Discussion of application of limit plasticity in soil mechanics by W. D. Liam Finn, ASCE Proc. SM2, March 1968, Reprint No. 68-7

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W. F. CHEN\textsuperscript{11}–Limit analysis has been used by the author to obtain upper and lower bounds for the active earth pressure against a vertical standing smooth retaining wall. The analysis assumes Coulomb's yield criterion with constant $c$ and $\phi$. It is shown by the coincidence of upper and lower bounds that the Coulomb solutions for frictionless walls are exact. The author extended the analysis by the upper-bound technique to include the effect of wall friction on the active-earth pressure. However, the application of this analysis requires that proper stress and velocity boundary conditions should be specified so that the meaning of each solution corresponding to a problem will be clear. These were not considered by the author; the writer's purpose is to show that they cannot be neglected. Several erroneous conclusions result when they are neglected.

It is helpful to summarize the behavior of the soil mass as load is increased or decreased to its maximum or minimum value. The behavior of a particular apparatus consisting of a large bin with a movable end section will be examined [see Fig. 16(a)]. By filling the bin with sand, a lateral pressure is developed against the end section which simulates the wall. This wall is constructed so that it can be held in a fixed position or moved inward or outward. A horizontal force $P_n$ must be applied to this wall to keep the apparatus in equilibrium in its initial position. As the force is increased to its maximum value, the soil mass goes through the successive stages of elastic action, contained plastic flow, and finally unrestricted plastic flow. The passive collapse load, $P_{pp}$, is then defined as the maximum load the soil mass can provide against the applied force, $P_n$, when changes in geometry are negligible.

When the applied force is steadily decreased, the soil mass will tend to slide down and force the wall to move to the left. An unrestricted plastic flow state will finally be reached and thus define the active collapse load, $P_{an}$. A typical force versus wall movement curve is shown in Fig. 16(b). The points marked A, B, and C represent the wall forces at rest, at passive collapse, at maximum load, and at active collapse, respectively.

\textsuperscript{a}September, 1967, by W. D. Liam Finn (Proc. Paper 5424).

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and at active collapse respectively. If the limit load theorems are applied to obtain bounds on the collapse loads, the stress (equilibrium) solutions will give force points which fall within the range BC. The velocity solutions will give force points falling outside the range of BC. When the active collapse load is sought, it is helpful to keep in mind that stress solutions give upper bounds (numerically) and velocity solutions give lower bounds.

**Movable Wall**

(a) Vertical Section Through Bin

(B) Collapse Load (Passive)

Unrestricted Plastic Flow

Contained Plastic Flow

Elastic Action

Wall At Rest

WALL MOVEMENT

(b) Load-Displacement Relationship

FIG. 16.—RESULTS OF RETAINING-WALL TESTS

Returning to the point, the active collapse load $P_{an}$, against a rough wall with angle of wall friction $\delta$ (Fig. 17) is computed by the velocity solution technique. Solutions will be obtained corresponding to: (1) different conditions imposed on the wall; (2) the way the force, $P_{an}$, is applied; and (3) the relative magnitude of $\delta$ and $\phi$. Consider the case where the wall is restrained in such a way that it can only move to the left or to the right and the applied force, $P_{an}$, is applied horizontally at a point one-third of the depth of the wall from
the bottom. Figure 17(a) then provides a kinematically admissible velocity field for the solution. In the notation of the paper, in which \( H \) is the wall depth, the sliding surface AC makes an angle \( \beta \) to the wall. The compatible velocity relations in Fig. 17(a) are shown in Fig. 17(b). Two discontinuous velocities are possible on the wall surface depending on relative magnitude of \( \delta \) and \( \phi \). For the case of \( \delta = \phi \), velocity diagram o-a-b [Fig. 17(b)] is expected where the vector \( \overrightarrow{ab} \) is the discontinuous velocity across the wall and the rigid triangle ABC [Fig. 17(a)]. For this case vector \( \overrightarrow{V_1} = \overrightarrow{ob} \). On the other hand, for \( \delta \neq \phi \) the perfect plastic idealization requires that the relative velocity of the wall to the triangle ABC (Vector \( \overrightarrow{ac} \)) must make a constant angle \( \phi \) with the slip-surface. Hence \( \overrightarrow{V_1} = \overrightarrow{ac} \).

