Stress intensity factors in a cracked elastic strip subjected to stamp loading.

Herman Frederick Nied

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STRESS INTENSITY FACTORS
IN A CRACKED ELASTIC STRIP
SUBJECTED TO STIFF LOADING

by
Herman Frederick Nied

A Thesis
Presented to the Graduate Committee
of Lehigh University
in Candidacy for the Degree of
Master of Science
in
Applied Mechanics

Lehigh University
1978
This thesis is accepted and approved in partial fulfillment of the requirements for the degree of Master of Science.

April 24, 1978

Professor in Charge

Chairman of Department
ACKNOWLEDGEMENTS

The author wishes to express sincere gratitude to Dr. Fazil Erdogan who directed this work to completion. His patience and understanding have been greatly appreciated.

Special thanks are also due to Dr. Lehmet Civelek whose suggestions, assistance and cheerful enthusiasm was of inestimable help.
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ABSTRACT

An infinite elastic strip with either an internal or edge crack, supported at two points and subjected to stamp loading is considered. The stamp may be either flat or curved, rigid or elastic.

The formulated problem is of a mixed boundary value type where a system of coupled singular integral equations is obtained to determine stress intensity factors and unknown contact stresses.

The integral equations are solved numerically using an approximate quadrature technique. Detailed numerical results are reported for various crack lengths and loading conditions due to the various types of stamps.
I. INTRODUCTION

Since the establishment of the field of Fracture Mechanics, an intensive effort has been made to analytically determine and catalogue stress intensity factors. These have been determined for an increasingly wide variety of geometries and loading conditions.

The purpose of this paper is to formulate and solve the problem of a cracked strip supported at two points and loaded by a punch (see Figure 1). This problem has significance beyond ordinary engineering applications, in that it models the three-point bend test used extensively to determine experimentally the material parameter $K_{IC}$. Thus, any improvement or extension over existing numerical solutions, such as in [6], would be most useful in better determining $K_{IC}$ values.

The problem as formulated, satisfactorily models a number of problems which have useful engineering applications such as cracked plates or rails loaded by rollers or wheels, cracked bearing races and simply supported beams carrying a center load.

Formulating the problem, two coupled singular integral equations are obtained by use of Fourier transforms. Two separate problems are involved; the infinite strip with a crack perpendicular to its boundaries and
the contact problem indenting an elastic strip. Both of these problems have been investigated extensively. For instance, in [11],[1],[5], and [9] the solution to the problem of a cracked strip has been determined using the techniques of Fourier transforms and singular integral equations. Most notably Erdogan in [7],[3], and Gupta and Erdogan in [9] have addressed the problem of edge cracks, through the concept of generalized Cauchy kernels. Various shaped punches indenting surfaces of elastic continuum were investigated by Muskhelishvili in [12] and more recently using Fourier transforms, Civelek and Erdogan in [2] and Ratwani and Erdogan in [13] have formulated and solved interesting punch contact problems.

In deriving the integral equations an attempt was made to keep the problem as general as possible. Thus a number of important and related problems have been solved. Two specific punch shapes were chosen for final numerical evaluation; Flat punches and Circular punches. As a special case, a very small flat punch was chosen to approximate a point load. The punch may be elastic if it has a rounded shape, but in the present formulation the flat punch is restricted to be rigid. Friction has been included in the mathematical model at the support
points, but in this paper the punch-strip interface has been considered to be frictionless since a satisfactory frictional relationship at the punch-strip interface, which takes into account the plowing effect of the punch, has yet to be developed. Both edge cracks and internal cracks of various lengths are investigated.

The system of singular integral equations is solved numerically using the techniques described in [7], [6], and [10]. For internal cracks, convergence is very rapid and though convergence is considerably slower for edge cracks, satisfactory results are easily obtained.

II. FORMULATION OF THE GENERAL PROBLEM

Figure 1 indicates the geometry and coordinate system used in formulating the general problem. The infinite strip has height $h$ and a crack on the $y$-axis which extends from $b$ to $c$. The contact width of the punch is given by $2a$ and the distance between the two support points is $2d$.

First, omitting the punch portion of the general problem and examining the cracked strip; by superposition the displacement component $u$ is given by $u_1+u_2$ and $v$ given by $v_1+v_2$. Where $u_1$ and $v_1$ are the displacement components in the $x$ and $y$ direction respectively, for
the uncracked strip and $u_2, v_2$ are the $x$ and $y$ displacement components for an infinite body with a through crack (Figure 2).

Assuming the displacements can be written as general Fourier integrals; for the uncracked strip the generalized Fourier transform may be simplified by recognizing that $u_1$ must be an odd function of $x$ and $v_1$ an even function (Figure 3).

For the cracked strip $u$ and $v$ are of the form

$$u = \frac{2}{\pi} \int_0^\infty F_1(\alpha, \gamma) \sin \alpha x \, d\alpha + \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(\beta, x) e^{i\beta y} \, d\beta$$

$$v = \frac{2}{\pi} \int_0^\infty G_1(\alpha, \gamma) \cos \alpha x \, d\alpha + \frac{1}{2\pi} \int_{-\infty}^{\infty} g_1(\beta, x) e^{i\beta y} \, d\beta$$

(1)

substituting into the field equations

$$\mu \nabla^2 u + (\lambda + \mu) \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0$$

$$\mu \nabla^2 v + (\lambda + \mu) \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0$$

(2)

and solving for $f_2$ and $g_2$, the integral representation of $u_2$ and $v_2$ becomes
\[ u_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( A_1 + A_2 x \right) e^{-|\beta|x} e^{i\beta y} d\beta \]  

\[ v_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} i \left( \frac{2\lambda+1}{\beta} A_2 - \frac{1}{\beta} A_1 - \frac{|\beta|}{\beta} A_2 x \right) e^{-|\beta|x} e^{i\beta y} d\beta \]

where \( C = \frac{\lambda}{\lambda + \mu} \) and \( \lambda, \mu \) are Lamé constants. Applying the boundary condition of a shear free surface

\[ T_{x'y} = 2\mu E \varepsilon_{x'y} = \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \bigg|_{x=0} = 0 \]

where \( \mu \) is the shear modulus.

\[ A_2 = \frac{|\beta| A_1}{c+1} \quad \text{and} \quad a = \frac{A_1}{c+1} \]

Rewriting \( u_2 \) and \( v_2 \)

\[ u_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} a(\beta) \left[ \frac{\kappa_1 + 1}{2} + |\beta|x \right] e^{-|\beta|x} e^{i\beta y} d\beta \]

\[ v_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} i a(\beta) \left[ \frac{|\beta| \kappa_1 - 1}{\beta} - \beta x \right] e^{-|\beta|x} e^{i\beta y} d\beta \]

Transforming the Biharmonic equation to get \( P_1 \) and \( G_1 \); \( u_1 \) and \( v_1 \) are expressed in terms of Fourier sine and cosine integrals
\[ u_1 = \frac{2}{\pi} \int_0^\infty \left\{ \frac{1}{\alpha} \left( f_1 + \frac{k_1+1}{2} g_1 + \alpha y g_2 \right) \sinh \alpha y \\
+ \frac{1}{\alpha} \left( f_2 + \frac{k_1+1}{2} g_2 + \alpha y g_1 \right) \cosh \alpha y \right\} \sin \alpha x \, d\alpha \] (5)

\[ u_1 = -\frac{2}{\pi} \int_0^\infty \left\{ \frac{1}{\alpha} \left( f_2 - \frac{k_1-1}{2} g_2 + \alpha y g_1 \right) \sinh \alpha y \\
+ \frac{1}{\alpha} \left( f_1 - \frac{k_1-1}{2} g_1 + \alpha y g_2 \right) \cosh \alpha y \right\} \cos \alpha x \, d\alpha \]

where \( k = 3 - 4\nu \) for plane strain and \( k = \frac{3-\nu}{1+\nu} \) for generalized plane stress. Poisson's ratio is given by \( \nu \) and \( f_1, f_2, g_1, \) and \( g_2 \) are all unknown functions of \( x \).

The displacement components \( u \) and \( v \) are now expressed by equations (4) and (5).

\[ u(x,y) = \frac{2}{\pi} \int_0^\infty \left\{ \frac{1}{\alpha} \left( f_1 + \frac{k_1+1}{2} g_1 + \alpha y g_2 \right) \sinh \alpha y \\
+ \frac{1}{\alpha} \left( f_2 + \frac{k_1+1}{2} g_2 + \alpha y g_1 \right) \cosh \alpha y \right\} \sin \alpha x \, d\alpha \] (6)

\[ + \frac{1}{2\pi} \int_{-\infty}^\infty \alpha(\beta) \left[ \frac{k_1+1}{2} + |\beta| x \right] e^{-|\beta| x} e^{i\beta y} \, d\beta \]

\[ v(x,y) = -\frac{2}{\pi} \int_0^\infty \left\{ \frac{1}{\alpha} \left( f_2 - \frac{k_1-1}{2} g_2 + \alpha y g_1 \right) \sinh \alpha y \\
+ \frac{1}{\alpha} \left( f_1 - \frac{k_1-1}{2} g_1 + \alpha y g_2 \right) \cosh \alpha y \right\} \cos \alpha x \, d\alpha \] (7)
At this point in the formulation \( T_{xy}(0,y) = 0 \) is the only condition satisfied. The unknown functions \( f_1(\alpha), f_2(\alpha), \quad e_1(\alpha), e_2(\alpha), \) and \( a(\alpha) \) are determined by applying the loading and crack boundary conditions for \( x > 0 \).

On the boundaries \( y = h \) and \( y = 0 \) of the elastic strip the boundary conditions as depicted in Figure 4 are:

\[
\begin{align*}
\left. \sigma_{xy} \right|_{y = 0} &= q \delta(x - x_0) \\
\left. \sigma_{yy} \right|_{y = 0} &= -\rho \delta(x - x_0) \quad (8)
\end{align*}
\]

\[
\begin{align*}
\left. \sigma_{xy} \right|_{y = h} &= Q \delta(x - d) \\
\left. \sigma_{yy} \right|_{y = h} &= -P \delta(x - d) \quad (9)
\end{align*}
\]

The fifth boundary condition is a mathematical dislocation on \( x = 0 \).

\[
\frac{\partial u_0(0,y)}{\partial y} = F \delta(y - y_0) \quad (10)
\]

In the friction boundary conditions in (8) and (9) \( q = \eta_1 \rho \) and \( \zeta = \eta_2 \rho \). The coefficients of friction are given by \( \eta_1 \) and \( \eta_2 \).

