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Line force in composite materials.

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LINE FORCE IN COMPOSITE MATERIALS

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ABSTRACT

Based on the generalized method of images, the elastic field of an in-plane line force acting in a two-phase orthotropic medium is analyzed. Several special cases of technological interest are deduced from the general solution, including the case of a line force applied on the free surface of a half space. Application of the results to the determination of the elastic field of an edge dislocation in a semi-infinite orthotropic medium is illustrated.

1. INTRODUCTION

Composite materials, for example the fiber-reinforced metals and ceramics, are widely used in modern technology. The understanding and improvement of their mechanical behavior are in ever-increasing demand. One of the critical regions where failure of composite materials often takes place is near or at the interphase boundary. On a microscopic scale, this region may be visualized as a two-phase medium of infinite extent, and such a model has been used in numerous analyses in both fundamental and applied research.

In this study, we shall be concerned with the elastic field of a line force in a two-phase anisotropic medium. A line force is the two-dimensional product of a continuous, uniform distribution of point forces (1) acting along a line. The analysis of line force in a two-phase isotropic medium was reported some years ago by Frasier and Rongved (2). An extension of their work to take account of the effect of elastic anisotropy is motivated by recent applications of line force to crack and inclusion problems in which the elastic anisotropy plays a dominant role (3)-(9).

Analyses of the elastic fields of defects in anisotropic single-phase media are in general complex. The complexity is enhanced in the case of a two-phase anisotropic medium, and to effect a solvable situation, it is necessary to assume that the anisotropic medium contains certain elements of symmetry. In this work we shall assume that both phases of the medium have two planes of symmetry orthogonal to each other (orthotropy).

The basic approach used in the present analysis will follow the work of Eshelby, Read and Shockley (10) and of Pande and Chou (11), as shown in sections 2 and 3. In section 4, we shall examine several special cases of technological importance, in particular the case of a line force near a free surface. A direct application of the present solutions could be made to the analysis of elastic inclusions in a two-phase anisotropic medium. However, such an analysis would be much too involved, and instead, we shall present a simple example in which the elastic field of an edge dislocation in the same medium is determined from the known solutions of a line force, as given in section 5.

2. ANALYSIS

2.1. Formulation of the Problem

We consider an anisotropic elastic medium comprising two welded half-spaces with phase I at $x>0$ and phase II at $x<0$ in reference to a Cartesian coordinates system Oxyz. A continuous distribution of point forces is applied at a distance "a" from the z axis on the xz plane in phase I. Since nothing varies with the z coordinate, the system can be treated as a two dimensional plane strain problem with a line force F acting at (a,0) (see Figure 1). In general, the elastic constants C_{ij} and other parameters pertaining to phases I and II will be denoted by superscripts (1) and (2), respectively. However, for a statement valid in both phases the superscripts may be omitted for simplicity.

To effect an analytical solution to the problem, we further assume that both phases are orthotropic with the elastic constants given by

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \qquad (1)$$

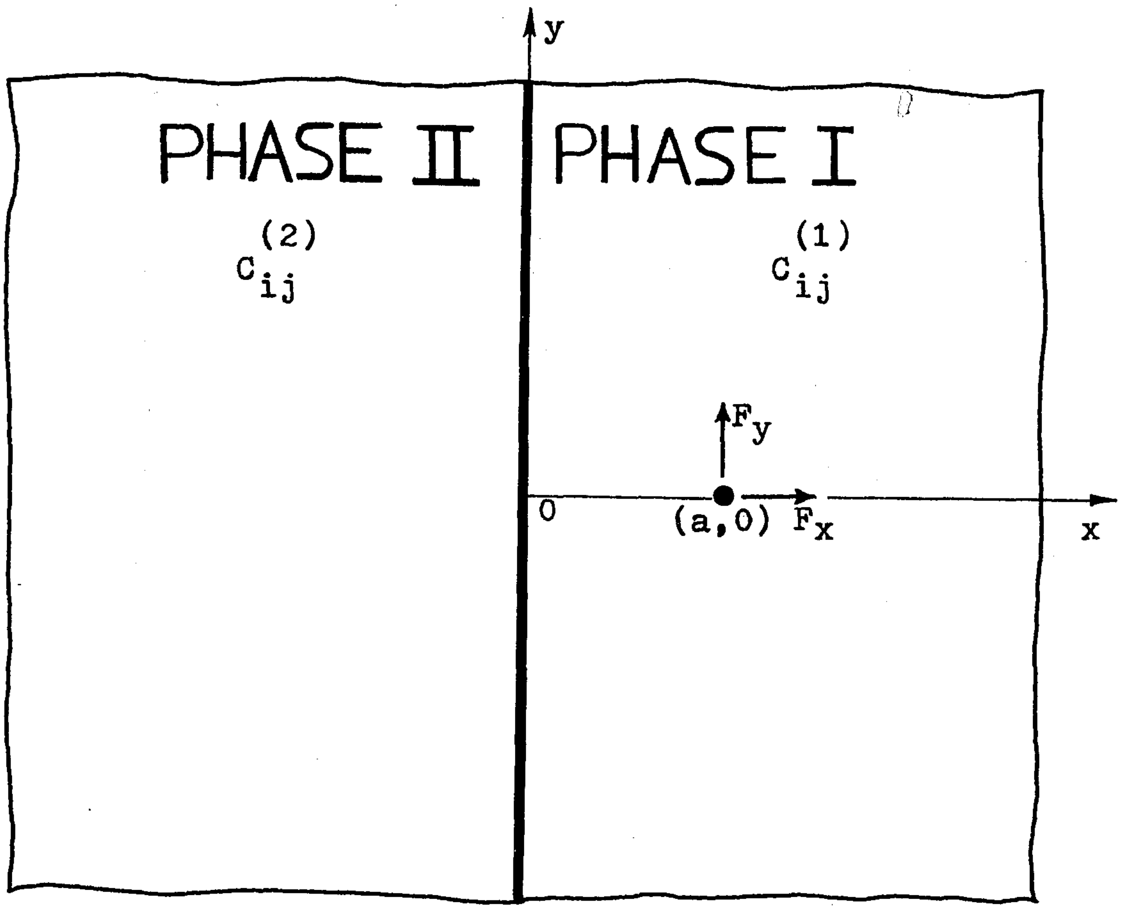

Fig. 1 An in-plane line force at (a,0) in a two-phase orthotropic medium

A line force in a homogeneous (single-phase) medium with condition (1) has been treated by Hirth (12). For a two-phase medium the solution is likely to consist of the "homogeneous" term plus an "image" term or terms. A simple approach based on the generalized image method was undertaken by Pande and Chou in the study of edge dislocations in a two-phase anisotropic medium (11). Their method will be used in the present analysis.

Following the work of Pande and Chou (11), the displacements u and v of the present system can be given as

$$u = \mathrm{Re}\left\{ D_1 \psi_+ + D_2 \psi_- \right\} \tag{2a}$$

$$v = \mathrm{Re}\left\{ - A \left(D_1 \psi_+ - D_2 \psi_- \right) \right\} \tag{2b}$$

where Re denotes the real part of the complex functions and D_1 and D_2 are complex constants. The constant A and the functions ψ_+ and ψ_- are defined as

$$A = \frac{\lambda\left(C_{66}\, e^{i\alpha} + \overline{C}_{12}\, e^{-i\alpha} \right)}{C_{12} + C_{66}} \tag{3}$$

and

$$\psi_{\pm} = f\left(\xi_{\pm} \right) = f\left(x + k_j \pm \lambda e^{i\alpha} y \right) \tag{4}$$

where k_j are unknown constants to be determined. In Eqs. (3) and (4)

$$\lambda = \left(\frac{C_{11}}{C_{22}} \right)^{1/4} = \left(\frac{\bar{C}_{12}}{C_{22}} \right)^{1/2} \tag{5}$$

$$\bar{C}_{12} = (C_{11} \; C_{22})^{1/2} \tag{6}$$

$$\alpha = \cos^{-1} \left[1/2 \quad (- C)^{1/2} \right] \quad \text{for } C<0 \tag{7}$$

$$= \cos^{-1} \left[i/2 \quad (C)^{1/2} \right] \quad \text{for } C>0$$

and

$$C = \frac{(\bar{C}_{12} + C_{12}) (\bar{C}_{12} - C_{12} - 2C_{66})}{\bar{C}_{12} \, C_{66}} \tag{8}$$

which has a lower limit of -4 and vanishes in the case of isotropy (13).

Under the condition of orthotropic symmetry the stress components are given by

$$\sigma_{xx} = \mathrm{Re} \left\{ E \, (D_1 \psi_+' + D_2 \psi_-') \right\} \tag{9a}$$

$$\sigma_{yy} = \mathrm{Re} \left\{ F \, (D_1 \psi_+' + D_2 \psi_-') \right\} \tag{9b}$$

$$\sigma_{xy} = \mathrm{Re} \left\{ G \, (D_1 \psi_+' - D_2 \psi_-') \right\} \tag{9c}$$

and

$$\sigma_{zz} = \frac{C_{22}C_{13} - C_{12}C_{23}}{\bar{C}_{12}^2 - C_{12}^2} \sigma_{xx} + \frac{C_{11}C_{23} - C_{12}C_{13}}{\bar{C}_{12}^2 - C_{12}^2} \sigma_{yy} \tag{9d}$$

where

$$E = C_{11} - C_{12} \lambda e^{i\alpha} A \tag{10a}$$

$$F = C_{12} - C_{22} \lambda e^{i\alpha} A \tag{10b}$$

$$G = C_{66} (\lambda e^{i\alpha} - A) \tag{10c}$$

and

$$\psi'_{\pm} = \frac{d\psi_{\pm}}{d\xi_{\pm}} \tag{10d}$$

For a welded boundary, the displacements u and v and the stresses σ_{xx} and σ_{xy} are continuous at the interface x=0, i.e.

$$u^{(1)} = u^{(2)} \tag{11a}$$

$$v^{(1)} = v^{(2)} \tag{11b}$$

$$\sigma_{xx}^{(1)} = \sigma_{xx}^{(2)} \tag{11c}$$

$$\sigma_{xy}^{(1)} = \sigma_{xy}^{(2)} \tag{11d}$$

However, for convenience we shall use an alternate version for the displacement continuity,

$$\frac{\partial u^{(1)}}{\partial y} = \frac{\partial u^{(2)}}{\partial y} \tag{11a)'$$

$$\frac{\partial v^{(1)}}{\partial y} = \frac{\partial v^{(2)}}{\partial y} \tag{11b)'$$

The use of Eqs. (11a)' and (11b)' may result in a difference in displacements by a constant at the boundary, but the constant can be eliminated by a suitable adjustment of the displacement functions.

