A numerical study of turbulent heat transfer in rotating rectangular ducts.

Abd El-Fattah Rizk
A NUMERICAL STUDY OF TURBULENT HEAT TRANSFER
IN ROTATING RECTANGULAR DUCTS

by

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<td>C_d</td>
<td>Empirical constant in equation (3.42).</td>
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<td>C_p</td>
<td>Specific heat.</td>
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<td>C_s</td>
<td>Empirical constant in triple correction tensor relation eq. (3.9).</td>
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<td>C_{r1}, C_{r2}</td>
<td>Constant related to Hanjalic and Launder [19].</td>
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<td>D_w</td>
<td>Hydrolic diameter, $D_w = \frac{4(2a)(2b)}{4(2a+2b)}$.</td>
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<td>D_{p}</td>
<td>Flow area for dependent variable, $p$.</td>
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<td>f</td>
<td>Friction factor based on pressure drop, $f = \frac{2(-\partial F/\partial x)D_w}{f^2W_b^2}$.</td>
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<td>Friction factor based on average wall shear stress, $f' = \frac{2(-\partial F/\partial x)D_w}{8 \overline{C_w}}$.</td>
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<td>F_{12}</td>
<td>Constant coefficient in Reynolds stresses equations.</td>
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<tr>
<td>Gr</td>
<td>Grashof number, $Gr = \frac{\Omega H \beta q_w D_w^4}{k_{th} \overline{\nu}^2}$.</td>
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<td>h</td>
<td>Heat transfer coefficient.</td>
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<td>H</td>
<td>Radial distance from axis of rotation to duct.</td>
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<td>k</td>
<td>Turbulent kinetic energy, $k = \frac{1}{2} \left( \overline{u^2} + \overline{v^2} + \overline{w^2} \right)$.</td>
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<td>k_{th}</td>
<td>Thermal conductivity.</td>
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<td>\ell</td>
<td>Characteristic Length scale of turbulent.</td>
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<td>\ell_b</td>
<td>Buleev's length scale.</td>
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<td>\ell_p</td>
<td>Prandtl's mixing length.</td>
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<td>\ell_{p}^*</td>
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<td>m_p</td>
<td>Mass source in differance equation.</td>
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Mass flow rate

Local Nusselt number, $NU_L = \frac{q_w D_H}{K_{th} (T_w - T_b)}$

Average Nusselt number, $\overline{NU} = \frac{\overline{q_w D_H}}{K_{th} (\overline{T_w} - T)}$

Pressure

Guessed value of pressure in difference equation

Instantaneous value of pressure in momentum equations. Correction value in difference equation

Space average pressure over cross-sectional plane

$p$-function of Jayatilleke, eq.(4.52)

Prandtl number

Turbulent Prandtl number

Wall heat flux

Gas constant

Reynolds number

Rotation number, $Ro = \frac{\Omega D_H}{W_b}$

Source term in the general form of conservation equation

Source term in difference equation for dependent variable

Temperature

Bulk fluid temperature

Wall temperature

Average wall temperature

Fluctuating components of temperature

Friction velocity, $u_\tau = \sqrt{\frac{\tau_w}{\rho}}$
u, v, w  Fluctuating components of velocity in x, y, z direction.
U, V, W  Mean velocity components in x, y, z direction.
U', V', W' Instantaneous components of velocity in x, y, z direction.
Correction values in difference equation.
W  Dimensionless axial velocity, \( W = \frac{W}{u_c} \).
W_b  Average axial velocity.
W_in  Inlet axial velocity.
x, y, z  Cartesian coordinates.
x^+, y^+ Dimensionless coordinate, \( x^+ = \frac{x u_c}{\nu} \), \( y^+ = \frac{y u_c}{\nu} \).
\( \beta \)  Coefficient of thermal expansion.
\( \Gamma^+ \)  Diffusion coefficient in the general conservation equation.
\( \alpha_1, \alpha_2 \)  Empirical constants in the Reynolds stresses equations.
\( \delta_x, \delta_y, \delta_z \)  Boundary layer thickness, figure(3.4).
\( \delta_{ij} \)  Kronecker delta.
\( \Delta x, \Delta y, \Delta z \)  Dimension of the main control volume.
\( \varepsilon \)  Isotropic dissipation rate.
\( \Theta \)  Dimensionless temperature, \( \Theta = \frac{(T - T_b) k_{th}}{q_{lw} D_m} \).
\( \kappa \)  Von Karman's constant.
\( \mu_v \)  Dynamic viscosity.
\( \mu_t \)  Turbulent viscosity.
\( \mu_e \)  Effective viscosity.
\( \nu \)  Kinematic viscosity.
\( \rho \)  Density.
\( \rho_b \)  Bulk density.
\( \overline{\tau}_w \) Average wall shear stress.

\( \Phi \) Flow property that can be transport.

\( \Omega \) Angular velocity.

Subscripts

- \textit{i, j, k} Rotational cartesian tensor notation.
- \textit{N, S, E, W, P} Nodes in finite differance grid.
Abstract

The use of cooling channels is very important in high performance electrical generators and high temperature gas turbines. Often these cooling channels rotate about an axis parallel to but displaced from its axis of rotation.

Finite difference solutions were made for turbulent flow of air in a rectangular duct of 2/1 aspect ratio, in which the duct wall is subjected to uniform heat flux. The solutions are obtained in both developing and fully developed regions of the duct for various Grashof and Reynolds numbers with a length scale model for turbulence.

The results show the effect of rotation on the development of velocity and temperature, flow patterns, heat transfer coefficient and friction factor variations. Large increases in the heat transfer coefficients with relatively small increases in the pressure drop were predicted in the Grashof number range $10^5$ to $10^8$ used here. This is a result of centrifugal acceleration, density gradients and Coriolis acceleration caused by the rotational body forces.
1. INTRODUCTION

Axial flow in cooling channels rotating parallel to but displaced from the axis of rotation occurs in a number of industrial processes. Two of the most important cases are cooling problems in the rotors of electrical machines and the rotor blades of high temperature turbines. High performance electrical generators with large rotors and high power densities are common in the industry today. The size of the rotor is quite often limited by heat transfer considerations. This generally results in larger than desired rotor sizes and large electrical and magnetic loadings. In order to minimize the rotor sizes without increasing the temperature limitation of the electrical insulation, it is necessary to better understand the cooling of the electrical windings with flow through channels in the windings. These coolant channels are generally rotating parallel to and displaced from the axis of rotation.

Operation of gas turbine at a high maximum cycle temperature has economic advantages. But the mechanical properties of the turbine rotor blade materials has limited these temperatures. Thus cooling channels are essential to protect the rotor blades from the effects of fatigue, thermal shock. So, in order to improve power equipment thermal design, it is important to have more knowledge about the effects of rotation on fluid mechanics and heat transfer.
The present work is concerned with an analytical study of turbulent flow in ducts of rectangular cross section, rotating parallel to the axis of rotation. With the duct wall subjected to a uniform heat flux in both axial and circumferential direction, finite difference solutions are presented for turbulent flow of air in a rectangular duct of 2/1 aspect ratio. The three-dimensional parabolic flow is solved by using a procedure developed by Patanker and Spalding [47]. Turbulence and mixing length models by Gessner and Emery [16] and [17] are used to calculate the Reynolds stresses in the momentum equations. For the large heat fluxes used here, there will be a large increase in the bulk temperature which will affect the bulk fluid properties such as the density, viscosity, thermal conductivity and specific heat quite significantly. Fluid properties are allowed to vary in the axial direction but are assumed constant in transverse plane.

The secondary flows which are generated in stationary ducts due to turbulent flow, will be modified by the density gradients caused by the heated walls in conjunction with the centrifugal acceleration and Coriolis acceleration in rotating ducts. Consequently, the heat transfer coefficient and friction factor in rotating ducts will be different from that of stationary ducts. The present work brings out these effects due to rotation. The results show that, there is an increase in the Nusselt number and friction factor in both developing and fully developed regions over a range of Reynolds
numbers and Grashof numbers.
2. LITERATURE REVIEW

2.1 ROTATING DUCT

Few studies are reported in the literature concerning parallel rotating rectangular ducts though there are some publications dealing with circular pipes and with square ducts. Theoretical studies of fully developed turbulent heated flow in circular duct were conducted by Nakayama [42]. The method used is very similar to the technique used for fully developed laminar flow by Mori and Nakayama [39]. The momentum integral method of solution was used with an assumed core flow region comprising a strong secondary flow together with a relatively thin boundary layer in the near wall vicinity. This solution is valid for fluids having a Prandtl number close to or larger than unity. He found that the influence of rotational buoyancy is not as marked as that in laminar flow, and the influence of Coriolis acceleration was found to be small for developed turbulent flow.

Majumder et al [38] extended the numerical method used for developing laminar flow by Skiadaressis and Spalding [56] in circular ducts to include the effects of turbulence. They used the two equation model \((k-\varepsilon)\) of turbulence to calculate the effective viscosity proposed by Harlow and Nakayama [21] modified by Launder and Spalding [33]. The influence of Coriolis acceleration on the developed flow was found to be small.
Experimental studies for turbulent flow in circular ducts have been reported by Woods [61]. He compared his results with the theoretical results of Nakayama [42], and Majumder et al [38]. But the experimental data was found to be significantly higher than that predicted.

Nakayama and Fuzoika [43] made experimental observations in the cooling channels of a prototype generator and also a simulated laboratory apparatus using water as the test fluid. Turbulent heat transfer in the entrance regions of circular ducts were also studied. Humphreys [24] experimentally investigated turbulent heat transfer in the entrance region of a circular duct. His results showed that the heat transfer increased with increase in the rotational Reynolds number.

Le Feuvre [36] reported mean heat transfer data for air flowing in internal circular passages in a model rotor used to simulate the coolant channels in a drum-type electrical armature.

Morris and Woods [41] attempted to correlate the effect of rotation by examining the difference in mean Nusselt number obtained with rotation to that at zero rotational speed and the same Reynolds number.

Few papers pertaining to turbulent heat transfer in non-
circular rotating ducts are reported in literature. The theoretical investigation described by Majumdar et al [38] included some aspects of flow and heat transfer in square sectioned tube but a systematic investigation of the physical parameters is not given. The Nusselt numbers measured experimentally agreed with those of the theoretical work.

Dias [12] and Morris and Dias [40] experimentally studied the laminar and turbulent heat transfer in square ducts with air as the test fluid. It was generally found that, the heat transfer tended to be improved with increases in the rotational speed in exactly the same manner as reported data taken with circular sectional tubes. They tried to correlate mean Nusselt numbers by using an approach similar to that of Morris and Woods [41] in a circular sectioned tube.

2.2 TURBULENCE MODELS

Turbulence modeling is a mathematical expression for the turbulent stresses in the time averaged conservation equations in terms of known or determinable quantities. There are two main types of models:

1. Eddy viscosity models.

2. Stress transport models.
2.2.1 Eddy viscosity models

Eddy viscosity models are built on Boussinesq's [6] concept, where the Reynolds stresses in the momentum equations are replaced by

\[ - \rho \overline{u v} \rightarrow \mu_t \frac{\partial U}{\partial y} \]

where, \( \mu_t \) is the turbulent viscosity which can be determined in various ways.

The most widely used model is Prandtl's mixing length hypothesis [49], in which a turbulent viscosity, \( \mu_t \), is related to the magnitude of the mean velocity gradient, and a characteristic length scale of turbulence, \( \ell_p \).

\[ \mu_t = \rho \ell_p^2 \left| \frac{\partial U}{\partial y} \right| \]

The mixing length, \( \ell_p \) must be prescribed algebraically. The mixing length hypothesis is considered to be the simplest model and gives good predictions for boundary layer flow, but is not successful in the prediction of recirculating flows. This could be because the mixing length hypothesis implies that the eddy viscosity vanishes whenever the mean velocity gradient is zero, and the model does not account the convection or the diffusion processes of turbulence.

In the differential models, the eddy viscosity, \( \mu_t \), is related
to the kinetic energy of turbulence, \( k \), the characteristic length scale, \( l \), and the density, \( \rho \). Prandtl [50] suggested that

\[
\mu_t = \sqrt{\frac{k}{l^3}}
\]

where, \( l \), \( k \) are to be determined either empirically or by solving certain conservation equations of the turbulence properties.

Models using one or two conservation equations are now well established [34]. The one-equation model requires the solution of a differential equation to determine the kinetic energy of turbulence, \( k \). This differential equation contains the rate of change due to convection, diffusion, production and dissipation of the turbulent kinetic energy, \( k \). While the length scale, \( l \), is prescribed algebraically. There are several other models which fall in this category such as, the model of Bradshaw et al [3], and the model of Nee and Kovasznay [44]. Prandtl's turbulence energy model is a good example of one-equation models. Though, \( \sqrt{\frac{\rho}{3}} \), is a better estimate of the velocity scale than, \( l_p \left| \frac{2U}{3} \right| \), perhaps the only case in which a one-equation model gives some advantages over the mixing length hypothesis is in axisymmetric flows in pipes and annuli. In this case, the one-equation model requires much less adjustment of constants to give good results.

Two-equation models differ from the one-equation models in the
way that they use differential transport equations for turbulent kinetic energy and a parameter related to length scale. There are several proposals for the dependent variable as the second variable. Kolmogorov [29] used, \( f = \frac{k^{1/2}}{l} \), the frequency of energy containing turbulence motions. Chou [8], Davidov [10], Harlow-Nakayama [22], and Jones-Launder [28] used, \( \varepsilon = \frac{k^{3/2}}{l} \), the turbulent energy dissipation rate. Rotta [52], and Spalding [57] used, \( \ell \), length scale. Rotta [53, 54], Rodi-Spalding [51], and Ng-Spalding [45] used, \( k \ell \). Spalding [58] used, \( k/\ell^2 \). Two-equation models have led to reasonable predictions of many two dimensional flows, including recirculation flows. But two-equation models have achieved little success in three dimensional flows and flows in non-circular ducts with secondary flows due to turbulence. Also, there are other disadvantages of the two-equation models that were discussed by Launder and Spalding [34] that limit their applicability, as they can not be used with a single set of constants for the prediction of axisymmetric round jets and plane free shear flows.

2.2.2 Stress transport models

Stress transport models entail the solution of differential transport equations for one or all the Reynolds stresses. These equations contain triple velocity correlations that must be determined. These correlations must be either represented in terms of other determinable quantities or in terms of another set of
transport equations of higher order correlations. The complexity of the model is increased with the addition of each new transport equation, since each contains additional unknown correlations. From a practical standpoint the triple correlations are small and can be approximated in terms of determinable quantities. Indeed, as the number of equations increases, the accuracy of the model can be improved only at the expense of its simplicity.