To apply the boundary conditions given by equations (8), (9), and (10) the stresses in the strip must be
determined from the Hooke's law.

\[ \sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij} \]

(11)

\[ \varepsilon_{ij} = \frac{1}{2} \left( u_{i,j} + u_{j,i} \right) \]

for example:

\[ \frac{\sigma_{xx}}{2\mu} = \frac{\kappa_1 + 1}{2(\kappa_1 - 1)} \frac{\partial u}{\partial x} + \frac{3 - \kappa_1}{2(\kappa_1 - 1)} \frac{\partial u}{\partial y} \]

From equations (6), (7), and (11) the stresses can be expressed as,

\[ \frac{\sigma_{xx}}{2\mu_1} = \frac{2}{\pi} \int_0^{\infty} \left\{ \left( f_1 + 2g_1 + \alpha y g_2 \right) \sinh \alpha y \right\} \cos \alpha x \, d\alpha \]

(12)

\[ + \left( f_2 + 2g_2 + \alpha y g_1 \right) \cosh \alpha y \right\} \cos \alpha x \, d\alpha \]

\[ - \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ |\beta| + \beta^2 x \right] a(\beta) e^{-1|\beta| x} e^{i\beta y} \, d\beta \]

\[ \frac{\sigma_{yy}}{2\mu_1} = -\frac{2}{\pi} \int_0^{\infty} \left\{ \left( f_1 + \alpha y g_2 \right) \sinh \alpha y \right\} \cos \alpha x \, d\alpha \]

(13)

\[ + \left( f_2 + \alpha y g_1 \right) \cosh \alpha y \right\} \cos \alpha x \, d\alpha \]

\[ - \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ |\beta| - \beta^2 x \right] a(\beta) e^{-1|\beta| x} e^{i\beta y} \, d\beta \]
\[
\frac{\sigma_{xy}}{2\mu_1} = \frac{2}{\pi} \int_0^\infty \left\{ \left( f_2 + g_2 + \alpha y g_1 \right) \sinh \alpha y \right. \\
\left. + \left( f_1 + g_1 + \alpha y g_2 \right) \cosh \alpha y \right\} \sin \alpha x \, d\alpha \\
+ \frac{i}{2\pi} \int_{-\infty}^\infty \alpha(\beta) |\beta| x e^{-|\beta|x} e^{i\beta y} \, d\beta
\]

The boundary conditions (5), (7), and (10) are imposed through the stress equations (12), (13), and (14), and the Fourier integrals inverted for the unknown functions. (See Appendix A for a table of integrals)

Thus from (10) we find

\[
\alpha(\beta) |\beta| = \frac{2 F}{i(k_1 + 1)} e^{-i\beta y_o}
\]

Substituting this value for \(\alpha(\beta)\) in the other boundary conditions and evaluating the remaining inversion integrals from equations (5) and (9) it follows that

\[
f_2(\alpha) = \frac{1}{2\mu_1} \rho \cos \alpha x_o + \frac{1}{1 + k_1} F y_o e^{-\alpha y_o}
\]

(16)

\[
\left( f_1 + \alpha h g_2 \right) \sinh \alpha h + \left( f_2 + \alpha h g_1 \right) \cosh \alpha h =
\]

(17)
\[
\frac{1}{2\mu_1} \rho \cos \alpha d - \frac{F}{\kappa_{1+1}} \alpha (h-y_0) e^{-\alpha(h-y_0)}
\]

\[
f_1 + g_1 = \frac{g}{2\mu_1} \sin \alpha x_0 - \frac{F}{\kappa_{1+1}} (1-\alpha y_0) e^{-\alpha y_0} \tag{16}
\]

\[
(f_2 + g_2 + \alpha h g_1) \sinh \alpha h + (f_1 + g_1 + \alpha h g_2) \cosh \alpha h
\]

\[
= \frac{Q}{2\mu_1} \sin \alpha d - \frac{F}{\kappa_{1+1}} \left[1 - \alpha (h-y_0)\right] e^{-\alpha(h-y_0)} \tag{19}
\]

Equations (16), (17), (18), and (19) are solved for the unknown functions \(f_1, f_2, g_1,\) and \(g_2\) giving

\[
f_1 = \frac{1}{\sinh^2(\alpha h) - \alpha^2 h^2} \left\{-[\alpha h + \cosh(\alpha h) \sinh(\alpha h)] A_1 - \alpha^2 h^2 A_2 + \left[\sinh(\alpha h) + \alpha h \cosh(\alpha h)\right] A_3 - \alpha h \sinh(\alpha h) A_4\right\} \tag{20}
\]

\[
f_2 = A_1 \tag{21}
\]

\[
g_1 = \frac{1}{\sinh^2(\alpha h) - \alpha^2 h^2} \left\{[\alpha h + \cosh(\alpha h) \sinh(\alpha h)] A_1 + \sinh^2(\alpha h) A_2 - \left[\sinh(\alpha h) + \alpha h \cosh(\alpha h)\right] A_3\right\} \tag{22}
\]
\[
\begin{align*}
&+ a h \sinh (a h) A_4 \\
&
g_2 = \frac{1}{\sinh^2(a h) - a^2 h^2} \left\{ -\sinh^2(a h) A_1 + \left[ a h \\
&- \cosh(a h) \sinh(a h) \right] A_2 + \left[ a h \sinh(a h) \right] A_3 \\
&+ \left[ \sinh(a h) - a h \cosh(a h) \right] A_4 \right\} 
\end{align*}
\]

with:

\[
A_1 = \frac{1}{2 \mu} \rho \cos \alpha x_0 + \frac{1}{1 + k_i} F y_0 \alpha e^{-\alpha y_0}
\]

\[
A_2 = \frac{1}{2 \mu} \rho \sin \alpha x_0 - \frac{F}{k_i + 1} (1 - \alpha y_0) e^{-\alpha y_0}
\]

\[
A_3 = \frac{1}{2 \mu} \rho \cos \alpha d - \frac{F}{k_i + 1} \alpha (h - y_0) e^{-\alpha (h - y_0)}
\]

\[
A_4 = \frac{Q}{2 \mu} \sin \alpha d - \frac{F}{k_i + 1} \left[ 1 - \alpha (h - y_0) \right] e^{-\alpha (h - y_0)}
\]

2.1) INTRODUCTION OF PUNCH BOUNDARY CONDITION

When the punch is introduced into the problem, it is assumed that the contact area between the punch and the strip is known and the contact stress \( \rho \) at the
interface is unknown. This is the inverse problem described by Kuskhelishvili in [12]. If the punch is a flat stamp, the problem is quite simple since the total stamp load $P$ is independent of contact area. Curved punches require an additional equation to inversely determine the total external load from the given contact area. This additional equation is:

$$P = \int_{-a}^{a} \rho(x_o) \, dx_o$$

Since $\rho$ is unknown in the inverse problem an additional boundary condition is necessary along the punch strip interface. If the punch is rigid this condition is given by:

$$\frac{\partial}{\partial x} \nu_1(x, 0) = \frac{\partial}{\partial x} \nu_2(x, 0^-) = f'(x)$$

where $\nu_1$ is the $y$ component of displacement in the strip, $\nu_2$ the $y$ component of displacement in the punch and $f'(x)$ is obtained by differentiating the equation $f(x)$ giving the stamp profile. Matching the derivatives of displacement along the interface will give an integral equation with Cauchy type singularities, whose properties and solution have been thoroughly investigated. If displacement conditions were to be matched along the interface instead, logarithmic singularities would occur which
would be less convenient to integrate.

If the punch is considered to be elastic and the local stamp radius is large compared to the contact area from \(-a\) to \(a\), the stamp may be approximated by an elastic half-space (Figure 5) and the general boundary condition becomes:

$$\frac{\partial}{\partial x} \left[ \nu_1(x,0^+) - \nu_2(x,0^-) \right] = f'(x) \quad (24)$$

Applying the derivative condition to the elastic strip:

$$\frac{\partial \nu_i}{\partial x} = \frac{1}{\pi} \int_0^\infty \left\{ \left( f_1 - \frac{k_i-1}{2} \right) \sinh \alpha y + \left( f_1 - \frac{k_i-1}{2} \right) \cosh \alpha y \right\} \sin \alpha x \, d\alpha$$

$$-\frac{2}{\pi (k_i+1)} \left[ \left( \frac{k_i+1}{2} \right) \frac{x}{x^2 + (y-y_0)^2} + \frac{x[(y-y_0)^2 - x^2]}{[x^2 + (y-y_0)^2]^2} \right]$$

Evaluation of \(\frac{\partial \nu_i}{\partial x}\) at the location of contact with the punch, requires that this be done as a limiting process to preserve the Cauchy type singularity. Terms with \(y\) as a product may immediately be set equal to zero, since in the limit the contribution of these terms will be zero. Also it can be shown that asymptotically as \(\alpha \to \infty\), \(e^{-\alpha y}\) terms cancel and \(e^{\alpha y}\) terms add together. Thus
we are justified in taking the \( \lim_{y \to 0} \) inside the integral and replacing the \( \cosh \alpha y \) term simply with the asymptotic behavior \( e^{-\alpha y} \). Applying this limit process, equation (25) is now written as:

\[
\frac{\partial \psi_i}{\partial x} (x^*, 0) = \lim_{y \to 0} \frac{1}{\pi} \int_0^\infty \left( f_1 - \frac{\kappa_i - 1}{2} \right) e^{-\alpha y} \sin \alpha x \, d\alpha 
\]

\[
- \frac{2 F}{\pi (\kappa_i + 1)} \left[ \left( \frac{\kappa_i + 1}{2} \right) \frac{x}{x^2 + y_o^2} + \frac{x (y^2 - x^2)}{(x^2 + y_o^2)^2} \right]
\]

Also, from equations (12) and (15) the stresses along the y axis at \( x = 0 \) are expressed as:

\[
\frac{\sigma_{xx}}{2 \mu_i} = \frac{2}{\pi} \int_0^\infty \left\{ \left( f_1 + 2 g_1 + \alpha y g_2 \right) \sinh \alpha y 
\right. \\
+ \left. \left( f_2 + 2 g_2 + \alpha y g_1 \right) \cosh \alpha y \right\} \, d\alpha 
\]

\[
- \frac{2 F}{\pi (\kappa_i + 1)} \left[ \frac{1}{y - y_o} \right]
\]

2.2) HALF - SPACE PUNCH MODEL

To fulfill the elastic punch portion of the boundary condition defined by equation (24), the component of displacement in the y direction and its derivative with respect to \( x \) are examined for the half-space depicted in Figure 5.
The boundary conditions imposed on the elastic half-space are given as

\[ \sigma_{y,y}(x,0) = -\rho \delta(x-x_0) \]

\[ \sigma_{x,y}(x,0) = q \delta(x-x_0) \]  

(28)

Assuming the \( x, y \)-components of displacement in the form

\[ u_2(x,y) = \sum_{n=1}^{\infty} F(n,y) \sin \alpha x \, d\alpha \]

\[ v_2(x,y) = \sum_{n=1}^{\infty} G(n,y) \cos \alpha x \, d\alpha \]  

(29)

and using the field equations (2) we get:

\[ u_2(x,y) = \sum_{n=1}^{\infty} \left\{ (A_1 + A_2 y) e^{\alpha y} + (A_3 + A_4 y) e^{-\alpha y} \right\} \sin \alpha x \, d\alpha \]  

(30)

\[ v_2(x,y) = \sum_{n=1}^{\infty} \left\{ \left( -A_1 + \left( \frac{\kappa}{\alpha} - y \right) A_2 \right) e^{\alpha y} + (A_3 + (\frac{\kappa}{\alpha} + y) A_4) e^{-\alpha y} \right\} \cos \alpha x \, d\alpha \]

Equations (30) can immediately be simplified for the half-space problem by recognizing that as \( y \to \infty \) the displacements must remain finite.

\[ u_2(x,y) = \frac{2}{\pi} \int_{0}^{\infty} \left\{ (A_1 + A_2 y) e^{\alpha y} \right\} \sin \alpha x \, d\alpha \]  

(31)
The relevant stress components in the subspace are:

\[
\frac{\sigma_{yy}}{2\mu_1} = \frac{2}{\pi} \int_{\alpha_0}^{\infty} \left\{ -\kappa (A_1 + A_2 \psi) + \frac{\kappa_1 + 1}{2} A_1 \right\} e^{\psi \alpha} \cos \alpha x \, d\alpha
\]

\[
\frac{\sigma_{xy}}{2\mu_2} = \frac{2}{\pi} \int_{\alpha_0}^{\infty} \left\{ \kappa (A_1 + \psi A_2) - \frac{\kappa_1 - 1}{2} A_1 \right\} e^{\psi \alpha} \sin \alpha x \, d\alpha
\]

Applying the boundary conditions from (28) and solving for \(A_1\) and \(A_2\):

\[
A_1 = \frac{1}{\kappa} \left[ \frac{\kappa_1 - 1}{2} \left( -\frac{1}{2\mu_2} \rho \cos \alpha x_o \right) + \frac{\kappa_1 + 1}{2} \left( \frac{1}{2\mu_2} \psi \sin \alpha x_o \right) \right]
\]

\[
A_2 = \frac{-1}{2\mu_2} \rho \cos \alpha x_o + \frac{1}{2\mu_2} \psi \sin \alpha x_o
\]

To determine the derivative of the y-component of displacement with respect to \(x\) at the location of contact with the strip, evaluation of the following equation is required:

\[
\frac{\partial \psi_2}{\partial x} (x, 0) = \lim_{\psi \to 0} \frac{2}{\pi} \int_{0}^{\infty} (\kappa A_1 - \kappa_1 A_2) e^{\psi \alpha} \sin \alpha x \, d\alpha
\]
Combining equations (35) and (36), evaluating the integrals and taking the limit expression (36) becomes:

\[ \frac{\partial \sigma_1}{\partial x}(x,-a) = -\frac{k_1-1}{4 \mu_1} q(x_0) \left[ \delta(x-x_0) + \delta(x+x_0) \right] \]

\[ + \frac{1}{\pi} \frac{k_1+1}{4 \mu_2} \left( \frac{P}{x+x_0} + \frac{P}{x-x_0} \right) \]