Conditions (11a)', (11b)', (11c) and (11d) can be rewritten as

$$\mathrm{Re}\,(\gamma^{(1)}\phi_-^{(1)}) = \mathrm{Re}\,(\gamma^{(2)}\phi_-^{(2)}) \tag{12a}$$

$$\mathrm{Re}(A^{(1)}\gamma^{(1)}\phi_+^{(1)}) = \mathrm{Re}(A^{(2)}\gamma^{(2)}\phi_+^{(2)}) \tag{12b}$$

$$\mathrm{Re}\,(E^{(1)}\phi_+^{(1)}) = \mathrm{Re}\,(E^{(2)}\phi_+^{(2)}) \tag{12c}$$

$$\mathrm{Re}\,(G^{(1)}\phi_-^{(1)}) = \mathrm{Re}\,(G^{(2)}\phi_-^{(2)}) \tag{12d}$$

where

$$\phi_\pm = D_1\psi'_+ \pm D_2\psi'_- \tag{13}$$

and

$$\gamma = \lambda e^{i\alpha} \tag{14}$$

To determine the four complex coefficients, $D_1^{(1)}$, $D_2^{(1)}$, $D_1^{(2)}$ and $D_2^{(2)}$ (eight unknowns) in Eqs. (2a) and (2b), we need four more conditions in addition to Eqs. (12). Two conditions are provided by the conditions of equilibrium,

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} = 0 \tag{15a}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \qquad (15b)$$

The other two conditions are obtained from the definition of the line force

$$\oint_c (\sigma_{xx} \, dy - \sigma_{xy} \, dx) = F_x \qquad (16a)$$

$$\oint_c (\sigma_{xy} \, dy - \sigma_{yy} \, dx) = F_y \qquad (16b)$$

with the integration contour c enclosing the singularity.*

2.2. Method of Solution

The solution of the problem is based on the generalized image method used previously by Head (14) and Pande and Chou (11). The solution consists of the "homogeneous" term plus the "image" term or terms. The position and strength of the image are determined by the boundary conditions at x=0.

Let us first construct the following trial functions for $\psi_{\pm}$,

---

* It was found that in the homogeous case (single-phase), $D_1=D_2$ for $F=F_x$ and $D_1=-D_2$ for $F=F_y$. Thus for a two-phase medium the unknown coefficients, D's, can be reduced to four and uniquely determined by the boundary conditions, Eqs. (12).

$$D^{(1)}\psi_{\pm}'^{(1)} = \frac{K_1}{x-a\pm\lambda^{(1)}e^{i\alpha^{(1)}}y} + \frac{K_2}{x+a\pm\lambda^{(1)}e^{i\alpha^{(1)}}y}$$

$$+ \frac{K_3 e^{2i\alpha^{(1)}}}{x-ae^{2i\alpha^{(1)}}\pm\lambda^{(1)}e^{i\alpha^{(1)}}y} + \frac{K_4 e^{2i\alpha^{(1)}}}{x+ae^{2i\alpha^{(1)}}\pm\lambda^{(1)}e^{i\alpha^{(1)}}y}$$

and (17)

$$D^{(2)}\psi_{\pm}'^{(2)} = \frac{P_1}{\varepsilon x-a\pm\lambda^{(1)}e^{i\alpha^{(1)}}y} + \frac{P_2}{\varepsilon x+a\pm\lambda^{(1)}e^{i\alpha^{(1)}}y}$$

$$+ \frac{P_3 e^{2i\alpha^{(1)}}}{\varepsilon x-ae^{2i\alpha^{(1)}}\pm\lambda^{(1)}e^{i\alpha^{(1)}}y} + \frac{P_4 e^{2i\alpha^{(1)}}}{\varepsilon x+ae^{2i\alpha^{(1)}}\pm\lambda^{(1)}e^{i\alpha^{(1)}}y} \tag{18}$$

where

$$\varepsilon = \gamma^{(1)}/\gamma^{(2)} \tag{19}$$

and K_i and P_i $(i=1,2,3,4)$ are unknown complex coefficients to be determined by the boundary conditions (12). In terms of K_i and P_i, Eqs. (12) become

$$\gamma^{(1)}(K_1\pm K_2)-\gamma^{(2)}(P_1\pm P_2)+\overline{\gamma^{(1)}}(\overline{K_3\pm K_4})-\overline{\gamma^{(2)}}(\overline{P_3\pm P_4}) = 0 \tag{20a}$$

$$A^{(1)}\gamma^{(1)}(K_1\mp K_2)-A^{(2)}\gamma^{(2)}(P_1\mp P_2)+\overline{A^{(1)}}\,\overline{\gamma^{(1)}}(\overline{K_3\mp K_4})$$

$$-\overline{A^{(2)}}\,\overline{\gamma^{(2)}}(\overline{P_3\mp P_4}) = 0 \tag{20b}$$

$$E^{(1)}(K_1 \mp K_2) - E^{(2)}(P_1 \mp P_2) + \overline{E^{(1)}}(\overline{K_3 \mp K_4}) - \overline{E^{(2)}}(\overline{P_3 \mp P_4}) = 0 \tag{20c}$$

$$G^{(1)}(K_1 \pm K_2) - G^{(2)}(P_1 \pm P_2) + \overline{G^{(1)}}(\overline{K_3 \pm K_4}) - \overline{G^{(2)}}(\overline{P_3 \pm P_4}) = 0 \tag{20d}$$

where "$\overline{(\ )}$" denotes the complex conjugate of "( )". The positive sign in $\pm$ refers to the case where $F = F_x$ and the negative sign refers to the case where $F = F_y$.

In the present study, K_4, P_2 and P_3 are set to be zero in order that the stresses may not have a singularity apart from that at the line force site.* (i.e. at $x = a$) (11). Furthermore, K_1 is determined from the homogeneous case (10) and is given by

$$K_1 = \frac{-F_x(\overline{C_{12}^{(1)}}e^{i\alpha^{(1)}} + C_{66}^{(1)}e^{-i\alpha^{(1)}})}{4\pi\lambda^{(1)}\overline{C_{12}^{(1)}}C_{66}^{(1)}\sin 2\alpha^{(1)}} \qquad \text{for } F = F_x \tag{21a}$$

---

* Strictly speaking, the vanishment of these constants depends on the range of the C values. For example $K_3 \neq 0$ and $K_4 = 0$ for $-2 < C < -4$, and $K_3 = 0$ and $K_4 \neq 0$ for $0 > C > -2$. However, the diffence in details does not alter the final solutions.

and

$$K_1 = \frac{F_y(C_{12}^{(1)}+C_{66}^{(1)})}{4\pi\overline{C_{12}^{(1)}}C_{66}^{(1)}\sin 2\alpha^{(1)}} \qquad \text{for } F = F_y \qquad (21b)$$

Once K_2, K_3, P_1 and P_4 are determined by Eqs. (20), the ψ functions and subsequently the displacement and stress fields can be obtained by straightforward substitutions.

3. RESULTS

The results of the problem will be presented in two parts, line force perpendicular to the interface ($F = F_x$) and that parallel to it ($F = F_y$). The elastic field of a line force in an arbitrary direction can be obtained by superposition of these two solutions, together with the one for an antiplane line force ($F = F_z$) reported previously (15).

For simplicity, only the elastic fields in phase I will be given below. We shall omit the superscripts for the elastic constants and parameters except where confusion may arise. The elastic fields in phase II can be obtained by the same approach (though algebraically more complex), and will not be reported in explicit form. Furthermore, a number of elastic parameters which will appear in the solutions are conveniently defined as follows:

$$Q_0=H_0^{-1}\Big[2(C_{12}^{(1)}+\overline{C_{12}^{(1)}})(C_{12}^{(2)}-\overline{C_{12}^{(2)}})+({C_{12}^{(2)}}^2-\overline{C_{12}^{(2)}}^2)(C_{66}^{(1)}+C_{12}^{(1)})/C_{66}^{(1)}-({C_{12}^{(1)}}^2-\overline{C_{12}^{(1)}}^2)(C_{66}^{(2)}+\overline{C_{12}^{(2)}})/C_{66}^{(2)}\Big] \quad (22)$$

$$Q_1=(\lambda^{(1)}\lambda^{(2)}H_0)^{-1}\Big[4\sin\alpha^{(1)}\sin\alpha^{(2)}\overline{C_{12}^{(1)}}\,\overline{C_{12}^{(2)}}({\lambda^{(1)}}^2-{\lambda^{(2)}}^2)\Big] \quad (23)$$

$$Q_2=(\overline{C_{12}^{(2)}}^2-C_{12}^{(2)^2})\overline{C_{12}^{(1)}}/H_0(\overline{C_{12}^{(1)}}+C_{12}^{(1)}) \tag{24}$$

$$N = \overline{C_{12}^{(1)}}(C_{12}^{(2)}-\overline{C_{12}^{(2)}})/H_0 \tag{25}$$

and

$$H_0= \frac{4(\lambda^{(1)^2}+\lambda^{(2)^2})}{\lambda^{(1)}\lambda^{(2)}} \sin\alpha^{(1)}\sin\alpha^{(2)}\overline{C_{12}^{(1)}}\,\overline{C_{12}^{(2)}}$$
$$+2(C_{12}^{(1)}-\overline{C_{12}^{(1)}})(C_{12}^{(2)}-\overline{C_{12}^{(2)}})+(\overline{C_{12}^{(2)}}^2-C_{12}^{(2)^2})(1+\overline{C_{12}^{(1)}}/C_{66}^{(1)})$$
$$+(\overline{C_{12}^{(1)}}^2-C_{12}^{(1)^2})(1+\overline{C_{12}^{(2)}}/C_{66}^{(2)}) \tag{26}$$

3.1. Displacement Fields ($x \geqslant 0$)

A. $F = F_x$ (line force perpendicular to the interface)

$$u=\frac{-F_x}{8\pi\lambda}\left\{\left(\frac{\overline{C}_{12}+C_{66}}{2\overline{C}_{12}C_{66}\sin\alpha}\ln\chi_1+\frac{\overline{C}_{12}-C_{66}}{\overline{C}_{12}C_{66}\cos\alpha}\right.\right.$$
$$\left.\cdot\tan^{-1}\frac{2\lambda^2y^2\sin2\alpha}{2(x-a)^2+(c+2)\lambda^2y^2}\right]-(Q_0+Q_1-CQ_2)\left(\frac{\overline{C}_{12}+C_{66}}{2\overline{C}_{12}C_{66}\sin\alpha}\right.$$
$$\left.\cdot\ln\chi_2+\frac{\overline{C}_{12}-C_{66}}{\overline{C}_{12}C_{66}\cos\alpha}\tan^{-1}\frac{2\lambda^2y^2\sin2\alpha}{2(x+a)^2+(C+2)\lambda^2y^2}\right]$$
$$+(Q_0-CQ_2)\left[\frac{(C+2)\overline{C}_{12}-2C_{66}}{C\overline{C}_{12}C_{66}\sin\alpha}\ln\chi_2+\frac{2}{C_{66}\cos\alpha}\right.$$

$$\cdot \tan^{-1}\frac{2\lambda^2 y^2 \sin 2\alpha}{2(x+a)^2+(C+2)\lambda^2 y^2}\Bigg] - (Q_0+CN)\left[\frac{2(C_{12}+C_{66})}{C\overline{C_{12}}C_{66}\sin\alpha}\ln\chi_3\right]\Bigg\}$$

