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THE STRENGTH AND BEHAVIOR
OF LATERALLY UNSUPPORTED COLUMNS

by
Lee Chong Lim

FRITZ ENGINEERING
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A Dissertation
Presented to the Graduate Faculty
of Lehigh University
in Candidacy for the Degree of
Doctor of Philosophy
in
Civil Engineering

Lehigh University

1970
Approved and recommended for acceptance as a dissertation in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

4/21/70

Date

Le-Wu Lu
Professor in Charge

Accepted: 5/5/70

Special Committee directing the doctoral work of Mr. Lee Chong Lim

Professor Ti Huang, Chairman

Professor Le-Wu Lu

Professor George C. Driscoll, Jr

Professor Robert T. Folk

Professor David A. VanHorn, Ex-Officio
ACKNOWLEDGMENTS

The author gratefully acknowledges the encouragement and advice that he received from Dr. Le-Wu Lu, the supervisor of this dissertation. He wishes to note his appreciation of the guidance he has received from the special committee directing his doctoral work. The committee comprises Professors T. Huang, L. W. Lu, G. C. Driscoll, Jr., R. T. Folk, and D. A. VanHorn.

The work described in this dissertation was produced as part of Project 329 "Design of Laterally Unsupported Columns" at Fritz Engineering Laboratory, Lehigh University. The author acknowledges the various sponsors of this project for the opportunities that they provided. These sponsors are: American Iron and Steel Institute, American Institute of Steel Construction, Naval Ship Engineering Center, Naval Facilities Engineering Command, and the Welding Research Council.

The research work of Project 329 is under the technical guidance of Column Research Council Task Group 10 of which Dr. Theodore V. Galambos (Washington University, St. Louis) is Chairman. Other members on CRC Task Group 10 committee are: Messrs. John A. Gilligan (U.S. Steel), William A. Milek, Jr. (AISC), Morris Ojalvo (Ohio State University), George C. Lee (State University of New York in Buffalo) and Le-Wu Lu. To the members of this guiding committee, the author wishes to express his gratitude.
The author would also like to thank Dr. Koichiro Yoshida (now Assistant Professor at Tokyo University) who initiated the project work of "Design of Laterally Unsupported Columns", and Mr. Sakda Santathadaporn for the use of his biaxial bending computer programs to obtain the solutions presented in Chapter 6 of this dissertation.

Professor David A. VanHorn is Chairman of the Civil Engineering Department and Professor Lynn S. Beedle is Director of Fritz Engineering Laboratory.

The dissertation was typed with great care by Miss Karen Philbin, and the drawings were prepared by Mr. John Gera and Mrs. Sharon Balogh.
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ABSTRACT

This dissertation describes a study of the behavior of laterally unsupported as-rolled steel W beam-columns under constant axial load and monotonically increasing strong-axis end moments. Because the beam-columns are laterally unsupported, they are prone to lateral-torsional buckling before their in-plane moment capacities are reached.

There are three parts in this dissertation. The first part presents the theoretical solutions for the occurrence of inelastic lateral-torsional buckling in as-rolled W beam-columns with restraints being provided at the column ends to resist weak-axis bending and warping of the section. The solutions consider both the tangent modulus and the reduced modulus concepts of unloading of the yielded portions in the column sections. The different degrees of restraints are studied and their beneficial effects are compared with the solutions for pinned-end beam-columns, free to warp and bend about the weak-axis. The theoretical solutions are then compared with the available test results and the CRC interaction formula for laterally unsupported beam-columns.

The second part of the dissertation extends the method previously developed for predicting the occurrence of inelastic lateral-torsional buckling in single-span beam-columns to continuous beam-columns with warping and weak-axis bending restraints at the joints. Only the tangent modulus solutions are considered in this case. Sample numerical results are obtained for three-span continuous beam columns.
They are used to check the accuracy of the method currently recommended by the AISI and AISC for predicting the buckling strength of an assembly of beams and beam-columns.

The last part of the dissertation presents a method for predicting the strength and behavior of laterally unsupported beam-columns with or without end restraints. The maximum strength capacities of unbraced beam-columns are compared with their tangent modulus and their reduced modulus buckling strength solutions, and then with the CRC interaction formula. The post-buckling rotation capacities of unbraced beam-columns are also examined.
1. INTRODUCTION

1.1 Introduction

In planar multi-story frames, columns are commonly erected with their strong axis perpendicular to the plane of the frame. For, these columns have been designed to carry only the axial loads and bending moments about their strong axes. The result of this orientation of the columns is an efficient use of the material especially when the columns are made of wide-flange or H sections. Methods are currently available for predicting the strength and behavior of columns subjected to axial load and strong axis bending moments.\(^{(1.1,1.2)}\) The methods assume that out-of-plane deformations can be effectively prevented by lateral bracing so that deformations are only confined in the plane of the applied moments. Design procedures for this type of building columns have been formulated.\(^{(1.3,1.4)}\) A number of high-rise buildings have been built whose columns were designed according to these procedures.\(^{(1.5)}\)

The assumption that out-of-plane deformations can be prevented by lateral bracing is justifiable for exterior columns which are usually held in position by the exterior walls. Most interior columns are, however, often left unsupported for architectural reasons. Such columns must be designed as unsupported columns.

When an unsupported (or unbraced) column is subjected to combined axial load and strong axis bending moments, its behavior is likely to be affected by lateral-torsional buckling—a type of
instability involving twisting and out-of-plane deformation of the column. Extensive theoretical studies on the initiation of lateral-torsional buckling in pinned-end beam-columns have been made.\(^{(1.6,1.7,1.8,1.9)}\)

Tests on beam-columns which were laterally supported only at their ends have been performed in the United States and abroad.\(^{(1.10,1.11)}\) However, there is no way the test results could be used to check the theoretical work as the test columns were always restrained to a certain degree at the ends. These restraints were the direct consequence of the end fixtures being used in order to effectively transfer the axial load from the testing machine to the column. Practically, it is extremely difficult to test beam-columns that are free to warp and at the same time free to bend about the weak-axis at the column ends. Nevertheless, an empirical formula has been proposed first by Hill and Clark\(^{(1.11)}\) and later on confirmed to some extent by the test results reported in Ref. 1.12. This formula relates the axial load on the column to the strong axis bending moment at which lateral-torsional buckling would occur. It has been adopted as a possible guide for design by the Column Research Council, ASCE in its manual on plastic design,\(^{(1.14)}\) the AISC Specification\(^{(1.15)}\) and Lehigh University.\(^{(1.3)}\) The limitation on strength prescribed in this formula means that the full in-plane strength of a beam-column may not be realized, and that no rotational deformation is permitted for the column beyond the rotation that corresponds to the initiation of lateral-torsional buckling. The imposed limitations are essential because there are no solutions available to predict the behavior of unbraced beam-columns after lateral-torsional buckling.

In a typical building frame, it is conceivable that reliance can be made on the floor beams and slabs to provide some restraints
to the columns at the joints, and thus delay the occurrence of lateral-torsional buckling. These restraints though small they may be, and the possible post-buckling strength that is inherent in many structural members may well contribute a significant "added" strength that is being neglected at the present time in the design of unbraced columns.

The plastic method of design of steel columns in a multi-story frame involves the arbitrary partitioning of a whole frame into a number of "subassemblages" as shown in Fig. 1.1. (1.3) A subassemblage is a structural system made up of a number of beams and columns. The "interior" subassemblage shown in Fig. 1.1 consists of a continuous three-span beam-column with two beams framing into the upper joint (Joint A) and the lower joint (Joint B). The system is under checker-board loading and failure of the beams is assumed to be caused by the formation of beam mechanism. At every joint, the moment transmitted from the severely loaded beam will be resisted by the upper column, the lower column, and the adjacent beam. The total resisting capacity of these three members can be obtained by graphically summing the moment-rotation relationships of every member as illustrated in Fig. 1.2 for the upper joint. If the columns were unbraced between joints, lateral-torsional buckling could occur before the attainment of the in-plane strength. There is no method available for calculating the inelastic lateral-torsional buckling strength of a continuous beam-column. Solutions for this problem are important if the strength of the whole structural subassemblage rather than an individual beam-column is sought.
1.2 Objective and Scope

The object of the investigation is to develop analytical methods for predicting the strength and behavior of laterally unsupported beam-columns.

The scope of this study is as follows:

1. Theoretical studies on the initiation of inelastic lateral-torsional buckling in pinned-end and restrained single-span beam-columns.

2. Theoretical studies on the initiation of inelastic lateral-torsional buckling in continuous beam-columns with or without joint restraints.


The lateral-torsional buckling behavior of an inelastic beam-column will first be discussed. It will be found from the discussion in Chapter 2 that the critical moment corresponding to the inception of lateral-torsional buckling is very much influenced by the magnitude of the virtual disturbing force. A small virtual disturbance will give rise to a lower bound solution, commonly known as the tangent modulus solution. Conversely a large virtual disturbance will cause the yielded sections to unload elastically, thus giving rise to an upper bound solution commonly known as the reduced modulus solutions.

The previous studies on the lateral-torsional buckling in beam-columns will be reviewed and their shortcomings will be stated in Chapter 2.
Three chapters are devoted to a study on inelastic lateral-torsional buckling in single-span beam-columns. Basic buckling and equilibrium equations will be formulated in Chapter 2. In Chapter 3 will be presented the buckling solutions for pinned-end or restrained beam-columns based on the tangent modulus concept. It will be shown that the theoretical solutions correlate exceptionally well with the available test results. The reduced modulus (upper bound) solutions will be given in Chapter 4. For the first time, the new solutions will provide an indication to what might be the maximum strength capacity of unbraced beam-columns after lateral-torsional buckling has occurred.

The stability of a continuous beam-column with joint restraints will be examined in Chapter 5. An analytical method will be developed to predict the inception of lateral-torsional buckling in such a continuous system under a variety of loading conditions. As a final contribution in this dissertation, a method will be developed in Chapter 6 to predict the strength and behavior of unbraced beam-columns after the occurrence of lateral-torsional buckling. It will be shown that certain commercially rolled W beam-columns have substantial post-buckling strength and rotation capacity which are being neglected in the current design practice.
2. FORMULATION OF THE GOVERNING DIFFERENTIAL EQUATIONS FOR INELASTIC LATERAL-TORSIONAL BUCKLING IN BEAM-COLUMNS

2.1 Introduction

The intent of this chapter is to formulate a set of governing differential equations for lateral-torsional buckling in beam-columns under general loading conditions, applicable to either pinned-end, restrained or continuous beam-columns. Although the problem of lateral-torsional buckling is not new, the differential equations derived by previous investigators are either incomplete or pertain to specific loading and boundary conditions. (1.6-1.9, 2.1-2.4)

A beam-column unbraced between the column ends may experience other forms of instability in addition to lateral-torsional buckling. One of these is local buckling of the flange in compression. The research work (2.5, 2.6, 2.7) on this problem has culminated in a design recommendation currently adopted for plastically designed steel structures. (1.13, 1.15) In essence, flange local buckling can be inhibited until the strain in the material in compression reaches strain-hardening strain by proper proportioning of the width-to-thickness ratio of each of the plate elements. Most of the commercially rolled wide-flange shapes of A36 steel have a width-to-thickness ratio smaller than the recommended value. (2.5)

In the absence of bending moments, a column under constant axial load may buckle about the strong or weak axis, depending largely on the slenderness ratio and the boundary conditions. This type of instability failure may occur when the column section is still elastic.
if the length of the column is sufficiently long. The investigation re-
ported herein is concerned mainly with the instability mode of lateral-
torsional buckling. However, it will be shown later that lateral-
torsional buckling will give way to either lateral or in-plane buckling
when the appropriate situation arises.

2.2 Buckling Behavior of an Inelastic Member

It is first necessary to review the fundamentals of the
buckling phenomenon. When a member subjected to an axial load
and strong-axis bending moments applied at the supports is given a
virtual lateral disturbance, it will either return towards, or depart
further from its configuration just prior to the application of the
disturbance. Under a constant axial load and monotonically increasing
bending moments, the beam-column will return towards its original
configuration for an initial range of loads. The first applied moment
at which the member fails to return to its original configuration is
termed the buckling moment, also commonly known as the critical moment
$M_{cr}$. Depending largely on the slenderness ratio, part of the beam-
column may be inelastic when buckling occurs. As the properties of
the yielded material are history dependent, the magnitude of the critical
moment will depend on the manner in which the disturbance is applied,
as illustrated in Ref. 2.8 for the case of an axially loaded column.

When the disturbance is applied during a flexural deformation
increment, the beam-column may buckle with no unloading across the
section. This is true if the disturbance is small with respect to
the flexural increment. In this case, the buckling moment is the
tangent modulus moment. It is the lowest bound of the buckling
moment for a perfect beam-column. On the other hand, if there is
no increase in flexural deformation during the disturbance, the buckling moment is known as the reduced modulus moment. The unloading of the yielded fibres contributes to the stiffening of the section. The reduced modulus moment thus provides the highest bound. It has been shown in Ref. 2.8 for an axially loaded column that a continuous series of buckling forces are possible between the highest and the lowest bounds, depending on the relative properties of axial and lateral strains. There is no reason not to believe that the same argument holds true for a beam-column.

Tests and recent analytical studies have shown that for an axially loaded column, lateral buckling would occur when the column is loaded to its tangent modulus load.\(^{(2.9,2.10)}\) The maximum load-carrying capacity of the column, however, lies somewhere between the tangent modulus and the reduced modulus loads. The reduced modulus load is therefore not a realistic buckling load, but it provides the only means of establishing the upper bound to the maximum load carrying capacity of the column. In an unbraced beam-column, it is believed that the same phenomenon exists in that the beam-column will buckle at the tangent modulus moment (lowest bound), and that its maximum moment-carrying capacity is somewhere between the tangent and the reduced modulus moment.\(^{(2.4)}\)

2.3 Previous Research

A comprehensive survey on the research on elastic lateral-torsional buckling in beams and beam-columns up to 1960 has been given in Ref. 2.11. Several papers on elastic buckling problems have been published since then, but the work was more complex in nature as solutions were attempted for restrained beams, continuous beams, rigid frames, and tapered members.\(^{(2.12-2.20)}\)
Analytical solutions for inelastic buckling problems are extremely difficult to obtain because of the non-linearity of the differential equations. A possible solution is by numerical methods, but for this a digital computer of large capacity is required to do the numerous calculations. Large capacity computers were not readily available until very recently.

