The laplace transform method for obtaining an open-loop model from time-domain data.

Jose P. Arencibia

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THE LAPLACE TRANSFORM METHOD FOR OBTAINING AN OPEN-LOOP MODEL FROM TIME-DOMAIN DATA

by

JOSE' P. ARENCIBIA, JR.

A THESIS

PRESENTED TO THE GRADUATE COMMITTEE OF LEHIGH UNIVERSITY IN CANDIDACY FOR THE DEGREE OF MASTER OF SCIENCE IN MECHANICAL ENGINEERING LEHIGH UNIVERSITY 1980
This thesis is accepted and approved in partial fulfillment of the requirements for the degree of Master of Science.

19 SEPT 1980

Date

Professor in Charge

Chairman of Department
ACKNOWLEDGEMENTS

The Author sincerely appreciates the guidance, assistance and encouragement of Professor Stanley H. Johnson. He allowed me the freedom to approach this work as I desired while always being ready with the advice and direction I needed.

My thanks, also, to my wife, Karen, for her patience, encouragement and support throughout all facets of this work.
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Discrete time domain data from aircraft short-period performance time-histories are used to solve for a fourth order open-loop standard configuration aircraft model. This model is suitable for use with the Rediess Model PI method[1] for the tuning of high order systems.

The fourth order model is identified by equating S-Domain analytical expressions, obtained from the time-domain data and approximately specified eigenvalues, to the Laplace Transform of the model step response. The result is sixteen non-linear algebraic equations in ten unknowns, which are approximately solved using a pattern search technique.

In the examples considered, well-behaved models were obtained which compared favorably with actual aircraft data. The designer can specify initial eigenvalues within acceptable domains. The final values are varied implicitly due to the approximate nature of the solution. The final eigenvalues of the examples considered were found to be reasonable.

*Numbers in brackets [ ] refer to references in the bibliography section.
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Section 1. Introduction
A recent development in Control System Design Techniques of particular interest is the Model Performance Index Method (Model PI) developed by Rediess[1]. The Model PI Technique provides a method by which a small number of parameters in a high order system can be adjusted to "approach" (in the Model PI sense) a known low order model having desirable performance characteristics.

This work concentrates on the development of a low-order open-loop model which could be used with the Model PI technique to tune higher order aircraft control systems. This model depicts lateral aircraft dynamics and is arrived at by using time-history data to solve for stability derivative parameters in the model's plant equation. In the case of longitudinal dynamics, the $C^*$ response, where

\[
\frac{C^*}{\frac{\phi}{\phi_e}} = k_1 \frac{\phi}{\phi_e} + k_2 \frac{\dot{\phi}}{\phi_e} + k_3 \ddot{\phi} \quad (k_1, k_2, k_3, \text{are dimensional constants})
\]

(as well as the $C^*$ response) must fall within a corresponding time history envelope, thus meeting an established test for longitudinal handling qualities[2].

Similar lateral handling quality criteria, and the corresponding time-history envelopes have been proposed[3]:

- Sideslip response to aileron command input (yaw axis):
This criterion is based on the MIL-F-8785B[4] limitation on maximum sideslip excursion occurring during aileron step command input as expressed in the form of limits of allowable $\Delta \varphi_{\text{max}}/k$.

- The roll rate response to aileron command input (roll axis): This criterion was generated using the lateral equations of motion presented in Appendix A with parameter selection in accordance with MIL-F-8785B[4] specifications as guidelines.

- The $D^*$ criterion:

This is a blend of sideslip angle and lateral force at the pilot station based on dynamic pressure. This criterion weights sideslip more heavily at the low flight speed conditions, conversely emphasizing lateral force at the pilot station at the higher flight speeds. Both parameters are equally weighted at the cross-over dynamic pressure.

In addition to the normalized forms of sideslip ($\beta_N$), roll rate ($\dot{p}_N$) and $D^*$ ($D^*_N$) lateral criteria, their first derivatives with respect to time are also utilized in order to prevent the use of models which would meet the time-history envelope criteria while being undesirable. This added constraint arises as a result of similar concern over the compatibility of non-linear dynamics and higher order effects with the $C^*$ criterion. In this case, the $C^*$ rate of change response was found adequate in preventing unacceptable behavior. Consequently, a further set of constraints, namely that the $\dot{\beta}_N$, $\dot{p}_N$ and $D^*_N$ responses fall within their corresponding envelopes, is imposed on the aircraft in arriving at the model.
From inspection of the plant equation (Appendix A, equation A-9.0-xxi), it is seen that the model to be developed is of the fourth order. Its solution form is that of a fourth order constant coefficient ordinary differential equation. Briefly, the solution form of the model with eigenvalues approximately specified:

\[ y_j(t) = c_{j1}e^{\lambda_1 t} + c_{j2}e^{\lambda_2 t} + c_{j3}e^{\lambda_3 t} + c_{j4}e^{\lambda_4 t} + c_{j5} \]

\[ j = 1, 2, 3 \]

is fitted to a selected time-history within each of the lateral time history envelopes \((p_N, q_N, D_N)\) using a least-squares error minimization method. The resulting response expressions are Laplace Transformed and used to solve for the values of the stability derivative parameters found in the model's plant equation and transfer function. In the \(S\) Domain:

\[ \bar{X} = (S\bar{I} - \bar{A})^{-1} \bar{B}/S \]

\[ \bar{Y} = \bar{G}\bar{X} + \bar{h}/S \]

combining (3) and (4)

\[ \bar{Y} = (\bar{G} (S\bar{I} - \bar{A})^{-1} \bar{B} + \bar{h})/S \]

Finding the values of the parameters in \(\bar{A}, \bar{B}, \bar{G}\) and \(\bar{h}\) completes the definition of the model. The detailed methodology is covered in the next section.
Section 2 - Problem Statement

In order to solve for the unknown parameters in the model's plant equation, data from the time-history envelopes shall be used in the form of response time-histories. The first step is to establish time-histories within the time-history envelopes. This may be done by arbitrarily sketching a desired response curve. The input form from these time-histories consists of a table of discrete values uniformly spaced in time $\hat{y}_j(t)$ where $j=1,2,3$. Each table consists of three sets of values.

Since the model to be used is of the fourth order, its solution must have the form expressed in (2):

$$\hat{y}_j(t) = c_{j1} e^{\lambda_1 t} + c_{j2} e^{\lambda_2 t} + c_{j3} e^{\lambda_3 t} + c_{j4} e^{\lambda_4 t} + c_{j5}$$

$$j = 1,2,3$$

The eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, are approximately specified to be within acceptable ranges. Initial guesses for the values of the eigenvalues are taken within the acceptable ranges.

$$\bar{\lambda}_1 = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix}$$

initial guesses

Given the initial eigenvalues, the coefficients $c_{j1},..., c_{j5}$ are found by least-squares error fitting the form of the solution (2) to the input tabular data.
At the end of this procedure, we have coefficient values that correspond to the initial eigenvalue guesses, the values of these coefficients are now the initial guess values; thus in vector form:

\[
(7) \hat{y}_I(t) = \begin{bmatrix} P_N(t) \\ \varepsilon_N(t) \\ D_N(t) \end{bmatrix} = \tilde{c}_I e^{\tilde{\lambda}_I(t)} + \tilde{c}_{0I}
\]

Using a Pattern Search Technique (see Appendix B) the fitted time-histories \(\hat{y}_I(t)\) can be made to further approach the input data \(\hat{y}_j(t_k)\) by minimizing:

\[
(8) \left( \hat{y}_j(t_k) - \hat{y}_I(t_k) \right)^2
\]

In this case both the \(C_{ji}\)'s and \(\lambda_i\)'s are adjusted to obtain optimized values. The \(\lambda_i\)'s are adjusted within the pre-determined allowable boundaries.

Finally, the "best fit" of the input data is obtained

\[
(9) \hat{y}(t) = \begin{bmatrix} P_N(t) \\ \varepsilon_N(t) \\ D^*_N(t) \end{bmatrix} = \tilde{c}_I e^{\tilde{\lambda}(t)} + \tilde{c}_0
\]

The two-step approach to arrive at (9) thus employs a "coarse" initial guess step followed by a "fine" or tuning step. The latter is an accurate procedure which requires significant computer time compared
to the former. The equations of motion of the fourth order open loop
model are derived in detail in Appendix A, from which we select the
plant equation:

\[ \dot{x} = \bar{A}x + \bar{b} \delta a \]  
(also equation A - 9.0 xxii)

where

\[ \begin{bmatrix} p \\ r \\ \phi \end{bmatrix} \]

\[ \dot{x} = \begin{bmatrix} p \\ r \\ \phi \end{bmatrix} \]  
(also equation A - 9.0 xxii)

and

\[ A = \begin{bmatrix} L_p & L_r & L_\phi & 0 \\ N_p & N_r & N_\phi & 0 \\ 0 & -1 & Y_\phi & g/V_e \\ 1 & 0 & 0 & 0 \end{bmatrix} \]  
(also equation A - 9.0 xxiii)

\[ \bar{A} = \begin{bmatrix} L_p & L_r & L_\phi & 0 \\ N_p & N_r & N_\phi & 0 \\ 0 & -1 & Y_\phi & g/V_e \\ 1 & 0 & 0 & 0 \end{bmatrix} \]

and

\[ \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ 0 \end{bmatrix} \]

\[ \bar{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ 0 \end{bmatrix} \]  
(also equation A - 9.0 xxv)

The output equation (normalized form):

\[ \tilde{y}_N(t) = \tilde{c}_N \bar{x} + \tilde{h}_N \delta a \]  
(also equation A - 11.0 - x)

where
\[
\begin{pmatrix}
1/pss & 0 & 0 & 0 \\
0 & 0 & 1/\xi_{ss} & 0 \\
G_{31} & G_{32} & G_{33} & G_{34} \\
D^*_{ss} & D^*_{ss} & D^*_{ss} & D^*_{ss}
\end{pmatrix}
\]

(15) \(\ddot{G}_N\) = (also equation A - 11.0 - xi)

with

(16) \(G_{31} = 1N_p\)

(17) \(G_{32} = 1N_r\)

(18) \(G_{33} = 1N_\xi + Y_\xi V_e + C_3 q_{co}\)

(19) \(G_{34} = g\)

and

(20) \(\ddot{h}_N = \begin{pmatrix} 0 \\ 0 \\ h_3/D^*_{ss} \end{pmatrix}\) (also equation A - 11.0 - xiii)

with

(21) \(h_3 = b_2 l + b_3 V_e\)

Equations (10) and (14) define the dynamics of the model, wherein the parameters in (12), (13), (15) and (20) are the unknowns whose values are sought.

The limitations of these equations, as developed in Appendix A, should be pointed out.

It is assumed that:

(22) The aircraft is a symmetric vehicle.

(23) The aircraft is flying straight and level.
The aircraft dynamic behavior is the result of small perturbations.

The earth is flat in the region near the aircraft.

Rotor gyroscopic effects are neglected.

Cross coupling terms, such as the derivatives of the symmetric forces and moments with respect to asymmetric motion variables are neglected.

In order to solve for the unknowns in equations (10) and (14), equations (9) - the data - and (14) - the model - are equated:

(28) \( \hat{y}(t) \approx \hat{y}_N(t) \)

Taking the Laplace Transform of the LHS of equation (28) we have

(29) \( \mathcal{L} [\hat{y}(t)] = \hat{y}(S) = \frac{\hat{N}(S)}{\hat{D}(S)} \)

where \( \hat{N}(S) \) and \( \hat{D}(S) \) are polynomials. Taking the Laplace Transform of equation (10), we have,

(30) \( S\bar{x}(S) = \bar{A}\bar{x} + \bar{b}/S \)

rearranging,

(31) \( \bar{x}(S) = (S\bar{I} - \bar{A})^{-1}\bar{b}/S = \phi(S)\bar{b}/S \)

where

(32) \( \phi(S) = (S\bar{I} - \bar{A})^{-1} \)

Taking the Laplace Transform of the RHS of (28), we have

(33) \( \hat{y}_N(S) = \hat{G}_N\bar{x} + \hat{H}_N/S \).
Combining (33) and (31), we have

\[ \tilde{Y}_N(S) = \frac{[\tilde{G}_N \tilde{F}(S) \tilde{D} + \tilde{h}_N]}{S} \]

Equation (34) can be expressed in the following form:

\[ \tilde{Y}_N(S) = \frac{\tilde{N}(S)}{D(S)} \]

where \( \tilde{N}(S) \) and \( D(S) \) are polynomials.

Substituting (29) and (35) into (28), we have

\[ \hat{\tilde{N}}(S) \Rightarrow \hat{\tilde{N}}(S) \]

\[ \frac{\hat{D}(S)}{D(S)} \]

By equating coefficient expressions for equal powers of \( S \), equation (36) yields 16 simultaneous non-linear algebraic equations in 10 unknowns. Highlights of the derivation follow:

Equation (29) can be expressed in the form

\[ \mathbb{L}[\hat{y}(t)] = \hat{Y}(S) = \frac{\hat{\tilde{N}}(S)}{\hat{D}(S)} \]

\[ \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} \end{bmatrix} \begin{bmatrix} 1/(S-\lambda_1) \\ 1/(S-\lambda_2) \\ 1/(S-\lambda_3) \\ 1/(S-\lambda_4) \end{bmatrix} \]

rearranging
(38) \( \hat{N}_j(S) = S^4(c_{j1} + c_{j2} + c_{j3} + c_{j4} + c_{j5}) \)
\[ + S^3 \left[ -c_{j1}(\lambda_2 + \lambda_3 + \lambda_4) - c_{j2}(\lambda_1 + \lambda_3 + \lambda_4) - c_{j3}(\lambda_1 + \lambda_2 + \lambda_4) \right. \]
\[ - c_{j4}(\lambda_1 + \lambda_2 + \lambda_3) - c_{j5}(\lambda_1 + \lambda_2 + \lambda_4) \]
\[ + S^2[c_{j1}(\lambda_2 \lambda_3 + \lambda_3 \lambda_4 + \lambda_2 \lambda_4) + c_{j2}(\lambda_1 \lambda_3 + \lambda_3 \lambda_4 + \lambda_1 \lambda_4) \]
\[ + c_{j3}(\lambda_1 \lambda_2 + \lambda_2 \lambda_4 + \lambda_1 \lambda_4) + c_{j4}(\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_1 \lambda_3) \]
\[ + c_{j5}(\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_1 \lambda_3 + \lambda_2 \lambda_4 + \lambda_1 \lambda_4) \]
\[ + [c_{j5}(\lambda_1 \lambda_2 \lambda_3 \lambda_4)] \]
\[ j = 1, 2, 3 \]

(39) \( \hat{N}_j(S) = S^4\hat{N}_{j4} + S^3\hat{N}_{j3} + S^2\hat{N}_{j2} + S\hat{N}_{j1} + \hat{N}_{j0} \)
\[ j = 1, 2, 3 \]

where

(40) \( \hat{N}_{j4} = c_{j1} + c_{j2} + c_{j3} + c_{j4} + c_{j5} \)

(41) \( \hat{N}_{j3} = -c_{j1}(\lambda_2 + \lambda_3 + \lambda_4) - c_{j2}(\lambda_1 + \lambda_3 + \lambda_4) - c_{j3}(\lambda_1 + \lambda_2 + \lambda_4) \)
\[ - c_{j4}(\lambda_1 + \lambda_2 + \lambda_3) - c_{j5}(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) \]