(27a)

$$v = \frac{F_x}{4\pi}\Bigg\{\frac{C_{12}+C_{66}}{\overline{C_{12}}C_{66}\sin 2\alpha}\Bigg[\tanh^{-1}\frac{2(x-a)\lambda y\cos\alpha}{(x-a)^2+\lambda^2 y^2} - (Q_0+Q_1-CQ_2)$$

$$\cdot \tanh^{-1}\frac{2(x+a)\lambda y\cos\alpha}{(x+a)^2+\lambda^2 y^2}\Bigg] + \frac{C_{12}+C_{66}}{\overline{C_{12}}C_{66}\sin 2\alpha}(Q_0-CQ_2)$$

$$\cdot \Bigg[\tanh^{-1}\frac{2(x+a)\lambda y\cos\alpha}{(x+a)^2+\lambda^2 y^2} - \frac{\sin\alpha}{\cos\alpha}\tan^{-1}\frac{2(x+a)\lambda y\sin\alpha}{(x+a)^2-\lambda^2 y^2}\Bigg]$$

$$+(Q_0+CN)\Bigg[\frac{\overline{C_{12}}+C_{66}}{\overline{C_{12}}C_{66}\sin 2\alpha}\tanh^{-1}\frac{2(x-a)\lambda y\cos\alpha}{(x+a)^2+\lambda^2 y^2+Cax}$$

$$-\frac{2(\overline{C_{12}}-C_{66})}{C\overline{C_{12}}C_{66}}\tan^{-1}\frac{2(x+a)\lambda y\sin\alpha}{(x+a)^2-\lambda^2 y^2+Cax}\Bigg]\Bigg\} \qquad (27b)$$

where

$$\chi_1 = \left[(x-a)^2+\lambda^2 y^2\right]^2 + C(x-a)^2\lambda^2 y^2 \qquad (28)$$

$$\chi_2 = \left[(x+a)^2+\lambda^2 y^2\right]^2 + C(x+a)^2\lambda^2 y^2 \qquad (29)$$

$$\chi_3 = \left[(x+a)^2+\lambda^2 y^2+Cax\right]^2 + C(x-a)^2\lambda^2 y^2 \qquad (30)$$

B. $F = F_y$ (line force parallel to the interface)

$$u = \frac{F_y}{4\pi}\left\{\frac{C_{12}+C_{66}}{\overline{C}_{12}C_{66}\sin 2\alpha}\left[\tanh^{-1}\frac{2(x-a)\lambda y\cos\alpha}{(x-a)^2+\lambda^2y^2} -(Q_0+Q_1-CQ_2)\right.\right.$$

$$\left.\cdot\tanh^{-1}\frac{2(x+a)\lambda y\cos\alpha}{(x+a)^2+\lambda^2y^2}\right] + \frac{C_{12}+C_{66}}{\overline{C}_{12}C_{66}\sin 2\alpha}(Q_0-CQ_2)$$

$$\cdot\left[\tanh^{-1}\frac{2(x+a)\lambda y\cos\alpha}{(x+a)^2+\lambda^2y^2} + \frac{\sin\alpha}{\cos\alpha}\tan^{-1}\frac{2(x+a)\lambda y\sin\alpha}{(x+a)^2-\lambda^2y^2}\right]$$

$$+(Q_0+CN)\left[\frac{\overline{C}_{12}+C_{66}}{\overline{C}_{12}C_{66}\sin 2\alpha}\tanh^{-1}\frac{2(x-a)\lambda y\cos\alpha}{(x+a)^2+\lambda^2y^2+Cax}\right.$$

$$\left.\left.+\frac{2(\overline{C}_{12}-C_{66})}{C\overline{C}_{12}C_{66}}\tan^{-1}\frac{2(x+a)\lambda y\sin\alpha}{(x+a)^2-\lambda^2y^2+Cax}\right]\right\} \qquad (31a)$$

$$v = \frac{-F_y}{8\pi\lambda}\left\{\left[\frac{\overline{C}_{12}+C_{66}}{2\overline{C}_{12}C_{66}\sin\alpha}\ln\chi_1 - \frac{\overline{C}_{12}-C_{66}}{\overline{C}_{12}C_{66}\cos\alpha}\right.\right.$$

$$\left.\cdot\tan^{-1}\frac{2\lambda^2y^2\sin 2\alpha}{2(x-a)^2+(C+2)\lambda^2y^2}\right] -(Q_0+Q_1-CQ_2)\left[\frac{\overline{C}_{12}+C_{66}}{2\overline{C}_{12}C_{66}\sin\alpha}\right.$$

$$\left.\cdot\ln\chi_2 - \frac{\overline{C}_{12}-C_{66}}{\overline{C}_{12}C_{66}\cos\alpha}\tan^{-1}\frac{2\lambda^2y^2\sin 2\alpha}{2(x+a)^2+(C+2)\lambda^2y^2}\right]$$

$$+(Q_0-CQ_2)\left[\frac{(C+2)\overline{C}_{12}-2C_{66}}{C\overline{C}_{12}C_{66}\sin\alpha}\ln\chi_2-\frac{2}{C_{66}\cos\alpha}\right.$$

$$\left.\cdot\tan^{-1}\frac{2\lambda^2y^2\sin2\alpha}{2(x+a)^2+(C+2)\lambda^2y^2}\right]-(Q_0+CN)\left[\frac{2(C_{12}+C_{66})}{C\overline{C}_{12}C_{66}\sin\alpha}\ln\chi_3\right]\Bigg\}$$

(31b)

3.2. Stress Fields (x⩾0)

A. $F = F_x$ (line force perpendicular to the interface)

$$\sigma_{xx} = \frac{-F_x\lambda}{4\pi\sin\alpha}\left\{\frac{(x-a)}{\chi_1}\left[(C+3+C_{12}/\overline{C}_{12})(x-a)^2-(C_{12}/\overline{C}_{12}-1)\lambda^2y^2\right]\right.$$

$$-\frac{(Q_0+Q_1-CQ_2)(x+a)}{\chi_2}\left[(C+3+C_{12}/\overline{C}_{12})(x+a)^2-(C_{12}/\overline{C}_{12}-1)\right.$$

$$\left.\left.\cdot\lambda^2y^2\right]\right\}-\frac{F_x\lambda}{2\pi C\sin\alpha}\left\{\frac{(Q_0-CQ_2)(x+a)}{\chi_2}\left[(C+2+2C_{12}/\overline{C}_{12})\right.\right.$$

$$\left.\cdot((x+a)^2+\lambda^2y^2)+(C+3+C_{12}/\overline{C}_{12})C(x+a)^2\right]-\frac{(Q_0+CN)}{\chi_3}$$

$$\cdot\left[(C+2+2C_{12}/\overline{C}_{12})(x+a)((x+a)^2+\lambda^2y^2)+C\langle C(C_{12}/\overline{C}_{12}-1)ax^2\right.$$

$$+x(x+a)((C_{12}/\overline{C}_{12}-1)x+(C+1+3C_{12}/\overline{C}_{12})a)+(C_{12}/\overline{C}_{12}-1)$$

$$+(Q_0-CQ_2)\left[\frac{(C+2)\overline{C}_{12}-2C_{66}}{C\overline{C}_{12}C_{66}\sin\alpha}\ln\chi_2-\frac{2}{C_{66}\cos\alpha}\right.$$

$$\left.\cdot\tan^{-1}\frac{2\lambda^2y^2\sin2\alpha}{2(x+a)^2+(C+2)\lambda^2y^2}\right]-(Q_0+CN)\left[\frac{2(C_{12}+C_{66})}{C\overline{C}_{12}C_{66}\sin\alpha}\ln\chi_3\right]\Bigg\}$$

(31b)

3.2. Stress Fields ($x \geqslant 0$)

A. $F = F_x$ (line force perpendicular to the interface)

$$\sigma_{xx}=\frac{-F_x\lambda}{4\pi\sin\alpha}\left\{\frac{(x-a)}{\chi_1}\left[(C+3+C_{12}/\overline{C}_{12})(x-a)^2-(C_{12}/\overline{C}_{12}-1)\lambda^2y^2\right]\right.$$

$$-\frac{(Q_0+Q_1-CQ_2)(x+a)}{\chi_2}\left[(C+3+C_{12}/\overline{C}_{12})(x+a)^2-(C_{12}/\overline{C}_{12}-1)\right.$$

$$\left.\left.\cdot\lambda^2y^2\right]\right\}-\frac{F_x\lambda}{2\pi C\sin\alpha}\left\{\frac{(Q_0-CQ_2)(x+a)}{\chi_2}\left[(C+2+2C_{12}/\overline{C}_{12})\right.\right.$$

$$\left.\cdot((x+a)^2+\lambda^2y^2)+(C+3+C_{12}/\overline{C}_{12})C(x+a)^2\right]-\frac{(Q_0+CN)}{\chi_3}$$

$$\cdot\left[(C+2+2C_{12}/\overline{C}_{12})(x+a)((x+a)^2+\lambda^2y^2)+C\langle C(C_{12}/\overline{C}_{12}-1)ax^2\right.$$

$$+x(x+a)((C_{12}/\overline{C}_{12}-1)x+(C+1+3C_{12}/\overline{C}_{12})a)+(C_{12}/\overline{C}_{12}-1)$$

$$\cdot\lambda^2 y^2 a \rangle \Big] \Big\} \tag{32a}$$

$$\sigma_{yy} = \frac{-F_x}{4\pi\lambda\sin\alpha}\left\{\frac{(x-a)}{\chi_1}\Big[(C_{12}/\overline{C}_{12}-1)(x-a)^2+(1+(C+3)C_{12}/\overline{C}_{12})\right.$$

$$\cdot\lambda^2y^2\Big]-\frac{Q_0+Q_1-CQ_2}{\chi_2}(x+a)\Big[(C_{12}/\overline{C}_{12}-1)(x+a)^2$$

$$\left.+(1+(C+3)C_{12}/\overline{C}_{12})\lambda^2y^2\Big]\right\}+\frac{F_x}{2\pi C\lambda\sin\alpha}\left\{\frac{(Q_0-CQ_2)(x+a)}{\chi_2}\right.$$

$$\cdot\Big[(C+2+2C_{12}/\overline{C}_{12})((x+a)^2+\lambda^2y^2)+(C_{12}/\overline{C}_{12}-1)C\lambda^2y^2\Big]$$

$$-\frac{(Q_0+CN)}{\chi_3}\Big[(C+2+2C_{12}/\overline{C}_{12})(x+a)((x+a)^2+\lambda^2y^2)+C\langle C(C+3$$

$$+C_{12}/\overline{C}_{12})a^2x+a(a+x)((2C+5+3C_{12}/\overline{C}_{12})x+(C+3+C_{12}/\overline{C}_{12})a)$$

$$\left.+(C+3+C_{12}/\overline{C}_{12})\lambda^2y^2x\rangle\Big]\right\} \tag{32b}$$

$$\sigma_{xy} = \frac{-F_x\lambda}{4\pi\sin\alpha}\left\{\frac{y}{\chi_1}\Big[(C+3+C_{12}/\overline{C}_{12})(x-a)^2-(C_{12}/\overline{C}_{12}-1)\lambda^2y^2\Big]\right.$$

$$- \frac{(Q_0+Q_1-CQ_2)y}{\chi_2}\Big[(C+3+C_{12}/\overline{C}_{12})(x+a)^2-(C_{12}/\overline{C}_{12}-1)\lambda^2y^2\Big]\Big\}$$

$$- \frac{F_x\lambda}{2\pi C\sin\alpha}\Big\{\frac{(Q_0-CQ_2)y}{\chi_2}\Big[(C+2+2C_{12}/\overline{C}_{12})((x+a)^2+\lambda^2y^2)$$

$$+C(C+3+C_{12}/\overline{C}_{12})(x+a)^2\Big] - \frac{(Q_0+CN)y}{\chi_3}\Big[(C+2+2C_{12}/\overline{C}_{12})$$

$$\cdot((x+a)^2+\lambda^2y^2)+C\langle(C_{12}/\overline{C}_{12}-1)x^2+(C+3+C_{12}/\overline{C}_{12})a^2\rangle\Big]\Big\} \quad (32c)$$

