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Stephen C. Tumminelli
Celal N. Kostem

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FINITE ELEMENTS FOR THE ELASTIC ANALYSIS
OF COMPOSITE BEAMS AND BRIDGES

by

Stephen C. Tumminelli

Celal N. Kostem

Fritz Engineering Laboratory Report
Department of Civil Engineering
Lehigh University
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ABSTRACT

This report presents formulations for general finite elements which can be used to perform elastic analyses of composite single or multibeam, simple or continuous bridge superstructures. The most sophisticated element can include the effects of slip between the bridge deck and the beams, shear deformations in the beams and shear lag in the deck.

The formulations begin by assuming internal displacement fields. Then compatibility at the nodes and constraining equations between the fields are applied to solve for the displacement field coefficients. Finally, the element stiffness matrix is formulated via the application of the principle of virtual work.

The shear connectors are mathematically incorporated as a continuous medium of constant stiffness. The shear deformations in the beam are handled by assuming separate displacement fields for the transverse deflection and rotation. The shear lag in the deck is achieved by using the well-known linear displacement fields for the in-plane displacements.

The resulting elements can be used in any displacement based finite element computer program. The solution procedure
is not iterative. Rather, the global equations are stacked and solved directly.

Numerical comparisons are made with solutions using other analytical techniques and test data. They show that the elements are accurate and monotonically convergent.

The intent of this work is to provide a finite element which can be used in a computer program to perform nonlinear analyses of bridge superstructures consisting of steel girders composite with a concrete deck. The eventual analysis scheme will solve the nonlinear problem via piecewise linearization. Thus, the solution will require a substantial number of linear analyses. Therefore, the number of global equations necessary to solve the structure has been kept to a minimum.

The elements are formulated on the basis of elastic behavior. However, recommendations are made on how to extend the formulations to include material nonlinearities.

This report is based on the doctoral research of the first author.
1. INTRODUCTION

1.1 Overview of Research

The material presented in this report is part of research into the behavior of highway bridges loaded beyond the elastic limit. The final result of this research will be a method to perform non-linear analysis of highway bridge superstructures consisting of steel beams or girders and a composite reinforced concrete deck (Fig. 1). Hereafter these bridges will be referred to as steel bridges.

A study on overweight permit operations indicates that over 700,000 overweight permits were issued in the forty-eight contiguous states in 1966 (Ref. 15). Assuming no changes in regulations and load limits this was expected to grow to approximately \(1,250,000\) in 1975. In addition, it is acknowledged that the real factors of safety for bridges are unknown. This is reflected in the widely varying levels of overload permitted by the individual states. Furthermore, recent information indicates that one in six bridges in the United States is structurally deficient (Ref. 11). Clearly a method to analyze bridges in the non-linear range is necessary.

The purpose of the present research is twofold. In the short term the methods, yet to be developed, will allow permit officers to analytically assess what damage, e.g., cracking of the slab or yielding of the girders, a bridge would sustain due to an
overload. Over the long term, experience with the analytical results and field tests may permit the correction of any deficiencies in new design.

Theoretical work has recently been completed on the non-linear analysis of highway bridge superstructures consisting of prestressed or reinforced concrete beams and a composite reinforced concrete deck (Refs. 20,22,23). These will subsequently be called concrete bridges. Although the steel bridges present their own special set of problems, use will be made of the previous work whenever possible.

The present analytical work is being done specifically for steel bridges but, so far, the methods developed may be applied to structures of other materials as long as they obey the assumptions used in the formulations.

It is envisioned that the non-linear solution scheme will be an incremental or an incremental-iterative process as in the previous work (Ref. 23). Both processes reduce the non-linear problem to a piecewise linear one. The forces are applied to the structure in increments. The basic difference between the two is that in the incremental process, the stiffness matrix for load step i is derived from stress levels at load step i-1; whereas, in the incremental-iterative approach, the stiffness matrix for load step i is converged within specified tolerances from the stress levels at load step i by iteration (Ref. 24). Depending upon such parameters
as the structural geometry, material properties, and load pattern, a satisfactory non-linear analysis can require one hundred or more elastic solutions.

The development of the new methods for steel bridges can be divided into two broad categories:

1. Linear elastic analysis. Since so many elastic analyses can be required, the method must be fast as well as accurate.

2. Nonlinear analysis. The elastic analysis will be upgraded to include material non-linearities. Rules must be formulated to gauge changes in stiffness as well as predict failure. Geometric linearity will be maintained as in the previous work (Ref. 23).

1.2 Linear Elastic Analysis

It is difficult to determine exactly which effects should be accounted for in the analysis, particularly when the eventual non-linear analysis scheme will be automatically adjusting the stiffnesses of the various elements as the load levels increase. Therefore, some judgments have been made regarding the phenomena which are of primary importance, secondary importance and not important. These judgments are based on a review of test results of composite beams, analytical studies of bridge superstructures and discussions with other researchers. Experience with the final analytical tool and/or
accumulation of more test results may show that some of the assumptions are incorrect.

In any event, those phenomena which are considered to have a pronounced effect on the internal force distribution in steel bridge superstructures are:

1. Slip between the steel beam and the concrete deck. The two components are held together with mechanical fasteners of finite stiffness (Fig. 2a).

2. Shear deformations of the girders. In deep girders, the deflections due to shear deformations can be a significant portion of the total (Fig. 2b).

3. Shear lag of the deck. Permitting the lag avoids the issue of trying to determine, a priori, that portion of the deck acting compositely with the beams (Fig. 3).

The following phenomena are thought to be of secondary importance:

1. Minor axis bending of the beams. This will affect the force in the bracing and hence the major axis bending moments (Fig. 3).

2. Torsion of the beam. This also will affect the bracing forces and deck twist (Fig. 3).

3. Wind bracing and diaphragms. Although these are not designed as major load carrying members, studies have
shown that they have an effect on the response of the major load carrying system.

It has been observed that vertical separation between the deck and beams does occur. Previous work on single T-beams has neglected this phenomenon with reasonably good results (Refs. 3, 9, 28, 29, 30, 37, 38). Consequently, it was decided that it was not worth the effort to include the effects of separation into the present formulations.

1.3 Non-Linear Analysis

The above elastic analysis will be the core solution scheme for the non-linear analysis. A procedure to analyze non-linear concrete decks has already been developed (Ref. 22). The theoretical work completed to date does not preclude the inclusion of that procedure in the yet to be developed analysis scheme.

The previous work dealt with concrete beams where the present work must handle steel beams. In addition to the obvious differences of material behavior, the steel beams present all the problems of thin walled members. Buckling of the various parts of a beam or girder, as well as the effects of residual stresses, are but a few of the known problems which must be investigated.
1.4 Scope of Investigation

This investigation addresses itself to a portion of the required linear elastic analysis procedure. In particular, a finite element is developed which will permit the analysis of bridges while including the effects of slip, shear deformation of the beams and shear lag in the deck. The finite element method has been chosen over other techniques for a variety of reasons already recognized by researchers (Ref. 23,35).

Minor axis bending and torsion effects are not included, however, the present work is formulated so that they can be added without altering the original equations. This can be done because the analysis is limited to linear geometry. Bracing and diaphragms can be added when the structure global equations are formed. Hence, they also will not affect the present formulation.

The formulations are presented in roughly the order in which they were developed. Each formulation is compared to solutions obtained using the SAP IV Program (Ref. 5) and the final formulation is compared to tests as well. The presentation begins with a description of composite beam models constructed from finite elements which are then used for numerical comparisons (Chapter 2). The first new formulations are for two beams fastened together by shear connectors (Chapter 3), then shear deformations are added to the bottom beam (Chapter 4). Finally, the top beam is replaced by the deck (Chapter 5) and recommendations are made to extend the work (Chapter 6).
1.5 Previous Work

Most investigations into the behavior of composite beams are based on either of two papers by N. M. Newmark. The first paper describes a method of analysis for a slab continuous over flexible beams (Ref. 21). The method is a distribution technique and is applicable to simply supported multi-beam bridges. However, the method assumes that the beams provide only vertical support for the slab. An approximation to composite analysis can be made by calculating the beam properties as a composite beam using the transformed section approach common in design. This requires that a judgment be made concerning the effective width of slab acting compositely with the beam. Newmark does not acknowledge it but the effects of shear lag in the slab and slip between the slab and beam have to be weighed in making such a judgment.

Newmark's second paper seeks to explicitly account for the effect of slip (Ref. 29). He derived a differential equation for the axial forces in the component parts of the composite beam. The derivation assumes that the composite beam is composed of two beams, one on top of another, connected by a continuous layer between them which accounts for the shear connectors. The equation is applicable only to determinate single T-beams. Since the slab is assumed to behave as a beam, if it is too wide, again some effective width must be used. This work is usually referred to as Newmark's incomplete or partial interaction theory.
Teraszkiewicz expanded on the partial interaction theory by devising a method to incorporate an effective width (Ref. 30). The solution procedure is iterative. An effective width is assumed and Newmark's equation is solved. The resulting inplane deformations along the beam are imposed on the slab and all the inplane deformations are computed. An effective width is then computed, the author does not say how, and Newmark's equation is solved again using the new effective width. This procedure is repeated until the effective width has converged. Teraszkiewicz also analyzes a two-span continuous T-beam by superimposing simple beam solutions. Neither Newmark nor Teraszkiewicz includes the shear deformations of the beams in their analyses.

Gustafson and Wright present a theory capable of analyzing multi-beam simply supported and continuous bridges using finite elements (Ref. 13). Wegmuller and Kostem rederived the basic formulations and showed favorable comparisons with field tests (Ref. 35). The theory incorporates shear lag but does not include slip or shear deformation in the beams. An attempt was made to incorporate slip into the theory by du Plesis (Ref. 10).

Recently, most research has been aimed at extending the previous elastic theories to include material non-linearities. Newmark's differential equation has been rederived in various discrete forms which allows for non-uniform connector spacing, initial strains, and non-linear material properties. The solution schemes
utilize an incremental load approach.

Algorithms developed by Proctor, Baldwin, Henry and Sweeney at the University of Missouri (Refs. 3, 27) and by Yam and Chapman at Imperial College (Ref. 38) handle the boundary value problem as an initial value problem where the equations are solved by successive approximations. The schemes developed by Dai, Thiruvengadam and Seiss at the University of Illinois (Ref. 9) and by Wu at Lehigh University (Ref. 37) use finite differences. Dai et al. use iteration to obtain a converged solution at each load increment. None of this work relieves the basic inadequacies of Newmark's partial interaction theory.

The finite element formulations have been extended into the non-linear range in a variety of ways. Kostem, Kulicki, Peterson and Wegmuller have done the work at Lehigh University using $J_2$ theory and a new theory for reinforced concrete. The structures analyzed have had $J_2$ beams and deck (Ref. 36), concrete beams and $J_2$ deck (Ref. 19) and concrete beams and deck (Refs. 20, 22, 23). They used both incremental and incremental-iterative techniques but the basic elastic finite element algorithm remains as derived by Wegmuller and Kostem (Ref. 35).

Several state-of-art papers discuss other aspects of composite beams and other theoretical work (Refs. 2, 16, 33). They indicate the lack of general sophisticated methods necessary to analyze steel bridges.
2. COMPOSITE BEAM MODEL

2.1 Introduction

The purpose of this chapter is to present a technique for modeling composite beams using a general purpose finite element program. The technique is used to model the various types of composite beams for which the new finite elements are developed so that numerical comparisons can be made. The reason for using this analytical technique as a basis for comparison rather than relying exclusively on test data is that [at this stage of development, the new formulations are not well suited for comparison to the available data.]

As mentioned earlier, Newmark's partial interaction theory is applicable only to determinate structures. Most researchers investigating composite beams have used this partial interaction theory and compared results to tests of simply supported single T-beams in which the shear lag is minimal. The new methods can handle the shear lag, other end conditions, continuous and multi-beam structures.

Some test data is available for two span continuous T-beams. However, the concrete slab cracks over the interior support very early in the test which results in non-linear behavior. The new methods are valid only for linear elastic analysis and can not predict non-linear behavior at this time.
A test on a steel bridge has been made and a comparison to that test will be presented. However, the analysis using the new methods assumes complete composite action, a valid assumption for the structure. Previously developed finite element theories could have been used to analyze the bridge (Ref. 35) since the effect of shear deformations is small in this case.

In view of all of the above, it appears that no test data is available that can demonstrate the versatility of the new formulations without forcing them to be extended into the non-linear range. Hence, resort was made to the construction of composite beam models from finite elements available in a general purpose program. The validity of the technique used to construct these models is verified via comparisons to Newmark's partial interaction theory and test data.

The ability to provide reliable values for comparison via an analytical technique has some advantages over comparisons made to test data. First of all, the models permit comparisons without confronting the problems of determining the effect of experimental errors. Second, the models allow the effects of slip, beam shear deformations and shear-lag to be separated whereas the experiments can not. Third and last, the models permit various configurations of cross-sections, materials, loads and support conditions. In some instances, properties are used which are outside the realm of practical civil engineering values. This is done to insure that the new
formulations are sound and not tied to structures of any particular material or geometry.

2.2 Finite Elements in the Models

The composite beams are modeled using the general purpose finite element program SAP IV (Ref. 5). The models are constructed from bar, beam and thin plate elements. This section describes the elements used in the models and Section 2.3 describes the models themselves.

The truss element is used to simulate the bars. The truss is formulated using the common linear displacement field for the axial deformation (Ref. 28). The SAP IV beam formulation is presented by Przemieniecki, and it includes deformations due to shear (Ref. 28). Przemieniecki derives the stiffness matrix directly from the differential equations. The shear deformations are handled using the Timoshenko beam approach, i.e. an area effective in shear is used to calculate an average shearing strain. The SAP IV Program is designed to model beams both with and without shearing deformations.

The SAP IV thin plate element is a general quadrilateral capable of simulating both in-plane (plane stress) and out-of-plane (plate bending) deformations. It is composed of four triangles with the internal degrees of freedom condensed out when the element stiffness matrix is formed (Ref. 5, Fig. 4). Each triangle is a constant strain triangle for the in-plane deformations and a Linear Curvature.
Compatible Triangle with nine degrees of freedom (LCCT9) for the out-of-plane deformations. The in-plane and out-of-plane deformations are not coupled.

The nodes are located at the four corners of the quadrilateral at the mid-height of the plate. Each node has five degrees of freedom. They are three displacements and the two out-of-plane rotations (Fig. 4).

Clough and Felippa show numerical comparisons for a quadrilateral element composed of only two LCCT9 elements (Ref. 7). Experience at Lehigh University shows that the SAP IV thin plate element is quite accurate and reliable for static analyses.

2.3 The Model

Three different finite element models are used. The first models two Bernoulli-Navier beams (i.e. plane sections remain plane-no shear deformations), one on top of the other, fastened together with shear connectors (Fig. 5a). (Note: the letter designations for the beams in Figure 5 will be explained in subsequent chapters.) The second adds the deformations due to shear to the bottom beam using the Timoshenko beam approach (Fig. 5b). The third replaces the top beam with a thin plate (Fig. 5c).

Each model is constructed by attaching shear connector linkage assemblies between beam elements (or plate and beam elements) at their centroids (or mid-plane and centroid) (Fig. 6). The shear
connector linkage assemblies are composed of truss and beam elements (Fig. 6a). The shear connector links are truss elements. They simulate the stiffness of the shear connectors which is done by specifying the axial stiffness \( \frac{AE}{L} \) of the link. The rigid links are beam elements which are used to hold the shear connector links at the interface while maintaining plane sections. They cannot be placed perpendicular to the beam elements because the shear connector links must have a finite length. However, they are very nearly perpendicular. The vertical links are very stiff truss elements which are used to force equal vertical deflections of the beams. The composite beam models constructed using these shear connector linkage assemblies will be subsequently referred to as assemblage models (Fig. 6b). Kaldjian reports on a similar model except he used some special elements (Ref. 17).

Care must be exercised when assigning numerical quantities to the elements of the shear connector linkage assemblies and when deciding upon the number of assemblies to use. The following conditions must be satisfied to insure that the analysis is accurate:

1. The linkage assemblies must be spaced close enough so that the solution will converge to that of a fully composite (no slip) beam when the stiffness of the shear connector links is large.

2. The stiffness of the vertical links must be large enough to enforce equal vertical deflections of the beams.
3. The stiffness of the rigid links must be large enough to force the end rotations of each link to be the same.

4. The stiffness of the rigid links and the assembly spacing must be such that the rotations of all four ends of the two rigid links in each assembly are the same. This result is not expected when shear deformations are permitted because the shear strain is not a rotation.

In the above, a stiffness is large when it is greater than the stiffnesses of the beam elements used to model the top and bottom beams that comprise the composite beam. Therefore, the stiffnesses of the various components of the linkage assemblies depend upon the material and geometry of the composite beam to be modeled and the type, size, and spacing of shear connectors. The stiffnesses can not be assigned arbitrarily high values because a numerical instability can occur if the stiffnesses are too high relative to the rest of the structure.

These models are not intended to account for the shear connectors individually, although it could be done. The new formulations presented in subsequent chapters and Newmark's theory assume that the stiffness of the connectors is spread uniformly along the interface. Comparisons with calculations done both ways show that there is no practical difference in the results, at least for the spacing of shear connectors considered. Indeed, Wu points out that when the spacing of the shear connectors is constant, the
formulations accounting for the connectors in a discrete fashion reduce to finite difference equations of Newmark's differential equation (Ref. 37).

2.4 Numerical Verification

Two beams taken from a report by Viest, Siess, Appleton and Newmark are used for numerical comparisons (Ref. 34). They are designated as test beams B24W and B21W in that report. Both are composite with reinforced concrete slabs and rolled shapes. They are simply supported and loaded at midspan. The geometry and structural properties are shown in Fig. 7 and Table 1. Some of the values in Table 1 are not reported in the reference and were taken from the AISC Manual (Ref. 1).

The number of shear connectors required for static strength of both beams were compared to current design practice (Ref. 40). The comparisons show that B24W has one more connector than current practice would recommend while B21W has eight fewer connectors (Table 1). Therefore, B24W could be considered fully composite, while B21W is partially composite.

The assemblage model used to analyze the beams is shown in Fig. 8. Parametric studies show that twenty-six shear connector linkage assemblies together with the properties given in Table 2 are more than adequate to insure an accurate analysis.

The shear area of the bottom beam is taken equal to the area of the web (Table 2). The area of the shear connector links is
calculated by concentrating the uniform stiffness at the links from the equation:

\[
A = \frac{k_{sc} SL_c}{E} \tag{2.1}
\]

where

- \( A \) = area of the shear connector link
- \( k_{sc} \) = uniform shear connector stiffness
- \( S \) = spacing of the shear connector linkage assemblies
- \( L_c \) = length of the shear connector links
- \( E \) = Young's modulus of the shear connector link

Plots of midspan load vs. midspan deflection are shown in Figs. 9 and 10. In both cases Newmark's theory and the assemblage analyses show excellent agreement for the case when shear deformations are not included. Both are somewhat stiffer than the test results. When the shear deformations are included, Newmark's theory predicts somewhat more deflection than the assemblage model and both are in good agreement with the tests. A discussion of why the two methods do not predict the same deflections when shear deformations are permitted will be included in Chapter 5.

The numerical comparisons show that the assemblage model can predict the response of a composite beam with partial interaction quite accurately. This technique is used to model composite beams of varying cross-sections, support conditions and load. The numerical results from the new finite elements presented in Chapters 3, 4 and 5 will be compared to results obtained using the assemblage model.
3. BERNOULLI-NAVIER COMPOSITE BEAM ELEMENTS

3.1 Introduction

This chapter presents the formulations of the stiffness matrix for the first of three different types of finite elements for composite beam analysis (Reference Section 1.4). This first type is capable of modeling two Bernoulli-Navier beams, one on top of the other and fastened together at their interface by shear connectors (Fig. 5a). Three variations of this type of element are presented. Only one formulation, which will be expanded in subsequent chapters, is presented in detail. The details of all three formulations are shown in Ref. 31.

3.2 The Finite Element Method

The analytical technique chosen to establish the global equations is the displacement-based finite element method which leads directly to the familiar set of node equilibrium equations (Ref. 39):

\[ \{F\} = [K] \{\delta\} \] (3.1)

where

- \( \{F\} \) = vector of applied generalized forces at the nodes
- \( [K] \) = structure stiffness matrix
- \( \{\delta\} \) = vector of generalized displacements at the nodes.
The structure stiffness matrix is obtained by stacking a null matrix with the element stiffness matrices:

\[ [K] = \Sigma [k]^e \]  

(3.2)

where the summation is over all the elements in the structure and \( [k]^e \) = element stiffness matrix for element \( i \).

This displacement-based method (Eqs. 3.1 and 3.2) is the most common finite element scheme in use today and it has shown itself to be a very reliable tool in previous research (Refs. 19, 20, 22, 23, 24, 35, 36).