Applying the symmetry conditions \( P(x_0) = P(-x_0) \), \( q(x_0) = q(-x_0) \) and considering (37) to be a Green's function for this portion of the problem, (37) is integrated along the length of contact from \(-a\) to \(a\). The derivative condition for the elastic stamp is expressed as

\[ \frac{\partial \sigma_2}{\partial x}(x,-a) = -\left( \frac{k_1-1}{4 \mu_2} \right) q(x) - \frac{1}{\pi} \frac{k_1+1}{4 \mu_2} \int_{-a}^{a} \frac{P(x_0)}{x_0-x} \, dx_0 \]

2.3) EXPRESSION OF THE SINGULAR INTEGRAL EQUATIONS

Before the first integral equation can be expressed from condition (24), the expressions for \( f_1 \) and \( \xi_1 \), (20) and (22), must be substituted into equation (26). After some algebraic manipulations, the necessary asymptotic analysis, and evaluation of limits for \( y \to 0 \), the Cauchy and Fredholm kernels are separated. Applying the symmetry conditions where appropriate and evaluating all other integrals we get:

\[ \text{(39)} \]
\[
\frac{\partial \nu_i}{\partial x}(x, + \epsilon) = -\frac{\kappa_{i-1}}{4}\gamma_i \cdot q(x) + \frac{1}{\pi} \frac{\kappa_i + 1}{4}\gamma_i \left[ \int_{-\alpha}^{\alpha} \frac{P(x)}{x_0 - x} \, dx_0 + \int_{-\alpha}^{\alpha} \frac{P(x)}{x_0 - x} \, dx_0 \right]
+ \int_{-\alpha}^{\alpha} \frac{P(x)}{x_0 - x} \, dx_0 + \int_{-\alpha}^{\alpha} \frac{P(x)}{x_0 - x} \, dx_0
- \int_{-\alpha}^{\alpha} \frac{P(x)}{x_0 - x} \, dx_0 - \int_{-\alpha}^{\alpha} \frac{P(x)}{x_0 - x} \, dx_0
\]
\]

where:
\[
\begin{align*}
K_{i1} &= \int_0^\infty \frac{4\alpha^2 h^2 + 4\alpha h + 2 - 2e^{-2\alpha h}}{e^{2\alpha h} - 4\alpha^2 h^2 - 2 + e^{2\alpha h}} \sin \alpha (x_0 - x) \, d\alpha \\
K_{i2} &= \int_0^\infty \frac{4\alpha^2 h^2}{e^{2\alpha h} - 4\alpha^2 h^2 - 2 + e^{2\alpha h}} \cos \alpha (x_0 - x) \, d\alpha \\
K_{i3} &= -2 \int_0^\infty \frac{e^{\alpha h} (1 + \alpha h) - e^{\alpha h} (1 - \alpha h)}{e^{2\alpha h} - 4\alpha^2 h^2 - 2 + e^{2\alpha h}} \sin \alpha (x_0 - x) \, d\alpha \\
K_{i4} &= 2 \int_0^\infty \frac{e^{\alpha h} (\alpha h) - e^{\alpha h} (\alpha h)}{e^{2\alpha h} - 4\alpha^2 h^2 - 2 + e^{2\alpha h}} \cos \alpha (x_0 - x) \, d\alpha
\end{align*}
\]
\[ k_{15} = \int_{0}^{\infty} \frac{4a^2h^2 + 4ah + 2 - 2e^{-2xh}}{e^{2xh} - 4a^2h^2 - 2 + e^{-2xh}} \alpha e^{-\alpha y_0} \sin \alpha x \, d\alpha \]

\[ k_{16} = \int_{0}^{\infty} \frac{4a^2h^2}{e^{2xh} - 4a^2h^2 - 2 + e^{-2xh}} (1 - \alpha y_0)e^{-\alpha y_0} \sin \alpha x \, d\alpha \]

\[ k_{17} = 2 \int_{0}^{\infty} \frac{\alpha h (1+\alpha h) - e^{-\alpha h}}{e^{2xh} - 4a^2h^2 - 2 + e^{-2xh}} \alpha (h-y_0) e^{-\alpha (h-y_0)} \sin \alpha x \, d\alpha \]

\[ k_{18} = 2 \int_{0}^{\infty} \frac{\alpha h (\alpha h) - e^{-\alpha h}}{e^{2xh} - 4a^2h^2 - 2 + e^{-2xh}} \left[ 1 - \alpha (h-y_0) \right] e^{-\alpha (h-y_0)} \sin \alpha x \, d\alpha \]

Substituting equations (38) and (39) into condition (24) and defining:

\[ y_1 = \frac{k_1 + 1}{\mu_1}, \quad y_2 = \frac{k_2 + 1}{\mu_2} \]

\[ \beta = \frac{y_1}{y_1 + y_2}, \quad A = \pi \eta_1 \left[ \frac{\mu_2 (k_1-1) - \mu_1 (k_2-1)}{\mu_2 (k_1+1) + \mu_1 (k_2+1)} \right] \]

and also observing that

\[ q(x_0) = \text{sgn } p(x_0) \eta_1 \quad Q(x_0) = \text{sgn } P(x_0) \eta_2 \]

we get the first integral equation.
where the relationship between \( p(x) \) and \( P(x) \) is given by the equilibrium condition:

\[
\int_{-a}^{a} \frac{P(x)}{x_0 - x} \, dx_0 + \beta \int_{-a}^{a} \left( k_{11} - s \gamma \eta_1 k_{12} \right) x_0 \, dx_0
\]

From equation (27) \( F \) and \( \varphi \) are separated and the other integral equation can be written as:

\[
\frac{2 \pi}{\pi (k_{i+1})} \left\{ \frac{1}{y_0 - y} + \frac{\varphi}{2 \mu_1} \right\} + \frac{\rho}{2 \mu_1} \left\{ \frac{2}{\pi} k_{\varphi A} (y, x_0) + \frac{2}{\pi} k_{\varphi B} (y, x_0) \right\} = 0
\]
Integrating the dislocation function $F$ from $b$ to $c$ and the contact lengths $-a$ to $a$, $-d$ to $d$, the equation becomes:

$$
\int_{b}^{c} \left[ \frac{1}{y_{o} - y} + K_{I}(y, y_{o}) \right] F(y_{o}) \, dy_{o}
$$

$$
+ \frac{k_{i+1}}{4\mu_{i}} \int_{-a}^{a} \rho(x_{o}) K_{II}^{}(y, x_{o}) \, dx_{o} + \frac{k_{i+1}}{4\mu_{i}} \int_{-d}^{d} \rho(x_{o}) K_{II}^{}(y, x_{o}) \, dx_{o} = 0
$$

where we again have the equilibrium condition relating $\rho(x_{o})$ and $\rho(x_{o})$:

$$
\int_{-a}^{a} \rho(x_{o}) \, dx_{o} = \int_{-d}^{d} \rho(x_{o}) \, dx_{o}
$$

The kernel $K_{I}$ is expressed in exponential form in appendix B. $K_{I}$ can now be broken into integrations which are all convergent as $\kappa \to \infty$. Writing $K_{I}$ as the sum of these integrals:

$$
K_{I} = \sum_{j=1}^{17} m_{j}
$$

Examining each integral separately, it can be seen that singularities occur whenever $y$ and $y_{o}$ simultaneously go to zero. This occurs in the special case of edge cracks and these singularities (generalized Cauchy kernels)
must be separated so that the integrals \( m_j \) remain convergent. As an example, from appendix B we have:

\[
m_1 = \int_0^\infty \frac{e^{2\alpha h} - e^{-2\alpha h}}{e^{2\alpha h} - 4\alpha h^2 - 2 + e^{-2\alpha h}} y\alpha e^{-\alpha y_o} \sinh ay \, d\alpha
\]

Separating the portion of the integrand which does not go to zero as \( \alpha \to \infty \), and evaluating this integral in closed form; (for integrals see appendix A)

\[
m_1 = \frac{y_o}{2} \left[ \frac{1}{(y_o - y)^2} - \frac{1}{(y_o + y)^2} \right]
\]

\[
+ \int_0^\infty \frac{4\alpha^2 h^2 + 4\alpha h + 2 - 2 e^{-2\alpha h}}{e^{2\alpha h} - 4\alpha h^2 - 2 + e^{-2\alpha h}} y\alpha e^{-\alpha y_o} \sinh ay \, d\alpha
\]

This remaining integral is absolutely convergent.

For some of the integrals there is a special case, when \( y_o = h \), for which the integrals become unbounded. This occurs when edge cracks open on the upper surface of the strip. The generalized Cauchy kernels from these integrals are separated by examining the asymptotic behavior of the integral and letting:

\[
m_j = k_s + \left\{ G(\alpha, y_o, y) - G_\infty(\alpha, y_o, y) \right\}
\]

where; \( G(\alpha, y_o, y) = m_j \),

\( k_s \) is the asymptotic part of the integral evaluated in
closed form, and \( G(\infty, y, y) \) is the portion of the integrand which asymptotically becomes unbounded and when integrated in closed form yields \( K_B \). The term in brackets when integrated will now be absolutely convergent.

Appendix C contains a tabulation of all \( m_j \)'s and the separated generalized Cauchy kernels.

Fredholm kernel \( K_{IIA} \) and \( K_{IIB} \) are now analysed in the same manner as \( K_I \). Appendix D contains \( K_{IIA} \) and \( K_{IIB} \) in exponential form. Following the same procedure as in \( K_I \), \( K_{II} \) can be broken into simple integrations which are all convergent as \( \alpha \to \infty \).

\[
K_{IIA} = \sum_{i=1,8} j_i
\]

\[
K_{IIB} = \sum_{i=7,16} j_i
\]

Expressing \( \cosh\alpha y \) and \( \sinh\alpha y \) in exponential form; combining like exponentials and examining the asymptotic behavior of the kernels, any singularities which occur (when \( y=h \) or \( y=0 \)) are separated and the remaining integrals can be shown to be absolutely convergent. Appendix E contains the integrals \( j_i \) and the separated terms.
evaluated in closed form.