B. $F = F_y$ (line force parallel to the interface)

$$\sigma_{xx} = \frac{-F_y\lambda^3}{4\pi\sin\alpha}\Big\{\frac{y}{\chi_1}\Big[(1+(C+3)C_{12}/\overline{C}_{12})(x-a)^2+(C_{12}/\overline{C}_{12}-1)\lambda^2y^2\Big]$$

$$- \frac{(Q_0+Q_1-CQ_2)y}{\chi_2}\Big[(1+(C+3)C_{12}/\overline{C}_{12})\cdot(x+a)^2$$

$$+(C_{12}/\overline{C}_{12}-1)\lambda^2y^2\Big]\Big\} + \frac{F_y\lambda^3}{2\pi C\sin\alpha}\Big\{\frac{(Q_0-CQ_2)y}{\chi_2}$$

$$\cdot \left[(C+2+2C_{12}/\overline{C}_{12})((x+a)^2+\lambda^2y^2)+C(C_{12}/\overline{C}_{12}-1)(x+a)^2 \right]$$

$$- \frac{(Q_0+CN)y}{\chi_3} \left[(C+2+2C_{12}/\overline{C}_{12})((x+a)^2+\lambda^2y^2)+C\langle (C+3 \right.$$

$$\left. \left. +C_{12}/\overline{C}_{12})x^2+(C_{12}/\overline{C}_{12}-1)a^2 \rangle \right] \right\} \qquad (33a)$$

$$\sigma_{yy} = \frac{F_y\lambda}{4\pi \sin\alpha} \left\{ \frac{y}{\chi_1} \left[(C_{12}/\overline{C}_{12}-1)(x-a)^2-(C+3+C_{12}/\overline{C}_{12})\lambda^2y^2 \right] \right.$$

$$\left. - \frac{(Q_0+Q_1-CQ_2)y}{\chi_2} \left[(C_{12}/\overline{C}_{12}-1)(x+a)^2-(C+3+C_{12}/\overline{C}_{12})\lambda^2y^2 \right] \right\}$$

$$- \frac{F_y\lambda}{2\pi C \sin\alpha} \left\{ \frac{(Q_0-CQ_2)y}{\chi_2} \left[(C+2+2C_{12}/\overline{C}_{12})((x+a)^2+\lambda^2y^2) \right. \right.$$

$$\left. +C(C+3+C_{12}/\overline{C}_{12})\lambda^2y^2 \right] - \frac{(Q_0+CN)y}{\chi_3} \left[(C+2+2C_{12}/\overline{C}_{12}) \right.$$

$$\cdot ((x+a)^2+\lambda^2y^2)+C\langle (C+4)C_{12}/\overline{C}_{12} \cdot a^2+2(C_{12}/\overline{C}_{12}-1)ax$$

$$\left. \left. +(C_{12}/\overline{C}_{12}-1)\lambda^2y^2 \rangle \right] \right\} \qquad (33b)$$

$$
\begin{aligned}
\sigma_{xy} = \frac{F_y\lambda}{4\pi\sin\alpha}\Bigg\{ & \frac{(x-a)}{\chi_1}\left[(C_{12}/\bar{C}_{12}-1)(x-a)^2-(C+3+C_{12}/\bar{C}_{12})\lambda^2y^2\right] \\
& -\frac{(Q_0+Q_1-CQ_2)(x+a)}{\chi_2}\Big[(C_{12}/\bar{C}_{12}-1)(x+a)^2-(C+3+C_{12}/\bar{C}_{12}) \\
& .\lambda^2y^2\Big]\Bigg\} - \frac{F_y\lambda}{2\pi C\sin\alpha}\Bigg\{\frac{(Q_0-CQ_2)(x+a)}{\chi_2}\Big[(C+2+2C_{12}/\bar{C}_{12}) \\
& .((x+a)^2+\lambda^2y^2)+C(C+3+C_{12}/\bar{C}_{12})\lambda^2y^2\Big] - \frac{(Q_0+CN)}{\chi_3} \\
& .\Big[(C+2+2C_{12}/\bar{C}_{12})(x+a)((x+a)^2+\lambda^2y^2)+C\big\langle C(C_{12}/\bar{C}_{12}-1) \\
& .a^2x+a(x+a)((C+1+3C_{12}/\bar{C}_{12})x+(C_{12}/\bar{C}_{12}-1)a) \\
& +(C_{12}/\bar{C}_{12}-1)\lambda^2y^2x\big\rangle\Big]\Bigg\} \qquad (33c)
\end{aligned}
$$

The results given have been checked for both the equilibrium and the boundary conditions. In the process of verification numerical computations, with the aid of a digital computer, were employed, especially for the data in phase II. For reference, the stress fields in phase II are given in complex functions as follows: $(x \leqslant 0)$

$$\sigma_{xx} = \mathrm{Re}\left\{E^{(2)}\left[P_1 \frac{2(\varepsilon x-a)\left[(\varepsilon x-a)^2-\lambda^{(1)^2} e^{-2i\alpha^{(1)}} y^2\right]}{\chi_4} - e^{2i\alpha^{(1)}} P_4 \frac{2(\varepsilon x+ae^{2i\alpha^{(1)}})\left[(\varepsilon x+ae^{-2i\alpha^{(1)}})^2-\lambda^{(1)^2} y^2 e^{-2i\alpha^{(1)}}\right]}{\chi_5}\right]\right\} \tag{34a}$$

$$\sigma_{yy} = \mathrm{Re}\left\{F^{(2)}\left[P_1 \frac{2(\varepsilon x-a)\left[(\varepsilon x-a)^2-\lambda^{(1)^2} e^{-2i\alpha^{(1)}} y^2\right]}{\chi_4} - e^{2i\alpha^{(1)}} P_4 \frac{2(\varepsilon x+ae^{2i\alpha^{(1)}})\left[(\varepsilon x+ae^{-2i\alpha^{(1)}})^2-\lambda^{(1)^2} y^2 e^{-2i\alpha^{(1)}}\right]}{\chi_5}\right]\right\} \tag{34b}$$

$$\sigma_{xy} = \mathrm{Re}\left\{G^{(2)}\left[P_1 \frac{-2\lambda^{(1)} e^{i\alpha^{(1)}} y\left[(\varepsilon x-a)^2-\lambda^{(1)^2} y^2 e^{-2i\alpha^{(1)}}\right]}{\chi_4} - e^{2i\alpha^{(1)}} P_4 \frac{-2\lambda^{(1)} e^{i\alpha^{(1)}} y\left[(\varepsilon x+ae^{-2i\alpha^{(1)}})^2-\lambda^{(1)^2} y^2 e^{-2i\alpha^{(1)}}\right]}{\chi_5}\right]\right\} \tag{34c}$$

where

$$\chi_4 = \left[(\varepsilon x-a)^2+\lambda^{(1)^2} y^2\right] + C^{(1)}\lambda^{(1)^2} y^2 (\varepsilon x-a)^2 \tag{35}$$

$$\chi_5 = \left\{\left[(\varepsilon x-a)^2+\lambda^{(1)^2} y^2\right] - C^{(1)} a\varepsilon x\right\}^2 + C^{(1)}\lambda^{(1)^2} y^2 (\varepsilon x+a)^2 \tag{36}$$

3.3. Numerical Illustration

To illustrate the characteristics of the elastic fields of a line force, we computed the stress contours produced by a line force of unit strength acting at (2,0) in the x direction in an α-iron bicrystal. We assumed that in phase I the crystal has its crystallographic orientations, [100], [010] and [001], parallel to the coordinate axes, x, y and z respectively, whereas in phase II the chosen orientations in parallel to x, y and z axes are $[\bar{1}10]$, [001] and [110] respectively.

The standard elastic constants of α-iron at room temperature were reported to be (16)

$$C^0_{11} = 23.31, \quad C^0_{12} = 13.54, \quad C^0_{44} = 11.78 \qquad (37)$$

in units of 10^{10} Pascals.

Hence, the elastic constants for phase I and phase II in reference to the coordinate system Oxyz are (17)

$$C^{(1)}_{11} = C^{(1)}_{22} = C^{(1)}_{33} = C^0_{11} = 23.31$$

$$C^{(1)}_{12} = C^{(1)}_{13} = C^{(1)}_{23} = C^0_{12} = 13.54$$

$$C^{(1)}_{44} = C^{(1)}_{55} = C^{(1)}_{66} = C^0_{44} = 11.78 \qquad (38)$$

and

$$C_{11}^{(2)} = C_{33}^{(2)} = C_{11}^{0} + \tfrac{1}{2}(\ 2C_{44}^{0}+C_{12}^{0}-C_{11}^{0}\ ) = 30.21$$

$$C_{22}^{(2)} = C_{11}^{0} = 23.31$$

$$C_{12}^{(2)} = C_{23}^{(2)} = C_{12}^{0} = 13.54$$

$$C_{13}^{(2)} = C_{12}^{0} - \tfrac{1}{2}(\ 2C_{44}^{0}+C_{12}^{0}-C_{11}^{0}\ ) = 6.65$$

$$C_{44}^{(2)} = C_{66}^{(2)} = C_{44}^{0} = 11.78$$

$$C_{55}^{(2)} = \tfrac{1}{2}(\ C_{11}^{0}-C_{12}^{0}\ ) = 4.89 \qquad (39)$$

in units of 10^{10} Pascals.

Numerical calculations of constant stress contours were made by a digital computer and the results are shown in Figures 2 to 5.