(42) \( \hat{N}_{j2} = c_{j1}(\lambda_2 \lambda_3 + \lambda_3 \lambda_4 + \lambda_2 \lambda_4) + c_{j2}(\lambda_1 \lambda_3 + \lambda_3 \lambda_4 + \lambda_1 \lambda_4) \)
\[ + c_{j3}(\lambda_1 \lambda_2 + \lambda_2 \lambda_4 + \lambda_1 \lambda_4) + c_{j4}(\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_1 \lambda_3) \]
\[ + c_{j5}(\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_1 \lambda_3 + \lambda_2 \lambda_4 + \lambda_3 \lambda_4) \]

(43) \( \hat{N}_{j1} = -c_{j1}(\lambda_2 \lambda_3 \lambda_4) - c_{j2}(\lambda_1 \lambda_3 \lambda_4) - c_{j3}(\lambda_1 \lambda_2 \lambda_4) \)
\[ - c_{j4}(\lambda_1 \lambda_2 \lambda_3) - c_{j5}(\lambda_1 \lambda_2 \lambda_3 + \lambda_1 \lambda_3 \lambda_4 + \lambda_2 \lambda_3 \lambda_4 + \lambda_1 \lambda_2 \lambda_4) \]

(44) \( \hat{N}_{j0} = c_{j5}(\lambda_1 \lambda_2 \lambda_3 \lambda_4) \)
and, similarly

\( D_j(S) = S_5^j + S_4^j + S_3^j + S_2^j + S_1^j \quad j=1,2,3 \)

where

\( D_{j5} = 1 \)

\( D_{j4} = -(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) \)

\( D_{j3} = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_1 \lambda_4 + \lambda_2 \lambda_4 + \lambda_3 \lambda_4 \)

\( D_{j2} = -(\lambda_1 \lambda_2 \lambda_3 + \lambda_1 \lambda_3 \lambda_4 + \lambda_2 \lambda_3 \lambda_4 + \lambda_1 \lambda_2 \lambda_4) \)

\( D_{j1} = \lambda_1 \lambda_2 \lambda_3 \lambda_4 \)

Expressing (37) in terms of (38) and (45), we have

\[
\hat{N}(S) = \begin{bmatrix}
\hat{N}_1(S)/\hat{D}_1(S) \\
\hat{N}_2(S)/\hat{D}_2(S) \\
\hat{N}_3(S)/\hat{D}_3(S)
\end{bmatrix}
\]

Equation 34 can be expressed in the form

\[
\hat{Y}_n(S) = \begin{bmatrix}
N_1(S)/D_1(S) \\
N_2(S)/D_2(S) \\
N_3(S)/D_3(S)
\end{bmatrix}
\]

where

\( N_1(S) = (\text{cof}_{11}b_1 + \text{cof}_{21}b_2 + \text{cof}_{31}b_3)/p_{ss} \)

\( N_2(S) = (\text{cof}_{13}b_1 + \text{cof}_{23}b_2 + \text{cof}_{33}b_3)/\theta_{ss} \)

\( N_3(S) = [(\text{cof}_{11}G_{31} + \text{cof}_{12}G_{32} + \text{cof}_{13}G_{33} + \text{cof}_{14}G_{34})b_1 \)

\( + (\text{cof}_{21}G_{31} + \text{cof}_{22}G_{32} + \text{cof}_{23}G_{33} + \text{cof}_{24}G_{34})b_2 \)

\( + (\text{cof}_{31}G_{31} + \text{cof}_{32}G_{32} + \text{cof}_{33} + \text{cof}_{34}G_{34})b_3 + h_3]/D^k_{ss} \)

and

\( D_1(S) = D_2(S) = D_3(S) = D(S) \)
(57) \( D(S) = S^5 + D_4 S^4 + D_3 S^3 + D_2 S^2 + D_1 S \)

where

(58) \( D_4 = -Lp - Nr - \psi_\ell \)

(59) \( D_3 = N_\psi + NrLp - LpNr + \psi_\ell (Lp + Nr) \)

(60) \( D_2 = \psi_\ell (LpNr - Lp) + LpN_\psi + LpN_\psi - (\frac{Lp}{\psi_\ell})L_\psi \)

(61) \( D_1 = \left(\frac{Lp}{\psi_\ell}\right)(LpN_\psi - LpNr) \)

(62) \( \text{cof}_{11} = S^3 + S^2 (-Nr - \psi_\ell) + S(Nr \psi_\ell + N_\psi) \)

(63) \( \text{cof}_{12} = S^2(Np) + S(-Np \psi_\ell) + N_\psi g/Ve \)

(64) \( \text{cof}_{13} = S(g/Ve - Np) - (g/Ve) Nr \)

(65) \( \text{cof}_{14} = S^2 + S(-Nr - \psi_\ell) + Nr \psi_\ell + N_\psi \)

(66) \( \text{cof}_{21} = S^2(Lr) - S(Y_\psi Lr + L_\psi) \)

(67) \( \text{cof}_{22} = S^3 - S^2(Lp + \psi_\ell) + S(\psi_\ell L_\psi) - L_\psi g/Ve \)

(68) \( \text{cof}_{23} = -S^2 + SLp + Lr g/Ve \)

(69) \( \text{cof}_{24} = S(Lr) - Lr \psi_\ell - L_\psi \)

(70) \( \text{cof}_{31} = S^2L_\psi + S(LrN_\psi - L_\psi Nr) \)

(71) \( \text{cof}_{32} = S^2(N_\psi) + S(L_\psi Np - LpN_\psi) \)

(72) \( \text{cof}_{33} = S^3 + S^2(-Lp-Nr) + S(LpN_\psi - LrNp) \)

(73) \( \text{cof}_{34} = S(L_\psi) + LrN_\psi - L_\psi Nr \)

Substituting equations (37) through (73) into equation (36) and equating coefficient expressions for equal power of \( S \), we obtain

(74) \( \hat{N}_{13} = b_1/p_{ss} \)

(75) \( \hat{N}_{12} = (b_1(-Nr-\psi_\ell) + b_2Lr + b_3L_\psi)/p_{ss} \)

(76) \( \hat{N}_{11} = (b_1(Nr\psi_\ell + N_\psi) - b_2(\psi_\ell Lr + L_\psi) + b_3(LrN_\psi - L_\psi Nr))/p_{ss} \)

(77) \( \hat{D}_{14} = D_4 \)
(78) $D_{13} = D_3$
(79) $D_{12} = D_2$
(80) $D_{11} = D_1$
(81) $\hat{N}_{23} = b_3/\xi_{ss}$
(82) $\hat{N}_{22} = (-b_2 - (Lp + Nr)b_3)/\xi_{ss}$
(83) $\hat{N}_{21} = ((g/Ve - Np)b_1 + Lp b_2 + (LpNr - LrNp)b_3)/\xi_{ss}$
(84) $\hat{N}_{20} = ((-g/Ve) N_r b_1 + L_r (g/Ve) b_2)/\xi_{ss}$
(85) $\hat{N}_{34} = h_3/D^*_s$
(86) $\hat{N}_{33} = (b_1 G_{31} + b_2 G_{32} + b_3 G_{33} + h_3 D_4)/D^*_s$
(87) $\hat{N}_{32} = G_{31}(b_1(-N_r-Y_\xi) + b_2(Lr + b_3 L_\xi) + G_{32}(b_1 N_p - b_2(Lp + Y_\xi + b_3 N_\xi) + G_{33}(-b_2 - b_3(Lp + Nr)) + G_{34} b_1 + h_3 D_3)/D^*_s$
(88) $\hat{N}_{31} = G_{31}(-b_1(N_r Y_\xi + N_\xi) - b_2(Y_\xi L_r + L_\xi) + b_3(LrN_\xi - L_\xi Nr))$
\hspace{1cm} + $G_{32}(-b_1 N_p Y_\xi + b_2 L_p Y_\xi + b_3(L_p N_p - L_p N_\xi))$
\hspace{1cm} + $G_{33}(b_1(g/Ve - N_p) + b_2 L_p + b_3(L_p N_r - L_r N_p))$
\hspace{1cm} + $G_{34}(b_1(-N_r Y_\xi) + b_2 L_r + b_3 L_\xi) + h_3 D_2)/D^*_s$
(89) $\hat{N}_{30} = G_{32}(b_1 N_\xi g/Ve - b_2 L_\xi g/Ve) + G_{33}(-b_1(g/Ve) N_r + b_2 L_r g/Ve)$
\hspace{1cm} + $G_{34}(b_1(N_r Y_\xi + N_\xi) - b_2(L_r Y_\xi + L_\xi) + b_3(L_r N_\xi - L_\xi Nr))$
\hspace{1cm} + $h_3 D_1)/D^*_s$

We thus have sixteen equations ((74) through (89), inclusive) and ten unknowns:

(90) $(Lp, Lr, L_\xi, N_p, Nr, N_\xi, Y_\xi, b_1, b_2, b_3)$
Section 3 - Solution Methodology

In order to solve equations (74) through (89) for unknowns (90) the following cost function is defined:

\[
(91) \text{COST} = \sum_{k=1}^{16} \text{Cost}_k
\]

where

\[
(92) \begin{align*}
(i) \quad \text{Cost}_1 &= ((\text{RHS equation (74)}) - \text{Re(LHS equation (74)))}^2 \\
(ii) \quad \text{Cost}_2 &= ((\text{RHS equation (75)}) - \text{Re(LHS equation (75)))}^2 \\
(iii) \quad \text{Cost}_3 &= ((\text{RHS equation (76)}) - \text{Re(LHS equation (76)))}^2 \\
(xvi) \quad \text{Cost}_{16} &= ((\text{RHS equation (89)}) - \text{Re(LHS equation (89)))}^2
\end{align*}
\]

It was found that Im (LHS equation (74)), Im (LHS equation (75)), ..., Im (LHS equation (89)) were very close to zero (1x10^-11), thus they are neglected. Using a Pattern Search Minimization Technique, the set (90) which approximates \text{COST} = 0 will be the solution set if the model time-histories generated with these values closely match the input data time-histories.

The description of the Search Technique follows:

The essence of the pattern search is the conjecture that any set of adjustments to the independent variables of a cost function, which have
been successful during early experiments, will be worth trying again. This concept was first introduced by Hooke and Jeeves[5]. Thus the search starts cautiously with short exploratory excursions about the starting point; the magnitudes of the steps (the "progression" of the search) increase with repeated successes. Subsequent failures indicate that shorter steps are in order. If a change in the direction of the search pattern is required, the procedure starts again (from the point where a change of direction was needed) with a new pattern. In the vicinity of a peak or valley, the resolution of the search becomes finer, thus being able to discern any promising direction to follow.

The search thus starts with an arbitrary point \( \mathbf{P}_1(x_i) \) \( i=1, \ldots, 10 \) (the coordinates of which are initial guesses for (90)). A step of size \( \delta_{1i} \) is subsequently chosen for each independent variable \( x_i \) \( i=1, \ldots, 10 \). Letting \( \mathbf{\Delta}_1 \) be a vector of dimension 10 (number of independent variables) whose \( i \)th component is \( \delta_{1i} \) while the rest are zero, the values of \( \text{COST} \) (as defined in (91)) at \( \mathbf{P}_1 \) and \( \mathbf{P}_1 + \mathbf{\Delta}_1 \) are determined. \( \mathbf{P}_1 \) and \( \mathbf{\Delta}_1 \) define an "arrow" in 10th dimensional space, the latter being the "head" and the former the "tail". The motion of the arrow follows the progression of the search. If the value of \( \text{COST} \) for \( \mathbf{P}_1 + \mathbf{\Delta}_1 \) is better than that for \( \mathbf{P}_1 \), \( \mathbf{P}_1 + \mathbf{\Delta}_1 \) is called the temporary head \( \mathbf{t}_{1,1} \), where the double subscript shows that we are developing the first pattern and that we have already perturbed the first variable \( x_1 \). If \( \mathbf{P}_1 + \mathbf{\Delta}_1 \) is not as good a point as \( \mathbf{P}_1 \), in other words \( \text{COST} (\mathbf{P}_1) \leq \text{COST} (\mathbf{P}_1 + \mathbf{\Delta}_1) \), we try
instead $\text{COST} \left( \bar{P}_1 - \bar{\Delta}_1 \right)$. If this new point is better (i.e., $\text{COST} \left( \bar{P}_1 - \bar{\Delta}_1 \right) < \text{COST} \left( \bar{P}_1 \right)$), then $\bar{P}_1 - \bar{\Delta}_1$ becomes the new temporary head $\bar{t}_{1,1}$, otherwise $\bar{t}_{1,1} = \bar{P}_1$ (no improvement shown by either $\bar{P}_1 + \bar{\Delta}_1$ or $\bar{P}_1 - \bar{\Delta}_1$); that is, $\bar{P}_1$ becomes the new temporary head.

The next independent variable, $x_2$, can now be perturbed, but this time about the temporary head $\bar{t}_{1,1}$. Generally speaking, the $k^{th}$ temporary head $\bar{t}_{j,k}$ is obtained from the preceding one as follows:

$$\begin{align*}
\bar{t}_{j,k} &= \begin{cases} 
\bar{t}_{j,k-1} + \Delta_k & \text{if } \text{COST} \left( \bar{t}_{j,k-1} + \Delta_k \right) < \text{COST} \left( \bar{t}_{j,k-1} \right) \\
\bar{t}_{j,k-1} - \Delta_k & \text{if } \text{COST} \left( \bar{t}_{j,k-1} - \Delta_k \right) < \text{COST} \left( \bar{t}_{j,k-1} \right) \\
\bar{t}_{j,k-1} & \text{if } \text{COST} \left( \bar{t}_{j,k-1} \right) < \text{MIN} \left( \text{COST} \left( \bar{t}_{j,k-1} \pm \Delta_k \right) \right)
\end{cases}
\end{align*}$$

In (93) it was assumed that

(94) $\bar{t}_{1,0} = \bar{P}_1$

Furthermore,

(95) $\bar{t}_{1,10} = \bar{P}_2$

And $\bar{P}_2$ is the new base point from which new perturbations will be performed on the independent variables.

The original starting point, $\bar{P}_1$, and the new starting point (or temporary base), $\bar{P}_2$, together establish the first pattern of search.
Reasoning that if a similar set of perturbations was started at \( \tilde{P}_2 \), the results are likely to be the same; therefore the local excursions about \( \tilde{P}_2 \) are by-passed, and a new temporary head is established as follows:

\[
(96) \quad \tilde{t}_{2,0} = \tilde{P}_1 + 2(\tilde{P}_2 - \tilde{P}_1) = 2\tilde{P}_2 - \tilde{P}_1
\]

In other words, the length of the search vector between \( \tilde{P}_1 \) and \( \tilde{P}_2 \) is doubled, assuming the successful nature of the first set of perturbations. Local explorations are then carried out about \( \tilde{t}_{2,0} \) using (93) with \( j = 2 \). Local recognizance is completed when all the variables have been perturbed (\( k = 10 \)). The last temporary head, \( \tilde{t}_{2,10} \), establishes the new temporary head, \( \tilde{t}_{3,0} \), after extrapolating.

Recalling \( \tilde{P}_3 = \tilde{t}_{2,10} \):

\[
(97) \quad \tilde{t}_{3,0} = 2\tilde{P}_3
\]

From \( \tilde{t}_{3,0} \), \( x_1 \) is perturbed and assuming a successful perturbation, \( \tilde{t}_{3,1} \) is reached ((93) with \( j = 3 \)). \( x_2 \) is now ready to be perturbed.