This dissertation is primarily concerned with the development of the element stiffness matrix in the element equilibrium equations:

\[ \{F\}^e = [k]^e \{\delta\}^e \]  

(3.3)

where \( \{F\}^e \) = vector of applied generalized forces at the element nodes

\( \{\delta\}^e \) = vector of generalized displacements at the element nodes.

In order to take maximum advantage of techniques developed in previous research, all of the finite elements derived in this dissertation are generalized coordinate elements (Refs. 20, 22, 23, 24).
The procedure used to formulate generalized coordinate finite elements can be divided into two parts. The first part is the derivation of shape functions from assumed displacement fields; the second is the derivation of the stiffness matrix from the shape functions. The second part is generally available in the literature but it will be presented in this section to further define terminology and to clarify assumptions. The first part is presented in the next section which describes in some detail the methods used to derive the shape functions from the displacement fields. However, the detailed formulations progress by assuming displacement fields, arriving at shape functions and then formulating the stiffness matrix.

The following formulation is presented by Zienkiewicz and it begins with shape functions for the displacement fields within the element as a function of the element node displacements (Ref. 39):

\[ \{f\} = [N] \{\delta\}^e \]  

(3.4)

where \( \{f\} \) = displacement field of the element  
\([N]\) = shape functions.

The shape functions are usually taken in the form of polynomials but the theory does not preclude the use of other functions.
Internal strains can be determined from the displacement fields by differentiation:

\[ \{\varepsilon\} = [B] \{\delta\}^e \]  

(3.5)

where \( \{\varepsilon\} \) = vector of generalized strains

\([B]\) = strain-displacement matrix.

Assuming no initial strains or initial stresses, the stresses can be obtained by using the appropriate constitutive relations:

\[ \{\sigma\} = [D] \{\varepsilon\} \]  

(3.6)

where \( \{\sigma\} \) = vector of generalized stresses

\([D]\) = stress-strain matrix for the material, sometimes referred to as the elasticity matrix.

The application of virtual work to the element results in the formation of the element stiffness matrix:

\[ [k]^e = \int_V \sigma \cdot \varepsilon \, dV \]  

(3.7a)

which can be rewritten in matrix form as:

\[ [k]^e = \int_V [B]^T [D][B] \, dV \]  

(3.7b)

This matrix results from the consideration of the internal work over the volume. The generalized force vector in Eq. 3.3 comes from considering the external work.
The successful element formulation is the one that uses the correct shape function to model the desired phenomena and considers all the appropriate internal work terms consistent with the shape functions.

3.3 Shape Functions

The shape functions are derived from assumed displacement fields. The composite beam is shown in Fig. 11. In order to preserve generality and reduce future computational effort, the displacement fields of the element and the node displacements are written at a reference plane which is parallel to, but at an arbitrary distance from the beam. This technique is valid as long as the strain-displacement matrix relates the displacements at the reference plane to the strains within the element. The displacements of the beam written at the reference plane are taken as polynomials. The following procedure is adapted from material presented by Peterson and Kostem and begins with the assumed displacement fields (Ref. 24):

\[ [f] = [P(x)] [\alpha] \quad (3.8) \]

where \([P(x)] = \text{functions of } x \text{ used to describe the shape of the displacement fields}

\([\alpha] = \text{vector of the coefficients in the displacement functions.}\)
The coefficients of the polynomials are defined in terms of the element node displacements via compatibility, i.e., at the nodes the internal displacement function is forced to be compatible with the node displacements. It is usual in this type of formulation to have the number of coefficients exactly equal to the number of node displacements, thereby allowing all the coefficients to be defined by the compatibility equations.

In the present formulations there are more coefficients than node displacements, so additional equations must be established. One way to obtain the additional equations is to introduce internal nodes with enough degrees of freedom to permit the compatibility equations to completely define the coefficients. These degrees of freedom are usually condensed out of the resulting stiffness matrix. Another way, which is not common, is to relate the polynomials directly to one another via equilibrium and/or compatibility equations without introducing any extra degrees of freedom. The present formulations employ this latter approach, hence equations can be written:

\[
\begin{bmatrix}
\{\delta\}^e \\
\{\alpha\}
\end{bmatrix} =
C
\begin{bmatrix}
\{\alpha\}
\end{bmatrix}
= 
\begin{bmatrix}
C_1 \\
C_2
\end{bmatrix}
\begin{bmatrix}
\alpha
\end{bmatrix}
\]

(3.9)

where \(\{\alpha\} = \text{vector of zeroes representing the left hand side of the equilibrium and/or compatibility equations}\)
\[
\begin{bmatrix}
C1
\end{bmatrix} = \text{matrix consisting of } P(x), \text{ evaluated at the appropriate nodes}
\]
\[
\begin{bmatrix}
C2
\end{bmatrix} = \text{coefficients of the equilibrium and/or compatibility equations}
\]

Inverting \([C]\) to solve for \([\alpha]\):

\[
[\alpha] = [C]^{-1} \begin{bmatrix}
[\delta]^e \\
[\alpha]
\end{bmatrix}
\]

and

\[
[\alpha] = [CC][\delta]^e
\]

where

\[
[CC] = \text{coefficient-displacement matrix consisting of the first } n \text{ columns of } [C]^{-1} \text{ where } n \text{ is the number of displacements in the vector } [\delta]^e.
\]

Therefore, the shape function is defined:

\[
[f] = [P(x)][\alpha] = [P(x)][CC][\delta]^e = [N][\delta]^e
\]

When performing the detailed derivations, the shape functions are not explicitly formed because they are inconvenient to use. The matrix \([CC]\) is not a function of \(x\) and, therefore, all derivatives with respect to \(x\) can be performed on \([P(x)]\) exclusively. All of the strains are functions of \(x\) only, therefore, only \([P(x)]\) will be differentiated. The operators necessary to define strains from the displacement fields will be
called $[\Gamma]$, hence:

$$[\varepsilon] = [\Gamma][f] = [\Gamma][P(x)][\alpha] = [Q][\alpha]$$  \hspace{1cm} (3.13a)

and now substituting for $[\alpha]$

$$[\varepsilon] = [Q][G][\delta] = [B][\delta]$$  \hspace{1cm} (3.13b)

This strain-displacement matrix and the stress-strain matrix can be substituted into Eq. 3.7b to formulate the element stiffness matrix.

3.4 Formulation of Element-I

The stiffness matrix developed in this section will be referred to as Element-I. The composite element consists of two Bernoulli-Navier beams, one on top of the other, fastened together at their interface by shear connectors (Fig. 5a). The upper beam is designated beam A and the lower one is B. The two acting together is the composite beam finite element.

The element, together with the degrees of freedom and sign conventions, is shown in Fig. 11. It is defined by two nodes each with four degrees of freedom. They are an axial displacement for each of the two beams, a vertical displacement, and a rotation. The position of the reference plane is arbitrary as long as it is parallel to the element. The quantities $Z_A$, $Z_B$, and $Z_{BB}$ are vector quantities, therefore, should the reference plane be located below the beams, they would be negative (Fig. 11).
All of the deformations, i.e., the node displacements, and the displacement fields are written at the reference plane.

All of the ordinary beam theory assumptions regarding plane sections apply, that is:

(a) no torsion or minor axis bending
(b) material homogeneity and isotropy
(c) prismatic geometry
(d) no shear deformations
(e) no initial curvature
(f) Hooke's Law

The shear connectors in the physical beam are at discrete points. In this mathematical model, the shear connectors are assumed to provide a uniform connection along the length. The details of the following formulation are given in Ref. 31.

The transverse displacement field (W) for both beams is taken as the common cubic polynomial for beams:

\[ W = c_1 + c_2 x + c_3 x^2 + c_4 x^3 \]  \hspace{1cm} (3.14a)

Assuming the same W-field for both beams does not permit separation between them and it forces the curvatures of both beams to be equal.

The axial displacement fields (U) are assumed to be, for beam A:

\[ U_A = a_1 + a_2 x + a_3 x^2 \]  \hspace{1cm} (3.14b)
and for beam B:

\[ U_B = b_1 + b_2 x + b_3 x^2 \]  \hspace{1cm} (3.14c)

Assuming separate axial displacement fields for each of the beams permits relative horizontal movement (slip) between them. The order of the axial displacement polynomials is dictated by the fact that they must be on the order of the first derivative of the transverse displacement polynomial. Putting the above equations into the matrix form of Eq. 3.8:

\[
\begin{pmatrix}
U_A \\
U_B \\
W
\end{pmatrix} =
\begin{bmatrix}
1 & x^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & x^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & x^2 & x^3
\end{bmatrix}
\begin{pmatrix}
\alpha
\end{pmatrix}
\]

(3.15)

where

\[ \{\alpha\}^T = \begin{bmatrix} a_1 & a_2 & a_3 & b_1 & b_2 & b_3 & c_1 & c_2 & c_3 & c_4 \end{bmatrix} \]

Enforcing compatibility of the displacement fields with the node displacements and noting that because of the sign convention:

\[ \theta = -\frac{dW}{dx} \]

at the nodes, the [C1] matrix of Eq. 3.9 is:

\[ \{\delta\}^e = [C1] \{\alpha\} \]
There are eight node displacements and ten coefficients, therefore two more equations must be established to define the coefficients. Considering the equilibrium of the axial forces and the interface shear flow results in the equations (Fig. 12):

\[
\frac{dN_A}{dx} = +s \quad (3.17a)
\]

and

\[
\frac{dN_B}{dx} = -s \quad (3.17b)
\]
where

\[ N_A = \text{axial force in beam A} \]
\[ N_B = \text{axial force in beam B} \]
\[ s = \text{interface shear flow} \]

Setting the interface shear flow equal to zero yields two separate equations. In order for the beams to behave compositely, the shear flow obviously cannot be zero. However, the polynomials do not have to satisfy equilibrium to yield acceptable results, although theoretically it would be preferable (Ref. 8). This point will be examined further in Section 3.5.

Equations 3.17 must be recast in terms of the polynomial coefficients. Expressing the axial deformation fields at the reference plane for beam A:

\[ U_{\text{in beam A}} = U_A - z \frac{dW}{dx} \tag{3.18} \]

The axial strain in the beam becomes:

\[ \varepsilon_{xA} = \frac{d(U_{\text{in beam A}})}{dx} = \frac{dU_A}{dx} - z \frac{d^2W}{dx^2} \tag{3.19} \]

Since the axial stress \( \sigma_{xA} = E_A \varepsilon_{xA} \), where \( E_A \) is Young's Modulus for beam A, and

\[ N_A = \int_A \sigma_{xA} \, dA, \text{ then:} \]

\[ -31- \]
\[
\frac{dN_A}{dx} = \int \frac{d \sigma_{xA}}{dx} \, dA = E_A \int \frac{d \varepsilon_{xA}}{dx} \, dA
\]  
(3.20a)

Substituting Eq. 3.19 for \( \varepsilon_{xA} \):

\[
\frac{dN_A}{dx} = E_A \int \left( \frac{d^2 U_A}{dx^2} - Z \frac{d^3 W}{dx^3} \right) \, dA
\]  
(3.20b)

Or, in terms of the polynomial coefficients:

\[
\frac{dN_A}{dx} = E_A \int (2a_3 - 26c_4) \, dA
\]  
(3.20c)

Performing the indicated integration:

\[
\frac{dN_A}{dx} = 2EA_A \, a_3 - 6ES_A \, c_4
\]  
(3.20d)

where

\( EA_A = E_A \times \text{area of beam } A \)

and

\( ES_A = E_A \times \text{first moment of inertia of beam } A \)

with respect to the reference plane.

Setting \( \frac{dN_A}{dx} \) equal to zero yields the first equation:

\[
0 = 2EA_A \, a_3 - 6ES_A \, c_4
\]  
(3.21a)
and similarly for beam B:

\[ 0 = 2EA_B \text{ } b_3 - 6ES_B \text{ } c_4 \]  

(3.21b)

Casting Eqs. 3.21 in matrix form yields the \([C2]\) matrix of Eq. 3.9:

\[ \{0\} = [C2] \{\alpha\} \]

therefore:

\[
\begin{pmatrix}
0 \\
0
\end{pmatrix} = \begin{bmatrix}
0 & 0 & 2EA_a & 0 & 0 & 0 & 0 & 0 & -6ES_A \\
0 & 0 & 0 & 0 & 2EA_B & 0 & 0 & 0 & -6ES_B
\end{bmatrix} \begin{pmatrix}
\alpha
\end{pmatrix}
\]

(3.22)

Combining Eqs. 3.16 and 3.22 yields the \([C]\) matrix of Eq. 3.9 which can then be solved to give the \([CC]\) matrix indicated in Eq. 3.11 where \([CC]\) consists of the first eight columns of \([C]^{-1}\).

It is convenient to consider the displacement fields as separate sets of equations, hence \([CC]\) is partitioned:

\[
[CC] = \begin{bmatrix}
CA \\
CB \\
CW
\end{bmatrix}
\]
where $[CA]$, $[CB]$ and $[CW]$ are the coefficient-displacement matrices for the $U_A$, $U_B$ and $W$ fields respectively.

The internal work of the element consists of two separate components. The first is the work done due to axial stresses and strains in each of the two beams; the second is the work due to the shear flow and slip at the interface. These work components are not coupled. Therefore, the stiffness matrix can be obtained by the consideration of each of these effects separately, hence:

$$[k]^e = [k]_b + [k]_u \quad (3.23)$$

where

$$[k]_b = \text{portion of the element stiffness matrix resulting from the consideration of the internal work due to axial stresses and strains}$$

$$[k]_u = \text{portion of the element stiffness matrix resulting from the consideration of the internal work due to shear flow and slip}$$

$$[k]^e = \text{element stiffness matrix in Eq. 3.3}$$

$$[F]^e = [k]^e \{\delta\}^e \quad (3.3)$$

$\{\delta\}^e$ is shown in Eq. 3.16

and

$$[F]^e = \begin{bmatrix} F_{LA} & F_{LB} & V_L & M_L & F_{MA} & F_{MB} & V_M & M_M \end{bmatrix}$$
where $F$, $V$ and $M$ are axial forces, shears and moments associated with the node displacements.

Considering the effects of axial components first, the strain-displacement matrix will be derived. Performing the operations indicated by $[\Gamma]$ in Eq. 3.13 on the internal fields in accordance with Eq. 3.19:

$$
\begin{bmatrix}
\varepsilon_{xA} \\
\varepsilon_{xB}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 2x \\
[CA] - Z[0 & 0 & 6x][CW]
\end{bmatrix}
\begin{bmatrix}
\delta^e \\
[B]_b \delta^e
\end{bmatrix}
$$

where

$$[B]_b = \text{the strain-displacement matrix for axial strains.}$$

Taking the stress-strain relations simply as:

$$
\begin{bmatrix}
\sigma_{xA} \\
\sigma_{xB}
\end{bmatrix} =
\begin{bmatrix}
E_A & 0 \\
0 & E_B
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{xA} \\
\varepsilon_{xB}
\end{bmatrix} = [D][\varepsilon]
$$

and performing the integral as in Eq. 3.7:

$$
[k]_b = \int_V [B]_b^T [D][B]_b \ dV
$$
results in:

\[
[k]_b = \begin{bmatrix}
\frac{E_A}{L} & 0 & 0 & \frac{12 \Sigma EI}{L^3} \\
0 & \frac{E_B}{L} & 0 & \frac{-6 \Sigma EI}{L^2} \\
\frac{E_A}{L} & \frac{E_B}{L} & \frac{-6 \Sigma EI}{L^2} & G_1 \\
0 & 0 & \frac{-12 \Sigma EI}{L^3} & \frac{6 \Sigma EI}{L^2} \\
\frac{-E_A}{L} & 0 & \frac{-E_B}{L} & \frac{EA_B}{L} \\
0 & \frac{-E_A}{L} & \frac{-E_B}{L} & 0 \\
0 & 0 & \frac{-6 \Sigma EI}{L^2} & \frac{ES_A}{L} \\
\frac{-6 \Sigma EI}{L^2} & \frac{ES_B}{L} & \frac{6 \Sigma EI}{L^2} & G_1 \\
\end{bmatrix}
\]

where

\[
(3.27)
\]

\[
\Sigma EI = EI_{OA} + EI_{OB}
\]

\[
EI_{OA} = E_A \times \text{second moment of inertia of beam A about its own centroidal axis}
\]

\[
EI_{OB} = E_B \times \text{second moment of inertia of beam B about its own centroidal axis}
\]
\[ G_1 = \frac{4}{L} \sum \frac{EI}{L} + \sum \frac{EA}{L} z^2 \]

\[ \Sigma EA z^2 = EA_A z_A^2 + EA_B z_B^2 \]

\[ H_1 = \frac{2}{L} \sum \frac{EI}{L} - \sum \frac{EA}{L} z^2. \]

An examination of this matrix reveals that \([k]_b\) is simply the sum of the stiffness matrices of the two beams without any composite action. Considering the work at the interface will provide the matrix to make the element composite.

The evaluation of the work done on the connectors at the interface requires that the integral be performed over the length of the element. Therefore, recasting Eq. 3.7a (Fig. 13):

\[ [k]_u = \int s \delta U \, dx \quad (3.28) \]

where \(\delta U\) = the slip between the beams.

The force-displacement relation at the interface is:

\[ s = k_{sc} \delta U \quad (3.29) \]

where

\[ k_{sc} \] = the stiffness of the uniform connection used to mathematically describe the shear connectors, therefore:
\[ k_{sc} = \frac{\sum k_c}{L} \]

where \( k_c \) = the stiffness of the individual shear connectors.

The summation is over all the connectors along the length of the element.

Expressing the slip in terms of the node displacements and noting that the slope of both beams is the same:

\[ \delta U = \left( U_A - Z_1 \frac{dW}{dx} \right) - \left( U_B - Z_1 \frac{dW}{dx} \right) = U_A - U_B \]

Expressed in terms of the polynomial coefficients:

\[ \delta U = [1 \times x^2 - 1 - x - x^2 0 0 0 0][\alpha] \]

or from Eq. 3.11:

\[ \delta U = [XU][\alpha] = [XU][CC][\delta]^e = [B]_u \{\delta\}^e \]

where \([B]_u\) = slip-displacement matrix.

Forming the expression for the internal work due to the shear flow and slip from Eq. 3.28 yields:

\[ [k]_u = \int_L \left[ B\right]_u^T k_{sc} [B]_u \, dx \]
Performing the integral gives:

\[
[k]_u = k_{sc}
\]

\[
\begin{bmatrix}
\frac{L}{3} & \frac{-L}{3} & 12d^2 & \frac{12d^2}{10L} \\
\frac{d}{2} & \frac{-d}{2} & \frac{3d^2}{5} & \frac{3d^2L}{10} \\
\frac{-dL}{4} & \frac{dL}{4} & \frac{-3d^2}{5} & \frac{3d^2L}{10} \\
\end{bmatrix}
\]

Adding Eq. 3.27 and 3.32 as indicated in Eq. 3.23 results in the element stiffness matrix for Element-I.

The recovery of internal stresses can be accomplished directly from equilibrium or by establishing the stress-displacement matrix in the conventional manner (Ref. 39).
3.5 Formulation of Elements-II and -III

The formulations for the two variations of Element-I are presented here in outline form; the details are available in Ref. 31. The first variation will be called Element-II and involves changing the polynomial constraining equations (Eq. 3.22). The second will be designated Element-III and involves assuming higher order polynomials for the displacement fields and a subsequent expansion of the polynomial constraining equations.

The derivation for Element-II is the same as for Element-I up to and including Eqs. 3.16. The first of the two polynomial constraining equations requires a reconsideration of the equilibrium at the interface. Rather than setting the interface shear flow equal to zero, Eqs. 3.17 can be added to yield:

\[ 0 = \frac{dN_A}{dx} + \frac{dN_B}{dx} \]  
(3.33a)

or in terms of the polynomial coefficients:

\[ 0 = 2EA_a a_3 - 6ES_A c_4 + 2EA_b b_3 - 6ES_B c_4 \]  
(3.33b)

The second constraining equation is obtained by considering the force-displacement relation of the connectors. Rearranging Eq. 3.29:

\[ 0 = k_{sc} \delta U - s \]  
(3.34a)
and substituting for $\delta U$ from Eq. 3.30a and for $s$ from Eq. 3.17a yields:

$$0 = k_{sc} (U_A - U_B) - \frac{dN_A}{dx}$$  \hspace{1cm} (3.34b)

The closed form stiffness matrix has not been derived. The formulations are carried out numerically starting with the $[C]$ matrix of Eq. 3.9; therefore, to avoid the possibility of numerically biasing the solution toward one beam in the element a substitution for $\frac{dN_A}{dx}$ is made. Rearranging Eq. 3.34b:

$$0 = k_{sc} (U_A - U_B) - \frac{1}{2} \left( \frac{dN_A}{dx} + \frac{dN_A}{dx} \right)$$  \hspace{1cm} (3.34c)

and substituting for one $\frac{dN_A}{dx}$ term from Eq. 3.33a yields:

$$0 = k_{sc} (U_A - U_B) - \frac{1}{2} \left( \frac{dN_A}{dx} - \frac{dN_B}{dx} \right)$$  \hspace{1cm} (3.34d)

Substituting Eqs. 3.14b and 3.14c for the axial displacements and Eq. 3.20d for $\frac{dN_A}{dx}$ and its counterpart for $\frac{dN_B}{dx}$ yields the force-displacement relation in terms of the polynomial coefficients and $x$:

$$0 = k_{sc} a_1 + k_{sc} x a_2 + (k_{sc} x^2 - EA_a) a_3$$

$$- k_{sc} b_1 - k_{sc} x b_2 - (k_{sc} x^2 - EA_B) b_3$$

$$+ 3 (EA_A - EA_B) c_4$$ \hspace{1cm} (3.34e)
Enforcing this condition for all \( x \) would yield too many constraining equations. Therefore, the force-displacement equation is enforced at the midlength of the element \( (x = \frac{L}{2}) \):

\[
0 = k_{sc} a_1 + k_{sc} \frac{L}{2} a_2 + (k_{sc} \frac{L^2}{4} - E_A) a_3
- k_{sc} b_1 - k_{sc} \frac{L}{2} b_2 - (k_{sc} \frac{L^2}{4} - E_B) b_3
+ 3 (E_{SA} - E_{SB}) c_4 \tag{3.34f}
\]

The new constraining equations (Eqs. 3.33b and 3.34f) are now recast in matrix form as in Eq. 3.22 and the formulation proceeds exactly as in Element-I.