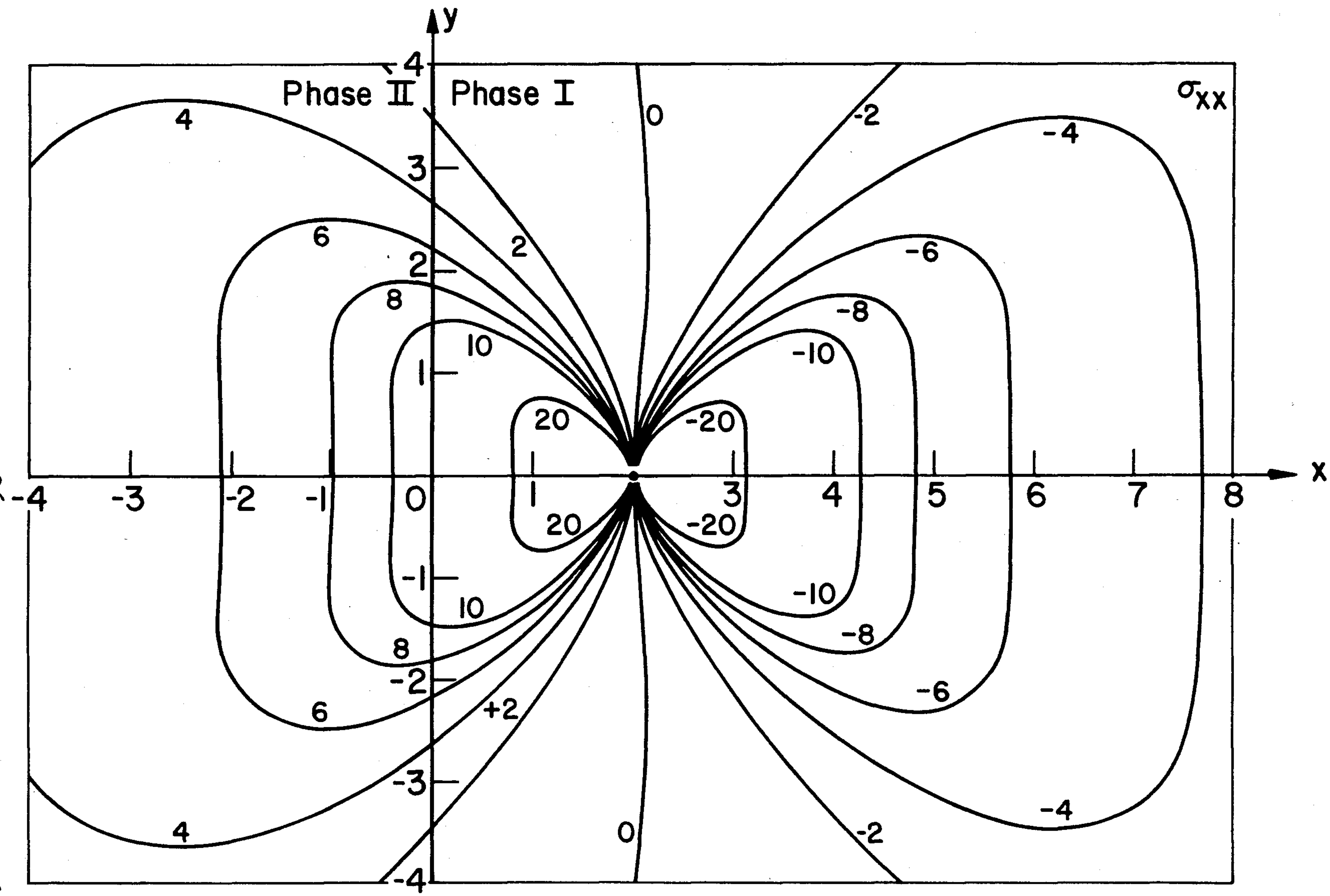

Fig. 2 σ_{xx} field of a unit line force normal to the interface in an α-iron bicrystal. The line force is located at (2,0). Units of stress : $1/40\pi$

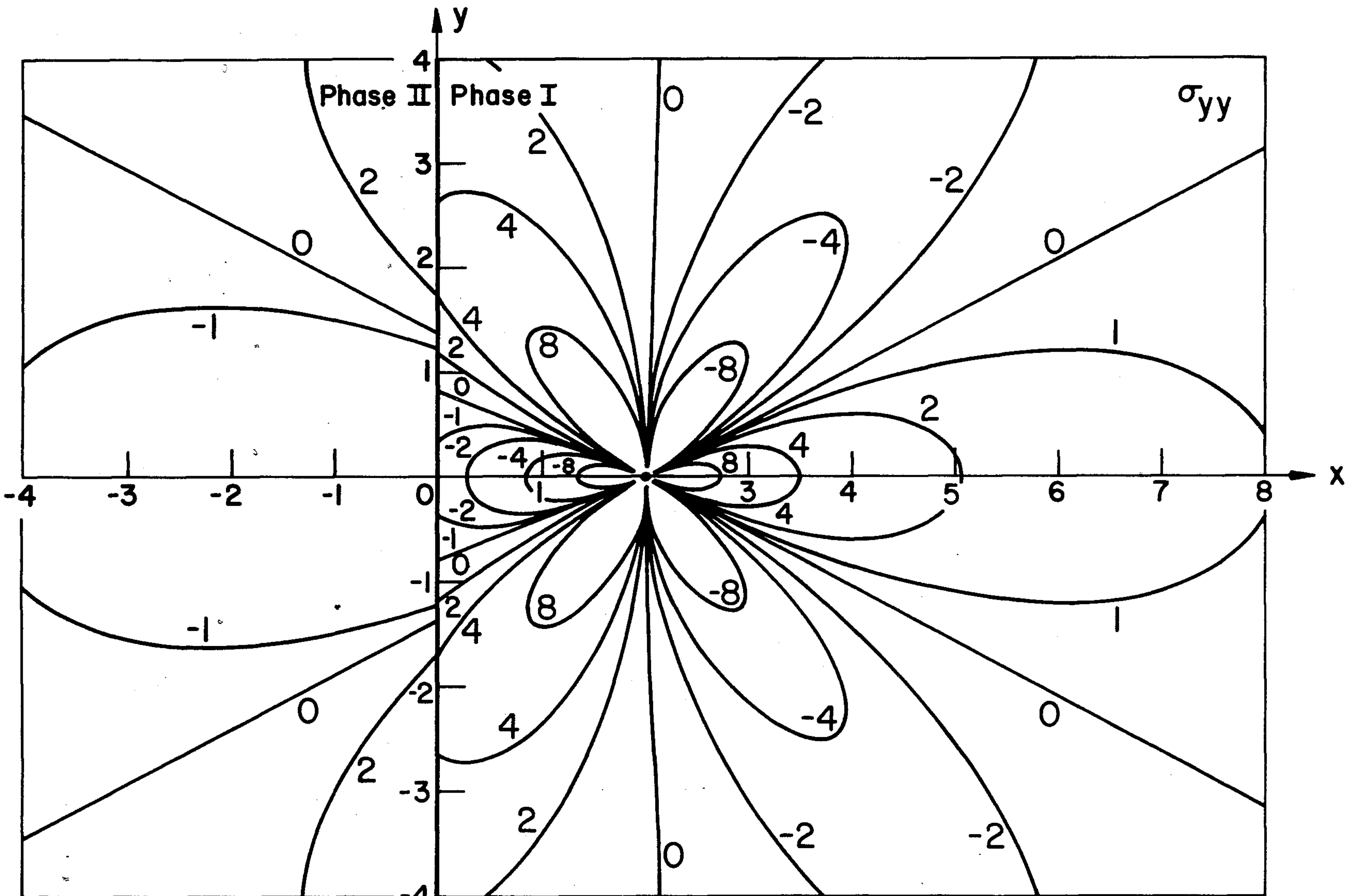

Fig. 3 σ_{yy} field of a unit line force normal to the interface in an α-iron bicrystal. The line force is located at (2,0). Units of stress : $1/40\pi$

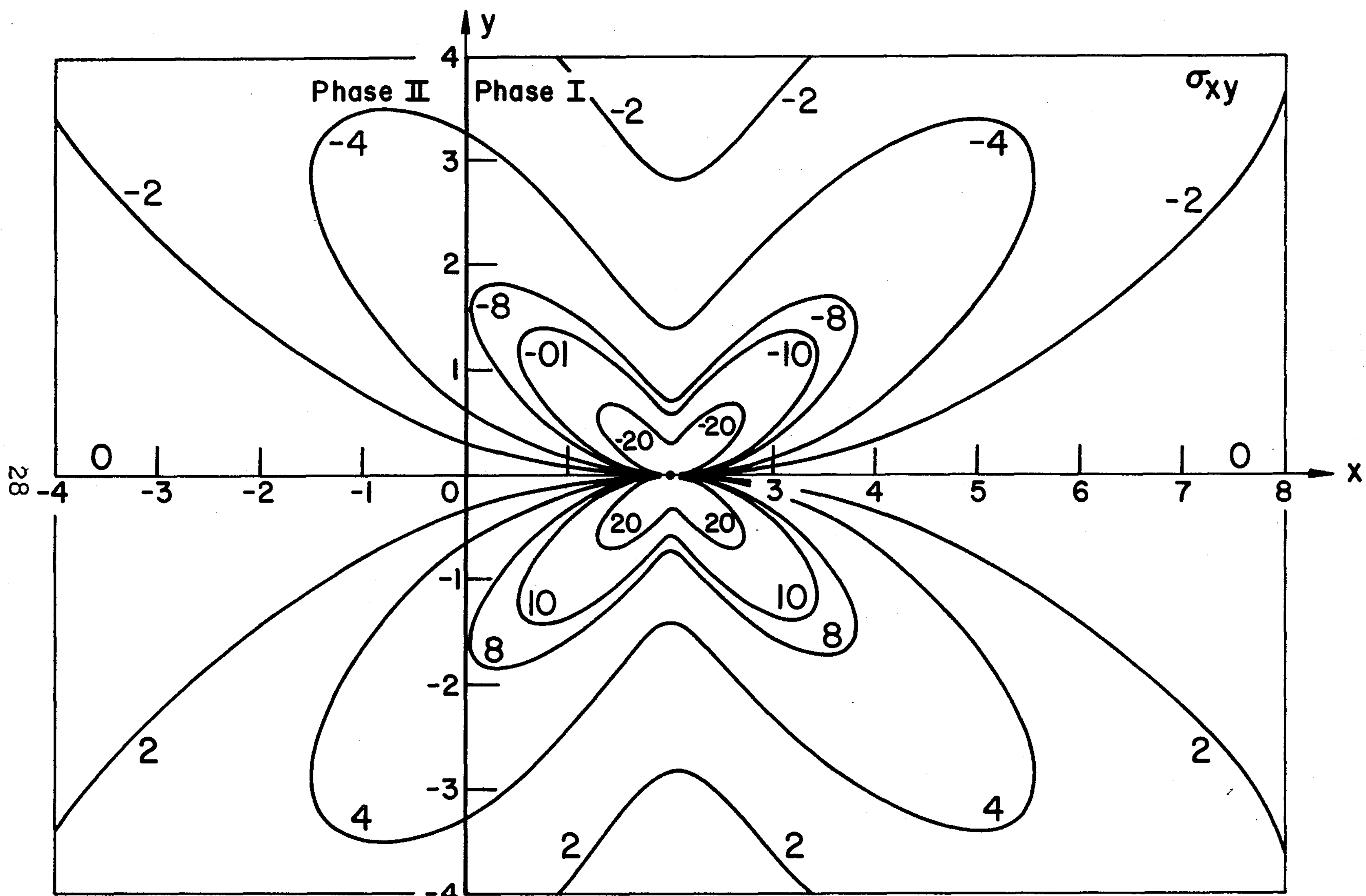

Fig.4 σ_{xy} field of a unit line force normal to the interface in an α-iron bicrystal. The line force is located at (2,0). Units of stress : $1/40\pi$