Suppose that the perturbation of \( x_2 \) about \( \tilde{t}_{3,1} \) fails to produce any improvement over \( \tilde{t}_{3,1} \) (\( \tilde{t}_{3,1} \) is still a better point) then

\[
(98) \quad \tilde{P}_4 = \tilde{t}_{3,2} = \tilde{t}_{3,1}
\]

and the pattern will find a new direction, though still growing, furthermore,

\[
(99) \quad \tilde{t}_{4,0} = 2\tilde{P}_4 - \tilde{P}_3
\]

For the fourth pattern, imagine that none of the perturbations about the initial temporary head \( \tilde{t}_{4,0} \) improve the value of COST, namely

\[
\text{COST} (\tilde{t}_{4,0}) > \text{COST} (\tilde{P}_4)
\]

then
(100) \( \bar{P}_5 = \bar{t}_{4,2} = \bar{t}_{4,1} = \bar{t}_{4,0} \)
and the pattern will maintain its direction and length without any
growth, thus
\[
(101) \quad \bar{P}_5 - \bar{P}_4 = 2\bar{P}_4 - \bar{P}_3 - \bar{P}_4 = \bar{P}_4 - \bar{P}_3
\]
Now suppose that none of the temporary heads, \( \bar{t}_{5,0}, \bar{t}_{5,1} \) or \( \bar{t}_{5,2} \), are any
better than \( \bar{P}_5 \),
then
\[
(102) \quad \bar{P}_6 = \bar{P}_5
\]
The pattern is destroyed. Now \( \bar{P}_6 \) becomes the new initial temporary head,
\( \bar{t}_{6,0} \), about which perturbations are made. If these perturbations do not
yield a better point, the steps \( \bar{\Delta}_k \) are shortened and a new pattern is
begun from the last successful starting point.

Appendix B contains a flow chart of subroutine DIRECT, which uses the
Pattern Search Technique to find the solution set (90).

The solution set (90) is used to generate \( P_N, \xi_N \) and \( D_N \) time-histories
which are plotted against the initial input data time-histories for com-
parison. A close match serves to check the validity of the solution.

Examples are presented in the next section.
Section 4 - Solution Examples

Three different cases are considered in the following sequence:
(a) - A trivial case corresponding to a USAF F4-E where all the answers are known a priori: In this example, the eigenvalue search step is by-passed; the input (known) eigenvalues are used in arriving at equation (9). The rest of the procedure remains unchanged: the Search Technique described in Section 3 is used to solve for the unknowns (90). These values are compared to the corresponding known values for the real airplane.

This case is tried to insure no errors in algebra or procedure have been committed.

(b) - A design case in which the same input time-history data is used as in example (a): The eigenvalue search step is used, consequently a different set of eigenvalues is employed in arriving at equation (9). In contrast to example (a), all the eigenvalues in equation (9) are constrained to have negative real components.

(c) - A design case in which arbitrary time-histories chosen and used to develop a model as prescribed in Section 3: The input rollrate data exhibits an oscillatory trace, as compared to the monotonically increasing traces of examples (a) and (b).
Example (a)

This example corresponds to a USAF F4E with the following parameters.

- **Weight** = 17568.54 Kg
- **Altitude** = 13716 m
- **Airspeed** = 634.3 m/sec
- **p**ss = 4.888
- **q**ss = -0.0158
- **D**ss = -206.0
- **l** = 4.94 m
- **C**3 = -2.03 x 10^{-3} m^3/N sec^2 deg
- **q**co = 16758 N/m^2

Values for the unknowns (90) are selected from reference [6] for a USAF F4E with the above flight parameters.

\[
\begin{align*}
L_p &= -1.036 \\
L_r &= 0.2365 \\
L_q &= -0.4700 \\
N_p &= 0.005972 \\
N_r &= -0.2228 \\
N_\xi &= 8.772 \\
Y_\xi &= -0.1328 \\
b_1 &= 4.991 \\
b_2 &= 0.1854 \\
b_3 &= -0.0007482
\end{align*}
\]
Using (103), time-histories are generated which are used as the input data as depicted in Table 1.

<table>
<thead>
<tr>
<th>Time (sec)</th>
<th>$P_N$</th>
<th>$\Theta_N$</th>
<th>$D^*_N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>-0.007</td>
</tr>
<tr>
<td>0.5</td>
<td>0.402</td>
<td>1.117</td>
<td>0.523</td>
</tr>
<tr>
<td>1.0</td>
<td>0.650</td>
<td>2.170</td>
<td>0.002</td>
</tr>
<tr>
<td>1.5</td>
<td>0.796</td>
<td>1.216</td>
<td>-0.514</td>
</tr>
<tr>
<td>2.0</td>
<td>0.876</td>
<td>0.020</td>
<td>-0.153</td>
</tr>
<tr>
<td>2.5</td>
<td>0.924</td>
<td>0.492</td>
<td>0.345</td>
</tr>
<tr>
<td>3.0</td>
<td>0.958</td>
<td>1.451</td>
<td>0.127</td>
</tr>
<tr>
<td>3.5</td>
<td>0.981</td>
<td>1.100</td>
<td>-0.324</td>
</tr>
<tr>
<td>4.0</td>
<td>0.989</td>
<td>0.128</td>
<td>-0.214</td>
</tr>
<tr>
<td>4.5</td>
<td>0.993</td>
<td>0.157</td>
<td>0.184</td>
</tr>
<tr>
<td>5.0</td>
<td>0.999</td>
<td>0.873</td>
<td>0.158</td>
</tr>
</tbody>
</table>

Using (103), the eigenvalues corresponding to the real airplane are obtained by substituting into (34):

\[
\begin{align*}
\lambda_1 &= -1.042 \\
\lambda_2 &= 0.0017 \\
\lambda_3 &= -0.1754 + i 2.9622 \\
\lambda_4 &= -0.1754 - i 2.9622
\end{align*}
\]
Using (104), initial guesses are made for values of (90) and used in the Pattern Search Method to arrive at final values for the unknowns. Table 2 depicts comparative values for this example.

**Table 2:**
Comparative Values For Unknowns (90), Example (a)

<table>
<thead>
<tr>
<th>Initial Guess</th>
<th>Final Value (Model)</th>
<th>Actual F4E Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lp</td>
<td>-1.361</td>
<td>-1.036</td>
</tr>
<tr>
<td>Lr</td>
<td>.3196</td>
<td>.2306</td>
</tr>
<tr>
<td>Lq</td>
<td>-6.494</td>
<td>-4.426</td>
</tr>
<tr>
<td>Np</td>
<td>.003032</td>
<td>.005978</td>
</tr>
<tr>
<td>Nr</td>
<td>-.3411</td>
<td>-.2228</td>
</tr>
<tr>
<td>Nq</td>
<td>12.11</td>
<td>8.771</td>
</tr>
<tr>
<td>Yq</td>
<td>-.2034</td>
<td>-.1332</td>
</tr>
<tr>
<td>b1</td>
<td>5.289</td>
<td>4.996</td>
</tr>
<tr>
<td>b2</td>
<td>.4108</td>
<td>.1854</td>
</tr>
<tr>
<td>b3</td>
<td>-.001155</td>
<td>-.0007649</td>
</tr>
</tbody>
</table>

In example (a) the final value of COST was .000039. Referring to figures 1, 2 and 3, we have in each
- discrete input data from Table 1, denoted by the x's.
- Time-history generated using (104) in the form of equation (2) where the C_j1, \ldots C_j5 coefficients were obtained from the input data and (104), as described in Section 2. This time-history is superimposed on the input data. The corresponding dY_j(t)/dt time-history is plotted.
Time-history generated using the final values in Table 2. This time-history is also superimposed on the input data. The corresponding first time derivative time-history is also superimposed on the \( \dot{Y}_j(t)/dt \) time-history.
FIG. 1 ROLL RATE
FIG. 2 SIDESLIP
FIG. 3 LATERAL CRITERION
Example (b)

This example represents a design problem.

Given an aircraft with the following physical parameters (chosen to be those of a USAF F4E):

Weight = 17569.54 Kg
Altitude = 13716 m
Airspeed = 634.3 m/sec
\( p_{ss} = 4.888 \)
\( \phi_{ss} = -0.0158 \)
\( D*_{ss} = -206.0 \)
\( l = 4.94 \) m
\( C_3 = -2.03 \times 10^{-3} \) m\(^3\)/N sec\(^2\) deg
\( q_{co} = 16758 \) N/m\(^2\)

And the time history performance data tabulated in Table 1, find the corresponding values for (90).

Note that in this case the same time-history data is used as in Example (a), although the final eigenvalues are not used as input.

Following the procedure described in Section 3, initial eigenvalues to be used in equation (9) are developed.
Subsequently, initial guesses for the values of (90) are made, and the Pattern Search Technique is used to arrive at the final values for the unknowns (90). Table 3 summarizes results for this example by comparing eigenvalues. Table 4 compares values for the unknowns (90).

**TABLE 3:**

<table>
<thead>
<tr>
<th>Initial Value (Equation(9))</th>
<th>Final Value (Model)</th>
<th>F4E</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1$</td>
<td>-1.369</td>
<td>-1.358</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>-.0020</td>
<td>-.0034</td>
</tr>
<tr>
<td>$\lambda_3$</td>
<td>-.1732 - i 2.9581</td>
<td>-.1466 - i 2.9711</td>
</tr>
<tr>
<td>$\lambda_4$</td>
<td>-.1732 + i 2.9581</td>
<td>-.1466 + i 2.9711</td>
</tr>
</tbody>
</table>

**TABLE 4:**

<table>
<thead>
<tr>
<th>Initial Guess</th>
<th>Final Value (Model)</th>
<th>F4E</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_p$</td>
<td>-1.361</td>
<td>-1.358</td>
</tr>
<tr>
<td>$L_r$</td>
<td>.3196</td>
<td>-.2115</td>
</tr>
<tr>
<td>$L_q$</td>
<td>-6.494</td>
<td>-2.763</td>
</tr>
<tr>
<td>$N_p$</td>
<td>.003032</td>
<td>.006492</td>
</tr>
<tr>
<td>$N_r$</td>
<td>-.3411</td>
<td>-.2732</td>
</tr>
<tr>
<td>$N_q$</td>
<td>12.11</td>
<td>8.843</td>
</tr>
<tr>
<td>$Y_q$</td>
<td>-.2034</td>
<td>-.0227</td>
</tr>
<tr>
<td>$b_1$</td>
<td>5.289</td>
<td>5.816</td>
</tr>
<tr>
<td>$b_2$</td>
<td>.4108</td>
<td>.1857</td>
</tr>
<tr>
<td>$b_3$</td>
<td>-.001155</td>
<td>-.000869</td>
</tr>
</tbody>
</table>

-31-
In Example (b), the final value of COST was .0085. Referring to Figures 4, 5 and 6, we have:

- Discrete input data from Table 1, denoted by the x's
- Time-history generated using initial values from Table 3, in the form of equation (2). This time-history is superimposed on the input data (denoted by 1 in Figure 4). The corresponding $\hat{Y}_j(t)/dt$ time-history is plotted.
- Time-history generated using the final values in Table 4. This time-history is also superimposed on the input data (and is denoted by 2 in Figure 4). The corresponding first time derivative time-history is also superimposed on the $\hat{Y}_j(t)/dt$ time-history.
FIG. 4 ROLL RATE
FIG. 5 SIDESLIP

TIME SECONDS

DBETA/DT

0.000
-5.000

7.000

BETAN

0.000
-5.000

TIME SECONDS

5.000
FIG. 6 LATERAL CRITERION
Example (C)
This example constitutes another design problem.

Given an aircraft with the following physical parameters (chosen to be those of a USAF F4E):

- Weight = 17569.54 Kg
- Altitude = 13716 m
- Airspeed = 634.3 m/sec
- Pss = 4.888
- \( \theta_{ss} \) = -0.0158
- \( D_{ss} \) = -206.0
- \( l \) = 4.94 m
- \( C_3 \) = \(-203 \times 10^{-3}\) m\(^3\)/N sec\(^2\) deg
- \( q_{CO} \) = 16758 N/m\(^2\)

And the time history performance data tabulated in Table 5, find the corresponding values for (90).

The time-history data in Table 5 was selected from arbitrarily "sketched" curves within the time-history envelopes.
<table>
<thead>
<tr>
<th>Time (sec)</th>
<th>$P_N$</th>
<th>$\xi_N$</th>
<th>$D^*_N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>0.5</td>
<td>0.368</td>
<td>0.800</td>
<td>0.500</td>
</tr>
<tr>
<td>1.0</td>
<td>0.700</td>
<td>2.200</td>
<td>0.100</td>
</tr>
<tr>
<td>1.5</td>
<td>0.750</td>
<td>1.000</td>
<td>-0.100</td>
</tr>
<tr>
<td>2.0</td>
<td>0.770</td>
<td>0.010</td>
<td>-0.200</td>
</tr>
<tr>
<td>2.5</td>
<td>0.800</td>
<td>0.300</td>
<td>0.200</td>
</tr>
<tr>
<td>3.0</td>
<td>0.900</td>
<td>0.900</td>
<td>0.170</td>
</tr>
<tr>
<td>3.5</td>
<td>0.800</td>
<td>1.100</td>
<td>-0.200</td>
</tr>
<tr>
<td>4.0</td>
<td>0.700</td>
<td>0.010</td>
<td>-0.100</td>
</tr>
<tr>
<td>4.5</td>
<td>0.800</td>
<td>0.200</td>
<td>0.200</td>
</tr>
<tr>
<td>5.0</td>
<td>0.900</td>
<td>0.900</td>
<td>0.300</td>
</tr>
</tbody>
</table>
Following the procedure described in Section 3, initial eigenvalues to be used in equation (9) are developed.

Subsequently, initial guesses for the values of (90) are made, and the Pattern Search Technique is used to arrive at the final values for the unknowns (90). Table 6 summarizes results for this example by comparing eigenvalues. Table 7 compares values for the unknowns (90).

**TABLE 6:**

Eigenvalues For Example (c)

<table>
<thead>
<tr>
<th>Initial Value (Equation (9))</th>
<th>Final Value (Model)</th>
<th>F4E</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1$</td>
<td>-1.369</td>
<td>-1.4271</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>-0.0020</td>
<td>0.0058</td>
</tr>
<tr>
<td>$\lambda_3$</td>
<td>-.1732 - i 2.9581</td>
<td>-.30634 - i 2.8897</td>
</tr>
<tr>
<td>$\lambda_4$</td>
<td>-.1732 + i 2.9581</td>
<td>-.30634 + i 2.8897</td>
</tr>
</tbody>
</table>
In example (c), the final value of COST was 2.004. Referring to figures 7, 8 and 9, we have:

- Discrete input data from Table 5, denoted by the x's.
- Time-history generated in the form of equation (2). This time-history is superimposed on the input data (denoted by ① in Figure 7). The corresponding $\hat{Y}_j(t)/dt$ time-history is plotted.
- Time-history generated using the final values in Table 7. This time-history is also superimposed on the input data (and is denoted by ② in Figure 7). The corresponding first time derivative time-history is also superimposed on the $\hat{Y}_j(t)/dt$ time-history.