From equations (40) and (41) the system of singular integral equations can be rewritten as:

\[
\int_b^c \left[ \frac{1}{y_* - y} + k_1(y, y_*) \right] F(y_*) \, dy_* + \int_{-a}^a \rho(x_0) k_{II_A}(y, x_0) \, dx_0 + \int_{-a}^a \frac{k_i + 1}{4} \mu_i \int_{-a}^a \rho(x_0) k_{II_B}(y, x_0) \, dx_0 = 0
\]  

(45)

\[
- \lambda \rho(x) + \int_{-a}^a \frac{\rho(x_0)}{x_0 - x} \, dx_0 + \beta \int_{-a}^a \rho(x_0) k_{III_A}(x, x_0) \, dx_0 + \beta \int_{-d}^d \rho(x_0) k_{III_B}(x, x_0) \, dx_0 + \frac{4}{\gamma_1 + \gamma_2} \int_b^c F(y_0) k_{IV}(x, y_0) \, dy_0
\]  

(46)

\[
= \frac{4 \pi}{\gamma_1 + \gamma_2} f'(x)
\]

subject to the equilibrium condition

\[
\int_{-a}^a \rho(x_0) \, dx_0 = \int_{-d}^d \rho(x_0) \, dx_0 ,
\]

and the single-valuedness condition

\[
\int_b^c F(y_0) \, dy_0 = 0
\]  

(47)

-25-
The punch loading force $P$ is included through the equation:

$$\int_{-a}^{a} p(x_0) \, dx_0 = P \quad (48)$$

The Fredholm kernels in (45) are given by:

$$K_1 = \sum_{j=1}^{17} m_j$$

$$K_{II_A} = \sum_{i=1}^{9} j_i$$

$$K_{II_B} = \sum_{i=9}^{16} j_i$$

In equation (46) the terms are defined:

$$A = \pi \eta_1 \left[ \frac{\mu_2 (\kappa_{i-1}) - \mu_1 (\kappa_{i+1})}{\mu_2 (\kappa_{i+1}) + \mu_1 (\kappa_{i+1})} \right]$$

$$\beta = \frac{\gamma_i}{\gamma_i + \gamma_2}$$

$$K_{III_A} = \kappa_{11} - \text{sgn} \eta_1 \kappa_{12}$$

$$K_{III_B} = \kappa_{13} - \text{sgn} \eta_2 \kappa_{14}$$

$$K_{IV} = \frac{-4 x y_0^2}{(x^2 + y_0^2)^2} - \gamma_0 \kappa_{16} + \kappa_{16} - \kappa_{17} + \kappa_{18}$$

$f'(x) =$ derivative of punch shape equation
2.4) SIMPLIFICATION OF THE GENERAL PROBLEM

The general problem described by equations (45) and (46) can be further simplified by considering the case of two concentrated loads applied to the layer at y=h, i.e. the support points at ±d. Here \( P(x) \) is given by

\[
P(x) = P \delta(x - d) + P \delta(x + d)
\]

The last integral in equation (45) is rewritten as

\[
\int_{-d}^{d} P(x) K_B \left( y, x \right) \, dx = P \left[ K_B \left( y, d \right) + K_B \left( y, -d \right) \right]
\]

and from the equilibrium equation (42) we can write

\[
\int_{-a}^{a} \rho(x) \, dx = 2P
\]

so (45) becomes:

\[
\int_{b}^{c} \left[ \frac{1}{y - y_o} + K \left( y, y_o \right) \right] F(y) \, dy_o
\]

\[
+ \frac{K_i + 1}{y \lambda_i} \int_{-a}^{a} \rho(x) K \left( y, x \right) \, dx = 0
\]

where

\[
K \left( y, x \right) = K_A \left( y, x \right) + \frac{1}{2} \left[ K_B \left( y, d \right) + K_B \left( y, -d \right) \right]
\]
Since we will exclude friction along the punch surface, \( A = 0 \) and (46) is reduced from a singular integral equation of the second kind to one of the first kind. With the same point load configuration used in writing (49) the third integral in (46), using equilibrium condition (42), becomes:

\[
\int_{-d}^{d} P(x_o) \kappa_{III}(x,x_o) \, dx_o = P \left[ \kappa_{III}(x,d) + \kappa_{III}(x,-d) \right]
\]

and (46) is reduced to

\[
\int_{-a}^{a} \left[ \frac{1}{x_o - x} + \beta \kappa_{III}(x,x_o) \right] P(x_o) \, dx_o + \frac{4}{\gamma_1 + \gamma_2} \int_{b}^{c} F(y_o) \kappa_{IV}(x,y_o) \, dy_o = \frac{4 \pi}{\gamma_1 + \gamma_2} f'(x)
\]

where:

\[
\kappa_{III}(x,x_o) = \kappa_{III}(x,x_o) + \frac{1}{2} \left[ \kappa_{III}(x,d) + \kappa_{III}(x,-d) \right]
\]

The Fourier integrals giving the Fredholm kernels \( K_{II}, K_{III} \) and \( K_{IV} \) may easily be evaluated by using Gauss-Legendre quadrature. However, if one examines the integrands of these terms it can be seen that these integrals, if treated separately will be divergent as \( \kappa \to 0 \). To
show that these kernels are truly Fredholm kernels, the
integrands for each particular kernel is expanded in a
Taylor series about zero. It can be shown that when
all the integrands for a particular kernel are summed
and equilibrium conditions are included, the coefficients
of the divergent terms will be zero. Though this is a
rather lengthy procedure it must be done to prove that
the infinite integrals $K_{II}$, $K_{III}$ and $K_{IV}$ are convergent
around zero.

III. SPECIAL CASES: SPECIFIC PROBLEMS

3.1) FLAT PUNCH LOADING

For the cracked strip loaded by a flat punch $f'(x)=0$.
The interval of integration for the system of coupled
singular equations (49),(50) is normalized by the
introduction of new variables $s,t,\tau$, and $\xi$.

$$y_0 = \frac{c-b}{2} s + \frac{c+b}{2}$$
$$y = \frac{c-b}{2} t + \frac{c+b}{2}$$
$$x_0 = a \tau$$
$$x = a \xi$$

From (49) and (50) the normalized expression is
normalizing the additional conditions (47) and (48)
\[ \int_{-1}^{1} F(s) \, ds = 0 \]
\[ \int_{-1}^{1} \rho(\tau) \, d\tau = \frac{P}{a} \]

dividing through all equations by \( \frac{P}{a} \) and multiplying by \( \frac{1}{\gamma_i} \); the normalized integral equations with auxiliary conditions, for the flat punch problem become
\begin{align*}
\int_{-1}^{1} \left[ \frac{1}{s-t} + \frac{c-b}{2} k_\pi(t,s) \right] F(s) \, ds \\
\int_{-1}^{1} \rho(\tau) a k_{\pi}(t,\tau) \, d\tau = 0
\end{align*}

(51)
\[
\int_{-1}^{1} \left[ \frac{1}{\tau - \xi} + \alpha \beta k_{3}(\xi, \tau) \right] \rho^{*}(\tau) d\tau
+ \beta \int_{-1}^{1} F^{*}(s) \left( \frac{c-b}{2} \right) k_{4}(\xi, s) ds = 0
\]

\[
\int_{-1}^{1} F^{*}(s) ds = 0
\]  

\[
\int_{-1}^{1} \rho^{*}(\tau) d\tau = 1
\]

\[
\rho^{*}(\tau) = \frac{\rho(\tau)}{\frac{P}{a}} \quad F^{*}(s) = \frac{q}{\gamma} \frac{F(s)}{\frac{P}{a}}
\]

For the flat punch (rigid) \( \gamma_{2} = 0 \)

\[ \beta = \frac{\gamma_{1}}{\gamma_{1} + \gamma_{2}} = 1 \]

3.2) NUMERICAL METHODS

The integral equations (51) and (52) both have integrable singularities at the end points for the case of a flat punch and therefore the index for both equations is given by \( \kappa = 1 \) \[7\]. Defining new functions
\[ F^* (s) = \phi (s) \omega (s) \]

\[ P^* (\tau) = \psi (\tau) \omega (\tau) \]

where the weights \( \omega (s) \) and \( \omega (\tau) \) are the weights of the Gauss-Chebyshev integration formulas.

\[ \omega (s) = (1 - s^2)^{-1/2} \]

\[ \omega (\tau) = (1 - \tau^2)^{-1/2} \]

Following the discussion by Krenk in [10] the solution of the system of singular integral equations becomes the solution of the following quadrature formula, which has the particularly useful feature of having abscissas located at the endpoints of the interval of integration.

\[ \frac{1}{2(n-1)} \left\{ \frac{1}{s_i - t_j} + \frac{c - b}{2} \kappa_1 (s_i, t_j) \right\} \phi (s_i) + \]

\[ \frac{1}{n-1} \sum_{i=2}^{n-1} \left\{ \frac{1}{s_i - t_j} + \frac{c - b}{2} \kappa_1 (s_i, t_j) \right\} \phi (s_i) + \]

\[ \frac{1}{2(m-1)} \left\{ \frac{1}{s_n - t_j} + \frac{c - b}{2} \kappa_1 (s_n, t_j) \right\} \phi (s_n) + \]

\[ \frac{1}{2(m-1)} \left\{ a \kappa_{\Pi} (t_j, t_1) \right\} \psi (t_1) + \]
\begin{align*}
&+ \frac{1}{m-1} \sum_{k=2}^{m-1} a_k \Psi(\tau_k) \\
&+ \frac{1}{2(m-1)} \left\{ a_k \Phi(\xi_k, s) \right\} \Phi(\xi_k) = 0 \\
&\quad (j = 1, \ldots, n-1)
\end{align*}

\begin{align*}
&\frac{1}{2(n-1)} \left\{ \frac{c-b}{2} \beta k_\text{IV}(\xi_k, s_k) \right\} \Phi(s_k) + \\
&\frac{1}{n-1} \sum_{i=1}^{n} \frac{c-b}{2} \beta k_\text{IV}(\xi_k, s_i) \Phi(s_i) + \\
&\frac{1}{2(n-1)} \left\{ \frac{1}{2} \beta k_\text{IV}(\xi_k, s_n) \right\} \Phi(s_n) + \\
&\frac{1}{2(m-1)} \left\{ \frac{1}{\tau_1 - \xi_k} + a \beta k_\text{III}(\xi_k, \tau_1) \right\} \Psi(\tau_1) + \\
&\frac{1}{m-1} \sum_{k=1}^{m} \left\{ \frac{1}{\tau_k - \xi_k} + a \beta k_\text{III}(\xi_k, \tau_k) \right\} \Psi(\tau_k) + \\
&\frac{1}{2(m-1)} \left\{ \frac{1}{\tau_m - \xi_k} + a \beta k_\text{III}(\xi_k, \tau_m) \right\} \Psi(\tau_m) = 0 \\
&\quad (\lambda = 1, \ldots, m-1)
\end{align*}
\[ \frac{1}{2} \phi(S_i) + \sum_{i=1}^{n-1} \phi(S_i) + \frac{1}{2} \phi(S_n) = 0 \]  

(58)

\[ \frac{1}{2} \psi(T_j) + \sum_{\kappa=2}^{\infty} \psi(T_\kappa) + \frac{1}{2} \psi(T_m) = \frac{n-1}{\pi} \]  

(59)

where:

\[ S_i = \cos \left( \frac{i-1}{n-1} \pi \right), \quad i = 1, 2, \ldots, n \]

\[ T_j = \cos \left( \frac{2j-1}{2n-2} \pi \right), \quad j = 1, 2, \ldots, n-1 \]

\[ T_\kappa = \cos \left( \frac{\kappa-1}{m-1} \pi \right), \quad \kappa = 1, 2, \ldots, m \]

\[ \xi_\lambda = \cos \left( \frac{2\lambda-1}{2m-2} \pi \right), \quad \lambda = 1, 2, \ldots, m-1 \]

The infinite, exponentially decaying, integrals which constitute Fredholm kernel \( K_I \) are evaluated using Gauss-Laguerre quadrature. The remaining Fredholm kernels \( K_{II}, K_{III}, \) and \( K_{IV} \), though infinite oscillating integrals, are evaluated between finite limits using Gauss-Legendre quadrature, where it can be shown that the remaining contribution of the integrals above the fixed upper limit is negligible.

Edge cracks are handled simply by modifying the
nature of the singularity at the ends of the crack. Using function-theoretic techniques Erdogan in [3], Erdogan, Gupta and Cook in [7] and Gupta and Erdogan in [9] have thoroughly investigated the nature of these singularities.

When there is an edge crack we can still use the $K=1$ quadrature with only a minor modification. The equations (56), (57), and (59) still apply, but equation (58) is replaced by $\phi(1)=0$ or $\phi(-1)=0$ depending on which surface the edge crack opens on. Doing this forces the condition that the crack slope must be finite and the stress intensity factor on the open edge must equal zero.