Neal (2.21) and then Wittrick (2.22) provided solutions for rectangular beams loaded into the inelastic range. Horne (2.23) expanded Neal’s work to solve the inelastic buckling strength of a steel wide-flange beam. About a decade later, Galambos extended Horne’s work to study the buckling of wide-flange beam-columns with due consideration given to the presence of residual stresses in the members. (1.6) Unlike beam problems, the bending and torsional stiffnesses of a beam-column vary along its length because of the "P-Δ" effect. Galambos' method can only solve beam-columns bent into single curvature by two equal but opposite end moments. The columns ends are assumed pinned which implies that the end sections are free to warp and to bend about the weak-axis. For simplicity, Galambos unconservatively assumed in his solutions a beam-column of uniform stiffnesses, and that the magnitudes of these stiffnesses were similar to those at the column ends. An upper bound solution was thus achieved. Technically it was possible to obtain a lower bound solution by computing the moment and thus the stiffnesses at the midspan of the beam-columns. But for this, one would have to resort to a numerical integration method such as Newmark's method which proves laborious and time-consuming. The idea of obtaining lower bound solutions was not realized until information on midspan moment of a beam-column became available through
the application of the concept of column deflection curves. \(^{(1.1)}\) Upper bound solutions were then obtained. Typical lower and upper bound solutions can be found in Refs. 1.8 and 1.10.

Recently in England, Horne derived theoretical solutions for an uniform beam-column with both ends completely restrained against warping but simply-supported about the strong and the weak axes. \(^{(2.24)}\) But his theory was based on the equivalent moment method which assumes that the behavior of a beam-column under an axial load \(P\) and two different end moments is the same as a beam-column subjected to an axial load \(P\) and an uniform strong-axis bending moment \(M/F\), where \(F\) is a function of the end-moment ratio. Fox improved Horne's theory by considering the actual loading conditions, that is, axial load \(P\) and two end moments. \(^{(2.25)}\) However, his solutions were based on oversimplifying assumptions that the weak-axis bending rigidity varied linearly with distance along the column in the yielded portion. Furthermore, the \(P\Delta\) effect, the warping rigidity and the Wagner effect were also neglected. The presence of residual stresses in the members was not considered in both studies.

More accurate solutions for lateral-torsional buckling in beam-columns have been development recently in the United States. \(^{(1.7,1.9)}\) In Ref. 1.7, the finite difference technique was used to determine the critical moment for the initiation of lateral-torsional buckling in as-rolled wide-flange beam-column bent in single curvature by either two equal but opposite end moments or one end moment. Only cooling residual stresses were considered presented in the members. The column ends were assumed pinned. An analytical approach
was used in Ref. 1.9 to detect the stage at which the inception of lateral-torsional buckling in annealed W beam-columns was imminent. All previous investigations except that described in Ref. 1.9 assumed that pre-buckling in-plane deformations were small and could thus be neglected in the derivation of the governing differential equations. This assumption proves to be unconservative for the annealed beam-columns reported in Ref. 1.9.

No "exact" solutions on inelastic buckling are yet available for the case in which end restraints exist at the ends of an individual beam-column or at the joints of a continuous beam-column. Furthermore no attempts have been made to develop a general method that can be used to determine the buckling strength of beam-columns, be they pinned-end or restrained and under any combination of end moments. As stated earlier, previous U.S. investigations were primarily concerned with pinned-end beam-columns subjected to an axial load and either two equal but opposite end moments or one end moment. These solutions are incomplete as one case neglects pre-buckling deformations (1.7) and the other case does not consider the effect of residual stresses. (1.9) Besides, the solutions presented were based on the tangent modulus concept which, as discussed in Sect. 2.2, does not take into account the stiffening effect of the unloading of the yielded portions of the section. A solution based on the reduced modulus concept would establish an upper bound to the maximum strength capacity of the column. The maximum strength capacity and the behavior of beam-columns after the initiation of lateral-torsional buckling are still unknown.
2.4 Formulation of Basic Equations of Lateral Torsional Buckling

2.4.1 Assumptions

The following assumptions are made in the derivation of the basic equations of lateral-torsional buckling:

1. No transverse loads are applied between the two ends of the column.

2. Only end moments which cause bending about the major axis are applied at the supports.

3. The column is an as-rolled wide-flange shape initially free of crookedness.

4. The axial load acts along the original centroidal axis of the column and retains this direction after buckling.

5. The cross-section is uniform over the column length and retains its original shape during the buckling process.

6. The displacements are small in comparison to the cross-sectional dimensions of the column.

7. The stress-strain relationship for the steel is ideally elastic-plastic as shown in Fig. 2.1.

8. Only cooling residual stresses are present in the member.

Figure 2.2 shows the assumed pattern of residual stresses in a typical W section used in this investigation. The maximum compressive stress $\sigma_{rc}$ in the tip of each flange is taken as $0.3 \sigma_y$. 
where $\sigma_y$ is the static yield stress of steel. The tensile residual stress $\sigma_{rt}$ in the web is:

$$\sigma_{rt} = \sigma_{rc} \left[ \frac{bt}{bt + w(d-2t)} \right]$$

(2.1)

where

- $b = \text{flange width}$,
- $t = \text{flange thickness}$,
- $d = \text{depth of section}$, and
- $w = \text{width of web}$.

This residual stress pattern is similar to that used by previous investigators. (1.6, 1.7, 1.8)

### 2.4.2 Derivation of Basic Equations of Lateral-Torsional Buckling

In this section, basic equations of lateral-torsional buckling will be derived for a single-span beam-column. The extension of the equations to the solution of continuous beam-columns will be discussed in Chapter 5.

In Fig. 2.3 is shown a beam-column subjected to axial load, bending moments and twisting moments at both ends. The axial force $P$ and strong axis bending moments $M_{Bx}$ and $M_{Tx}$ are externally applied. The weak-axis bending moments $M_{By}$ and $M_{Ty}$ and the twisting moment $M_z$ are the induced moments due to the restraints at the column supports.

The coordinate system chosen for this study is also shown in Fig. 2.3. The letters $x$, $y$ and $z$ denote the co-ordinates of the undeformed member. When strong-axis bending moments are applied, the beam-column deforms in the plane of the web and continues to do so until lateral-torsional buckling occurs. The position of a typical
cross-section of the beam-column prior to buckling is marked 2 in Fig. 2.4. Position 1 is when the column is not yet loaded. After buckling, the column section deforms to position 3. The letters $\xi$, $\eta$, and $\zeta$ denote the co-ordinate axes of the deformed member. The lateral displacement, vertical displacement and the twist about the shear center of the deformed member are expressed by $u$, $v$ and $\beta$ respectively.

The quantity $v_o$ is the pre-buckling deformation in the plane of bending. The co-ordinates of the shear center are denoted by $x_o$ and $y_o$. The diagrams in Fig. 2.4 are for the case in which the unloading of the yielded portions of the cross-section is neglected. In this case, tangent modulus concept is assumed and the sectional properties are computed based on the elastic portions of the yielded section prior to buckling. In the reduced modulus concept, unloading of the yielded portions is taken into consideration and thus the shear center will be sometimes located outside the web of the cross-section as illustrated in Fig. 2.5. The procedures for determining the location of shear center, the corresponding bending and warping stiffnesses during the different stages of loading will be discussed in Sect. 3.1 and 4.1.

In Fig. 2.6 are shown the positions of the centroid $C$ and the shear center $S$ of a cross-section during the various stages of deformations. Before any force is applied on the column, the section is undeformed and elastic. The centroid and the shear center coincide at the same point. At the inception of lateral-torsional buckling and assuming that part of the section is inelastic, the centroid has moved to $C'$ and the shear center to $S'$. The vertical displacement of the centroid is denoted by $v_o$ and the location of $S'$ with respect to the centroid $C'$ is denoted by $(x_o, y_o)$ as shown. After buckling,
the centroid moves to $C''$ and the shear center moves to $S''$. The displacements due to buckling are $u$, $v$ and $\beta$. It can be found from geometry that the lateral displacement $u_c$ and the transverse displacement $v_c$ of the centroid are:

\[ u_c = u + y_0 \beta \]  \hspace{1cm} (2.2)
\[ v_c = v_0 + v - x_0 \beta \]  \hspace{1cm} (2.3)

In the case where the tangent modulus concept is used, the yield pattern is symmetrical about the $y$ axis and therefore $x_0 = 0$. Equation 2.3 then reduces to

\[ v_c = v_0 + v \]  \hspace{1cm} (2.4)

The relationships for the direction cosines between $x$, $y$, $z$ axes and $\xi$, $\eta$, $\zeta$ axes are:

\[ \begin{array}{ccc}
\xi & 1 & \beta & -u' \\
\eta & -\beta & 1 & -v' \\
\zeta & u' & v' & 1 \\
\end{array} \]  \hspace{1cm} (2.1)

Using the above cosine transformation relationships, it is possible to relate the moments with respect to the $\xi$, $\eta$ and $\zeta$ axes to those with respect to the $x$, $y$ and $z$ axes as follows:

\[ M_\xi = M_x + \beta M_y - u' M_z \]  \hspace{1cm} (2.5)
\[ M_\eta = -\beta M_x + M_y - v' M_z \]  \hspace{1cm} (2.6)
\[ M_\zeta = u' M_x + v' M_y + M_z \]  \hspace{1cm} (2.7)

The prime in the above equations denotes first derivative with respect to $z$. 
The internal resisting moments $M_x$ and $M_y$ at a distance $z$ from the bottom end of the column can be expressed in terms of the axial load, end moments, and the displacements of the centroid by considering separately the equilibrium of the forces in the $z-x$ and $z-y$ planes:

\[
M_x = -M_{bx} + \frac{z}{L} (M_{bx} + M_{tx}) + P \nu_c \quad (2.8)
\]

\[
M_y = -M_{by} + \frac{z}{L} (M_{by} + M_{ty}) - P \mu_c \quad (2.9)
\]

The internal twisting moment $M_z$ is simply

\[
M_z = M_{zo} \quad (2.10)
\]

In addition to $M_{c1}$, the torsional moment $M_c$ has three other components which can be expressed as:

\[
M_{c2} = P y_o u' - P x_o v' \quad (2.11)
\]

\[
M_{c3} = -\beta' \int_A \sigma a^2 \, dA \quad (2.12)
\]

\[
M_{c4} = -\frac{1}{L} (M_{bx} + M_{tx}) \left( u + x_o - x_{o,z=0} \right) - \frac{1}{L} (M_{by} + M_{ty}) \left( v + v_o + y_o - y_{o,z=0} \right) \quad (2.13)
\]

where $a$ is the distance between the shear center and any point in the section. $M_{c2}$ is the torsional moment due to the components of axial thrust $P$ in the $\xi$ and $\eta$ directions about the shear center (Fig. 2.7). $M_{c3}$ is the torsional moment due to the component of the normal stress on the warped cross-section (2.4) (See Fig. 2.8). Finally the torsional moment $M_{c4}$ is the contribution due to the end shears (Fig. 2.9).

The internal resisting moments $M_x$ and $M_y$ given by Eqs. 2.5 and 2.6 can now be expressed in terms of the end forces:
\[ M_z = -M_{Bx} + \frac{Z}{L} (M_{Bx} + M_{Tx}) + P \nu c + \]
\[ \beta \left[-M_{By} + \frac{Z}{L} (M_{By} + M_{Ty}) - P u c \right] - u' M_{zo} \quad (2.14) \]

\[ M_\eta = -\beta \left[-M_{Bx} + \frac{Z}{L} (M_{Bx} + M_{Tx}) + P \nu c \right] - M_{By} + \frac{Z}{L} (M_{By} + M_{Ty}) - P u_c - v' M_{zo} \quad (2.15) \]

The internal twisting moment is
\[ M_\zeta = M_{\zeta 1} + M_{\zeta 2} + M_{\zeta 3} + M_{\zeta 4} \]
\[ = u' \left[-M_{Bx} + \frac{Z}{L} (M_{Bx} + M_{Tx}) + P \nu c \right] + v' \left[-M_{By} + \frac{Z}{L} (M_{By} + M_{Ty}) - P u_c \right] + M_{zo} + P y_o u' - P x_o v' \]
\[ - \beta' \int_a^2 A \int_A -\frac{1}{L} (M_{Bx} + M_{Tx}) (u + x_o - x_o z=0) \]
\[ - \frac{1}{L} (M_{By} + M_{Ty}) (v + v_o + y_o - y_o z=0) \quad (2.16) \]

In terms of stiffness and deformation, the internal moments
\[ M_z, M_\eta \text{ and } M_\zeta \text{ are approximately equal to} \]
\[ M_z \approx -EI_x (v'' + v'_o) \quad (2.17) \]
\[ M_\eta \approx EI_y u'' \quad (2.18) \]
\[ M_\zeta \approx G K \beta' - (EI_w s'')' \quad (2.19) \]

where
\[ E = \text{elastic modulus}, \]
\[ I_x = \text{moment of inertia about x axis}, \]
\[ I_y = \text{moment of inertia about y axis}, \]
\[ G = \text{shearing modulus}, \]
The torsion constant, and

I_w = warping moment of inertia.