<table>
<thead>
<tr>
<th>Initial Guess</th>
<th>Final Value (Model)</th>
<th>F4E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lp</td>
<td>-1.361</td>
<td>-1.430</td>
</tr>
<tr>
<td>Lp</td>
<td>.3196</td>
<td>.4931</td>
</tr>
<tr>
<td>Lq</td>
<td>-6.494</td>
<td>3.616</td>
</tr>
<tr>
<td>Np</td>
<td>.003032</td>
<td>-.004325</td>
</tr>
<tr>
<td>Nr</td>
<td>-.3411</td>
<td>-.1176</td>
</tr>
<tr>
<td>Nq</td>
<td>12.11</td>
<td>8.38</td>
</tr>
<tr>
<td>Yq</td>
<td>-.2034</td>
<td>-.4867</td>
</tr>
<tr>
<td>b1</td>
<td>5.289</td>
<td>5.788</td>
</tr>
<tr>
<td>b2</td>
<td>.4108</td>
<td>.1857</td>
</tr>
<tr>
<td>b3</td>
<td>-.001155</td>
<td>-.000869</td>
</tr>
</tbody>
</table>

In example (c), the final value of COST was 2.004. Referring to figures 7, 8 and 9, we have:

- Discrete input data from Table 5, denoted by the x's.
- Time-history generated in the form of equation (2). This time-history is superimposed on the input data (denoted by ① in Figure 7). The corresponding $\hat{Y}_j(t)/dt$ time-history is plotted.
- Time-history generated using the final values in Table 7. This time-history is also superimposed on the input data (and is denoted by ② in Figure 7). The corresponding first time derivative time-history is also superimposed on the $\hat{Y}_j(t)/dt$ time-history.
FIG. 7 ROLL RATE
FIG. 8 SIDESLIP
FIG. 9 LATERAL CRITERION
6.0 Analysis and Conclusions

In considering Example (a) it is noted that using the eigenvalues corresponding to the solution yields the values of (90) which are very close to those of the aircraft, as seen in Table 2. Note that in this particular example the real aircraft has an eigenvalue with a positive real component.

Attempts made to completely eliminate the requirement for "approximately specifying" the initial input eigenvalues, as described in Section 3, proved unsuccessful. The eigenvalues obtained by numerical fitting of the input time-history data proved to have unrealistic magnitudes.

As a result a range of "typical" eigenvalue magnitudes was obtained from the F4E data in reference [6]. From this data, one of the eigenvalues is typically a very small positive real number. A negative real number range of the same relative magnitude was selected in this case to be used for input eigenvalue selection in the model design algorithm.

With the input eigenvalues constrained to be within a specified domain, initial guess values within this domain are selected in example (b). As prescribed in Section 3, these values are used to arrived at the \( C_j \)'s in equation (2) by a least-squares error fit of equation (2) to the input time-history data, yielding equation (7). At this point, \( \bar{C}_I \), \( \bar{C}_{DI} \) and \( \bar{X}_I \) are further adjusted in the least-squares error sense using the Pattern Search Technique discussed in Chapter 4.
The initial values of Table 3, which correspond to $\tilde{\lambda}$ in equation 9, remained unchanged from the corresponding $\tilde{\lambda}_I$.

The fact that $\tilde{\lambda}_I = \tilde{\lambda}$ was not only observed in Example (b) but also in Example (c).

This result is, however, not surprising as the sensitivity of eigenvalues to time-history deviations is the reason for having input eigenvalues approximately specified. The designer's initial eigenvalue guesses will thus remain as those to be used in equation (9). It is important to note that $\tilde{\lambda}$ in equation (9) does not represent the final eigenvalues of the model. The model's eigenvalues will be different because the value of COST (equation (91)) will not be zero at the end of the pattern search algorithm.

Although in solving for the unknowns (90) the eigenvalues are not perturbed as are the unknown variables, any final set (90) implies a set of eigenvalues different from those in equation (9) because of the approximate nature of the search algorithm.

The overall accuracy of the final values of (90) in Example (b), Table 4, is best observed in Figures 4, 5 and 6 where it is seen how well the model time-histories match the input data.
In examples (a) & (b), it was seen that the roll rate time-histories are monotonically increasing for these particular F4E flight conditions. In Example c, the roll rate input data is (arbitrarily) chosen to be oscillatory.

Table 6 contains the initial guesses and final model eigenvalues for Example c. Note that $\lambda_2$ is positive and real, like the F4E.

In Figure 7, it is seen that the roll rate time-history of the model (curve 2) is monotonically increasing. This model time-history compares favorably to those in Example (a) and (b), the former corresponding to an F4E. In example (c), the model time-history does not track the oscillating input data exactly; the model time-history "smooths" the input data.

It is important to note that in all those examples the performance envelopes do not play a roll in arriving at the model; they are only provided as a reference or guideline in selecting input time-history data. The actual model is referenced entirely to the input data.

Flexibility in eigenvalue selection is provided for as far as equation (9), the final form of the input data. As previously stated, beyond that point the final eigenvalues will vary inasmuch as the final solu-
tion form of (90) in only a "best solution". In all the cases considered, the accuracy of $\lambda$ in equation (9) provided well-behaved and reasonable models.

Further flexibility of initial eigenvalue selection is provided by adjusting the values of $\lambda_1$ in equation (7) and repeating the search procedure leading to equation (9) and subsequently the final values of (90). This is an option not found necessary in this work which would add to computer time.
Conclusions:

The techniques developed in this work provide for the synthesis of a fourth order open-loop aircraft model from arbitrary time-history input data. Lateral handling criteria time-history envelopes are used as guidelines in data selection.

The resulting models, in all the instances considered, are well-behaved and compare favorably in performance with typical aircraft.

The basic principle, that of matching matrix plant equation parameters to time-history data, depends for effectiveness on the numerical solution of non-linear algebraic equations. Consequently, the efficiency and accuracy of the numerical search algorithm is of paramount importance. The Pattern Search Method utilized in this work has proven successful in arriving at the solution sought.
APPENDIX A

THE EQUATIONS OF MOTION

A-1.0) Using the notation utilized in reference [7], the following frames of reference are defined:

A-1.1) Earth-Surface Reference Frame, \( F_E \) (O\(E_xE_yE_z \) axes) see Figure 1. In this frame of reference, \( O_{Ez} \) points towards the center of the earth, \( O_{E}\times E_y \) is the local horizontal plane, \( O_{E}\times E_x \) points north and \( O_{E}\times E_y \) points east. Particular \( x_E \) and \( y_E \) coordinates are known as latitude (\( \lambda \)) and longitude (\( \mu \)) when \( O_E \) lays on the equator and the Greenwich Meridian.

A-1.2) Earth-Center Reference Frame, \( F_{EC} \) (O\(EC_xEC_yEC_z \) axes) see Figure 1. In this frame \( O_{EC} \) is the center of the Earth, \( O_{EC}\times EC_y \) points north along the axes of rotation of the Earth, and \( O_{EC}\times EC_y \) is in the equatorial plane.

A-1.3) Vehicle-Carried Vertical Frame, \( F_V \) (O\(VyVxVz \) axes)

In this frame \( O_V \) is usually attached at the vehicle's center of mass \( C \), \( O_{Vz} \) is directed along the gravity vector \( \ddot{g} \), \( O_{Vx} \) points to the north and \( O_{Vy} \) to the east.

If the origins of \( F_E \) and \( F_V \) are near enough, the curvature of the Earth can be neglected and both frames can be considered to be parallel. When this criterion does not apply, both
FIGURE 1
EARTH AXES. $(\lambda, \mu) =$ LATITUDE, LONGITUDE
frames can be made parallel by a rotation \(-\Delta \lambda\) around \(O_Ey_E\) and a rotation \(\Delta \mu\) around \(O_Ez_E\) where
\[
\Delta \lambda = \lambda - \lambda_E \\
\Delta \mu = \mu - \mu_E
\]
with
\[
(\lambda, \mu) = \text{latitude and longitude of } O_V \\
(\lambda_E, \mu_E) = \text{latitude and longitude of } O_E
\]
At this point, the reader is reminded of the fact that Newton's second law, namely
\[
\sum F = m\ddot{a}
\]
applies when the system of axes with respect to which \(\ddot{a}\) is determined must have a constant orientation with respect to the stars and their origin must either be attached to the center of mass of the solar system or move with a constant velocity with respect to it; herein the definition of an inertial frame of reference. However, in most flight dynamics problems, a frame of reference attached to the Earth can be assumed to be inertial without introducing appreciable error.

**A-1.4) Atmosphere-Fixed Reference Frame, \(F_A\) (\(O_Ax_Ay_Az_A\) axes)**

This frame is centered in the local atmosphere near the flight vehicle. The atmosphere can also be in relative motion w.r.t. \(F_E\), with a velocity \(\mathbf{W}\) (constant on a local basis), therefore, given the velocity of the flight vehicle w.r.t. \(F_A\) to be \(\mathbf{V}\), the velocity of the flight vehicle w.r.t. \(F_A\) to be \(\mathbf{V}\), the velocity of the flight vehicle w.r.t. \(F_E\) becomes

i) \(\mathbf{V}^E = \mathbf{V} + \mathbf{W}\)
A-1.5) **Wind Axes, $F_w$ (O$x_y_z_w$ axes) See Figure 2**

The origin of this frame is fixed at the flight vehicle's mass center, $C$, and $O_{w_x_w}$ is directed along the velocity vector $\vec{V}$ of the flight vehicle relative to the atmosphere. $O_{w_z_w}$ lays in the plane of symmetry of the vehicle, if one exists, otherwise is chosen to be along the gravity vector $\vec{g}$. If the atmosphere were at rest w.r.t. $F_E$ ($\vec{W} = 0$), $O_w$ would trace the trajectory of the flight vehicle w.r.t. $F_E$ and $O_{w_x_w}$ would always be tangent to it. The angular velocity of $F_w$ is $\vec{W}^w$, relative to an inertial frame of reference.

A-1.6) **Body Axes, $F_B$ (O$xyz$ axes) See Figure 3**

This frame is centered at the mass center $C$, with the plane $C_xz$ being a plane of symmetry and $z$ being downward. The angular velocity of $F_B$, relative to an inertial frame of reference is $\vec{\omega}$ and has components $p$ (roll rate), $q$ (pitch rate) and $r$ (yaw rate) depicted in Figure 3. The velocity of the flight vehicle relative to $F_B$ is $\vec{V}_B$ and has components $u$, $v$ and $w$ depicted also in Figure 3.
Figure 2
(a) Plane of symmetry \( C_{xz} \)

(b) Plane \( C_{xw} y_{w} \)
FIGURE 2
c & d
$L = \text{ROLLING MOMENT}$

$M = \text{PITCHING MOMENT}$

$N = \text{YAWING MOMENT}$

$p = \text{RATE OF ROLL}$

$q = \text{RATE OF PITCH}$

$r = \text{RATE OF YAW}$

$[X, Y, Z] = \text{COMPONENTS OF RESULTANT AERODYNAMIC FORCE}$

$[u, v, w] = \text{COMPONENTS OF VELOCITY OF C RELATIVE TO ATMOSPHERE}$

**FIGURE 3**

**BODY AXES**
It is often necessary to know the orientation of one particular reference frame with respect to another. Of special interest are the transformations which rotate the vehicle-carried vertical frame, $F_v$ into coincidence with either the body axes, $F_B$ or the wind axes, $F_W$. These transformations involve consecutive rotations about the axes, $z_v$, $y_v$ and $x_v$, in that order that carry $F_v$ into coincidence with $F_B$ or $F_W$. Referring to Figures 4 and 5, we have:

i) A rotation $\psi$ about $0_vz_v$, carrying the axes $0_vx_vy_vz_v$ to $0_vx_2y_2z_2$ ($0_vz_2$ being, of course $0_vz_v$) defines the azimuth angle,

ii) A rotation $\Theta$ about $0_vy_v$, carrying the axes $0_vz_2y_2z_2$ to $0_vx_3y_3z_3$, where $\Theta$ is the elevation angle. $0_vy_2$ and $0_vy_3$ coincide.

iii) A rotation $\phi$ about $0_vx_3$ carrying the axes $0_vx_3y_3z_3$ to their final position: the axes $0xyz$ (or $0_wx_wy_wz_w$), defines the bank angle, $\phi$. $0_vx_3$ and $0_x$ (or $0_xw$) coincide.

If the final axes are $0_wx_wy_wz_w$, the rotation becomes $\psi_w$, $\Theta_w$, $\phi_w$, respectively.

The ranges of the azimuth angle $\psi$, the elevation angle $\Theta$ and the bank angle $\phi$ are defined as follows (same definition applies to ($\psi_w$, $\Theta_w$ and $\phi_w$):
FIGURE 4
THE EULER ANGLES
\[
\begin{align*}
-\pi & \leq \psi \leq \pi \\
-\frac{\pi}{2} & \leq \theta \leq \frac{\pi}{2} \\
-\pi & \leq \phi \leq \pi
\end{align*}
\]

or

\[
0 \leq \psi \leq 2\pi \\
-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \\
0 \leq \phi \leq 2\pi
\]

\((\psi, \theta, \phi)\) or \((\psi_w, \theta_w, \phi_w)\) are referred to as the Euler angles in the literature.

A-2.1) Just like the Euler angles previously defined carry \(F_V\) to \(F_B\) or \(F_W\), it is also important to know the transformation which carries \(F_B\) to \(F_W\). Referring again to Figure 2, we have

1) the angle of attach
\[
\alpha_x = \tan^{-1} \left(\frac{w}{u}\right) \quad -\pi \leq \alpha_x \leq \pi
\]

2) the sideship angle
\[
\theta = \sin^{-1} \left(\frac{v}{|\vec{v}|}\right) \quad -\pi \leq \theta \leq \pi
\]

Recall that \(\vec{v}\) has the components \((u, v, w)\). Expressing the components of \(V\) in terms of \(\alpha\) and \(\theta\), we have:

3) \(u = |\vec{v}| \cos \theta \cos \alpha_x\)
\(v = |\vec{v}| \sin \theta\)
\(w = |\vec{v}| \cos \theta \sin \alpha_x\)

Therefore, to carry \(F_W\) into \(F_B\), the rotation \((-\theta, \alpha_x, 0)\) have to be effectuated.
In order to transform a vector $\vec{R}$ from being referenced w.r.t. one reference frame $F_a$ into reference w.r.t. another reference frame, $F_b$, the matrix containing the nine direction cosines $L_{ba}$ is utilized:

1) $\vec{R}_b = L_{ba} \vec{R}_a$
2) $\vec{R}_a = L_{ba}^{-1} \vec{R}_b$

Being that $\vec{R}_a$ and $\vec{R}_b$ are the same vector, it follows that

1iii) $|\vec{R}_a|^2 = |\vec{R}_b|^2$
1iv) $|\vec{R}_b|^2 = \vec{R}_b^T \vec{R}_b$

Using A-2.2-i

1v) $|\vec{R}_b|^2 = \vec{R}_b^T \vec{R}_b = \vec{R}_a^T L_{ba}^T L_{ba} \vec{R}_a = \vec{R}_a^T \vec{R}_a = |\vec{R}_a|^2$

Therefore

1vi) $L_{ba}^T L_{ba} = I$, the orthogonality condition and
1vii) $L_{ba}^T = L_{ba}^{-1} = L_{ab}$

In light of the present discussion, we will now derive the transformation that carries $F_V$ into $F_B$ and $F_B$ in $F_w$, respectively.