The quantities of main importance are the stress intensity factors. These are defined by

$$\kappa(b) = \lim_{\gamma \to b} \sqrt{\gamma (b - \gamma)} \sigma_{xx} (0, \gamma)$$

$$\kappa(c) = \lim_{\gamma \to c} \sqrt{\gamma (c - \gamma)} \sigma_{xx} (0, \gamma)$$

The stress intensity factors are directly related to $\phi(\gamma)$ at the endpoints.

$$\kappa(b) = \phi(-1) \sqrt{\frac{c-b}{2}} \frac{P}{a}$$

$$\kappa(c) = -\phi(1) \sqrt{\frac{c-b}{2}} \frac{P}{a}$$

(appendix P contains the derivation of the stress intensity factor expression)
3.3) CURVED ELASTIC STAMP

For the cracked strip loaded by a symmetric curved stamp \( f'(x) = -x/R \), where \( R \) is the local radius of curvature. After normalization, equations (49) and (50) are expressed as:

\[
\int_{-1}^{1} \left[ \frac{1}{s-t} + \frac{c-b}{2} \kappa_{I}(t,s) \right] F(s) \, ds \\
+ \frac{k_{1}+1}{q \mu_{1}} \int_{-1}^{1} \rho(\tau) \alpha \kappa_{II}(t,\tau) \, d\tau = 0
\]

\[
\int_{-1}^{1} \left[ \frac{1}{\tau - \xi} + a \beta \kappa_{III}(\xi, \tau) \right] \rho(\tau) \, d\tau \\
+ \frac{4}{\gamma_{1} + \gamma_{2}} \int_{-1}^{1} F(s) \left( \frac{c-b}{2} \right) \kappa_{IV}(\xi, s) \, ds = \frac{4 \pi}{\gamma_{1} + \gamma_{2}} \left( \frac{-a \xi}{R} \right)
\]

with the auxiliary conditions; single-valuedness and equilibrium written as:

\[
\int_{-1}^{1} F(s) \, ds = 0 \\
\int_{-1}^{1} \rho(\tau) \, d\tau = \frac{p}{a}
\]

Further normalization and rearrangement is accomplished by dividing by \( \frac{4}{\gamma_{1} + \gamma_{2}} \frac{a}{R} \) and defining:

\[
\rho^{*}(\tau) = \rho(\tau) \frac{\gamma_{1} + \gamma_{2}}{4} \frac{R}{a}
\]

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\[ F^*(s) = F(s) \frac{R}{a} \]

After further manipulation the system of equations becomes

\[ \int_{-1}^{1} \left[ \frac{1}{s-t} + \frac{c-b}{2} \kappa_{II}(t,s) \right] F^*(s) \, ds \\
+ \beta \int_{-1}^{1} \rho^*(\tau) a \kappa_{II}(t,\tau) \, d\tau = 0 \\
\]

(62)

\[ \int_{-1}^{1} \left[ \frac{1}{\tau-\xi} + \alpha \beta \kappa_{III}(\xi,\tau) \right] \rho^*(\tau) \, d\tau \\
+ \int_{-1}^{1} F^*(s) \left( \frac{c-b}{2} \right) \kappa_{IV}(\xi,\tau) \, ds = -D \xi \\
\]

(63)

\[ \int_{-1}^{1} F^*(s) \, ds = 0 \]

(64)

\[ \int_{-1}^{1} \rho^*(\tau) \, d\tau = \lambda \]

(65)

where \[ \lambda = \frac{P}{a} \left( \gamma_1 + \gamma_2 \right) \frac{R}{a} \]

Unlike the flat punch problem, here the unknown function \(\rho^*(\tau)\) is bounded at the endpoints \(x = \pm a\). Hence
the index for the singular integral equation (63) is -1.

The curved punch contact width 2a is an unknown, as is $\rho^*(\tau)$, and it is clear that the contact width will not be independent of the magnitude of the external stamp load $P$ as it was for the flat stamp. The problem is solved in an inverse manner; namely, it is assumed that instead of the load $P$, the contact half-width $a$, is specified and the integral equations (62)-(64) are solved for various crack lengths [13]. After determining $\rho^*(\tau)$ for each given crack length and contact half-width $a$, $P$ is determined from (65).

3.4) NUMERICAL SOLUTION FOR CURVED ELASTIC STAMP

Again following the technique as outlined by Krenk [10]:

$$ F^*(s) = \phi(s) \omega(s) \quad \text{where} \quad \omega(s) = (1 - s^2)^{-1/2} $$

$$ \rho^*(\tau) = \psi(\tau) \omega(\tau) \quad \text{where} \quad \omega(\tau) = (1 - \tau^2)^{-1/2} $$

where $\omega(s), \omega(\tau)$ is the weight function of the Gauss-Chebyshev quadrature. The general method for numerical solution from (56) and (57) still applies, with the appropriate rearrangement of material constant terms and the inclusion of $-\frac{\xi}{1}$ on the right hand side of (57).
The extra conditions for equation (63), which has an index of −1, are
\[ \psi(T_i) = 0 \quad \psi(T_n) = 0 \]
Equation (59) is omitted and again the same auxiliary equation given by (58) is used. It should be noted that the index of (62) remains +1 since the nature of this equation's singularities remain unbounded.

Since there is one more equation than unknown, m is always chosen to be even and equation m/2 + 1 is omitted [7], from (57). The quadrature abscissa points remain the same as previously defined.

Finally, once \( \phi(\tau) \) is determined from the equilibrium condition (65), \( \lambda \) is calculated:
\[
\lambda = \int_{-1}^1 \frac{\psi(T_k)}{\sqrt{1 - \frac{T_k^2}{1}}} \, d\tau \approx \sum_{k=1}^n \omega_k \psi(T_k)
\]
with \( \psi(T_i) = 0, \psi(T_n) = 0 \) and \( \omega_k = \frac{\pi}{n-1} \)

It can be shown, following a derivation similar to that for the flat stamp loading, that the stress intensity factor for the cracked strip loaded by a curved elastic stamp is given by:
\[
K(b) = \phi(-1) \sqrt{\frac{c-b}{2}} \frac{P}{\alpha} \frac{1}{\beta \lambda}
\]
\[
K(c) = -\phi(1) \sqrt{\frac{c-b}{2}} \frac{P}{\alpha} \frac{1}{\beta \lambda}
\]

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IV. NUMERICAL RESULTS

The stress intensity factors for an infinite strip subjected to various stamp loading conditions, are given for both edge and internal cracks in tables 1-7.

For convenience in comparing the data, the crack length is expressed as

\[ l = c - b \]

Also, numerical results for stress intensity factors have been normalized with respect to \( P/h \), so that the form presented in the tables becomes

\[ K(b) = \phi^*(\gamma) \sqrt{\frac{l}{2}} \frac{P}{h} \]

with \( \phi^* = \phi \frac{h}{a} \)

for edge cracks and

\[ \lambda^* = \frac{P}{h} \gamma_1 + \gamma_2 \frac{R}{h} \]

for internal cracks. In the same manner, for convenience, in the curved elastic punch loading results, \( \lambda \) is normalized with respect to \( R/h \) instead of \( a/h \)

Table 1 gives the stress intensity factors for a strip with edge cracks loaded by a rigid flat stamp with a small \( a/h \) ratio. Thus, in this case the flat stamp very closely approximates a point load and the situation is assumed to be 3-point loading.
The results for two loading configurations, \( d/h = 2.0 \) and \( d/h = 4.0 \), are compared with the edge crack stress intensity factors computed from the results by Gross and Srawley [8]; solutions obtained using boundary collocation techniques.

Since convergence becomes quite slow for edge cracks with a large \( l/h \) ratio, the final results given for \( l/h = 0.6 \), \( l/h = 0.7 \), and \( l/h = 0.8 \) were obtained through a 3-point quadratic extrapolation [11]. Figure 6 graphically represents the results given in table 1.

The \( a/h \) ratios for flat stamp loading are restricted somewhat; for once separation occurs between the stamp and the strip, indicated by a tensile stamp load, the original assumptions used in the stamp-strip interface formulation are no longer valid. The \( a/h \) ratio at which separation occurs is a function of crack length and \( d/h \). Table 2 shows the stress intensity factors for edge cracks with \( d/h = 4.0 \) till separation quickly occurs.

Table 3 and figure 7 shows the variation of stress intensity factors with increasing \( a/h \) ratios till separation occurs at \( a/h \approx 0.35 \). In this case \( d/h = 2.0 \).

Stress intensity factors for symmetric central internal cracks and eccentric internal cracks are given in tables 4 and 5, figures 8 and 9. The flat stamp loading
used is $a/h=0.01$, with $d/h=2.0$ and $d/h=4.0$. Convergence is very rapid for the internal crack, and solutions are easily obtained with a high degree of accuracy. It should be noted, that even though the negative stress intensity factors given in table 4 are physically not acceptable there may be axial loads superimposed on bending to make $k > 0$.

Table 6 contains the stress intensity factors for an infinite strip with an edge crack loaded by a curved stamp. Curved stamps with three types of elastic properties are tabulated. These are: rigid curved stamp with $\beta = 1$, elastic curved stamp with $\beta = 0.5$, and elastic curved stamp with $\beta = 5.0 \times 10^{-5}$.

When $\beta = 0.5$ the stamp and the infinite strip are made of the same material. When $\beta = 5.0 \times 10^{-5}$ a very soft punch indents the strip and is useful as an approximation, for example, of rubber loading on a concrete strip.

For each curved stamp with different elastic properties the stress intensity factors are given for various stamp half-contact lengths ($a/h$) and various crack lengths ($l/h$). From these results it's noted that the stress intensity factors vary only slightly for various ($a/h$) ratios.

Table 6 also contains the parameter $\lambda \left( \frac{\pi}{h} \frac{R}{h} \right)$.
which is used to determine the applied load in the inverse problem. Figures 10, 11, and 12 plot $\lambda^* vs. a/h$ for the three different types of elastic punches, completing the inverse problem. For the given bi-elastic constant, $a/h$ can now be determined from a given load $P$ and stamp radius $R$. Once $a/h$ is determined from these plots the stress intensity factors are obtained directly from table 6. Presentation of the results in this manner allows one to solve directly what otherwise would have been an iterative type problem for a given load $P$.

Table 7 shows the contribution to the stress intensity factors due to a shear surface traction applied at $1/h=2.0$.

With $a/h=0.01$ the stamp load very closely approximates a point load and the results, when compared with the information in table 1, give the stress intensity factors $k(b)_f$ due to the shear traction alone, where $h=1.0$ and $Q = \eta_2 P/2$.

In actual engineering applications the contribution to the stress intensity factor due to friction $k(b)_f$, from table 7, is simply multiplied by the actual coefficient of friction and the final stress intensity factor is obtained by superposition with the values given in table 1 for stress intensity factors without friction.

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V. CONCLUSION

The mathematical model outlined in this paper is a complete treatment of three point bending in a cracked elastic strip. The results obtained from the numerical solution of the system of coupled singular integral equations not only agrees well with results obtained by other techniques [8], but is, in addition a significant improvement and extension over these earlier results. Certainly the stress intensity factors obtained will be useful in better determination of $K_{IC}$ values.

Inclusion of elasticity in the stamp and friction at support points also widens the scope of engineering applicability of the computed results.

Future investigations of this type of problem should deal with an elastic strip of finite length and examine the plausibility of including a reasonable frictional theory at the stamp-strip interface.

Another important area of future investigation is plasticity. Plastic analysis about the crack tip would be of great interest and of course this would be coupled with any plastic effects associated with the stamp indentation.

