{\alpha}{\alpha + \beta}$. Finally,
the probability of having \( x \) generators working out of \( m \) generators is:

\[
\binom{m}{x}(1 - \pi)^x \pi^{m-x}.
\]

Therefore, combining the results above, we have:

\[
\alpha' = \sum_{x=1}^{m} \frac{\binom{m}{x}(1 - \pi)^x \pi^{m-x}}{1 - \pi^m} \alpha^x (1 - \beta)^{m-x}
\]

\[
= \frac{1}{1 - \pi^m} \sum_{x=1}^{m} \binom{m}{x} (\alpha - \alpha \pi)^x (\pi - \pi \beta)^{m-x}
\]

\[
= \frac{1}{1 - \pi^m} \left( (\alpha - \alpha \pi + \pi - \pi \beta)^m - (\pi - \pi \beta)^m \right)
\]

\[
= \frac{1}{1 - \pi^m} \left( (\alpha - (\alpha + \beta) \pi + \pi)^m - \pi^m (1 - \beta)^m \right)
\]

\[
= \frac{1}{1 - \pi^m} (\pi^m - \pi^m (1 - \beta)^m)
\]

\[
= (1 - (1 - \beta)^m) \frac{\pi^m}{1 - \pi^m}
\]

\[\square\]

**Proof of Lemma 2.3**  Since \( 0 \leq 1 - \beta \leq 1 \), for any integer \( m \geq 1 \), \((1 - \beta)^m \leq 1 - \beta\) which implies that \( \beta \leq 1 - (1 - \beta)^m \). Therefore, by (2.33) we have \( \beta \leq \beta' \). Now, let \( \zeta = 1 - \beta \); then by (2.34),

\[
\alpha' = \frac{(1 - \zeta^m) \alpha^m}{(\alpha - \zeta + 1)^m - \alpha^m},
\]

and \( \alpha' \leq \alpha \) if and only if

\[
f(m) = \frac{(1 - \zeta^m) \alpha^{m-1}}{(\alpha - \zeta + 1)^m - \alpha^m} \leq 1.
\]

We use mathematical induction on \( m \) to prove that \( f(m) \leq 1 \).
**Base Case:** If $m = 1$, then

\[
f(1) = \frac{(1 - \zeta)}{(\alpha - \zeta + 1) - \alpha} = 1 \leq 1.
\]

**Induction Step:** Suppose the claim holds for $m$. Then,

\[
f(m) = \frac{(1 - \zeta^m)\alpha^{m-1}}{(\alpha - \zeta + 1)^m - \alpha^m} \leq (1 - \zeta^m)\alpha^{m-1} + \alpha^m \leq (\alpha - \zeta + 1)^m
\]

\[
\Rightarrow (\alpha - \zeta + 1)(1 - \zeta^m)\alpha^{m-1} + \alpha^m \leq (\alpha - \zeta + 1)^{m+1}
\]

since $\alpha - \zeta + 1 \geq 0$.

The claim holds for $m + 1$ if

\[
(1 - \zeta^{m+1})\alpha^m + \alpha^{m+1} \leq (\alpha - \zeta + 1)(1 - \zeta^m)\alpha^{m-1} + \alpha^m.
\]

Suppose not (for a contradiction); then

\[
(1 - \zeta^{m+1})\alpha^m + \alpha^{m+1} > (\alpha - \zeta + 1)(1 - \zeta^m)\alpha^{m-1} + \alpha^m
\]

\[
\Rightarrow (1 - \zeta^{m+1})\alpha^2 > (\alpha + 1 - \zeta)(\alpha + 1 - \zeta^m)
\]

\[
\Rightarrow \alpha(\alpha + 1) - \alpha\zeta^{m+1} > (\alpha + 1)^2 - (\alpha + 1)\zeta^m - (\alpha + 1)\zeta + \zeta^{m+1}
\]

\[
\Rightarrow \alpha(\alpha + 1) - (\alpha + 1)^2 + (\alpha + 1)\zeta^m + (\alpha + 1)\zeta > (\alpha + 1)\zeta^{m+1}
\]

\[
\Rightarrow -1 + \zeta^{m+1} + \zeta > \zeta^{m+1} \Rightarrow \zeta - 1 > \zeta^m(\zeta - 1)
\]