Physically, Eqs. 3.33 permit the interface shear to assume any value it wants to, while Eqs. 3.21 force it to zero. In view of this refinement, together with the fact that the shear connector force-displacement relation is also enforced, it would appear that Element-II is superior to Element-I. However, the restrictions on the interface shear are at the lowest level in the formulations, i.e., the polynomials. The formulation for Element-I waits until the formation of the internal work to account for the fact that there is a connection and a shear between the beams. The numerical comparisons at the end of this chapter will show that Element-II is no more accurate than Element-I. Finally, Element-I has advantages in that the matrices are uncoupled. This point will be further explained in Chapter 5.
The formulation for an Element-III was prompted by an examination of the analyses using the assemblage models which showed that the interface shear flow was not constant along the length, nor was it linear. The assumption was made that the shear flow \( s = fcn(x^2) \) where the notation \( fcn \) indicates that \( s \) is a function of \( x^2 \). Since \( s = \frac{dN}{dx} \) then \( N = fcn(x^3) \) and since \( N = fcn(\frac{dU}{dx}) \) then \( U = fcn(x^4) \) which implies that \( W = fcn(x^5) \). Hence, the following polynomials for the displacement fields were assumed:

\[
U_A = a_1 + a_2 x + a_3 x^2 + a_4 x^3 + a_5 x^4 \quad (3.35a)
\]

\[
U_B = b_1 + b_2 x + b_3 x^2 + b_4 x^3 + b_5 x^4 \quad (3.35b)
\]

\[
W = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4 + c_6 x^5 \quad (3.35c)
\]

Solving for the coefficients in terms of the node displacements requires 16 equations. Eight equations are provided by the compatibility equations which are similar to Eqs. 3.16. Enforcing the equilibrium condition (Eq. 3.33a) for all \( x \) results in three equations. Finally, enforcing the force-displacement relation (Eq. 3.34a) for all \( x \) yields five more equations for a total of sixteen. The derivations proceed in a manner similar to that for Element-I except that the matrices are larger to account for the higher powers of \( x \).
3.6 Limitations

An examination of the stiffness matrix for Element-I reveals that as the shear connector stiffness $k_{sc}$ increases the element gets stiffer. If no upper bound is placed on $k_{sc}$, the elements of the $[k]_u$ matrix (Eq. 3.32) can get large enough to mask the contribution of the $[k]_b$ matrix (Eq. 3.27). Numerical tests show that $k_{sc}$ can get large enough to render useless results and, in some cases, numerical problems result which prevent the completion of the analysis.

The upper bound for $k_{sc}$ is that quantity that makes the stiffness matrix of Element-I equivalent to the stiffness matrix for a fully composite beam. The fully composite stiffness matrix is derived in Ref. 31 using principles and techniques similar to those used to derive the matrix for Element-I. It is Fig. 11 with $U_L = U_{LA} = U_{LB}$ and $U_M = U_{MA} = U_{MB}$.
\[ \begin{align*}
M_L &= \begin{bmatrix}
\frac{E_A}{L} & 0 & -6 \frac{E_I}{L^2} & G_2 \\
0 & \frac{6 E_I}{L^3} & 12 \frac{E_I}{L^3} & 0 \\
-6 \frac{E_I}{L^2} & 12 \frac{E_I}{L^3} & \frac{H_2}{L} & \frac{6 E_I}{L^2} & G_2 \\
\end{bmatrix}
\end{align*} \]

\[ \begin{align*}
F_L &= \begin{bmatrix}
\frac{E_A}{L} \\
0 \\
-\frac{E_A}{L} \\
-\frac{E_A}{L} \\
\end{bmatrix}
\end{align*} \]

\[ \begin{align*}
V_L &= \begin{bmatrix}
0 & 12 \frac{E_I}{L^3} \\
-6 \frac{E_I}{L^2} & \frac{H_2}{L} \\
12 \frac{E_I}{L^3} & \frac{6 E_I}{L^2} \\
0 & \frac{6 E_I}{L^2} & G_2 \\
\end{bmatrix}
\end{align*} \]

\[ \begin{align*}
M_M &= \begin{bmatrix}
\frac{E_A}{L} & 0 & -6 \frac{E_I}{L^2} & G_2 \\
0 & \frac{6 E_I}{L^3} & 12 \frac{E_I}{L^3} & 0 \\
-6 \frac{E_I}{L^2} & 12 \frac{E_I}{L^3} & \frac{H_2}{L} & \frac{6 E_I}{L^2} & G_2 \\
\end{bmatrix}
\end{align*} \]

where

\[ \begin{align*}
E_A &= E_{A_1} + E_{A_2} \\
E_S &= E_{S_1} + E_{S_2} \\
E_I &= E_{I_{OA}} + E_{I_{OB}} + E_A (Z_A - e)^2 + E_B (Z_B - e)^2 \\
\end{align*} \]
Equating stiffness coefficients of the Element-I matrix to those of Eq. 3.36 for the V-W relation, the M-θ relation and the V-θ relation all result in the same expression for an upper bound to \( k_{sc} \):

\[
k_{\text{max}} = \frac{10}{d^2L^2} (EA_A (Z_A - e)^2 + EA_B (Z_B - e)^2)
\]  

(3.37)

where

\( k_{\text{max}} \) = largest numerical value that should be permitted for \( k_{sc} \).

Numerical tests on Elements II and III have not uncovered any difficulties with the value of \( k_{sc} \). They behave well for a range of values from zero to values much greater than that given by Eq. 3.37.

3.7 Numerical Comparisons

Numerical comparisons for the deflection at the load are shown in Tables 3 to 14 for four different beams each with
three different cross sections. Three values of shear connector stiffness are used for each cross-section. The lower value approximates no composite action and the upper value approximates full composite action. The deflections are given for analyses using Elements I, II and III and the assemblage model and/or beam theory.

The models using Elements I, II and III were all constructed using four elements along the length. The assemblage models were constructed using twenty-six shear connector linkage assemblies. Some deflections for the assemblage models had to be approximated because a node was not placed at the midspan. The midspan deflections were approximated using the node deflections on both sides of the midspan.

The deflections predicted by all of the new elements compare favorably with the deflections predicted by the assemblage models and/or beam theory. Where \( k_{sc} = 25 \text{ MN/m}^2 \) the deflections are equal to or less than those given by the beam theory. Since the beam theory assumes no composite action, the results predicted by the new formulations are on the correct side. When \( k_{sc} = 250,000 \text{ MN/m}^2 \) Elements II and III and the assemblage model predict deflections that are somewhat higher than those predicted by the beam theory. Since the beam theory assumes full composite action, a model that uses a finite shear connector stiffness should be more flexible. On the other hand, Element-I
is usually stiffer than the beam theory. This is because the $k_{sc}$ values exceed the value necessary to make Element-I fully composite (Eq. 3.37). However, the model is not particularly sensitive to the value for $k_{max}$. For example, the $k_{max}$ exceeded by 56% for the configuration shown in Tables 3 to 6 and yet the deflections still compare favorably with the beam theory prediction.

Examples were analyzed using Element-I and limiting the $k_{sc}$ to that value given by Eq. 3.37. Tables 15 and 16 show the results for these models using 1, 2 and 4 elements. In all cases, when the value of $k_{sc}$ was lowered to $k_{max}$ the deflection increased to values which are extremely close to those predicted by the beam theory.

Several examples using Elements I, II and III were also analyzed where the reference plane location was changed. These analyses show that the results are the same (to five decimal places printed) for all reference plane locations.

In summary, the following conclusions can be drawn from the numerical data:

- Elements I, II and III all predict deflections which compare favorably with the assemblage models and/or beam theory. This verifies the procedures and assumptions used in their formulations.
• Elements II and III are no more accurate than Element I.
• The value of $k_{\max}$ from Eq. 3.37 is valid for the fully composite analysis. However, even if $k_{\max}$ is exceeded by a modest amount, the results are still good.
• The results are insensitive to the reference plane location.

The concepts presented in this chapter will be expanded and additional concepts will be introduced in subsequent chapters to formulate composite elements that can account for shear deformations in beam B and shear lag in the deck. Element-I was chosen as the basis for these subsequent elements for reasons that will be explained in Chapter 5.
4. BERNOULLI-NAVIER AND TIMOSHENKO

COMPOSITE BEAM ELEMENT

4.1 Introduction

This chapter presents the formulation for an element which accounts for the shear deformation of beam B in Element-I (Fig. 5b). Conventional techniques for incorporating shear deformation, i.e., forming a flexibility matrix and transforming it to a stiffness matrix (Ref. 12) proved unsatisfactory because the effects on beam A and the slip could not be properly included. Therefore, the assumed polynomial approach was again used. In order to establish some basic principles which will be used in the composite beam element formulation an ordinary Timoshenko beam element is derived.

4.2 Timoshenko Beam Finite Element

The intent of this formulation is to establish principles relating to transverse deformations, therefore, the axial deformations are omitted and the reference plane is fixed at the centroidal axis of the beam. The element is defined by two nodes each with two degrees of freedom (Fig. 14a). The only restriction removed from the Bernoulli-Navier beam theory assumptions given in Section 3.4 is that shear deformation is permitted. The
formulation follows the outline presented in Sections 3.2 and 3.3 and the details are provided in Ref. 31.

The cubic polynomial is again assumed for the transverse displacement ($W$):

$$ W = c_1 + c_2x + c_3x^2 + c_4x^3 \quad (4.1) $$

The total transverse displacement is assumed to be composed of two components (1) the displacement due to bending and (2) the displacement due to shear. In order to include shear deformations, the rotation field due to bending ($\Theta$) must be separated from the transverse displacement field because the slope of the deformed beam ($\frac{dW}{dx}$) is not equal to the rotation ($\Theta$) (Fig. 15). The transverse displacement due to rotation is on the order of the integral of the rotation field with respect to $x$ and the transverse displacement due to shear is a linear function of $x$. Therefore, the rotation field polynomial is assumed to be the same order as the first derivative of $W$.

$$ \Theta = d_1 + d_2x + d_3x^2 \quad (4.2) $$

Enforcing compatibility of the displacement fields with displacements at the nodes yields the $[C1]$ matrix of Eq. 3.9:

$$ [\delta]^e = [C1][\alpha] $$
or
\[
\begin{bmatrix}
W_L \\
\theta_L \\
W_M \\
\theta_M \\
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
1 & L & L^2 & L^3 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -L & -L^2 \\
\end{bmatrix} \{\alpha\} \quad (4.3)
\]

where
\[
\{\alpha\}^T = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 & d_1 & d_2 & d_3 \end{bmatrix}
\]

Three more equations are necessary to solve for the coefficients. They can be derived by enforcing compatibility between the displacement fields. First, the shear strain (\(\gamma\)) must be expressed in terms of the polynomial coefficients. Equilibrium requires (Fig. 14b):
\[
- \frac{dM}{dx} = V \quad (4.4)
\]

where
\[
M = \text{the moment in the beam} \\
V = \text{the shear in the beam}
\]

Expressing the moment in terms of the rotation field:
\[
M = EI \frac{d\theta}{dx} \quad (4.5)
\]

therefore, from Eq. 4.4:
\[
- EI \frac{d^2\theta}{dx^2} = V \quad (4.6)
\]
The average shear strain at any section is:

\[ \gamma = \frac{V}{A_s G} \]  \hspace{1cm} (4.7)

where

\[ A_s = \text{a shear area calculated such that the work} \]

\[ \text{done by the shear moving through a displacement } \gamma \, dx \text{ is equal to the work done} \]

\[ \text{by the actual shear stresses moving through the actual displacements.} \]

and

\[ G = \text{shear modulus of the material} \]

Substituting Eq. 4.6 into Eq. 4.7 results in an equation for the shear strain in terms of the rotation field:

\[ \gamma = -\frac{EI}{A_s G} \frac{d^2 \theta}{dx^2} \]  \hspace{1cm} (4.8)

Compatibility between the displacement fields requires that the slope of the transverse displacement field equals the rotation plus the shear strain (Fig. 15):

\[ 0 = -\frac{dW}{dx} + \theta + \gamma \]  \hspace{1cm} (4.9a)

Or, in terms of the polynomial coefficients:

\[ 0 = -c_2 - 2c_3 x - 3c_4 x^2 + d_1 + d_2 x + (x^2 - \frac{2EI}{A_s G}) d_3 \]  \hspace{1cm} (4.9b)
Enforcing Eq. 4.9b for all x yields the set of equations:

\[ 0 = -c_2 + d_1 - \frac{2EI}{A_sG}d_3 \]  
(4.10a)

\[ 0 = -2c_3 + d_2 \]  
(4.10b)

\[ 0 = -3c_4 + d_3 \]  
(4.10c)

Casting Eqs. 4.10 in matrix form gives the \([C2]\) matrix of Eq. 3.9:

\[
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} =
\begin{bmatrix}
0 & -1 & 0 & 0 & 1 & 0 & \frac{2EI}{A_sG} \\
0 & 0 & -2 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -3 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\alpha
\end{bmatrix}
\]  
(4.11)

Combining Eqs. 4.3 and 4.11 yields the \([C]\) matrix of Eq. 3.9 which is solved to give the \([CC]\) matrix, where \([CC]\) consists of the first four columns of \([C]^{-1}\).

The \([CC]\) matrix is partitioned in order to handle the displacement fields separately:

\[
[CC] = \begin{bmatrix}
CW \\
\hline
CD
\end{bmatrix}
\]

where \([CW]\) and \([CD]\) are the coefficient-displacement matrices for the \(W\) and \(\theta\) fields respectively.

The internal work consists of two separate components; the work due to bending and the work due to shear. These components are not coupled hence, the element stiffness matrix can
be derived by considering each component separately and adding the results:

\[ [k]^e = [k]_b + [k]_s \]  \hspace{1cm} (4.12)

where

\[ [k]_b = \text{portion of the stiffness matrix due to bending} \]

and

\[ [k]_s = \text{portion of the stiffness matrix due to shear}. \]

The \([k]_b\) matrix will be formed first. The axial strain due to bending is \(Z \frac{d\theta}{dx}\); therefore, the strain-displacement matrix is:

\[ e = Z \begin{bmatrix} 0 & 1 & 2x \end{bmatrix} [CD][\delta]^e = [B]_b[\delta]^e \]  \hspace{1cm} (4.13)

Therefore, with \(e = E\sigma\), the stiffness matrix due to bending is:

\[ [k]_b = \int_V [B]_b^T E [B]_b \, dV \]  \hspace{1cm} (4.14)

The evaluation of the stiffness matrix due to shear involves the integral over the volume of the internal work due to shear, hence:

\[ [k]_s = \int_V \gamma \, dV \]  \hspace{1cm} (4.15)

where

\( \tau = \text{shear stress} \).
The stress-strain relation for shear is:

\[ \tau = G \gamma \]  \hspace{1cm} (4.16)

Expressing the shear strain in terms of the displacement fields:

\[ \gamma = \frac{dW}{dx} - \Theta \]  \hspace{1cm} (4.17a)

and substituting the displacement fields:

\[ \gamma = \begin{bmatrix} 0 & 1 & 2x & 3x^2 \end{bmatrix} \{\omega\} - \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \{\phi\} \]  \hspace{1cm} (4.17b)

or

\[ \gamma = [B]_s \{\delta\}^e \]  \hspace{1cm} (4.17c)

where

\[ [B]_s = \text{shear strain-displacement matrix}. \]

Finally, forming the expression for the internal work as in Eq. 3.7b:

\[ [k]_s = \int_V [B]_s^T G [B]_s \, dV \]  \hspace{1cm} (4.18)

When performing the integration indicated in Eq. 4.18, the area of the beam is taken as \( A_s \).

Adding Eqs. 4.14 and 4.18 as indicated in Eq. 4.12 gives the element stiffness matrix for the equilibrium equations:

\[ \{F\}^e = [k]^e \{\delta\}^e \]
where

\[ [F]^e_T = [V_L \ M_L \ V_M \ M_M] \]

4.3 Numerical Comparisons

The conventional method of forming a stiffness matrix for a beam with shear deformations is to form the flexibility matrix and then transform it to a stiffness matrix (Ref. 12). Beam theory can accommodate shear deformations by adding the deflection due to shear directly to the deflection due to bending:

\[ \delta_{\text{total}} = \delta_{\text{bending}} + \delta_{\text{shear}} \]

Table 17 shows numerical comparisons for the deflection at the load between the new element, the element using the conventional stiffness matrix and beam theory including the deflection due to shear.

Only one element is used to model the beam using both the new and the conventional elements. The deflection using the beam theory is calculated from:

\[ \delta = \frac{PL^3}{3EI} + \frac{PL}{A_sG} \]

The deflections predicted by all three methods are the same. Further studies modelling the beam with 2, 4 and 8 elements using both the new and conventional elements predict the same
deflections as the one element model. This data establishes the validity of the assumptions and procedures used to formulate the stiffness matrix of the new element.

4.4 Alternate Formulation

This variation of the Timoshenko beam element presented in Section 4.2 involves changing the constraining equations of the polynomials (Eqs. 4.11). The polynomials are sufficiently well behaved to allow the compatibility equation (Eq. 4.9a) to be enforced at discrete points rather than for all x without changing the results. Enforcing Eq. 4.9b at x = 0, \( \frac{L}{2} \), and L yields the following set of constraining equations:

\[
\{0\} = \begin{bmatrix}
0 & -1 & 0 & 0 & 1 & 0 & \frac{-2EI}{A_sG}
\end{bmatrix}
\]

\[
\{0\} = \begin{bmatrix}
0 & -1 & -L & \frac{-3L^2}{4} & 1 & \frac{L}{2} & \left(\frac{L^2}{4} - \frac{2EI}{A_sG}\right)
\end{bmatrix}\{\alpha\}
\]

\[
\{0\} = \begin{bmatrix}
0 & -1 & -2L & \frac{-3L^2}{2} & 1 & L & \left(L^2 - \frac{2EI}{A_sG}\right)
\end{bmatrix}
\]

(4.19)

Using these equations instead of Eqs. 4.11 and proceeding with the formulation exactly as before produces a stiffness matrix that gives the same numerical results as before (Table 17).
4.5 Formulation of Element-IV

The stiffness matrix developed in this section will be referred to as Element-IV. The intent of this element is to introduce shear deformations in beam B of Element-I. The composite element consists of an elastic Bernoulli-Navier beam on top (beam A), and a Timoshenko beam on the bottom (beam B) fastened together at their interface by shear connectors (Fig. 5b). The two beams acting together is the composite beam element.

The element, together with the degrees of freedom and sign conventions, is shown in Fig. 16. It is defined by two nodes each with five degrees of freedom, a vertical displacement for both beams and an axial displacement and rotation for each beam. However, in an effort to reduce future computational effort, two reference planes are used, one for beam A and one for B. This causes the nodes at each end to split. The axial displacements and rotations associated with a beam are at the reference plane for that beam, however, the vertical displacement is the same for both beams. The positions of the reference planes are arbitrary as long as they are parallel to the element.

The quantities \( \bar{Z}, \bar{Z}_A, \bar{Z}_B, Z_{1A} \) and \( Z_{1B} \) are all vector quantities as shown in the figure. All of the deformations, i.e., node displacements and displacement fields are written at the reference planes.
Separate node rotations are provided for each of the beams because these rotations are associated with bending. The slopes of the surfaces defined by the centroidal axes of each beam will be the same, however, the node rotations will be different. The difference between the two will be the shear strain in the bottom beam (See Section 4.2).

All of the beam theory assumptions and the assumption regarding the shear connectors stated in Section 3.4 apply except that shear deformations are permitted in beam B. The details of the following formulation are given in Ref. 31.