The detailed derivation of the total internal twisting moment M_ξ is given in Appendix 1. If the following substitutions are made:

\[ B_x = E I_x = \text{bending rigidity about x axis,} \]
\[ B_y = E I_y = \text{bending rigidity about y axis,} \]
\[ C_T = G K_T = \text{St. Venant constant, and} \]
\[ C_w = E I_w = \text{warping rigidity,} \]

Equations 2.17 - 2.19 can be written simply:

\[ M_\xi = - B_x (v'' + v_0'') \quad (2.21) \]

\[ M_\eta = B_y u'' \quad (2.22) \]

\[ M_\zeta = C_T \beta' - (C_w \beta'')' \quad (2.23) \]

Equating Eq. 2.14 with Eq. 2.21, 2.15 with 2.22, and 2.16 with 2.23, and rearranging, the following equations are obtained:

\[ B_x (v'' + v_0'') - M_{Bx} + \frac{z}{L} (M_{Bx} + M_{Tx}) + P v_c + \beta [- M_{By} + \frac{z}{L} (M_{By} + M_{Ty}) - P u_c] - u' M_{zo} = 0 \quad (2.24) \]

\[ B_y u'' + M_{By} - \frac{z}{L} (M_{By} + M_{Ty}) + P u_c + \beta [- M_{Bx} + \frac{z}{L} (M_{Bx} + M_{Tx}) + P v_c] + v' M_{zo} = 0 \quad (2.25) \]
Substituting the values of $u_c$ and $v_c$ in the above equations with those from Eqs. 2.2 and 2.3, and differentiating with respect to $z$ twice Eqs. 2.24 and 2.25, and once Eq. 2.26, the following expressions are obtained:

\[
\begin{align*}
(C_w \beta'')' - C_T \beta' + u' & \left[ - M_{Bx} + \frac{z}{L} (M_{Bx} + M_{Tx}) + P v_c \right] \\
+ v' & \left[ - M_{By} + \frac{z}{L} (M_{By} + M_{Ty}) - P u_c \right] + M_{zo} + \\
P y_o u' - P x_o v' - \beta' \int_A \sigma a^2 \, dA - \\
\frac{1}{L} (M_{Bx} + M_{Tx}) (u + x_o - x_o) \bigg|_{z=0} - \\
\frac{1}{L} (M_{By} + M_{Ty}) (v + v_o + y_o - y_o) \bigg|_{z=0} = 0 \quad (2.26)
\end{align*}
\]

\[
\begin{align*}
(B_x u'')'' + P (v'' + v_o'' - v o') & + \beta'' [- M_{By} + \\
\frac{z}{L} (M_{By} + M_{Ty})] + 2 \beta' [\frac{1}{L} (M_{By} + M_{Ty})] - u'''' M_{zo} = 0 \\ (2.27)
\end{align*}
\]

\[
\begin{align*}
(B_y u'')'' + P [u + (y_o + v_o)\beta]'' & + \beta'' [- M_{Bx} + \\
\frac{z}{L} (M_{Bx} + M_{Tx})] + 2 \beta' [\frac{1}{L} (M_{Bx} + M_{Tx})] + v'''' M_{zo} = 0 \\ (2.28)
\end{align*}
\]

\[
\begin{align*}
(C_w \beta'')'' & = [(C_T + \int_A \sigma a^2 \, dA) \beta']' + \\
u'' & \left[ - M_{Bx} + \frac{z}{L} (M_{Bx} + M_{Tx}) + P (v_o + y_o) \right] \\
+ u' & \left[ P (v_o + y_o)' + \frac{1}{L} (M_{Bx} + M_{Tx}) \right] + \\
v'' & \left[ - M_{By} + \frac{z}{L} (M_{By} + M_{Ty}) - P x_o' \right] + v' \left[ \frac{1}{L} (M_{By} + M_{Ty}) - P x_o' \right] - \\
\end{align*}
\]
\[
\begin{align*}
(v_0 + v + y_o - y_o z=0)' \cdot \frac{(M_{By} + M_{Ty})}{L} - (u + x_o - x_o z=0)' \cdot \frac{(M_{Bx} + M_{Tx})}{L} &= 0 \tag{2.29}
\end{align*}
\]

The cross terms of \(u, v, \) and \(\beta\) have been neglected in the above equations. The unknown parameters in the above equations are the deformations \(u, v, \) and \(\beta,\) and the induced moments \(M_{By}, M_{Ty}, \) and \(M_{zo}.\) Procedures to determine these induced moments are described in Appendix 2. It will be shown that:

\[
\begin{align*}
M_{By} &\approx - (B_y u'')' z=0 - P (u + y_o \beta)' z=0 + \beta z=0 M_{Bx} \tag{2.30} \\
M_{Ty} &\approx (B_y u'')' z=L + P (u + y_o \beta)' z=L + \beta z=L M_{Tx} \tag{2.31} \\
M_{zo} &\approx - (C_w \beta'')' z=0 + (C_T + \int_A \sigma a^2 dA) z=0 \beta z=0 \\
&\quad - u'(z=0 - M_{Bx} + P y_o) + \frac{1}{L} (M_{Bx} + M_{Tx}) u z=0 \tag{2.32}
\end{align*}
\]

Substituting for the values of \(M_{By}, M_{Ty}, \) and \(M_{zo}\) in Eqs. 2.27, 2.28, and 2.29 with those from Eqs. 2.30, 2.31, and 2.32, and neglecting again the cross terms of \(u, v, \) and \(\beta,\) the following expressions are obtained:

\[
\begin{align*}
[B_x (v'' + v_o '')]' + P (v + v_o - x_o \beta)' &= 0 \tag{2.30} \\
(B_y u'')' + P [u + (y_o + v_o) \beta]' + \beta'' [- M_{Bx} + \\
&\quad \frac{Z}{L} (M_{Bx} + M_{Tx})] + 2 \beta' \left[ \frac{1}{L} (M_{Bx} + M_{Tx}) \right] = 0 \tag{2.31} \\
(C_w \beta'')' - [(C_T + \int_A \sigma a^2 dA) \beta']' + \\
&\quad u'' [- M_{Bx} + \frac{Z}{L} (M_{Bx} + M_{Tx}) + P (v_o + y_o) 
\end{align*}
\]
It has been stated earlier that in the tangent modulus concept of buckling, \( x_o = 0 \) because of the symmetry of the yield pattern in the section about the \( y \) axis. Thus, by eliminating \( x_o \) in the above three equations, the following equations are obtained:

\[
+ u' \left[ P (v_o + y_o)' + \frac{1}{L} (M_{Bx} + M_{Tx}) \right] - \\
\begin{align*}
& P x_o v'' - P x_o' v' - (u + x_o \beta)' \left( \frac{M_{Bx} + M_{Tx}}{L} \right) = 0 \\
& \text{(2.32)}
\end{align*}
\]

Equation 2.33 does not contain the lateral and twisting deformations \( u \) and \( \beta \). It is independent of Eqs. 2.34 and 2.35. It is simply an expression for in-plane strength and is a non-homogeneous equation. Equations 2.34 and 2.35 are, however, complex and homogeneous. They form the set of basic equations for lateral-torsional buckling in beam-columns.

In the reduced modulus concept, the unloading effect of the yielded fibres is taken into consideration in calculating the bending and torsional stiffnesses of the section. It is conceivable that the
resulting 'elastic' portion of the cross-section will not be symmetrical about the x or the y axis. Thus the distance \( x \) may not be zero in some instances when parts of the section have yielded. When this situation exists, the three buckling equations (Eqs. 2.30, 2.31 and 2.32) are coupled. They then represent the expressions for the biaxial bending of the beam-column. Therefore conceptually, there exists a conflicting case in which in one instance (when the beam-column is elastic) the differential equations are for buckling of the beam-column, and in another instance (when the beam-column is inelastic) they are associated with biaxial bending. It will be shown in Chapter 4 that the accuracy of the method used in determining the location of shear center depends on the accuracy of the assumption that a beam-column buckles laterally first before twisting occurs. Because of the uncertainties described in above, and since the reduced modulus solutions that are being sought are still buckling problems, it is henceforth assumed that \( x = 0 \) and that Eqs. 2.34 and 2.35 are valid for both concepts (tangent modulus and reduced modulus) to be used to determine the critical moment for lateral-torsional buckling.

### 2.4.3 Expressions in Finite Differences

The basic equations for lateral-torsional buckling are given by Eqs. 2.34 and 2.35. These two equations are highly nonlinear and cannot be solved directly. Therefore an approximate numerical solution must be considered. A numerical solution based on center finite differences is used in this study.
The length of a beam-column is arbitrarily cut into \( n \) segments each of length \( \delta \). Using the notation shown in Fig. 2.10, the basic equations for lateral-torsional buckling (Eqs. 2.34, 2.35) can be expressed as:

\[
\left( \frac{1}{2} B y_{i-1} + B y_i - \frac{1}{2} B y_{i+1} \right) u_{i-2} +
\]

\[
(- 6 B y_i + 2 B y_{i+1} + P \delta^2) u_{i-1} +
\]

\[
(- 2 B y_{i-1} + 10 B y_i - 2 B y_{i+1} - 2 P \delta^2) u_i +
\]

\[
(2 B y_{i-1} - 6 B y_i + P \delta^2) u_{i+1} +
\]

\[
(- \frac{1}{2} B y_{i-1} + B y_i + \frac{1}{2} B y_{i+1}) u_{i+2} +
\]

\[
\left\{ \delta^2 \left[- M_{Bx} + \frac{z_i}{L} (M_{Bx} + M_{Tx}) + \frac{P}{2} (y_o + v_o)_{i-1} \right] - \frac{\delta^3}{L} (M_{Bx} + M_{Tx}) \right\} \beta_{i-1} +
\]

\[
P (y_o + v_o)_i - \frac{P}{2} (y_o + v_o)_{i+1} - \frac{\delta^3}{L} (M_{Bx} + M_{Tx}) \beta_{i+1} = 0
\]

(2.36)
\[ [2 \delta^2 \left( - \frac{M_{Bx}}{L} + \frac{z_i}{L} (M_{Bx} + M_{Tx}) - P (y_o + v_o) \right) ] u_i + \]

\[ \{ \delta^2 \left( - \frac{M_{Bx}}{L} + \frac{z_i}{L} (M_{Bx} + M_{Tx}) - \frac{P}{4} (y_o + v_o) \right) ] u_{i+1} + \]

\[ \left( \frac{1}{2} c_{w_{i-1}} + c_{w_i} - \frac{1}{2} c_{w_{i+1}} \right) \beta_{i-2} + \]

\[ \{ - 6 c_{w_i} + 2 c_{w_{i+1}} - \frac{\delta^2}{4} (C_T + \int_A \sigma a^2 \, dA)_{i-1} - \]

\[ \delta^2 (C_T + \int_A \sigma a^2 \, dA)_i + \frac{\delta^2}{4} (C_T + \int_A \sigma a^2 \, dA)_{i+1} \} \beta_{i+1} \]

\[ + \{ - 2 c_{w_{i-1}} + 10 c_{w_i} - 2 c_{w_{i+1}} \} \beta_i + \]

\[ 2 \delta^2 (C_T + \int_A \sigma a^2 \, dA)_i \} \beta_i + \]

\[ [2 c_{w_{i-1}} - 6 c_{w_i} + \frac{\delta^2}{4} (C_T + \int_A \sigma a^2 \, dA)_{i-1} - \]

\[ \delta^2 (C_T + \int_A \sigma a^2 \, dA)_i - \frac{\delta^2}{4} (C_T + \int_A \sigma a^2 \, dA)_{i+1} \} \beta_{i+2} = 0 \quad (2.37) \]

The subscript \( i \) refers to a pivotal point in the beam-column. In matrix notation, this set of simultaneous equations may be written as:

\[ [ C ] \begin{bmatrix} u \\ \beta \end{bmatrix} = 0 \quad (2.38) \]
where the matrix \([C]\) is a set of the coefficient \(C_{ij}\) representing combinations of the cross-sectional properties \((B_y, C_T, C_w, P_y, \text{ and } \int_A \sigma a^2 \, \text{d}A)\) and the length of the beam-column under investigation.

It will be shown in the next section that in order to eliminate the displacement terms of the imaginary points, a few coefficients \(C_{ij}\) will contain restraint parameters. It should be realized that the cross-sectional properties are themselves functions of the load parameters. Eigenvalues can be obtained from the condition that the determinant \([C]\) be zero. The eigenvalue that corresponds to the smallest combination of the end moments is considered as the one which causes the lateral-torsional buckling in the beam-column.

2.5 Formulation of Equilibrium Equations at Column Ends

2.5.1 Assumptions

A beam-column in a typical building frame is restrained at the floor level by the floor slabs, beams, and to certain extent, the shear stiffeners in the joint. The types of restraints at the floor joint may be classified into the following groups:

1. Lateral restraints which prevent the column joint from moving laterally.

2. Twisting restraints which prevent the column joint from twisting.

3. Bending restraints which prevent the column from bending about the plane of the web.

4. Warping restraints which prevent the column section from warping at the floor level.
In this investigation, it will be assumed that the lateral and twisting restraints at the column joint are infinitely stiff so that the lateral displacement and the twist are zero:

\[ u_{z=0} = u_{z=L} = 0 \]  \hspace{1cm} (2.39)
\[ \beta_{z=0} = \beta_{z=L} = 0 \]  \hspace{1cm} (2.40)

It is assumed that the bending and warping restraints can be represented by elastic springs as shown in Figs. 2.11 and 2.12. The spring constants for the bending restraints are denoted by \( K_{OB} \) and \( K_{OT} \) (subscripts B and T denote bottom end and top end of the column, respectively). The spring constants for the warping restraints are denoted by \( K_{UB} \), \( K_{UT} \), \( K_{LB} \), and \( K_{LT} \). The first subscripts U and L denote the upper and the lower flange respectively. The second subscripts B and T denote the bottom and the top end of the column, respectively.

2.5.2 Weak-Axis Bending Restraints

In Fig. 2.13 is shown the deformed shape of a beam-column in the x-z plane. The spring at each column end produces a couple to resist the column from buckling into the configuration as shown. This spring moment is, in fact, the induced moment \( M_{By} \) or \( M_{Ty} \) previously derived in Sect. 2.4.2. Thus the equilibrium equations at the columns ends may be written as (from Eqs. 2.30 and 2.31):

\[
(- B_y u'')_{z=0} - P (u + \beta y'_o)_{z=0} + \beta_{z=0} M_{Bx} + K_{OB} u'_{z=0} = 0 \]  \hspace{1cm} (2.41)
\[
(B_y u'')_{z=L} + P (u + \beta y'_o)_{z=L} + \beta_{z=L} M_{Tx} + K_{OT} u'_{z=L} = 0 \]  \hspace{1cm} (2.42)
For no lateral displacement and no twisting deformation at the supports, Eqs. 2.41 and 2.42 reduce to:

\[- (B_y u'')_{z=0} + K_{0B} u'_{z=0} = 0 \]  \hspace{1cm} (2.43)

\[(B_y u'')_{z=L} + K_{0T} u'_{z=L} = 0 \]  \hspace{1cm} (2.44)

Expressing the above equations in finite differences, it can be shown that the lateral displacements of the imaginary points 0 and \(J + 1\) are:

\[u_0 = - \left( \frac{B_y(1) - K_{0B} \frac{\delta}{2}}{B_y(1) + K_{0B} \frac{\delta}{2}} \right) u_2 \]  \hspace{1cm} (2.45)

\[u_{J+1} = - \left( \frac{B_y(J) - K_{0T} \frac{\delta}{2}}{B_y(J) + K_{0T} \frac{\delta}{2}} \right) u_{J-1} \]  \hspace{1cm} (2.46)

2.5.3 Warping Restraints

In Fig. 2.14a is shown a column subjected to a pair of twisting moments at the column ends. Each twisting moment can be represented by a pair of shear forces acting on the top and bottom flanges of the column section as illustrated in Fig. 2.14b. The warping restraints are represented by a pair of moments acting on each column flange.