The numerical procedure in the present work will use an algebraic equation for Reynolds stresses developed by Gessner and Emery [16,17]. They developed these equations from the Reynolds stress transport equations for corner flow. The model contains two empirical constants and a procedure for determining the turbulent mixing length. The derivation is presented in a later chapter.
3. DEFINITION OF THE PROBLEM AND GOVERNING EQUATIONS

3.1 DEFINITION OF THE PROBLEM

The problem of interest here is the numerical solution of turbulent heat transfer in rotating rectangular ducts parallel to but displaced from the axis of rotation. Following Patanker and Spalding [47] and Patanker [46], this kind of flow can be written in terms of a one-way coordinate. The conditions under which a space coordinate can be treated as a one-way coordinate are:

1. The existence of predominant unidirectional velocity in that direction.
2. The diffusion can be neglected in that direction since the convection which is one-way overpowers the diffusion which is a two-way coordinate.
3. The pressure variation in the cross-stream plane can be neglected for the primary flow.

Under these conditions the significant influences travel only from upstream to downstream, and the conditions at a point are then affected largely by upstream conditions and very little by the downstream ones. The kinds of flows which satisfy the one-way coordinate lead to parabolic differential equations and can be called three dimensional parabolic flows.
3.2 CONSERVATION EQUATIONS

The conservation equations that need to be solved for convective heat transfer for three dimensional parabolic flow in a rotating rectangular duct are continuity, three momentum equations and the energy equation. The equation of state is also needed to calculate the bulk density of the fluid at a temperature corresponding to the local bulk temperature.

The geometry of interest is a rectangular duct with an aspect ratio 2/1. Figure (3.1) shows the duct geometry and the axis of rotation, with the larger side of the rectangle in the radial direction. The equations in cartesian coordinates x,y,z are:

continuity

$$\frac{\partial}{\partial x} (\int U) + \frac{\partial}{\partial y} (\int V) + \frac{\partial}{\partial z} (\int W) = 0 \quad (3.1)$$

x-momentum

$$\frac{\partial}{\partial x} (\int U^2) + \frac{\partial}{\partial y} (\int UV) + \frac{\partial}{\partial z} (\int UW) = -\frac{\partial}{\partial x}$$

$$+ \frac{\partial}{\partial x} (\mu \frac{\partial U}{\partial x}) + \frac{\partial}{\partial y} (\mu \frac{\partial U}{\partial y}) + \frac{\partial}{\partial z} (- \int U^2)$$
Fig. 3.1 Duct geometry and coordinate system
\[ + \frac{3}{\delta y} (-f u \overline{v}) + (-2 f \Omega \overline{V}) \]

\[ \text{(3.2)} \]

**y-momentum**

\[ \frac{\partial}{\partial x} (fUV) + \frac{3}{\partial y} (fV^2) + \frac{3}{\partial z} (fWV) = -\frac{3\bar{P}}{\partial y} \]

\[ + \frac{3}{\partial x} (\mu V \frac{\partial V}{\partial x}) + \frac{3}{\partial y} (\mu V \frac{\partial V}{\partial y}) + \frac{3}{\partial z} (-f \overline{uV}) \]

\[ + \frac{3}{\partial y} (-f \overline{V}) + f \Omega^2 H + 2 f \Omega U \]

\[ \text{(3.3)} \]

**z-momentum**

\[ \frac{3}{\partial x} (fUW) + \frac{3}{\partial y} (fVW) + \frac{3}{\partial z} (fW^2) = \frac{3\bar{P}}{\partial z} \]

\[ + \frac{3}{\partial x} (\mu V \frac{\partial W}{\partial x}) + \frac{3}{\partial y} (\mu V \frac{\partial W}{\partial y}) \]

\[ + \frac{3}{\partial x} (-f \overline{wu}) + \frac{3}{\partial y} (-f \overline{wV}) \]

\[ \text{(3.4)} \]

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Energy equation

\[ \frac{\partial}{\partial x} (f \mu T) + \frac{\partial}{\partial y} (f \nu T) + \frac{\partial}{\partial z} (f \omega T) = \]

\[ \frac{\partial}{\partial x} \left( \frac{k_{th}}{c_p} \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{k_{th}}{c_p} \frac{\partial T}{\partial y} \right) + \]

\[ \frac{\partial}{\partial x} (-f \mu \omega T) + \frac{\partial}{\partial y} (-f \nu \omega T) - \nabla^2 \] \( \text{(3.5)} \)

In these equations we have neglected the shear stresses and the diffusion fluxes acting in the lateral plane \((x-y\) plane\). This is to insure that no influence from downstream can affect the upstream stresses and fluxes on the lateral plane. Also, the pressure, \( \bar{p} \), in \( z \)-momentum equation and, \( p \), in \( x \) and \( y \) momentum equations are calculated separately. We can think of, \( \bar{p} \), as a space averaged pressure over the cross section, and \( \frac{\partial \bar{p}}{\partial z} \) is assumed known before \( \frac{\partial p}{\partial x} \) and \( \frac{\partial p}{\partial y} \) are calculated. This results in the longitudinal and lateral pressure gradients being uncoupled. By this procedure, the elliptic equations are converted to parabolic equations, and thus permitting the use of two dimensional computer storage of the dependent variables.

The last terms in equations \((3.2)\) and \((3.3)\) are the Coriolis forces. And the term before the last term in equation \((3.3)\) is centrifugal force. This term can be written in the form

\[ \int \frac{\Omega^2}{r} H = \Omega^2 \int (f - f_b) + \Omega^2 \int f_b \]
since

$$\beta = - \frac{1}{f} \left( \frac{3 f}{\beta} \right)_p$$

$$\approx - \frac{1}{f} \left( \frac{f - f_b}{T - T_b} \right)$$

$$\rho \Omega^2 \psi = - \Omega^2 \psi \beta f (T - T_b) + f_b \Omega^2 H$$ (3.6)

where $\beta$ = coefficient of expansion, $T_b$ local bulk temperature, $f_b$ local bulk density based on the local bulk temperature.

The first term in the right hand side of equation (3.6) represents a force due to density gradients caused by the heated walls.

In the present rotating duct problem with high wall heat fluxes, the density of the fluid and other properties such as viscosity, thermal conductivity and specific heat can vary quite significantly in the axial direction with the bulk temperature. We assume the fluid properties to be constant in the transverse direction and allow them to vary with $z$-direction as a function of bulk temperature.

$$f_b = \frac{P}{R \cdot T_b}$$

$$\mu \nu = \mu \nu (T_b)$$
where $R$ is a gas constant, $T_b$ can be calculated from the uniform wall heat flux condition

$$2 q_w (2a + 2b) = \int_b W_b (2b)(2a) C_P \frac{dT_b}{dZ}$$

3.3 TURBULENCE MODEL

3.3.1 Turbulence model for corner flow

The turbulence model to be used in the present work is a length scale model developed by Gessner and Emery [16]. Gessner and Emery proposed the model for local turbulence structure in the flow along a streamwise corner. They developed an algebraic Reynolds stress model from the Reynolds stress transport equations. These algebraic equations for Reynolds stresses depend on two empirical constants and the turbulence mixing length. Details of the model can be found in [16,17].

To describe the model we shall start from the exact form of the Reynolds stress transport equations for steady, incompressible flow.
The terms in this equation are labeled as:

(1) = convection term

(2) = generation term

(3) = dissipation term

(4) = redistribution term

(5) = diffusion term

where, \( \overline{u_i u_j} \) is the Reynolds stress tensor, and \( \overline{u_i u_j u_k} \) is the triple correlation tensor.

For high Reynolds numbers equation (3.7) can be simplified by neglecting the viscous diffusion term [19]. Also, based on experimental work in symmetric plane channels [20], the diffusion term resulting from pressure-velocity fluctuations can be
neglected. For high Reynolds number, it may also be assumed that the smallest scales of motion, which are responsible for the correlation \( \frac{\partial u_i}{\partial x_k} \) \( \frac{\partial u_j}{\partial x_k} \), are isotropic. Then the dissipation term can be written in the form

\[
2 \nu \left( \frac{\partial u_i}{\partial x_k} \right) \left( \frac{\partial u_j}{\partial x_k} \right) = \frac{2}{3} \delta_{ij} \varepsilon
\]  

(3.8)

where \( \varepsilon \) is the isotropic dissipation rate.

Hanjalic and Launder [19] showed that the triple velocity correlation tensor, can be replaced by a relationship containing only second-order correlations

\[
\overline{u_i u_j u_k} = - C_S \frac{k}{\varepsilon} \left( \overline{u_i u_k} \frac{\partial u_j}{\partial x_k} + \overline{u_j u_k} \frac{\partial u_i}{\partial x_k} + \overline{u_k u_l} \frac{\partial u_i u_j}{\partial x_k} \right) \]  

(3.9)

where, \( C_S \) is an empirical constant, and, \( k \), is the turbulent kinetic energy.

Hanjalic and Launder [19] extended the Chou [8], and Rotta [52] work, and developed an expression for the pressure-strain in the form
\[
\frac{\partial}{\partial \xi_j} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = -C \eta \frac{E}{k} \left( u_i u_j - \frac{2}{3} \delta_{ij} k \right) \\
+ \frac{\partial U e}{\partial x_m} a_{ij}^m + \frac{\partial U e}{\partial x_m} a_{ij}^m - - - - - (3.10)
\]

where,

\[
a_{ij}^m = \alpha \frac{u_m u_i \delta_{ij} + \epsilon \left( u_m u_i \delta_{ij} + u_m u_j \delta_{il} + u_i u_j \delta_{ml} + u_i u_l \delta_{mj} \right) + \lambda \left( \delta_{mj} \delta_{ij} \right)}{k} + \frac{\lambda (u_m u_j \cdot u_i u_m)}{k} + C_{\eta_2} \left( \frac{u_m u_i \cdot u_l u_j}{k} \right)
\]

--- (3.11)

where, \( \alpha , \epsilon , \delta , \gamma , \lambda , C_{\eta_1} , C_{\eta_2} \) are constants. Rotta [52] observed that

\[
a_{ij}^m = 2 \frac{u_m u_i}{k} - - - - - - - - (3.12)
\]

But equation (3.11) does not satisfy equation (3.12) if \( \ell = j \).
Thus they replaced the correlation, \((u_m u_j \cdot u_i u_j)\) by \((u_m u_i \cdot u_l u_j)\).
With this substitution, the constants may be expressed in terms of \(C_{\eta_2}\).
\[ \alpha = \frac{(10 - 8 C \varphi_1)}{11} \]
\[ \beta = \frac{-2 - 6 C \varphi_1}{11} \]
\[ \gamma = \frac{-4 - 12 C \varphi_1}{55} \]
\[ \zeta = \frac{6 - 18 C \varphi_1}{55} \]
\[ \lambda = -C \varphi_2 \]

where, \( C \varphi_1 \) and \( C \varphi_2 \) are the same empirical constants mentioned above.

Equation (3.11) is an approximate equation, and an alternate expression has been suggested by Launder et al [32]

\[ a_{ij}^{mi} = \alpha' u_m u_i s_{le} + \beta' (u_m u_e s_{ij} + u_m u_j s_{il}) \]
\[ + u_i u_j s_{ml} + u_i u_e s_{mj} + \left[ s' s_{mi} s_{lj} + \gamma' (s_{ml} s_{ij} + s_{mj} s_{il}) \right] K + C \varphi_3 u_m u_j s_{mi} \]

where, \( \alpha' \), \( \beta' \), \( \gamma' \), \( \gamma' \) are constants. These constants defined by

\[ \alpha' = \frac{(4 + C \varphi_3 + 10)}{11} \]
\[ \beta' = \frac{-2 + 3 C \varphi_3}{11} \]
\[ \delta' = -\left(\frac{50 C_{\theta_3} + 4}{55}\right) \]
\[ \gamma' = \left(\frac{20 C_{\theta_3} + 6}{55}\right) \]

where, \( C_{\theta_3} \) is an empirical constant.

Although equation (3.14) is less complicated than equation (3.11), the model being used here uses equation (3.11) in order to develop relationships which are consistent with Hanjalic and Launder's [19] formulation.

Substituting equations (3.8), (3.9), (3.10), (3.11) into equation (3.7), and using the above assumptions we get

\[ \frac{U_k}{x_k} \frac{\partial \bar{u}_i \bar{u}_j}{\partial x_k} = - \left( \bar{u}_j \frac{\partial U_i}{\partial x_k} + \bar{u}_i \frac{\partial U_j}{\partial x_k} \right) - \frac{2}{3} \delta_{ij} \varepsilon - \]

\[ C_{\theta_i} \varepsilon \left( \frac{\partial u_i u_j}{\partial x_k} - \frac{2}{5} \delta_{ij} \varepsilon \right) + \frac{\partial U_i}{\partial x_m} \alpha^{mj}_{ij} + \frac{\partial U_j}{\partial x_m} \alpha^{mj}_{ij} \]

\[ + C_{\theta_s} \frac{\partial}{\partial x_k} \left[ \frac{k}{\bar{E}} \left( \frac{\partial u_i u_j}{\partial x_k} + \frac{\partial u_l u_k}{\partial x_\ell} \right) + \frac{\partial u_j u_k}{\partial x_\ell} \frac{\partial u_l}{\partial x_\ell} \right] \]

\[ - - - - - - - - - - -(3.16) \]

The instantaneous velocity vector at each point in the flow can be resolved into three scalar mean components \( U, V, W \) and three fluctuating components, \( u, v, w \), in the \( x, y, z \) direction.
Fig. 3.2 Coordinate system and velocity components
respectively, figure (3.2).