Since $\zeta = 1 - \beta$, we have

\[
-\beta > -\beta(1 - \beta)^m \Rightarrow (1 - \beta)^m > 1,
\]
which is a contradiction. Thus,

$$(1 - \zeta^{m+1})\alpha^m + \alpha^{m+1} \leq (\alpha - \zeta + 1)(1 - \zeta^m)\alpha^{m-1} + \alpha^m).$$

Since the claim holds for $m$, we have:

$$(1 - \zeta^{m+1})\alpha^m + \alpha^{m+1} \leq (\alpha - \zeta + 1)^{m+1} \Rightarrow \frac{(1 - \zeta^{m+1})\alpha^m}{(\alpha - \zeta + 1)^{m+1} - \alpha^{m+1}} \leq 1.$$

Therefore, $f(m + 1) \leq 1$. \hfill \Box

**Proof of Lemma 2.4** By Lemma 2.3, $0 \leq \alpha' \leq \alpha \leq 1$, and we know $0 \leq \beta \leq \beta' \leq 1$,

thus

$$\frac{(1 - \beta')^k}{(1 - \beta)^k} \leq 1, \quad \text{(B.6)}$$

and

$$\alpha'\beta \leq \alpha\beta' \Rightarrow \alpha' + \alpha' \beta \leq \alpha + \alpha' \beta \Rightarrow \frac{\alpha\alpha' + \alpha'\beta}{\alpha\alpha' + \alpha'\beta} \leq 1. \quad \text{(B.7)}$$

(B.6) and (B.7) imply that

$$\left(\frac{\alpha\alpha' + \alpha'\beta}{\alpha\alpha' + \alpha'\beta}\right) \frac{(1 - \beta')^k}{(1 - \beta)^k} \leq 1 \Rightarrow \frac{\alpha' (1 - \beta')^k}{\alpha (1 - \beta)^k} \leq 1$$

$$\Rightarrow \frac{\alpha'}{\alpha'} (1 - \beta')^k \leq \frac{\alpha}{\alpha + \beta} (1 - \beta)^k$$

$$\Rightarrow 1 - \frac{\alpha'}{\alpha'} (1 - \beta')^k \geq 1 - \frac{\alpha}{\alpha + \beta} (1 - \beta)^k$$

$$\Rightarrow F(k) \leq F'(k).$$

\hfill \Box

**Proof of Lemma 2.5** By Lemma 2.4, for any integer $k$, $F(k) \leq F'(k)$. Since $F(\cdot)$ and $F'(\cdot)$ are both monotonically increasing functions, $F^{-1}(x) \geq F'^{-1}(x)$ for $0 \leq x \leq 1$. 

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Thus,

$$
\sum_{i \in G} S_i^* = \sum_{i \in G} \bar{d}_i \left[ 1 + F^{-1} \left( \frac{p}{h} \right) \right] = D \left[ 1 + F^{-1} \left( \frac{p}{h} \right) \right] \geq D \left[ 1 + F'^{-1} \left( \frac{p}{h} \right) \right] = S^*.
$$

\[\square\]