Using the sign conventions in Fig. 2.15, the displacement of the top flange \(u_{UB}\) and that of the bottom flange \(u_{LB}\) of the column at section \(z = 0\) are:

\[u_{UB} = d_{UB} \beta_{z=0} \]  \hspace{1cm} (2.47)

\[u_{LB} = - d_{LB} \beta_{z=0} \]  \hspace{1cm} (2.48)
where $d_{UB}$ is the distance between the shear center and the upper flange at section $z = 0$, and $d_{LB}$ is the distance from the shear center to the lower flange. At the other end of the column i.e. $z = L$, the displacements of the flanges are:

$$u_{UT} = d_{UT} \beta_{z=L}$$  \hspace{1cm} (2.49)$$
$$u_{LT} = -d_{LT} \beta_{z=L}$$  \hspace{1cm} (2.50)$$

where $d_{UT}$ and $d_{LT}$ are the respective distances from the shear center to the upper and the lower flanges.

In Fig. 2.16 are shown the distorted shapes of the upper and the lower flanges of the beam-column. The moments about the $y$ axis acting on the flanges at both ends are in fact the induced moments due to the restraints provided by the springs, that is, for the upper flange:

$$M_{UB} = K_{UB} u'_{UB} = K_{UB} d_{UB} \beta'_{z=0}$$  \hspace{1cm} (2.51)$$
$$M_{UT} = K_{UT} u'_{UT} = K_{UT} d_{UT} \beta'_{z=L}$$  \hspace{1cm} (2.52)$$

and for the lower flange:

$$M_{LB} = K_{LB} (-u'_{LB}) = K_{LB} d_{LB} \beta'_{z=0}$$  \hspace{1cm} (2.53)$$
$$M_{LT} = K_{LT} (-u'_{LT}) = K_{LT} d_{LT} \beta'_{z=L}$$  \hspace{1cm} (2.54)$$

The bi-moment of a section is the sum of the product of the flange moment and the distance from the shear center to the flange. This has been shown in detail in Appendix 3. Mathematically the expression for bi-moment is:
With proper substitution for the values of the flange moments given
by Eqs. 2.51-2.54, the above two equations become:

\[
M_{BB} = (K_{UB} d_{UB}^2 + K_{LB} d_{LB}^2) \beta'_{z=0}
\]

\[
M_{BT} = - (K_{UT} d_{UT}^2 + K_{LT} d_{LT}^2) \beta'_{z=L}
\]

Using the following notations:

\[
K_{WB} = K_{UB} d_{UB}^2 + K_{LB} d_{LB}^2
\]

\[
K_{WT} = K_{UT} d_{UT}^2 + K_{LT} d_{LT}^2
\]

Equations 2.57 and 2.58 can be written simply:

\[
M_{BB} = K_{WB} \beta'_{z=0}
\]

\[
M_{BT} = - K_{WT} \beta'_{z=L}
\]

From Ref. 2.3, and using the sign conventions adopted in this
report, the bi-moment of a section is simply:

\[
M_B = C_w \beta''
\]

Therefore the equilibrium equations for the bi-moments at \( z = 0 \)
and \( z = L \) are:

\[
(C_w \beta'')_{z=0} - K_{WB} \beta'_{z=0} = 0
\]

\[
(C_w \beta'')_{z=L} + K_{WT} \beta'_{z=L} = 0
\]
In finite difference expressions, the twists of the imaginary point 0 and \((J + 1)\) are:

\[
\beta_0 = -\frac{\left[ C_w(1) - K w_B \frac{\delta}{2} \right]}{\left[ C_w(1) + K w_B \frac{\delta}{2} \right]} \beta_2
\]  

\[
\beta_{J+1} = -\frac{\left[ C_w(J) - K w_T \frac{\delta}{2} \right]}{\left[ C_w(J) + K w_T \frac{\delta}{2} \right]} \beta_{J-1}
\]  

2.6 Procedures for Determining the Buckling Strength of Beam-Columns

In an inelastic beam-column, the extent of plastification in a section varies along the length of the column. Thus the strong axis bending rigidity \(B_x\) is no longer constant and analytical solutions for the in-plane behavior of a beam-column are not possible unless some assumptions are made on the moment-thrust-curvature relationships of the column section. (2.26) Perhaps, an easier way to obtain the in-plane behavior is to use a numerical method similar to the one described in Ref. 1.1. For this, general computer programs have been developed. (1.2)

In order to solve the finite difference equations for lateral-torsional buckling in beam-columns given by Eqs. 2.36 and 2.37, it is imperative that the various sectional properties and the pre-buckling in-plane deformation at every pivotal point of a beam-column are known. The sectional properties are \(B_y\), \(C_T\), \(C_w\), \(\gamma_0\), and \(\int_0^\infty \sigma \cdot a^2 \, dA\) and the pre-buckling deformation is \(v_0\). The computer programs documented in Ref. 1.2 have been modified to include the computations of the sectional properties and the pre-buckling deformation. A complete documentation of the modified program has been made in Ref. 2.27. In Fig. 2.17 is shown a
simplified flow-chart for the modified program. Essentially there are six stages of computations. In stage (1), the user must supply all the information listed in the flow-chart. This information consists of the dimensions of the cross-section b, t, d and w, the mechanical properties of steel $\sigma_y$ and $E$, the magnitude of the residual stress ratio $\frac{\sigma_{rc}}{\sigma_y}$, the axial load ratio $\frac{P}{P_y}$, the end moment ratio $\frac{M_{Tx}}{M_{Bx}}$, the slenderness ratio $\frac{L}{r_x}$, the curvature increment $\Delta \left( \frac{\theta}{\theta_{pc}} \right)$ the increment for the initial slope $\Delta \tau_0$, the initial maximum slope $\tau_{0\text{max}}$ in a column deflection curve (CDC), and lastly the control number JTAN which will instruct the computer to do the subsequent calculations based on either the tangent modulus or the reduced modulus concept. Once the computer has the above information, it will proceed to stage (2) whereby the moment-curvature $M-\theta$ relationships and the corresponding sectional properties $B_y$, $C_T$, $C_w$, $y_o$ and $\int_a^b x^2 dA$ are computed for every $\frac{\theta}{\theta_{pc}}$ value. When all the $\frac{\theta}{\theta_{pc}}$ values have been executed, data of the sectional properties will be printed and punched on cards (stage (3)).

As pointed out in Ref. 1.1, the essential parameter for constructing a CDC is a complete set of the $M-\theta$ relationships. The $M-\theta$ relationships developed in stage (2) will be used to generate CDC's. To perform this, the computer picks a $\tau_0$ value and then generates a quarter wave length of a CDC (Stage (4)). Once this quarter wave has been generated, and with known values of $\frac{L}{r_x}$ and $\frac{M_{Tx}}{M_{Bx}}$, the computer interprets from the CDC, the moment-rotation relationships of the beam-column under investigation. This computation is performed in stage (5). In addition, the computer interprets the pre-buckling deformation $\nu_{o_1}$ and the bending moment $M_{o_1}$ at every pivotal point of the beam-column which has been arbitrarily cut into 20 equal segments. Figure 2.18 is provided to demonstrate how the interpretation from
a CDC of the M-θ relationships, the pre-buckling in-plane deformation \( v_{o_i} \) and the bending moment \( M_i \) are performed. Having done the above, the computer in Stage (6) prints and punches data of M-θ, \( v_{o_i} \), and \( M_i \). The whole cycle of computations in stages (4), (5) and (6) is repeated until the last initial slope value \( \tau^\text{max}_o \) is executed.

The detailed procedures for obtaining the M-P-θ relationships and subsequently a quarter wave length of a CDC have been given in Ref. 1.2. Sample outputs will be given in Sects. 3.1 and 4.1.

The punched data from the previous program is then fed into a second program, a general flow-chart of which is given in Fig. 2.19. This second program computes by iteration the value of the critical moment \( M_{cr} \). A detailed documentation for this program is also made in Ref. 2.27. The reason for breaking the whole computation into two separate parts is the limitation of the field capacity of the CDC6400 computer being used for this investigation.

The various stages of computations in Program 2 are briefly discussed below. In addition to the punched data from Program 1, the following information is required.

1. Slenderness \( \frac{L}{r_x} \), radius of gyration \( r_x \), and number of segment SEG.

2. Initial end rotation of column \( \theta_i \), and the end rotation increment \( \Delta\theta \).

3. Number of cycle of computations to determine \( M_{cr} \).

4. Number of sets of M-P-θ or the sectional properties NEND, number of sets of M-θ relationships NOSET, number of pivotal points PT, and size of coefficient matrix MATR.
(5) AISC identification for the column under investigation  
ISEC, LBS.

(6) End moment ratio MRATIO.

(7) Bending and warping restraints at column ends.

(8) The values $\frac{P}{P_y}$, $P_y$,$\ M_y$,$\ M_{pc}$, and $C_T$.

The controlling parameter in this program is the end rotation. Almost all computations begin with $\theta_i = 0$. The rotation increment $\Delta \theta$ in each case is chosen such that the corresponding moment increment is about 0.1-1.0 percent of the reduced plastic moment $M_{pc}$. Since every beam-column has been assumed to be cut into 20 segments, the size of the coefficient matrix is $38 \times 38$. The number of cycles of computations to reach $M_{cr}$ varies from about 50 to 200, depending largely on the values of $\theta$ and $\Delta \theta$. In reality, it is impossible to obtain zero value for the determinant of the coefficient matrix $[C]$. A procedure has been adopted in the program whereby a critical moment is assumed if there is no appreciable difference (within 0.1%) between two moment values, one of which exhibits a positive determinant and the other a negative determinant. The average of these two moment values is then taken as the critical moment $M_{cr}$.

2.7 Summary

This chapter has examined the phenomenon of lateral-torsional buckling in an inelastic $W$ beam-column. The governing differential equations for lateral-torsional buckling based on either the tangent modulus or the reduced modulus concept of unloading of the yielded
portions of the column sections have been formulated. The concept of representing the warping and weak-axis bending restraints at the column ends by elastic springs has been advanced. Based on the spring models, equilibrium equations at the column ends have been established.

Much of the equations presented herein are new. These equations take into account the variation of flexural and torsional stiffnesses along the length of the member. They form the basis for the solutions to be presented in the next two chapters.
3. TANGENT MODULUS SOLUTIONS FOR INELASTIC LATERAL-TORSIONAL BUCKLING IN BEAM-COLUMNS

In this chapter are presented the results of either pinned-end or restrained beam-columns based on the tangent modulus concept as discussed in Section 2.2.

3.1 Cross-Sectional Properties

It is first necessary to study the various types of yield patterns for an as-rolled W section under constant axial load and subjected to a monotonically increasing strong-axis bending moment. Depending mainly on the magnitude of the axial load ratio $P/P_y$, it has been shown in Ref. 3.1 that five yield patterns are very likely to occur as illustrated in Fig. 3.1:

1. Whole section remains elastic (Fig. 3.1a)
2. Compression flange is partially yielded. The tension flange and the web remain elastic (Fig. 3.1b)
3. The compression flange, the tension zones of the web, and the tension flange are all partially yielded (Fig. 3.1c)
4. The whole compression flange and part of the compression zones of the web are yielded (Fig. 3.1d)
5. The compression and tension flanges are fully yielded. In addition, yielding occurs in the compression zones and in the tension zones of the web (Fig. 3.1e)

3.1.1 Weak-Axis Bending Rigidity, $B_y$

The weak-axis bending rigidity $B_y$ is the stiffness of the unyielded portion of the cross-section. It is equal to the elastic modulus $E$ times the moment of inertia of the unyielded portion about
the y axis. The equations for $B_y$ for the five cases of yielding are:

\[(3.1)\]

<table>
<thead>
<tr>
<th>Yield Pattern</th>
<th>$B_y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fig. 3.1(a)</td>
<td>$E I_y$</td>
</tr>
<tr>
<td>Fig. 3.1(b)</td>
<td>$\left[\frac{1 + (1 - 2\alpha)^3}{2}\right] E I_y$</td>
</tr>
<tr>
<td>Fig. 3.1(c)</td>
<td>$\frac{1}{2} [1 + (1 - 2\alpha)^3 - 8 \psi^3] E I_y$</td>
</tr>
<tr>
<td>Fig. 3.1(d)</td>
<td>$\frac{1}{2} E I_y$</td>
</tr>
<tr>
<td>Fig. 3.1(e)</td>
<td>0</td>
</tr>
</tbody>
</table>

In the above tabulation, $\alpha$ and $\psi$ are the ratios of the yielded length as defined in Fig. 3.1. For a section cut into a large number of small elements as discussed in Section 2.6, the weak-axis bending rigidity may be computed as:

\[B_y = E \sum_i (I_{yi} + A_i y_i^2) \alpha_{oi} \]

where $I_{yi}$ is the moment of inertia about y axis of finite element $i$, $A_i$ the area and $y_i$ the y ordinate of the element. The term $\alpha_{oi}$ equals to unity if the stress $\sigma_i$ in the element is less than $\sigma_y$, otherwise it is zero. The stress $\sigma_i$ is the combined stress due to the axial loading, residual stresses and the strong-axis end moments.