The method of formulation and solution used in this problem is applicable to many mixed boundary value
problems in elasticity which arise when considering cracked elastic media and surfaces indented by a stamp over some finite region.
TABLE 1
Stress Intensity Factors for a Strip with an Edge Crack
Loaded by a Flat Stamp.  a/h=0.01

<table>
<thead>
<tr>
<th>$\lambda/h$</th>
<th>d/h=2.0</th>
<th>d/h=4.0</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$k(b)$</td>
<td>$k(b)$</td>
</tr>
<tr>
<td></td>
<td>$\sqrt{\lambda} \frac{P}{h}$</td>
<td>$\sqrt{\lambda} \frac{P}{h}$</td>
</tr>
<tr>
<td>0.01</td>
<td>6.345</td>
<td>6.434</td>
</tr>
<tr>
<td>0.10</td>
<td>5.893</td>
<td>5.910</td>
</tr>
<tr>
<td>0.20</td>
<td>5.867</td>
<td>5.882</td>
</tr>
<tr>
<td>0.30</td>
<td>6.243</td>
<td>6.255</td>
</tr>
<tr>
<td>0.40</td>
<td>7.041</td>
<td>7.042</td>
</tr>
<tr>
<td>0.50</td>
<td>8.448</td>
<td>8.467</td>
</tr>
<tr>
<td>0.60</td>
<td>10.92</td>
<td>10.96</td>
</tr>
<tr>
<td>0.70</td>
<td>15.77</td>
<td>-</td>
</tr>
<tr>
<td>0.80</td>
<td>27.40</td>
<td>-</td>
</tr>
</tbody>
</table>

* Reference [6]
TABLE 2

Stress Intensity Factors for a Strip with an Edge Crack
Loaded by a Flat Stamp.  \( a/h = 0.2 \quad d/h = 4.0 \)

<table>
<thead>
<tr>
<th>( l/h )</th>
<th>( \frac{K(b)}{\sqrt{a/b}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>12.84</td>
</tr>
<tr>
<td>0.10</td>
<td>12.03</td>
</tr>
<tr>
<td>0.20</td>
<td>(separation)</td>
</tr>
</tbody>
</table>

TABLE 3

Variation of Stress Intensity Factors with Respect to
Punch Half-Contact Width for Flat Punches till Separation
Occurs.  Edge Crack \( l/h = 0.1 \quad d/h = 2.0 \)

<table>
<thead>
<tr>
<th>( a/h )</th>
<th>( \frac{K(b)}{\sqrt{a/h}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>5.883</td>
</tr>
<tr>
<td>0.05</td>
<td>5.876</td>
</tr>
<tr>
<td>0.10</td>
<td>5.856</td>
</tr>
<tr>
<td>0.15</td>
<td>5.819</td>
</tr>
<tr>
<td>0.20</td>
<td>5.762</td>
</tr>
<tr>
<td>0.25</td>
<td>5.681</td>
</tr>
<tr>
<td>0.30</td>
<td>5.570</td>
</tr>
<tr>
<td>0.35</td>
<td>(separation)</td>
</tr>
</tbody>
</table>
TABLE 4
Stress Intensity Factors for a Central Internal Crack.
Flat Punch Loading.  a/h=0.01  d/h=4.0

<table>
<thead>
<tr>
<th>( \lambda / h )</th>
<th>b</th>
<th>( \frac{k(b)}{\sqrt{\frac{h}{P}}} )</th>
<th>( \frac{k(c)}{\sqrt{\frac{h}{P}}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.45</td>
<td>-0.287634</td>
<td>0.789191</td>
</tr>
<tr>
<td>0.2</td>
<td>0.40</td>
<td>-0.820499</td>
<td>1.33645</td>
</tr>
<tr>
<td>0.3</td>
<td>0.35</td>
<td>-1.55395</td>
<td>1.89537</td>
</tr>
<tr>
<td>0.4</td>
<td>0.30</td>
<td>-1.89747</td>
<td>2.47803</td>
</tr>
<tr>
<td>0.5</td>
<td>0.25</td>
<td>-2.47138</td>
<td>3.10946</td>
</tr>
<tr>
<td>0.6</td>
<td>0.20</td>
<td>-3.11769</td>
<td>3.84049</td>
</tr>
<tr>
<td>0.7</td>
<td>0.15</td>
<td>-3.92939</td>
<td>4.78222</td>
</tr>
<tr>
<td>0.8</td>
<td>0.10</td>
<td>-5.15519</td>
<td>6.22814</td>
</tr>
<tr>
<td>0.9</td>
<td>0.05</td>
<td>-7.81569</td>
<td>9.35474</td>
</tr>
<tr>
<td>0.95</td>
<td>0.025</td>
<td>-11.6675</td>
<td>13.7382</td>
</tr>
</tbody>
</table>
TABLE 5

Internal Cracks. Stress Intensity Factors for Eccentric Cracks. Eccentricity $e=1-(c+b)/h$. $e=-0.5$ $a/h=0.01$

d/$h=2.0

<table>
<thead>
<tr>
<th>$l/h$</th>
<th>$b$</th>
<th>$k(b)$</th>
<th>$k(c)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.725</td>
<td>2.61221</td>
<td>2.87842</td>
</tr>
<tr>
<td>0.10</td>
<td>0.7</td>
<td>2.51674</td>
<td>3.05622</td>
</tr>
<tr>
<td>0.20</td>
<td>0.650</td>
<td>2.39859</td>
<td>3.54291</td>
</tr>
<tr>
<td>0.30</td>
<td>0.6</td>
<td>2.40183</td>
<td>4.34923</td>
</tr>
<tr>
<td>0.40</td>
<td>0.550</td>
<td>2.64703</td>
<td>6.10014</td>
</tr>
</tbody>
</table>

d/$h=4.0

<table>
<thead>
<tr>
<th>$l/h$</th>
<th>$b$</th>
<th>$k(b)$</th>
<th>$k(c)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.725</td>
<td>5.47486</td>
<td>6.04227</td>
</tr>
<tr>
<td>0.10</td>
<td>0.7</td>
<td>5.26626</td>
<td>6.41548</td>
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<tr>
<td>0.20</td>
<td>0.650</td>
<td>4.99470</td>
<td>7.42653</td>
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<tr>
<td>0.30</td>
<td>0.6</td>
<td>4.96808</td>
<td>9.08946</td>
</tr>
<tr>
<td>0.40</td>
<td>0.550</td>
<td>5.43617</td>
<td>12.6981</td>
</tr>
</tbody>
</table>
TABLE 6

Stress Intensity Factors for an Infinite Strip with an Edge Crack Loaded by a Curved Stamp. \( \lambda^* = \frac{P}{h} \frac{\gamma_i + \gamma_a}{\beta} \frac{R}{h} \)

where \( \gamma = \frac{\gamma_i}{\mu} \), \( \beta = \frac{\gamma_i}{\gamma_i + \gamma_a} \)

<table>
<thead>
<tr>
<th>( \lambda / h )</th>
<th>( a / h )</th>
<th>( \sqrt{\lambda \frac{K(b)}{P}} )</th>
<th>( \lambda^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.01</td>
<td>12.17</td>
<td>1.567x10(^{-4})</td>
</tr>
<tr>
<td>0.1</td>
<td>0.03</td>
<td>12.17</td>
<td>1.387x10(^{-3})</td>
</tr>
<tr>
<td>0.1</td>
<td>0.05</td>
<td>12.17</td>
<td>3.725x10(^{-3})</td>
</tr>
<tr>
<td>0.1</td>
<td>0.10</td>
<td>12.16</td>
<td>1.292x10(^{-2})</td>
</tr>
<tr>
<td>0.1</td>
<td>0.15</td>
<td>12.14</td>
<td>2.383x10(^{-2})</td>
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<tr>
<td>0.5</td>
<td>0.01</td>
<td>17.44</td>
<td>1.549x10(^{-4})</td>
</tr>
<tr>
<td>0.5</td>
<td>0.03</td>
<td>17.44</td>
<td>1.254x10(^{-3})</td>
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<tr>
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<td>17.44</td>
<td>2.902x10(^{-3})</td>
</tr>
<tr>
<td>0.7</td>
<td>0.01</td>
<td>32.11</td>
<td>1.469x10(^{-4})</td>
</tr>
<tr>
<td>$\lambda/h$</td>
<td>$a/h$</td>
<td>$K(b)$</td>
<td>$\frac{\sqrt{\lambda}}{P/h}$</td>
</tr>
<tr>
<td>----------</td>
<td>--------</td>
<td>--------</td>
<td>-------------------------------</td>
</tr>
<tr>
<td>0.1</td>
<td>0.01</td>
<td>12.17</td>
<td>1.569x10^{-4}</td>
</tr>
<tr>
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<td>0.03</td>
<td>12.17</td>
<td>1.400x10^{-3}</td>
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<tr>
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<td>1.418x10^{-2}</td>
</tr>
<tr>
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<td>4.403x10^{-2}</td>
</tr>
<tr>
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<td>0.03</td>
<td>17.44</td>
<td>1.329x10^{-3}</td>
</tr>
<tr>
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<td>0.05</td>
<td>17.44</td>
<td>3.338x10^{-3}</td>
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<tr>
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<td>17.34</td>
<td>1.759x10^{-2}</td>
</tr>
<tr>
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<td>0.03</td>
<td>32.11</td>
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<td>2.127x10^{-3}</td>
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<tr>
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<td>0.10</td>
<td>32.03</td>
<td>3.707x10^{-3}</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\lambda/h$</th>
<th>$a/h$</th>
<th>$K(b)$</th>
<th>$\frac{\sqrt{\lambda}}{P/h}$</th>
<th>$\lambda^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.10</td>
<td>12.16</td>
<td>1.571x10^{-2}</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>0.50</td>
<td>11.88</td>
<td>0.3926</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>1.0</td>
<td>11.27</td>
<td>1.569</td>
<td></td>
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<tr>
<td>0.5</td>
<td>0.10</td>
<td>17.42</td>
<td>1.571x10^{-2}</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.50</td>
<td>16.94</td>
<td>0.3923</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>1.0</td>
<td>16.05</td>
<td>1.566</td>
<td></td>
</tr>
</tbody>
</table>
TABLE 7
Contribution to Stress Intensity Factor due to Shear Surface Traction (Friction), in Strip with Edge Crack and Flat Stamp Loading.

\( \frac{a}{h}=0.01 \), \( \frac{d}{h}=2.0 \), \( \eta_2=1.0 \), \( \frac{P}{2} \eta_2=C \)

<table>
<thead>
<tr>
<th>( \frac{l}{h} )</th>
<th>( \frac{k(b)}{\sqrt{\frac{l}{h}}} \frac{P}{h} )</th>
<th>( \frac{k(b)_f}{\sqrt{\frac{l}{h}}} \frac{P}{h} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>8.570</td>
<td>2.225</td>
</tr>
<tr>
<td>0.1</td>
<td>8.049</td>
<td>2.166</td>
</tr>
<tr>
<td>0.2</td>
<td>8.135</td>
<td>2.268</td>
</tr>
<tr>
<td>0.3</td>
<td>8.761</td>
<td>2.518</td>
</tr>
<tr>
<td>0.4</td>
<td>9.988</td>
<td>2.947</td>
</tr>
</tbody>
</table>
Figure 1  Elastic Strip with internal crack indented by circular punch.
Figure 2  Formulation of the problem by superposition

Figure 3  Formulation of the problem by superposition
Figure 4  The boundary conditions
Figure 5  The elastic stamp boundary condition
Figure 6  Stress intensity factor in a strip with an edge crack loaded by a rigid flat stamp, $a/h=0.01$, $d/h=4.0$, and $d/h=2.0$
Figure 7  Stress intensity factor vs. stamp half-contact length in a strip with an edge crack loaded by a flat rigid stamp.  $1/h=0.1$  $d/h=2.0$
Figure 8  Stress intensity factors for a symmetric central crack. Flat stamp loading. \( a/h = 0.01 \)  \( d/h = 4.0 \).
Figure 9  Stress intensity factors for eccentric cracks in an infinite strip with flat stamp loading. 
\( e = -0.5 \quad a/h = 0.01 \quad d/h = 4.0 \) \& \( d/h = 2.0 \)
Figure 10  $\lambda^* vs. a/h$ for curved punch loading. Rigid punch; $\beta = 1.0$  \( d/h = 4.0 \)
Figure 11 \( \lambda^* \) vs. \( a/h \) for curved punch loading. Elastic punch same material as cracked strip; \( \beta = 0.5 \), \( d/h = 4.0 \).
Figure 12  $\lambda^* vs. a/h$ for curved punch loading with very soft elastic punch. $\beta = 5.0 \times 10^{-5}$  $d/h=4.0$
REFERENCES