The term, \( \delta = \phi \) will be discussed first in the following paragraphs. The rate of dissipation of the energy due to the Coulomb sliding friction between the soil and the wall can be computed by multiplying the discontinuity in velocity across the surface by \( \tan \delta \) times the normal force acting on this surface. The total rate of dissipation of the energy due to friction in this problem is then found to be

\[
(P_{an} \tan \delta)(|\overrightarrow{ab}|)H = P_{an} \tan \delta V_o \sin(90^\circ - \beta - \phi)H
\]

The total rate of dissipation of the energy is obtained by adding the dissipation of the energy due to all the discontinuities in the mass of the soil. Thus, equating the rate of the external work to the rate of the internal dissipation gives

\[
-P_{an} V_o \cos(90^\circ - \beta - \phi) + \frac{1}{2} \gamma H^2 \tan \beta V_o \sin(90^\circ - \beta - \phi)
\]

or

\[
P_{an} = \frac{1}{1 + \tan \delta \tan(90^\circ - \beta - \phi)} \left[ \frac{1}{2} \gamma H^2 \tan \beta \tan(90^\circ - \beta - \phi) - cH \frac{\cos \phi}{\cos \beta \cos(90^\circ - \beta - \phi)} \right]
\]

FIG. 17.—COMPUTATION OF ACTIVE COLLAPSE LOAD BY THE VELOCITY SOLUTION TECHNIQUE
By Theorem 2, $P_{an}$ is a lower bound for the active collapse value. The function has a maximum value when $dP_{an}/d\beta = 0$. For the special case, when $c = 0$, the condition for the best choice of angle $\beta$ is

$$1 + \tan \delta \tan (90 - \beta - \phi) = \frac{\sin 2 \beta}{\sin 2 (\beta + \phi)} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (73)$$

It is difficult to solve for $\beta$ in terms of given values of $\phi$ and $\delta$. Alternatively, solve for $\delta$ in terms of $\beta$ and $\phi$ by assuming any value of $\beta$. Hence,

$$\tan \delta = \frac{\sin \phi \cos (180 - 2\beta - \phi)}{\cos^2 (\beta + \phi)} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (74)$$

Since $\delta \geq 0$, this requires $\beta \geq 45^\circ - \frac{\phi}{2}$. For the case $\delta = 0$, the best choice of $\beta$ is $45^\circ - \frac{\phi}{2}$.

The total active earth pressure, $P_a$, is usually defined in soil mechanics to be the resultant of $P_{an}$ and $P_{an} \tan \delta$. Hence,

$$P_a = \left[P_{an}^2 + (P_{an} \tan \delta)^2\right]^{1/2} = \sec \delta P_{an} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (75)$$

when $\delta = 0$, (smooth wall), Eq. 75 reduces to Eq. 39, agreeing with the value obtained by the author.

When the angle $\beta$ is arbitrarily chosen equal to $45 - \frac{\phi}{2}$, the total active collapse pressure $P_a$ of Eq. 75 can be reduced to the following form

$$P_a = \frac{1}{1 + \tan \delta \tan \left(45 - \frac{\phi}{2}\right)} \left[\frac{1}{2} \sqrt{H} \tan \left(45 - \frac{\phi}{2}\right) - 2 c H \tan \left(45 - \frac{\phi}{2}\right)\right] \quad (76)$$

For the particular case of cohesionless soil, in which $c = 0$, the value of $P_a$ depends solely on the values of the angles $\phi$ and $\beta$. For $\phi = \delta = 30^\circ$, the difference between the total active earth pressure $P_a$ corresponding to Eq. 74 where $\beta = 35.8^\circ$, and the arbitrary choice of $\beta = \frac{\pi}{4} - \frac{\phi}{2}$ is less than 3%. In connection with practical problems, this error is insignificant. With decreasing values of $\delta$, the error decreases further until for $\delta = 0$, the approximate solution is exact. It is interesting to note that the values obtained are identical with the classical Coulomb’s solution which is based on the assumption that the cohesion of the soil is equal to zero.

There is no classical solution available for the case where $c \neq 0$. Eq. 75 or Eq. 76 do provide useful information, although the answers may not be exact.

For the case $\delta \geq \phi$, Coulomb shear, instead of Coulomb sliding, is another possible kinematically admissible choice for a solution. Now the energy dissipated along the wall is the energy given by Eq. 27.