3.1.2 St. Venant Constant $C_T$

Based on some experimental evidence and a theoretical study, Neal has suggested that the St. Venant torsional constant $C_T$ of a
column section is unaffected, to the first order, by yielding under direct bending stresses.\(^{(2.21)}\) Although Massey has produced contrary evidence to the effect that \(C_T\) is significantly reduced by yielding under direct stresses, the limited number of experimental results and the errors in his assumptions could well contribute to the differences that he observed.\(^{(3.2)}\) Recent investigators have followed Neal's assumptions.\(^{(1.7,1.8,1.9,2.25)}\) It is therefore also assumed in this investigation that \(C_T\) is unaffected by yielding:

\[
C_T = G K_T
\]

\[
= \frac{1}{3} G \left[ 2 b t^3 + (d-2t) w^3 \right] \quad (3.2)
\]

### 3.1.3 Warping Rigidity \(C_w\)

The warping rigidity \(C_w\) is the resistance of the section to cross bending. It is equal to the elastic modulus \(E\) times the warping moment of inertia of the elastic core. For the five yield patterns, the equations for \(C_w\) are:\(^{(2.4,3.1)}\)

<table>
<thead>
<tr>
<th>Yield Pattern</th>
<th>(C_w)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fig. 3.1(a)</td>
<td>(E I_w = E \left[ \frac{1}{4} (d-t)^2 I_y \right] )</td>
</tr>
</tbody>
</table>
| Fig. 3.1(b)   | \[
\left[ \frac{2}{1 + \frac{1}{(1-2\alpha)^3}} \right] E I_w
\]
| Fig. 3.1(c)   | \[
\left[ \frac{2 (1-8\psi^3)(1-2\alpha)^3}{1 + (1-2\alpha)^3 - 8\psi^3} \right] E I_w
\]
| Fig. 3.1(d)   | 0       |
| Fig. 3.1(e)   | 0       |
3.1.4 Shear Center Distance, \( y_o \)

As shown in Fig. 2.4, the shear center distance \( y_o \) is measured from the original centroid to the shear center of the elastic core of the yielded section. The equations for \( y_o \) are:

**Yield Pattern**

<table>
<thead>
<tr>
<th>Fig. 3.1(a)</th>
<th>( y_o )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

| Fig. 3.1(b) | \[
\frac{1}{2} - \frac{(1-2\alpha)^3}{1 + (1-2\alpha)^3} \]
| (d-t) |

| Fig. 3.1(c) | \[
\frac{1}{2} - \frac{(1-2\alpha)^3}{1 + (1-2\alpha)^3 - 8\gamma^3} \]
| (d-t) |

| Fig. 3.1(d) | \[
\frac{(d-t)}{2} \]
|  |

| Fig. 3.2(e) | \[
\frac{(\gamma_1 - \gamma_2)}{2} \]
|  |

The yielded portion of the web has been neglected in establishing the relationship for yield pattern (c).

3.1.5 Determination of the Coefficient, \( \int_A \sigma \ a^2 \ dA \)

Since the M-P-\( \theta \) relationships for a column section are obtained in this investigation by a numerical procedure which involves the dissection of the whole section into 420 elements, the coefficient \( \int_A \sigma \ a^2 \ dA \) is best obtained by numerical summation of the contributions of individual elements. In numerical notation

\[
\int_A \sigma \ a^2 \ dA = \sum_{i=1}^{420} \sigma_i \ a_i^2 \ dA_i \quad (3.3)
\]

As explained earlier, the stress \( \sigma_i \) in every finite element is the combined stress due to axial loading, residual stresses, and end bending moments. The distance \( a_i \) is simply:
Analytical expressions for the coefficient \[ \int_{A} \sigma a^2 \, dA \]
have been derived in Ref. 3.1 for the different yield patterns. In all cases the expressions are very lengthy.

3.1.6 Graphical Representation of the Sectional Properties

In Figs. 3.2 - 3.5 are shown for \( P/P_y = 0.2, 0.4, \) and 0.6 the results of the sectional properties of an 8W31 shape as variations of the non-dimensionalized term \( \frac{M}{M_y} \). Also plotted in these figures are the corresponding curves of Fukumoto. \((3.1)\)

The slight differences between the results of the present investigation and those of Fukumoto are due to the following three reasons:

1. The elastic modulus \( E \) used in this investigation is 29,600 ksi, \((1.13)\) whereas \( E \) was taken as 30,000 ksi by Fukumoto.

2. The yield stress \( \sigma_y \) is assumed to be 36 ksi (for A36 steel) as opposed to \( \sigma_y = 33 \) ksi used by Fukumoto (apparently for A7 steel).

3. The computer accuracy and the number of small elements into which a section is arbitrarily cut. It is believed that Fukumoto adopted less number of elements for a section because of the smaller computer he used in his computations.

The value of \( G \) used in this investigation is taken as 11,500 ksi. The sectional properties for other structural shapes have also been obtained, but are not presented here for the reason that the intent of this section is to show typical curves as compared with solutions previously published.
3.2 Lateral-Torsional Buckling Strength of Pinned-end Beam-Columns

The equilibrium equations for the column ends have been derived in Sects. 2.5.2 and 2.5.3. The pertinent equations are:

\[-(B_y \ u'')_{z=0} + K_{OB} \ u'_{z=0} = 0\]  \hspace{1cm} (2.43)

\[(B_y \ u'')_{z=L} + K_{OT} \ u'_{z=L} = 0\]  \hspace{1cm} (2.44)

\[(C_w \ \beta'')_{z=0} - K_{WB} \ \beta'_{z=0} = 0\]  \hspace{1cm} (2.64)

\[(C_w \ \beta'')_{z=L} + K_{WT} \ \beta'_{z=L} = 0\]  \hspace{1cm} (2.65)

In finite differences, the imaginary points or stations are:

\[u_0 = - \left[ \frac{B_y(1) - K_{OB} \delta}{B_y(1) + K_{OB} \delta} \right] u_2\]  \hspace{1cm} (2.45)

\[u_{J+1} = - \left[ \frac{B_y(J) - K_{OT} \delta}{B_y(J) + K_{OT} \delta} \right] u_{J-1}\]  \hspace{1cm} (2.46)

\[\beta_0 = - \left[ \frac{C_w(1) - K_{WB} \delta}{C_w(1) + K_{WB} \delta} \right] \beta_2\]  \hspace{1cm} (2.66)

\[\beta_{J+1} = - \left[ \frac{C_w(J) - K_{WT} \delta}{C_w(J) + K_{WT} \delta} \right] \beta_{J-1}\]  \hspace{1cm} (2.67)

If the restraining springs are removed from both ends of the beam-column, the end sections are then free to warp and bend about the weak-axis. The condition of no restraining springs at the column end is identical to the condition of column ends having restraining springs of zero stiffnesses, that is,

\[K_{OB} = K_{OT} = K_{WB} = K_{WT} = 0\]  \hspace{1cm} (3.5)
The equilibrium equations (Eqs. 2.43, 2.44, 2.64 and 2.65) at the ends become:

\[ u''_{z=0} = u''_{z=L} = \beta''_{z=0} = \beta''_{z=L} = 0 \] (3.6)

The four finite difference equations reduce to:

\[
\begin{align*}
    u_0 &= -u_2 \\
    u_{J+1} &= -u_{J-1} \\
    \beta_0 &= -\beta_2 \\
    \beta_{J+1} &= -\beta_{J-1}
\end{align*}
\] (3.7)

The above equations are identical to those used in Refs. 1.6, 1.7, 1.8 and 1.9 for pinned-end beam-columns. A pinned-end beam-column therefore represents the simplest case of a restrained beam-column.

It was stated in earlier chapters that the available solutions for pinned-end beam-columns have not been accurate because of the simplifying assumptions made by different investigators. The results to be presented in this section are thus aimed at:

1. providing a means to check the computational procedures developed for this investigation.
2. finding the difference in results by comparing the present solutions with the existing ones, and
3. demonstrating the versatility of the general computational procedures presented herein.

3.2.1 Comparisons with Galambos' Solutions

It was mentioned in Sect. 2.3 that Galambos' method of determining the buckling strength of a beam-column was based on the
concept of uniform stiffnesses for the beam-column.\(^{(1.6)}\) Thus an upper bound and a lower bound solution can be established using his method. In Fig. 3.6 are shown Galambos' upper and lower bound curves, and the solution of this investigation. These curves were computed for pinned-end beam-columns of \(8W31\) shape, subjected to a constant axial load of \(P/P_y = 0.4\), and equal but opposite end moments. The original curves of Galambos are for \(\sigma_y = 33\) ksi. Those plotted in Fig. 3.6 have been adjusted to \(\sigma_y = 36\) ksi by using the following correction factor:\(^{(3.1)}\)

\[
\left(\frac{L}{r_y}\right)\sigma_y = 33 = \left(\frac{L}{r_x}\right)\sigma_y \sqrt{\frac{\sigma_y}{33}} \quad (3.8)
\]

As it should be, the 'exact' solution lies somewhere between the upper and lower bound curves. Thus the method of solution reported herein appears valid. It should be pointed out that all three buckling curves in Fig. 3.6 were obtained by neglecting the term \(v_o\) in the differential equations. This is the only means whereby the solution of the current investigation can be effectively checked with Galambos' solutions which also neglect the pre-buckling deformation term \(v_o\).

### 3.2.2 Comparisons with Fukumoto's Solutions

Fukumoto presented solutions for pinned-end beam-columns subjected to an axial load and one end bending moment.\(^{(3.1)}\) The \(M/M_{pc}\) vs. \(L/r_x\) curve for \(8W31\) shape with \(P/P_y = 0.4\) from Ref. 3.1 is replotted in Fig. 3.7. The original curve has been adjusted for \(\sigma_y = 36\) ksi by the use of Eq. 3.8. Also plotted in Fig. 3.7 is the solution of the present investigation. The pre-buckling deformation term \(v_o\) has been neglected in the differential equations for both curves.
There is a large difference in the solutions. Fukumoto's results show that for the range of $L/r_x$ at which lateral-torsional buckling can occur, the critical moment for the initiation of lateral-torsional buckling is higher than that obtained in the current investigation. It is believed that Fukumoto errs in his computation. His solution was obtained by interpolation from a few $M/M_y$ values each of a sizable difference in magnitude from the other. It has been mentioned in Sect. 2.6 that the present investigation uses end rotation as the controlling parameter. The computation begins with zero end rotation, that is, the initial end moment is equal to zero. The rotation increment is chosen such that the increment in the end moment ratio $M/M_{pc}$ is never greater than 0.005.

A small moment increment is essential for accurate results since it has been found from theoretical studies and tests on curved members that various lateral-torsional buckling modes can occur at loads very close to one another.\footnote{3.3-3.5} In order to see if the same observation is true for initially straight members from the analytical standpoint, computer program No. 2 has been modified to output $M/M_{pc}$ values corresponding to the different buckling modes. The results are presented in Table 3.1 for $L/r_x = 30, 50,$ and 80. It can be seen that different buckling modes do, in fact, occur at loads close to each other. Also listed below the table are the corresponding $M/M_{pc}$ values obtained by Fukumoto. Except for $L/r_x = 80$, Fukumoto's solutions are for modes higher than the first mode. As an example, the solution for $L/r_x = 30$ corresponds to mode 9.
3.2.3 Effect of Pre-Buckling Deformation

The effect of neglecting the pre-buckling deformation term \( v_0 \) in the differential equations has been found significant for annealed beam-columns.\(^{(1.9)}\) In order to study what might be the effect for as-rolled beam-columns, two cases are considered; beam-columns subjected to two equal but opposite end moments, and beam-columns subjected to one end moment. Members of 8W31 shape with axial load ratio \( P/P_y \) = 0.4 are considered. The results of this study are presented in Fig. 3.8. In both cases, the effect of neglecting the pre-buckling deformation term \( v_0 \) in the differential equations does not result in any noticeable change in the critical moment ratio \( M/M_p \) for slenderness ratio \( L/r_x \leq 40 \). The slenderness ratio \( L/r_x \) around 40 is common for practical building columns. For larger \( L/r_x \), the effect becomes significant. As an example, for \( L/r_x = 80 \), the discrepancy in neglecting the term \( v_0 \) is about 39 percent on the unconservative side for the beam-column under two equal end moments. Miranda and Ojalvo reported a value of 37.3 percent for a beam-column of identical shape and slenderness ratio, but free of residual stresses.\(^{(1.9)}\) Thus the results of present investigation and those of Miranda and Ojalvo indicate that the effect of pre-buckling deformations can be significant for long \( W \) beam-columns, with or without the presence of residual stresses.

3.2.4 Effect of Residual Stresses

In Fig. 3.9 are shown the buckling solutions for beam-columns either with or without the presence of residual stresses. Both curves are computed for the 8W31 shape with axial load ratio \( P/P_y \) = 0.4. The pre-buckling deformation term \( v_0 \) is included in both analyses.
Also plotted is the in-plane strength curves for as-rolled \( \text{W31} \) beam-columns.

There is a large difference in the solutions. The buckling curve of Miranda and Ojalvo for no residual stresses permits a critical moment some 20 percent higher than that based on the current investigation for beam-columns of \( L/r_x = 20 \). This percentage increases to about 78 for \( L/r_x = 70 \). Between the slenderness ratios of 4.5 and 74, the buckling strength for beam-columns without residual stresses is higher than the in-plane strength for beam-columns with residual stresses. In practice, all commercially rolled \( \text{W} \) shapes have residual stresses although the pattern may not be exactly identical to that used in the present study (see Fig. 2.2). It has been demonstrated that the residual stress pattern shown in Fig. 2.2 is a good approximation of the actual pattern in most of the commercially rolled shapes.\(^{3.6}\) Thus the results in Fig. 3.9 reflect the inaccuracy that might incur in predicting the critical moment if the columns of \( \text{W} \) shape are assumed to have no residual stresses.

Figure 3.10 shows the composite results of the various investigations. All curves are plotted for \( \sigma_y = 36 \text{ ksi} \). It can be seen that the solution of the present study follows closely Galambos' lower bound curve. These two curves would not be close to one another for large \( L/r_x \) values had Galambos' lower bound solution included the pre-buckling deformation term \( v_o \).

### 3.2.5 Effect of Variation in Yield Stress Level

It has been stated in Ref. 3.7 that the residual stress level does not change appreciably with an increase in the yield stress. The maximum compressive stress \( \sigma_{rc} \) in the tip of the flanges for A36
steel has been assumed to be 0.3 \( \sigma_y \). Thus for A441 steel member, with a yield stress level of 50 ksi, the magnitude of \( \sigma_{rc} \) is around 0.2 \( \sigma_y \).

In Fig. 3.11 are shown the in-plane strength curves and the lateral-torsional buckling curves for A36 and A441 members of 8W31 shape with a \( P/P_y \) ratio of 0.4. The lateral-torsional buckling strength for the two types of members is almost identical for slenderness ratio \( L/r_x \) less than 60 even though the in-plane strength for A441 steel is, for the range of \( L/r_x \), less than that for A36 steel. This can be explained by the fact that the reduction in bending and torsional rigidities is less severe for beam-columns with smaller \( \sigma_{rc}/\sigma_y \) ratio. For slenderness ratio \( L/r_x \) greater than 60, lateral-torsional buckling will occur earlier for A441 than A36 steel members. This is because the different residual stress ratios do not appreciably influence the elastic buckling of the beam-columns. Similar types of results are presented in Ref. 1.8 for steel members of \( \sigma_y = 33 \), 50 and 100 ksi. However, the analysis in this case is based on Galambos' upper bound approach.