Table of Integrals

\[ \int_{0}^{\infty} e^{-\beta x} \sin \alpha x \, dx = \frac{\alpha}{\alpha^2 + \beta^2} \]

\[ \int_{0}^{\infty} e^{-\beta x} \cos \alpha x \, dx = \frac{\beta}{\beta^2 + \alpha^2} \]

\[ \int_{0}^{\infty} (\beta x e^{-\beta x}) \sin \alpha x \, dx = \frac{2 \beta^2 \alpha}{(\alpha^2 + \beta^2)^2} \]

\[ \int_{0}^{\infty} (1 - \beta x) e^{-\beta x} \cos \alpha x \, dx = \frac{2 \beta \alpha^2}{(\alpha^2 + \beta^2)^2} \]

\[ \frac{2}{\pi} \int_{0}^{\infty} \frac{2 \beta^2 \alpha}{(\alpha^2 + \beta^2)^2} \sin \alpha x \, d\alpha = \beta x e^{-\beta x} \]

\[ \frac{2}{\pi} \int_{0}^{\infty} \frac{2 \beta \alpha^2}{(\alpha^2 + \beta^2)^2} \cos \alpha x \, d\alpha = (1 - \beta x) e^{-\beta x} \]

\[ \int_{0}^{\infty} e^{-\alpha y_0} \sinh ay \, d\alpha = \frac{1}{2} \left[ \frac{1}{y_0 - y} - \frac{1}{y_0 + y} \right] \]

\[ \int_{0}^{\infty} \alpha e^{-\alpha y_0} \sinh ay \, d\alpha = \frac{1}{2} \left[ \frac{1}{(y_0 - y)^2} - \frac{1}{(y_0 + y)^2} \right] \]
\[
\int_0^\infty \alpha^2 e^{-\alpha y_0} \sinh \alpha y \, d\alpha = \left[ \frac{1}{(y_0-y)^3} - \frac{1}{(y_0+y)^3} \right] \\
\int_0^\infty (1-\alpha y_0) e^{-\alpha y_0} \sinh \alpha y \, d\alpha = \frac{1}{2} \left[ \frac{1}{y_0-y} - \frac{1}{y_0+y} \right] \\
- \frac{y_0}{2} \left[ \frac{1}{(y_0-y)^2} - \frac{1}{(y_0+y)^2} \right] \\
\int_0^\infty \alpha^3 e^{-\alpha y_0} \sinh \alpha y \, d\alpha = 3 \left[ \frac{1}{(y_0-y)^4} - \frac{1}{(y_0+y)^4} \right] \\
\int_0^\infty e^{-\alpha y_0} \cosh \alpha y \, d\alpha = \frac{1}{2} \left[ \frac{1}{y_0-y} + \frac{1}{y_0+y} \right] \\
\int_0^\infty \alpha e^{-\alpha y_0} \cosh \alpha y \, d\alpha = \frac{1}{2} \left[ \frac{1}{(y_0-y)^2} + \frac{1}{(y_0+y)^2} \right] \\
\int_0^\infty \alpha^2 e^{-\alpha y_0} \cosh \alpha y \, d\alpha = \left[ \frac{1}{(y_0-y)^3} + \frac{1}{(y_0+y)^3} \right] \\
\int_0^\infty (1-\alpha y_0) e^{-\alpha y_0} \cosh \alpha y \, d\alpha = \frac{1}{2} \left[ \frac{1}{y_0-y} + \frac{1}{y_0+y} \right] \\
- \frac{y_0}{2} \left[ \frac{1}{(y_0-y)^2} + \frac{1}{(y_0+y)^2} \right] \\
\int_0^\infty \alpha^3 e^{-\alpha y_0} \cosh \alpha y \, d\alpha = 3 \left[ \frac{1}{(y_0-y)^4} + \frac{1}{(y_0+y)^4} \right]
\]
APPENDIX B

\( K_x(y, y_o) \) expressed in exponential form:

\[
K_x(y, y_o) = \int_0^\infty \left[ e^{2\alpha h} - 4\alpha^2 h^2 - 2 + e^{-2\alpha h} \right] \times
\times \left\{ \left[ (e^{2\alpha h} + 4\alpha h - e^{-2\alpha h}) y_o + (-2e^{2\alpha h} + 4\alpha^2 h^2 + 4 - 2e^{-2\alpha h}) \times
\times (1 - \alpha y_o) - \alpha y \left[ (e^{2\alpha h} - 2 + e^{-2\alpha h}) y_o + (-e^{2\alpha h} + 4\alpha h + e^{-2\alpha h})(1 - \alpha y_o) \right] e^{-\alpha y_o}
+ \left[ (2(1+\alpha h)e^{\alpha h} + 2(\alpha h-1)e^{-\alpha h})(\alpha [h-y_o])
+ (-2\alpha h e^{\alpha h} + 2\alpha h e^{-\alpha h})(1 - \alpha [h-y_o])
- \alpha y \left[ (2\alpha h e^{\alpha h} - 2\alpha h e^{-\alpha h})(\alpha [h-y_o])
+ (2(1-\alpha h)e^{\alpha h} - 2(1+\alpha h)e^{-\alpha h})(1 - \alpha [h-y_o]) \right] \right\} \times
\times e^{-\alpha [h-y_o]} \right\} \times
\times \sinh \alpha y

+ \left[ e^{2\alpha h} - 4\alpha^2 h^2 - 2 + e^{-2\alpha h} \right] \left\{ \left[ (e^{2\alpha h} - 4\alpha^2 h^2 - 2 + e^{-2\alpha h}) \times
\times (1 - \alpha y_o) - \alpha y \left[ (e^{2\alpha h} - 2 + e^{-2\alpha h}) y_o + (-e^{2\alpha h} + 4\alpha^2 h^2 + 4 - 2e^{-2\alpha h}) \times
\times (1 - \alpha y_o) \right] e^{-\alpha y_o}
+ \left[ (2(1+\alpha h)e^{\alpha h} + 2(\alpha h-1)e^{-\alpha h})(\alpha [h-y_o])
+ (-2\alpha h e^{\alpha h} + 2\alpha h e^{-\alpha h})(1 - \alpha [h-y_o])
- \alpha y \left[ (2\alpha h e^{\alpha h} - 2\alpha h e^{-\alpha h})(\alpha [h-y_o])
+ (2(1-\alpha h)e^{\alpha h} - 2(1+\alpha h)e^{-\alpha h})(1 - \alpha [h-y_o]) \right] \right\} \times
\times e^{-\alpha [h-y_o]} \right\} \times
\times \sinh \alpha y
\]

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\[\begin{align*}
&\ast y_0 \ast + (-2 e^{2ah} + 4 - 2 e^{-2ah}) y_0 \ast \\
&+ (2 e^{-2ah} - 8ah - 2 e^{-2ah}) (1 - ay_0) + ay \left[ (e^{2ah} + 4ah - e^{-2ah}) y_0 \ast + (-e^{2ah} + 2 - e^{-2ah}) (1 - ay_0) \right] e^{-ay_0} \\
&+ \left[ (-4ah e^{ah} + 4ah e^{-ah}) (\alpha [h-y_0]) \right] \\
&+ (4 (ah-1) e^{ah} + 4 (1+ah) e^{-ah}) (1 - \alpha [h-y_0]) \\
&+ ay \left[ (2 (1+ah) e^{ah} + 2 (ah-1) e^{-ah}) (\alpha [h-y_0]) \right] e^{-\alpha (h-y_0)} \ast \\
&\ast \} \cosh ay \, d\alpha
\end{align*}\]
APPENDIX C

\[ K_i = \sum m_j \] ; \( m_j \) is given in the following table minus the separated terms \( K_{iS} \), which occur when \( y \) and \( y_0 \) either go to zero or \( h \) simultaneously. These generalized Cauchy kernels which compose \( K_{iS} \) are included at the end of Appendix C.

\[
m_1 = \int_0^\infty \frac{4a^2h^2 + 4ah + 2 - 2e^{-2ah}}{e^{2ah} - 4a^2h^2 - 2 + e^{-2ah}} \cdot y_0 e^{-ay} \sinh ay \, da
\]

\[
m_2 = \int_0^\infty \frac{-4a^2h^2}{e^{2ah} - 4a^2h^2 - 2 + e^{-2ah}} \cdot (1 - ay) e^{-ay} \sinh ay \, da
\]

\[
m_3 = -\int_0^\infty \frac{4a^2h^2}{e^{2ah} - 4a^2h^2 - 2 + e^{-2ah}} \cdot y_0 ay^2 e^{-ay} \sinh ay \, da
\]

\[
m_4 = \int_0^\infty \frac{-4a^2h^2 + 4ah - 2 + 2e^{-2ah}}{e^{2ah} - 4a^2h^2 - 2 + e^{-2ah}} \cdot \alpha y (1 - ay) e^{-ay} \sinh ay \, da
\]

\[
m_5 = \int_0^\infty \left\{ \frac{2(1+ah)e^{-ah}}{e^{2ah} - 4a^2h^2 - 2 + e^{-2ah}} - 2(1+ah)e^{-ah} \right\} \star \alpha \left[ h - y_0 \right] e^{-\alpha \left[ h - y_0 \right]} \sinh ay \, da
\]

\[
m_6 = \int_0^\infty \left\{ \frac{-2ah e^{-ah} + 2ah e^{-ah}}{e^{2ah} - 4a^2h^2 - 2 + e^{-2ah}} + 2ah e^{-ah} \right\} \star \left( 1 - \alpha \left[ h - y_0 \right] \right) e^{-\alpha \left[ h - y_0 \right]} \sinh ay \, da
\]
\[ m_7 = -y \sum_{0}^{\infty} \left\{ \frac{2ae^{-h} - 2ae^{-h} - a e^{-h}}{e^{2ah} - 4a^2h^2 - 2 + e^{-2ah} - 2ae^{-h}} \right\} \times \]
\* \* \left[ a^2 \left[ h-y_0 \right] e^{-a \left[ h-y_0 \right]} \right] \sinh ay d\alpha

\[ m_8 = -y \sum_{0}^{\infty} \left\{ \frac{z(1-ah)e^{-h} - 2(1+ah)e^{-h} - z(1-ah)e^{-h}}{e^{2ah} - 4a^2h^2 - 2 + e^{-2ah} - 2ah} \right\} \times \]
\* \* \alpha \left( 1- a \left[ h-y_0 \right] \right) e^{-a \left[ h-y_0 \right]} \sinh ay d\alpha

\[ m_9 = \int_{0}^{\infty} \frac{-8a^2h^2}{e^{2ah} - 4a^2h^2 - 2 + e^{-2ah}} y_0 \alpha e^{-a y_0} \cosh ay d\alpha \]

\[ m_{10} = \int_{0}^{\infty} \frac{8a^2h^2 - 8ah + 4 - 4e^{-2ah}}{e^{2ah} - 4a^2h^2 - 2 + e^{-2ah}} \left( 1- ay_0 \right) e^{-a y_0} \cosh ay d\alpha \]

\[ m_{11} = \int_{0}^{\infty} \frac{4a^2h^2 + 4ah + 2 - 2e^{-2ah}}{e^{2ah} - 4a^2h^2 - 2 + e^{-2ah}} y_0 a^2 e^{-a y_0} \cosh ay d\alpha \]

\[ m_{12} = \int_{0}^{\infty} \frac{-4a^2h^2}{e^{2ah} - 4a^2h^2 - 2 + e^{-2ah}} \alpha \left( 1- ay_0 \right) y e^{-a y_0} \cosh ay d\alpha \]

\[ m_{13} = \int_{0}^{\infty} \left\{ \frac{-4ahe^{-h} + 4ahe^{-h} + 4ah e^{-h}}{e^{2ah} - 4a^2h^2 - 2 + e^{-2ah}} \right\} \times \]
\* \* \left[ a \left[ h-y_0 \right] e^{-a \left[ h-y_0 \right]} \right] \cosh ay d\alpha

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The separated expressions evaluated in closed form which include the generalized Cauchy kernels complete the expression for $K_I$.

$$K_{15} = \int_{0}^{\infty} \left\{ \frac{q(1 + x h)}{e^{2xh} - 4x^2h^2 - 2 + e^{-2xh}} \left( 1 - \alpha [h - y_o] \right) e^{-\alpha [h - y_o]} \cosh \alpha y \, d\alpha \right\}$$

$$K_{16} = \int_{0}^{\infty} \left\{ \frac{-2ax e^{-\alpha}}{e^{2xh} - 4x^2h^2 - 2 + e^{-2xh}} + 2ax e^{-\alpha} \right\} \left( 1 - \alpha [h - y_o] \right) e^{-\alpha [h - y_o]} \cosh \alpha y \, d\alpha$$

The separated expressions evaluated in closed form which include the generalized Cauchy kernels complete the expression for $K_I$. 