According to various experimental studies [15, 5, 59], U and V are two orders of magnitude smaller than W, except in the viscous sublayer. \( \frac{\partial W}{\partial x} \) and \( \frac{\partial W}{\partial y} \) are of the same order of magnitude, and the other mean velocity component derivatives are negligible by comparison. Reynolds stress components, \( \overline{u^2}, \overline{v^2}, \overline{w^2}, \overline{uv}, \overline{uw}, \overline{vw} \) are of the same order of magnitude in the corner region. Due to this we can neglect the convective and diffusive terms as a first approximation. Then equation (3.16) can be written in the form

\[
C_{p_{1}} \frac{\varepsilon}{K} \overline{w^2} - \frac{2}{3} (C_{p_{1}} - 1) \varepsilon - \frac{\partial W}{\partial y} \left[ (2 \alpha + 4 f - 2) \overline{wv} - \frac{2 C_{p_{2}}}{K} (\overline{w^2} \cdot \overline{wv}) \right] = 0 \quad (3.17)
\]

\[
C_{p_{1}} \frac{\varepsilon}{K} \overline{v^2} - \frac{2}{3} (C_{p_{1}} - 1) \varepsilon - \frac{\partial W}{\partial y} \left[ 4 f \overline{wv} - \frac{2 C_{p_{2}}}{K} (\overline{v^2} \cdot \overline{wv}) \right] = 0 \quad (3.18)
\]
\[ C_\varphi, \frac{\varepsilon}{K} \bar{u}^2 - \frac{2}{3} (C_\varphi - 1) \varepsilon - \frac{3W}{\alpha y} \left[ 2 \int \bar{w} v - \right. \]

\[ \left. \frac{2 C_\varphi}{K} (\bar{u}^2 \cdot \bar{w} v) \right] - \frac{3W}{\alpha x} \left[ 4 \int \bar{w} u - \right. \]

\[ \left. \frac{2 C_\varphi}{K} (\bar{u}^2 \cdot \bar{w} u) \right] = 0 \quad -(3.19) \]

\[ C_\varphi, \frac{\varepsilon}{K} \bar{w} v - \frac{3W}{\alpha y} \left[ (\alpha + \frac{f}{2} - 1) v^2 + \frac{f}{2} w^2 + (\gamma + \eta) K \right. \]

\[ \left. - \frac{2 C_\varphi}{K} (\bar{w} v)^2 \right] - \frac{3W}{\alpha x} \left[ (\alpha + \frac{f}{2} - 1) v u - \right. \]

\[ \left. \frac{2 C_\varphi}{K} (\bar{w} v \cdot \bar{w} u) \right] = 0 \quad -(3.20) \]

\[ C_\varphi, \frac{\varepsilon}{K} \bar{w} u - \frac{3W}{\alpha y} \left[ (\alpha + \frac{f}{2} - 1) v u - \frac{2 C_\varphi}{K} (\bar{w} v \cdot \bar{w} u) \right. \]

\[ \left. - \frac{3W}{\alpha x} \left[ (\alpha + \frac{f}{2} - 1) \bar{u}^2 + \int \bar{w}^2 - \right. \]

\[ \left. (\gamma + \eta) K - \frac{2 C_\varphi}{K} (\bar{w} u)^2 \right] \right] = 0 \quad -(3.21) \]

\[ C_\varphi, \frac{\varepsilon}{K} \bar{w} u - \frac{3W}{\alpha y} \left[ \int \bar{w} u - \frac{2 C_\varphi}{K} (\bar{w} v \cdot \bar{w} u) \right] - \]

26
\[
\frac{\partial W}{\partial x} \left[ \int \overline{wv} - \frac{2 C_{f_2}}{K} (\overline{w} \cdot \overline{u}) \right] = 0 \quad - -(3.22)
\]

We now have six equations from (3.17) to (3.22), for Reynolds stress with seven unknowns, \( \overline{u^2} , \overline{v^2} , \overline{w^2} , \overline{uw} , \overline{vw} , \overline{uw} , \varepsilon \) since turbulent kinetic energy, \( k \), is related to \( \overline{u^2} , \overline{v^2} , \overline{w^2} \) by \( k = \frac{1}{2}(\overline{u^2} + \overline{v^2} + \overline{w^2}) \). Thus they are not enough to solve for the Reynolds stresses.

Adding equations (3.17), (3.18) and (3.19) we get

\[
C_{f_1} \frac{E}{K} \left( \overline{u^2} + \overline{v^2} + \overline{w^2} \right) - 2(C_{f_1} - 1) \varepsilon - \frac{\partial W}{\partial y} \left[ (2 \alpha + 4 f - 2) \overline{wv} - \frac{2 C_{f_2}}{K} (\overline{w^2} \cdot \overline{wv}) + 4f \overline{wv} - \frac{2 C_{f_2}}{K} (\overline{v^2} \cdot \overline{wv}) \right] - \frac{\partial W}{\partial x} \left[ (2 \alpha + 4 f - 2) \overline{wu} - \frac{2 C_{f_2}}{K} (\overline{w^2} \cdot \overline{wu}) + 2f \overline{wu} - \frac{2 C_{f_2}}{K} (\overline{u^2} \cdot \overline{wu}) \right] + 2f \overline{wu} - \frac{2 C_{f_2}}{K} (\overline{v^2} \cdot \overline{wu}) + 4f \overline{wu} - \frac{2 C_{f_2}}{K} (\overline{u^2} \cdot \overline{wu}) = 0 \quad -(3.23)
\]

and by using equation (3.13) we get

\[
\varepsilon = - \left( \overline{wv} \frac{\partial W}{\partial y} + \overline{wu} \frac{\partial W}{\partial x} \right) \quad -(3.24)
\]
substituting these into equations (3.17) to (3.22) we get the

\[ \bar{\omega}^2 = e_1 k \quad \mbox{(3.25)} \]

\[ \bar{\nu}^2 = -e_2 e_4 \frac{k^3}{\varepsilon^2} \left( \frac{\partial W}{\partial y} \right)^2 + e_3 k \quad \mbox{(3.26)} \]

\[ \bar{u}^2 = -e_2 e_4 \frac{k^3}{\varepsilon^2} \left( \frac{\partial W}{\partial x} \right)^2 + e_3 k \quad \mbox{(3.27)} \]

\[ \bar{\omega} \nu = -e_4 \frac{k^2}{\varepsilon} \frac{\partial W}{\partial y} \quad \mbox{(3.28)} \]

\[ \bar{\omega} u = -e_4 \frac{k^2}{\varepsilon} \frac{\partial W}{\partial x} \quad \mbox{(3.29)} \]

\[ \bar{\nu} u = -e_2 e_4 \frac{k^3}{\varepsilon^2} \left( \frac{\partial W}{\partial y} \right) \left( \frac{\partial W}{\partial x} \right) \quad \mbox{(3.30)} \]

where
\[ e_1 = \frac{22(c_{\theta} - 1) - 6(4c_{\varphi_2} - 5)}{33(c_{\varphi_1} - 2c_{\varphi_2})} \]  \hspace{1cm} (3.31)

\[ e_2 = \frac{4(3c_{\varphi_2} - 1)}{11(c_{\varphi_1} - 2c_{\varphi_2})} \]  \hspace{1cm} (3.32)

\[ e_3 = \frac{22(c_{\varphi_1} - 1) - 12(3c_{\varphi_2} - 1)}{33(c_{\varphi_1} - 2c_{\varphi_2})} \]  \hspace{1cm} (3.33)

\[ e_4 = \frac{44c_{\varphi_1} - 22c_{\varphi_1}c_{\varphi_2} - 128c_{\varphi_2} - 36c_{\varphi_2}^2 + 10}{165(c_{\varphi_1} - 2c_{\varphi_2})^2} \]  \hspace{1cm} (3.34)

Equations (3.25) to (3.30) can be used in conjunction with the transport equations for \( k \) and \( \varepsilon \) to obtain the Reynolds stresses. But the numerical solution of these equations is likely to be a major effort. To simplify the model further, we note from equation (3.25) that

\[ \frac{\overline{\omega^2}}{K} = e_1 = \alpha_1 \]  \hspace{1cm} (3.35)

where, \( \alpha_1 \) is function of \( c_{\varphi_1} \), \( c_{\varphi_2} \). Also by squaring equation (3.28) and equation (3.29) and adding
\[
\frac{w^2}{v^2} + \frac{w^2}{u^2} = e^2 \frac{k^4}{\varepsilon^2} \left[ \left( \frac{\partial W}{\partial y} \right)^2 + \left( \frac{\partial W}{\partial x} \right)^2 \right] - \quad -(3.36)
\]

\[
\frac{\overline{w^2} + \overline{u^2}}{k^2} = e^2 \frac{k^2}{\varepsilon^2} \left[ \left( \frac{\partial W}{\partial y} \right)^2 + \left( \frac{\partial W}{\partial x} \right)^2 \right] - \quad -(3.37)
\]

Equation (3.37) can be simplified, if equations (3.25), (3.26) and (3.27) are summed

\[
\overline{\omega^2} + \overline{v^2} + \overline{u^2} = e_1 k + 2 e_3 k -
\]

\[
e_2 k^2 \frac{k^3}{\varepsilon^2} \left[ \left( \frac{\partial W}{\partial y} \right)^2 + \left( \frac{\partial W}{\partial x} \right)^2 \right] - \quad -(3.38)
\]

\[
\frac{2 - e_1 - 2 e_3}{e_2 e_4} = - \frac{k^2}{\varepsilon^2} \left[ \left( \frac{\partial W}{\partial y} \right)^2 + \left( \frac{\partial W}{\partial x} \right)^2 \right] - \quad -(3.39)
\]

and substituting equations (3.31), (3.32) and (3.33) into (3.39)

\[
\frac{k^2}{\varepsilon^2} \left[ \left( \frac{\partial W}{\partial y} \right)^2 + \left( \frac{\partial W}{\partial x} \right)^2 \right] = \frac{1}{e_4} - \quad -(3.40)
\]

Then from equation (3.37) and equation (3.40)

\[
\left( \frac{\overline{w^2} + \overline{u^2}}{k} \right)^{1/2} = \frac{e^{1/2}}{e_4} = \alpha_2 - \quad -(3.41)
\]
where, $\alpha_2$ is a function of $C_1, C_2$.

At high Reynolds numbers the dissipation of turbulent kinetic energy is due to transfer of energy from larger eddies to smaller eddies. This process depends on the turbulent kinetic energy, $k$, and the characteristic length scale of turbulence, $\ell$, which is proportional to the size of the large energy containing eddies of the turbulent motion, and can be used to relate, $k$, and, $\ell$ as follows

$$\varepsilon = C_d \frac{k^{3/2}}{\ell}$$

where, $C_d$ is an empirical constant.

Solving equation (3.40) in conjunction with (3.42) we get $k, \varepsilon$

$$k = e_4 \frac{\ell^2}{C_d^2} \left[ \left( \frac{\partial W}{\partial y} \right)^2 + \left( \frac{\partial W}{\partial x} \right)^2 \right]$$

$$\varepsilon = \frac{C_d}{\ell} \frac{e_4^3 \rho^3}{C_d^3} \left[ \left( \frac{\partial W}{\partial y} \right)^2 + \left( \frac{\partial W}{\partial x} \right)^2 \right]$$

Then from equation (3.28) using the $k, \varepsilon$ values from equations (3.43) and (3.44)
$$\overline{w'v'} = -e_4 \frac{\ell K^2}{C_d K^{3/2}} = -e_4 \frac{\ell}{C_d} K^{1/2}$$

$$\overline{w'v'} = -e_4 \frac{\ell}{C_d} \left[ \frac{e_4}{C_d} \left( \frac{\partial W}{\partial y} \right)^2 + \left( \frac{\partial W}{\partial x} \right)^2 \right]$$

$$\overline{w'v'} = -e_4 \frac{3/2}{C_d} \ell^2 \left[ \left( \frac{\partial W}{\partial y} \right)^2 + \left( \frac{\partial W}{\partial x} \right)^2 \right] \quad (3.45)$$

For two dimensional flow (z-y) plane, equation (3.45) is reduced to

$$\overline{w'v'} = -\alpha_2^3 \frac{\ell^2}{C_d^2} \left| \frac{\partial W}{\partial y} \right| \frac{\partial W}{\partial y} \quad (3.46)$$

Another expression for $\overline{w'v'}$, can be written in terms of Prandtl's mixing length in two dimensional flow (z-y) plane

$$\overline{w'v'} = -\ell_p^2 \left| \frac{\partial W}{\partial y} \right| \frac{\partial W}{\partial y} \quad (3.47)$$

where, $\ell_p$ is Prandtl's mixing length, which is the distance traveled, on the average, by the turbulent lumps of fluid in a direction normal to the mean flow. Equation (3.47) can be written in the near wall region by
\[
\bar{u}v = - \kappa \left( \frac{\partial W}{\partial \eta} \right)^2 \frac{\partial W}{\partial \eta} \quad \text{(3.48)}
\]

where, \( \ell_p = k y \), \( \kappa \) is Von Karman's constant.

Comparing equations (3.46), (3.47) and (3.48) gives

\[
C_d = \frac{\alpha_2^{3/2}}{H} \quad \text{--- (3.49)}
\]

and

\[
\ell_p = H \ell \quad \text{--- (3.50)}
\]

Then the equations for \( k, \varepsilon \) may be written in the form

\[
k = \frac{\ell_p^2}{\alpha_2} \left[ \left( \frac{\partial W}{\partial \eta} \right)^2 + \left( \frac{\partial W}{\partial \chi} \right)^2 \right] \quad \text{--- (3.51)}
\]

\[
\varepsilon = \frac{(\alpha_2^2 k)^{3/2}}{\ell_p} \quad \text{--- (3.52)}
\]

Sustituting equations (3.51) and (3.52) into equations (3.25) to (3.30), and expressing, \( e_1, e_2, e_3, e_4 \) in terms of, \( \alpha_1, \alpha_2 \) then the Reynolds stress equations can be put in the
\[
\overline{\omega}^2 = \frac{\alpha_1}{\alpha_2} \ell_p^2 \left[ \left( \frac{\partial W}{\partial y} \right)^2 + \left( \frac{\partial W}{\partial x} \right)^2 \right] \quad (3.53)
\]

\[
\overline{\nu}^2 = \left( \frac{2F_{12} + \alpha_1 - 2}{\alpha_2} \right) \ell_p^2 \left( \frac{\partial W}{\partial y} \right)^2 + \frac{F_{12}}{\alpha_2} \ell_p^2 \left[ \left( \frac{\partial W}{\partial y} \right)^2 + \left( \frac{\partial W}{\partial x} \right)^2 \right] \quad (3.54)
\]

\[
\overline{u}^2 = - \left( \frac{2F_{12} + \alpha_1 - 2}{\alpha_2} \right) \ell_p^2 \left( \frac{\partial W}{\partial x} \right)^2 + \frac{F_{12}}{\alpha_2} \ell_p^2 \left[ \left( \frac{\partial W}{\partial y} \right)^2 + \left( \frac{\partial W}{\partial x} \right)^2 \right] \quad (3.55)
\]

\[
\overline{\omega \nu} = - \ell_p^2 \left( \frac{\partial W}{\partial y} \right) \left[ \left( \frac{\partial W}{\partial y} \right)^2 + \left( \frac{\partial W}{\partial x} \right)^2 \right]^{1/2} \quad (3.56)
\]

\[
\overline{\omega u} = - \ell_p^2 \left( \frac{\partial W}{\partial x} \right) \left[ \left( \frac{\partial W}{\partial y} \right)^2 + \left( \frac{\partial W}{\partial x} \right)^2 \right]^{1/2} \quad (3.57)
\]

\[
\overline{\nu u} = \left( \frac{2F_{12} + \alpha_1 - 2}{\alpha_2} \right) \ell_p^2 \left( \frac{\partial W}{\partial y} \right) \left( \frac{\partial W}{\partial x} \right) \quad (3.58)
\]

where
\[ F_{12} = 2(11 - 18 \alpha_3) \left[ \alpha_6 + \left( 4 \alpha_5 \alpha_7 + \alpha_6^{1/2} \right) \right] + 4 \alpha_5(18 \alpha_4 - 5) / \]

\[ 33(1 - 2 \alpha_3) \left[ \alpha_6 + \left( 4 \alpha_5 \alpha_7 + \alpha_6^{1/2} \right) \right] + 132 \alpha_4 \alpha_5 \quad (3.59) \]

and

\[ \alpha_3 = \frac{(33 \alpha_1 - 22)}{(66 \alpha_1 - 24)} \]

\[ \alpha_4 = \frac{4}{(33 \alpha_1 - 12)} \]

\[ \alpha_5 = 165 \alpha_2^2 + 660 \alpha_2 \alpha_3 + 22 \alpha_3^2 + 36 \alpha_3^4 + 660 \alpha_2 \alpha_3^3 \]

\[ \alpha_6 = 44 + 22 \alpha_4 - 128 \alpha_3 - 72 \alpha_3 \alpha_4 - 660 \alpha_2 \alpha_4^2 + 1320 \alpha_2^2 \alpha_3 \alpha_4 \]

\[ \alpha_7 = 128 \alpha_4 - 36 \alpha_4^2 - 660 \alpha_2 \alpha_4^2 + 10 \]