### 3.2.6 Typical Solutions for Beam-Columns Subjected to Different End Moment Ratios

This section presents the solutions for beam-columns subjected to four different end-moment ratios: -1, -0.5, 0, and 1. The computer programs developed for this study can handle beam-columns subjected to an end-moment ratio between -1 and 1 inclusive. The results presented herein are intended to demonstrate the versatility of the computer programs.
Beam-columns of the $8W31$ shape were chosen for this study. The results are shown in Fig. 3.12. It can be seen that for small $L/r_x$, the curves bunch together. As $L/r_x$ increases, these curves diverge before converging at $L/r_x = 82.5$ which corresponds to the weak-axis lateral buckling length if the beam-column. In no case does the beam-column reach its in-plane strength $M_{\text{max}}$ as demonstrated in Fig. 3.13.

### 3.2.7 Effect of $D_T$ Variation

The parameter $D_T$ is a non-dimensional ratio expressed as follows:

$$D_T = \frac{K_T \times 10^6}{A d^2}$$  \hspace{1cm} (3.9)

It has been stated in Ref. 1.8 that for constant values of $L/r_y$ and $P/P_y$ and the same material, the critical moment in both the elastic and inelastic range is primarily proportional to the parameter $D_T$. No reduction in strength due to inelastic lateral-torsional buckling needs to be considered for beam-columns with $L/r_y \leq 60$ and $D \geq 1500$. This conclusion has been based on Galambos' upper bound approach.

In Fig. 3.14 is shown a non-dimensionalized $M_{cr}/M_{\text{max}}$ vs. $D_T$ plot for beam-columns $L/r_y = 60$ and $P/P_y = 0.4$. All beam-columns considered herein are column sections listed in the AISC Manual. (3.8)

It can be seen that the critical moment is approximately proportional to $D_T$. But the $D_T$ value above which no lateral-torsional buckling can occur is 2400 instead of $D_T = 1500$ given in Ref. 1.8. Members of $14W/43$ shape represent the weakest in torsion among the rectangular shapes listed in
the AISC Manual. (3.8) The reduction in $M_{cr}$ for this case is not much more than that for square columns (8W31) even though the $D_T$ for the 8W31 shape is about twice that for the 14W43 shape.

### 3.2.8 Comparison with CRC Interaction Formula

It has been stated in Sect. 1.1 that an empirical formula originally introduced by Hill and Clark for laterally unbraced column has been adopted by the AISC and the CRC as a possible design guide. This formula, commonly known as the CRC Interaction Formula is:

\[
P = P_0 + \frac{C_m M_{cr}}{M_o} \left[ \frac{1}{1 - \frac{P}{P_e}} \right] \leq 1
\]  

In the above equation $P$ is the applied axial load and $M_{cr}$, the critical moment, is the larger of the two end moments. The quantity $M_o$ is the maximum bending moment which the column can sustain if no axial force is present. It is dependent on the column slenderness ratio $L/r_y$ and its relationship is given in Ref. 1.3. The maximum axial load $P_0$ which the column can support if no bending moment is present can be calculated from

\[
P_o = P_y \left[ 1 - \frac{\sigma_y}{4\pi^2 E} \left( \frac{KL}{r_y} \right)^2 \right]
\]  

where $K$ is the effective length factor.

The quantity $P_e$ in Eq. 3.10 is the elastic buckling load in the plane of bending and is equal to

\[
P_e = \frac{\pi^2 E I_x}{L^2}
\]
The end moment ratio factor $C_m$ is given by

\[ C_m = 0.6 - 0.4q \pm 0.4 \]  \hspace{1cm} (3.13)

where $q$ is the ratio of the smaller end moment to the larger end moment and lies between -1 and +1. As an example, if a beam-column is bent into single curvature by equal but opposite end moments, the value of $q$ is -1.

The comparison of the various interaction curves is made in Fig. 3.15. The use of the CRC interaction formula can err on the unconservative side in predicting the value of $M_{cr}$ for beam-columns of $L/r_x \leq 60$ which includes the range of practical column lengths. Thus if the limit of usefulness of an unbraced beam-column were defined by its buckling strength, the CRC interaction formula would have to be modified to offset the difference as shown in Fig. 3.15.

### 3.3 Lateral-Torsional Buckling Strength of Restrained Beam-Columns

If the bending and warping restraining springs at the column ends are infinitely stiff, the beam-column would behave like a fixed-end beam-column. This will be shown to be true in the manipulation of the equilibrium equations for the column ends given below.

In Eqs. 2.45, 2.46, 2.66, and 2.67, if the spring stiffnesses are infinitely large, that is,

\[ K_{oB} = K_{oT} = K_{wB} = K_{wT} = \infty \]

the equations reduce to

\[ u_0 = u_2 \]

\[ u_{j+1} = u_{j-1} \]
The above equations are simply the conditions for a fixed end beam-column.

For an elastic beam subjected to pure bending, the relationship between the end moment $M$ and the end rotation $\theta$ is:

$$M = \frac{2EI}{L} \theta$$

where $I$ is the moment of inertia. The quantity $\frac{2EI}{L}$ is a stiffness parameter. It may be used as a means to express the degree of restraints at the column ends. Two new notations are thus introduced:

$$\lambda_b = \frac{2EI}{L} = \frac{2B_y}{L} \quad (3.15)$$

$$\lambda_w = \frac{2EI_w}{L} = \frac{2C_w}{L} \quad (3.16)$$

### 3.3.1 Effect of Warping Restraints

For the purpose of studying the influence of warping restraints alone on the lateral-torsional buckling strength of beam-columns, it is arbitrarily assumed that there are no bending restraints at the column ends. In Fig. 3.16 are shown the lateral-torsional buckling strength curves for beam-columns of 8W31 shape with warping restraint parameters $K_{WB}$ and $K_{WT}$ equal to 0, $\lambda_w$, $5\lambda_w$, $10\lambda_w$, and $\infty$ respectively. It can be seen that in general, the buckling strength is improved by increasing restraining stiffness. However, the increase
in buckling strength is small even for the case of infinite stiffness. At $L/r_x = 82.5$, the beam-column buckles laterally about the weak-axis. The presence of warping restraints has no effect on this form of instability.

### 3.3.2 Effect of Weak-Axis Bending Restraints

To study the effect of weak-axis bending restraints, the warping restraints at the column ends have been assumed zero. In Fig. 3.17 are shown the buckling curves for beam-columns of $8W31$ shape with bending restraints $K_{OB}$ and $K_{OT}$ equal to $0$, $5\lambda_b$, $10\lambda_b$, and $\infty$. The axial load ratio $P/P_y$ in this study is taken as $0.4$. Unlike the previous case in which only warping restraints are considered, the improvement in the buckling strength becomes very significant with increasing slenderness ratio $L/r_x$. For $L/r_x \leq 20$, the increase in $M_{cr}$ due to the presence of bending restraints is almost negligible. It can also be seen that for large $L/r_x$, the buckling strength approaches very rapidly with increasing stiffnesses to that of a fully restrained beam-column. This is demonstrated by the closeness of the curve for $K_{OB} = K_{OT} = 5\lambda_b$ and that for $K_{OB} = K_{OT} = \infty$.

The buckling curve for $K_{OB} = K_{OT} = \infty$ terminates at $L/r_x = 142.2$ which corresponds to the critical slenderness ratio for strong axis buckling in a pinned-end beam-column. It should be realized that all beam-columns under investigation are assumed pinned in the plane of bending. The critical slenderness ratio for weak-axis buckling is greater than 142.2 for columns with high bending restraints. As an example, for a column with $K_{OB} = K_{OT} = \infty$, the critical $L/r_x$ for weak-axis buckling is 165.0. Thus in this case,
the buckling strength of the column is governed by strong-axis buckling instead of weak-axis buckling.

3.3.3 Combined Effect of Weak-Axis Bending and Warping Restraints

In Fig. 3.18 are shown typical buckling curves for 8W31 beam-columns with different combinations of restraints. The shapes of these curves are similar to those in Fig. 3.17. It can be stated, in general, that the increase in the \( M_{cr} \) is significant for beam-columns of large \( L/r_x \) values. For very long columns, it may be possible for an unbraced beam-column to reach its in-plane strength if it is fully restrained at both ends.

Figures 3.19, 3.20, 3.21 and 3.22 show the lateral-torsional buckling strength curves for beam-columns of different end moment ratios or of different \( D_T \) values. There are four buckling strength curves and an in-plane strength curve in each figure. The four buckling strength curves represent the following end conditions respectively:

(1) no restraints,
(2) only full warping restraints present,
(3) only full weak-axis bending restraints present, and
(4) full warping and bending restraints present at both column-ends.

In all cases it is seen that weak-axis bending restraints are the major cause for increase in \( M_{cr} \) for long columns. For short columns, warping restraints are the influencing factor.

To study the effect of different \( D_T \) values on the lateral-torsional buckling strength of unbraced restrained beam-columns, the buckling curves in Figs. 3.19 and 3.22 are replotted in Fig. 3.23
as \( \frac{M_{\text{max}}}{M_{\text{max}}} \) vs. \( L/r_x \). The quantity \( M_{\text{max}} \) is the in-plane strength of a beam-column. It can be seen that the beam-column with higher \( D_T \) value exhibits higher \( \frac{M_{\text{max}}}{M_{\text{max}}} \) for a given \( L/r_x \), be it pinned-end or fully restrained.

### 3.3.4 Comparisons with Test Results

It is almost impossible to test beam-columns that are free to warp and bend about the weak-axis. The series of tests performed in Belgium actually had warping fully restrained although the beam-columns were free to bend about the weak-axis. (1.11) The hemispherical seatings at the top and bottom ends of the column also permitted the column to twist at the column ends. Yet previous investigators have erroneously used the test results reported in Ref. 1.11 to compare with their theoretical predictions which are based on pinned-end conditions, that is, column ends are free to warp and bend about the weak-axis, but are not permitted to twist or displace laterally. (1.6,3.1)

In Ref. 1.10 test results were reported for three unbraced beam-columns and one unbraced restrained beam-column. The end conditions for these beam-columns were reported as follows:

"------the beam-column ends were essentially fixed about their weak-axis and pinned about their strong axis. Warping of the end section was fully restrained by end plates------."

The test results are reproduced in Figs. 3.24-3.27. In Ref. 1.10, upper and lower bound solutions based on Galambos' analysis were given and they are also replotted in Figs. 3.24-3.27. In these figures are also shown the in-plane strength curves and the solutions developed in the present study. The present solutions are computed
based on the actual sectional and mechanical properties of the beam-columns. It can be seen that there is good agreement between the reported buckling moments and the present theoretical predictions.

Shown in Fig. 3.28 is the test result of T-31 beam-column reported in Ref. 3.6. This beam-column was subjected to one end-moment. The end fixtures permitted to column ends to bend freely about the strong and weak-axes, but warping was fully restrained. The predicted buckling moment is also shown in the figure. It is slightly below the value from which the experimental curve starts to deviate away from the in-plane moment-rotation curve. It will be shown later in Chapter 6 that a beam-column essentially exhibits in-plane response immediately after lateral-torsional buckling before it starts to unload. Based on this phenomenon, it is concluded here that there is good agreement between the test result and the theoretical prediction on the initiation of inelastic lateral-torsional buckling in "T-31" beam-column.

3.4 Summary

This chapter has presented the tangent modulus solutions for inelastic lateral-torsional buckling in either pinned-end or restrained beam-columns under a variety of loading conditions. These results are new and are compared favorably with existing but inaccurate solutions for pinned-end beam-columns.

It is found that the CRC interaction formula (Eq. 3.10) errs on the unconservative side in predicting the buckling strength for beam-columns of $L/r_x \leq 60$. The presence of warping restraints slightly improves the buckling strength of pinned-end beam-columns.
Although weak-axis bending restraints do not have significant effect on the buckling strength of short columns, they do become significant as column length becomes larger.

The solutions developed in this analysis are found to be in excellent correlation with the test results of restrained W beam-columns.
4. REDUCED MODULUS SOLUTIONS FOR INELASTIC LATERAL-TORSIONAL BUCKLING IN BEAM-COLUMNS

The purpose of this chapter is to present solutions of inelastic lateral-torsional buckling in unbraced beam-columns based on the reduced modulus concept. These solutions are, therefore, the upper bound solutions.

4.1 Cross-Sectional Properties:

4.1.1 Assumption:

It has been stated in Sect. 2.3 that to date, there has not been any research work done on the reduced modulus solutions for lateral-torsional buckling. This is attributed to the fact that the science of lateral-torsional buckling is too complex to understand. A good knowledge of the interaction between the weak axis bending moment $M_y$ and the twisting moment $M_z$, and their combined influence on the unloading of the yielded fibres of the cross-section is essential for the development of a method to calculate the various sectional properties. Since such a knowledge is still lacking, it is assumed herein that $M_y$ is more predominant than $M_z$. This assumption implies that the beam-column buckles laterally first, and is then followed by twisting although the two actions may occur almost simultaneously. This is quite a fair assumption since tests performed on unbraced columns have shown that in almost all cases, columns exhibited lateral deformations long before twists occurred. (1.10, 4.1)
The fundamentals of the buckling phenomenon have been previously discussed in Sect. 2.2. It has been stated that at buckling or bifurcation, a beam-column having been given a slight excitation continues to deform from its original configuration without any increase in the applied load and end moments. Thus in considering the sectional properties by taking into account the unloading effect of the yielded fibres, the following two conditions must be satisfied:

\[
\begin{align*}
\delta P &= 0 \\
\delta M_\xi &= 0
\end{align*}
\] (4.1)

These two imposed conditions mean that there should be no increase in the axial load and the strong axis bending moment when the yielded fibres of the section unload. It has been assumed that lateral buckling occurs first before torsional buckling. Thus the resultant "elastic" cores of the yielded section for the five prevalent yield patterns are as shown on the right-hand side of Fig. 4.1. On the left are the corresponding yield patterns from Fig. 3.1.

4.1.2 Weak-Axis Bending Rigidity

Mathematically it is possible to write analytical expressions for the bending rigidity \( B_y \) of the yielded sections. However, since the computations in this investigation have been computerized, it is easier to find \( B_y \) by the use of Eq. 3.2 written as follows:

\[
B_y = \frac{1}{2} E \left[ \sum_{i} \left( I_{y_i} + A_i y_i^2 \right) \alpha_{i1} + I_Y \right]
\] (3.2)

The summation is taken over 420 elements into which a section is assumed arbitrarily cut. All the terms in the above equation are as defined in Sect. 3.1.1.
Shown in Fig. 4.2 are the \( \frac{M}{M_y} \) vs. \( B \) curves for \( P/P_y = 0.2, 0.4 \) and 0.6 of 8W31 shape. There is no difference between the tangent modulus values and those of reduced modulus when the section is elastic. However, as soon as yielding commences the loss in stiffness \( B \) is smaller if the computations are based on the reduced modulus concept. At complete plastification of the section, the reduced modulus method still gives half the elastic value whereas by the tangent modulus method, the stiffness is zero.