This formulation follows the outline given in Sections 3.2 and 3.3 and it begins by assuming the cubic polynomial for the transverse displacement (W) for both beams:

\[ W = c_1 + c_2 x + c_3 x^2 + c_4 x^3 \]  \hspace{1cm} (4.20a)

As before, this assumption does not permit separation between the beams.

In order to permit shearing deformations in beam B, a separate rotation field must be used:

\[ \theta_B = d_1 + d_2 x + d_3 x^2 \]  \hspace{1cm} (4.20b)

Finally, the axial displacement fields permitting slip between the beams are:

\[ U_A = a_1 + a_2 x + a_3 x^2 \]  \hspace{1cm} (4.20c)
and

\[ U_B = b_1 + b_2 x + b_3 x^2 \tag{4.20d} \]

Enforcing compatibility between the node displacements and the displacement fields and noting that \( \theta = -\frac{dW}{dx} \) generates the [C1] matrix of Eq. 3.9:

\[ \delta^e = [C1]\alpha \]

or

\[ \begin{bmatrix} U_{LA} \\ U_{LB} \\ W_L \\ \theta_{LA} \\ \theta_{LB} \\ U_{MA} \\ U_{MB} \\ W_M \\ \theta_{MA} \\ \theta_{MB} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & L & L^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & L & L^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & L & L^2 & L^3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2L & -3L^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -L \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ b_1 \\ b_2 \\ b_3 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix} \]

\[ (4.21) \]
The sign conventions for node displacements and the internal displacement fields for Elements-I and IV and the Timoshenko beam element are all the same (Figs. 11, 14b, and 16). Therefore, the equilibrium equations and strain compatibility equations developed for Element-I and the Timoshenko element also apply to Element-IV without modification except for Eq. 4.8 which must be rederived because the reference plane for beam B of Element-IV is not fixed at its centroidal axis.

There are ten node displacements and thirteen coefficients, therefore three more equations must be established. Enforcing equilibrium (Eqs. 3.17) and setting the shear flow equal to zero, yields two constraining equations similar to Eqs. 3.21:

\[
0 = 2EA_A a_3 - 6ES_A c_4 \tag{4.22a}
\]

and

\[
0 = 2EA_B b_3 - 2ES_B d_3 \tag{4.22b}
\]

The difference between Eqs. 3.21 and 4.22 is that the axial strain in beam B is derived from the rotation field (Eq. 4.20b) instead of the transverse displacement field (Eq. 4.20a).

Compatibility between the rotation fields will be enforced to arrive at the last constraining equation. First, the shear strain in beam B (\(\gamma_B\)) must be expressed in terms of the
polynomial coefficients. Consider the equilibrium of an element of beam B (Fig. 17). In order to maintain some generality, the interface shear flow \( s \) will not be set to zero until after the equilibrium equations are formed.

Summing the moments about point 0 (Fig. 17):

\[
0 = -s \int_{z_B} Z \, dx + V_B \, dx + dM_B \tag{4.23a}
\]

Substituting for \( s \) from Eq. 3.17b and dividing through by \( dx \) yields the shear in beam B:

\[
V_B = - \frac{dM_B}{dx} - Z \int_{z_B} \frac{dN_B}{dx} \tag{4.23b}
\]

The average shear strain from Eq. 4.7 is:

\[
\gamma_B = \frac{V_B}{A u_B G_B} \tag{4.24}
\]

Setting \( \frac{dN_B}{dx} = 0 \) (i.e., \( s = 0 \)) and substituting the polynomial expression for \( \frac{dM_B}{dx} \) from Eq. 4.5 yields the expression for the shear strain in terms of the polynomial coefficients:

\[
\gamma_B = - \frac{2E I_B}{A u_B G_B} d_3 \tag{4.25}
\]

where

\[ E I_B = E_B x \] second moment of inertia of beam B about the reference plane for beam B.
Compatibility between the transverse displacement field and the rotation field requires (as in Eq. 4.9a):

\[ 0 = -\frac{dW}{dx} + \theta + \gamma_B \]  

(4.26a)

Or, in terms of the polynomial coefficients:

\[ 0 = -c_2 - 2c_3x - 3c_4x^2 + d_1 + d_2x + (x^2 - \frac{2EI_B}{A_{SB}G_B})d_3 \]  

(4.26b)

Enforcing Eq. 4.26b for all \( x \) as in Eqs. 4.10 would yield too many constraining equations. However, the polynomials are sufficiently well behaved that compatibility can be required only at \( x = \frac{L}{2} \) which yields the last constraining equation:

\[ 0 = -c_2 - Lc_3 - \frac{3}{4} L^2 c_4 + d_1 + \frac{L}{2} d_2 + (\frac{L^2}{4} - \frac{2EI_B}{A_{SB}G_B})d_3 \]  

(4.26c)

This is similar to the alternate formulation presented in Section 4.4.

Casting the three constraining equations in matrix form gives the \([C2]\) matrix of Eq. 3.9:

\[
\begin{bmatrix}
0 & 0 & 0 & 2E_A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6E_A & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2E_B & 0 & 0 & 0 & 0 & 0 & 0 & -2E_B & \end{bmatrix}
\begin{bmatrix}
\alpha \\
\end{bmatrix}
\]

(4.27)
Combining Eqs. 4.21 and 4.27 gives the \([C]\) matrix of Eq. 3.9 which is solved to give the \([CC]\) matrix indicated in Eq. 3.11 where \([CC]\) consists of the first ten columns of \([C]^{-1}\).

Therefore as in Eq. 3.11:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{4} & 0 & \frac{6}{2} & -\frac{6}{2} & 0 & \frac{6}{2} & \frac{6}{2} & 0 & 0 & 0 \\
0 & 0 & \frac{6}{2} & -\frac{6}{2} & 0 & 0 & 0 & \frac{6}{2} & \frac{6}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{6}{2} & \frac{6}{2} & \frac{6}{2} & \frac{6}{2} & \frac{6}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{6}{2} & -\frac{6}{2} & 0 & 0 & 0 & \frac{6}{2} & \frac{6}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{6}{2} & \frac{6}{2} & \frac{6}{2} & \frac{6}{2} & \frac{6}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

(4.28)
where

\[ J_1 = \frac{4A_{SB} G_B}{L^2 A_{SB} G_B + 8EI_B} \]

The \([CC]\) matrix is then partitioned to be able to handle the displacement fields separately:

\[
[CC] = \begin{bmatrix}
CA \\
\cdots\\
CB \\
\cdots\\
CW \\
\cdots\\
CD
\end{bmatrix}
\]

where \([CA]\), \([CB]\), \([CW]\) and \([CD]\) are the coefficient-displacement matrices for the \(U_A\), \(U_B\), \(W\) and \(\Theta_B\) fields respectively.

The internal work of the element consists of four separate and uncoupled components. The first two are the work due to axial stresses and strains in beams \(A\) and \(B\), the third is the work due to shear stresses and strains in beam \(B\) and the fourth is the work due to the shear flow and slip at the interface. Therefore, the element stiffness matrix can be formed as:

\[
[k]^e = [k_A]_b + [k_B]_b + [k_B]_s + [k]_u \quad (4.29)
\]

where \([k_A]_b\), \([k_B]_b\), \([k_B]_s\) and \([k]_u\) are the portions of the element stiffness matrix resulting from the consideration of the internal work due to axial stresses and strains in beam \(A\), axial stresses
and strains in beam B, shear stresses and strains in beam B and shear flow and slip at the interface respectively.

Performing the required operations on the displacement fields indicated by \([\Gamma]\) in Eq. 3.13a and substituting the coefficient-displacement matrices for \([\alpha]\) as in Eq. 3.13b results in the following strain-displacement matrices. Axial strain in beam A in accordance with Eq. 3.19:

\[
\varepsilon_{xA} = \frac{dU_A}{dx} - Z \frac{d^2W}{dx^2} \quad (4.30a)
\]

therefore,

\[
\varepsilon_{xA} = \begin{bmatrix} 0 & 1 & 2x \end{bmatrix} \begin{bmatrix} CA \end{bmatrix} - Z \begin{bmatrix} 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} CW \end{bmatrix} \begin{bmatrix} \delta \end{bmatrix}^e \quad (4.30b)
\]

or

\[
\varepsilon_{xA} = [B_A]_b \begin{bmatrix} \delta \end{bmatrix}^e \quad (4.30c)
\]

Axial strain in beam B:

\[
\varepsilon_{xB} = \frac{dU_B}{dx} - Z \frac{d\Theta_B}{dx} \quad (4.31a)
\]

therefore,

\[
\varepsilon_{xB} = \begin{bmatrix} 0 & 1 & 2x \end{bmatrix} \begin{bmatrix} CB \end{bmatrix} - Z \begin{bmatrix} 0 & 1 & 2x \end{bmatrix} \begin{bmatrix} CD \end{bmatrix} \begin{bmatrix} \delta \end{bmatrix}^e \quad (4.31b)
\]

or

\[
\varepsilon_{xB} = [B_B]_b \begin{bmatrix} \delta \end{bmatrix}^e \quad (4.31c)
\]
Shear strain in beam B in accordance with Eq. 4.26:

\[ \gamma_B = \frac{dW}{dx} - \Theta_B \]  \hfill (4.32a)

therefore,

\[ \gamma_B = \left[ \begin{array}{c} 0 & 1 & 2x & 3x^2 \end{array} \right] [CW] - \left[ \begin{array}{c} 1 & x^2 \end{array} \right] [CD] \right] \{\delta\}^e \]  \hfill (4.32b)

or

\[ \gamma_B = [B_B]^T \{\delta\}^e \]  \hfill (4.32c)

Slip at the interface - similar to Eq. 3.30a:

\[ \delta U = (U_A - Z_{A} \frac{dW}{dx}) - (U_B - Z_{B} \Theta_B) \]  \hfill (4.33a)

therefore,

\[ \delta U = \left[ \begin{array}{cccc} 1 & x & x^2 & -1 & -x & -x^2 \end{array} \right] \left[ \begin{array}{c} 0 & -Z_{iA} & -2Z_{iA}x & -3Z_{iA}x^2 \\
Z_{iB} & Z_{iB}x & Z_{iB}x^2 \end{array} \right] [CC] \{\delta\}^e \]  \hfill (4.33b)

or

\[ \delta U = [XU][CC]\{\delta\}^e = [B]_u \{\delta\}^e \]  \hfill (4.33c)

Forming the expressions for the internal work as in Eq. 3.7a results in the component stiffness matrices of Eq. 4.29:

Axial stresses and strains in beam A - see Eq. 3.26:

\[ [k_A]_b = \int_Y \left[ \left( \left[ B_A \right]_b^T E_A \left[ B_A \right]_b \right) dV \]  \hfill (4.34)
Axial stresses and strains in beam B - see Eq. 3.26:

\[
[k_B]_b = \int_V [B_B]^T E_B [B_B]_b \, dV
\]  

(4.35)

Shear stresses and strains in beam B - see Eq. 4.18:

\[
[k_S]_s = \int_V [B_B]^T G_B [B_B]_s \, dV
\]  

(4.36)

Shear flow and slip at the interface - see Eq. 3.31:

\[
[k_u] = \int_L [B]^T k_{sc} [B]_u \, dx
\]  

(4.37)

These matrices are shown in Appendix A. Summing them yields the stiffness matrix for Element-IV (Eq. 4.29).

Element-IV is an intermediary element on the way to the final element presented in Chapter 5. All of the concepts which went into the formulation are validated by previous numerical comparisons. Hence, no numerical results for this element will be presented. Rather, the final element which makes extensive use of Element-IV is presented with numerical results in Chapter 5.
5. PLATE AND TIMOSHENKO BEAM COMPOSITE ELEMENT

5.1 Introduction

The steel bridges of interest in this research consist of two or more beams spanning side by side each acting compositely with the deck (Fig. 1). Elements-I through IV can not be used to analyze such a structure because they are all limited to the analysis of single composite beams. This chapter presents the formulation of an element that replaces the upper beam (beam A) of Element-IV with a plate bending element (Fig. 5c). This will permit analyses of steel bridges which will account for the interaction of the beams resulting from the fact that they are all connected to the deck. The effects of slip and shear deformations in beam B will be maintained. In addition, the plate will account for the effects of shear lag. This new element will be referred to as Element-V and it is the final finite element presented in this dissertation. This is the element referred to in Chapter 1 which will be employed in future research into the non-linear behavior of steel bridges.

No new derivations based on assumed displacement fields are presented. Rather, Element-V is literally put together from Element-IV and a plate bending element. Numerical comparisons with test data and the assemblage models are presented to
deemonstrate the accuracy, numerical stability and convergence properties of Element-V. The assemblage models are modified to the extent that the beam elements for the upper beam are replaced with the LCCT9 element from the SAP IV library (see Section 2.2).

5.2 Formulation of Element-V

5.2.1 Characteristics of Element-IV

An examination of the component matrices of Element-IV (Eqs. 4.29 and 4.34 to 4.37) shown in Appendix A reveals that the stiffness matrix for beam A is not coupled to the stiffness matrices for beam B in so far as a dependence upon structural properties is concerned, i.e., $[k_A^b]$ is not a function of the properties of beam B and $[k_B^b]$ and $[k_B^s]$ are not functions of the properties of beam A. The $[k]^u$ matrix is dependent upon the properties of beam B. It is also dependent upon the properties of beam A but only for the vertical location of the centroidal axis and vertical physical dimensions.

In order to achieve this uncoupling of the stiffness matrices, the displacement fields (Eqs. 4.20) have to remain uncoupled after the compatibility and constraining equations are enforced. The compatibility equations (Eqs. 4.21) do not relate the displacement fields to each other and the constraining
equations (Eqs. 4.27) were chosen to leave the fields uncoupled as evidenced by the coefficient-displacement matrix (Eqs. 4.28). An examination of this matrix shows that $U_A$ is not a function of beam B properties, $U_B$ and $\theta_B$ are not functions of beam A properties and $W$ is not a function of either beam A or B properties. Continuing the derivation through to the stiffness matrices (Eqs. 4.34 to 4.37) does not result in any manipulations that couple them except for the vertical dimensions of both beams A and B appearing in $[k]_u$.

Element-IV was patterned after Element-I, i.e., the equilibrium condition (Eqs. 3.17) with the shear flow set equal to zero is enforced for both elements. A formulation for an element that adds the shear deformations to beam B of Element-II was derived numerically, i.e., numbers were used in the establishment of the $[C]$ matrix (Eq. 3.9) and the stiffness matrices were formed by matrix manipulations using the computer. The actual stiffness matrices were not derived in algebraic form. That formulation is the same as for Element-IV except that the constraining equations (Eqs. 4.27) are changed. The constraining equations used instead were:

1) Equilibrium - Eq. 3.33a
2) Force-displacement for slip - Eq. 3.34d @ $x = \frac{L}{2}$
3) Compatibility of rotations - Eq. 4.26a @ $x = \frac{L}{2}$
The numerical results predicted by this element are close to those predicted by Element-IV. However, the component matrices of the element are coupled. The advantage that Element-IV holds is that the uncoupling of the component stiffness matrices permits the $[k_A^b]$ matrix to be replaced by the appropriate terms of a plate element stiffness matrix by direct substitution.

If a plate element were to be added to the element patterned after Element-II, the derivation would have to begin at the polynomial level in order to enforce the constraining equations. Such a formulation would be very involved which may prevent the establishment of the stiffness matrices in algebraic form. The logistics of implementing the element into the final nonlinear analysis program would be complicated. The program would take longer to execute than one using Element-V and the numerical results would be no more accurate. In view of the above, Element-V was developed using Element-IV and the alternate formulation patterned after Element-II was not used.

5.2.2. Incorporation of the Plate Element

In order for a plate element to successfully replace beam A in Element-IV, the following requirements regarding the displacement fields and node displacements must be met:

1) The transverse displacement field for the plate must reduce to the transverse
displacement field for Element-IV along the Element-IV axis.

2) The axial displacement field for the plate and beam A must be the same along the beam A axis.

3) The degrees of freedom of the plate nodes and beam A nodes must be of the same form, i.e., displacements and rotations.

These requirements are imposed so that the \([k_B]_b\), \([k_B]_s\) and \([k]\) matrices can remain intact. If a plate element were used that did not meet the above requirements, and the above three matrices were used as is, the final element would contain incompatible displacement fields. The result would be that the internal work would not be properly formulated which would lead to theoretically incorrect stiffness matrices. This may result in an inaccurate or nonconvergent element.

An element composed of the ACM plate bending element originally proposed by Adini, Clough and Melosh and a plane stress element originally proposed by Clough does satisfy the above requirements provided that the reference plane for beam A is fixed at the mid-height of the plate. This element has been successfully used in previous research and has shown itself to be both accurate and reliable (Refs. 22, 23, 24, 35, 36). The general aspects of its formulation are well known in the literature and the details are provided by Wegmuller and Kostem (Ref. 35).
The element has five degrees of freedom per node (Fig. 4). They are a transverse displacement, two in-plane displacements and two out-of-plane rotations. The transverse displacement, one in-plane displacement and one out-of-plane rotation replace the corresponding degrees of freedom of beam A (Fig. 16). The remainder of this section is devoted to showing that the displacement fields of the plate meet requirements one and two above.

The polynomial for the transverse displacement field \( W \) of the plate element is given as (Ref. 35):

\[
W = c_1 + c_2 x + c_3 y + c_4 x^2 + c_5 xy + c_6 y^2 \\
+ c_7 x^3 + c_8 x^2 y + c_9 xy^2 + c_{10} y^3 \\
+ c_{11} x^3 y + c_{12} xy^3 
\]  

(5.1)

When \( y \) is held constant, this field reduces to the form:

\[
W \bigg|_{y=\text{constant}} = c_1 + c_2 x + c_3 x^2 + c_4 x^3 
\]  

(5.2)

The polynomial (Eq. 5.1) takes on the same form when \( x \) is held constant. This form is the same as that chosen for the transverse displacement field of Element-IV (Eq. 4.20a). Hence the transverse displacement field of the plate reduces to that of Element-IV along the element axis.
The plate in-plane displacement fields are given as (Ref. 35):

\[ u = a_1 + a_2 x + a_3 xy + a_4 y \]  \hspace{1cm} (5.3a)

and

\[ v = b_1 + b_2 x + b_3 xy + b_4 y \]  \hspace{1cm} (5.3b)

Working with the \( U \) field and holding \( y \) constant results in:

\[ U \bigg|_{y=\text{constant}} = a_1 + a_2 x \]  \hspace{1cm} (5.4)

A similar form results if \( x \) is held constant; and the \( V \) field behaves exactly the same way.

The axial displacement \( (U) \) at the centroidal axis of beam A in Element-IV is given by:

\[ U \bigg|_{\text{CG beam A}} = U_A - \bar{Z}_A \frac{dW}{dx} \]  \hspace{1cm} (5.5a)

or in terms of the polynomials (Eqs. 4.20a and c):

\[ U \bigg|_{\text{CG beam A}} = a_1 + a_2 x + a_3 x^2 - \bar{Z}_A (c_2 + 2c_3 x + 3c_4 x^2) \]  \hspace{1cm} (5.5b)

If the reference plane is fixed at the centroidal axis of beam A (mid-height of the plate), then \( \bar{Z}_A = 0 \) and Eq. 5.5b reduces to:
\[ U = a_1 + a_2x + a_3x^2 \]  
\text{CG beam A} \tag{5.5c}

Furthermore, since \( \bar{z}_A = 0 \), then \( E_S A = 0 \) and from Eq. 4.22a:

\[ 0 = 2EA_A a_3 \]  
\text{Since } EA_A \neq 0, \text{ then } a_3 = 0 \text{ and Eq. 5.5c reduces to:}  
\[ U = a_1 + a_2x \]  
\text{CG beam A} \tag{5.7}

This is the same form that the plate polynomials reduced to (Eq. 5.4). Hence, when the reference plane is fixed at the centroidal axis of beam A (mid-height of the plate), the axial displacement fields are the same.

Since the plate element does satisfy the requirements set forth at the beginning of this section, the appropriate terms of its stiffness matrix can be used to replace the \([k_A]_b\) matrix in Element-IV. Furthermore, the additional degrees of freedom of the plate along with the rest of the plate element stiffness matrix can be added directly which completes the replacement of beam A by the plate. This aspect of the formulation will be explained in detail in the next section.
5.3 Implementation

Element-V can be considered as a T-beam element (Fig. 5c). Using only this element, a bridge model could be constructed (Fig. 18b). However, if a finer mesh for the deck is desired, then the T-beam element must be used in conjunction with the plate element (Fig. 18c). On the other hand, if a coarser mesh is desired with only one plate element between the beams, the T-beam element could not be used at all (Fig. 18d).

The arguments presented in the previous section demonstrate that if the \([k_B]_b\), \([k_B]_s\) and \([k]_u\) matrices are added to the plate element stiffness matrix, then the effects of beam B would be properly included and the whole model would behave as one element. The replacement of \([k_A]_b\) is not limited to one plate element. Indeed, to form a T-beam element two plate elements must be used.

There is no need to put together the stiffness matrix of such a T-beam element. In fact, it is preferable not to do so. Rather, Element-V can be implemented as follows (Fig. 19):

1) Stack the global equations for the deck degrees of freedom with the plate element stiffness matrices.