**Proof of Theorem 2.1** First observe that

$$
g(S^*) = \sum_{x=0}^{\infty} \psi_x [h(S^* - (x + 1)D)^+ + p((x + 1)D - S^*)^+] \\
= \sum_{x=0}^{\infty} \psi_x [h(k'D - (x + 1)D)^+ + p((x + 1)D - k'D)^+] \\
= \sum_{x=0}^{\infty} D\psi_x \left[ h(k' - (x + 1))^+ + p((x + 1) - k')^+ \right] \\
= \sum_{x=0}^{k'-1} Dh\psi_x [k' - (x + 1)] + \sum_{x=k'}^{\infty} D\psi_x ((x + 1) - k') \\
= D \left[ h \sum_{x=0}^{k'-1} \psi_x (k' - (x + 1)) + p \sum_{x=k'}^{\infty} \psi_x ((x + 1) - k') \right]. \tag{B.8}
$$

Now, consider

$$
g(S^* + \Delta S) = \sum_{x=0}^{\infty} \psi_x [h(S^* + \Delta S - (x + 1)D)^+ + p((x + 1)D - S^* - \Delta S)^+] \\
= \sum_{x=0}^{\infty} \psi_x \left[ (h'(k' + \Delta k) - (x + 1))^+ + p((x + 1) - k' - \Delta k)^+ \right] \\
= \sum_{x=0}^{k'+\Delta k-1} Dh\psi_x (k' + \Delta k - (x + 1)) + \sum_{x=k'+\Delta k}^{\infty} D\psi_x ((x + 1) - k' - \Delta k) \\
= D \left[ h \sum_{x=0}^{k'+\Delta k-1} \psi_x (k' - x - 1) + \psi_x \Delta k \right] + p \sum_{x=k'+\Delta k}^{\infty} \psi_x ((x + 1) - k') - \psi_x \Delta k] \\
= D \left[ h \sum_{x=0}^{k'+\Delta k-1} \psi_x (k' - x - 1) + p \sum_{x=k'+\Delta k}^{\infty} \psi_x (x + 1 - k') + \Delta k \mathcal{T}_1 \right],
$$

where \( \mathcal{T}_1 = h \sum_{x=0}^{k'+\Delta k-1} \psi_x - p \sum_{x=k'+\Delta k}^{\infty} \psi_x = (h + p)F'(k' + \Delta k - 1) - p, \) and \( F'(k) = \sum_{x=0}^{k} \psi_x \)
is the cdf corresponding to the pmf $\psi_x$. Thus,

$$g(S^* + \Delta S) = D \left( h \sum_{x=0}^{k'-1} \psi_x(k' - x - 1) + p \sum_{x=k'}^{\infty} \psi_x(x + 1 - k') + \Upsilon_2 + \Delta k \Upsilon_1 \right),$$

where

$$\Upsilon_2 = h \sum_{x=k'}^{k' + \Delta k - 1} \psi_x(k' - x - 1) - p \sum_{x=k'}^{k' + \Delta k - 1} \psi_x(x + 1 - k') = (h + p) \sum_{x=k'}^{k' + \Delta k - 1} \psi_x(k' - x - 1).$$

Using (B.8), we get

$$g(S^* + \Delta S) = g(S^*) + D (\Upsilon_2 + \Delta k \Upsilon_1). \quad (B.9)$$

We know that $k' = 1 + F'^{-1}\left(\frac{p}{p+h}\right)$, thus $F'(k' - 1) \geq \frac{p}{p+h}$, by definition. Let $\epsilon = F'(k' - 1) - \frac{p}{p+h}$; then

$$\Upsilon_1 = (h + p)F'(k' + \Delta k - 1) - p$$

$$= (h + p)F'(k' - 1) + (h + p) \sum_{x=k'}^{k' + \Delta k - 1} \psi_x - p$$

$$= (h + p)(\frac{p}{p+h} + \epsilon) + (h + p) \sum_{x=k'}^{k' + \Delta k - 1} \psi_x - p$$

$$= (h + p)\epsilon + (h + p) \sum_{x=k'}^{k' + \Delta k - 1} \psi_x.$$

Hence,

$$\Upsilon_2 + \Delta k \Upsilon_1 = (h + p) \sum_{x=k'}^{k' + \Delta k - 1} \psi_x(k' - x - 1) + \Delta k \left((h + p)\epsilon + (h + p) \sum_{x=k'}^{k' + \Delta k - 1} \psi_x \right)$$