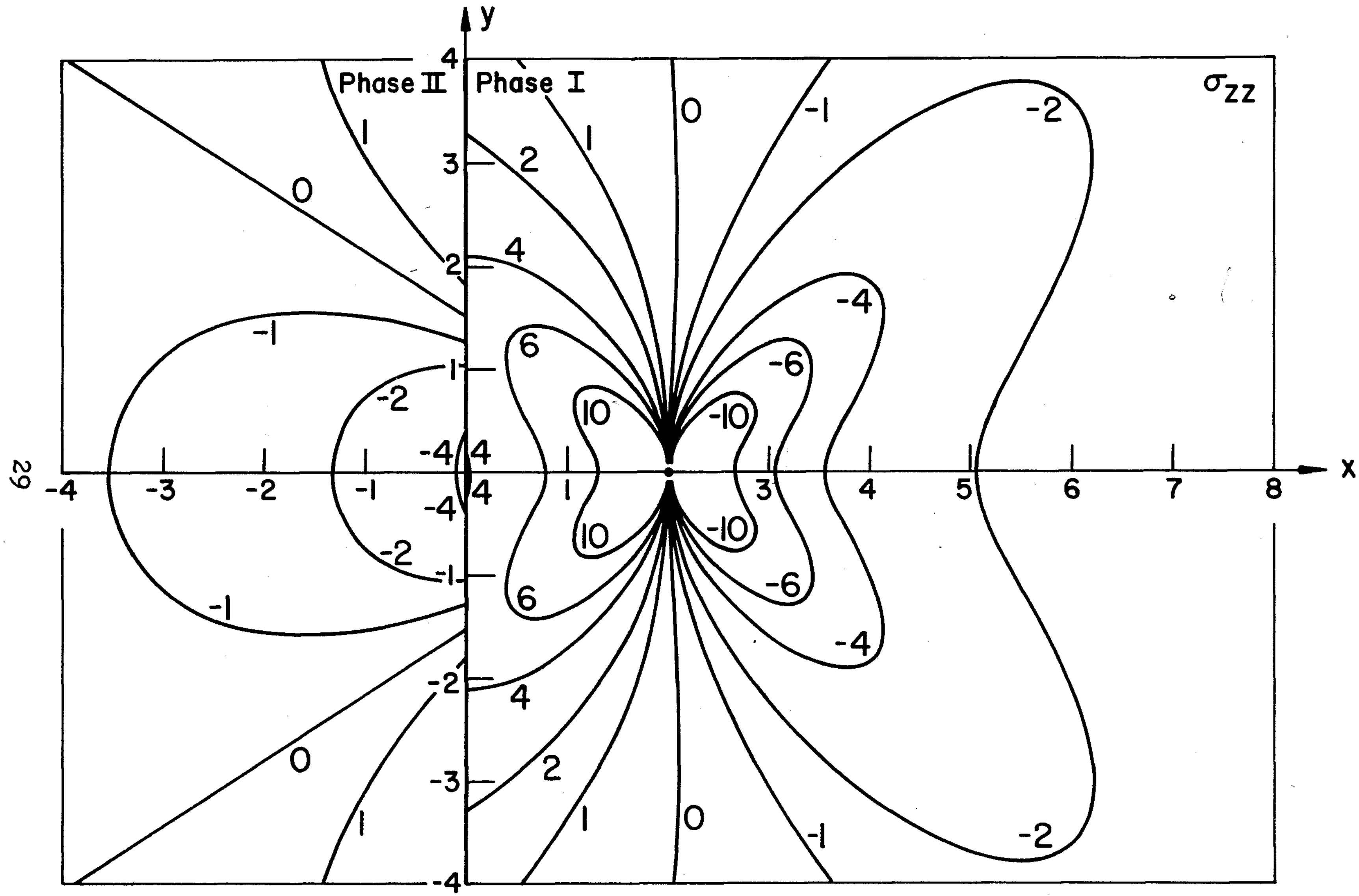

Fig.5 σ_{zz} field of a unit line force normal to the interface in an α-iron bicrystal.
The line force is located at (2,0). Units of stress : $1/40\pi$

4. SPECIAL CASES

There are several simple examples which can be obtained from the general problem analyzed in the preceding section. First of all, if we let $C_{ij}^{(1)} = C_{ij}^{(2)}$, then $Q_1=0$, $Q_0-CQ_2=0$ and $Q_0+CN=0$. The general results then reduce to those for a homogeneous, orthotropic medium treated by Hirth and Hirth and Lothe (12,18). Explicit expressions can also be deduced for other special cases as shown in the following.

4.1. The Elastic Field of a Line Force in a Two-Phase Isotropic Medium

In the case of isotropy,

$$C_{11} = \frac{2\mu(1-\nu)}{1-2\nu} \tag{40}$$

$$C_{12} = \frac{2\mu\nu}{1-2\nu} \tag{41}$$

$$C_{66} = C_{55} = C_{44} = \mu \tag{42}$$

and

$$C = 0 \tag{43}$$

where μ is the shear modulus and ν is the Poisson's ratio. The general solutions given in section 3 will be simplified considerably.

A. $F = F_x$ (line force perpendicular to the interface)$(x \geq 0)$

$$u = \frac{-F_x}{16\pi\mu(1-\nu)} \left\{ (3-4\nu) \ln\left[(x-a)^2+y^2\right] + \frac{2y^2}{(x-a)^2+y^2} \right.$$

$$\left. +(Q_0+M/4) \ln\left[(x+a)^2+y^2\right] + 2Q_0(3-4\nu) \frac{y^2}{(x+a)^2+y^2} \right\} \quad (44a)$$

$$v = \frac{F_x}{16\pi\mu(1-\nu)} \left\{ \frac{2(x-a)y}{(x-a)^2+y^2} + 2Q_0(3-4\nu) \frac{(x-a)y}{(x+a)^2+y^2} \right.$$

$$\left. +Q_0 \frac{8ax(x+a)y}{\left[(x+a)^2+y^2\right]^2} - M \cdot \tan^{-1} \frac{2(x+a)y}{(x+a)^2-y^2} \right\} \quad (44b)$$

$$\sigma_{xx} = \frac{-F_x}{4\pi(1-\nu)} \left\{ \frac{(x-a)\left[3(x-a)^2+y^2\right]}{\left[(x-a)^2+y^2\right]^2} + Q_0 \frac{(x+a)\left[3(x+a)^2+y^2\right]}{\left[(x+a)^2+y^2\right]^2} \right.$$

$$+2Q_0 x \frac{(x+3a)(x+a)^3 - 6a(x+a)y^2 - y^4}{\left[(x+a)^2+y^2\right]^3} + M \frac{x+a}{(x+a)^2+y^2}$$

$$\left. -2\nu \left[\frac{x-a}{(x-a)^2+y^2} + Q_0 \frac{x+a}{(x+a)^2+y^2} + 2Q_0 x \frac{(x+a)^2-y^2}{\left[(x+a)^2+y^2\right]^2} \right] \right\} \quad (45a)$$

$$\sigma_{yy} = \frac{-F_x}{4\pi(1-\nu)} \left\{ \frac{-(x-a)\left[3(x-a)^2+y^2\right]}{\left[(x-a)^2+y^2\right]^2} + Q_0 \frac{(x+a)\left[3(x+a)^2+y^2\right]}{\left[(x+a)^2+y^2\right]^2} \right.$$

$$+2Q_0\frac{a(x+3a)(x+a)^3+2(x+a)(2x^2+7ax+2a^2)y^2+(4x+a)y^4}{[(x+a)^2+y^2]^3}$$

$$-M\frac{x+a}{(x+a)^2+y^2}+2\nu\left[\frac{x-a}{(x-a)^2+y^2}-Q_0\frac{x+a}{(x+a)^2+y^2}\right.$$

$$\left.\left.-2Q_0\frac{a(x+a)^2+(2x+a)y^2}{[(x+a)^2+y^2]^2}\right]\right\} \tag{45b}$$

$$\sigma_{xy}=\frac{-F_x}{4\pi(1-\nu)}\left\{\frac{y[3(x-a)^2+y^2]}{[(x-a)^2+y^2]^2}-Q_0\frac{y[3(x+a)^2+y^2]}{[(x+a)^2+y^2]^2}\right.$$

$$+4Q_0xy\frac{(2x+5a)(x+a)^2+(2x+a)y^2}{[(x+a)^2+y^2]^3}+M\frac{y}{(x+a)^2+y^2}$$

$$\left.-2\nu\left[\frac{y}{(x-a)^2+y^2}-Q_0\frac{y}{(x+a)^2+y^2}+4Q_0xy\frac{x+a}{[(x+a)^2+y^2]^2}\right]\right\} \tag{45c}$$

where

$$M=\frac{4\mu^{(2)}(1-\nu^{(1)})[\mu^{(1)}(1-2\nu^{(2)})-\mu^{(2)}(1-2\nu^{(1)})]}{[\mu^{(1)}+\mu^{(2)}(3-4\nu^{(1)})][\mu^{(2)}+\mu^{(1)}(3-4\nu^{(2)})]} \tag{46}$$

$$Q_0=\frac{\mu^{(1)}-\mu^{(2)}}{\mu^{(1)}+\mu^{(2)}(3-4\nu^{(1)})} \tag{47}$$

B. $F = F_y$ (line force parallel to the interface)($x \geq 0$)

$$u = \frac{F_y}{16\pi\mu(1-\nu)}\left\{\frac{2(x-a)y}{(x-a)^2+y^2} + 2Q_0(3-4\nu)\frac{(x-a)y}{(x+a)^2+y^2}\right.$$

$$\left. -Q_0\frac{8ax(x+a)y}{[(x+a)^2+y^2]^2} + M\cdot\tan^{-1}\frac{2(x+a)y}{(x+a)^2-y^2}\right\} \quad (48a)$$

$$v = \frac{-F_y}{16\pi\mu(1-\nu)}\left\{(3-4\nu)\ln\left[(x-a)^2+y^2\right] - \frac{2y^2}{(x-a)^2+y^2}\right.$$

$$\left. +(Q_0+M/4)\ln\left[(x+a)^2+y^2\right] - 2Q_0(3-4\nu)\frac{y^2}{(x+a)^2+y^2}\right\} \quad (48b)$$

$$\sigma_{xx} = \frac{-F_y}{4\pi(1-\nu)}\left\{\frac{y\left[(x-a)^2-y^2\right]}{\left[(x-a)^2+y^2\right]^2} - Q_0\frac{y\left[(x+a)^2-y^2\right]}{\left[(x+a)^2+y^2\right]^2}\right.$$

$$+4Q_0xy\frac{(2x-a)(x+a)^2+(2x+3a)y^2}{\left[(x+a)^2+y^2\right]^3} - M\frac{y}{(x+a)^2+y^2}$$

$$\left. +2\nu\left[\frac{y}{(x-a)^2+y^2} - Q_0\frac{y}{(x+a)^2+y^2} - 4Q_0xy\frac{(x+a)}{\left[(x+a)^2+y^2\right]^2}\right]\right\}$$