### 4.1.3 Warping Rigidity \( C_w \)

The warping rigidity of the yielded section based on reduced modulus concept is determined by a numerical procedure as presented in Ref. 2.4. It is assumed in the computations that shear force can flow through the yielded zones although these zones do not in any way offer resistance to warping. Computer programs have been written to compute \( C_w \) for any yield pattern. These programs are documented in Ref 2.27.

Figure 4.3 shows the \( \frac{M}{M_y} \) vs. \( \frac{C_w}{C_T} \) curves for \( P/P_y = 0.2, 0.4, \) and 0.6 of 8W31 shape. The reduced modulus value of \( C_w/C_T \) is higher for a given \( \frac{M}{M_y} \) value. At full plastification the section still has some warping resistance if unloading of the yielded portions is taken into consideration.

### 4.1.4 Shear Center Distance \( y_o \)

In Fig. 4.4 are shown typical \( \frac{M}{M_y} \) vs. \( y_o/d \) curves based on both the tangent and reduced modulus concepts. Again 8W31 shape is considered. The shear center distance \( y_o \) based on the reduced modulus concept does not move as far away from the original centroid as the
corresponding tangent modulus value. The actual shear center distance \( x_0 \) is not always zero. In fact, for yield pattern (b), (c), and (d), the shear center distance \( x_0 \) is never zero. However, for the reasons discussed in Sect. 2.4.2, this investigation neglects the \( x_0 \) term in the buckling equations.

### 4.1.5 Determination of the Coefficient \( \int_A \sigma \, a^2 \, dA \)

It has been stated in Sect. 3.1.5 that the coefficient \( \int_A \sigma \, a^2 \, dA \) is best obtained by numerical summation of the contribution of individual elements. Equation 3.3 is used for this purpose, and it is rewritten as follows:

\[
\int_A \sigma \, a^2 \, dA = \sum_{i=1}^{420} \sigma_i \, a_i^2 \, dA_i \quad (3.3)
\]

The term \( a_i \) is now equal to

\[
a_i = \sqrt{(x_i - x_{o_i})^2 + (y_i - y_{o_i})^2} \quad (4.2)
\]

The above equation is different from Eq. 3.4 in that there is an additional term \( x_{o_i} \) in Eq. 4.2. In the tangent modulus concept the elastic core of the yielded section is symmetrical about the y axis, thus the shear center is always located in the y axis. The term \( x_{o_i} \) is equal to zero. In the reduced modulus concept, the values of \( x_{o_i} \) and \( y_{o_i} \) are easily obtained from the general computer program for the determination of \( C_w \).

In Fig. 4.5 are shown the non-dimensionalized curves of \( M/M_y \) vs. \( 1 - \frac{\int_A \sigma \, a^2 \, dA}{C_T} \). For three selected \( P/P_y \) values, the reduced modulus curves never exhibit negative values. This is due
to the fact that the quantity \( \int_A \sigma a^2 \, dA / C_T \) is smaller because of the smaller \( a_1 \) value.

4.2 Lateral-Torsional Buckling Strength of Pinned-End Beam-Columns

To study the lateral-torsional buckling strength of pinned-end beam-columns based on the reduced modulus concept, beam-columns of 8W31 and 14W142 shapes are selected so that they may be readily compared with the available tangent modulus solutions given in Chapter 3. The axial load ratio \( P/P_y \) is taken as 0.4. Five types of beam-columns are considered, the results of which are given in Figs. 4.6-4.10. In all instances, the inelastic lateral-torsional buckling strength of a beam-column based on the reduced modulus concept is higher than that based on the tangent modulus concept. As a matter of fact, the reduced modulus results show the possibility of a beam-column of \( L/r_x \leq 40 \) reaching its full in-plane strength before lateral-torsional buckling can occur. Since the \( P/P_y \) ratio for the case study is 0.4, long beam-columns have the same buckling strength because they are elastic. The sectional properties are identical irrespective of the concept used to compute them.

Shown in Fig. 4.11 are the reduced modulus buckling strength curves for 8W31 beam-columns plotted as variations of \( M/M_{\text{max}} \) with \( L/r_x \). These curves demonstrate a remarkable improvement in buckling strength over the tangent modulus curves as shown in Fig. 3.13.

To study the effect of \( D_T \) on the reduced modulus buckling solutions, the solutions of the 8W31 and 14W142 are compared in Fig. 4.12.
It can be seen that both the reduced and tangent modulus curves for beam-columns of higher $D_T$ ($14W142$ shape) give higher critical moments. For a $8W31$ beam-column with $D_T = 925$, the reduced modulus concept permits the beam-column to be loaded to its in-plane strength if its slenderness ratio $L/r_x$ is less than 43. For a beam-column of $14W142$ shape, the cut-off point at which the lateral-torsional buckling curve intersects the in-plane strength curve moves from $L/r_x = 22$ for the tangent modulus solution to $L/r_x = 57$ for the reduced modulus solution.

4.3 Lateral-Torsional Buckling Strength of Restrained Beam-Columns

4.3.1 Effect of Warping Restraints

The effect of warping restraints on the lateral-torsional buckling strength is exemplified by $8W31$ beam-columns with the axial load ratio $P/P_y = 0.4$. The tangent modulus solutions have previously been obtained in Sect. 3.3.1. Plotted in Fig. 4.13 are typical buckling strength curves and the in-plane strength curve for the type of beam-columns under investigation. Three values of restraining parameters are considered, that is, $K_{WB} = K_{WT} = 0, 5\lambda_w, and \infty$. The bending restraints are assumed zero. The reduced modulus solutions in the inelastic range are consistently higher in value. For the case of full warping restraints, beam-columns of $L/r_x \leq 60$ can reach their in-plane strength before the initiation of lateral-torsional buckling.

4.3.2 Effect of Weak-Axis Bending Restraints

In Fig. 4.14 are shown typical curves demonstrating the effect of weak-axis bending restraints alone on the initiation of inelastic lateral-torsional buckling. The beam-columns considered are of $8W31$ shape with $P/P_y = 0.4$. The tangent modulus curves indicate that weak-
axis bending restraints have very little effect on short beam-columns. This observation is not upheld if the reduced modulus concept is employed to compute the buckling solutions. For the case of full restraints, that is, \( K_{oB} = K_{oT} = \infty \), no lateral-torsional buckling can occur before the beam-column reaches its full in-plane value on the basis of reduced modulus analysis.

### 4.3.3 Effect of Combined Restraints

Examples of the effect of combined warping and weak-axis bending restraints on lateral-torsional buckling strength are given in Fig. 4.15. The 8W3l beam-columns are assumed to be fully restrained against warping at the column ends. Three values of weak-axis bending restraints chosen for this study are \( K_{oB} = K_{oT} = 0 \), \( 2\lambda_b \), and \( \infty \). The improvement in buckling strength is even more remarkable here than for the case in which only bending restraints are present. For relatively small bending restraints, that is, \( K_{oB} = K_{oT} = 2\lambda_b \), it is possible to load a relatively long unbraced beam-column to its in-plane strength.

### 4.4 Summary

This chapter has presented the reduced modulus solutions for inelastic lateral-torsional buckling in either pinned-end or restrained beam-columns under a variety of loading conditions. The reduced modulus solution which is the highest bound solution, gives an indication as to what might be the maximum strength of an unbraced beam-column.
In the tangent modulus analysis presented in the previous chapter, it has been found that for an 8W31 pinned-end beam-column, lateral-torsional buckling will always occur irrespective of the slenderness ratio and the end moment ratio. In the present chapter in which the reduced modulus method is discussed, the same beam-column is predicted to attain its full in-plane strength under the most severe loading condition if its length does not exceed $40 \tau_x$. The maximum strength solution should lie between the tangent and the reduced modulus solutions.

The buckling solutions for restrained beam-columns have also been examined. It is believed that all the solutions reported in this chapter are new.
5. **INELASTIC LATERAL-TORSIONAL BUCKLING IN CONTINUOUS BEAM-COLUMNS**

It has been stated in Sect. 1.1 that the research work previously done on lateral-torsional buckling in continuous beams or beam-columns has been primarily concerned with elastic buckling. (2.13, 2.15, 2.16) This chapter presents solutions for inelastic lateral-torsional buckling in continuous beam-columns, either pinned or restrained with respect to weak-axis bending and warping of the section at the supports.

### 5.1 Assumptions

The following assumptions are made in the derivation of the basic equations of lateral-torsional buckling:

1. No transverse loads are applied between the supports.
2. Only strong-axis bending moments and axial loads are applied at the supports as shown in Fig. 5.1.
3. The column is an as-rolled wide-flange shape initially free of crookedness.
4. The axial loads act along the original centroidal axis of the continuous column and retain this direction after buckling.
5. The cross-section is uniform over the column length and retains its original shape during the buckling process.
6. The displacements are small in comparison to the cross-sectional dimensions of the column.
7. The stress-strain relationship for the steel is as assumed in Chapter 2 for individual beam-column problems, that is, elastic-plastic as shown in Fig. 2.1.

8. The residual stress pattern and the magnitude of the maximum compressive residual stress \( \sigma_{rc} \) are similar to those described in Sect. 2.4.1.

9. The beam-column cannot displace laterally or twist at every support.

5.2 Differential Equations for Lateral-Torsional Buckling

To derive a set of differential equations for lateral-torsional buckling in continuous beam-columns, it is first necessary to consider the equilibrium of a span of a beam-column subjected to end moments and axial load similar to the one shown in Fig. 2.3. The moments \( M_{By} \), \( M_{Ty} \), and \( M_{z0} \) in this instance are the restraining moments at the column ends of every span of the continuous beam-columns. Continuity and equilibrium compatibilities at the joints are then established to connect the individual spans together. The differential equations for lateral-torsional buckling in a single-span beam-column have previously been derived in Chapter 2. Applying these equations to span \( n \) of a continuous beam-column, they have the following relationships:

\[
(B_y u'')^{(n)} + P^{(n)} [u + (y_o + v_o) \beta]^n_{(n)} + \\
\beta''_{(n)} \left[ - \frac{M_{By}}{L} + \frac{z}{L} (M_{By} + M_{Tx}) \right]_{(n)} + \\
2 \beta'_{(n)} \left[ \frac{1}{L} (M_{By} + M_{Tx}) \right]_{(n)} = 0
\]  

\text{(5.1)}
\[(C_w \beta')''(n) - \left[ (C_T + \int_A \sigma a^2 dA) \beta' \right]'(n) \]

\[+ u''(n) \left[ - M_{Bx} + \frac{z}{L} (M_{Bx} + M_{Tx}) \right] + \]

\[P (v_o + y_o)'(n) + \]

\[P(n) (v_o + y_o)'(n) u'(n) = 0 \quad (5.2) \]

The above equations are similar to Eqs. 2.34 and 2.35 except for the subscript \(n\) which denotes span \(n\) of a continuous beam-column. In finite difference expressions, Eqs. 5.1 and 5.2 become:

\[\left[ \frac{1}{2} B y_{i-1} + B y_i - \frac{1}{2} B y_{i+1} \right](n) u_{(i-2)}(n) + \]

\[\left[ - 6 B y_i + 2 B y_i + P \delta^2 \right](n) u_{(i-1)}(n) + \]

\[\left[ - 2 B y_i + 10 B y_i - 2 B y_{i+1} - 2 P \delta^2 \right](n) u_i(n) + \]

\[\left[ 2 B y_{i-1} - 6 B y_i + P \delta^2 \right](n) u_{i+1}(n) + \]

\[\left[ - \frac{1}{2} B y_{i-1} + B y_i + \frac{1}{2} B y_{i+1} \right](n) u_{(i+2)}(n) + \]

\[\left[ \delta^2 \left[ - M_{Bx} + \frac{z_i}{L} (M_{Bx} + M_{Tx}) + \frac{P}{2} (y_o + v_o)_{i-1} \right] + \right. \]

\[P (y_o + v_o)_{i-1} - \frac{P}{2} (y_o + v_o)_{i+1} \right] - \]

\[\frac{\delta^3}{L} (M_{Bx} + M_{Tx}) \right] (n) \beta_{(i-1)}(n) + \]
\[ 2 \delta^2 [M_{bx} - \frac{z_i}{L} (M_{bx} + M_{tx}) - \frac{P}{2} (y_o + v_o)_{i-1} - \\
3 P (y_o + v_o)_i + P (y_o + v_o)_{i+1}] (n) \beta_i(n) + \\
\delta^2 [-M_{bx} + \frac{z_i}{L} (M_{bx} + M_{tx}) - \frac{P}{2} (y_o + v_o)_{i-1} + \\
P (y_o + v_o)_i + \frac{P}{2} (y_o + v_o)_{i+1}] + \\
\frac{\delta^3}{L} (M_{bx} + M_{tx}) (n) \beta_{(i+1)}(n) = 0 \quad (5.3) \\
\delta^2 [-M_{bx} + \frac{z_i}{L} (M_{bx} + M_{tx}) + \frac{P}{4} (y_o + v_o)_{i+1} + \\
P (y_o + v_o)_i - \frac{P}{4} (y_o + v_o)_{i+1}] (n) u_{(i-1)}(n) + \\
\delta^2 [-M_{bx} + \frac{z_i}{L} (M_{bx} + M_{tx}) + \frac{P}{4} (y_o + v_o)_{i+1} + \\
P (y_o + v_o)_i + \frac{P}{4} (y_o + v_o)_{i+1}] (n) u_{(i+1)}(n) + \\
[\frac{1}{2} c_{w_{i-1}} + c_{w_i} - \frac{1}{2} c_{w_{i+1}}] (n) \beta_{(i-2)}(n) + \\
[-6 c_{w_i} + 2 c_{w_{i+1}} - \frac{\delta^2}{4} (C_T + \int_A \sigma a^2 \, dA)_{i-1} - \\
\delta^2 (C_T + \int_A \sigma a^2 \, dA)_i + \frac{\delta^2}{4} (C_T + \\
\int_A \sigma a^2 \, dA)_{i+1}] (n) \beta_{(i-1)}(n) + \\
[-2 c_{w_{i-1}} + 10 c_{w_i} - 2 c_{w_{i+1}} + \\
2 \delta^2 (C_T + \int_A \sigma a^2 \, dA)_i] (n) \beta_i(n) + 
\]
\[
\begin{bmatrix}
2 C_{wi-1} - 6 C_{wi} + \frac{\delta^2}{4} (C_T + \int_A \sigma a^2 dA)_{i-1} - \\
\delta^2 (C_T + \int_A \sigma a^2 dA)_{i} - \frac{\delta^2}{4} (C_T + \\
\int_A \sigma a^2 dA)_{i+1} \end{bmatrix}_{(n)} \beta_{(i+1)(n)} + \\
\left[- \frac{1}{2} C_{wi-1} + C_{wi} + \frac{1}{2} C_{wi+1} \right]_{(n)} \beta_{(i+2)(n)} = 0
\]

(5.4)

The subscript \( i \) refers to a pivotal point in the beam-column.