Equations (3.53) to (3.58) give the Reynolds stresses as a function of Prandtl's mixing length, and mean rate of strain \[ \frac{\partial W}{\partial x} , \frac{\partial W}{\partial y} \]. These equations together with the equation of motion (3.1) to (3.5) give the complete set of equations which can be used in turbulent corner flow. Solution of the overall system of the equations requires specification of two empirical constants \[ \alpha_1, \alpha_2 \], and a global representation of the Prandtl's mixing length, \( \ell_p \).
3.3.2 Selection of constants $\alpha_1$, $\alpha_2$ and calculation of mixing length

(a) Fully developed flow in rectangular duct

The length scale and mixing length used in the present work are calculated based on a method developed by Buleev [7]. The constants $\alpha_1$ and $\alpha_2$ in the algebraic equations for the Reynolds stresses are based on the experimental data of Gessner and Po [18], Gessner [15], and Tracy [59]. Buleev's expression for mixing length can be written as

$$\frac{1}{\ell_b} = \frac{1}{2} \int_c^{2\pi} \frac{d\psi}{S}$$

where, $\ell_b$, Buleev's mixing length at point $P$, $\psi$ and $S$ are defined in figure (3.3).

Gessner and Po [18], write Prandtl's mixing length as $\ell_P = \kappa' \ell_b$, where, $\kappa'$ may be related to Karman's constant ($\sim 0.4$), and depends on near wall behavior. They compared the distribution of mixing length based on equation (3.60) for different values of $\kappa'$, with experimental data of Gessner [15]. Based on the experimental data, their expression for the mixing length takes the form
Fig. 3.3 Notation for calculating length scale at point P
Fig. 3.3 Notation for calculating length scale at point P
$$\ell_p = \frac{2 \kappa' y' x' x}{y' x' \left( \frac{y'^2}{x'^2} + \frac{x'^2}{y'^2} \right)^{\alpha'}} + \alpha' x' \left( \frac{y'^2}{x'^2} + \frac{x'^2}{y'^2} \right)^{\alpha'} + \beta' x' \left( \frac{y'^2}{x'^2} + \frac{x'^2}{y'^2} \right)^{\beta'}.$$  

(3.61)

where, $\kappa' = 0.45$, $y' = 2b - y$, $x' = 2b - x$, and $x = 1.6b$ when $1.6 < x/b < a/b$, so that, $\ell_p = \ell_p(y)$ only.

The behavior of mixing length near the wall can be obtained in the manner suggested by Launder and Priddin [31].

$$\ell_p' = \ell_p \left[ 1 - \exp \left( \frac{y'^*}{26} \left( 1 - \frac{y'^*}{a'^*} \right) \right) \right].$$  

(3.62)

where, $\ell_p$ is given by equation (3.61), $\ell_p'$ is the mixing length near the wall, $y'^* = \frac{y' u_z}{\nu}$, $u_z$ the friction velocity, $r$, is an exponent which must be specified. For $r = 0$ the equation (3.62) reduces to Van Driest's formulation [60], and for $r = 2$ the equation (3.62) corresponds to an apparent optimum value for low Reynolds number duct flow [31].

Gessner and Po [18] specify the values of, $\alpha_1$, $\alpha_2$ as 1.2 and 0.25, respectively for fully developed flow. These values give the best overall fit between the predicted and measured Reynolds stress values by Gessner [15]. But the corresponding values prescribed by
Hanjalic and Launder [19] for two dimensional boundary layer flows are, $\alpha_1 = 0.94$, $\alpha_2 = 0.26$, and those specified by Launder and Ying [35] for fully developed square duct flow are, $\alpha_1 = 0.92$, $\alpha_2 = 0.3$, corresponding to the values of, $C_{\varphi_1}$, $C_{\varphi_2}$ in [19] and [35] respectively. These values tend to underestimate normal stress distributions.

(b) Developing turbulent flow in rectangular duct.

Gessener and Emery [17] developed a three dimensional mixing length model which is applicable to both developing and fully developed turbulent flow in rectangular ducts. The model can be applied to 90-degree corner flows with moderate streamwise pressure gradients. From the equations (3.56) and (3.57) for Reynolds stresses for turbulent corner flow [16], they developed an equation to calculate the mixing length values experimentally.

$$\ell_p = \frac{\sqrt{\left(\frac{2u'}{v} + \frac{2\bar{u}}{v}\right)^{1/4}}}{\left[\left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right]^{1/2}}$$

(3.63)

The data of Po [48] for entrance region of Formica-lined square duct included axial mean velocity distributions and Reynolds stress distributions measured at different axial locations, $(Z/D_H)$. By using equation (3.63) and the data of Po [48], they computed the mixing length values at various locations. They compared the mixing.
Fig. 3.4 Boundary layer thicknesses and the three regions for mixing length.
length values calculated from experimental data [48] to the data of Dean [11], Comete – Bellot [9] and Hussain and Reynolds [25], for developing two dimensional duct flows. The mixing length behavior in the corner region (region (1)) and in the regions adjacent to the boundary walls (regions (2) and (3)), figure (3.4), must satisfy the following requirements:

1. For each mixing length profile, the slope near the wall must approximately equal 0.4 for all combinations x, y, z.

2. As long as the flow is developing, there exists a local peaking of mixing length profiles in the vicinity of the corner bisector.

3. The mixing length profiles must approach an asymptotic value approximately equal to 0.08 in the outer region of the boundary layer, and before the boundary layers begin to merge.

4. In the outer region of the boundary layer, the asymptotic value of mixing length must increase from 0.08 to 0.13 as fully developed conditions are approached.

Based on the above requirements an expression for the mixing length takes the form

\[ \ell_p^* = \frac{2H}{F(x^*, y^*) + K(z^*)[G(x^*, y^*) - H(x^*, y^*)]} \]  

(3.64)

where
\[ F(x^*, y^*) = \left[ \frac{1}{y^*} + \frac{1}{x^*} \right]^{1/2} + \left[ \frac{1}{y^*} + \frac{1}{x^*} \right]^{1/2} + \left[ \frac{1}{y^*} + \frac{1}{x^*} \right]^{1/2} \]

\[ K(z^*) = 1 - C_{ii} \left\{ 1 + \tanh \left[ C_{12} \left( z^* - Z_0 \right) \right] \right\} \]

\[ G(x^*, y^*) = \sum_{n=0}^{\infty} C_n \left[ (y^*)^n + (z^*)^n \right] \]

\[ H(x^*, y^*) = \frac{C_0}{\sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \left( \frac{y^*}{\sigma_0} + x^* - 1 \right)^2 + (2 - C_0)(y^* - x^*)^2 \right\} \]

with

\[ Z^* = Z/D_H \]

\[ u_{y_1}^* = 2 - u_y^* \]

\[ x_1^* = 2 - x^* \]

and

\[ \ell_p^* = \frac{\ell_p}{\sigma_c}, \quad y_* = \frac{y}{\sigma_y}, \quad x_* = \frac{x}{\sigma_c} \quad \text{For} \]

\[ 0 \leq y_* \leq \sigma_c, \quad 0 \leq x_* \leq \sigma_c, \quad \text{region (1)} \]

\[ \ell_p^* = \frac{\ell_p}{\sigma_y}, \quad y_* = \frac{y}{\sigma_y}, \quad x_* = 1 \quad \text{For} \quad (x \geq \sigma_c, \text{region (2)}) \]
In which, \( S_{y} = S_{y}(x) \), \( S_{x} = S_{x}(y) \), and \( c_{0}, c_{1}, c_{2}, \ldots, c_{12}, z^{*} \) are empirical coefficients.

The function, \( F \), is based on Buleev's [7] mixing length, for fully developed rectangular ducts. The function, \( F \), satisfies the first requirements with \( \kappa = 0.45 \). The functions \( G \) and \( H \) are much less than the function \( F \) in the fully developed region. In this case the equation (3.64) is reduced to equation (3.61).

The function, \( G \), is constructed to satisfy, the third requirement, the asymptotic behavior of, \( \ell_{p}^{*} \) in the outer region of boundary layer before the boundary layer begins to merge.

The function, \( H \), is designed to cause local peaking of, \( \ell_{p}^{*} \) in the vicinity of a corner bisector, and satisfy requirement number 2.

The function, \( K \), enables the increase of the asymptotic value of, \( \ell_{p}^{*} \) monotonically in the outer region of boundary layer from 0.08 to 0.13 as a fully developed flow is approached.

The constants, \( c_{0}, c_{1}, \ldots, c_{12}, z^{*} \) are selected to give
the best overall fit with the data these constants are

\[
\begin{align*}
    c_0 &= -1.29982 & c_5 &= 7.29654 & c_{10} &= 0.3 \\
    c_1 &= 1.15718 & c_6 &= 16.0888 & c_{11} &= 0.4 \\
    c_2 &= 21.32706 & c_7 &= -10.3638 & c_{12} &= 0.11 \\
    c_3 &= -20.60664 & c_8 &= 6.0 & z^* &= 46.0 \\
    c_4 &= -10.65843 & c_9 &= 20.0 & \kappa &= 0.45
\end{align*}
\]

It is found that the predicted and measured mixing length profiles are in reasonable agreement for the flow from entrance to the fully-developed condition.

In order to complete the model, we have to modify equation (3.64) in the region close to the wall (viscous sublayer). To get the viscous damping effects in that region, additional mean velocity and shear stress measurements in the corner region are necessary. However, since this data is not available, it may be possible to replace, \( y^* \), and, \( x^* \), in the function, \( F(x^*, y^*) \) by

\[
\begin{align*}
    y^*_1 &= [1 - \exp\left(-\frac{y^+}{\Lambda^+}\right)] \\
    x^*_1 &= [1 - \exp\left(-\frac{x^+}{\Lambda^+}\right)]
\end{align*}
\]

where

\[
\begin{align*}
    y^+ &= \frac{u y^*}{\nu} , \quad x^+ &= \frac{x u^*}{\nu} , \quad \Lambda^+ &= 2.9
\end{align*}
\]
the best overall fit with the data these constants are

\[ c_0 = -1.29982 \quad c_5 = 7.29654 \quad c_{10} = 0.3 \]
\[ c_1 = 1.15718 \quad c_6 = 16.0888 \quad c_{11} = 0.4 \]
\[ c_2 = 21.32706 \quad c_7 = -10.3638 \quad c_{12} = 0.11 \]
\[ c_3 = -20.60664 \quad c_8 = 6.0 \quad z^*_o = 46.0 \]
\[ c_4 = -10.65843 \quad c_9 = 20.0 \quad \kappa = 0.45 \]

It is found that the predicted and measured mixing length profiles are in reasonable agreement for the flow from entrance to the fully - developed condition.

In order to complete the model, we have to modify equation (3.64) in the region close to the wall (viscous sublayer). To get the viscous damping effects in that region, additional mean velocity and shear stress measurements in the corner region are necessary. However, since this data is not available, it may be possible to replace, \( y^* \), and, \( x^* \), in the function, \( F(x^*, y^*) \) by

\[ y^*_o \left[ 1 - \exp \left( -\frac{y^+}{A^+} \right) \right] \]
\[ x^*_o \left[ 1 - \exp \left( -\frac{x^+}{A^+} \right) \right] \]

where

\[ y^+ = \frac{y u_*}{\nu}, \quad x^+ = \frac{x u_*}{\nu}, \quad A^+ = 29 \]
Then equation (3.64) would include the Van Driest's [60] damping factor.

The Reynolds stress predicted with the mixing length, prescribed by equation (3.64) are then used to obtain the Reynolds stress distribution at each streamwise location. The values of, \( \alpha_1, \alpha_2 \), are 0.94, 0.26, respectively. These values correspond to the values of, \( C_{f1} \), \( C_{f2} \), chosen by Hanjalic and Launder [19], and \( F_{12} = 0.564 \), which is function of, \( \alpha_1, \alpha_2 \), [16]. Gessner and Emery [17] compared this value of, \( F_{12} = 0.564 \), with, \( F_{12} = 0.6 \), for fixed, \( \alpha_1, \alpha_2 \), and found that for, \( F_{12} = 0.6 \), predicted the Reynolds stresses agreed better with their experimental data. But Emery, Neighbors and Gessner [14] changed the value of, \( F_{12} \), from 0.6, to the value of 0.535, in order to avoid multiple secondary flow cells in an octant of the duct and overly distorted isotach patterns. Thus a value of, \( F_{12} = 0.535 \), is used in the present work.