(49a)

$$\sigma_{yy} = \frac{-F_y}{4\pi(1-\nu)}\left\{\frac{y\left[(x-a)^2+3y^2\right]}{\left[(x-a)^2+y^2\right]^2} - Q_0\frac{y\left[(x+a)^2+3y^2\right]}{\left[(x+a)^2+y^2\right]^2}\right.$$

$$+4Q_0y\frac{(3ax+2y^2)(x+a)^2+(2y^2-ax)y^2}{[(x+a)^2+y^2]^3}+M\frac{y}{(x+a)^2+y^2}$$

$$-2\nu\left[\frac{y}{(x-a)^2+y^2}-Q_0\frac{y}{(x+a)^2+y^2}+4Q_0y\frac{(a^2+ax+y^2)}{[(x+a)^2+y^2]^2}\right]\Bigg\}$$

(49b)

$$\sigma_{xy}=\frac{-F_y}{4\pi(1-\nu)}\left\{\frac{(x-a)[(x-a)^2+3y^2]}{[(x-a)^2+y^2]^2}-Q_0\frac{(x+a)[(x+a)^2+3y^2]}{[(x+a)^2+y^2]^2}\right.$$

$$+2Q_0\frac{a(3x+a)(x+a)^3+2(x+a)(2x^2+ax+2a^2)y^2+(4x+3a)y^4}{[(x+a)^2+y^2]^3}$$

$$+M\frac{(x+a)}{(x+a)^2+y^2}-2\nu\left[\frac{(x-a)}{(x-a)^2+y^2}-Q_0\frac{(x+a)}{(x+a)^2+y^2}\right.$$

$$\left.\left.+2Q_0\frac{(2x+a)y^2+a(x+a)^2}{[(x+a)^2+y^2]^2}\right]\right\} \qquad (49c)$$

The above stress equations have been checked numerically with the results given by Frasier and Rongved (2) after the latter had been converted from the plane-stress state to the plane-strain state.

4.2. The Elastic Field of an Interfacial Line Force

In some circumstances, the line force may lie at the boundary, i.e. $a = 0$. In this case, the expressions are much simpler. We obtained the following results for phase

I.

A. $F = F_x$ (line force perpendicular to the interface)

$$u = \frac{-F_x}{8\pi\lambda}\left\{\left[R\frac{C_{66}+\overline{C}_{12}}{2\overline{C}_{12}C_{66}\sin\alpha} + RS\frac{C_{66}-\overline{C}_{12}}{2\overline{C}_{12}C_{66}\sin\alpha} + (Q_0+CN)\right.\right.$$

$$\left.\cdot\frac{(C+2)\overline{C}_{12}-2C_{12}-4C_{66}}{C\overline{C}_{12}C_{66}\sin\alpha}\right] \ln\left[(x^2+\lambda^2y^2)^2+Cx^2\lambda^2y^2\right]$$

$$+\left[\left(R-\frac{C}{C+4}RS\right)\frac{\overline{C}_{12}-C_{66}}{\overline{C}_{12}C_{66}\cos\alpha} + (Q_0+CN)\frac{2}{C_{66}\cos\alpha}\right]$$

$$\left.\cdot\tan^{-1}\frac{2\lambda^2y^2\sin2\alpha}{2x^2+(C+2)\lambda^2y^2}\right\} \qquad (50a)$$

$$v = \frac{F_x}{4\pi}\left\{\left[(R+2RS)\frac{C_{12}+C_{66}}{\overline{C}_{12}C_{66}\sin2\alpha} + (Q_0+CN)\frac{\overline{C}_{12}+C_{12}+2C_{66}}{\overline{C}_{12}C_{66}\sin2\alpha}\right]\right.$$

$$\cdot\tanh^{-1}\frac{2x\lambda y\cos\alpha}{x^2+\lambda^2y^2} - \left[RS\frac{2(C_{12}+C_{66})}{\overline{C}_{12}C_{66}(C+4)} - (CN+Q_0)\right.$$

$$\left.\left.\cdot\frac{2(\overline{C}_{12}-C_{12}-2C_{66})}{C\overline{C}_{12}C_{66}}\right]\tan^{-1}\frac{2x\lambda y\sin\alpha}{x^2-\lambda^2y^2}\right\} \qquad (50b)$$

$$\sigma_{xx} = \frac{-F_x\lambda}{4\pi \sin\alpha} \cdot \frac{x}{(x^2+\lambda^2y^2)^2+Cx^2\lambda^2y^2} \left\{ R\left[(C+3+C_{12}/\overline{C}_{12})x^2 - (C_{12}/\overline{C}_{12}-1)\lambda^2y^2\right] - RS\left[(C+1+C_{12}/\overline{C}_{12})x^2+(C_{12}/\overline{C}_{12}+1) \cdot \lambda^2y^2\right] + 2(C+4)(Q_0+CN)x^2 \right\} \quad (51a)$$

$$\sigma_{yy} = \frac{-F_x}{4\pi\lambda \sin\alpha} \cdot \frac{x}{(x^2+\lambda^2y^2)^2+Cx^2\lambda^2y^2} \left\{ R\left[(C_{12}/\overline{C}_{12}-1)x^2 + (1+(C+3)C_{12}/\overline{C}_{12})\lambda^2y^2\right] + RS\left[(C_{12}/\overline{C}_{12}+1)x^2 + (1+(C+1)C_{12}/\overline{C}_{12})\lambda^2y^2\right] + 2\ (C+4)(Q_0+CN)\lambda^2y^2 \right\} \quad (51b)$$

$$\sigma_{xy} = \frac{-F_x\lambda}{4\pi \sin\alpha} \cdot \frac{y}{(x^2+\lambda^2y^2)^2+Cx^2\lambda^2y^2} \left\{ R\left[(C+3+C_{12}/\overline{C}_{12})x^2 - (C_{12}/\overline{C}_{12}-1)\lambda^2y^2\right] - RS\left[(C+1+C_{12}/\overline{C}_{12})x^2+(C_{12}/\overline{C}_{12}+1) \cdot \lambda^2y^2\right] + 2(C+4)(Q_0+CN)x^2 \right\} \quad (51c)$$

where

$$R = 1 - Q_0 - Q_1 - CN \quad (52)$$

$$S = (Q_2 + N) (C + 4) / R \tag{53}$$

B. $F = F_y$ (line force parallel to the interface)

$$u = \frac{F_y}{4\pi}\left\{\left[(R+2RS)\frac{C_{12}+C_{66}}{\overline{C}_{12}C_{66}\sin 2\alpha} + (Q_0+CN)\frac{\overline{C}_{12}+C_{12}+2C_{66}}{\overline{C}_{12}C_{66}\sin 2\alpha}\right]\right.$$

$$\cdot \tanh^{-1}\frac{2x\lambda y\cos\alpha}{x^2+\lambda^2y^2} + \left[RS\frac{2(C_{12}+C_{66})}{\overline{C}_{12}C_{66}(C+4)} - (Q_0+CN)\right.$$

$$\left.\left.\cdot\frac{2(\overline{C}_{12}-C_{12}-2C_{66})}{C\overline{C}_{12}C_{66}}\right]\tan^{-1}\frac{2x\lambda y\sin\alpha}{x^2-\lambda^2y^2}\right\} \tag{54a}$$

$$v = \frac{-F_y}{8\pi\lambda}\left\{\left[R\,\frac{\overline{C}_{12}+C_{66}}{2\overline{C}_{12}C_{66}\sin\alpha} - RS\,\frac{\overline{C}_{12}-C_{66}}{2\overline{C}_{12}C_{66}\sin\alpha}\right.\right.$$

$$\left.+(Q_0+CN)\,\frac{(C+2)\overline{C}_{12}-2C_{12}-4C_{66}}{C\overline{C}_{12}C_{66}\sin\alpha}\right]\ln\left[(x^2+\lambda^2y^2)^2+Cx^2\lambda^2y^2\right]$$

$$-\left[(R-\frac{C}{C+4}RS)\frac{\overline{C}_{12}-C_{66}}{\overline{C}_{12}C_{66}\cos\alpha} + \frac{2}{C_{66}\cos\alpha}\,(Q_0+CN)\right]$$

$$\left.\cdot\ \tan^{-1}\frac{2\lambda^2y^2\sin 2\alpha}{2x^2+(C+2)\lambda^2y^2}\right\} \tag{54b}$$

$$\sigma_{xx} = \frac{-F_y\lambda^3}{4\pi\sin\alpha}\cdot\frac{y}{(x^2+\lambda^2y^2)^2+Cx^2\lambda^2y^2}\Big\{R\Big[(1+(C+3)C_{12}/\overline{C}_{12})x^2$$

$$+(C_{12}/\overline{C}_{12}-1)\lambda^2y^2\Big]+RS\Big[((C+1)C_{12}/\overline{C}_{12}+1)x^2+(C_{12}/\overline{C}_{12}$$

$$+1)\lambda^2y^2\Big]+2(C+4)(Q_0+CN)x^2\Big\} \qquad (55c)$$

$$\sigma_{yy} = \frac{F_y\lambda}{4\pi\sin\alpha}\cdot\frac{y}{(x^2+\lambda^2y^2)^2+Cx^2\lambda^2y^2}\Big\{R\Big[(C_{12}/\overline{C}_{12}-1)x^2$$

$$-(C+3+C_{12}/\overline{C}_{12})\lambda^2y^2\Big]+RS\Big[(C_{12}/\overline{C}_{12}+1)x^2+(C+1+C_{12}/\overline{C}_{12})$$

$$\cdot\lambda^2y^2\Big]-2(C+4)(Q_0+CN)\lambda^2y^2\Big\} \qquad (55b)$$

$$\sigma_{xy} = \frac{F_y\lambda}{4\pi\sin\alpha}\cdot\frac{x}{(x^2+\lambda^2y^2)^2+Cx^2\lambda^2y^2}\Big\{R\Big[(C_{12}/\overline{C}_{12}-1)x^2$$

$$-(C+3+C_{12}/\overline{C}_{12})\lambda^2y^2\Big]+RS\Big[(C_{12}/\overline{C}_{12}+1)x^2+(C+1+C_{12}/\overline{C}_{12})$$

$$\cdot\lambda^2y^2\Big]-2(C+4)(Q_0+CN)\lambda^2y^2\Big\} \qquad (55c)$$

Since the line force is situated at the interface, the corresponding elastic fields in phase II can be obtained by interchanging the superscripts (1) and (2) in all elastic constants and parameters. It is seen that the present results are consistent with those for an interfacial edge dislocation in a two-phase orthotropic medium (19).

4.3. The Elastic Field of a Line Force in a Semi-Infinite Medium

Another special case of great interest occurs when the second phase is a void. The elastic system consists of a line force acting at distance "a" from the free surface x=0 in a semi-infinite medium. The displacement and stress equations can be obtained from Eqs. (27) - (33) simply by setting $Q_0=1$ and $Q_1=Q_2=N=0$.