Referring to Figure 5-a, we have

- $z_2 = z_v$
- $y_2 = -x_v \sin \psi + y_v \cos \psi$
- $x_2 = x_v \cos \psi + y_2 \sin \psi$
a) ROTATION ABOUT $O_v z_v$

b) ROTATION ABOUT $O_v x_2$

c) ROTATION ABOUT $O_v x_3$

FIGURE 5
$F_v$ TO $F_B$ TRANSFORMATION
or in matrix form
\[
\begin{bmatrix}
\cos \psi & \sin \psi & 0 \\
-sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_v \\
y_v \\
z_v
\end{bmatrix}
=
\begin{bmatrix}
x_2 \\
y_2 \\
z_2
\end{bmatrix}
\]

Referring to Figure 5-b

\[z_3 = x_2 \sin \theta + z_2 \cos \theta\]
\[y_3 = y_2\]
\[x_3 = x_2 \cos \theta - z_2 \sin \theta\]

or in matrix form
\[
\begin{bmatrix}
\cos \theta & 0 & -\sin \theta \\
0 & 1 & 0 \\
\sin \theta & 0 & \cos \theta
\end{bmatrix}
\begin{bmatrix}
x_2 \\
y_2 \\
z_2
\end{bmatrix}
=
\begin{bmatrix}
x_3 \\
y_3 \\
z_3
\end{bmatrix}
\]

Referring to Figure 5-C

\[z = -y_3 \sin \phi + z_3 \cos \phi\]
\[y = y_3 \cos \phi + z_3 \sin \phi\]
\[x = x_3\]

or in matrix form
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \phi & \sin \phi \\
0 & -\sin \phi & \cos \phi
\end{bmatrix}
\begin{bmatrix}
x_3 \\
y_3 \\
z_3
\end{bmatrix}
=
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\]
combining viii, ix and x, we have

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta & \sin \theta \\
0 & -\sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
\cos \psi & 0 \\
0 & \cos \psi & \sin \psi \\
-\sin \psi & 0 & \cos \psi
\end{bmatrix}
\begin{bmatrix}
x, \\
y, \\
z,
\end{bmatrix}
\begin{bmatrix}
x' \\
y' \\
z'
\end{bmatrix}
\]

expanding (xi), we have

\[
\begin{align*}
x_{\text{iv}} & = x \\
y_{\text{iv}} & = y \\
z_{\text{iv}} & = z
\end{align*}
\]

where

\[
\begin{bmatrix}
\cos \theta & \cos \psi & \cos \theta \sin \psi & -\sin \theta \\
\sin \theta & \sin \psi & \sin \theta \sin \psi & \sin \theta \cos \psi \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Using \( \phi_w, \Theta_w \) and \( \Psi_w \) we have \( L_{wv} \).

Now we seek to derive the transformation which carries \( F_B \) into \( F_W, L_{wB} \). Recalling that \( L_{wB} = L_{wB}^{-1} = L_{wB}^T \), we can now find \( L_{wB} \) by starting with \( F_W \) and making rotations \((-\theta, \xi, \theta, 0)\) successively.
Refering to Figures 2-b and 6-a, we have

\[ x_2 = x_w \cos(-\beta) + y_w \sin(-\beta) = x_w \cos(\beta) - y_w \sin(\beta) \]

\[ y_2 = -x_w \sin(-\beta) + y_w \cos(-\beta) = x_w \sin(\beta) - y_w \cos(\beta) \]

\[ z_2 = z_w \]

or in matrix form

\[
\begin{bmatrix}
\cos \beta & -\sin \beta & 0 \\
\sin \beta & \cos \beta & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_w \\
y_w \\
z_w
\end{bmatrix}
= 
\begin{bmatrix}
x_2 \\
y_2 \\
z_2
\end{bmatrix}
\]

Referring to Figures 2-a and 6-b, we have

\[ x = x_2 \cos \alpha_x - z_2 \sin \alpha_x \]

\[ y = y_2 \]

\[ z = x_2 \sin \alpha_x + z_2 \cos \alpha_x \]

or in matrix form

\[
\begin{bmatrix}
\cos \alpha_x & 0 & -\sin \alpha_x \\
0 & 1 & 0 \\
\sin \alpha_x & 0 & \cos \alpha_x
\end{bmatrix}
\begin{bmatrix}
x_2 \\
y_2 \\
z_2
\end{bmatrix}
= 
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\]
FIGURE 6 a) ROTATION ABOUT $O_w z_w$

FIGURE 6 b) ROTATION ABOUT $O_w y$

FIGURE 6
Combining (xiv) and (xv), we have
\[
\begin{bmatrix}
\cos \alpha_x & 0 & -\sin \alpha_x \\
0 & 1 & 0 \\
\sin \alpha_x & 0 & \cos \alpha_x
\end{bmatrix}
\begin{bmatrix}
\cos \beta & -\sin \beta & 0 \\
\sin \beta & \cos \beta & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_w \\
y_w \\
z_w
\end{bmatrix}
=
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\]

Expanding (xvi) we obtain
\[
\begin{bmatrix}
x_w \\
y_w \\
z_w
\end{bmatrix}
=
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\]

where
\[
\begin{bmatrix}
\cos \alpha_x & \cos \beta & -\cos \alpha_x \sin \beta & -\sin \alpha_x \\
\sin \beta & \cos \beta & 0 \\
\sin \alpha_x & \cos \beta & -\sin \alpha_x \sin \beta & \cos \alpha_x
\end{bmatrix}
\]

therefore
\[
\begin{bmatrix}
x_w \\
y_w \\
z_w
\end{bmatrix}
=
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\]

where
\[
\begin{bmatrix}
\cos \alpha_x & \cos \beta & \sin \beta & \sin \alpha_x \cos \beta \\
-\cos \alpha_x \sin \beta & \cos \beta & -\sin \alpha_x \sin \beta \\
-\sin \alpha_x & 0 & \cos \alpha_x
\end{bmatrix}
\]

We are now able to transform between $F_V$, $F_B$ and $F_W$. 

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At this point, we are ready to begin to develop the equations which relate the interaction of forces in the flight vehicle (external forces, that is, given Newton's third law), assuming a rigid vehicle. We have, in the wind axes

1) \( \bar{f}_w = \bar{A}_w + m \bar{g}_w = \bar{a}_{cw} \)

where

\[
\bar{A}_w = \begin{bmatrix} D \\ C \\ L \end{bmatrix} + \begin{bmatrix} T_{xw} \\ T_{yw} \\ T_{zw} \end{bmatrix}
\]

with \( D = \text{drag} \)
\( C = \text{crosswind} \)
\( L = \text{lift} \)

and

\( T_{xw} \)
\( \bar{T}_w = T_{yw} = \text{thrust vector} \)
\( T_{zw} \)

\( \bar{g}_w = \bar{W}_w \bar{g}_v, \bar{g}_v^T [0, 0, g] \)

using A-2.2-xiii with \( \phi_w, \theta_w \) and \( \psi_w \)

\[
g_w = mg \\
\begin{bmatrix} -\sin \theta_w \\ \cos \theta_w \sin \phi_w \\ \cos \theta_w \cos \phi_w \end{bmatrix}
\]

\[
iii) m \bar{g}_w = mg \\
\begin{bmatrix} -\sin \theta_w \\ \cos \theta_w \sin \phi_w \\ \cos \theta_w \cos \phi_w \end{bmatrix}
\]
Combining ii and iii, we have

\[ \mathbf{v}) \quad - \left[ \begin{array}{c} \mathbf{D} \\ \mathbf{C} \\ \mathbf{L} \end{array} \right] + \mathbf{T}_{xw} + mg \left[ \begin{array}{cc} -\sin \theta_w & \sin \phi_w \\ \cos \theta_w & \sin \phi_w \\ \cos \theta_w & \cos \phi_w \end{array} \right] = \mathbf{m} \ddot{\mathbf{a}}_{cw} \]

Now, \( \ddot{\mathbf{a}}_{cw} \) is sought. The absolute acceleration of a point \( P \) moving in space (see Figure 7) is found to be

\[ \mathbf{v}) \quad \ddot{\mathbf{a}}_P = \ddot{\mathbf{a}}_A + \ddot{\mathbf{r}}_{P/A} + (\dddot{\mathbf{r}}_{P/A})_{Axyz} + \ddot{\omega} \ddot{\mathbf{r}}_{P/A} + 2 \ddot{\omega} \dot{\mathbf{r}}_{P/A} \]

where:
- \( \ddot{\mathbf{a}}_A \) = acceleration of the origin of the moving frame \( Axyz \)
- \( \ddot{\mathbf{r}}_{P/A} \) = acceleration due to the rotational acceleration of \( Axyz \)
- \( \dddot{\mathbf{r}}_{P/A} \) = acceleration of \( P \) relative to the moving frame \( Axyz \)
- \( \ddot{\omega} \ddot{\mathbf{r}}_{P/A} \) = centripedal acceleration
- \( 2 \ddot{\omega} \dot{\mathbf{r}}_{P/A} \) = coriolis acceleration

Assuming that the earth's axis is fixed in inertial space, \( \ddot{\omega} = 0 \), \( \ddot{\mathbf{a}}_A \) is therefore the centripedal acceleration of the axis \( Axyz \) due to the rotation of the earth, which is approximately: \( .0335 \text{ m/sec}^2 \). Therefore it is negligible compared to \( g = 9.81 \text{ m/sec}^2 \) at the equator and zero at the poles. The same argument holds true for the centripedal acceleration term, \( \ddot{\omega} \ddot{\mathbf{r}}_{P/A} \), namely, it is usually negligible compared to \( g \). Therefore, (v) becomes
vii) \( \ddot{a}_{CE} = \ddot{v}^E + 2\dot{\omega}^E \ddot{v}^E \)

\[ \ddot{a}_{CW} = l_{WE} \ddot{a}_{CE} = \ddot{v}^E + (\dot{\omega}_W^W - \dot{\omega}^E) \ddot{v}^E + 2l_{WE} \dot{\omega}^E \ddot{v}^E \]

\[ = \ddot{v}^E + (\dot{\omega}_W^W - \dot{\omega}^E) \ddot{v}^E + 2\dot{\omega}_W^W \ddot{v}^E \]

viii) \[ = \ddot{v}^E + (\dot{\omega}_W^W + \dot{\omega}^E) \ddot{v}^E \]

We have

ix) \( \ddot{v}^E_w = \begin{bmatrix} V \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} w_{xw} \\ w_{yw} \\ w_{zw} \end{bmatrix} \) from (A 1.4-i)

x) \( \dot{\omega}_w^W = \begin{bmatrix} p_w \\ q_w \\ r_w \end{bmatrix} \)

xi) \( \dot{\omega}_w^E = \begin{bmatrix} p_w^E \\ q_w^E \\ r_w^E \end{bmatrix} \)
Assuming that the atmosphere is at rest w.r.t. the Earth, substituting ix, x and xi into viii, we have

\[
x_{ii}) \quad \ddot{a}_{cw} = \begin{bmatrix} V \\ V (r_w + r_w^E) \\ -V (q_w + q_w^E) \end{bmatrix}
\]

Therefore, combining xii and iv,

(a) \( T_{xw} - D - mg \sin \theta_w = m\dot{V} \)

xiii) (b) \( T_{yw} - C + mg \cos \theta_w \sin \phi_w = mV (r_w + r_w^E) \)

(c) \( T_{zw} - L + mg \cos \theta_w \cos \phi_w = mV (q_w + q_w^E) \)

Refering to Figure 8, we have

\[
\begin{align*}
\vec{T}_w &= \vec{I}_{WB} \vec{T}_B = \vec{I}_{WB} T \begin{bmatrix} \cos \alpha_T \\ \sin \alpha_T \end{bmatrix} \\
\end{align*}
\]

using A-2.2-xx, we have

\[
x_{iv}) \quad \vec{T}_w = \begin{bmatrix} T_{xw} \\ T_{yw} \\ T_{zw} \end{bmatrix} = \begin{bmatrix} T(\cos \alpha_x \cos \phi \cos \alpha_t - \sin \alpha_x \cos \phi \sin \alpha_t) \\ T(-\cos \alpha_x \sin \phi \cos \alpha_t + \sin \alpha_x \sin \phi \sin \alpha_t) \\ T(-\sin \alpha_x \cos \alpha_t - \cos \alpha_x \sin \alpha_t) \end{bmatrix}
\]

In the body axes we have

\[
x_{v}) \quad \vec{\dot{r}}_B = \vec{\dot{a}}_B + \vec{mg}_B = m \ddot{a}_{cB}
\]
where
\[
\vec{A}_B = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}
\]

\[\text{xvi)} \quad \vec{g}_B = g \begin{bmatrix} -\sin \theta \\ \cos \theta \sin \phi \\ \cos \theta \cos \phi \end{bmatrix}\]

using A-2.2-xiii

and

\[\text{xvii)} \quad \vec{a}_{CB} = \vec{v}_B^E + (\vec{\omega}_B + \vec{\omega}_B^E) \vec{v}_B^E\]

with

\[\text{xviii)} \quad \vec{v}_B^E = \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix}\]

\[\text{xix)} \quad \vec{\omega}_B = \begin{bmatrix} p \\ q \\ r \end{bmatrix}\]

\[\text{xx)} \quad \vec{\omega}_B^E = \begin{bmatrix} p \\ q \\ r \end{bmatrix}\]
Assuming that the atmosphere is at rest w.r.t. Earth, and substituting xviii, xix and xx into xvii, we have

\[
\vec{a}_{cB} = \begin{bmatrix}
    u + (q + q^E) w - (r + r^E_B) v \\
    v + (r + r^E_B) u - (p + p^E_B) w \\
    w + (p + p^E_B) v - (q + q^E_B) u
\end{bmatrix}
\]

xxi) a) \( X = m(\ddot{u} + (q + q^E_B)w - (r + r^E_B)v) \)

xxii) b) \( Y = mg \sin \Theta \sin \phi = m(\ddot{v} + (r + r^E_B)u - (p + p^E_B)w) \)

\( c) Z = mg \cos \Theta \cos \phi \theta = m(\ddot{w} + (p + p^E_B)v - (q + q^E_B)u) \)

In xxii and xiii, the terms \( p^E_B, q^E_B, r^E_B, q^E_W, r^E_W \) respectively vanish when the rotation of the earth can be neglected.
In the previous section, we started with the fundamental equation,
\[ \ddot{f}_c = m \ddot{a}_c \]
and the force equations of motion A-3.0-xxii and A-3.0-viii for body and wind axes, respectively. In this section, we will start with the fundamental equation

\[ \tilde{G}_I = \dot{H}_I \]

where

\( \tilde{G}_I \) = moment about C in \( F_I \)
\( \dot{H}_I \) = angular momentum about C in \( F_I \)

In the body axes, we have

\[ \tilde{G}_B = L_B I \tilde{G}_I = \dot{H}_B + \omega_B H_B \]

where

\[ \tilde{G}_B = \begin{bmatrix} L \\ M \\ N \end{bmatrix} \]

see figure 3

and

\[ H_B = \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} \]

with

\[ \omega_B = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \]
ii becomes

a) \( L = H_x + qH_z - rH_y \)

vi) b) \( M = H_y + rH_x - pH_z \)

c) \( N = H_z + pH_y - qH_x \)

The assumption is made that the flight vehicle is in a rigid body, consequently, \( v \) becomes

\[
\begin{bmatrix}
L \\
M \\
N
\end{bmatrix} =
\begin{bmatrix}
I_x & -I_{xy} & -I_{xz} \\
-I_{xy} & I_y & -I_{yz} \\
-I_{xz} & -I_{yz} & I_z
\end{bmatrix}
\begin{bmatrix}
p \\
q + r \\
r -q
\end{bmatrix}
\begin{bmatrix}
I_x & -I_{xy} & -I_{xz} \\
-I_{xy} & I_y & -I_{yz} \\
-I_{xz} & -I_{yz} & I_z
\end{bmatrix}
\begin{bmatrix}
p \\
q \\
r
\end{bmatrix}
\]

a) \( L = I_x \ddot{\phi} - I_{yz} (q^2 - r^2) - I_{zx} (p + q) - I_{xy} (q - rp) - (I_y - I_z) qr \)

vii) b) \( M = I_y \ddot{\theta} - I_{zx} (r^2 - p^2) - I_{xy} (p + qr) - I_{yz} (r - pq) - (I_z - I_x) rp \)

c) \( N = I_z \ddot{\gamma} - I_{xy} (p^2 - q^2) - I_{yz} (q + rp) - I_{zx} (p - qr) - (I_x - I_y) pq \)

Since \( Cxz \) is usually a plane of symmetry, in which case \( I_{xy} = I_{yz} = 0 \), vii becomes

a) \( L = I_x \ddot{\phi} - I_{zx} (p + q) - (I_y - I_z) qr \)

viii) b) \( M = I_y \ddot{\theta} - I_{zx} (r^2 - p^2) - (I_z - I_x) rp \)

c) \( N = I_z \ddot{\gamma} - I_{zx} (p - qr) - (I_x - I_y) pq \)
A-5.0) The angular velocities \( \omega^V \) and \( \omega^B \) of \( F_w \) and \( F_B \), respectively, w.r.t. \( F \) are given by \( \dot{\psi}, \dot{\theta}, \dot{\phi} \) (subscript \( w \) for \( F_w \)).