In summary the constants used in the present work are

\[
\begin{align*}
\alpha_1 &= 0.94 & \alpha_2 &= 0.26 & F_{12} &= 0.535 \\
C_a &= -1.29982 & C_1 &= 1.15718 \\
C_2 &= 21.32706 & C_3 &= -20.60664
\end{align*}
\]
\[ c_4 = -10.65843 \quad c_5 = 7.29654 \]
\[ c_6 = 16.0888 \quad c_{M} = -10.3638 \]
\[ c_8 = 6.0 \quad c_9 = 20.0 \]
\[ c_{10} = 0.3 \quad c_{11} = 0.4 \]
\[ c_{12} = 0.11 \quad z_0^* = 46.0 \]
\[ k = 0.45 \]

It is hoped that the above mixing length model and the constants would yield reasonable results for the rotating rectangular duct problem also.
4. NUMERICAL MODEL

A general numerical procedure for the calculation of transport processes in three dimensional parabolic flows has been presented by Patanker and Spalding [47]. The conservation equations for steady three dimensional ducts rotating parallel to the axis of rotation are

\[
\frac{\partial}{\partial x} (f U) + \frac{\partial}{\partial y} (f V) + \frac{\partial}{\partial z} (f W) = 0 \quad \text{(4.1)}
\]

\[
\frac{\partial}{\partial x} (f U^2) + \frac{\partial}{\partial y} (f V U) + \frac{\partial}{\partial z} (f W U) = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial x} (\mu_y \frac{\partial U}{\partial x}) + \frac{\partial}{\partial y} (\mu_x \frac{\partial U}{\partial y}) + \frac{\partial}{\partial z} (-f u^3)
\]

\[
+ \frac{\partial}{\partial y} (-f u V) - 2f \Omega V \quad \text{(4.2)}
\]

\[
\frac{\partial}{\partial x} (f U V) + \frac{\partial}{\partial y} (f V^2) + \frac{\partial}{\partial z} (f W V) = -\frac{\partial P}{\partial y} + \frac{\partial}{\partial x} (\mu_y \frac{\partial V}{\partial x}) + \frac{\partial}{\partial y} (\mu_x \frac{\partial V}{\partial y}) + \frac{\partial}{\partial z} (-f v^3)
\]

\[
+ \int f \Omega^2 h + 2f \Omega U + (\Omega^2 h \beta' f (T - T_b)) \quad \text{(4.3)}
\]
\[ \frac{\partial}{\partial x} (f U W) + \frac{2}{\partial y} (f V W) + \frac{2}{\partial z} (f W^2) = \frac{\partial P}{\partial z} \]

\[ + \frac{3}{\partial x} (\mu_e \frac{\partial W}{\partial x}) + \frac{3}{\partial y} (\mu_e \frac{\partial W}{\partial y}) \quad (4.4) \]

\[ \frac{\partial}{\partial x} (f U T) + \frac{3}{\partial y} (f V T) + \frac{3}{\partial z} (f W T) = \frac{3}{\partial x} \left( \frac{K_{th}}{C_p} \frac{\partial T}{\partial x} \right) \]

\[ + \frac{3}{\partial y} \left( \frac{K_{th}}{C_p} \frac{\partial T}{\partial y} \right) + \frac{3}{\partial x} (-f u t) + \frac{3}{\partial y} (-f v t) \quad (4.5) \]

The Reynolds stresses, \( \overline{u^2} \), \( \overline{v^2} \), \( \overline{u v} \), \( \overline{u w} \), \( \overline{v w} \) are evaluated using the algebraic stress model of Gessner and Emery [16] described in an earlier section.

\[ \overline{u^2} = - \left( \frac{2F_{12} + \alpha_1 - 2}{\alpha_2} \right) \ell_p^2 \left( \frac{\partial W}{\partial x} \right)^2 + \]

\[ \frac{F_{12}}{\alpha_2} \ell_p^2 \left[ \left( \frac{\partial W}{\partial x} \right)^2 + \left( \frac{\partial W}{\partial y} \right)^2 \right] \quad (4.6) \]

\[ \overline{v^2} = - \left( \frac{2F_{12} + \alpha_1 - 2}{\alpha_2} \right) \ell_p^2 \left( \frac{\partial W}{\partial y} \right)^2 + \]

\[ \frac{F_{12}}{\alpha_2} \ell_p^2 \left[ \left( \frac{\partial W}{\partial x} \right)^2 + \left( \frac{\partial W}{\partial y} \right)^2 \right] \quad (4.7) \]
\[
\overline{uv} = - \frac{(2F_2 + \alpha_1 - 2)}{\alpha_2} \ell_P^2 \left( \frac{\partial W}{\partial x} \right) \left( \frac{\partial W}{\partial y} \right) \quad \text{(4.8)}
\]

\[
\overline{uw} = - \ell_P^2 \left( \frac{\partial W}{\partial x} \right) \left[ \left( \frac{\partial W}{\partial x} \right)^2 + \left( \frac{\partial W}{\partial y} \right)^2 \right]^{1/2} \quad \text{(4.9)}
\]

\[
\overline{vw} = - \ell_P^2 \left( \frac{\partial W}{\partial y} \right) \left[ \left( \frac{\partial W}{\partial x} \right)^2 + \left( \frac{\partial W}{\partial y} \right)^2 \right]^{1/2} \quad \text{(4.10)}
\]

As mentioned earlier the values of, \( \alpha_1, \alpha_2, F_2, \) used here are, 0.94, 0.26, 0.535, respectively. \( \mu_e \) is effective viscosity where

\[
\mu_e = \mu_v + \int \ell_P^2 \left[ \left( \frac{\partial W}{\partial x} \right)^2 + \left( \frac{\partial W}{\partial y} \right)^2 \right]^{1/2} \quad \text{(4.11)}
\]

and, \( \mu_v \) is the dynamic viscosity.

The three dimensional variation of Prandtl's mixing length, \( \ell_P \), used here is that proposed by Gessner and Emery's [17] equation (3.64).

The turbulent correlation terms in the energy equation, \( \overline{ut}, \overline{vt} \) are related to, \( \overline{uw}, \overline{vw} \), respectively, and to the use of the turbulent Prandtl number.
\[ \frac{\partial}{\partial t} \left( \frac{\nu w}{Pr} \right) \frac{1}{(\frac{\partial W}{\partial y})} \frac{\partial T}{\partial y} = (4.12) \]

\[ \frac{\partial}{\partial t} \left( \frac{u w}{Pr} \right) \frac{1}{(\frac{\partial W}{\partial x})} \frac{\partial T}{\partial x} = (4.13) \]

With the above turbulent Prandtl number the energy equation can be put in the form

\[ \frac{\partial}{\partial x} \left( \int U T \right) + \frac{\partial}{\partial y} \left( \int V T \right) + \frac{\partial}{\partial z} \left( \int W T \right) = \]

\[ \frac{\partial}{\partial x} \left[ \left( \frac{K_{th}}{c_p} + \int \frac{u w}{Pr} \frac{1}{(\frac{\partial W}{\partial x})} \frac{\partial T}{\partial x} \right) \right] + \]

\[ \frac{\partial}{\partial y} \left[ \left( \frac{K_{th}}{c_p} + \int \frac{v w}{Pr} \frac{1}{(\frac{\partial W}{\partial y})} \frac{\partial T}{\partial y} \right) \right] \quad (4.14) \]

The general form of the conservation equations is

\[ \frac{\partial}{\partial x} \left( \int U \Phi \right) + \frac{\partial}{\partial y} \left( \int V \Phi \right) + \frac{\partial}{\partial z} \left( \int W \Phi \right) = \]

\[ \frac{\partial}{\partial x} \left( \int \frac{\Phi}{\partial x} \right) + \frac{\partial}{\partial y} \left( \int \frac{\Phi}{\partial y} \right) + S^\Phi \quad (4.15) \]
where, $\Phi$ may stand for the components of velocity, $W, U, V$ or the temperature, $T$. $\Gamma^\Phi$, $S^\Phi$ are the diffusion coefficient and the source term respectively, and pertain to the particular, $\Phi$.

For the $W$-momentum equation

$$\Phi = W$$

$$\Gamma^\Phi = \mu e$$

$$S^\Phi = -\frac{\partial P}{\partial Z}$$

For the energy equation

$$\Phi = T$$

$$\Gamma^\Phi = \frac{K_{th}}{c_p} - \frac{f \ell_p^2}{Pr} \left[ \left( \frac{\partial W}{\partial x} \right)^2 + \left( \frac{\partial W}{\partial y} \right)^2 \right]^{1/2}$$

$$S^\Phi = 0$$

For the $U$-momentum equation

$$\Phi = U$$

$$\Gamma^\Phi = \mu \nu$$

$$S^\Phi = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial x} (-f \bar{u}^2) + \frac{\partial}{\partial y} (-f \bar{uv})$$

$$-2f \Omega \nu$$
And for the $V$-momentum equation

$$
\Phi = V
$$

$$
\Gamma^1 \Phi = \mu_v
$$

$$
S^\Phi = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x} \left(-\int u \bar{v}\right) + \frac{2}{\partial y} \left(-\int \bar{v}^2\right)
$$

$$
+ \int_b \Omega^2 H + 2 \int \Omega U - \Omega^2 H \beta f (T - T_b)
$$

The problem at hand then involves the solution of the continuity, three momentum and energy equations together with the algebraic equations for the Reynolds stresses. Following Patanker and Spalding [47] a marching integration along the direction of primary flow is used to solve the differential equations. This technique is valid since for the present problem the main flow direction is a one-way coordinate, i.e., equations are parabolic. To incorporate the marching integration, the shear stresses and diffusion fluxes acting on the lateral plane are neglected, and longitudinal and lateral pressures are uncoupled as illustrated in earlier section.
other variable \( c\cdot v \)

\( V \cdot c\cdot v \)

\( U \cdot c\cdot v \)

Fig. 4.1 Portion of a uniform grid size
Fig. 4.2 (a) Main control volume for W, p, T
(b) U control volume
(c) V control volume
4.1 GRID SIZE AND CONTROL VOLUME

Figure (4.1) shows a portion of a uniform grid size used in the present work. The figure shows the points at which the dependent variables are stored. For U and V a staggered control volume will be used similar to that of Harlow and Welch [23], and Patanker and Spalding [47]. For other dependent variables the control volume will be around the main grid point. With the use of such staggered grid,

1. U and V velocities are located at the points at which we need to calculate the convective contribution to the, W, T and p.

2. The pressures are stored so as to make it easy to calculate the pressure gradients which drive U and V.

Around the main grid point, P, there are four neighboring points, E, W, N, S. Figure (4.2) shows the three control volumes around the same main grid point, P.

4.2 CALCULATION PROCEDURE

The overall calculation procedure is described below with the use of a typical finite difference equation, and the derivation of this finite difference equation is presented in the next section. The difference equation for a general variable, \( \phi \), can be obtained by integrating the differential equation for, \( \phi \) over the control volume (\( \Delta x \Delta y \Delta z \)) and can be written as:

\[ \text{Equation} \]
\[ \Phi_P = A_N \Phi_N + A_S \Phi_S + A_E \Phi_E + A_W \Phi_W + S_U \]  

where \( N, S, E, W \) are the neighboring points around point, \( P \), and \( U \) denotes the upstream value. In equation (4.16) could represent one of the variables \( W, U, V, p, \) and \( T \). The coefficients \( A_N, A_S, A_E, \) and \( A_W \) contain mass fluxes, viscosities, diffusion coefficient, etc. The coefficient \( A_P \) is equal to

\[ A_P = A_N + A_S + A_E + A_W - S_U \]

and \( S_U, S_P \) are the source terms. These coefficients \( A's, S's \) are specific to a particular, \( \Phi \). The difference equations for, \( \Phi = W, U, V, p, \) will all be solved simultaneously, and then the equation for, \( T, \) is solved. Our solution procedure starts with the \( W \)-momentum equation. The difference equation for \( W \)-velocity is

\[ A_P W_P = A_N W_N + A_S W_S + A_E W_E + A_W W_W + D \left( \frac{\partial \Phi}{\partial z} \right) \]

where, \( D \) represents the flow area. Since we do not know \( \left( \frac{\partial \Phi}{\partial z} \right) \), a priori, we use guessed values of \( \left( \frac{\partial \Phi}{\partial z} \right) \), denoted by \( \left( \frac{\partial \Phi}{\partial z} \right) \) and we solve for the approximate \( W^* \) field from
\[ A_P W_P = A_N W_N + A_S W_S + A_E W_E + A_W W_W + D \left( \frac{\partial P}{\partial Z} \right) \] (4.18)

This, \( W^* \), field will give us a mass flow rate, \[ \sum \sum \int \Delta x \Delta y W^* \] (taken over the cross-section), which may not be equal to the actual mass flow rate, \( M \). The actual mass flow rate can be computed from the inlet and boundary conditions. If we write the new value of, \( \left( \frac{\partial P}{\partial Z} \right) \) as

\[ \left( \frac{\partial P}{\partial Z} \right) = \left( \frac{\partial P}{\partial Z} \right)^* + \left( \frac{\partial P}{\partial Z} \right)' \] (4.19)

where, \( \left( \frac{\partial P}{\partial Z} \right)' \) a correction value. In the same manner we can write

\[ W_P = W_P^* + W_P' \] (4.20)

Subtracting equation (4.18) from equation (4.17) we have

\[ A_P W'_P = A_N (W_N - W_N^*) + A_S (W_S - W_S^*) + \]

\[ A_E (W_E - W_E^*) + D \left( \frac{\partial P}{\partial Z} \right)' \] (4.21)
By dropping the first four terms on the right hand side of equation (4.21), (since they are small), we get

$$W'_P = \frac{D^W}{A^W} (\frac{\partial P}{\partial z})' \quad \text{(4.22)}$$

Then equation (4.20) becomes

$$W_P = W^*_P + \frac{D^W}{A^W} (\frac{\partial P}{\partial z})' \quad \text{(4.23)}$$

Then by equating the actual mass flow rate, $M$, and the mass flow rate calculated with, $W^*$

$$M = \sum \int \Delta x \Delta y W^*_P \quad \text{(4.24)}$$

Substituting equation (4.23) into equation (4.24) we can get the correction value for $$(\frac{\partial P}{\partial z})'$$.

$$M = \sum \int \Delta x \Delta y \left( W^*_P - \frac{D^W}{A^W} (\frac{\partial P}{\partial z})' \right)$$

$$(\frac{\partial P}{\partial z})' = \frac{M - \sum \int \Delta x \Delta y W^*_P}{\sum \int \Delta x \Delta y \frac{D^W}{A^W}} \quad \text{(4.25)}$$
The correct value of \( \frac{\partial P}{\partial z} \) is now used to calculate the new value of \( W \)-field. The above procedure is repeated using \( \frac{\partial P}{\partial z} \) as \( \frac{\partial P}{\partial z}^* \) and \( W \) as \( W^* \) until a converged solution is obtained.

After solving for the correct \( W \)-field at downstream and \( \frac{\partial P}{\partial z} \), the next step is to obtain \( U, V \) velocities. The difference equations for \( U, V \)-momentum equations are

\[
A_P^U U_P = A_N^U U_N + A_S^U U_S + A_E^U U_E + A_W^U U_W \\
+ D^U (P_w - P) \quad (4.26)
\]

\[
A_P^V V_P = A_N^V V_N + A_S^V V_S + A_E^V V_E + A_W^V V_W \\
+ D^V (P_s - P) \quad (4.27)
\]

where, \( D^U, D^V \) again represent the flow areas for \( U, V \) velocities respectively. Since the pressure field, \( p \), is not known, we shall again use a guessed value denoted by, \( p^* \), and solve for the approximate velocity, \( U^*, V^* \) from

\[
A_P^U U_P^* = A_N^U U_N^* + A_S^U U_S^* + A_E^U U_E^* + A_W^U U_W^* \\
+ D^U (P_w^* - P^*) \quad (4.28)
\]
\[ A^V_P V^*_P = A^V_N V^*_N + A^V_S V^*_S + A^V_E V^*_E + A^V_W V^*_W \]
\[ + D^V ( P^*_S - P^*_P ) \quad \text{(4.29)} \]

The starred velocities, \( U^*, V^* \) may not satisfy the continuity equation, and may produce a net mass source, \( m_P \), for point, \( P \). Our aim is to correct the pressure and velocities to eliminate this mass source.