2) Stack the appropriate global equations with the beam stiffness matrices.

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Using this technique all three cases in Fig. 18 are handled in the same fashion. The number of plate elements between the beams does not matter. Furthermore, the resulting algorithms make efficient use of the routines used to establish the plate element and beam matrices and the logistics of stacking the global equations are simplified.

5.4 Numerical Comparisons

A computer program based on the technique outlined in the previous section was written and implemented on the CDC 6400 at the Lehigh University Computing Center (Ref. 32). This program was used to analyze a variety of structures, four of which are presented here to verify the formulation.

The first structure is a deep steel girder composite with a concrete deck. It is adapted from a design example presented in a bridge design manual published by the California Department of Transportation and will hereafter be referred to as the California Girder (Ref. 6). This structure is used to show convergence properties and other characteristics of Element-V. Numerical comparisons are made with the assemblage model since there are no test results available.

The next two structures are the two beams used to verify the assemblage models in Chapter 2, referred to as beams B24W and
B21W (Ref. 34). The analytical results are compared to test data for these beams. The last structure is bridge 3B from the AASHO Road Test and numerical comparisons are made with a modified assemblage model and test data (Ref. 14).

5.4.1 California Girder

The composite girder is composed of a 2438 mm (96 in.) deep steel girder, with a 292 mm (11.5 in.) thick concrete deck on a 24384 mm (960 in.) span (Fig. 20 and Table 18). The load for all analyses was 1.78 kN (400 kips) at the midspan. Four base configurations were used for analysis, they are:

1) Simple supports - without deformation due to shear.
2) Simple supports - with deformation due to shear.
3) Fixed supports - without deformation due to shear.
4) Fixed supports - with deformation due to shear.

Three values of the shear connector stiffness \( k_{sc} \) were used with each base configuration.

The girder was analyzed using both the assemblage model and Element-V. Both models take advantage of quarter symmetry; and to study convergence, progressively finer meshes were used.
The meshes for the assemblage models are shown in Fig. 21 where meshes A through D get progressively finer, while maintaining the use of one element to model the deck width. Mesh E is the finest mesh longitudinally and it uses two elements to model the width of the deck. The Element-V meshes are shown in Fig. 22 where meshes I through L get finer and mesh M has the same configuration of plate elements as mesh E.

The deflection at the load predicted by all the analyses are presented in Tables 19 through 22. The deflections predicted by meshes E and M agree with one another to within 1% for configurations 1 and 2, 2% for configuration 3, and 4% for configuration 4 which indicates that Element-V does predict the correct deflections. The fixed end predictions do not agree as well as the simple support predictions which is expected since the deflected shape of a fixed end beam is more complex than that for a simple beam.

The degrees of freedom (DOF) shown in the tables is the total number of unknowns in the analysis and the quantity R is a non-dimensional parameter which can be used to examine convergence. Plots of R vs. DOF are shown in Figs. 23 and 24 for the highest values of ksc which is the worst case. The term "with γ" means that shear deformations were permitted. These plots show that Element-V is monotonically convergent from the stiffer side of R equal to one. This is a desirable property for the element to possess since the stiffness of the analytical model should always be larger than that of the real structure (Ref. 8).
The assemblage model predictions converge from the flexible side of \( R \) equals one. It is suspected that this is caused by the fact that the internal work due to slip is not properly included as the meshes get coarser.

The plots of \( R \) for Element-V in Fig. 24 show that the element is very stiff for the coarse meshes. This is because the value of \( k_{sc} \) is too large, i.e., exceeds \( k_{\text{max}} \) (Eq. 3.37). If the value of \( k_{\text{max}} \) is used, the properties of Element-V are much improved (Fig. 24 and Table 23). The derivation of \( k_{\text{max}} \) does not include the effects due to shear deformation and shear lag (Ref. Section 3.6). However, including these effects should not greatly change the value of \( k_{\text{max}} \). The numerical data presented in Chapter 3 show that if the value chosen is in the vicinity of \( k_{\text{max}} \), the results are reasonable. Furthermore, if the value of \( E_{A} \) in Eq. 3.37 is calculated from a full width of slab disregarding the effect of shear lag, the value of \( k_{\text{max}} \) should be on the high side. This is desirable since the intent of \( k_{\text{max}} \) is to insure that no slip occurs. The results shown in Fig. 24 are in agreement with the above since the element is still converging from the stiff side when \( k_{\text{max}} \) is used.

The deflections predicted by Element-V are somewhat affected by the choice of reference planes. Examples were run for configurations 3 and 4 using two sets of reference planes (Tables 24 and 25). The deflections for configuration 3 are all within
1% of each other and for configuration 4 within 3.5% for the coarser meshes (J and K), dropping to 2% for mesh L and 1% for M. The reason for the discrepancy is due to the nature of the constraining equations (Eqs. 4.26) and the shear variation in the real structure.

Compatibility between the beam B displacement fields and the slope of the transverse displacement is only enforced at a point and for only one value of the shear strain (Ref. Eq. 4.26c). The adjustment of the polynomial coefficients is a function of the location of the displacement fields, i.e., the positions of the reference planes. However, the coefficients will adjust themselves to give the best solution within the confines of the displacement fields chosen (Ref. 8). This, coupled with the fact that the shear in beam B is not a constant (see Section 3.5), means that the adjustment is not always the same which results in slight changes in the stiffness matrix. However, the results between the analyses using the two sets of reference planes are all within acceptable limits. Furthermore, additional studies using widely varying positions of reference planes all produce results similar to those presented here. Hence, although the deflections are affected by the choice of reference plane locations, they are all within acceptable limits and the variations are no cause for concern.
5.4.2 Test Beams

These two beams are described in Section 2.4 and pertinent geometry and structural properties are shown in Fig. 7 and Table 1. The Element-V models took advantage of quarter symmetry and the meshes are similar to mesh L (Fig. 22).

The analyses predicted the same deflections as predicted by the assemblage models described in Chapter 2. The plots show good agreement between the test results, Newmark's Theory, and Element-V (Figs. 9 and 10). The capability of Element-V to account for the additional deflection due to shear lag does not appear because of the geometry of the test specimens. The maximum width-thickness ratio of the beams is 5.22 which is well within the limit of 8 prescribed as an effective width (Ref. 1). Also, the width of the slab is less than one-fourth the span length (Ref. 1). Therefore, for these specimens, shear lag is not expected to contribute to the overall deflection and the analyses confirm this.

The plots show that Newmark's Theory and Element-V are in close agreement when the shear deformations are not included. However, when the shear deformations are included, Element-V is stiffer than Newmark's Theory. The reason for this is twofold. First, Element-V converges from the stiffer side. Second, Newmark's Theory overestimates the deflection due to shear because
it does not account for the fact that the beam is more flexible when shear deformation is included. Rather, the shear in the beam is computed from an analysis that does not include the shear deformations. Then the deflection due to shear is computed and added to the deflection due to bending. On the other hand, Element-V does account for the change in stiffness due to shear in the beam. The beam becomes more flexible which tends to lower the shear in it. The result is less deflection due to shear than predicted by Newmark's Theory. However, the plots indicate that for the specimens tested, the discrepancy between the two analytical methods is not significant. This may not be true for deep girders loaded into the non-linear range.

Further analyses and comparisons were made with beams tested by McGarraugh and Baldwin (Ref. 41). These beams were built using stud shear connectors while beams B24W and B21W used channels. The results are presented in Ref. 31 and are similar to those presented here.

5.4.3 AASHO Bridge 3B

The AASHO Bridge 3B is a 15240 mm (600 in.) long simple span composed of three steel beams composite with a concrete deck (Fig. 25, Table 26, Ref. 14). The middle 5639 mm (220 in.) of the beams have cover plates. The bridge was analyzed for a truck located to produce the maximum moment on the span.
No slip between the deck and the beams was reported, therefore, the assemblage model was not used. Rather, rigid links were placed between the mid-plane of the deck and the centroidal axes of the beams. This model will subsequently be referred to as the SAP IV model and the mesh is shown in Fig. 26.

The mesh utilizing Element-V is shown in Fig. 27. In order to prevent slip a maximum \( k_{\text{max}} \) of \( 14010 \text{ MN/m}^2 \) (2032 ksi) from Eq. 3.37 was calculated for the shortest element with the widest width of deck. (The \( E_A \) value was calculated using the spacing between the beams.) This value was then used for all the elements. Due to programming problems related to the operating system and the core size of the computer, the current version of the program using Element-V can handle only prismatic beams (Ref. 32). This restriction can be removed in subsequent programs since the formulation requires that only each beam element be prismatic. However, the model using Element-V has the cover plates extending over the full length of the beams.

In order to estimate the effect of extending the cover plates, resort was made to the SAP IV model. Analyses were performed with the cover plates as is, and with them extended over the full length. Based on the average of the vertical deflections across the bridge at the middle set of loads (load line A, Fig. 26), extending the cover plates stiffens the structure by roughly 4\% (Table 27).
In addition the comparisons show that the SAP IV models and the Element-V model are in close agreement.

The tests reported in Ref. 14 state that bridge 3B has a stiffness of $44480 \frac{\text{mm-kN}}{\text{mm}} (10000 \frac{\text{in-k}}{\text{in}})$ when the maximum moment is at midspan. The stiffness predicted by the Element-V model adjusted by the 4% to account for the fact that the cover plates were extended for the analysis is $42970 \frac{\text{mm-kN}}{\text{mm}} (9660 \frac{\text{in-k}}{\text{in}})$. The two values are within 4% of each other. This is within acceptable limits given the variabilities associated with test procedures and the fact that the location of the maximum moment is not at the same place. In the test it is at midspan; in the analysis it is at load line A. Still, the two locations are close to each other and given the shape of the moment diagrams the discrepancy is small.

5.5 Concluding Remarks

The numerical results have shown that Element-V is accurate, monotonically convergent from the stiffer side of the actual behavior, i.e., under-estimates the deformations, and numerically stable. It can accurately predict the effects of slip, shear deformation in the beam and shear lag in the deck. The element can be used to analyze simple and multiple span T-beams and multi-beam bridges.

The reason for its development is to provide an efficient means of analysis for use in the final computer program to perform
non-linear analysis of steel bridges. This dissertation is but the first step in the development of that program. The non-linear analysis program could be written using the assemblage model instead of Element-V. However, Element-V is preferable to the assemblage model for two reasons:

1) The Element-V model predicts deformations to the same degree of accuracy as the assemblage model while using fewer equations. Rough estimates based on the degrees of freedom necessary and the amount of computer central processor time required indicate that the assemblage model does seven to ten times the work as the Element-V model.

The computer program used to implement the previous research usually requires 1000 or more central processor seconds on the CDC 6400 to solve the concrete bridges of interest (Refs. 18, 23, 25, 26). Roughly one-half of that time is devoted to solving for the deformations. It is expected that the program using Element-V will take longer to execute than the previous program but a program based on the assemblage model could take so long to execute that the cost would be prohibitive. Furthermore, when effects such as minor axis bending etc. are added to Element-V, the ratio of
the work done by the assemblage model to that of
Element-V will increase.

2) When the mesh gets coarse, the deflections pre-
dicted by the assemblage model begin to increase
dramatically. On the other hand, as long as the
limiting value of $k_{sc}$ is observed, Element-V will
predict reasonable deflections, even for the coarse
mesh. Since the final non-linear analysis program
will be adjusting stiffness parameters automatically
and since the permissible coarseness of the
assemblage model mesh is a function of the structure
stiffness, it is questionable whether a reliable
non-linear analysis program could be built using the
assemblage model.

The other effects mentioned in Chapter 1 should be added
to Element-V before the modifications are made to include non-
linear behavior. Chapter 6 contains some remarks about the way in
which the additions should be made as well as some comments on
modifying Element-V for non-linearities.
6. CONSIDERATIONS FOR FUTURE RESEARCH

6.1 Introduction

This report has presented various finite elements for the analysis of composite beams and bridges (Table 28). Element-V can be used to analyze steel bridges and include the effects thought to be of primary importance, i.e., slip, shear deformations in the beams and shear lag in the deck (Sect. 1.2). These bridges can be simple or multiple span continuous structures with single or multiple beams built fully composite, partially composite, or noncomposite.

The technique used for the analysis is the displacement based finite element method and the generality of that method has been maintained. All of the formulations are founded on first principles and no special considerations were used which would prevent the inclusion of any of the elements into a general displacement based finite element analysis program.

As Element-V is refined and its capability of analyzing steel bridges is extended, its range of applicability to other problems is narrowed. Although the element is general, that is not a property which must be maintained. Indeed, if the accuracy
of the solution and/or speed of execution can be increased then
the generality of the element should be sacrificed.

This chapter presents some remarks on the current for-
mulations and some suggestions and considerations for extending
the work. The next three sections present discussions on
extending the elastic analysis to add the secondary effects,
modeling, and extending the technique for nonlinear analysis.
All of the discussions will refer to Element-V, however, it will
be readily apparent when they are applicable to the other elements
as well.

6.2 Extension of the Elastic Analysis Technique

The main reason for adding the secondary effects, i.e.,
torsion and minor axis bending, to the beam is to include the
effect of wind bracing on the bridge superstructure. The tor-
sional stiffness of the beams alone does not significantly affect
the response of the bridge and even with some wind bracing the
response is not greatly affected. However, wind bracing can be
installed in a configuration where the response would approximate
that of a box beam bridge which is substantially different from
that of the I-beam bridge. More study is needed to determine
just what geometries of bracing affect the bridge response to the
extent that it must be explicitly included in the analysis.
It may be that if most steel bridges are constructed such that the bracing need not be explicitly included, then Element-V does not have to be extended. This would not necessarily mean that the forces in the bracing could not be found. The possibility exists that the bridge could be analyzed using Element-V and some form of back substitution, modified by structural properties, used to calculate the bracing forces.

If it is determined that the effect of the bracing should be explicitly included in the analysis, then more degrees of freedom (DOF) must be added to Element-V. A beam and plate element with the existing Element-V DOF and the additional DOF for the secondary effects are shown in Fig. 28. The DOF transverse to the beam (V) is the one required to include the bracing stiffness. The other two DOF are necessary because V is coupled with them. Note that, unless the bracing stiffness is expressed with respect to an arbitrary reference plane, the V displacement must be at the level of the bracing. Hence, the \( Z \) quantity will be fixed (Fig. 16).

Several options are available to develop the stiffness coefficients that relate the new DOF to each other and to the existing ones. The most rigorous alternative is to assume polynomials for displacement fields and proceed in a manner similar to that outlined in Sections 3.2 and 3.3. This will require a great deal of research and since the effects on the total
structure are thought to be small, it does not appear to be a worthwhile approach at this time.

Another approach is to use the plate bending element stiffness matrix (Ref. 35) for the contribution of the web and assume the flange to behave as a beam (Fig. 29). The beam element stiffness matrix is well known and its displacement field is compatible with that of the plate, so the stiffness coefficients of the two can be added directly.

A third approach is to make assumptions regarding effective widths of web and generate all of the stiffness coefficients from the beam element stiffness matrix (Fig. 30). This is the least rigorous of the three alternatives and would require a good deal of judgment. Parametric studies appear to be necessary to determine if and how the web can be approximated by beam elements.

Regardless of the approach taken, the researcher should be cognizant of the problems of torsion, slip, stiffeners and the nonlinear analysis scheme.

6.2.1 Torsion

The torsion due to warping is a significant portion of the total for the beam shapes used in steel bridges. However, it is a function of the second derivative of the twist angle which is not presently a DOF of the element. It is preferable not to
include this DOF so that the problem of making the second derivative of the twist angle compatible with the twist angle itself via the differential equation does not have to be confronted. Rather, it is suggested that some approximations be made where the warping torsion is included based on the twist angle.

6.2.2 Transverse Slip

In the formulation it will be necessary to relate the transverse DOF of the beam (V) to that of the plate (Fig. 28). Hence, some assumption regarding the stiffness of the shear connectors transverse to the beam must be made. The stiffness coefficients generated by considering the behavior of the web are likely to be small relative to the stiffness of the shear connectors. Hence, the stiffness of the connectors will not affect the total stiffness between the transverse DOF of the beam and the plate very much. Therefore, assuming that no slip occurs is probably realistic and should not introduce any intolerable inaccuracies into the formulation.

6.2.3 Stiffeners

The effects of stiffeners should be included in both Element-V and in the new work to add the secondary effects. Vertical stiffeners on both sides of the web will not have any effect on the Element-V stiffness matrix. A vertical stiffener on one side will affect the stiffness but its effect will be both local and small. Therefore, it can be neglected. Longitudinal
stiffeners on one or both sides of the web can be included directly into the beam cross-section properties.

The effect of stiffeners on the stiffness coefficients for the new DOF are likely to be substantial. If the plate bending matrix approach is used, the coefficients could be adjusted by using fictitious dimensions for the plate or by using some correction factors. If the beam approach is used, the properties of the stiffeners can be included directly into the calculations of the stiffness coefficients. Also, the presence of stiffeners will affect the effective width of web acting as a beam.

6.2.4 Nonlinear Analysis

The entire formulation will be extended into the nonlinear range using piecewise linearization (Sect. 6.4). Therefore, it is preferable to have the total element stiffness matrix in an algebraic form in order to obtain an efficient solution scheme. Failure and yield criteria, numerical properties of the global equations and other considerations will probably require that a somewhat fine mesh be used for analysis. This situation combined with the fact that the expected effects are of a secondary nature suggest that crude approximations resulting in stiffness coefficients in algebraic form would be better than more sophisticated techniques which would require numerical integration.
6.3 Modeling with Element-V

All of the models using Element-V in this report have been for single span structures without diaphragms. Diaphragms built compositely or not can be modeled by placing beam B only (no plate) between the appropriate nodes. The discussion in Chapter 5 showed that beam B is compatible with the plate element along any edge. Therefore, no special considerations are required.

Continuous beams and bridges can be modeled by removing the equations for the support DOF. If the reference plane for the beams is located near the bottom flange, the longitudinal support can be included there as well. Also, the effect of bearing pads can be included either by applying a force at the DOF or by attaching a member with the appropriate stiffness (Fig. 31).

Frequently, continuous bridges are haunched over the interior support (Fig. 32a). These can be modeled by assuming the haunched section to be a series of prismatic elements (Fig. 32b). To fix the interior support longitudinally, the reference plane for the beam is moved to that support elevation. Now, if axial forces or stiffnesses are to be included to model bearing pads at the ends, they must be transformed to the level of the beam reference plane. In all of the analyses, but in particular this one, it must be remembered that the DOF for the beam are written at the beam reference plane. Each element has its stiffness properties transformed to that plane. Therefore, compatibility between elements
of different cross sections is not a problem. When formulating the stiffness coefficients for the DOF for the secondary effects (Sect. 6.2) it is desirable to preserve this feature.

6.4 Extension to the Non-linear Range

There are two kinds of non-linearities usually considered: geometric and material. Geometric non-linearity implies that the effect of secondary deflections are explicitly included in the analysis. However, steel bridges are primarily structures in bending and the stability of the overall structure is not usually a problem. Therefore, to include the effects of secondary deflections explicitly does not seem to be a worthwhile approach. On the other hand, it appears that the stability of some of the components of the bridge, such as the web, longitudinal stiffeners, and the compression flange in a negative moment region, should be included. These components present local buckling problems and the element stiffness matrix is singular when formulated using first order geometry. However, it is suggested that the effect of buckling be included by adjusting the stiffness coefficients of the components within the confines of first order geometry.

Material non-linearity has two aspects: local yielding and general yielding. Local yielding occurs at stress concentrations and the regions of yielded material is small and does not spread throughout the member. General yielding occurs at points
of maximum gross stresses and the zone of yielded material can pro-
gress across the cross section eventually resulting in failure of
the member.

Local yielding has little effect on the overall struc-
tural response and an attempt to include it in the current formu-
lations should not be made. Rather, if local stresses are re-
quired, it is suggested that they be obtained by substructuring
after the global deformations and internal forces are calculated.

General yielding affects the entire cross section of
members and significantly alters the global response of the
structure. The following remarks are directed to this aspect of
the non-linear analysis.

6.4.1 Beams

The beams can be modeled using the layering technique
presented by Kulicki and Kostem (Ref. 20) and included into the
concrete bridge analysis scheme by Peterson and Kostem (Ref. 23).
In this technique the beam is divided into layers (Fig. 33) and
the changes in stresses, strains, and moduli are recorded for
each layer as the load is incremented (Sect. 1.1). The element
properties are summed over the layers at each load level. There-
fore, the appropriate equations for the beam, adapted from Ref.
23 are:
\[ E_{A_B} = \sum_i E_i A_i \quad (6.1a) \]
\[ E_{S_B} = \sum_i E_i S_i \quad (6.1b) \]
\[ E_{I_B} = \sum_i E_i I_i \quad (6.1c) \]

where \( A_i \), \( S_i \) and \( I_i \) are the area, first moment of inertia and second moment of inertia of layer \( i \) about the reference plane and \( E_i \) is the modulus.