1) \( \ddot{\omega} - \dot{\omega}^V = i \dot{\phi} + j_3 \dot{\theta} + k_2 \dot{\psi} \)

Referring to figure 4, by inspection

ii) \( i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad j_3 = \begin{bmatrix} 0 \\ \cos \phi \\ -\sin \phi \end{bmatrix}, \quad k_2 = \begin{bmatrix} -\sin \theta \\ \cos \theta \sin \phi \\ \cos \theta \cos \phi \end{bmatrix} \)

we have

iii) \( \ddot{\omega} - \dot{\omega} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin \theta \\ 0 & \cos \phi & \cos \theta \sin \phi \\ 0 & -\sin \theta & \cos \theta \sin \phi \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \)

solving for \( (\dot{\phi}, \dot{\theta}, \dot{\psi})^T \)

iv) \( \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin \theta \\ 0 & \cos \phi & \cos \theta \sin \phi \\ 0 & -\sin \theta & \cos \theta \sin \phi \end{bmatrix}^{-1} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \)

or

\[
\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi \sec \theta & \cos \phi \sec \theta \end{bmatrix} \begin{bmatrix} P \\ Q \\ R \end{bmatrix}
\]
but since
\[
\begin{bmatrix}
P \\
Q \\
R
\end{bmatrix} = \bar{\omega} - \bar{\omega}
\]
and
\[
\bar{\omega} = \begin{bmatrix}
p \\
q \\
r
\end{bmatrix}
\]
(iv) becomes
\[
\begin{bmatrix}
\dot{\phi} \\
\dot{\theta} \\
\dot{\psi}
\end{bmatrix} = \begin{bmatrix}
1 & \sin \phi \tan \Theta & \cos \phi \tan \Theta \\
0 & \cos \phi & -\sin \phi \\
0 & \sin \phi \sec \Theta & \cos \phi \sec \Theta
\end{bmatrix} \begin{bmatrix}
p \\
q \\
r
\end{bmatrix} - \bar{L}_{BV} \bar{\omega}_V
\]
For the wind axes, we have \( \phi_W, \theta_W, \psi_W, \dot{\phi}_W, \dot{\theta}_W, \dot{\psi}_W \) and \( \bar{L}_{W} \) instead of \( \bar{L}_{BV} \) in (v). Clearly \([P Q R]^T = [p q r]^T\) when \( \bar{\omega}_V \) and \( \bar{\omega}_F \) are negligible.

The angular velocity of \( F_B \) relative to \( F_W \) can be expressed as
(see figure 2)
vi) \( \bar{\omega}_{REL} = \bar{\omega} - \bar{\omega}_W = -k_W \ddot{\phi} + \bar{j}_W \alpha_x \)
where \( \vec{k}_w \) is a unit vector along \( \text{Czw} \) and \( \vec{j} \) is a unit vector along \( \text{Cy} \), therefore, transforming all the components of \( \vec{v}_i \) into \( \vec{F}_w \), we have

\[
\text{vii)} \quad \mathbf{l}_{\text{WB}} \begin{bmatrix} p \\ q \\ r \end{bmatrix} - \begin{bmatrix} p_w \\ q_w \\ r_w \end{bmatrix} = -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{\phi} + \mathbf{l}_{\text{WB}} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \dot{\alpha}_x
\]

rearranging and using A 2.2 xx

\[
\text{viii)} \quad \begin{bmatrix} p_w \\ q_w \\ r_w \end{bmatrix} = \begin{bmatrix} \cos \alpha_x \cos \phi & \sin \phi & \sin \alpha_x \cos \phi \\ -\cos \alpha_x \sin \phi & \cos \phi & -\sin \alpha_x \sin \phi \\ -\sin \alpha_x & 0 & \cos \alpha_x \end{bmatrix} \begin{bmatrix} p \\ q - \dot{\alpha}_x \\ r \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]

therefore

\[
\begin{align*}
\text{a)} & \quad p_w = p \cos \alpha_x \cos \phi + (q - \dot{\alpha}_x) \sin \phi + r \sin \alpha_x \cos \phi \\
\text{b)} & \quad q_w = -p \cos \alpha_x \sin \phi + (q - \dot{\alpha}_x) \cos \phi - r \sin \alpha_x \sin \phi \\
\text{c)} & \quad r_w = -p \sin \alpha_x + r \cos \alpha_x + \dot{\phi}
\end{align*}
\]
A-6.0) Since $F_E$ and $F_V$ have parallel axes when $\omega^E$ and $\omega^Y$ are neglected, we have, using A-2.2-xiii

\[
\begin{bmatrix}
\dot{x}_E \\
\dot{y}_E \\
\dot{z}_E
\end{bmatrix} = \tilde{L}_{VW}^T \begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix}
\]

i) $\dot{x}_E = V \cos \theta_w \cos \psi_w$

ii) $\dot{y}_E = V \cos \theta_w \sin \psi_w$

iii) $\dot{z}_E = -V \sin \theta_w$

Similarly, using A-2.2-xiii

\[
\begin{bmatrix}
\dot{x}_E \\
\dot{y}_E \\
\dot{z}_E
\end{bmatrix} = \tilde{L}_{BV}^T \begin{bmatrix} u \\ v \\ w \end{bmatrix}
\]

expanding

a) $\dot{x}_E = (\cos \theta \cos \psi)u + ((\sin \phi \sin \theta \cos \psi) - (\cos \phi \sin \psi))v$

\[+ ((\cos \phi \sin \theta \cos \psi))
\]

b) $\dot{y}_E = (\cos \theta \sin \psi)u + ((\sin \phi \sin \theta \sin \psi) - (\cos \phi \cos \psi))v$

\[+ ((\cos \phi \sin \theta \sin \psi) - (\sin \phi \cos \psi))w
\]

c) $\dot{z}_E = -(\sin \theta)u + (\sin \phi \cos \theta)v + (\cos \phi \cos \theta)w$
A-7.0) In sections A-1.0 thru A-6.0, inclusive, the force, moment and kinematic equations that apply to the airplane have been developed w.r.t. various reference frames, namely $F^E$, $F^B$, $F^W$. Transformations that relate the reference frames in question were also developed. At this point, the aforementioned equations will be collected to reflect the following assumptions:

i) Rotation of the earth can be neglected
i.e., $\Omega^E = 0$

ii) Curvature of the earth can be neglected
i.e., $\Omega^E = 0$

iii) The $C_{xz}$ plane (body axes) of the airplane is a plane of symmetry.

iv) The airplane is a rigid body.

v) The atmosphere is at rest w.r.t. earth ($w=0$).

We have:

\[ \begin{bmatrix}
  x_E \\
  y_E \\
  z_E
\end{bmatrix} = \bar{L}_{VB} \begin{bmatrix}
  u \\
  v \\
  w
\end{bmatrix} \]

\[ \begin{bmatrix}
  x_E \\
  y_E \\
  z_E
\end{bmatrix} = \bar{L}_{BV} \begin{bmatrix}
  v \\
  0 \\
  0
\end{bmatrix} \]
a) \[ p_w = p \cos \alpha_x \cos \beta + (q - \alpha_x) \sin \beta + r \sin \alpha_x \cos \beta \]

\[ \dot{x}_w = q - q_w \sec \beta - p \cos \alpha_x \tan \beta - r \sin \alpha_x \tan \beta \]

\[ \dot{\beta} = r_w + p \sin \alpha_x - r \cos \alpha_x \]

a) \[ \phi_w = p_w + q_w \sin \phi_w \tan \theta_w + r_w \cos \phi_w \tan \theta_w \]

A-5.0-V)

b) \[ \dot{\phi}_w = q_w \cos \phi_w - r_w \sin \phi_w \]

c) \[ \psi_w = (q_w \sin \phi_w + r_w \cos \phi_w) \sec \theta_w \]

Without the subscript \( w \), A-5.0-V applies to the body axes.

a) \[ L = I_x \dot{\phi} - I_{zx} (\dot{r} + pq) - (I_y - I_z) qr \]

A-4.0-viii(b) \[ M = I_y \dot{\theta} - I_{zx} (r^2 - p^2) - (I_z - I_x) rp \]

c) \[ N = I_z \dot{\psi} - I_{zx} (q - qr) + (I_x - I_y)pq \]

a) \[ X - mg \sin \theta = m(\dot{u} + qw - rv) \]

A-3.0-xxii(b) \[ Y + mg \cos \theta \sin \phi = m(\dot{v} + ru - pw) \]

c) \[ Z + mg \cos \theta \cos \phi = m(\dot{w} + pv - qu) \]

a) \[ T_{xw} - D - mg \sin \theta_w = m \dot{V} \]

A-3.0-xii(b) \[ T_{yw} - C + mg \cos \theta_w \sin \phi_w = mV_r \]

c) \[ T_{zw} - L + mg \cos \theta_w \cos \phi_w = -m V_q \]
The collection of equations presented in section A-7.0 are non-linear. A linearized model which is predicated upon small disturbances about a reference condition of steady symmetric recti-linear flight over a flat earth is utilized as follows:

i) Denoting the steady state values by the subscript $e$ and the changes from them by the prefix $\Delta$, we have:

$$
\begin{align*}
\mathbf{v} &= \mathbf{v}_e + \Delta \mathbf{v} \\
\dot{\phi} &= \dot{\phi}_e + \Delta \phi \\
p &= p_e + \Delta p \\
L &= L_e + \Delta L \quad \text{etc.}
\end{align*}
$$

ii) The steady state of the airplane is that of symmetric (no sideslip) wings-level translation, which means that at most, the following state variables are non-zero.

$$
\mathbf{v}_e, \alpha_{xe}, \theta_{we}, \psi_{we}
$$

iii) Since initial heading has no particular significance given the Flat-Earth approximation (A-7.0-ii), we select $\psi_{we} = 0$.

iv) Since the reference states for the state variables other than those mentioned in A-8.0-ii, above are zero, the prefix $\Delta$ is not needed, this pertains to the following $p, q, r, p_w, q_w, r_w, u, v, w, \beta, x_e, y_e, z_e, \phi, \theta, \psi, \phi_w$.

v) The stability axes, $F_S$ are defined as a special set of body axes where $\mathbf{\bar{V}}$ lies in the plane of symmetry $C_{xz}$ and $F_S$ coincides with $F_W$ in the reference condition,
but departs from \( F_w \) moving with \( F_B \) during the disturbance.

vi) All perturbation quantities are small; their squares and products can be neglected.

Furthermore: \( \cos \Delta \approx 1, \sin \Delta \approx \Delta \)

vii) In view of A-8.0-v, \( \alpha_x = \omega_e = 0 \) and instead of \( \alpha_x \) (angle of attack of the \( x \) axis), the angle of attack variable is chosen to be that of the zero lift line, usually \( \alpha \). See figures 8 and 2-a.

The nomenclature hereinafter to be used for \( \theta_w \), the angle of climb is \( \gamma \), as indicated in figures 2-a and 8.

In view of A-8.0-i thru A-8.0-vii, we are now ready to linearize the equation of section A-7.0.

Operating on A-6.0-iiia we have

\[
\dot{x}_E = (V_e + \Delta V) \cos (\gamma_e + \Delta \gamma) \cos \psi_w
\]

\[
= (V_e + \Delta V) (\cos \gamma_e \cos \Delta \gamma - \sin \gamma_e \sin \Delta \gamma)
\]

\[
\dot{x}_E = (V_e + \Delta V) \cos \gamma_e - (V_e + \Delta V) (\sin \gamma_e) (\Delta \gamma)
\]

viii) \( \dot{x}_E = (V_e + \Delta V) \cos \gamma_e - V_e \sin \gamma_e \Delta \gamma \)

Also, from A-6.0-iv-b

\[
\dot{y}_E = u (\cos \theta \sin \psi) + v (\sin \phi \sin \theta \sin \psi - \cos \phi \cos \psi)
\]

\[
+ w ((\cos \phi \sin \theta \sin \psi) - (\sin \phi \cos \psi))
\]
= u\psi + v

But \( u = V_e \)

therefore

ix) \( \dot{z}_E = V_e (\cos \gamma_e)\psi + v \)

from A-6.0-ii-c, we have

\[
\dot{z}_E = -(V_e + \Delta V)\sin (\gamma_e + \Delta \gamma_e) \\
= -(V_e + \Delta V)(\sin \gamma_e \cos \Delta \gamma_e + \cos \gamma_e \sin \Delta \gamma_e) \\
= -(V_e + \Delta V)(\sin \gamma_e + \Delta \gamma_e \cos \gamma_e)
\]

x) \( \dot{z}_E = -V_e (\sin \gamma_e + \Delta \gamma_e \cos \gamma_e) - \Delta V \sin \gamma_e \)

From A-5.0-ixb

\[
\dot{\alpha} = q - q_w \sec \beta - p \cos \alpha \tan \beta - r \sin \alpha \tan \beta \\
= q - q_w \\
q_w = q - \dot{\alpha}
\]

From A-5.0-v-a

\[
\dot{\phi} = p + q \sin \phi \tan \theta + r \cos \phi \tan \theta \\
= p + r \tan \theta
\]

but \( \tan \theta = \tan \gamma_e \) since \( F_s \) coincides with \( F_w \) during the reference state, therefore

xi) \( \dot{\phi} = p + r \tan \gamma_e \)

From A-5.0-v-b

\[
\dot{\phi}_w = q_w \cos \phi_w - r_w \sin \phi_w
\]
xii) \[ \dot{\psi} = q_w \]
From A-5.0-v-c
\[ \dot{\psi} = (q \sin \phi + r \cos \phi) \sec \theta \]
\[ = r \sec \theta \]
but since \( F_S \) coincides with \( F_W \) during the reference state

xiii) \[ \dot{\psi} = r \sec \psi_e \]
From A-4.0-viii-a, we have

xiv) \[ \Delta L = I_x \dot{\phi} - I_{zx} \dot{\psi} \]
From A-4.0-viii-b, we have

xv) \[ \Delta M = I_y q \]
From A-4.0-viii-c, we have

xvi) \[ \Delta N = I_z \dot{\phi} - I_{zx} \psi \]
From A-3.0-xiii a and A-3.0-xiv, we have

xvii) \[ \Delta T \cos \alpha_t - \Delta \alpha_t T_e \sin \alpha_t - \Delta D - mg \cos \psi_e \Delta \psi_e = m \dot{\psi} \]
From A-2.1-iiib, A-3.0-viib and A-3.0-xiv, we have

xviii) \[ \Delta Y = mg \cos \psi_e \psi = m(\dot{\psi} + r) \psi_e \]
From A-3.0-xiiiic and A-3.0-xiv, we have

xix) \[ \Delta T \sin \alpha_t + \Delta \alpha_t T_e \cos \alpha_t + \Delta L + mg \sin \psi_e \Delta \psi_e = m \dot{\psi}_w \]
Since the state variables \( (\Delta \alpha, \Delta V, q, \Delta Y, x_E, \text{and} \, z_E) \) are known as longitudinal variables and the state variables \( (\dot{\psi}, p, r, \dot{\phi}, \psi, \psi_E) \) as lateral variables, equations A-8.0-viii thru A-8.0-xix inclusive can be classified as longitudinal equations and lateral equations, as follows.