$$= (h + p) \sum_{x=k'}^{k' + \Delta k - 1} \psi_x(k' + \Delta k - x - 1) + \Delta k(h + p)\epsilon.$$
By plugging $\Upsilon_2 + \Delta k \Upsilon_1$ into (B.9), we get

$$g(S^* + \Delta S) = g(S^*) + D(h + p) \left( \epsilon \Delta k + \sum_{x=k'}^{k' + \Delta k - 1} \psi_x(k' + \Delta k - x - 1) \right).$$  \hspace{1cm} (B.10)

\[ \square \]

**Proof of Corollary 2.1**  The percentage increase in cost is defined as

$$\frac{g(S^* + \Delta S) - g(S^*)}{g(S^*)}.$$

Using equations (B.8) and (B.9) in the proof of Theorem 2.1, we have

$$\frac{g(S^* + \Delta S) - g(S^*)}{g(S^*)} = \frac{D(h + p) \left( \epsilon \Delta k + \sum_{x=k'}^{k' + \Delta k - 1} \psi_x(k' + \Delta k - x - 1) \right)}{D \left( h \sum_{x=0}^{k'-1} \psi_x(k' - (x + 1)) + p \sum_{x=k'}^{\infty} \psi_x((x + 1) - k') \right)}$$

$$\frac{(h + p) \left[ \epsilon \Delta k + \sum_{x=k'}^{k' + \Delta k - 1} \psi_x(k' + \Delta k - x - 1) \right]}{h \sum_{x=0}^{k'-1} \psi_x(k' - (x + 1)) + p \sum_{x=k'}^{\infty} \psi_x((x + 1) - k')}$$

which is independent of the total demand.  \[ \square \]
Appendix C

AC/DC Power Flow Analysis

A power network consists of a set of nodes, called buses, connected via transmission lines. Each bus may be connected to equipment which will either supply power to or consume power from the network. In power system terminology, power refers to complex power with both real and reactive components. Power flow in a network is determined by the voltage values of each bus, the power supply quantities and the complex power loads, given constraints on the voltage values, generation quantities and impedances of the lines between buses. According to the laws of physics, the complex power flow into the network at bus $k$ is:

$$S_k = P_k + iQ_k = V_k I_k^*,$$  \hspace{1cm} (C.1)

where $V_k$ is the voltage of bus $k$ and $I_k$ is the current injected at bus $k$. $P_k$ and $Q_k$ are net real and reactive power in bus $k$, respectively, and are defined as:

$$P_k = P_k^e - P_k^d,$$

$$Q_k = Q_k^e - Q_k^d.$$
Here, $P_k^p$ and $Q_k^q$ are the real and reactive power generation at bus $k$, respectively, and $P_k^d$ and $Q_k^d$ are the real and reactive power demands at bus $k$, respectively. Let $n$ be the number of buses in the power network; then due to Kirchhoff’s current law, the injected current at bus $k$ is:

$$I_k = \sum_{j=1}^{n} \frac{V_j}{Z_{kj}},$$

(C.2)

where $Z_{jk} = Z_{kj}$ is the impedance of the line connecting node $j$ to $k$. Impedance is a measure of the opposition that a line presents to a current when a voltage is applied. Admittance is the inverse of impedance, $Y_{jk} = \frac{1}{Z_{jk}}$, and is a measure of how easily a transmission line will allow a current to flow. In power flow analysis, it is easier to work with admittance instead of impedance. From (C.1) and (C.2), we have:

$$P_k - iQ_k = V_k I_k^* = \sum_{j=1}^{n} \frac{V_j V_k^*}{Z_{kj}} = \sum_{j=1}^{n} Y_{jk} V_j V_k^*.$$  (C.3)