A straightforward calculation gives

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\[-P_{an} \left[ V_o \cos (90 - \beta - \phi) + V_o \sin (90 - \beta - \phi) \tan \phi \right] + \frac{1}{2} \gamma H^2 \tan \beta V_o \sin (90 - \beta - \phi) = c V_o \cos \phi \frac{H}{\cos \beta} + c H \frac{V_o \sin (90 - \beta - \phi)}{\cos \phi} \cos \phi \]............. (77)

or \[P_{an} = \frac{1}{1 + \tan \phi \tan (90 - \beta - \phi)} \left[ \frac{1}{2} \gamma H^2 \tan \beta \tan (90^\circ - \beta - \phi) - c H \tan (90 - \beta - \phi) \right] \] ............. (78)

Bear in mind that when the active collapse load is sought, velocity solutions give lower bounds. Hence, the results of this evaluation are summarized in Table 1 by comparing Eq. 72 with Eq. 78. (Maximizing the positive term, and minimizing the negative terms on the right of these equations.)

It is clear that if a different velocity and stress boundary are imposed on the wall and on the loading, a different pattern of velocity field may govern the solution. Hence, different answers can be expected.

TABLE 1.—EQUATIONS GOVERNING THE ACTIVE COLLAPSE LOAD

| Value of \( \delta \) | \( c = 0, \phi \neq 0 \) | \( c \neq 0, \phi \neq 0 \) |
|-----------------|----------------|-----------------
| \( \delta < \phi \) | Eq. 72 governs | Eq. 72 governs |
| \( \delta = \phi \) | Eq. 72 Eq. 78 | Eq. 72 governs |
| \( \delta > \phi \) | Eq. 78 governs | Eq. 72 or Eq. 78 governs depending on the magnitude of \( c \) and \( \phi \) |

Strictly speaking, the limit load theorems are not applicable in general to any process in which energy is dissipated by friction.\(^\text{13}\) Nevertheless, for a soil, there is a strong temptation to ignore all these considerations and to compute the upper bounds as discussed above. The results do provide useful information, if not the full answer.

The author’s purpose, as stated, is to present the theory of limiting plasticity to soil mechanicians, hence, many known problems\(^\text{2,3,4}\) have been selected for illustrative purposes. It is rather surprising to note that the vital assumptions the theory is based upon are not discussed thoroughly. For instance, the theory has rested largely on the assumption that the plastic strain increment vector is normal to the yield surface. If normality holds, then the strain increment vector must indicate a volume increase. Numerous experiments give evidence that this predicted dilation is much larger than that found in practice,\(^\text{14,15}\) even though other predictions based on this idealization


are remarkably good. Considerable care, therefore, is required in attempting to correlate the theoretical results with the experimental data.

A. C. Palmer has tentatively concluded that the Coulomb yield criterion represents a lower yield condition for real soils. More recently, J. L. Dais formulated an isotropic frictional theory as contrasted with a plastic idealization for a granular medium. His predictions on the shape and the extent of the deformed region for a wedge indentation problem agree with experiments.

Finally, the writer wishes to note that the author has discussed two mechanisms for the upper bound computations of ultimate bearing capacity in soil. One, by L. Prandtl (Fig. 14), contains a rigid region which acts as an extension of the punch; there is no relative motion between the footing and the contacted soil. The other, by R. Hill (Fig. 15), assumes zero friction, and appreciable slip does take place.

Both solutions give the same answer as Eq. 66. Shield has shown that by extension of the classical Prandtl’s plastic-stress field into the remaining rigid regions, Eq. 66 is also a lower bound for each φ less than 75° without violating the Coulomb’s yield criterion. Therefore, the limit pressure, P, of Eq. 66 is exact for all possible finite sliding friction between the footing and the soil mass, according to Drucker’s frictional theorems.

The writer would like to point out that the determination of an admissible-stress field for a lower-bound solution is not merely a matter of guess. Excellent techniques have been developed using the physical intuition developed by engineers for obtaining lower bounds on plastic-limit loads. The general strip-foundation problem has been solved. Further research is now under way at Lehigh University.

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20 Shield, R. T., "The Bearing Capacity of a Footing on Soil (plane Strain)," Report No. 93, Brown University, Providence, R.I., Aug., 1953.