Longitudinal equations:

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The classification of the above equations arise from the fact that in the longitudinal equations only longitudinal variables appear explicitly and in the lateral equations only lateral variables appear explicitly. It is important to point out that the longitudinal and lateral equations (xx and xxi above, respectively) are completely decoupled into two independent sets.
A-9.0) Aerodynamic forces and moments are, in the strictest sense of the word, functions of the state variables, which in turn are functions of time. In other words, the aerodynamic forces and moments depend significantly on the time-histories of the state variables.

One assumption of linear aerodynamic theory is that the aerodynamic forces and moments can be expressed as functions of the state variables and their derivatives.

Using lift, \( L(t) \) as an example, we have

i) \( L(t) = L(\alpha(t)) \quad -\alpha \leq t \leq \alpha \)

which is an exact functional relation. But when \( \alpha(t) \) can be expressed as a convergent Taylor series around \( t \), namely

ii) \( \alpha(t) = \alpha(t) + (t-t) \alpha'(t) + \\
\,(1/2) (t-t)^2 \alpha''(t) + ...... \)

then (ii) can be substituted in (i) thus,

iii) \( L(t) = L(\alpha, \alpha', \alpha'', ...) \)

where \( \alpha, \alpha', \alpha'', ... \) are values at time \( t \). Further series expansion of iii around \( \alpha = 0 \) yields

iv) \( L(t) = \frac{\partial L}{\partial \alpha} + \frac{1}{2} \frac{\partial^2 L}{\partial \alpha^2} (\Delta \alpha)^2 + .... + \frac{3}{2} \frac{\partial^3 L}{\partial \alpha^3} \)

\( + (1/2) \frac{\partial^2 L}{\partial \alpha^2} (\Delta \alpha)^2 + ...... \)
Expressions such as (v) are known as stability or aero-
dynamic derivatives. Most of the time only the first term
is kept, though sometimes higher order terms are maintained
for added accuracy.

Recalling the assumptions A-8.0-ii, A-8.0-iii, and A-8.0-vii
it is pointed out that in a truly symmetric, wings level
flight configuration, Y; L, the rotating moment, N, p, r, \( \phi \), \( \psi \) and \( y_E \) are zero. Therefore, the derivatives of
the lateral forces and moments, Y, L, N, w.r.t. the longi-
tudinal state variables \( \Delta V, \Delta \alpha, \Delta \gamma, q, x_E \) and \( z_E \) are
zero.

Further assumptions are made:

vi) the derivatives of the longitudinal forces and
moment \( D, L \) (lift), \( \gamma \) and \( M \), w.r.t. the lateral
state variables \( \delta \), p, r, \( \phi \), \( \psi \) and \( y_E \) can be
neglected.

vii) All the derivatives w.r.t. the rates of change
of the state variables can be neglected, except
for \( \frac{\partial L}{\partial q} \) and \( \frac{\partial M}{\partial \alpha} \) [8].

viii) The derivative \( \frac{\partial D}{\partial q} \) is negligibly small [9].
Applying the aforementioned assumptions, we can now write the expression for the linear forces and moments:

Longitudinal:

a) \( \Delta D = \frac{2}{3} \frac{D}{V} \Delta V + \frac{2}{3} \frac{D}{\alpha} \Delta \alpha + \frac{2}{3} \frac{D}{z_E} \Delta z_E + \Delta D_c \)

b) \( \Delta L = \frac{2}{3} \frac{1}{V} \Delta V + \frac{2}{3} \frac{1}{\alpha} \Delta \alpha + \frac{2}{3} \frac{1}{z_E} \Delta z_E + \frac{2}{3} q \)

\[ + \frac{1}{z_E} z_E + L_c \]

ix) c) \( \Delta M = \frac{2}{3} \frac{M}{V} \Delta V + \frac{2}{3} \frac{M}{\alpha} \Delta \alpha + \frac{2}{3} \frac{M}{z_E} \Delta z_E + \Delta M_c \)

\[ + \frac{2}{3} M \Delta z_E + \Delta M_c \]

d) \( \Delta T = \frac{2}{3} \frac{T}{V} \Delta V + \frac{2}{3} \frac{T}{z_E} \Delta z_E + \Delta T_c \)

Lateral:

a) \( Y = \frac{2}{3} \frac{Y}{p} q + \frac{2}{3} \frac{Y}{p} p + \frac{2}{3} \frac{Y}{r} r + \Delta Y_c \)

ix) b) \( L = \frac{2}{3} \frac{L}{p} q + \frac{2}{3} \frac{L}{p} p + \frac{2}{3} \frac{L}{r} r + \Delta L_c \)

c) \( N = \frac{2}{3} \frac{N}{p} q + \frac{2}{3} \frac{N}{p} p + \frac{2}{3} \frac{N}{r} r + \Delta N_c \)

In equations (ix) and x) above, the quantities with the subscript \( c \) (\( \Delta Y_c, \Delta L_c, \Delta D_c \), etc.) denote incremental forces and moments that result from the actuation of the control systems of the plane. It should also be pointed out that in equation ix-b, \( L \) pertains to "lift" and in equation x-b \( L \) pertains to "Rolling moment". The ambiguity will be resolved as long as it is remembered that "lift" is a longitudinal force and "rolling moment" a lateral moment.
Substituting equations six and x above into equations A-8.0-xx and A-8.0-xxi, respectively and re-arranging, we have:

**Longitudinal equations:**

*a)* \( \dot{m}V = \left( \frac{\partial T}{\partial V} \cos \alpha_t - \frac{\partial D}{\partial V} \right) \Delta V - \left( T_e \sin \alpha_t + \frac{\partial D}{\partial \alpha} \right) \Delta \alpha \\
- mg \cos \gamma_e \Delta \gamma + \left( \frac{\partial T}{\partial \gamma} \cos \alpha_t - \frac{\partial D}{\partial \gamma} \right) \Delta \gamma_e \\
+ \Delta T_c \cos \alpha_t - \Delta D_c \)

*b)* \(- (mV_e + \frac{1}{2} l I) \dot{\alpha} = \left( \frac{\partial T}{\partial \alpha} \sin \alpha_t + \frac{\partial D}{\partial \alpha} \right) \Delta V \\
+ \left( T_e \cos \alpha_t + \frac{1}{2} l \right) \dot{\alpha} + \left( \frac{\partial T}{\partial \alpha} \sin \alpha_t + \frac{\partial D}{\partial \alpha} \right) \Delta \gamma_e \\
+ mg \sin \gamma_e \Delta \gamma + \left( \frac{\partial T}{\partial \gamma} \sin \alpha_t + \frac{\partial D}{\partial \gamma} \right) \Delta \gamma_e \\
+ \Delta T \sin \alpha_t + \Delta L_c \)

*c)* \( I_{y} \dot{\gamma} + \frac{3M}{2V} \dot{\alpha} = \frac{3M}{2V} \Delta V + \frac{3M}{2q} \Delta \alpha + \frac{3M}{2q} \Delta q + \frac{3M}{2z} \Delta \gamma_e + \Delta M_c \)

*d)* \( \alpha + \dot{\gamma} = q \)

*e)* \( \dot{x}_e = V_e \cos \gamma_e + \cos \gamma_e \Delta V - V_e \sin \gamma_e \Delta \gamma \\
\dot{y}_e = -V_e \sin \gamma_e - \sin \gamma_e \Delta V - V_e \cos \gamma_e \)

**Lateral equations:**

*a)* \( m \ddot{V}_e = \frac{\partial V}{\partial p} p + \frac{\partial V}{\partial r} (-m V_e) r + mg \cos \gamma_e \phi + \Delta Y \)

*b)* \( I_{xp} \dot{p} - I_{xz} \dot{r} = \frac{2N}{2p} \phi + \frac{2N}{2r} \phi + \frac{2N}{2r} r + \Delta L_c \)

*c)* \(-I_{xerp} + I_{z} \dot{r} = \frac{2N}{2p} \phi + \frac{2N}{2r} \phi + \frac{2N}{2r} r + \Delta N_c \)

*xii)* \( d)* \phi = p + r \tan \gamma_e \)

*e)* \( \psi = r \sec \gamma_e \)

*f)* \( \dot{y}_e = V_e (\cos \gamma_e) \psi + v \)
Rearranging xi, we have

\[ a) \dot{p} = \left( \frac{1}{2} \frac{d}{dp} + \frac{1}{2} \frac{d}{dR} \right) p + \left( \frac{1}{2} \frac{d}{dI_x} + \frac{1}{2} \frac{d}{dI_z} \right) r + \frac{1}{2} \frac{d}{dI_x} \left( \frac{d}{dI_x} + \Delta N_c \right) \]

\[ b) \dot{r} = \left( \frac{1}{2} \frac{d}{dp} + \frac{1}{2} \frac{d}{dI_z} \right) p + \left( \frac{1}{2} \frac{d}{dI_x} + \frac{1}{2} \frac{d}{dI_z} \right) r + \frac{1}{2} \frac{d}{dI_z} \left( \frac{d}{dI_z} + \Delta N_c \right) \]

c) \[ \dot{\theta} = \frac{1}{mV} \left( \frac{3}{2} \frac{d}{dp} p + \frac{3}{2} \frac{d}{dR} r + \frac{3}{2} \frac{d}{dI_x} \theta + mg \cos \gamma_e \Delta Y_e \right) \]

d) \[ \dot{\phi} = p + r \tan \gamma_e \]

where

\[ I_x' = \frac{(I_x I_z - I_{z x}^2)}{I_z} \]

\[ I_z' = \frac{(I_x I_z - I_{z x}^2)}{I_x} \]

\[ I_{z x} = \frac{I_{z x}}{(I_x I_z - I_{z x}^2)} \]

Referring to equations A-2.1-iiib, A-5.0-ix-c, A-3.0-xii-b and A-9.0-x-a we have

\[ Y + mg \cos \theta \sin \phi = m (V \cos \beta \Delta \phi + ru - pw) \]

\[ xiv) \quad Y = -mg \cos \theta \sin \phi + m (V \cos \beta (r_w + p \sin \alpha_x - r \cos \alpha_x) - ru-pw) \]

\[ \frac{3Y}{dp} = mV \cos \theta \sin \alpha_x - w \]

or applying the foregoing assumptions

\[ xv) \quad \frac{3Y}{dp} = mV_e \alpha \]

Similarly, from xiv)

\[ \frac{3Y}{dr} = m (V \cos \theta (- \cos \alpha_x) + u) \]
but

\[ u = V \cos \alpha \cos \beta \]

therefore

\[ \text{xvi) } \frac{\partial v}{\partial t} = 0 \]

from xv and vvi, xiii-c becomes

\[ \text{xvii) } \frac{\dot{\mathbf{e}}}{\dot{\mathbf{e}}} = \frac{1}{mV_e} \left( (mV_e \alpha) p + (-mV_e) r + \frac{\partial v}{\partial e} + mg(\cos \gamma_e) \phi + \Delta Y_c \right) \]

Rearranging xiii a, b, d and xvii in matrix form, with the assumption that the body axes are principal axes \((I_x' - I_x', I_y' - I_y', I_z' - I_z, \text{and } I_{2x} = 0)\), we have:

\[ \begin{bmatrix}
\frac{\dot{p}}{p} \\
\frac{\dot{r}}{r} \\
\frac{\dot{\theta}}{\theta} \\
\frac{\dot{\phi}}{\phi}
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{I_x} & \frac{1}{I_x} & \frac{1}{I_x} & 0 \\
\frac{1}{I_y} & \frac{1}{I_y} & -1 & 0 \\
1 & \tan \gamma_e & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
p \\
r \\
\theta \\
\phi
\end{bmatrix} +
\begin{bmatrix}
L_{Sa} \\
N_{Sa} \\
Y_{Sa} \\
0
\end{bmatrix}
\]

where:

\[ L_{Sa} = \frac{\Delta L_c}{I_x} \]

\[ N_{Sa} = \frac{\Delta N_c}{I_y} \]

\[ Y_{Sa} = \frac{\Delta Y_c}{mV_e} \]

Equation xviii can always be applied, since it is always possible to make a choice of body axes so that they are principal axes.
Since the equilibrium state assumes symmetric wing level flight, $\nu_e = 0$, therefore, xviii becomes

\[
\begin{bmatrix}
\dot{p} \\
\dot{r} \\
\dot{\theta} \\
\dot{\phi}
\end{bmatrix} =
\begin{bmatrix}
L_p & L_r & L_\phi & 0 \\
N_p & N_r & N_\phi & 0 \\
\alpha & -1 & 0 & g/Ve \\
1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
p \\
r \\
\theta \\
\phi
\end{bmatrix} +
\begin{bmatrix}
L_{sa} \\
N_{sa}
\end{bmatrix}
\]

where

\begin{enumerate}
\item \( L_p = \frac{1}{I_x} \frac{\partial L}{\partial \theta} \), \( L_r = \frac{1}{I_r} \frac{\partial L}{\partial \phi} \), \( L = \frac{1}{I_x} \frac{\partial L}{\partial \phi} \)
\item \( N_p = \frac{1}{I_z} \frac{\partial N}{\partial \theta} \), \( N_r = \frac{1}{I_z} \frac{\partial N}{\partial \phi} \), \( N = \frac{1}{I_z} \frac{\partial N}{\partial \phi} \)
\item \( Y_{\phi} = \frac{\partial Y}{\partial \phi} \)
\end{enumerate}

Equation (xix) can be expressed in the plant form

\[
\dot{x} = \bar{A} x + \bar{b} \bar{u}
\]

with

\[
\begin{bmatrix}
\dot{p} \\
\dot{r} \\
\dot{\theta} \\
\dot{\phi}
\end{bmatrix} =
\begin{bmatrix}
L_p & L_r & L_\phi & 0 \\
N_p & N_r & N_\phi & 0 \\
\alpha & -1 & Y_{\phi} & g \\
1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
p \\
r \\
\theta \\
\phi
\end{bmatrix}
\]
\[ xxiv) \quad \ddot{x} = \begin{bmatrix} p \\ r \\ q \\ \phi \end{bmatrix} \]

\[ xxv) \quad \ddot{b} = \begin{bmatrix} L_{\delta_a} \\ N_{\delta_a} \\ Y_{\delta_a} \\ 0 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \]

\[ xxvi) \quad \delta_a = \text{step input at ailerons (radians)} \]
In the same manner as equation A-9.0-x, we can write

\[ \dot{p} = \frac{\partial L}{\partial p} \dot{p} + \frac{\partial L}{\partial r} \dot{r} + \frac{\partial L}{\partial \phi} \dot{\phi} + \Delta L_c \]

\[ \dot{r} = \frac{\partial N}{\partial p} \dot{p} + \frac{\partial N}{\partial r} \dot{r} + \frac{\partial N}{\partial \phi} \dot{\phi} + \Delta N_c \]

\[ \dot{\phi} = \frac{\partial Y}{\partial p} \dot{p} + \frac{\partial Y}{\partial r} \dot{r} + \frac{\partial Y}{\partial \phi} \dot{\phi} + \Delta Y_c \]

where \( \Delta L_c, \Delta N_c, \Delta Y_c \) denote incremental forces and moments that result from the actuation of the aircraft controls (rudder and aileron for this particular lateral case). Assuming a "rudder free" condition, we can write

\[ \Delta L_c = \frac{\partial L}{\partial \delta_a} \delta_a \]

\[ \Delta N_c = \frac{\partial N}{\partial \delta_a} \delta_a \]

\[ \Delta Y_c = \frac{\partial Y}{\partial \delta_a} \delta_a \]

where

\[ \mathcal{L}[\delta_a] = K_i/s = \delta_a \]

and \( S = \) Laplace operator

\( K_i = \) step size

for level flight \( Ye = 0 \), therefore

\[ \phi = \int \dot{\phi} \, dt \]

and

\[ \Psi = \int \phi \, dt \]

also, as shown in section A-9.0

\[ \frac{\dot{Y}}{r} = 1.0 \]
vii) \( \frac{\partial Y}{\partial P} = \alpha = \text{very small} \approx 0 \)

viii) \( \frac{\partial Y}{\partial V} = g/V_e \approx 0 \quad V_e \gg 0 \)

ix) Furthermore, it can be shown that \( \frac{\partial Y}{\partial a} = 0 \) [10]