Voltage $V_k$ at bus $k$ is a complex number with voltage magnitude $|V_k|$ and voltage angle $\theta_k$, and

$$V_k = |V_k|e^{i\theta_k}.  \quad \text{(C.4)}$$

Now, let $Y_{kj} = G_{kj} + iB_{kj}$; then we have:

$$P_k = |V_k| \sum_{j=1}^{n} |V_j| \left( G_{jk} \cos(\theta_j - \theta_k) + B_{jk} \sin(\theta_j - \theta_k) \right) \quad \text{ (C.5)}$$

$$Q_k = |V_k| \sum_{j=1}^{n} |V_j| \left( G_{jk} \sin(\theta_j - \theta_k) - B_{jk} \cos(\theta_j - \theta_k) \right). \quad \text{ (C.6)}$$

These equations are called the AC power flow equations. In power flow analysis, there are four quantities of interest associated with bus $k$: real power, $P_k$, reactive power, $Q_k$, voltage magnitude, $|V_k|$, and voltage angle, $\theta_k$. At every bus of the system, two of these four quantities will be known and the others need to be determined. Depending on which quantities are known, we have different bus types in the power system, such
as:

1. Slack bus: A single bus for which the voltage magnitude and angle are known.

2. Load bus (demand): Any bus of the system for which the real and reactive power are known.

3. Voltage controlled bus (generator): Any bus for which the voltage magnitude and the real power are known.

The AC power flow equations, (C.5) and (C.6), are nonlinear, which makes power flow analysis difficult. In power systems analysis, it is common to linearize equations (C.5) and (C.6) to estimate the unknown power system variables. The DC approximation simplifies the nonlinear AC model to a linear form by assuming:

1. Line resistances (active power losses) are negligible, i.e., $G_{jk} \approx 0$.

2. Voltage angle differences are small, i.e., $\sin(\theta_j - \theta_k) \approx \theta_j - \theta_k$ and $\cos(\theta_j - \theta_k) \approx 1$.

3. Magnitudes of bus voltages are known parameters.

Based on the above assumptions, the AC power flow equations simplify to the DC power flow equations:

$$P_k = |V_k| \sum_{j=1}^{n} |V_j| B_{jk}(\theta_j - \theta_k) \quad \text{(C.7)}$$

$$Q_k = -|V_k| \sum_{j=1}^{n} |V_j| B_{jk}. \quad \text{(C.8)}$$

Note that the DC approximation makes some restrictive assumptions, and researchers differ about its accuracy and usefulness. This issue has been studied by Purchala et al. [83].
Appendix D

Scenario Based Chance Constrained Optimization

One approach to write the deterministic equivalent of a chance constrained optimization problem is by defining a set of binary variables corresponding to the set of possible scenarios. Let $\Pi_j = \{\pi^1_j, \pi^2_j, \ldots, \pi^m_j\}$ be the set of possible scenarios for $\epsilon'_j$ in time $t$, $\forall j \in \tilde{G}$. We define binary variables $\bar{z}_{ts}$, $\hat{z}_{ts}$ and $\tilde{z}_{ts}$ for $s = 1, 2, \ldots, m$ and $t = 1, 2, \ldots, T$ as:

$$\bar{z}_{ts} = \begin{cases} 0 & \text{if } -\bar{f}_k \leq \sum_{i \in \bar{G}} \bar{B}_{ki} \left( \bar{p}_i + \frac{1}{|\bar{G}|} \sum_{j \in \bar{G}} x^t_j \pi^t_j \right) + \sum_{j \in \bar{G}} \hat{B}_{kj} x^t_j \pi^t_j - \sum_{i \in \bar{D}} \hat{B}_{ki} d^t_i \leq \bar{f}_k \\ 1 & \text{otherwise,} \end{cases}$$