In particular, if a=0, the line force is acting on the free surface. This case has many practical applications. The simple results for $F=F_x$ are given in the following:

$$u = \frac{-F_x}{8\pi\lambda}\left\{\frac{(C+2)\overline{C}_{12}-2C_{12}-4C_{66}}{C\overline{C}_{12}C_{66}\sin\alpha}\ \ln\left[(x^2+\lambda^2y^2)^2+Cx^2\lambda^2y^2\right]\right.$$

$$\left. + \frac{2}{C_{66}\cos\alpha}\tan^{-1}\frac{2\lambda^2y^2\sin2\alpha}{2x^2+(C+2)\lambda^2y^2}\right\} \tag{56a}$$

$$v = \frac{F_x}{4\pi}\left\{\frac{\overline{C}_{12}+C_{12}+2C_{66}}{\overline{C}_{12}C_{66}\sin2\alpha}\tanh^{-1}\frac{2x\lambda y\cos\alpha}{x^2+\lambda^2y^2}\right.$$

$$\left. + \frac{2\overline{C}_{12}-C_{12}-2C_{66}}{C\overline{C}_{12}C_{66}}\tan^{-1}\frac{2x\lambda y\sin\alpha}{x^2-\lambda^2y^2}\right\} \tag{56b}$$

$$\sigma_{xx} = \frac{-2F_x\lambda\sin\alpha}{\pi}\cdot\frac{x^3}{(x^2+\lambda^2y^2)^2+Cx^2\lambda^2y^2} \tag{57a}$$

$$\sigma_{yy} = \frac{-2F_x\sin\alpha}{\pi\lambda}\cdot\frac{x\lambda^2y^2}{(x^2+\lambda^2y^2)^2+Cx^2\lambda^2y^2} \tag{57b}$$

$$\sigma_{xy} = \frac{-2F_x\lambda\sin\alpha}{\pi}\cdot\frac{x^2y}{(x^2+\lambda^2y^2)^2+Cx^2\lambda^2y^2} \tag{57c}$$

which agree with the results of Lekhnitskii (20).

5. APPLICATION

The basic information expressed in the preceding sections is useful in various applications. For the sake of simplicity, we shall apply the results to the determination of the elastic field of an edge dislocation in a semi-infinite orthotropic medium (21). The derivation is based on the principle of virtual work through a virtual work cycle. The cycle may begin by introducing a line force F_x at the position (x,y), as shown in Figure 6. As a result, the system gains an amount of work W_f.

A cut is then made from x=0 to a along the plane y=0. In order to maintain the equilibrium, it is necessary to apply surface tractions on the cut planes with a magnitude equal to the stresses produced by the line force at (x,y). The two cut planes are next displaced relative to each other by an amount b_x in the x direction, and then rejoined together. In this way an edge dislocation is formed at (a,0) with a strain energy W_d. However, during the process of forming the dislocation, two interaction work terms are also involved. The first is the work done by the surface tractions on the cut planes due to the stress field of the line force, $W_I = \int_0^a \sigma_{xy} b_x \, dx$. The second term is the work done by the line force at (x,y) in the

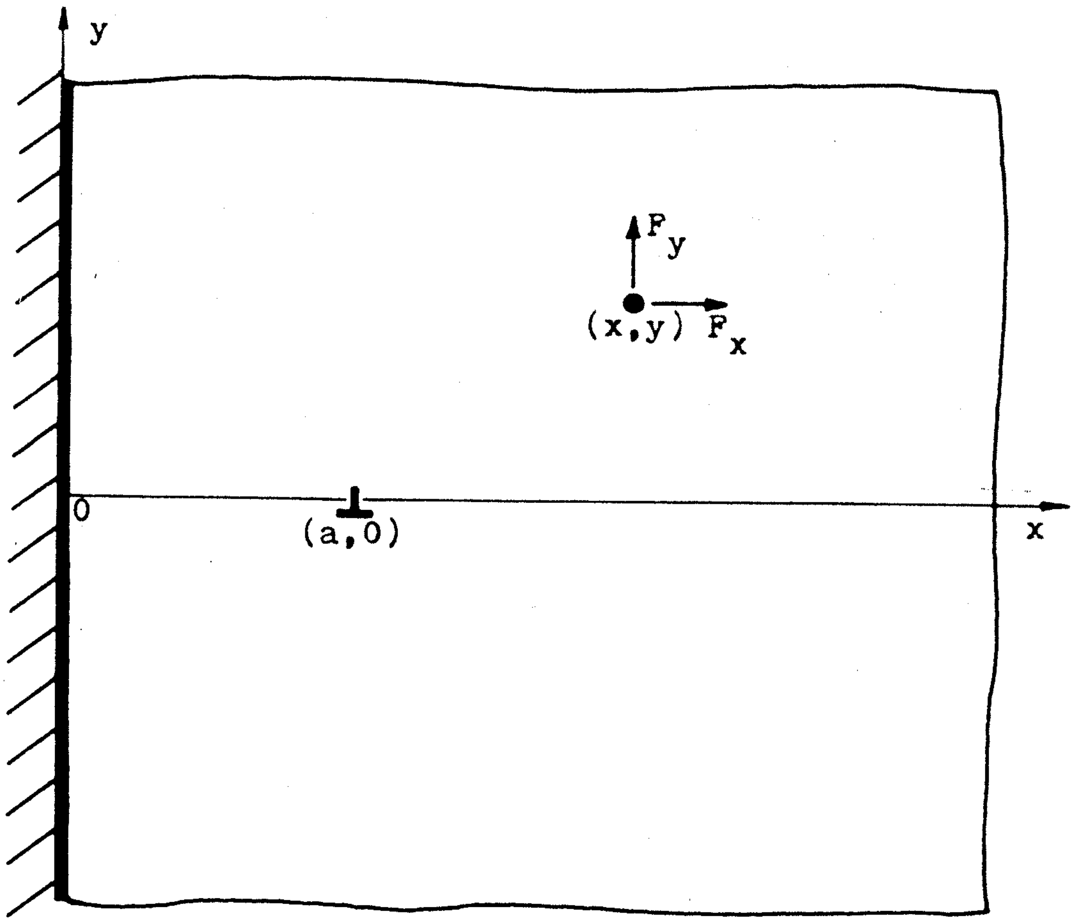

Fig. 6 Interaction of a line force at (x,y) and an edge dislocation at (a,0) in an orthotropic half-space

displacement field of the edge dislocation, $W_{II}=F_x \cdot u_e$. The total atrain energy in the system up to this stage is $W_f + W_d + W_I + W_{II}$.

Let us now remove the line force, and then the edge dislocation, from the system. An amount of strain energy $W_f + W_d$ is released accordingly. The system is back to its original state, and by the law of conservation of energy, we have

$$W_f + W_d + W_I + W_{II} - W_f - W_d = 0 \qquad (58)$$

or

$$W_I = - W_{II} \qquad (59)$$

Consequently

$$u_e = \frac{-b_x}{F_x}\int_0^a \sigma_{xy}(\bar{x},x,\bar{y},y)\, d\bar{x} \qquad (60)$$

evaluated at $\bar{y}=0$, where σ_{xy} can be obtained from Eq. (32c). Using a similar cycle yields

$$v_e = \frac{-b_x}{F_y}\int_0^a \sigma_{xy}(\bar{x},x,\bar{y},y)\, d\bar{x} \qquad (61)$$

evaluated at $\bar{y}=0$ with σ_{xy} derived from Eqs. (33c). The final results are

$$u_e = \frac{b_x}{4\pi}\left\{\frac{C_{12}-\overline{C}_{12}\cos 2\alpha}{2\overline{C}_{12}\sin 2\alpha}\ln\frac{(x-a)^2+\lambda^2y^2+2(x-a)\lambda y\cos\alpha}{(x-a)^2+\lambda^2y^2-2(x-a)\lambda y\cos\alpha}\right.$$

$$+ \tan^{-1} \frac{2(x-a)\lambda y \sin\alpha}{(x-a)^2 - \lambda^2 y^2}$$

$$- \frac{\sin\alpha}{2\cos\alpha} \ln \frac{(x+a)^2 + \lambda^2 y^2 + 2(x+a)\lambda y \cos\alpha}{(x+a)^2 + \lambda^2 y^2 - 2(x+a)\lambda y \cos\alpha}$$

$$+ \frac{C_{12} - \overline{C}_{12}\cos 2\alpha}{2\overline{C}_{12}\cos^2\alpha} \tan^{-1} \frac{2(x+a)\lambda y \sin\alpha}{(x+a)^2 - \lambda^2 y^2}$$

$$- \frac{C_{12} - \overline{C}_{12}}{2\overline{C}_{12}\sin 2\alpha} \ln \frac{(x+a)^2 + \lambda^2 y^2 + Cax + 2(x-a)\lambda y \cos\alpha}{(x+a)^2 + \lambda^2 y^2 + Cax - 2(x-a)\lambda y \cos\alpha}$$

$$- \frac{C_{12} + \overline{C}_{12}}{2\overline{C}_{12}\cos^2\alpha} \tan^{-1} \frac{2(x+a)\lambda y \sin\alpha}{(x+a)^2 + Cax - \lambda^2 y^2} \Bigg\} \qquad (62a)$$

$$v_e = \frac{-b_x \lambda}{4\pi} \Bigg\{ \frac{-(C_{12} - \overline{C}_{12})}{4\overline{C}_{12}\sin\alpha} \ln \left[\langle (x-a)^2 + \lambda^2 y^2 \rangle^2 + C(x-a)^2 \lambda^2 y^2 \right]$$

$$- \frac{C_{12} + \overline{C}_{12}}{2\overline{C}_{12}\cos\alpha} \tan^{-1} \frac{\lambda^2 y^2 \sin 2\alpha}{(x-a)^2 - \lambda^2 y^2 \cos 2\alpha}$$

$$- \frac{(C_{12} + \overline{C}_{12})\sin\alpha}{4\overline{C}_{12}\cos^2\alpha} \ln \left[\langle (x+a)^2 + \lambda^2 y^2 \rangle^2 + C(x+a)^2 \lambda^2 y^2 \right]$$

$$+ \frac{C_{12} - \overline{C}_{12}}{2\overline{C}_{12}\cos\alpha} \tan^{-1} \frac{\lambda^2 y^2 \sin 2\alpha}{(x+a)^2 - \lambda^2 y^2 \cos 2\alpha}$$

$$+ \frac{C_{12}-\overline{C_{12}}\cos 2\alpha}{2\overline{C_{12}}\sin 2\alpha \cos\alpha} \ln\left[\langle\!\langle (x+a)^2+\lambda^2y^2+Cax\rangle^2+C(x-a)^2\lambda^2y^2\right]$$

$$\left. + \frac{1}{\cos\alpha} \tan^{-1}\frac{2ax\sin 2\alpha - a^2\sin 4\alpha + \lambda^2y^2\sin 2\alpha}{x^2-2ax\cos 2\alpha + a^2\cos 4\alpha - \lambda^2y^2\cos 2\alpha} \right\} \tag{62b}$$

which agree with the previous analysis (21).

6. CONCLUSIONS

(1) Eplicit expressions were obtained for the elastic fields of a line force acting in a two-phase orthotropic medium. The stress fields, on approaching isotropy, agree with Frasier and Rongved's results.

(2) The general solutions were simplified to several special cases including the solutions for an interfacial line force and a line force in a half space of orthotropic medium.

(3) A simple application of the results was made to determine the elastic fields of an edge dislocation in an orthotropic half space.

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