We shall write the \( U, V, p \) quantities in terms of guessed values (stars) and correction values (primes) as

\[
\begin{align*}
P &= P^* + p' \\
U &= U^* + U' \\
V &= V^* + V'
\end{align*}
\quad \text{(4.30)}
\]

As in the case of the \( W \)-equation we subtract equation (4.28) from equation (4.26) and equation (4.29) from equation (4.27) and get

\[ A^V_P U'_P = \left( A^V_N U'_N + A^V_S U'_S + A^V_E U'_E + A^V_W U'_W \right) \]
\[ + D^V ( P'_W - P'_P ) \quad \text{(4.31)} \]
\[ A_P V_P' = (A_N V_N' + A_S V_s' + A_E V_E' + A_W V_W') \]

\[ + D^v (P_s' - P_P') \]  \hspace{1cm} (4.32)

Dropping the first four terms in the right hand side of equations (4.31) and (4.32) results in

\[ U_P' = \frac{D^u}{A_P} (P_w' - P_P') \]  \hspace{1cm} (4.33)

\[ V_P' = \frac{D^v}{A_P} (P_s' - P_P') \]  \hspace{1cm} (4.34)

The velocities will now be corrected using

\[ U_P = U_P^* + \frac{D^u}{A_P} (P_w' - P_P') \]  \hspace{1cm} (4.35)

\[ V_P = V_P^* + \frac{D^v}{A_P} (P_s' - P_P') \]  \hspace{1cm} (4.36)

The pressure correction \((p')\) equation can now be found from continuity equation by integrating the continuity equation over the
control volume( $\Delta x \Delta y \Delta z$ ), and substituting velocity correction equations (4.35) and (4.36)

$$A_P p_P' = A_N p_N' + A_S p_S' + A_E p_E' + A_W p_W' + S_U$$  \hspace{1cm} (4.37)

The mass source, $m_p$, has been incorporated into, $S_U$. Thus $U, V, p$ can be calculated for each grid point. After calculating $U, V, W, p$ fields, we calculate the temperature field, $T$, from its difference equation given below

$$A_P T_P = A_N T_N + A_S T_S + A_E T_E + A_W T_W - S_U$$  \hspace{1cm} (4.38)

The numerical procedure for solving the partial differential equations is non-iterative. The calculation of the coefficients in the finite difference equations is done with values at the upstream station, thus we force the equations to be linear.

The difference equations are solved by a line by line method using a tri-diagonal matrix algorithm (TDMA). Details on the TDMA scheme are presented in appendix A.
4.3 DERIVATION OF FINITE DIFFERENCE EQUATIONS

The assumptions that are used to derive the finite difference solutions are as follows:

1. We use fully implicit schemes, which means that the downstream values of the dependent variable, \( \Phi \), are applicable over the interval z-upstream to z-downstream except at z-upstream.

2. In the x-y plane, the value of \( \Phi \), is assumed to remain uniform over the control volume surrounding (i,j).

3. Hybrid differencing used, which is a combination of central and upwind differencing.

The general form of the transport equation for a dependent variable, \( \Phi \), is

\[
\frac{\partial}{\partial t} (\int U \Phi) + \frac{\partial}{\partial x} (\int V \Phi) + \frac{\partial}{\partial z} (\int W \Phi) = 0
\]

(1)

\[
\frac{\partial}{\partial x} \left( \int \Phi \nabla \phi \right) + \frac{\partial}{\partial y} \left( \int \Phi \nabla \phi \right) + S = 0 \quad (4.39)
\]

(4)

If, \( \Phi = W, T \), The control volume shown in figure(4.2a) will be used. Integrating each of these terms over the control volume \((\Delta x \Delta y \Delta z)\)

\[
(1) = \int_{c.v.} \frac{\partial}{\partial x} (\int U \Phi) \, dx \, dy \, dz \equiv (\int U \Phi)_{e} \Delta x \Delta y \Delta z
\]

\[
= [(\int U)_{e} \Phi_{e} - (\int U)_{w} \Phi_{w}] \Delta y \Delta z
\]

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using the average values between the grid points then

\[
\begin{align*}
= & \left[\left(\frac{f_E + f_P}{2}\right) u_e \left(\frac{\Phi_E + \Phi_P}{2}\right) - \left(\frac{f_W + f_P}{2}\right) u_w \left(\frac{\Phi_W + \Phi_P}{2}\right)\right] \Delta y \Delta z \\
\end{align*}
\]

let

\[
CE = \left(\frac{f_E + f_P}{4}\right) u_e \Delta y & \quad CW = \left(\frac{f_W + f_P}{4}\right) u_w \Delta y
\]

then term

\[
(1) = \left[ CE \left(\frac{\Phi_E + \Phi_P}{2}\right) - CW \left(\frac{\Phi_W + \Phi_P}{2}\right) \right] \Delta z
\]

Similar expressions for other terms (2) to (6) can be written

\[
(2) = \int_{c.v} \frac{3}{\Delta y} \left(\int \nabla \Phi \right) d\mathbf{x} d\mathbf{y} d\mathbf{z} = \int_{c.v} \nabla \Phi = \int_{c.v} \nabla \Phi = \int_{c.v} \nabla \Phi = \int_{c.v} \nabla \Phi = \int_{c.v} \nabla \Phi
\]

let

\[
CN = \left(\frac{f_N + f_P}{4}\right) V_n \Delta x & \quad CS = \left(\frac{f_S + f_P}{4}\right) V_s \Delta x
\]
\[(2) = \left[ CN(\phi_n + \phi_p) - CS(\phi_s + \phi_p) \right] \Delta z \]

\[(3) = \int_{c.v}^{} \left( f \cdot w \phi \right) dx dy dz \equiv (f \cdot w \phi \Delta x \Delta y)_{up}^{dn} \]

where, UP, upstream, DN, downstream.

\[= (f \cdot w)_{dn} \phi_{p_{dn}} \Delta x \Delta y - (f \cdot w)_{up} \phi_{p_{up}} \Delta x \Delta y \]

Integration of the convective term requires the consideration of the overall mass flow through control volume. The mass flow balance through the control volume can be written as

\[(f \cdot w)_{up} \Delta x \Delta y + (f \cdot u)_{w_{up}} \Delta z \Delta y + (f \cdot v)_{s_{up}} \Delta z \Delta x =
(f \cdot w)_{dn} \Delta x \Delta y + (f \cdot u)_{e_{up}} \Delta z \Delta y + (f \cdot v)_{n_{up}} \Delta z \Delta x \]

then

\[(f \cdot w)_{dn} \Delta x \Delta y = (f \cdot w)_{up} \Delta x \Delta y - [ (f \cdot v)_{n_{up}} - (f \cdot v)_{s_{up}} ] \Delta z \Delta x \]

\[\Delta x \Delta z - [ (f \cdot u)_{e_{up}} - (f \cdot u)_{w_{up}} ] \Delta y \Delta z \]

let
\[ CU = (f \mathcal{W} )_{up} \frac{\Delta x \Delta y}{\Delta z} \]
\[ CD = (f \mathcal{W} )_{DN} \frac{\Delta x \Delta y}{\Delta z} \]
\[ CD = CU - 2CN + 2CS - 2CE + 2CW \]

(3) \[ \int \frac{\partial}{\partial x} \left( \Gamma^\phi \frac{\partial \phi}{\partial x} \right) dx dy dz \approx \left( \Gamma^\phi \frac{\partial \phi}{\partial x} \Delta y \Delta z \right) \]

(4) \[ = \left[ \nu \left( \frac{\partial \phi}{\partial x} \right)_e - \Gamma_w \left( \frac{\partial \phi}{\partial x} \right)_w \right] \Delta y \Delta z \]

\[ = \left[ \left( \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial x} \right)(\phi_e - \phi_p) - \left( \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial x} \right)(\phi_p - \phi_w) \right] \Delta y \Delta z \]

let

\[ DE = \left( \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial x} \right) \frac{\Delta y}{\partial x} \]

\[ DW = \left( \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial x} \right) \frac{\Delta y}{\partial x} \]

(4) \[ = \left[ DE(\phi_e - \phi_p) - DW(\phi_p - \phi_w) \right] \Delta Z \]

(5) \[ = \int \frac{\partial}{\partial y} \left( \Gamma^\phi \frac{\partial \phi}{\partial y} \right) dx dy dz \approx \left( \Gamma^\phi \frac{\partial \phi}{\partial y} \Delta x \Delta z \right) \]

\[ = \left[ \left( \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial y} \right)(\phi_n - \phi_p) - \left( \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial y} \right)(\phi_p - \phi_s) \right] \Delta x \Delta z \]

let
\[
DN = \left(\frac{\Gamma_N^\phi + \Gamma_P^\phi}{2}\right) \frac{\Delta x}{\delta y_{dn}} \quad \text{&} \quad DS = \left(\frac{\Gamma_S^\phi + \Gamma_P^\phi}{2}\right) \frac{\Delta x}{\delta y_{ds}}
\]

(5) \[= \left[ DN(\Phi_N - \Phi_P) - DS(\Phi_P - \Phi_S) \right] \Delta z \]

(6) \[= \int_{c.v} S_p^\phi dx dy dz \equiv \left( S_p^\phi \Delta x \Delta y \Delta z \right) \]

\[= \left( S_U^\phi + S_P^\phi \Phi_{p,\text{DN}} \right) \Delta z \]

\(SU, SP\), arise from the source term linearizations. For, \(W\), the source terms include the pressure term and the energy equation needs no source term.

Adding all the six terms we get

\[
\Phi_p (CE - CW + CN - CS + CD + DE + DW + DN + DS - SP)
\]

\[= \Phi_N (DN - CN) + \Phi_S (DS + CS) + \Phi_E (DE - CE) \]

\[+ \Phi_W (DW + CW) + CU \Phi_{p,\text{up}} + SU^\phi \quad \text{(4.40)} \]

Substituting the value for, \(CD\), from the below equation

\[
CD = CU - 2CN + 2CS - 2CE + 2CW
\]
then
\[ \phi_p \left[ (DE - CE) + (DW + CW) + (DN - CN) + 
(DS + CS) + CU - SP^e \right] = \phi_n (DN - CN) + 
\phi_s (DS + CS) + \phi_e (DE - CE) + \phi_w (DW + CW) 
+ CU \phi_{B,up} + SU^e \] 
(4.41)

By using the hybrid differencing, the coefficients take the
form (in FORTRAN)
\[ \begin{align*}
AN &= \text{MAX}(0.5|CN|, DN) - 0.5CN \\
AS &= \text{MAX}(0.5|CS|, DS) + 0.5CS \\
AE &= \text{MAX}(0.5|CE|, DE) - 0.5CE \\
AW &= \text{MAX}(0.5|CW|, DW) + 0.5CW \\
AU &= CU \\
SU &= AU + SU^e \\
AP &= (AN + AS + AE + AW + AU - SP) \tag{4.42}
\end{align*} \]

Then the finite difference equation is
AP \phi_p = AE \phi_E + AW \phi_w + AN \phi_n + AS \phi_s + SU \hspace{1em} (4.43)

In U - momentum equation, and V - momentum equation we have to use the staggered control volume, figure (4.2b) and figure (4.2c). The source terms in both equations include, Reynolds stresses, and body forces.

4.4 THE PRESSURE CORRECTION EQUATION

The continuity equation is

$$\frac{\partial}{\partial x} (\int U) + \frac{\partial}{\partial y} (\int V) + \frac{\partial}{\partial z} (\int W) = 0 \hspace{1em} (4.44)$$

Integrating over the control volume, figure (4.2a) we get

$$\left[ (\int U)_e - (\int U)_w \right] \Delta y \Delta z + \left[ (\int V)_n - (\int V)_s \right] \Delta x \Delta z$$

$$+ \left[ (\int W)_{BuP} - (\int W)_{P, DN} \right] \Delta x \Delta y = 0 \hspace{1em} (4.45)$$

By using the velocity correction equation presented earlier, the correct velocities can be written as

$$U_e^* = U_e + \frac{D_e}{A_e} (P_p' - P_E')$$

$$U_w^* = U_w + \frac{D_w}{A_w} (P_w' - P_P')$$
\[ V_n = V_n^* + \frac{D_n^V}{A_n^V} (P_p' - P_N') \]
\[ V_s = V_s^* + \frac{D_s^V}{A_s^V} (P_s' - P_p') \]

Substituting the equation (4.46) into the equation (4.45) we have

\[
\left[ \int_e \left[ U_e^* + \frac{D_e^V}{A_e^V} (P_p' - P_E') \right] - \int_w \left[ U_w^* + \frac{D_w^V}{A_w^V} (P_w' - P_p') \right] \right] \\
\Delta y \Delta z + \left[ \int_n \left[ V_n^* + \frac{D_n^V}{A_n^V} (P_p' - P_N') \right] \right] - \\
\int_s \left[ V_s^* + \frac{D_s^V}{A_s^V} (P_s' - P_p') \right] \Delta x \Delta z \\
+ (CU - CD) \Delta z = 0 \quad \text{(4.47)}
\]

Now let us make the following substitutions for brevity

\[
AN = \frac{\int_n D_n^V \Delta x}{A_n^V} \\
AS = \frac{\int_s D_s^V \Delta x}{A_s^V} \\
AE = \frac{\int_e D_e^V \Delta y}{A_e^V} \\
AW = \frac{\int_w D_w^V \Delta y}{A_w^V} \\
CN = \frac{\int_n V_n^* \Delta x}{A_n^V} \\
CS = \frac{\int_s V_s^* \Delta x}{A_s^V} \\
CE = \frac{\int_e U_e^* \Delta y}{A_e^V}
\]
\[ CW = \int_{w} U_w^* \Delta y \]
\[ CU = (\int_{w} \nu_p) \frac{\Delta x \Delta y}{\Delta z} \]
\[ CD = (\int_{w} \eta_{PN}) \frac{\Delta x \Delta y}{\Delta z} \]
\[ SU = CS - CN + CW - CE + CU - CD \]
\[ AP = AN + AS + AE + AW \]

Then the pressure correction equation becomes

\[ \Delta P'_{p} = AN'_{p} + AS'_{s} + AE'_{p} + AW'_{w} + SU \quad (4.48) \]

The interface densities \( \left( f_e, f_w, f_n, f_s \right) \) may be calculated by interpolation. We can see that the term, \( SU \), is evaluated in terms of the starred velocities. If \( SU \) is zero, it means that the starred velocities do satisfy the continuity equation, and no pressure correction is needed. The term, \( SU \), thus represents a (mass source) which the pressure correction must annihilate.