Substituting equation ii and iv thru xiii, inclusive into i-c, we obtain

x) \( r = - \dot{\phi} + \frac{\partial Y}{\partial \phi} \dot{\phi} + \frac{\partial Y}{\partial \psi} \int p \, dt \)

Substituting equation ii and iv thru x inclusive into i-a and i-b and rearranging, we obtain

xi) \( p - \frac{\partial L}{\partial P} = - \frac{\partial L}{\partial P} \frac{\partial Y}{\partial \psi} \int p \, dt = \left( \frac{\partial L}{\partial \phi} + \frac{\partial L}{\partial \phi} \frac{\partial Y}{\partial \phi} \right) \dot{\phi} - \frac{\partial L}{\partial \phi} + \frac{\partial L}{\partial \phi} \dot{\psi} \)

xii) \( \dot{\phi} = \frac{\partial N}{\partial \psi} \dot{p} + \left( \frac{\partial N}{\partial \phi} + \frac{\partial N}{\partial \phi} \frac{\partial Y}{\partial \phi} \right) = \frac{\partial Y}{\partial \phi} - \frac{\partial N}{\partial \phi} \dot{p} \)

F-4 data indicates that [10]:

xiii) \( \left( - \frac{\partial L}{\partial \psi} / \left( \frac{\partial L}{\partial \phi} + \frac{\partial L}{\partial \phi} \right) \right) \approx 0 \)

and

xiv) \( \frac{\partial L}{\partial \phi} \frac{\partial Y}{\partial \phi} \approx 0 \)

letting

a) \( \dot{P} = - \frac{\partial L}{\partial P} \)

b) \( \dot{\phi} = - \frac{\partial L}{\partial \phi} + \frac{\partial L}{\partial \phi} \frac{\partial Y}{\partial \phi} \)

c) \( \dot{\phi} = \frac{\partial L}{\partial \phi} \)

dx) \( \dot{N} = - \frac{\partial N}{\partial \phi} + \frac{\partial N}{\partial \phi} \frac{\partial Y}{\partial \phi} \)

e) \( \dot{N} = - \frac{\partial N}{\partial \phi} + \frac{\partial N}{\partial \phi} \frac{\partial Y}{\partial \phi} \)

e) \( \dot{N} = - \frac{\partial N}{\partial \phi} + \frac{\partial N}{\partial \phi} \frac{\partial Y}{\partial \phi} \)

f) \( \dot{N} = - \frac{\partial N}{\partial \phi} + \frac{\partial N}{\partial \phi} \frac{\partial Y}{\partial \phi} \)

g) \( \dot{N} = - \frac{\partial N}{\partial \phi} + \frac{\partial N}{\partial \phi} \frac{\partial Y}{\partial \phi} \)
h) $nS_a = \frac{\partial N}{\partial S_a}$

Substituting xv into xi and xii and using xiii and xiv, we have

xvi) $p - 1p \phi = I_\phi + I_a \delta_a$

xvii) $\phi - n_r \phi + n_p \Phi = -n_p \phi - n_\phi \phi - n_s \delta_a$

taking the Laplace transform of xvi and xvii

xviii) $(S - 1p) \bar{p} = I_\phi + I_s \delta_a$

xix) $(S^2 - n_r S + n_p) \Phi = -n_p \bar{p} - n_\phi \phi - n_s \delta_a$

Now, we will proceed to normalize equations (xviii) and (xix). Dividing the former by $-1p$ and the latter by $n_\phi$, we obtain respectively

\[
\left( -\frac{S}{1p} + 1 \right) \bar{p} = \left( -\frac{I_\phi}{1p} \right) \bar{\phi} + \left( -\frac{I_s}{1p} \right) \bar{\delta_a}
\]

\[
\left( \frac{S^2}{n_\phi} - \frac{n_r}{n_\phi} S + 1 \right) \bar{\Phi} = \left( -\frac{n_p}{n_\phi} \right) \bar{p} + \left( -\frac{n_\phi}{n_\phi} \right) \bar{\phi} + \left( -\frac{n_s}{n_\phi} \right) \bar{\delta_a}
\]

arbitrarily settling $n_\phi = 0$, we have

Substituting for $\bar{\Phi}$ in the first of the above, we have

\[
\bar{p} \left( \left( -\frac{S}{1p} + 1 \right) - \frac{(1p)(n_p)}{(n_\phi)} \right) = \bar{\delta_a} \left( \frac{(1p)(n_\phi)}{(n_\phi)} \right) - \frac{I_s}{1p}
\]

\[
\bar{\Phi} \left( \frac{S^2}{n_\phi} - \frac{n_r}{n_\phi} S + 1 \right) = \bar{\delta_a} \left( \frac{S^2}{n_\phi} - \frac{n_r}{n_\phi} S + 1 \right)
\]

rearranging

\[
\bar{\Phi} = \frac{-s^2 I_s + I_s \bar{\delta_a} n_r - I_a n_p + I_\phi n_s \delta_a}{s^3 + s^2 (n_r + 1p) s (n_\phi + n_r 1p) + (1p n_\phi - I_\phi n_p)}
\]

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therefore, the gain is

\[ K = \lim_{s \to 0} \frac{\bar{p}}{s} = \frac{1}{p} \frac{n_\phi}{n_\phi} - \frac{1}{p} \frac{n_q}{n_q} \]

the amplitude normalization factor (final value of the output, or steady state) for \( \bar{p} \) is

xx) \[ p_{ss} = \frac{1}{p} \frac{n_\phi}{n_\phi} - \frac{1}{p} \frac{n_q}{n_q} Ki \]

Similarly, the amplitude normalization factor (final value of the output or steady state) for \( \phi \) is

xxi) \[ \phi_{ss} = \frac{1}{p} \frac{n_\phi}{n_\phi} - \frac{1}{p} \frac{n_q}{n_q} Ki \]

Using xx and xxii in xviii and xix we obtain respectively

xxii) \( (T_0 S + 1) \bar{p}_n = X_1 \bar{\theta}_n + (1 - X_1) \Delta_a \)

xxiii) \( \frac{S^2}{w_o^2} + \frac{2Z_0 S + 1}{w_o} \) \[ n = X_2 \bar{p}_n + (1 - X_2) a + X_3 \phi_n \]

where

\( \bar{p}_n = p/p_{ss} \)
\( \bar{\theta}_n = \phi/\phi_{ss} \)
\( \Delta_a = \delta a/Ki \)
\( \phi_n = \phi/p_{ss} \)
\( t_0 = 1/-1p \) time constant, roll axis
\( Z_0 = -n_r / (s\sqrt{n_q}) \) damping ratio, roll axis
\( w_o = \sqrt{n_q} \) undampened frequency, roll axis
\( X_1 = (-1\phi_{ss})/p_{ss} \)
\( X_2 = (-n_p p_{ss})/(n_\phi \phi_{ss}) \)
\( X_3 = (-n_p p_{ss})/(n_\phi \phi_{ss}) \)
The $\tilde{p}_n/\tilde{A}_a$ and $\tilde{f}_n/\tilde{A}_a$ transfer functions are easily obtained from xxii and xxiii
A-11.0 The $D^*$ Criterion.

As indicated in reference [10]

1) $D^* = \Delta n_{yp} + K_3$

where

11) $\Delta n_{yp} =$ incremental lateral load factor at pilot station
    $\theta =$ sideslip angle
    $K_3 = C_3q_{co}$ sideslip gain constant
    $C_3 =$ dimensional constant
    $q_{co} =$ cross-over dynamic pressure

Equation (ii) can be expressed as

111) $n_{yp} = l + \hat{\theta}V_e + rV_e$
    $l =$ distance between pilot station and c.g.

substituting (iii) and (i) with A-9.0-xix, we obtain

iv) $D^* = (IN_{p} + \alpha V_{e}) p + (IN_{r}) r + (IN_{p} + Y_{\phi} V_{e} + C_3q_{co}) \theta$
    $\quad + g\phi + N_{ea} l + Y_{sa} V_e$

the output vector $y$ is

v) $y = \begin{bmatrix} p \\ \theta \\ D^* \end{bmatrix}$

and

vi) $\ddot{y} = \ddot{G} \ddot{x} + \ddot{h} \delta_a$

where

vii) $\ddot{x} = \begin{bmatrix} p \\ r \\ \theta \\ \phi \end{bmatrix}$ equation A-9.0-xxiv
viii) \[
\mathbf{G} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
(lNp+Ve) & (lN_r+Ve) & (lN_p+Ve+C_3q_{CO}) \\
\end{bmatrix}
\]

Equation viii is arrived at by inspection from (iv), (v), (vi) and (vii).

\[
h = \begin{bmatrix}
0 \\
0 \\
N_{sa} l + Y_{sa} Ve \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
b_2 l + b_3 Ve \\
\end{bmatrix}
\text{from A-9.0-xxiv}
\]

With \(D^*\) available (as well as \(P_{ss}\) and \(B_{ss}\)) equation vi can be expressed in normalized form thus

x) \[
\tilde{y}_N = \mathbf{G}_N \tilde{x} + \mathbf{h}_N \delta_a
\]

where

xi) \[
\tilde{y}_N = \begin{bmatrix}
p/p_{ss} \\
\theta_{q_{ss}} \\
D^*/D^*_{ss}
\end{bmatrix}
\]

xii) \[
\mathbf{G}_N = \begin{bmatrix}
1/p_{ss} & 0 & 0 & 0 \\
0 & 0 & 1/\theta_{ss} & 0 \\
(lNp+Ve) & (lN_r-Ve) & (lN_p+Ve+C_3q_{CO}) \\
\frac{D^*_{ss}}{D^*} & \frac{D^*_{ss}}{D^*} & \frac{D^*_{ss}}{D^*} \\
\end{bmatrix}
\]

xiii) \[
\mathbf{h}_N = \begin{bmatrix}
0 \\
0 \\
b_2 l + b_3 Ve \\
\frac{D^*_3}{D^*_{ss}}
\end{bmatrix}
\]
$D^{*}_{ss}$ is found in reference [10] with the aid of reference [4].

In the former, the data available is of the form

\[ xiv) \quad D\star_{N} = K + \frac{\Delta n_{VD}}{K_{3}K} = \frac{D\star_{K}}{K_{3}K} = \frac{D\star}{K_{3}K} \]

therefore

\[ xv) \quad D\star_{ss} = K_{3}K \]

where $K = \text{ratio of "commanded roll performance" to "applicable roll performance requirement"}$

as defined in 3.3.4 or 3.3.4.1 of reference [5].

The reader should note the $D\star_{ss}$ is NOT a steady state value of $D\star$ but a desired (or selected) constant of reference.
The search method is embodied in the subroutine DIRECT, shown in the ensuing flow chart.

The notation used in the search method is identified in terms of the pertinent state variables and constants.

i) XVEC: a vector containing the state variable of equation (90)
   XVEC (1) = Lp
   XVEC (2) = Lr
   XVEC (3) = Lq
   XVEC (4) = Np
   XVEC (5) = Nr
   XVEC (6) = Nq
   XVEC (7) = \theta
   XVEC (8) = b_1
   XVEC (9) = b_2
   XVEC (10) = b_3

ii) STEP: the perturbation quantity \delta_j
iii) COST: current values of the objective function as well as final value attained

iv) c: a vector containing $\tilde{c}$ and $\tilde{c}_0$ of equation (9)

v) EIGEN: a vector containing the eigenvalues $\tilde{\lambda}$ of equation (9)

vi) VO: a constant $= V_e$

vii) XL: a constant $= 1$

viii) PSS: $p_{ss}$ normalization factor for roll rate, a constant

ix) BETASS: $\theta_{ss}$ normalization factor for sideslip angle, a constant

x) DSTARSS = $D^*_{ss}$ normalization factor for $D^*$, a constant
DIRECT

INPUT: NP, XVEC, STEP, NPASS, MAXIT, IOUTP, IO, C, EIGEN, V_0, X_L, C_0, q_00, p_00, A_0, D^2_0

CALL BOUNDS (XVEC, IOUT, NP)

BOUND: CHECK FOR BOUNDARY VIOLATION

IOUT: 0 > STOP

NP: 20 > STOP

NUMIT = 1

CALL FUNCT (XVEC, C, EIGEN)

FUNCT: CALCULATES COST USING FOR GIVEN XVEC

COLD = COST

IOUTP: 2 <

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OUTPUT: XVEC (I2)
XVEC (I2) = 0

I2 = 1

I2: NP
D

IXCELL = 0
NCTRAN = 2

NPOO = 0
NFAIL = 0

I3 = 1
NBOU = 0

L
Q
E

11: NP

I1 = I1 + 1

D

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NIMP = 0

XVEC (i3) = B3 (i3) + STEP (i3)

CALL BOUNDS (XVEC, IOUT, NP)

IF IOUT = 0

NB2 = NB2 + 1

IF NB2 > 1

STEP (i3) = -STEP (i3)

CALL F4NC4 (NUMIT, MAXIT, ABCDE)

OUTPUT: NUMIT, COST, "NO CONVERGENCE"

G

H

CALL ITCHECK (IO, NUMIT, MAXIT, ABCDE)

OUTPUT: I3, XVEC

ABCDE = 1

IF ABCDE = 1

CALL ITCHECK (IO, NUMIT, MAXIT, ABCDE)

OUTPUT: NUMIT, COST, "MIN VAL OF COST",

"FUNCT. EVAL SOFAR"
ABS (COST): ABS (COLD)

B3 (I3) = XVEC (I3)

NIMP = NIMP + 1

COLD = COST

STEP (I3) = STEP (I3)

NFAIL = NFAIL + 1

XVEC (I3) = B3 (I3)

STEP (I3) = STEP (I3)

I3: NP

I3 = I3 + 1

NFAIL: NP

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\begin{align*}
\text{AM} &= 1.0 \\
I6 &= 1 \\
B3(I6) &= B2(I6) + AM(B2(I6)) - B1(I6) \\
\text{XVEC}(I6) &= B3(I6) \\
I6: NP &\geq 0 \\
\text{AM} &= \text{AM} - (1.1) \\
\text{CALL BOUNDS} \\
\text{IOUT} &= 0 \\
NPOO &= NPOO + 1
\end{align*}

\begin{align*}
R &\text{ OUTPUT: } I9, B2(I9) \\
I9 &= \text{NP} \\
J &= 1 \\
\text{XVEC}(J) &= B2(J) \\
J: NP &\geq 0 \\
\text{CALL FUNCT} \\
\text{OUTPUT: } \text{XVEC} \\
\text{STOP}
\end{align*}
BIBLIOGRAPHY


8. Ibid, page 160 ii.


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    ii) "Pourable Superinsulation" (Pending).