It may be that another approach, which could be more efficient, can be taken. If the steel exhibits a stress-strain curve with a definite knee in it at yield, then the element properties could be calculated using the elastic core of the section (Fig. 34).

Regardless of the approach used, the effect of residual stresses due to manufacture and fabrication should be investigated.

It is expected that the slip of the connectors will cause a shift in the location of the point of zero strain due to bending (the centroid for elastic analysis). This is the point where the response due to axial force and that due to bending are uncoupled and its location on the beam is defined by \( Z_B \) (Fig. 16). Therefore, \( Z_B \) should be calculated at each load level by:
The shift will affect quantities in the beam B matrices (Appendix A).

The beam has the capability to model a moment gradient which means that Eqs. 6.1 will give different values of stiffness properties as a function of where they are applied along the length of the element. However, only one value for the entire element can be used in the analysis for each load level. The previous research used the average of the values calculated at the ends (Ref. 23). This appears to be the most reasonable approach rather than using the values calculated at the midlength and it is recommended that this approach be taken in the current research.

Nonlinearities in shear can be included by adjusting the shear modulus as the load increases as long as the web remains planar. Once a diagonal tension field occurs the GA_{SB} quantity can be adjusted. Note that the GA_{SB} quantity is an equivalent work term and should remain so throughout the formulations (Eq. 4.7). The effects of stiffeners should be investigated as well.
6.4.2 Deck

The nonlinearities of the deck can be included as formulated and applied to concrete bridges by Peterson, Kostem and Kulicki (Refs. 22, 23). A layering technique similar to that described for the beams is used (Fig. 35). That formulation does not account for a shift in the location where the in-plane and transverse bending responses are uncoupled. Since the deck is relatively thin, it is expected that the shift will be small enough so that it can be neglected. It is recommended, therefore, that the plate element matrices not be reformulated to include the shift at this time.

6.4.3 Connectors

In order to successfully complete the nonlinear analysis scheme some data on the load-slip characteristics of the various types of connectors is required. The value of $k_{sc}$ for a particular load level will vary along the length of the element. Some study is required to determine the appropriate value for the whole element. An equivalent work term is a possibility.

6.5 Concluding Remarks

The material presented in this dissertation represents the first step in the development of the nonlinear analysis program for steel bridges. The discussions in this chapter present
some, but by no means all, of the possibilities available to extend the work.

Much more research is required before the algorithms for the nonlinear analysis scheme can be developed. It is suggested that extensive parametric studies be carried out to determine just what effects have to be included since the formulations are likely to be extensive.

Finally, research into failure and serviceability criteria for the superstructure and its components should be initiated so that the results of the analyses can be properly applied to the real structure.
7. TABLES
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* Adjusted for reinforcing steel

+ Number of connectors provided

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<tr>
<td></td>
<td>(1 ksi)</td>
<td>(1 ksi)</td>
</tr>
<tr>
<td>Interior Links</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>2172 mm$^2$</td>
<td>526 mm$^2$</td>
</tr>
<tr>
<td></td>
<td>(3.366 in$^2$)</td>
<td>(0.816 in$^2$)</td>
</tr>
<tr>
<td>E</td>
<td>6.895 MN/m$^2$</td>
<td>6.895 MN/m$^2$</td>
</tr>
<tr>
<td></td>
<td>(1 ksi)</td>
<td>(1 ksi)</td>
</tr>
<tr>
<td>$d_t$</td>
<td>79 mm</td>
<td>78 mm</td>
</tr>
<tr>
<td></td>
<td>(3.125 in)</td>
<td>(3.055 in)</td>
</tr>
<tr>
<td>$d_b$</td>
<td>305 mm</td>
<td>268 mm</td>
</tr>
<tr>
<td></td>
<td>(11.995 in)</td>
<td>(10.565 in)</td>
</tr>
<tr>
<td>$P/2$</td>
<td>155.7 kN</td>
<td>111.2 kN</td>
</tr>
<tr>
<td></td>
<td>(35 kips)</td>
<td>(25 kips)</td>
</tr>
</tbody>
</table>
TABLE 3
NUMERICAL COMPARISONS - CANTILEVER BEAM -
RECTANGULAR CROSS SECTION

<table>
<thead>
<tr>
<th>k_{sc}</th>
<th>ELEM-I</th>
<th>ELEM-II</th>
<th>ELEM-III</th>
<th>ASSEM.</th>
<th>BM. THEORY</th>
</tr>
</thead>
<tbody>
<tr>
<td>25.0 MN/m^2</td>
<td>42.52</td>
<td>42.52</td>
<td>42.52</td>
<td>42.52</td>
<td>43.79*</td>
</tr>
<tr>
<td>(3.625 ksi)</td>
<td>(1.674)</td>
<td>(1.674)</td>
<td>(1.674)</td>
<td>(1.674)</td>
<td>(1.724)</td>
</tr>
<tr>
<td>2500 MN/m^2</td>
<td>17.63</td>
<td>17.70</td>
<td>17.70</td>
<td>17.70</td>
<td>----</td>
</tr>
<tr>
<td>(362.5 ksi)</td>
<td>(0.694)</td>
<td>(0.697)</td>
<td>(0.697)</td>
<td>(0.697)</td>
<td></td>
</tr>
<tr>
<td>250000 MN/m^2</td>
<td>10.90</td>
<td>11.05</td>
<td>11.05</td>
<td>11.05</td>
<td>10.95+</td>
</tr>
<tr>
<td>(36250 ksi)</td>
<td>(0.429)</td>
<td>(0.435)</td>
<td>(0.435)</td>
<td>(0.435)</td>
<td>(0.431)</td>
</tr>
</tbody>
</table>

* Assumes k_{sc} = 0
+ Assumes k_{sc} \to \infty

E_A = E_B = 200000 MN/m^2 (29000 ksi)
### TABLE 4

**NUMERICAL COMPARISONS - PROPPED CANTILEVER - RECTANGULAR CROSS SECTION**

![Diagram of a cantilever with loads and deflection calculations]

<table>
<thead>
<tr>
<th>$k_{sc}$</th>
<th>ELEM-I</th>
<th>ELEM-II</th>
<th>ELEM-III</th>
<th>ASSEM.</th>
<th>BM.THEORY</th>
</tr>
</thead>
<tbody>
<tr>
<td>25.0 MN/m²</td>
<td>1.194</td>
<td>1.194</td>
<td>1.194</td>
<td>-----</td>
<td>1.196*</td>
</tr>
<tr>
<td>(3.625 ksi)</td>
<td>(0.0470)</td>
<td>(0.0470)</td>
<td>(0.0470)</td>
<td></td>
<td>(0.0471)</td>
</tr>
<tr>
<td>2500 MN/m²</td>
<td>0.925</td>
<td>0.925</td>
<td>0.925</td>
<td>0.919</td>
<td>-----</td>
</tr>
<tr>
<td>(362.5 ksi)</td>
<td>(0.0364)</td>
<td>(0.0364)</td>
<td>(0.0364)</td>
<td>(0.0362)</td>
<td></td>
</tr>
<tr>
<td>250000 MN/m²</td>
<td>0.284</td>
<td>0.323</td>
<td>0.323</td>
<td>0.325</td>
<td>0.300+</td>
</tr>
<tr>
<td>(36250 ksi)</td>
<td>(0.0112)</td>
<td>(0.0127)</td>
<td>(0.0127)</td>
<td>(0.0128)</td>
<td>(0.0118)</td>
</tr>
</tbody>
</table>

* Assumes $k_{sc} = 0$

+ Assumes $k_{sc} \rightarrow \infty$

$E_A = E_B = 200000 \text{ MN/m}^2$ (29000 ksi)

---

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### TABLE 5
NUMERICAL COMPARISONS - FIXED END BEAM - RECTANGULAR CROSS SECTION

<table>
<thead>
<tr>
<th>$k_{sc}$</th>
<th>ELEM-I</th>
<th>ELEM-II</th>
<th>ELEM-III</th>
<th>ASSEM.</th>
<th>BM. THEORY</th>
</tr>
</thead>
<tbody>
<tr>
<td>25.0 MN/m²</td>
<td>0.683</td>
<td>0.683</td>
<td>0.683</td>
<td>-----</td>
<td>0.683*</td>
</tr>
<tr>
<td>(3.625 ksi)</td>
<td>(0.0269)</td>
<td>(0.0269)</td>
<td>(0.0269)</td>
<td></td>
<td>(0.0269)</td>
</tr>
<tr>
<td>2500 MN/m²</td>
<td>0.582</td>
<td>0.582</td>
<td>0.582</td>
<td>-----</td>
<td>-----</td>
</tr>
<tr>
<td>(362.5 ksi)</td>
<td>(0.0229)</td>
<td>(0.0229)</td>
<td>(0.0229)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>250000 MN/m²</td>
<td>0.1588</td>
<td>0.1915</td>
<td>0.1910</td>
<td>-----</td>
<td>0.1712+</td>
</tr>
<tr>
<td>(36250 ksi)</td>
<td>(0.00625)</td>
<td>(0.00754)</td>
<td>(0.00752)</td>
<td></td>
<td>(0.00674)</td>
</tr>
</tbody>
</table>

* Assumes $k_{sc} = 0$
+ Assumes $k_{sc} \rightarrow \infty$

$E_A = E_B = 200000 \text{ MN/m}^2 \text{ (29000 ksi)}$

---
TABLE 6
NUMERICAL COMPARISONS - SIMPLE BEAM -
RECTANGULAR CROSS SECTION

![Diagram of simple beam with rectangular cross section]

<table>
<thead>
<tr>
<th>$k_{sc}$</th>
<th>ELEM-I</th>
<th>ELEM-II</th>
<th>ELEM-III</th>
<th>ASSEM.</th>
<th>BM.THEORY</th>
</tr>
</thead>
<tbody>
<tr>
<td>25.0 MN/m² (3.625 ksi)</td>
<td>2.72 (0.107)</td>
<td>2.72 (0.107)</td>
<td>2.72 (0.107)</td>
<td>----</td>
<td>2.74* (0.108)</td>
</tr>
<tr>
<td>2500 MN/m² (362.5 ksi)</td>
<td>1.709 (0.0673)</td>
<td>1.717 (0.0676)</td>
<td>1.717 (0.0676)</td>
<td>1.709 (0.0673)</td>
<td>----</td>
</tr>
<tr>
<td>250000 MN/m² (36250 ksi)</td>
<td>0.673 (0.0265)</td>
<td>0.706 (0.0278)</td>
<td>0.706 (0.0278)</td>
<td>0.706 (0.0278)</td>
<td>0.683⁺ (0.0269)</td>
</tr>
</tbody>
</table>

* Assumes $k_{sc} = 0$
⁺ Assumes $k_{sc} \to \infty$

$E_A = E_B = 200000$ MN/m² (29000 ksi)
TABLE 7
NUMERICAL COMPARISONS - CANTILEVER BEAM - INVERTED T SECTION

![Diagram of cantilever beam with inverted T section]

DEFLECTION $\text{mm (in)}$

<table>
<thead>
<tr>
<th>$k_{sc}$</th>
<th>ELEM-I</th>
<th>ELEM-II</th>
<th>ELEM-III</th>
<th>ASSEM.</th>
<th>BM.THEORY</th>
</tr>
</thead>
<tbody>
<tr>
<td>$25.0 \text{ MN/m}^2$ (3.625 ksi)</td>
<td>28.63</td>
<td>28.63</td>
<td>28.63</td>
<td>-----</td>
<td>29.18*</td>
</tr>
<tr>
<td></td>
<td>(1.127)</td>
<td>(1.127)</td>
<td>(1.127)</td>
<td></td>
<td>(1.149)</td>
</tr>
<tr>
<td>$2500 \text{ MN/m}^2$ (362.5 ksi)</td>
<td>13.69</td>
<td>13.74</td>
<td>13.74</td>
<td>13.74</td>
<td>-----</td>
</tr>
<tr>
<td></td>
<td>(0.539)</td>
<td>(0.541)</td>
<td>(0.541)</td>
<td>(0.541)</td>
<td></td>
</tr>
<tr>
<td>$250000 \text{ MN/m}^2$ (36250 ksi)</td>
<td>7.95</td>
<td>8.05</td>
<td>8.05</td>
<td>8.05</td>
<td>7.95+</td>
</tr>
<tr>
<td></td>
<td>(0.313)</td>
<td>(0.317)</td>
<td>(0.317)</td>
<td>(0.317)</td>
<td>(0.313)</td>
</tr>
</tbody>
</table>

* Assumes $k_{sc} = 0$
+ Assumes $k_{sc} \rightarrow \infty$

$E_A = E_B = 200000 \text{ MN/m}^2$ (29000 ksi)
### TABLE 8
**NUMERICAL COMPARISONS - PROPPED CANTILEVER - INVERTED T SECTION**

![Diagram](image)

<table>
<thead>
<tr>
<th>$k_{sc}$</th>
<th>ELEM-I</th>
<th>ELEM-II</th>
<th>ELEM-III</th>
<th>ASSEM.</th>
<th>BM. THEORY</th>
</tr>
</thead>
<tbody>
<tr>
<td>25.0 MN/m²</td>
<td>0.798</td>
<td>0.798</td>
<td>0.798</td>
<td>-----</td>
<td>0.798*</td>
</tr>
<tr>
<td>(3.625 ksi)</td>
<td>(0.0314)</td>
<td>(0.0314)</td>
<td>(0.0314)</td>
<td></td>
<td>(0.0314)</td>
</tr>
<tr>
<td>2500 MN/m²</td>
<td>0.663</td>
<td>0.663</td>
<td>0.663</td>
<td>-----</td>
<td>-----</td>
</tr>
<tr>
<td>(362.5 ksi)</td>
<td>(0.0261)</td>
<td>(0.0261)</td>
<td>(0.0261)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>250000 MN/m²</td>
<td>0.2144</td>
<td>0.2398</td>
<td>0.2395</td>
<td>-----</td>
<td>0.2177+</td>
</tr>
<tr>
<td>(36250 ksi)</td>
<td>(0.00844)</td>
<td>(0.00944)</td>
<td>(0.00943)</td>
<td></td>
<td>(0.00857)</td>
</tr>
</tbody>
</table>

* Assumes $k_{sc} = 0$

+ Assumes $k_{sc} \to \infty$

$$E_A = E_B = 200000 \text{ MN/m}^2 \ (29000 \text{ ksi})$$
### TABLE 9

**NUMERICAL COMPARISONS - FIXED END BEAM - INVERTED T SECTION**

<table>
<thead>
<tr>
<th>$k_{sc}$</th>
<th>ELEM-I</th>
<th>ELEM-II</th>
<th>ELEM-III</th>
<th>ASSEM.</th>
<th>BM. THEORY</th>
</tr>
</thead>
<tbody>
<tr>
<td>25.0 MN/m² (3.625 ksi)</td>
<td>0.455 (0.0179)</td>
<td>0.455 (0.0179)</td>
<td>0.455 (0.0179)</td>
<td>-----</td>
<td>0.457 (0.0180)</td>
</tr>
<tr>
<td>2500 MN/m² (362.5 ksi)</td>
<td>0.406 (0.0160)</td>
<td>0.406 (0.0160)</td>
<td>0.406 (0.0160)</td>
<td>-----</td>
<td>-----</td>
</tr>
<tr>
<td>250000 MN/m² (36250 ksi)</td>
<td>0.1214 (0.00478)</td>
<td>0.1433 (0.00564)</td>
<td>0.1430 (0.00563)</td>
<td>-----</td>
<td>0.1245 (0.00490)</td>
</tr>
</tbody>
</table>

* Assumes $k_{sc} = 0$

+ Assumes $k_{sc} \to \infty$

$E_A = E_B = 200000$ MN/m² (29000 ksi)
TABLE 10
NUMERICAL COMPARISONS - SIMPLE BEAM - INVERTED T SECTION

<table>
<thead>
<tr>
<th>k_{sc}</th>
<th>ELEM-I</th>
<th>ELEM-II</th>
<th>ELEM-III</th>
<th>ASSEM.</th>
<th>BM. THEORY</th>
</tr>
</thead>
<tbody>
<tr>
<td>25.0 MN/m² (3.625 ksi)</td>
<td>1.816 (0.0715)</td>
<td>1.816 (0.0715)</td>
<td>1.816 (0.0715)</td>
<td>-----</td>
<td>1.824* (0.0718)</td>
</tr>
<tr>
<td>2500 MN/m² (362.5 ksi)</td>
<td>1.233 (0.0505)</td>
<td>1.285 (0.0506)</td>
<td>1.285 (0.0506)</td>
<td>-----</td>
<td>-----</td>
</tr>
<tr>
<td>250000 MN/m² (36250 ksi)</td>
<td>0.495 (0.0195)</td>
<td>0.518 (0.0204)</td>
<td>0.518 (0.0204)</td>
<td>-----</td>
<td>0.498⁺ (0.0196)</td>
</tr>
</tbody>
</table>

* Assumes k_{sc} = 0
⁺ Assumes k_{sc} → ∞

E_A = E_B = 200000 MN/m² (29000 ksi)


**TABLE 11**

**NUMERICAL COMPARISONS - CANTILEVER BEAM - T SECTION**

<table>
<thead>
<tr>
<th>( k_{sc} )</th>
<th>ELEM-I</th>
<th>ELEM-II</th>
<th>ELEM-III</th>
<th>ASSEM.</th>
<th>BM.THEORY</th>
</tr>
</thead>
<tbody>
<tr>
<td>25.0 MN/m(^2) (3.625 ksi)</td>
<td>22.25 (0.876)</td>
<td>22.25 (0.876)</td>
<td>22.25 (0.876)</td>
<td>-----</td>
<td>22.61* (0.890)</td>
</tr>
<tr>
<td>2500 MN/m(^2) (362.5 ksi)</td>
<td>11.48 (0.452)</td>
<td>11.51 (0.453)</td>
<td>11.51 (0.453)</td>
<td>-----</td>
<td>-----</td>
</tr>
<tr>
<td>250000 MN/m(^2) (36250 ksi)</td>
<td>6.20 (0.244)</td>
<td>6.27 (0.247)</td>
<td>6.27 (0.247)</td>
<td>-----</td>
<td>6.20+ (0.244)</td>
</tr>
</tbody>
</table>

* Assumes \( k_{sc} = 0 \)
+ Assumes \( k_{sc} \rightarrow \infty \)

\( E_A = E_B = 200000 \text{ MN/m}^2 \) (29000 ksi)
### TABLE 12

NUMERICAL COMPARISONS - PROPPED CANTILEVER

<table>
<thead>
<tr>
<th>$k_{sc}$</th>
<th>ELEM-I</th>
<th>ELEM-II</th>
<th>ELEM-III</th>
<th>ASSEM.</th>
<th>BM.THEORY</th>
</tr>
</thead>
<tbody>
<tr>
<td>25.0 MN/m$^2$ (3.625 ksi)</td>
<td>0.617 (0.0243)</td>
<td>0.617 (0.0243)</td>
<td>0.617 (0.0243)</td>
<td>-----</td>
<td>0.617* (0.0243)</td>
</tr>
<tr>
<td>2500 MN/m$^2$ (362.5 ksi)</td>
<td>0.531 (0.0209)</td>
<td>0.533 (0.0210)</td>
<td>0.533 (0.0210)</td>
<td>-----</td>
<td>-----</td>
</tr>
<tr>
<td>250000 MN/m$^2$ (36250 ksi)</td>
<td>0.1722 (0.00678)</td>
<td>0.1908 (0.00751)</td>
<td>0.1905 (0.00750)</td>
<td>-----</td>
<td>0.1692+ (0.00666)</td>
</tr>
</tbody>
</table>

* Assumes $k_{sc} = 0$

+ Assumes $k_{sc} \rightarrow \infty$

$E_A = E_B = 200000$ MN/m$^2$ (29000 ksi)
TABLE 13
NUMERICAL COMPARISONS - FIXED END BEAM - T SECTION

REF. PLANE

2540 mm (100 in)

508 mm (20 in)

508 mm (20 in)

Section

127 mm (5 in)

762 mm (30 in)

DEFORMATION mm (in)

<table>
<thead>
<tr>
<th>( k_{sc} )</th>
<th>ELEM-I</th>
<th>ELEM-II</th>
<th>ELEM-III</th>
<th>ASSEM.</th>
<th>BM. THEORY</th>
</tr>
</thead>
<tbody>
<tr>
<td>25.0 MN/m(^2) (3.625 ksi)</td>
<td>0.353 (0.0139)</td>
<td>0.353 (0.0139)</td>
<td>0.353 (0.0139)</td>
<td>-----</td>
<td>0.353(*) (0.0139)</td>
</tr>
<tr>
<td>2500 MN/m(^2) (362.5 ksi)</td>
<td>0.323 (0.0127)</td>
<td>0.323 (0.0127)</td>
<td>0.323 (0.0127)</td>
<td>-----</td>
<td>-----</td>
</tr>
<tr>
<td>250000 MN/m(^2) (36250 ksi)</td>
<td>0.0993 (0.00391)</td>
<td>0.1151 (0.00453)</td>
<td>0.1148 (0.00452)</td>
<td>-----</td>
<td>0.0968(+) (0.00381)</td>
</tr>
</tbody>
</table>