$$K_{15} = \frac{-1}{y_o + y} + \frac{6y}{(y_o + y)^2} - \frac{4y^2}{(y_o + y)^3}$$

$$+ \frac{-1}{y_o - (zh - y)} + \frac{-6(h - y)}{[y_o - (zh - y)]^2} + \frac{-4(h - y)^2}{[y_o - (zh - y)]^3}$$

$$+ \frac{-2}{2h - y_o + y} + \frac{4h - y_o + y}{(2h - y_o + y)^2}$$

$$+ \frac{-12h[y_o - y] - 4yh}{(2h - y_o + y)^2} + \frac{24yh[y_o - y]}{(2h - y_o + y)^4}$$
\[ K_{II_A} = \int_0^\infty \left[ e^{2\alpha h} - 4\alpha^2 h^2 - 2 + e^{-2\alpha h} \right]^{-1} \left\{ \left[ (e^{2\alpha h} + \alpha h - e^{-2\alpha h}) \right] \cos \alpha x_0 \ight. \\
+ \left. \eta_1 \left[ -4\alpha^2 h^2 + 2e^{2\alpha h} - 4 + 2e^{-2\alpha h} \right. \right. \\
+ \left. \alpha y \left( e^{2\alpha h} + 4\alpha h + e^{-2\alpha h} \right) \right] \sin \alpha x_0 \right\} \sinh \alpha y \\
+ \left[ e^{2\alpha h} - 4\alpha^2 h^2 - 2 + e^{-2\alpha h} \right]^{-1} \left\{ \left[ (e^{2\alpha h} - 4\alpha^2 h^2 - 2 + e^{-2\alpha h}) + (-2e^{2\alpha h} + 4 - 2e^{-2\alpha h}) \right. \right. \\
+ \left. \alpha y \left( e^{2\alpha h} + 4\alpha h - e^{-2\alpha h} \right) \right] \cos \alpha x_0 \\
+ \left. \eta_1 \left[ (-2e^{2\alpha h} + 8\alpha h + 2e^{-2\alpha h}) \right. \right. \\
+ \left. \alpha y \left( e^{2\alpha h} - 2 + e^{-2\alpha h} \right) \right] \sin \alpha x_0 \right\} \cosh \alpha y \, d\alpha \]
\[
K_{IB} = \int_0^\infty \left[ e^{2\alpha h} - 4\alpha^2 h^2 - 2 + e^{-2\alpha h} \right]^{-1} \times
\left\{ \left[ (-2(1+\alpha h) e^{\alpha h} + 2(1-\alpha h) e^{-\alpha h}) + \alpha y (2\alpha h e^{\alpha h} - 2\alpha h e^{-\alpha h}) \right] \cos \alpha d + \eta_2 \left[ \left( 2\alpha h e^{\alpha h} - 2\alpha h e^{-\alpha h} \right) + \alpha y (2(1-\alpha h) e^{\alpha h} - 2(1-\alpha h) e^{-\alpha h}) \right] \sin \alpha d \right\} \sinh \alpha y
+ \left[ e^{2\alpha h} - 4\alpha^2 h^2 - 2 + e^{-2\alpha h} \right]^{-1} \left\{ \left[ (4\alpha h e^{\alpha h} - 4\alpha h e^{-\alpha h}) - \alpha y (2(1+\alpha h) e^{\alpha h} - 2(1-\alpha h) e^{-\alpha h}) \right] \cos \alpha d + \eta_2 \left[ (4(1-\alpha h) e^{\alpha h} - 4(1+\alpha h) e^{-\alpha h}) + \alpha y (2\alpha h e^{\alpha h} - 2\alpha h e^{-\alpha h}) \right] \sin \alpha d \right\} \cosh \alpha y \right. 
\left. \right. 
\]
APPENDIX E

\( j_1 \) is given in the following table minus the separated terms \( K_{115} \), which occur when \( y=h \) or \( y=0 \). These terms are included at the end of Appendix E.

\[
\begin{align*}
    j_1 &= \frac{1}{2} \int_0^\infty \left( -\frac{4\alpha^2 h^2 + 4\alpha h + 2 - 2e^{-2\alpha h}}{e^{2\alpha h} - 4\alpha^2 h^2 - 2 + e^{-2\alpha h}} \right) e^{\alpha y} \cos \alpha x_0 \, d\alpha \\
    j_2 &= -\frac{1}{2} \int_0^\infty \left( \frac{12\alpha^2 h^2 + 4\alpha h + 2 - 2e^{-2\alpha h}}{e^{2\alpha h} - 4\alpha^2 h^2 - 2 + e^{-2\alpha h}} \right) e^{-\alpha y} \cos \alpha x_0 \, d\alpha \\
    j_3 &= \frac{1}{2} \int_0^\infty \alpha y \left( \frac{4\alpha h + 2 - 2e^{-2\alpha h}}{e^{2\alpha h} - 4\alpha^2 h^2 - 2 + e^{-2\alpha h}} \right) e^{\alpha y} \cos \alpha x_0 \, d\alpha \\
    j_4 &= \frac{1}{2} \int_0^\infty \alpha y \left( \frac{8\alpha^2 h^2 + 4\alpha h + 2 - 2e^{-2\alpha h}}{e^{2\alpha h} - 4\alpha^2 h^2 - 2 + e^{-2\alpha h}} \right) e^{-\alpha y} \cos \alpha x_0 \, d\alpha \\
    j_5 &= \frac{\eta_1}{2} \int_0^\infty \frac{-4\alpha^2 h^2 + 8\alpha h - 4 + 4e^{-2\alpha h}}{e^{2\alpha h} - 4\alpha^2 h^2 - 2 + e^{-2\alpha h}} e^{\alpha y} \sin \alpha x_0 \, d\alpha \\
    j_6 &= \frac{\eta_1}{2} \int_0^\infty \frac{-12\alpha^2 h^2 + 8\alpha h - 4 + 4e^{-2\alpha h}}{e^{2\alpha h} - 4\alpha^2 h^2 - 2 + e^{-2\alpha h}} e^{-\alpha y} \sin \alpha x_0 \, d\alpha \\
    j_7 &= \frac{\eta_1}{2} \int_0^\infty \alpha y \left( \frac{4\alpha h - 2 + 2e^{-2\alpha h}}{e^{2\alpha h} - 4\alpha^2 h^2 - 2 + e^{-2\alpha h}} \right) e^{\alpha y} \sin \alpha x_0 \, d\alpha
\end{align*}
\]
\[ j_8 = \frac{\eta_2}{2} \int_0^\infty \alpha y \left( \frac{8 \alpha^2 h^2 - 4 \alpha h + 2 - 2 e^{-2 \alpha h}}{e^{2 \alpha h} - 4 \alpha^2 h^2 - 2 + e^{-2 \alpha h}} \right) e^{-\alpha y} \sin \alpha d\alpha \]

\[ j_9 = \frac{1}{2} \int_0^\infty \left\{ \frac{2 (\alpha h - 1) e^{\alpha h} + 2 (1 - 3 \alpha h) e^{-\alpha h}}{e^{2 \alpha h} - 4 \alpha^2 h^2 - 2 + e^{-2 \alpha h}} \right. \\
\left. - 2 (\alpha h - 1) e^{-\alpha h} \right\} e^{-\alpha y} \cos \alpha d\alpha \]

\[ j_{10} = \frac{1}{2} \int_0^\infty \left( \frac{2 (1 + 3 \alpha h) e^{\alpha h} - 2 (1 + \alpha h) e^{-\alpha h}}{e^{2 \alpha h} - 4 \alpha^2 h^2 - 2 + e^{-2 \alpha h}} \right) e^{-\alpha y} \cos \alpha d\alpha \]

\[ j_{11} = \frac{1}{2} \int_0^\infty \left( \frac{-2 e^{\alpha h} + 2 (1 - 2 \alpha h) e^{-\alpha h}}{e^{2 \alpha h} - 4 \alpha^2 h^2 - 2 + e^{-2 \alpha h}} + 2 e^{-\alpha h} \right) \]
\[ \times e^{-\alpha y} \cos \alpha d\alpha \]

\[ j_{12} = \frac{1}{2} \int_0^\infty \left( \frac{-2 (1 + 2 \alpha h) e^{\alpha h} + 2 e^{-\alpha h}}{e^{2 \alpha h} - 4 \alpha^2 h^2 - 2 + e^{-2 \alpha h}} \right) \]
\[ \times e^{-\alpha y} \cos \alpha d\alpha \]

\[ j_{13} = \frac{\eta_2}{2} \int_0^\infty \left\{ \frac{2 (2 - \alpha h) e^{\alpha h} - 2 (2 + 3 \alpha h) e^{-\alpha h}}{e^{2 \alpha h} - 4 \alpha^2 h^2 - 2 + e^{-2 \alpha h}} \right. \\
\left. - 2 (2 - \alpha h) e^{-\alpha h} \right\} e^{-\alpha y} \sin \alpha d\alpha \]

\[ j_{14} = \frac{\eta_2}{2} \int_0^\infty \left( \frac{(2 - 3 \alpha h) e^{\alpha h} - (2 + \alpha h) e^{-\alpha h}}{e^{2 \alpha h} - 4 \alpha^2 h^2 - 2 + e^{-2 \alpha h}} \right) e^{-\alpha y} \sin \alpha d\alpha \]
\[
\begin{align*}
\psi_{15} &= \frac{\eta_2}{2} \int_0^\infty \alpha y \left( \frac{2e^{\alpha h} - 2(1+2\alpha h) e^{-\alpha h}}{e^{2\alpha h} - 4\alpha^2 h^2 - 2 + e^{-2\alpha h}} \right) \frac{e^{-\alpha h}}{e^{-\alpha h}} \, \sin \alpha \, d\alpha \\
\psi_{16} &= \frac{\eta_2}{2} \int_0^\infty \alpha y \left( \frac{2(2\alpha h - 1) e^{\alpha h} + 2 e^{-\alpha h}}{e^{2\alpha h} - 4\alpha^2 h^2 - 2 + e^{-2\alpha h}} \right) \frac{e^{-\alpha h}}{e^{-\alpha h}} \, \sin \alpha \, d\alpha
\end{align*}
\]

\( K_{II} \) is completed with the following separated terms, which compose \( K_{II S} \):

\[
K_{II S} = \frac{-y - \eta_1 2x_0}{x_0^2 + y^2} + \frac{y(y^2 - x_0^2) + \eta_1^2 y^2 x_0}{(x_0^2 + y^2)^2}
\]

\[
+ \frac{-(h-y)}{(h-y)^2 + d^2} + \frac{(h-y)[(h-y)^2 - d^2]}{[(h-y)^2 + d^2]^2}
\]

\[
+ \frac{2d\eta_2}{(h-y)^2 + d^2} + \frac{-2(h-y)^2 d\eta_2}{[(h-y)^2 + d^2]^2}
\]
APPENDIX P

Derivation of the stress intensity factor expression.

\[ K(b) = \lim_{y \to b} \sqrt{2(y-b)} \sigma_{xx}(0,y) \]
\[ K(c) = \lim_{y \to c} \sqrt{2(y-c)} \sigma_{xx}(0,y) \]
\[ K(b) = \lim_{y \to b} \sqrt{2(y-b)} \frac{4\mu_1}{K_{1+1}} \frac{\partial u}{\partial y} \]
\[ = \lim_{y \to b} \sqrt{2(y-b)} \frac{4\mu_1}{K_{1+1}} F(t) \]
\[ F^*(t) = \frac{4\mu_1}{K_{1+1}} \frac{F(t)}{P/a} = \frac{\phi(t)}{\sqrt{1-t^2}} \]
\[ F(t) = \frac{\phi(t)}{\sqrt{1-t^2}} \frac{K_{1+1}}{4\mu_1} \frac{P}{a} \]

\[ y \to b : \quad \phi(t) \to \phi(-1) \]
\[ K(b) = \lim_{y \to b} \sqrt{2(y-b)} \frac{\phi(-1)}{\sqrt{1-t^2}} \frac{P}{a} \]
\[ y = \frac{c-b}{2} t + \frac{c+b}{2} \quad t = \frac{2y-c-b}{c-b} \]
\[ K(b) = \lim_{y \to b} \phi(-1) \frac{c-b}{\sqrt{2(c-y)}} \frac{P}{a} \]
\[ K(b) = \phi(-1) \frac{\sqrt{c-b/2}}{2} \frac{P}{a} \]

in a likewise manner:

\[ K(c) = -\phi(1) \sqrt{\frac{c-b}{2}} \frac{P}{a} \]
Vita

The author was born in Cocoa Beach, Florida on December 2, 1953 to Herman A. and Joan Y. Nied.

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