4.5 BOUNDARY CONDITIONS AND CALCULATIONS CLOSE TO THE WALL

Special care was needed to calculate the effective viscosity, \( \mu_e \), close to the wall. The law of the wall based on experimental data was used as a near-wall boundary condition with a relatively coarse nodal distribution. In this method, the wall shear stress is used to define an effective viscosity at the wall.
for bottom and top walls we used
\[ \mu_e = \frac{\tau_w}{\partial W/\partial y} \]
and for side walls we used
\[ \mu_e = \frac{\tau_w}{\partial W/\partial x} \]
where, \( \mu_e = (\mu_v + \mu_t) \), and, \( \tau_w \), the wall shear stress

with \( y^+ < 11.65 \) (laminar sublayer) \( \mu_e = \mu_v \)
and for \( y^+ > 11.65 \) (turbulent layer) \( \mu_e = \mu_t \)

In the turbulent region the wall shear stress is written in terms of
the friction velocity, \( u_\tau \)

\[ u_\tau = \sqrt{\frac{\tau_w}{\rho}} \]

which is evaluated from the law of the wall, with empirical
coefficients based on recently obtained near-wall experimental
data in a square duct [37]

\[ W^+ = \frac{W}{u_\tau} = 2.5 \ln \frac{y_{u_\tau}}{\nu} + 5.2 \]

\[ u_\tau = \frac{W}{2.5 \ln \frac{y_{u_\tau}}{\nu} + 5.2} \quad (4.49) \]

where, \( W \), the axial velocity and the nodal points closest to the
twall. Then
The energy equation is solved for flow in a smooth duct for constant heat flux. The universal semi-logarithmic law is applied at the nodal points near the wall to calculate the wall temperature. The wall temperature and the nodal temperature difference is written as

\[
(T_w - T) = \frac{q_w}{\rho C_p U} \left[ W^+ + P \right] \quad (4.51)
\]

where the parameter, \( P \), is the so called "p-function" of Jayatilleke [26] which expresses the relative resistance of the viscous sublayer to heat and momentum transport.

\[
P = 9.24 \left( \left( \frac{P_t}{P_{Pr}} \right)^{3/4} - 1 \right) \quad (4.52)
\]

where, \( P_{Pr} \), turbulent Prandtl number assumed to be 0.9.

Then to calculate the Nusselt number at any point along the wall

\[
NUL = \frac{\mu e}{K_{th}} = \frac{q_w D_H}{K_{th}(T_{wL} - T_b)} \quad (4.53)
\]
The average Nusselt number can be written as

$$\overline{NU} = \frac{q_w D_h}{k_{th} (T_w - T_b)}$$

(4.54)
5. RESULTS AND CONCLUSIONS

5.1 RESULTS

The computational scheme and the numerical procedure used to solve the governing equations have been discussed in earlier sections. Like most other numerical procedures, the present computational procedure is also subjected to truncation and round off errors and instabilities. To satisfy the stability criteria, the step size was limited to 0.125 of the duct hydraulic diameter in the marching direction for the 12 x 24 grid size in the transverse plane. Convergence was determined by requiring the sum of the residuals which are the absolute sum of the differences between the left and right hand-sides of the difference equation for the whole grid, to be less than $10^{-6}$.

The computation was started with a uniform axial velocity at the duct inlet. The computer program was tested by solving the turbulent flow in square duct. Since Gessner and Emery models [16, 17] are used to calculate the Reynolds stresses, the results are compared with Emery et al [14] for $Re = 75000$. The differences between the present computations and those of Emery et al are small as shown in figures (5.1, 5.2, 5.3) which illustrate the variation of axial center line velocity, axial variation of friction factor, which in turn depends on the average wall shear stress, and axial variation of Stanton number, respectively. Friction factor,
Fig. 5.1 Axial variation of center line velocity

Re = 75000

- present work
- Emery et al. [14]

Stationary Square duct
Re = 75000

--- present work
--- Emery et al. [14]

○ Launder & Ying[35]

Stationary Square duct

Fig. 5.2 Axial variation of wall shear stress
\[ \frac{(\bar{z}^2 / 1/2 \int w^2) / 2}{2 \times 10^3} \]

Fig. 5.2 Axial variation of wall shear stress

Stationary Square Duct

Re = 75000

- Present work
- Emery et al. [14]
- Launder & Ying [35]
Fig. 5.3 Axial variation of Stanton number

Re = 75000

- Present work
- Emery et al. [14]
- Launder & Ying [35]

Stationary Square duct
Fig. 5.3  Axial Variation of Station number

St x 10^3
0 10 20 30 40 50 60 70 80 90 100
Hg/Z

Stationary Square duct

Lauder & Ting [35]

--- Emery et al. [14]

Re = 75000

Present work
and Stanton number for fully developed turbulent flow in square duct by Launder and Ying [35] are also shown in figures (5.2, 5.3).

As a second stage of the development, calculations were obtained for a stationary rectangular duct for three different Reynolds numbers, \( Re = 8000, 12000, 20000 \), for constant as well as variable properties. In this case Coriolis and centrifugal forces were not included in the governing equations. These calculations were made to clarify the difference between the stationary and rotating duct results, and to bring out the differences between the constant and variable property solutions. The axial variations of center line velocity for a stationary rectangular duct of constant properties are shown in figure (5.4) and the variations of friction factor, and Nusselt number for both constant and variable properties are illustrated in figures (5.5, 5.6). These results are similar to those of the square duct represented by Emery et al [14]. The values of friction factor given by Jones [27] for a rectangular duct of 2/1 aspect ratio, and the Nusselt number given by the empirical equation of Dittus and Boelter [13] for fully developed flow are also shown in these figures.

Secondary flows are known to play an important role for turbulent flow through straight ducts of a non-circular cross
Fig. 5.4 Axial variation of center line velocity
Fig. 5.5 Axial variation of wall shear stress

Stationary Rectangular duct (2/1)
Fig. 5.5 Axial variation of wall shear stress

\[ (\overline{\tau_w}/1/2 \rho w_b^2) \times 10^3 \]

- Re=20000
- Re=12000
- Re=8000

© 20000 © 12000 © 8000

Stationary Rectangular duct (2/1)

Jones, constant properties

constant properties

variable properties

© 20000 © 12000 © 8000
Fig. 5.6 Axial variation of average Nusselt number
Figure (5.7) represents the velocity vectors in the transverse plane for a stationary 2/1 rectangular duct with Re = 8000. Secondary flows in a quarter of the duct at z = 50 \( D_H \), 75 \( D_H \) are shown and they are similar to those in square ducts where the flow is toward the diagonal.

The Coriolis and buoyancy terms were added to the equations to get the effect of the rotation on the development of the flow. Tables (I, II, III) list the various computational runs considered here over a range of Grashof numbers and three Reynolds numbers. Also shown are the values of Re, Gr, and Ro, at the inlet and at z = 75 \( D_H \), which are varying along the duct axis. The results indicate a significant effect of rotation on both wall shear stress as well as Nusselt number as illustrated in figures (5.8, 5.9, 5.10). For higher Grashof numbers the transverse velocities in the lateral plane are reorganized from those in the stationary ducts as shown in figures (5.11, 5.12, 5.13). This reorganization of the transverse flow pattern is due to the Coriolis and centrifugal effects. The peaks in the axial variation of friction factor and Nusselt number, figures (5.8, 5.9, 5.10) are also due to the reorganization of the flow from one boundary layer development to body force dominated.

Figures (5.14 through 5.19) show the severe effect of rotation on the flow development, circumferential variation of friction factor, Nusselt number and wall temperature for a Reynolds number.
Fig. 5.7 Secondary flow velocity for stationary 2/1 rectangular duct at $Re=8000$ and $Z/D_H=6.25, 50, 75$
<table>
<thead>
<tr>
<th>Run</th>
<th>A1</th>
<th>B1</th>
<th>C1</th>
<th>D1</th>
<th>E1</th>
<th>F1</th>
<th>G1</th>
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</thead>
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<tr>
<td>Re(75)</td>
<td>7303</td>
<td>7306</td>
<td>7304</td>
<td>7306</td>
<td>7306</td>
<td>7306</td>
<td>7306</td>
</tr>
<tr>
<td>$\Omega_{\text{rad/s}}$</td>
<td>27.5</td>
<td>62.8</td>
<td>94.2</td>
<td>200.0</td>
<td>338.0</td>
<td>618.0</td>
<td>874.0</td>
</tr>
<tr>
<td>Gr(0)</td>
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<td>5</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>7</td>
<td>10</td>
</tr>
<tr>
<td>Gr(75)</td>
<td>$5 \times 10^4$</td>
<td>$2.38 \times 10^5$</td>
<td>$5.25 \times 10^5$</td>
<td>$2.41 \times 10^6$</td>
<td>6.88</td>
<td>$2.29 \times 10^7$</td>
<td>$4.58 \times 10^7$</td>
</tr>
<tr>
<td>Ro(0)</td>
<td>$2.25 \times 10^{-3}$</td>
<td>$5.14 \times 10^{-3}$</td>
<td>$7.72 \times 10^{-3}$</td>
<td>$1.64 \times 10^{-2}$</td>
<td>$2.77 \times 10^{-2}$</td>
<td>$5.10 \times 10^{-2}$</td>
<td>$7.16 \times 10^{-2}$</td>
</tr>
<tr>
<td>Ro(75)</td>
<td>$1.87 \times 10^{-3}$</td>
<td>$4.33 \times 10^{-3}$</td>
<td>$6.46 \times 10^{-3}$</td>
<td>$1.38 \times 10^{-2}$</td>
<td>$2.33 \times 10^{-2}$</td>
<td>$4.25 \times 10^{-2}$</td>
<td>$6.01 \times 10^{-2}$</td>
</tr>
<tr>
<td>$f(75)$</td>
<td>$4.33 \times 10^{-2}$</td>
<td>$4.33 \times 10^{-2}$</td>
<td>$4.33 \times 10^{-2}$</td>
<td>$4.45 \times 10^{-2}$</td>
<td>$4.55 \times 10^{-2}$</td>
<td>$4.60 \times 10^{-2}$</td>
<td>$4.68 \times 10^{-2}$</td>
</tr>
<tr>
<td>$f'(75)$</td>
<td>$3.86 \times 10^{-2}$</td>
<td>$3.86 \times 10^{-2}$</td>
<td>$3.87 \times 10^{-2}$</td>
<td>$3.95 \times 10^{-2}$</td>
<td>$4.08 \times 10^{-2}$</td>
<td>$4.17 \times 10^{-2}$</td>
<td>$4.22 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

Table (I). Tabulation of Re, $\Omega$, Gr, Ro, NU, $f$, and $f'$ for

$Re = 8000$, $q_w = 7500 \text{ W/m}^2$ at $z/D_H = 0$ and $z/D_H = 75$
<table>
<thead>
<tr>
<th>Run</th>
<th>A2</th>
<th>B2</th>
<th>C2</th>
<th>D2</th>
<th>E2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Re(75)</td>
<td>11008</td>
<td>11008</td>
<td>11008</td>
<td>11008</td>
<td>11008</td>
</tr>
<tr>
<td>$\Omega_{rad/s}$</td>
<td>94.24</td>
<td>200.0</td>
<td>338.0</td>
<td>618.0</td>
<td>874.0</td>
</tr>
<tr>
<td>Gr(0)</td>
<td>1.18x10^6</td>
<td>5.32x10^6</td>
<td>1.52x10^7</td>
<td>5x10^7</td>
<td>10x10^7</td>
</tr>
<tr>
<td>Gr(75)</td>
<td>6.04x10^5</td>
<td>2.72x10^6</td>
<td>7.76x10^6</td>
<td>2.58x10^7</td>
<td>5.13x10^7</td>
</tr>
<tr>
<td>Ro(0)</td>
<td>5.15x10^-3</td>
<td>1.10x10^-2</td>
<td>1.84x10^-2</td>
<td>3.37x10^-2</td>
<td>4.77x10^-2</td>
</tr>
<tr>
<td>Ro(75)</td>
<td>4.40x10^-3</td>
<td>9.33x10^-3</td>
<td>1.57x10^-2</td>
<td>2.87x10^-2</td>
<td>4.05x10^-2</td>
</tr>
<tr>
<td>Nu(75)</td>
<td>37.17</td>
<td>37.18</td>
<td>38.0</td>
<td>39.98</td>
<td>41.03</td>
</tr>
<tr>
<td>f (75)</td>
<td>3.87x10^-2</td>
<td>3.88x10^-2</td>
<td>3.97x10^-2</td>
<td>4.07x10^-2</td>
<td>4.14x10^-2</td>
</tr>
<tr>
<td>f'(75)</td>
<td>3.435x10^-2</td>
<td>3.436x10^-2</td>
<td>3.479x10^-2</td>
<td>3.585x10^-2</td>
<td>3.675x10^-2</td>
</tr>
</tbody>
</table>

Table (II). Tabulation of $Re$, $\Omega$, $Gr$, $Ro$, $Nu$, $f$, and $f'$ for $Re = 12000$, $q_w = 7500 \text{ w/m}^2$ at $z/D_H = 0$ and $z/D_H = 75$.
<table>
<thead>
<tr>
<th>RUN</th>
<th>A3</th>
<th>B3</th>
<th>C3</th>
<th>D3</th>
<th>E3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Re(75)</td>
<td>18362</td>
<td>18362</td>
<td>18362</td>
<td>18362</td>
<td>18362</td>
</tr>
<tr>
<td>Ω rad/15</td>
<td>68.8</td>
<td>217.2</td>
<td>500.0</td>
<td>870.0</td>
<td>1545.0</td>
</tr>
<tr>
<td>Gr(0)</td>
<td>10</td>
<td>10</td>
<td>5.32x10</td>
<td>1.6x10</td>
<td>5x10</td>
</tr>
<tr>
<td>Gr(75)</td>
<td>4.1x10</td>
<td>3.87x10</td>
<td>2.16x10</td>
<td>6.42x10</td>
<td>1.98x10</td>
</tr>
<tr>
<td>Ro(0)</td>
<td>2.25x10</td>
<td>7.11x10</td>
<td>1.63x10</td>
<td>2.85x10</td>
<td>5.00x10</td>
</tr>
<tr>
<td>Ro(75)</td>
<td>1.71x10</td>
<td>5.29x10</td>
<td>1.24x10</td>
<td>2.14x10</td>
<td>3.70x10</td>
</tr>
<tr>
<td>Nu(75)</td>
<td>55.12</td>
<td>55.23</td>
<td>55.53</td>
<td>58.56</td>
<td>60.39</td>
</tr>
<tr>
<td>f(75)</td>
<td>3.99x10</td>
<td>3.99x10</td>
<td>3.99x10</td>
<td>4.14x10</td>
<td>4.19x10</td>
</tr>
<tr>
<td>f'(75)</td>
<td>3.00x10</td>
<td>3.00x10</td>
<td>3.02x10</td>
<td>3.11x10</td>
<td>3.15x10</td>
</tr>
</tbody>
</table>