* Assumes \( k_{sc} = 0 \)
+ Assumes \( k_{sc} \to \infty \)

\[ E_A = E_B = 200000 \text{ MN/m}^2 \text{ (29000 ksi)} \]
**TABLE 14**

**NUMERICAL COMPARISONS - SIMPLE BEAM - T SECTION**

![Diagram of beam with labeled dimensions and loads](image)

<table>
<thead>
<tr>
<th>k_{sc}</th>
<th>ELEM-I</th>
<th>ELEM-II</th>
<th>ELEM-III</th>
<th>ASSEM.</th>
<th>BM. THEORY</th>
</tr>
</thead>
<tbody>
<tr>
<td>25.0 MN/m² (3.625 ksi)</td>
<td>1.407 (0.0554)</td>
<td>1.407 (0.0554)</td>
<td>1.407 (0.0554)</td>
<td>-----</td>
<td>1.412* (0.0556)</td>
</tr>
<tr>
<td>2500 MN/m² (362.5 ksi)</td>
<td>1.057 (0.0416)</td>
<td>1.057 (0.0416)</td>
<td>1.057 (0.0416)</td>
<td>-----</td>
<td>-----</td>
</tr>
<tr>
<td>250000 MN/m² (36250 ksi)</td>
<td>0.389 (0.0153)</td>
<td>0.406 (0.0160)</td>
<td>0.406 (0.0160)</td>
<td>-----</td>
<td>0.386⁺ (0.0152)</td>
</tr>
</tbody>
</table>

* Assumes k_{sc} = 0

⁺ Assumes k_{sc} → ∞

E_A = E_B = 200000 MN/m² (29000 ksi)
**TABLE 15**

NUMERICAL COMPARISONS - ELEMENT - I
WITH $k_{max}$ - RECTANGULAR SECTION

<table>
<thead>
<tr>
<th>CONFIG.</th>
<th>$k_{sc}$ (ksi)</th>
<th>NO. ELEM</th>
<th>DEFLEC. (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>10000* (1450)</td>
<td>1</td>
<td>11.02 (0.434)</td>
</tr>
<tr>
<td></td>
<td>160000* (23200)</td>
<td>4</td>
<td>10.95 (0.431)</td>
</tr>
<tr>
<td></td>
<td>250000 (36250)</td>
<td>4</td>
<td>10.90 (0.429)</td>
</tr>
<tr>
<td>b</td>
<td>40000* (5800)</td>
<td>2</td>
<td>0.302 (0.0119)</td>
</tr>
<tr>
<td></td>
<td>160000* (23200)</td>
<td>4</td>
<td>0.302 (0.0119)</td>
</tr>
<tr>
<td></td>
<td>250000 (36250)</td>
<td>4</td>
<td>0.284 (0.0112)</td>
</tr>
<tr>
<td>c</td>
<td>40000* (5800)</td>
<td>2</td>
<td>0.1709 (0.00673)</td>
</tr>
<tr>
<td></td>
<td>160000* (23200)</td>
<td>4</td>
<td>0.1725 (0.00679)</td>
</tr>
<tr>
<td></td>
<td>250000 (36250)</td>
<td>4</td>
<td>0.1588 (0.00625)</td>
</tr>
<tr>
<td>d</td>
<td>40000* (5800)</td>
<td>2</td>
<td>0.688 (0.0271)</td>
</tr>
<tr>
<td></td>
<td>160000* (23200)</td>
<td>4</td>
<td>0.686 (0.0270)</td>
</tr>
<tr>
<td></td>
<td>250000 (36250)</td>
<td>4</td>
<td>0.673 (0.0265)</td>
</tr>
</tbody>
</table>

* Value is $k_{max}$ from EQ. 3.37

---

**Figure Description:**

- (a) Diagram showing load and deformation with a label of 4.45 MN (1000 kips).
- (b) Diagram showing load and deformation with a label of 4.45 MN.
- (c) Diagram showing load and deformation with a label of 4.45 MN.
- (d) Diagram showing a 2540 mm (100 in) section with labels 127 mm (5 in) and 508 mm (20 in) for REF. PLANE and SECTION respectively.

---
### TABLE 16

**NUMERICAL COMPARISONS - ELEMENT - I WITH $k_{\text{max}}$ - INVERTED T SECTION**

<table>
<thead>
<tr>
<th>CONFIG.</th>
<th>$k_{sc}$ (ksi)</th>
<th>NO. ELEM</th>
<th>DEFLEC. (in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>13300* (1933)</td>
<td>1</td>
<td>8.03 (0.316)</td>
</tr>
<tr>
<td></td>
<td>213300* (30933)</td>
<td>4</td>
<td>7.98 (0.314)</td>
</tr>
<tr>
<td></td>
<td>250000 (36250)</td>
<td>4</td>
<td>7.95 (0.313)</td>
</tr>
<tr>
<td>b</td>
<td>53300* (7733)</td>
<td>2</td>
<td>0.2189 (0.00862)</td>
</tr>
<tr>
<td></td>
<td>213300* (30933)</td>
<td>4</td>
<td>0.2189 (0.00862)</td>
</tr>
<tr>
<td></td>
<td>250000 (36250)</td>
<td>4</td>
<td>0.2144 (0.00844)</td>
</tr>
<tr>
<td>c</td>
<td>53300* (7733)</td>
<td>2</td>
<td>0.1245 (0.00490)</td>
</tr>
<tr>
<td></td>
<td>213300* (30933)</td>
<td>4</td>
<td>0.1252 (0.00493)</td>
</tr>
<tr>
<td></td>
<td>250000 (36250)</td>
<td>4</td>
<td>0.1237 (0.00487)</td>
</tr>
<tr>
<td>d</td>
<td>53300* (7733)</td>
<td>2</td>
<td>0.500 (0.0197)</td>
</tr>
<tr>
<td></td>
<td>213300* (30933)</td>
<td>4</td>
<td>0.498 (0.0196)</td>
</tr>
<tr>
<td></td>
<td>250000 (36250)</td>
<td>4</td>
<td>0.495 (0.0195)</td>
</tr>
</tbody>
</table>

* Value is $k_{\text{max}}$ from EQ. 3.37

**Legend:**
- (a) 4.45MN (1000 Kips)
- (b) 4.45MN
- (c) 4.45MN
- (d) 2540mm (100 in)

**Diagram:**
- REF. PLANE
- 127mm (5 in)
- 508mm (20 in)
- A
- 508mm
- B
- 508mm
- 254mm (10 in)
- SECTION

---

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TABLE 17
NUMERICAL COMPARISONS - TIMOSHENKO
BEAM ELEMENT

DEFLECTION (mm) (in)

<table>
<thead>
<tr>
<th>SHEAR AREA $A_s$ (in²)</th>
<th>NEW ELEMENT</th>
<th>CONVENTIONAL ELEMENT</th>
<th>BEAM TH. INCL. SHEAR</th>
</tr>
</thead>
<tbody>
<tr>
<td>645 (1.0)</td>
<td>238.671 (9.3965)</td>
<td>238.671 (9.3965)</td>
<td>238.671 (9.3965)</td>
</tr>
<tr>
<td>6452 (10.0)</td>
<td>33.721 (1.3276)</td>
<td>33.721 (1.3276)</td>
<td>33.721 (1.3276)</td>
</tr>
<tr>
<td>64516 (100.0)</td>
<td>13.2255 (0.52069)</td>
<td>13.2255 (0.52069)</td>
<td>13.2255 (0.52069)</td>
</tr>
</tbody>
</table>

$E = 200000 \text{ MN/m}^2$ (29000 ksi)
$G = 76900 \text{ MN/m}^2$ (11154 ksi)

* See Sections 4.2 and 4.4
TABLE 18
PROPERTIES OF CALIFORNIA COMPOSITE GIRDER
(See Figure 20)

SLAB PROPERTIES

\[ E = 22753 \text{ MN/m}^2 \ (3300 \text{ ksi}) \]
\[ v = 0.15 \]

GIRDER PROPERTIES

\[ E = 200000 \text{ MN/m}^2 \ (29000 \text{ ksi}) \]
\[ G = 76900 \text{ MN/m}^2 \ (11154 \text{ ksi}) \]
\[ A = 35710 \text{ cm}^2 \ (60 \text{ in}^2) \]
\[ A_{SB} = 23226 \text{ mm}^2 \ (36 \text{ in}^2) \]
\[ I = 3.332 \times 10^{10} \text{ mm}^4 \ (80050 \text{ in}^4) \]
### TABLE 19
**DEFLECTION AT LOAD - CALIFORNIA GIRDER - SIMPLE SUPPORTS**
**SHEAR DEFORMATION NOT PERMITTED - SEE FIG. 20**

**DEFLECTION (mm)**

<table>
<thead>
<tr>
<th>MESH and DOF (see Figs. 21 and 22)</th>
<th>$k_{sc}$ MN/m(^2)</th>
<th>and $R^*$</th>
<th>$R$</th>
<th>$R$</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>200000 (290000)</td>
<td>2000 (290)</td>
<td>20  (2.9)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>J 23</td>
<td>25.9 (1.02)</td>
<td>.94</td>
<td>.96</td>
<td>59.9 (2.36)</td>
<td>1.00</td>
</tr>
<tr>
<td>K 33</td>
<td>26.7 (1.05)</td>
<td>.97</td>
<td>.98</td>
<td>59.9 (2.36)</td>
<td>1.00</td>
</tr>
<tr>
<td>L 43</td>
<td>27.2 (1.07)</td>
<td>.99</td>
<td>.99</td>
<td>59.9 (2.36)</td>
<td>1.00</td>
</tr>
<tr>
<td>M 156</td>
<td>27.4 (1.08)</td>
<td>----</td>
<td>28.4 (1.12)</td>
<td>----</td>
<td></td>
</tr>
<tr>
<td>A 57</td>
<td>29.2 (1.15)</td>
<td>1.06</td>
<td>30.0 (1.18)</td>
<td>1.04</td>
<td></td>
</tr>
<tr>
<td>B 74</td>
<td>28.4 (1.12)</td>
<td>1.03</td>
<td>29.5 (1.16)</td>
<td>1.03</td>
<td></td>
</tr>
<tr>
<td>C 91</td>
<td>28.2 (1.11)</td>
<td>1.02</td>
<td>29.0 (1.14)</td>
<td>1.01</td>
<td></td>
</tr>
<tr>
<td>D 108</td>
<td>27.9 (1.10)</td>
<td>1.01</td>
<td>29.0 (1.14)</td>
<td>1.01</td>
<td></td>
</tr>
<tr>
<td>E 229</td>
<td>27.7 (1.09)</td>
<td>----</td>
<td>28.7 (1.13)</td>
<td>----</td>
<td></td>
</tr>
</tbody>
</table>

$* R = \frac{\text{DEFLECTION FROM MESHES J, K OR L}}{\text{DEFLECTION FROM MESH M}}$ or

$\frac{\text{DEFLECTION FROM MESHES A, B, C OR D}}{\text{DEFLECTION FROM MESH E}}$ whichever is applicable.

---

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TABLE 20
DEFLECTION AT LOAD - CALIFORNIA GIRDER - SIMPLE SUPPORTS
SHEAR DEFORMATION PERMITTED - SEE FIG. 20

DEFLECTION \( \text{mm} \) (in)

<table>
<thead>
<tr>
<th>MESH and DOF (see Figs. 21 and 22)</th>
<th>( k_{sc} ) ( \text{MN/m}^2 ) (ksi) and ( R^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>200000 (2900)</td>
</tr>
<tr>
<td>J 23 (1.14)</td>
<td>29.0 .95</td>
</tr>
<tr>
<td>K 33 (1.22)</td>
<td>31.0 .93</td>
</tr>
<tr>
<td>L 43 (1.25)</td>
<td>31.3 .93</td>
</tr>
<tr>
<td>M 156 (1.28)</td>
<td>32.5 ----</td>
</tr>
<tr>
<td>A 57 (1.35)</td>
<td>34.3 1.05</td>
</tr>
<tr>
<td>B 74 (1.32)</td>
<td>33.5 1.02</td>
</tr>
<tr>
<td>C 91 (1.31)</td>
<td>33.3 1.02</td>
</tr>
<tr>
<td>D 108 (1.30)</td>
<td>33.0 1.01</td>
</tr>
<tr>
<td>E 229 (1.29)</td>
<td>32.8 ----</td>
</tr>
</tbody>
</table>

* See note - Table 19
### TABLE 21

DEFLECTION AT LOAD - CALIFORNIA GIRDER - FIXED SUPPORTS

SHEAR DEFORMATION NOT PERMITTED - SEE FIG. 20

<table>
<thead>
<tr>
<th>MESH and DOF (see Figs. 21 and 22)</th>
<th>k_{s_\alpha C} MN/m^2 (ksi)</th>
<th>R</th>
<th>200000 (29000)</th>
<th>R</th>
<th>2000 (290)</th>
<th>R</th>
<th>20 (2.9)</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>J 23</td>
<td>5.31 (.209)</td>
<td>.75</td>
<td>6.83 (.269)</td>
<td>.86</td>
<td>18.01 (.709)</td>
<td>1.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>K 33</td>
<td>6.27 (.247)</td>
<td>.89</td>
<td>7.52 (.296)</td>
<td>.94</td>
<td>18.03 (.710)</td>
<td>1.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>L 43</td>
<td>6.63 (.261)</td>
<td>.94</td>
<td>7.75 (.305)</td>
<td>.97</td>
<td>18.03 (.710)</td>
<td>1.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>M 156</td>
<td>7.06 (.273)</td>
<td>---</td>
<td>7.98 (.314)</td>
<td>---</td>
<td>18.06 (.711)</td>
<td>---</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A 57</td>
<td>8.53 (.336)</td>
<td>1.13</td>
<td>9.17 (.361)</td>
<td>1.14</td>
<td>18.01 (.709)</td>
<td>1.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>B 74</td>
<td>7.95 (.313)</td>
<td>1.10</td>
<td>8.66 (.341)</td>
<td>1.07</td>
<td>17.98 (.708)</td>
<td>1.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C 91</td>
<td>7.62 (.300)</td>
<td>1.05</td>
<td>8.38 (.330)</td>
<td>1.04</td>
<td>17.98 (.708)</td>
<td>1.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D 108</td>
<td>7.44 (.293)</td>
<td>1.03</td>
<td>8.23 (.324)</td>
<td>1.02</td>
<td>17.98 (.708)</td>
<td>1.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>E 229</td>
<td>7.24 (.285)</td>
<td>---</td>
<td>8.08 (.318)</td>
<td>---</td>
<td>17.98 (.708)</td>
<td>---</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* See note - Table 19

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TABLE 22

DEFLECTION AT LOAD - CALIFORNIA GIRDER - FIXED SUPPORTS
SHEAR DEFORMATION PERMITTED - SEE FIG. 20

DEFLECTION mm (in)

<table>
<thead>
<tr>
<th>MESH and DOF (see Figs. 21 and 22)</th>
<th>k sc (ksi)</th>
<th>and R* (2.9)</th>
</tr>
</thead>
<tbody>
<tr>
<td>J 23 (29000)</td>
<td>7.19 (.253)</td>
<td>11.02 (.434)</td>
</tr>
<tr>
<td>K 33 (290)</td>
<td>9.45 (.372)</td>
<td>12.14 (.478)</td>
</tr>
<tr>
<td>L 43 (290)</td>
<td>10.52 (.414)</td>
<td>12.41 (.491)</td>
</tr>
<tr>
<td>M 156 (290)</td>
<td>11.51 (.465)</td>
<td>12.88 (.507)</td>
</tr>
<tr>
<td>A 57 (290)</td>
<td>13.72 (.540)</td>
<td>14.38 (.566)</td>
</tr>
<tr>
<td>B 74 (290)</td>
<td>13.08 (.515)</td>
<td>13.82 (.544)</td>
</tr>
<tr>
<td>C 91 (290)</td>
<td>12.73 (.501)</td>
<td>13.54 (.533)</td>
</tr>
<tr>
<td>D 108 (290)</td>
<td>12.52 (.493)</td>
<td>13.36 (.526)</td>
</tr>
<tr>
<td>E 229 (290)</td>
<td>12.29 (.484)</td>
<td>13.18 (.519)</td>
</tr>
</tbody>
</table>

* See note - Table 19
TABLE 23
DEFLECTION AT LOAD - CALIFORNIA GIRDER - FIXED SUPPORTS
SHEAR DEFORMATION PERMITTED - USING LIMITING
VALUES OF $k_{sc}$ - SEE FIG. 20

<table>
<thead>
<tr>
<th>MESH (See Fig. 22)</th>
<th>$k_{sc}$</th>
<th>DEFL.</th>
<th>R</th>
<th>$k_{max}$</th>
<th>DEFL.</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>J 200000 (29000)</td>
<td>7.19 (.263)</td>
<td>.61</td>
<td>1765 (256)</td>
<td>11.40 (.449)</td>
<td>.96</td>
<td></td>
</tr>
<tr>
<td>K 200000 (29000)</td>
<td>9.45 (.372)</td>
<td>.80</td>
<td>3971 (576)</td>
<td>11.63 (.458)</td>
<td>.97</td>
<td></td>
</tr>
<tr>
<td>L 200000 (29000)</td>
<td>10.52 (.414)</td>
<td>.89</td>
<td>7060 (1024)</td>
<td>11.73 (.462)</td>
<td>.98</td>
<td></td>
</tr>
<tr>
<td>M 200000 (29000)</td>
<td>11.81 (.465)</td>
<td>---</td>
<td>44106 (6397)</td>
<td>11.94 (.470)</td>
<td>---</td>
<td></td>
</tr>
</tbody>
</table>

* See note - Table 19
+ From EQ. 3.37
### TABLE 24

DEFLECTION AT LOAD - CALIFORNIA GIRDER - FIXED SUPPORTS
SHEAR DEFORMATION NOT PERMITTED - TWO SETS OF
REFERENCE PLANES - SEE FIG. 20

<table>
<thead>
<tr>
<th>MESH (See Fig. 22)</th>
<th>DEFLECTION mm (in)</th>
<th>k\textsuperscript{sc} ( \text{MN/m}^2 ) (ksi)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>200000 (29000)</td>
<td>2000 (290)</td>
</tr>
<tr>
<td></td>
<td>1*</td>
<td>2</td>
</tr>
<tr>
<td>J</td>
<td>5.31 (.209)</td>
<td>5.31 (.209)</td>
</tr>
<tr>
<td>K</td>
<td>6.27 (.247)</td>
<td>6.27 (.247)</td>
</tr>
<tr>
<td>L</td>
<td>6.63 (.261)</td>
<td>6.63 (.261)</td>
</tr>
<tr>
<td>M</td>
<td>7.06 (.278)</td>
<td>7.06 (.278)</td>
</tr>
</tbody>
</table>

* Reference Plane

<table>
<thead>
<tr>
<th>Position (See Fig. 16)</th>
<th>( \overline{Z} ) mm (in)</th>
<th>( \overline{Z}_B ) mm (in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0 (0.0)</td>
<td>1447.8 (57.0)</td>
</tr>
<tr>
<td>2</td>
<td>2533.7 (99.75)</td>
<td>-1085.9 (-42.75)</td>
</tr>
</tbody>
</table>

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TABLE 25
DEFLECTION AT LOAD - CALIFORNIA GIRDER - FIXED SUPPORTS
SHEAR DEFORMATION PERMITTED - TWO SETS OF
REFERENCE PLANES - SEE FIG. 20

<table>
<thead>
<tr>
<th>MESH (See Fig. 22)</th>
<th>DEFLECTION mm (in)</th>
<th>( k_{sc} ) MN/m² (ksi)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>200000 (29000)</td>
<td>2000 (290)</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>J</td>
<td>7.44 (.293)</td>
<td>7.19 (.283)</td>
</tr>
<tr>
<td>K</td>
<td>9.75 (.384)</td>
<td>9.45 (.372)</td>
</tr>
<tr>
<td>L</td>
<td>10.72 (.422)</td>
<td>10.52 (.414)</td>
</tr>
<tr>
<td>M</td>
<td>11.81 (.465)</td>
<td>11.81 (.465)</td>
</tr>
</tbody>
</table>

* See note - Table 24
TABLE 26
PROPERTIES OF AASHO BRIDGE 3B
(See Figure 25)

SLAB PROPERTIES

\[ E = 37230 \text{ MN/m}^2 (5400 \text{ ksi}) \]
\[ \nu = 0.0 \]