(D.1)

for the probabilistic constraints (3.24) and (3.25);

$$\tilde{z}_{ts} = \begin{cases} 0 & \text{if } \bar{i}^t_{\text{min}} \leq \bar{i}^t_{\tilde{r}} - \frac{1}{|\bar{G}|} \sum_{j \in \bar{G}} x^t_j \pi^t_j + \frac{1}{|\bar{G}|} \sum_{j \in \bar{G}} x^{t-1}_j \pi^{t-1}_j \leq \bar{i}^t_{\text{max}} \\ 1 & \text{otherwise,} \end{cases}$$

(D.2)
for the probabilistic constraints (3.28) and (3.29); and

\[
\bar{z}_t^s = \begin{cases} 
0 & \text{if } \bar{p}_i^t - \frac{1}{|\hat{G}|} \sum_{j \in \hat{G}} \bar{L}_{j}^s \leq \bar{p}_i^t - \frac{1}{|\hat{G}|} \sum_{j \in \hat{G}} \bar{L}_{j}^s \leq P_{i}^{\text{max}} \\
1 & \text{otherwise,}
\end{cases}
\]  

(D.3)

for the probabilistic constraints (3.26) and (3.27). Therefore, we rewrite (3.24), (3.25), (3.28), (3.29), (3.26) and (3.27) as the following constraints:

\[
- \sum_{i \in D} \hat{B}_d^d i + \sum_{i \in \hat{G}} \hat{B}_d^d i \left( \bar{p}_i^t - \frac{1}{|\hat{G}|} \sum_{j \in \hat{G}} \bar{L}_{j}^s \right) + \sum_{i \in \hat{G}} \hat{B}_d^d i \bar{L}_{j}^s \leq \bar{f}_k \forall t, k \in L
\]

\[
\sum_{i \in D} \bar{B}_d^d i - \sum_{i \in \hat{G}} \bar{B}_d^d i \left( \bar{p}_i^t - \frac{1}{|\hat{G}|} \sum_{j \in \hat{G}} \bar{L}_{j}^s \right) + \sum_{i \in \hat{G}} \bar{B}_d^d i \bar{L}_{j}^s \leq \bar{f}_k \forall t, k \in L
\]

\[
\bar{p}_i^t - \frac{1}{|\hat{G}|} \sum_{j \in \hat{G}} \bar{L}_{j}^s + \frac{1}{|\hat{G}|} \sum_{j \in \hat{G}} \bar{L}_{j}^s \leq r_i^\text{max} \forall t, i \in \hat{G}
\]

\[
- \bar{p}_i^t + \frac{1}{|\hat{G}|} \sum_{j \in \hat{G}} \bar{L}_{j}^s - \frac{1}{|\hat{G}|} \sum_{j \in \hat{G}} \bar{L}_{j}^s \leq -r_i^\text{min} \forall t, i \in \hat{G}
\]

\[
\bar{p}_i^t - \frac{1}{|\hat{G}|} \sum_{j \in \hat{G}} \bar{L}_{j}^s \leq P_i^\text{max} \forall t, i \in \hat{G}
\]

\[
- \bar{p}_i^t + \frac{1}{|\hat{G}|} \sum_{j \in \hat{G}} \bar{L}_{j}^s \leq -P_i^\text{min} \forall t, i \in \hat{G}
\]

\[
\sum_{s=1}^{m} \bar{z}_t^s \leq m \gamma \forall t
\]

\[
\sum_{s=1}^{m} \bar{z}_t^s \leq m \gamma \forall t
\]

\[
\sum_{s=1}^{m} \bar{z}_t^s \leq m \gamma \forall t
\]
Finally, we can write the deterministic version of the stochastic ramping optimization model as follows:

\[
\begin{align*}
\min & \sum_{i \in G} \sum_{t=1}^{T} \sum_{s=1}^{m} C_p (\bar{p}_t^i - \frac{1}{|G|} \sum_{j \in G} x_j^i \pi_{jt}^i) + C_r (\bar{r}_t^i - \frac{1}{|G|} \sum_{j \in G} \pi_{jt}^i - \pi_{jt-1}^i) \\
\text{s.t.} & \sum_{i \in D} \bar{d}_t^i - \sum_{j \in G} x_j^i \mu_j^i - \sum_{i \in G} \bar{p}_t^i = 0 \quad \forall t \quad (D.5) \\
& - \sum_{i \in D} \hat{B}_{ki}^l d_t^i + \sum_{i \in G} \hat{B}_{ki}^l (\bar{p}_t^i - \frac{1}{|G|} \sum_{j \in G} x_j^i \pi_{jt}^i) + \sum_{j \in G} \hat{B}_{kj}^l x_j^i \pi_{jt}^i - M \bar{z}_t^i \leq \bar{f}_k \quad \forall t, k \in L \quad (D.6) \\
& \sum_{i \in G} \hat{B}_{ki}^l d_t^i - \sum_{i \in G} \hat{B}_{ki}^l (\bar{p}_t^i - \frac{1}{|G|} \sum_{j \in G} x_j^i \pi_{jt}^i) - \sum_{j \in G} \hat{B}_{kj}^l x_j^i \pi_{jt}^i - M \bar{z}_t^i \leq \bar{f}_k \quad \forall t, k \in L \quad (D.7) \\
& \bar{p}_t^i - \frac{1}{|G|} \sum_{j \in G} x_j^i \pi_{jt}^i + \frac{1}{|G|} \sum_{j \in G} \pi_{jt}^i - \pi_{jt-1}^i - M \bar{z}_t^i \leq \bar{r}_t^i \quad \forall t, i \in \bar{G} \quad (D.8) \\
& -\bar{p}_t^i + \frac{1}{|G|} \sum_{j \in G} x_j^i \pi_{jt}^i - \frac{1}{|G|} \sum_{j \in G} \pi_{jt}^i - \pi_{jt-1}^i - M \bar{z}_t^i \leq -\bar{r}_t^i \quad \forall t, i \in \bar{G} \quad (D.9) \\
& \bar{p}_t^i - \frac{1}{|G|} \sum_{j \in G} x_j^i \pi_{jt}^i - M \bar{z}_t^i \leq \bar{P}_{\max} \quad \forall t, i \in \bar{G} \quad (D.10) \\
& -\bar{p}_t^i + \frac{1}{|G|} \sum_{j \in G} x_j^i \pi_{jt}^i - M \bar{z}_t^i \leq -\bar{P}_{\max} \quad \forall t, i \in \bar{G} \quad (D.11) \\
& \sum_{j=1}^{m} \bar{z}_t^i \leq m \gamma \quad \forall t \quad (D.12) \\
& \sum_{j=1}^{m} \bar{z}_t^i \leq m \gamma \quad \forall t \quad (D.13)
\end{align*}
\]
\[ \sum_{s=1}^{m} \hat{z}_s \leq m\gamma \quad \forall t \quad (D.14) \]

\[ \bar{p}_i^t = \bar{p}_i^{t-1} + \bar{r}_i^t \quad \forall t, i \in \tilde{G} \quad (D.15) \]

\[ 0 \leq x_j^t \leq 1 \quad \forall t, j \in \tilde{G} \quad (D.16) \]
Biography

Mohsen Moarefdoost was born in Kerman, Iran in 1984. In 2006, he received his B.Sc. degree in Industrial Engineering from Iran University of Science and Technology, Tehran, Iran. In 2009, he completed his M.Sc. in Industrial Engineering at Sharif University of Technology, Tehran, Iran. Mohsen joined the Industrial and Systems Engineering Department at Lehigh University in 2011 to pursue a doctoral degree. He was the Ph.D. Student of the Year in 2013 and a recipient of the Rossin Doctoral Fellowship at Lehigh in 2014. He served as the vice president of the INFORMS student chapter at Lehigh University in 2013.