Table (III). Tabulation of Re, Ω, Gr, Ro, Nu, f and f' for

Re = 20000, q_w = 12000 w/m^2 at z/D_H = 0 and z/D_H = 75
Fig. 5.8 Axial variation of $\overline{\text{NU}}$ and wall shear stress for rotating 2/1 rectangular duct
Fig. 5.8 Axial variation of $\overline{U}$ and wall shear stress for rotating 2/1 rectangular duct

$\frac{Z/H}{(\overline{U}/1/2 f_b w_b^2) \times 10^3}$

- $Re = 8000$
- $Gr = 5.25 \times 10^5$
- $Gr = 5.3 \times 10^6$
- $Gr = 5.0 \times 10^7$
- $Gr = 1.8 \times 10^5$
- $Gr = 1.5 \times 10^6$
- $Gr = 10^8$
Fig. 5.9 Axial variation of $\overline{NU}$ and wall shear stress for rotating 2/1 rectangular buct.
For Rotating 2/1 Rectangular duct

Fig. 5.9 Axial variation of Nu and wall shear stress

\( \frac{\bar{w}}{1/2 \bar{v}} \) x 10^3

- A2, B2
- C2
- D2
- E2

- GR = 108
- GR = 1.5 x 10^7
- GR = 5.0 x 10^7
- GR = 1.8 x 10^6
- GR = 5.3 x 10^6

\( \text{Re} = 12000 \)
Fig. 5.10 Axial variation of $\overline{\text{NU}}$ and wall shear stress for rotating 2/1 rectangular duct
Fig. 5.10 Axial variation of $\overline{NU}$ and wall shear stress for rotating 2/1 rectangular duct.
Fig. 5.11 Development of transverse velocity for rotating 2/1 rectangular duct at Re=8000
Fig. 5.12 Development of transverse velocity for rotating 2/1 rectangular duct at $Re=12000$
Fig. 5.13 Development of transverse velocity for rotating 2/1 rectangular duct at $Re = 20000$
of 8000 and two different Grashof numbers, \( Gr = 5.3 \times 10^6 \) and \( Gr = 5 \times 10^7 \). Figures (5.14, 5.15, 5.16) give the variation of axial velocity along the vertical and horizontal center lines, and the temperature along the vertical center line. It is clear that the maximum axial velocity and the temperature are moving outward with increasing Grashof number. This movement of the maximum is the effect of centrifugal force which increases with RPM. Figures (5.17, 5.18) represent the circumferential variation of wall shear stress and of the local Nusselt number based on bulk properties and local wall temperature. It is interesting to note the similarities in the variation of these two parameters. This indicates at least a qualitative relationship between friction factor and Nusselt number. Figure (5.19) gives the circumferential variation of wall temperature and the value of bulk temperature of the flow, which exhibit a typical mixed convection behavior as shown in the transverse flow patterns, figures (5.11, 5.12, 5.13).

The development of Nusselt number and fully developed flow friction factor versus Grashof numbers are given in figures (5.20 through 5.25) for three different Reynolds numbers. These figures illustrate the amount of influence of rotation on the Nusselt number with up to about 25% improvement at larger Grashof numbers. The computational results for Nusselt numbers are consistently lower than the experimental data. Friction factor data in figures (5.23, 5.24, 5.25) also show an increase with Grashof number but not
Fig. 5.14 Development of axial velocity along line C-D for rotating 2/1 rectangular duct
Fig. 5.15 Development of axial velocity along line A - B for rotating 2/1 rectangular duct
Fig. 5.16 Development of temperature along C-D for rotating 2/1 rectangular duct

Re = 8000
Gr = 5 \times 10^7

Re = 8000
Gr = 5.3 \times 10^6
Fig. 5.17 Circumferential variation of wall shear stress at $Z/D_H = 75$ for rotating 2/1 rectangular duct.
Fig. 5.17 Circumferential variation of wall shear stress at Z/DH = 75 for rotating 2/1 rectangular duct.

\[
\frac{\bar{\tau}_w}{1/2 \rho_b w_b^2} \times 10^3
\]

Re = 8000
Gr = 5.3 \times 10^6
Gr = 5 \times 10^7
Fig. 5.18 Cicumferential variation of NU at $Z/D_H=75$
for rotating 2/1 rectangular duct

Re = 8000

- $Gr = 5.3 \times 10^6$
- $Gr = 5 \times 10^7$
Fig. 5.19  Circumferential variation of wall temperature at $Z/D_H=75$ for rotating 2/1 rectangular duct.
Experimental work, Bayat[2]
Fig. 5.21 Development of $\overline{\text{NU}}$ vs $\text{Gr}$

- Experimental work, Bayat [2]

- $Z/D_H = 15$
- $Z/D_H = 25$
- $Z/D_H = 37.5$
- $Z/D_H = 75$

$Re = 12000$
Fig. 5.22 Development of $\overline{NU}$ vs Gr

Experimental work, Bayat [2]

Re = 20000

$Z/D_H$

- - - - 15

25

- - - - 37.5

- - - - 75

Experimental work, Bayat [2]
Experimental work, Bayat [2]

Re = 8000

$Z/D_H = 75$

Present constant properties at $Z/D_H = 80$

Jones [ ]

stationary 2/1 rectangular duct

Fig. 5.23 Friction factor vs Gr
Experimental work, Bayat[2]

Present constant properties at $Z/D_H = 75$

- Jones[27]
- Stationary 2/1 rectangular duct

Friction factor vs $Gr$

Fig. 5.24
Present constant properties at $Z/D_H = 80$

Jones [19]

stationary 2/1 rectangular duct

Fig. 5.25 Friction factor vs Gr
Fig. 5.25 Friction factor vs Gr

Friction factor $\times 10^2$

- Stationary 2/1 rectangular duct
- Present constant properties at $\frac{Z}{DH} = 80$
- $Re = 20000$
- $Z/DH = 75$
as much as heat transfer. Thus, there is a definite benefit in the overall heat transfer without a considerable increase in the pumping requirements for these rotating duct turbulent flows. Experimental results for Nusselt number and friction factor for, Re = 8000, 12000, 20000 given by Bayat [2] are also shown in figures (5.20 through 5.24).

The data presented indicate a discrepancy between theoretical and experimental results. The reasons for this may be the following:

1. The constants for the turbulence model, $\alpha_1$, $\alpha_2$ which we used here, were for stationary square duct, and may not be quite appropriate for 2/1 rectangular ducts. In addition they may not be valid for rotating ducts.

2. The effect of buoyancy forces is not included in the Reynolds stress model, one would have to start with the exact Reynolds stress transport equation which contains the body force terms, Launder [30].

3. The effect of buoyancy force is not included in the length scale calculation either. This effect can be accounted for by an empirical equation, as given by Monin-Obukhov relation [1] or Keyps formula [5].

4. The length scale algebraic Reynolds stress turbulence model used in the present computations is independent of Re. Experimental results indicate a strong effect of Re on the enhancement particularly at high Gr.

It is conceivable that with the implementation of the above changes the computational results could be closer to the experimental data. The purpose of this work was to evaluate the validity of a very simple turbulence model for complex rotating flows. It appears the
effect of the body forces are significant and further changes to the turbulence model as listed above are necessary for better results.

5.2 CONCLUSIONS

1. A finite difference solution has been carried out for a turbulent flow in a 2/1 rectangular duct rotating parallel to but displaced from the axis of rotation.

2. The flow is treated as a three-dimensional parabolic flow and a numerical procedure developed by Patanker and Spalding [47] is used.

3. The Reynolds stresses have been calculated by using a mixing length model developed by Gessner and Emery [16] for corner flow. The effect of the buoyancy force was not included in the model. And the constants used here are the ones for stationary square ducts.

4. The influence of rotation on the heat transfer in rotating rectangular ducts (as compared to the stationary rectangular ducts) is different at various Reynolds numbers. The minimum Grashof number for any noticeable changes are:

   for Re = 8000  \( Gr = 5 \times 10^5 \)
   for Re = 12000  \( Gr = 4 \times 10^6 \)
   and for Re = 20000  \( Gr = 6 \times 10^6 \)

5. The transverse flow patterns are markedly rearranged at the larger Grashof numbers. And as a consequence, the circumferential variation of friction factor, Nusselt number, and temperature around the duct are correspondingly different too.

6. The axial variation of averaged friction factor and Nusselt number are calculated for different Grashof number. Significant increases in the Nusselt number with moderate increases in friction factor are seen as a function of buoyancy force.

7. The axial velocity and temperature distributions also changed with the Grashof number. The maximum axial
velocity and temperature locations move outward in the direction of the centrifugal force.

8. Friction factor and Nusselt number results have been compared with the experimental data of Bayat [2]. Computational results are consistently lower. The disagreement possibly due to the very simple turbulence model used in the present work. The model used is applicable only to stationary square duct flows.

9. Incorporation of the body forces into the Reynolds stress equations and the length scale formulation may improve the computational results.
Tri-Diagonal Matrix Algorithm - TDMA

The difference equations are solved by TDMA. The designation of this algorithm refers to the fact that when the matrix of the coefficients of these equations is written, all the nonzero coefficients align themselves along three diagonals of the matrix.

The general form of the difference equation for the dependant variable, \( \Phi \), at the point \((i,j)\) is

\[
A_P \Phi_{i,j} = A_N \Phi_{i,j+1} + A_S \Phi_{i,j-1} + A_E \Phi_{i+1,j} + A_W \Phi_{i-1,j} + B \quad \text{(A-1)}
\]

This equation can be solved by the successive use of the TDMA in the \(x\) and \(y\) directions. For the \(y\)-direction sweep, we write

\[
A_P \Phi^I_{i,j} = A_N \Phi^I_{i,j+1} + A_S \Phi^I_{i,j-1} + (A_E \Phi^I_{i+1,j} + A_W \Phi^I_{i-1,j} + B) \quad \text{(A-2)}
\]

where the expression in the parenthesis is assumed known by using the previous solution. The superscript, \( I \), denotes the values obtained from the first phase of solution.
Let
\[ A_P = a_j \]
\[ A_N = b_j \]
\[ A_S = c_j \]
\[ A_E \phi_{i+1,j}^{up} + A_W \phi_{i-1,j}^{up} + B = d_j \]

Then equation (A - 2) becomes
\[ a_j \phi_{i,j}^I = b_j \phi_{i,j+1}^I + c_j \phi_{i,j-1}^I + d_j \] (A - 3)

For \( j = 1, 2, 3, \ldots, M \), with 1 and M denoting the boundary points. From equation (A - 3), the value of \( \phi_{i,j}^I \) is related to the neighboring values, \( \phi_{i,j+1}^I, \phi_{i,j-1}^I \). For \( j = 1 \), we have to set, \( c_1 = 0 \), \( \phi_{i,0}^I \) does not exist, and also for \( j = M \) we have to set, \( b_M = 0 \), for, \( \phi_{i,M+1}^I \). So, the difference equations become
\[ a_1 \phi_{i,1}^I = b_1 \phi_{i,2}^I + d_1 \]
\[ a_2 \phi_{i,2}^I = b_2 \phi_{i,3}^I + c_2 \phi_{i,1}^I + d_2 \]

\[ a_j \phi_{i,j}^I = b_j \phi_{i,j+1}^I + c_j \phi_{i,j-1}^I + d_j \] (A - 4)
From these equations we can deduce that, $\Phi^I_{i,1}$, is known in terms of, $\Phi^I_{i,2}$, and, $\Phi^I_{i,2}$, is known in terms of, $\Phi^I_{i,3}$, 

\ldots\ldots\ldots, and, $\Phi^I_{i,M}$, is known in terms of, $\Phi^I_{i,M+1}$. But, since $\Phi^I_{i,M+1}$, does not exist, we use back-substitution.

Suppose, in the forward-substitution process, we can have the relation for $(i, j-1)$ as

$$\Phi^I_{i,j-1} = P_{j-1} \Phi^I_{i,j} + Q_{j-1} \quad (A-5)$$

Then to get the relation for $(i, j)$ to be in the form

$$\Phi^I_{i,j} = P_j \Phi^I_{i,j+1} + Q_j$$

substituting equation $(A-5)$ into equation $(A-4)$ we get

$$a_j \Phi^I_{i,j} = b_j \Phi^I_{i,j+1} + c_j \left( P_{j-1} \Phi^I_{i,j} + Q_{j-1} \right) + d_j \quad (A-6)$$

which can be put in the form

$$\Phi^I_{i,j} = P_j \Phi^I_{i,j+1} + Q_j \quad (A-7)$$
where

\[
\begin{align*}
P_j &= \frac{b_j}{a_j - c_j P_{j-1}} \\
Q &= \frac{c_j \phi_{j-1} + d_j}{a_j - c_j P_{j-1}}
\end{align*}
\]

(A - 8)

These are recurrence relations for \( P_j, Q_j \). To start the recurrence process, for \( j = 1 \), the values of \( P, Q \) are given by

\[
P_1 = \frac{b_1}{a_1} \quad \text{and} \quad Q_1 = \frac{d_1}{a_1}
\]

The other \( P_j, Q_j \) can be computed from the recurrence form (A - 8).

For \( j = M' \), we note that \( P_M = 0 \), because \( b_M = 0 \). Then

\[
\phi_{i, M} = Q_M \quad \text{(A - 9)}
\]

using equation (A - 7) for \( j = M-1, M-2, \ldots, 3, 2, 1 \) to obtain

\[
\phi_{i, M-1}, \phi_{i, M-2}, \ldots, \phi_{i, 3}, \phi_{i, 2}, \phi_{i, 1}
\]

For the x-direction sweep we can do the same procedure. The difference equation (A - 1) becomes

\[
A_p \phi_{i, j} = A_E \phi_{i+1, j} + A_W \phi_{i-1, j} + \left( A_N \phi_{i, j+1} + A_S \phi_{i, j-1} + B \right) \quad \text{(A - 10)}
\]
where superscript, \( II \), denote the values obtained from the second phase of the solution.
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VITA

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