BEAM PROPERTIES

\[ E = 206840 \text{ MN/m}^2 (30000 \text{ ksi}) \]
\[ G = 79550 \text{ MN/m}^2 (11538 \text{ ksi}) \]
\[ A = 11387 \text{ mm}^2 (17.65 \text{ in}^2) \text{ without cover plate} \]
\[ A = 12943 \text{ mm}^2 (20.07 \text{ in}^2) \text{ with cover plate} \]
\[ A_S = 4897 \text{ mm}^2 (7.59 \text{ in}^2) \]
\[ I = 4.096 \times 10^8 \text{ mm}^4 (984 \text{ in}^4) \text{ without cover plate} \]
\[ I = 5.028 \times 10^8 \text{ mm}^4 (1208 \text{ in}^4) \text{ with cover plate} \]
TABLE 27
NUMERICAL COMPARISONS - AASHO BRIDGE 3B -
DEFLECTIONS AT LOAD LINE A
SEE FIGURES 26 AND 27

![Bridge model diagram]

<table>
<thead>
<tr>
<th>MODEL</th>
<th>BEAM NUMBER</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>SAP IV</td>
<td>16.51</td>
<td>16.81</td>
<td>16.66</td>
</tr>
<tr>
<td></td>
<td>(0.650)</td>
<td>(0.662)</td>
<td>(0.656)</td>
</tr>
<tr>
<td>SAP IV*</td>
<td>15.80</td>
<td>16.10</td>
<td>15.95</td>
</tr>
<tr>
<td></td>
<td>(0.622)</td>
<td>(0.634)</td>
<td>(0.629)</td>
</tr>
<tr>
<td>ELEM.-V*</td>
<td>15.62</td>
<td>15.98</td>
<td>15.82</td>
</tr>
<tr>
<td></td>
<td>(0.615)</td>
<td>(0.629)</td>
<td>(0.623)</td>
</tr>
</tbody>
</table>

* Cover plate is extended over full length of beam
TABLE 28
SUMMARY OF ELEMENTS

<table>
<thead>
<tr>
<th>ELEMENT</th>
<th>TYPE OF BEAM A</th>
<th>TYPE OF BEAM B</th>
<th>TOTAL DOF</th>
<th>ORDER OF W-FIELD POLY.</th>
<th>CONSTRAINING EQUATIONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>-I</td>
<td>B-N*</td>
<td>B-N</td>
<td>8</td>
<td>3</td>
<td>Shear flow = 0 at interface</td>
</tr>
<tr>
<td>-II</td>
<td>B-N</td>
<td>B-N</td>
<td>8</td>
<td>3</td>
<td>Equilibrium at interface Force-displ. at interface at X = ( \frac{L}{2} )</td>
</tr>
<tr>
<td>-III</td>
<td>B-N</td>
<td>B-N</td>
<td>8</td>
<td>5</td>
<td>Equilibrium and force-displ. at interface for all X</td>
</tr>
<tr>
<td>TIMO.</td>
<td>T+</td>
<td>N.A.</td>
<td>4</td>
<td>3</td>
<td>Compatibility between rotation fields for all X</td>
</tr>
<tr>
<td>ALT. TIMO.</td>
<td>T</td>
<td>N.A.</td>
<td>4</td>
<td>3</td>
<td>Compatibility between rotation fields at X = 0, ( \frac{L}{2} ), L</td>
</tr>
<tr>
<td>-IV</td>
<td>B-N</td>
<td>T</td>
<td>10</td>
<td>3</td>
<td>Shear flow = 0 at interface Compatibility between rotation fields at X = ( \frac{L}{2} )</td>
</tr>
<tr>
<td>-V</td>
<td>ACM+</td>
<td>T</td>
<td>34(^\circ)</td>
<td>3</td>
<td>N.A.</td>
</tr>
</tbody>
</table>

* Bernoulli-Navier
+ Timoshenko
± ACM plate bending with in-plane displacements
° For a T-beam element composed of two plate elements and one beam
8. FIGURES
Fig. 1  Steel Bridge Superstructure
Fig. 2(a) Composite Beam With and Without Slip
(b) Deflections Due to Bending and Shear
Differences in stress due to shear lag

Longitudinal stress distribution in deck

Elevation

Deflection due to minor axis bending of bottom flange and torsion

Section

Fig. 3 Longitudinal Stress Distribution in Deck and Transverse Deflection of Beams
Fig. 4 LCCT9 and ACM Plate Bending Finite Elements
BERNOULLI–NAVIER BEAMS
(a)

BERNOULLI–NAVIER BEAMS

TIMOSHENKO BEAM
(b)

THIN PLATE
TIMOSHENKO BEAM
(c)

Fig. 5 Types of Composite Beams
Shear connector linkage assembly

(a)

Shear connector linkage assemblies (equally spaced)

Composite beam and assemblage model

(b)

Fig. 6 Assemblage Model
Fig. 7 Beams B24W and B21W
MODELING AS A TWO-DIMENSIONAL STRUCTURE AND UTILIZING THE HALF SYMMETRY OF THE PROBLEM - THE FOLLOWING DEGREES OF FREEDOM ARE HELD EQUAL TO ZERO:

- X DEFLECTIONS - ALL NODES
- Y DEFLECTIONS - NODES AT CL
- Z DEFLECTIONS - NODES AT SUPPORT
- XX ROTATIONS - NODES AT CL
- YY ROTATIONS - ALL NODES
- ZZ ROTATIONS - ALL NODES

SEE TABLE 2 FOR OTHER DIMENSIONS, MATERIAL AND GEOMETRIC PROPERTIES AND LOADS

Fig. 8 Details of Assemblage Model for Beams B24W and B21W
Fig. 9 Numerical Comparisons for Beam B24W

- TEST
- NEWMARK INCL. SHEAR DEF.
- ASSEMBLAGE * INCL. SHEAR DEF.
- ASSEMBLAGE * AND NEWMARK NO SHEAR DEF.

* ELEMENT-Ⅲ PREDICTS THE SAME DEFLECTIONS AS THE ASSEMBLAGE MODEL
Fig. 10 Numerical Comparisons for Beam B21W
Fig. 11 Elements-I, -II, -III and Sign Convention
(positive as shown)
Fig. 12 Equilibrium and Sign Convention for the Composite Element (positive as shown)
Fig. 13 Internal Deformations of the Composite Element

(positive as shown)
Fig. 14(a) Timoshenko Beam Element

(b) Internal Forces

Sign Conventions are positive as shown
Fig. 15(a) Rotation Due to W Field
(b) Rotation Due to θ Field
(c) Slope Due to Shear
Sign Conventions are Positive as Shown
Fig. 16 Element-IV and Sign Convention
(positive as shown)
Fig. 17 Internal Forces for Beam B of Element-IV

Sign Convention is positive as shown
Fig. 18 Finite Element Models Using Element-V
Fig. 19 Bridge Model Using Element-V and Degrees of Freedom
Fig. 20 California Girder
Fig. 21 Assemblage Model Meshes of California Girder
Fig. 22 Element-V Meshes of California Girder

- MESH J, K, and L
  - 2235 mm (88 in)
  - MESH M - NOT SHOWN

- MESH J: 2 @ 6036 mm, 12192 mm (480") NOT SHOWN
- MESH K: 3 @ 4064 mm, 12192 mm NOT SHOWN
- MESH L: 4 @ 3048 mm, 12192 mm NOT SHOWN
- MESH M: 10 @ 1219 mm

- SUPPORT

- 1448 mm (57 in)

- 445 kN (100 Kips)

- Symmetry indicators: X, Y, Z
Fig. 23 Convergence Study on California Girder - Simple Supports

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Fig. 24 Convergence Study on California Girder - Fixed Supports
Fig. 25 AASHO Bridge 3B
Fig. 26 SAP IV Mesh for AASHO Bridge 3B
Fig. 27 Element-V Mesh for AASHO Bridge 3B
Fig. 28 Element-V and DOF for Secondary Effects
Fig. 29  Model of Web as a Plate Bending Element
Fig. 30 Model of Web as Beam Elements
Fig. 31 Model of Continuous Bridge and Bearing Pad End Conditions
Fig. 32 Model of Continuous Bridge Haunched over the Interior Support
Fig. 33 Layered Beam Element
Fig. 34(a) Stress-Strain Curve for Steel
(b) Elastic Core
Fig. 35 Layered Plate Bending Element
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10. NOMENCLATURE

A = Cross-sectional area

A_s = Area effective in shear

E = Modulus of elasticity

EA = Axial rigidity (ExA)

EI = Flexural rigidity (ExI)

ES = Modulus of elasticity x first moment of inertia (ExS)

F = Generalized nodal forces

G = Shear modulus

G_i = Stiffness coefficient for M-θ relation where M and θ are at the same node \( i = 1,2 \)

H_i = Stiffness coefficient for M-θ relation where M and θ are at different nodes \( i = 1,2 \)

I = Second moment of inertia

J_l = Parameter used in formulation of Element-IV

L = Length of element

L_c = Length of shear connector link

M = Moment

N = Axial force in beam

P = Applied force
R = Dimensionless parameter used to study convergence

S = Spacing of shear connector links or first moment of inertia

U = Axial displacement for beams or in-plane displacement in X direction for plate

V = Shear in beams or in-plane displacement in Y direction for plate

W = Displacement in Z direction

X, Y, Z = Coordinate axes - right hand rule

Z_A = Vector from reference plane to centroid of beam A - for Elements-I, -II, -III

Z_BB = Vector from reference plane to centroid of beam B - for Elements-I, -II, -III

Z_I = Vector from reference plane to interface for Elements-I, -II, -III

Z = Vector from reference plane for beam A to reference plane for beam B

Z_A', Z_B = Vector from reference planes for beams A and B to centroids of beams A and B

Z_iA', Z_iB = Vectors from reference planes for beams A and B to interface
\( a_i \) = Coefficients in polynomials for axial displacement field in beam A or in-plane displacement field in X direction for plate \( i = 1, \ldots, 5 \)

\( b_i \) = Coefficients in polynomials for axial displacement field in beam B or in-plane displacement field in Y direction for plate \( i = 1, \ldots, 5 \)

\( c_i \) = Coefficients in polynomials for transverse displacement fields \( i = 1, \ldots, 12 \)

\( c_{tB} \) = Distance from centroid to top of beam B

\( c_{bA} \) = Distance from centroid to bottom of beam A

\( d \) = Distance between centroids of beams A and B

\( d_b \) = Same as \( c_{tB} \) - used in assemblage model

\( d_t \) = Same as \( c_{bA} \) - used in assemblage model

\( d_i \) = Coefficients in polynomial for rotation field in beam B \( i = 1, 2, 3 \)

\( e \) = Distance from reference plane to point on fully composite beam where the bending and axial responses are uncoupled

\( k_c \) = Stiffness of a shear connector

\( k_{\text{max}} \) = Maximum value of \( k_{sc} \)

\( k_{sc} \) = Stiffness of uniform medium used to model the shear connectors
s = Shear flow at interface
θ = Rotation
γ = Shear strain
δ = Difference operator
e = Axial strain
σ = Axial stress
τ = Shear stress

[B] = Strain-displacement matrix

[C] = Matrix relating polynomial coefficients to element node displacements using compatibility and constraining equations

[C1] = Compatibility matrix = [P(X)] evaluated at the nodes

[C2] = Matrix of constraining equations

[CA],[CB] = Coefficient-displacement matrices for axial fields in beams A and B

[CC] = Coefficient-displacement matrix for all displacement fields

[CD] = Coefficient-displacement matrix for rotation field

[CW] = Coefficient-displacement matrix for transverse displacement field

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\[ D \] = Stress-strain or elasticity matrix
\[ F \] = Vector of generalized nodal forces
\[ f \] = Vector of shape functions
\[ K \] = Global stiffness matrix
\[ k \] = A component matrix of the element stiffness matrix
\[ N \] = Matrix defining the shape functions
\[ P(X) \] = Matrix of powers of X defining polynomials
\[ \alpha \] = Vector of polynomial coefficients
\[ \delta \] = Vector of generalized node displacement
\[ \varepsilon \] = Vector of strains
\[ \Gamma \] = Operator matrix to compute strains from the displacement fields
\[ \sigma \] = Vector of stresses

Notes:

1. The use of A or B as a subscript on single variables or within the brackets on matrices indicates that the parameter is applicable to beam A or B. Example: \( U_A \) is the axial displacement field for beam A.

2. The use of C as a subscript indicates the
variable is for a fully composite beam, i.e., no slip.

3. The use of 0 as a subscript indicates that the quantity is referenced to the centroid rather than the reference plane.

4. The use of L or M as a subscript indicates that the quantity is at node L or M. Example: \( w_L \) is the transverse displacement at node L. These are also combined with the beam subscripts (see note 1). Example: \( u_{LA} \) is the axial displacement for beam A at note L.

5. The use of b, s or u as a subscript outside the brackets on matrices indicates that the matrix is derived from the consideration of axial and bending deformation (b), shear deformation (s) or slip (u).

6. The use of e as a superscript on vectors or matrices indicates that the quantities are applicable to the element.
11. APPENDIX A

Stiffness matrices for Element-IV
\[ [k_A]_{b} = \begin{bmatrix}
  \frac{E_A}{L} & 0 & 0 & \frac{E_A}{L} & 0 & 0 & 0 & 0 & 0 \\
  0 & \frac{32}{L^3} (E_I - E_A \bar{z}^2) & \frac{6}{L^2} (-E_I + E_A \bar{z}^2) & 4E_I \frac{A}{L} - \frac{3E_A \bar{z}^2}{L} & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & \frac{6}{L^2} (-E_I + E_A \bar{z}^2) & \frac{2E_I}{L} - \frac{3E_A \bar{z}^2}{L} & 0 & \frac{E_A}{L} & 0 & 0 & \frac{12}{L^3} (E_I - E_A \bar{z}^2) \\
  0 & 0 & 0 & \frac{6}{L^2} (-E_I + E_A \bar{z}^2) & 4E_I \frac{A}{L} - \frac{3E_A \bar{z}^2}{L} & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & \frac{E_A}{L} & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & \frac{E_A}{L} & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & \frac{E_A}{L} & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{E_A}{L} & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{E_A}{L}
\end{bmatrix}\]
\[
[k_{B_s}] = G_{B_s B_s}^{-1}
\]

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**Symmetric**
\[
[k]_u = k_{sc} \begin{bmatrix}
[k_{11}] & [k_{12}] \\
--- & + & --- \\
[k_{21}] & [k_{22}] 
\end{bmatrix}
\]

and

\[
M_1 = - \frac{3Z_A^2}{5} + \frac{7Z_A Z_{iA}}{10} - \frac{Z_{iA}^2}{10} - \frac{J_{1L}^2}{20} C_{bA} C_{tB} - \frac{J^2}{80} C_{tB}^2 + \frac{3Z_A J_{1L}^2}{20} C_{tB} - \frac{Z_{iA} J_{1L}^2}{40} C_{tB}
\]

\[
M_2 = \frac{Z_A Z_{iB}}{2} - \frac{Z_{iA} Z_{iB}}{2} - \frac{J_{1L}^2}{10} C_{bA} C_{tB} - \frac{Z_{iB} J_{1L}^2}{8} C_{tB} - \frac{J^2}{40} C_{tB}^2
\]

\[
M_3 = \frac{3Z_A^2}{10} - \frac{Z_A Z_{iA} L}{10} + \frac{2Z_{iA}^2}{15} - \frac{Z_A J_{1L}^3}{20} C_{tB} + \frac{Z_{iA} J_{1L}^3}{120} C_{tB} + \frac{J_{1L}^5}{480} C_{tB}^2
\]

\[
M_4 = - \frac{Z_A Z_{iB} L}{4} - \frac{Z_{iA} Z_{iB} L}{12} - \frac{Z_A J_{1L}^3}{20} C_{tB} + \frac{Z_{iA} J_{1L}^3}{120} C_{tB} + \frac{Z_{iB} J_{1L}^3}{48} C_{tB} + \frac{J_{1L}^5}{240} C_{tB}^2
\]

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\[
[k_{11}] = \begin{bmatrix}
\frac{L}{3} & -\frac{L}{3} & \frac{C_{BA}}{2} & -\frac{Z_{BA} L}{4} + \frac{Z_{BA} L}{12} & -\frac{Z_{BA} L}{3} \\
-\frac{L}{3} & \frac{L}{3} & -\frac{C_{BA}}{2} & -\frac{Z_{BA} L}{4} + \frac{Z_{BA} L}{12} & -\frac{Z_{BA} L}{3} \\
\frac{C_{BA}}{2} & -\frac{C_{BA}}{2} & 0 & \frac{6}{5} C_{BA} & 0 \\
\frac{J_{BA}}{8} C_{CB} & -\frac{J_{BA}}{8} C_{CB} & \frac{J_{BA}}{8} - \frac{Z_{BA} L}{12} & 0 & 0 \\
\frac{Z_{BA} L}{4} + \frac{Z_{BA} L}{12} & -\frac{Z_{BA} L}{4} - \frac{Z_{BA} L}{12} & 0 & 0 & 0 \\
\frac{Z_{BA} L}{3} & -\frac{Z_{BA} L}{3} & \frac{Z_{BA} L}{3} & 0 & 0 \\
-\frac{J_{BA}}{24} C_{CB} & \frac{J_{BA}}{24} C_{CB} & \frac{J_{BA}}{24} - \frac{Z_{BA} L}{24} & 0 & 0 \\
\frac{J_{BA}}{24} C_{CB} & -\frac{J_{BA}}{24} C_{CB} & \frac{J_{BA}}{24} - \frac{Z_{BA} L}{24} & 0 & 0 \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
L / 6 & -L / 6 & -C_{bA} / 2 & -J_{1L}^2 / 8 C_{tB} & -Z_{1bL} / 4 - Z_{1aL} / 12 \\
- L / 6 & L / 6 & C_{bA} / 2 & -J_{1L}^2 / 8 C_{tB} & -J_{1L}^3 / 48 C_{tB} \ \\
- L / 6 & L / 6 & C_{bA} / 2 & -J_{1L}^2 / 8 C_{tB} & -J_{1L}^3 / 48 C_{tB} \ \\

\frac{C_{bA}}{2} & \frac{C_{bA}}{2} & -\frac{6}{5} C_{bA} & -\frac{4}{3} C_{bA} C_{tB} & -\frac{3}{5} J_{1L}^2 C_{bA} C_{tB} - \frac{3}{5} J_{1L}^2 L \frac{C_{tB}}{40} - C_{tB}^2 \ \\
+ \frac{J_{1L}^2}{8} C_{tB} & - \frac{J_{1L}^2}{8} C_{tB} & - \frac{J_{1L}^3}{3} C_{bA} C_{tB} & -\frac{3}{5} J_{1L}^2 C_{bA} C_{tB} & -\frac{3}{5} J_{1L}^2 L \frac{C_{tB}}{40} - C_{tB}^2 \ \\
- \frac{Z_{1L}}{4} - \frac{Z_{1aL}}{12} & - \frac{Z_{1L}}{4} + \frac{Z_{1aL}}{12} & \frac{Z_{1aL}}{4} & \frac{Z_{1aL}}{4} + \frac{Z_{1bL}}{6} & \frac{Z_{1aL}}{4} + \frac{Z_{1bL}}{6} + \frac{Z_{1L} J_{1L}^3}{12} C_{tB} \ \\
- \frac{Z_{1aL}}{4} - \frac{Z_{1bL}}{12} & - \frac{Z_{1bL}}{6} + \frac{Z_{1aL}}{12} & \frac{Z_{1bL}}{6} & \frac{Z_{1bL}}{6} + \frac{Z_{1aL} J_{1L}^3}{12} C_{tB} & \frac{Z_{1bL}}{6} + \frac{Z_{1aL} J_{1L}^3}{12} C_{tB} + \frac{Z_{1L} J_{1L}^3}{12} C_{tB} \ \\
\end{bmatrix}
\]
\[ [k_{21}] = \begin{bmatrix}
\frac{L}{6} & -\frac{L}{6} & \frac{c_{BA}}{2} & \frac{\bar{z}_L}{4} - \frac{z_{1A}L}{12} & -\frac{z_{1B}L}{6} \\
-\frac{L}{6} & \frac{L}{6} & -\frac{c_{BA}}{2} & -\frac{\bar{z}_L}{4} + \frac{z_{1A}L}{12} & -\frac{z_{1B}L}{6} \\
-\frac{L}{6} & \frac{L}{6} & -\frac{c_{BA}}{2} & -\frac{\bar{z}_L}{4} - \frac{z_{1A}L}{12} & -\frac{z_{1B}L}{6} \\
\end{bmatrix}
\]
$$
\begin{array}{|c|c|c|c|}
\hline
k \cup k' & \frac{z}{4L} - \frac{3}{4L} \cdot \frac{1}{\sin \alpha} & \frac{z}{4L} + \frac{3}{4L} \cdot \frac{1}{\sin \alpha} & \frac{3}{4L} - \frac{1}{\sin \alpha} \\
\hline
-\mu & \frac{-2}{4L} \cdot \frac{1}{\sin \alpha} & \frac{-2}{4L} \cdot \frac{1}{\sin \alpha} & \frac{-2}{4L} \cdot \frac{1}{\sin \alpha} \\
\hline
+\mu & \frac{2}{4L} \cdot \frac{1}{\sin \alpha} & \frac{2}{4L} \cdot \frac{1}{\sin \alpha} & \frac{2}{4L} \cdot \frac{1}{\sin \alpha} \\
\hline
\end{array}
$$
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