In matrix notation, Eqs. 5.3 and 5.4 may be written as

\[
\begin{bmatrix}
C
\end{bmatrix}
\begin{bmatrix}
u \\
\beta
\end{bmatrix} = 0
\]

As in the single-span beam-column problem, the matrix \([C]\) is a set of the coefficients \( C_{ij} \) representing the combinations of the cross-sectional properties \( B_y, C_T, C_w, P y_o, \) and \( \int_A \sigma a^2 dA \) and the length of a span of the continuous beam-column. Critical moments are obtained when the determinant \([C]\) is zero.

5.3 Continuity and Equilibrium Equations at Joints

It is first necessary to distinguish the two types of joints in a continuous beam-column. These are the exterior joints and the interior joints. Referring to Fig. 5.1, the exterior joints are those located at the two ends of the continuous beam-column, that is, at supports \( A \) and \( N \). The interior joints are those located between the exterior joints. Every interior joint has two spans framing into it, one above and the other below the joint.
For an exterior joint, two boundary conditions and two equilibrium equations are sufficient to completely define the unknown parameters for the solution of the buckling equations. For an interior joint, in addition to the two boundary conditions and two equilibrium equations, two continuity equations are also required to solve the buckling equations. In this section, equations will first be derived for an interior joint. It will be shown later that the equations so obtained can easily be extended to the exterior joints.

5.3.1 Continuity Conditions at an Interior Joint

It has been assumed in Sect. 5.1 that there are no lateral displacements or twists at the supports. Thus

\[ u_{\text{support}} = \beta_{\text{support}} = 0 \]  

Consider two spans of a continuous beam-column joined at a support. These two spans are shown in Fig. 5.2 in an offset position in order to clarify the designation of the pivotal points adjacent to the joint. The pivotal point at the joint is designated J. The conditions of no lateral displacements and no twists at the support may be expressed in finite differences as follows:

\[ u_J = 0 \quad (5.5) \]
\[ \beta_J = 0 \quad (5.6) \]

For geometrical compatibility, it is conceivable that the following continuity conditions must hold:
\[ u'_z = L(n) = u'_z = 0(n+1) \] (5.7)

\[ \beta'_z = L(n) = \beta'_z = 0(n+1) \] (5.8)

In establishing the above two conditions, it is implicitly assumed that there is no plastic hinge developed at the support. This assumption is justifiable since this investigation is primarily concerned with lateral-torsional buckling before the attainment of in-plane strength. The in-plane strength can only be developed before the formation of a hinge at the support.

### 5.3.2 Equilibrium Equation At An Interior Joint

The assumption that bending and warping restraints at the two ends of a beam-column can be represented by elastic springs is again used in this chapter. It is further assumed for simplicity of analysis that the restraints at an interior support may be represented by two sets of springs, one below the joint and the other above the joint. The total stiffness of the restraining spring at each support is the sum of the stiffnesses of the two sets of springs.

**Weak-Axis Bending Restraints**

The equilibrium equations for the column ends of a single-span beam-column have been derived in Sect. 2.5.2. These equations when applied to a span of a continuous beam-column are:

\[ - (B_z u'')_z = 0(n) + K_{OB}(n) u'_z = 0(n) = 0 \] (5.9)

\[ (B_z u'')_z = L(n) + K_{OT}(n) u'_z = L(n) = 0 \] (5.10)
Consider the top end of span $n$ and the bottom end of span $(n+1)$ as shown in Fig. 5.2. The two appropriate equilibrium equations are

\[
(B_y u'') z=L(n) + K_{oT}(n) \quad u'_z=L(n) = 0
\]

\[
-(B_y u'') z=0(n+1) + K_{oB(n+1)} \quad u'_z=0(n+1) = 0
\]

Adding the above two equations and using the continuity condition of Eq. 5.7, the following equilibrium equation is obtained:

\[
(B_y u'') z=L(n) - (B_y u'') z=0(n+1) + K_{o(J)} \quad u'_z=0(n+1) = 0
\]

(5.11)

where $K_{o(J)} = K_{oT(n)} + K_{oB(n+1)}$ = the stiffness of the spring at joint $J$ for restraining the beam-column against weak-axis buckling.

Equations 5.7 and 5.11 are sufficient to establish the lateral displacement relationships in finite difference expressions for the pivotal points $I(J+1)(n)$ and $I(J-1)(n+1)$. These relationships are:

\[
u_{I(J+1)}(n) = \rho_1 \quad u_{(J+1)}(n+1) + \rho_2 \quad u_{(J-1)}(n)
\]

(5.12)

\[
u_{I(J-1)}(n+1) = \rho_3 \quad u_{(J+1)}(n+1) + \rho_4 \quad u_{(J-1)}(n)
\]

(5.13)

where

\[
\rho_1 = \frac{2 \gamma^2 B_y J(n+1)}{[B_y J(n) + \gamma B_y J(n+1) + K_{o(J)} \frac{\delta(n+1)}{2}]}
\]

(5.14)

\[
\rho_2 = \frac{[\gamma B_y J(n) + \gamma B_y J(n+1) + K_{o(J)} \frac{\delta(n)}{2}]}{[B_y J(n) + \gamma B_y J(n+1) + K_{o(J)} \frac{\delta(n)}{2}]}
\]

(5.15)
\[ \rho_3 = \frac{B_y J(n) - \gamma B_y J(n+1) + K_o(J) \frac{\delta(n)}{2}}{\left[B_y J(n) + \gamma B_y J(n+1) + K_o(J) \frac{\delta(n)}{2}\right]} \]

\[ \rho_4 = \frac{2 B_y J(n)}{\gamma \left[B_y J(n) + \gamma B_y J(n+1) + K_o(J) \frac{\delta(n)}{2}\right]} \]

\[ \gamma = \frac{\delta(n)}{\delta(n+1)} \]

The detailed derivations of Eqs. (5.12) and (5.13) are given in Appendix 4.

Warping Restraints

The equilibrium equations for the column ends of a beam-column restrained against warping by springs have been derived in Sect. 2.5.2. For the top end of span \( n \) and the bottom end of span \( (n+1) \) in a continuous beam-column, Eqs. 2.64 and 2.65 can be appropriately used by proper identification of the span concerned. Thus for span \( n \),

\[ (C_w \beta')_{z=L(n)} + (K_w T \beta')_{z=L(n)} = 0 \]  \hfill (5.19)

For the bottom end of span \( (n+1) \):

\[ -(C_w \beta'')_{z=0(n+1)} + (K_w \beta')_{z=0(n+1)} = 0 \]  \hfill (5.20)

The restraining parameters \( K_{wB} \) and \( K_{wT} \) have been defined by Eqs. 2.59 and 2.60. Combining Eqs. 5.19 and 5.20 to yield the following:

\[ (C_w \beta'')_{z=L(n)} - (C_w \beta'')_{z=0(n+1)} + K_w(J) \beta'_{z=0(n+1)} = 0 \]  \hfill (5.21)
The quantity $K_w(J)$ is equal to

$$K_w(J) = K_wB(n+1) + K_wT(n)$$

$$= [K_{UB} d_{UB}^2 + K_{LB} d_{LB}^2][n+1] +$$

$$[K_{UT} d_{UT}^2 + K_{LT} d_{LT}^2][n]$$

equation (5.22)

It represents the combined warping restraining effect at the joint $J$.

Equations 5.8 and 5.21 can be used to establish the twisting relationships for the imaginary points $I(J+1)(n)$ and $I(J-1)(n+1)$. The method of establishing such relationships is also given in Appendix 4.

In finite difference expressions, the twisting relationships are:

$$\beta_{I(J+1)}(n) = \Omega_1 \beta_{(J+1)}(n+1) + \Omega_2 \beta_{(J-1)}(n)$$

(5.23)

$$\beta_{I(J-1)}(n+1) = \Omega_3 \beta_{(J+1)}(n+1) + \Omega_4 \beta_{(J-1)}(n)$$

(5.24)

where

$$\Omega_1 = \frac{2 \gamma^2 C_w}{[C_wJ(n) + \gamma C_wJ(n+1) + K_w(J) \frac{\delta(n)}{2}]}$$

(5.25)

$$\Omega_2 = \frac{2 \gamma^2 C_w}{[C_wJ(n) + \gamma C_wJ(n+1) + K_w(J) \frac{\delta(n)}{2}]}$$

(5.26)

$$\Omega_3 = \frac{2 \gamma^2 C_w}{[C_wJ(n) - \gamma C_wJ(n+1) + K_w(J) \frac{\delta(n)}{2}]}$$

(5.27)
5.3.3 Extension of the Equilibrium Equations to the Exterior Joints

Consider first the conditions at support A (Fig. 5.1). In this case only Eqs. 5.13 and 5.24 are applicable since the span below the joint does not exist. Substituting for \( J=1 \) at support A, the two displacement functions for the imaginary point immediately below A are

\[
\begin{align*}
\beta_{I(0)} &= -\beta_3 \beta(2) \\
\tilde{\beta}_{I(0)} &= \tilde{\beta}_3 \beta(2)
\end{align*}
\]

where

\[
\begin{align*}
\beta_3 &= -\frac{\gamma B_y(1)(n+1) + K_0(1) \frac{\delta(n)}{2}}{\gamma B_y(1)(n+1) + K_0(1) \frac{\delta(n)}{2}} \\
\tilde{\beta}_3 &= -\frac{\gamma C_w(1)(n+1) + K_w(1) \frac{\delta(n)}{2}}{\gamma C_w(1)(n+1) + K_w(1) \frac{\delta(n)}{2}}
\end{align*}
\]

Multiplying the numerator and the denominator of each of the above equations by \( \frac{1}{\gamma} \), and noting that span \((n+1)\) now refers to the first span, the above two equations can now be written as:

\[
\begin{align*}
\beta_3 &= -\frac{B_y(1)(span 1) + K_0(1) \frac{\delta(span 1)}{2}}{B_y(1)(span 1) + K_0(1) \frac{\delta(span 1)}{2}} \\
\tilde{\beta}_3 &= -\frac{C_w(1)(span 1) + K_w(1) \frac{\delta(span 1)}{2}}{C_w(1)(span 1) + K_w(1) \frac{\delta(span 1)}{2}}
\end{align*}
\]
The quantity $K_{o(1)}$ is equal to $K_{oB}$ and $K_{w(1)}$ is $K_{WB}$ for span 1. For clarity, the subscript (span 1) is hereafter deleted in the above equations. Equation 5.29 and 5.30 may now be written simply

$$u_{I(0)} = \frac{[B_{y(1)} - K_{oB} \frac{\delta}{2}]}{[B_{y(1)} + K_{oB} \frac{\delta}{2}]} u_{(2)} \quad (5.33)$$

$$b_{I(0)} = \frac{[C_{w(1)} - K_{WB} \frac{\delta}{2}]}{[C_{w(1)} + K_{WB} \frac{\delta}{2}]} b_{(2)} \quad (5.34)$$

These two equations are exactly identical to Eqs. 2.45 and 2.66 for the bottom end of a single-span restrained beam-column as derived in Sect. 2.5.

For the other exterior joint (joint N), only Eqs. 5.12 and 5.23 are valid since the span above the joint does not exist. With proper substitutions, it can be shown that the imaginary point above joint N has the following displacement relationships:

$$u_{I(J+1)} = \frac{[B_{y(J)} - K_{oT} \frac{\delta}{2}]}{[B_{y(J)} + K_{oT} \frac{\delta}{2}]} u_{(J-1)} \quad (5.35)$$

$$b_{I(J+1)} = \frac{[C_{w(J)} - K_{wT} \frac{\delta}{2}]}{[C_{w(J)} + K_{wT} \frac{\delta}{2}]} b_{(J-1)} \quad (5.36)$$

These two equations refer to the last span in the continuous beam-column. The subscript (last span) has been discarded in the above equations for clarity purpose. The equations are identical to Eqs. 2.46 and 2.67 for the top end of a single-span beam-column.
5.4 Computational Procedures

The procedures for computing the critical moment for a continuous beam-column with joint restraints are in some ways similar to those for single-span beam-columns. First, it is necessary to compute the sectional properties \( (B_y, C_w, C_T, y_o, \int_A \sigma \, a^2 \, dA) \), and the pre-buckling deformations \( v_o \) for all spans of the continuous beam-column. Second, the coefficient matrix \( [C_{ij}] \) is set up and the critical moment is obtained by iteration when the determinant of the coefficient matrix \( [C_{ij}] \) is zero.

Before the sectional properties and pre-buckling deformations can be computed, it is first necessary to define the end moment ratio for every span of the continuous beam-column. Once the end moment ratio is known, it is possible to construct the in-plane M-0 curve for every span of the continuous beam-column. The M-0 characteristics of a joint are found by graphically compounding the M-0 curves of the columns immediately above and below the joint. This technique of determining the M-0 relationships for a joint insure rotational compatibility.

In the present study, every span of a continuous beam-column is assumed to be arbitrarily cut into 20 equal segments. Therefore for a continuous beam-column with \( n \) number of spans, the number of pivotal points is \( [2ln - (n-1)] \) and the size of the coefficient matrix \( [C_{ij}] \) is \( 38n \times 38n \). As an example, for a 3 span beam-column, the size of \( [C_{ij}] \) is \( 114 \times 114 \).

The general flow-charts shown in Figs. 2.17 and 2.19 are also applicable to continuous beam-column problems. For a beam-column with \( n \) spans, Program 1 is repeated \( (n-1) \) times in order to
generate all the necessary data for Program 2. Again a joint rotation increment technique is used to determine the value of critical moment $M_{cr}$.

### 5.5 Lateral-Torsional Buckling Strength of Continuous Beam-Columns

The method developed in this chapter can be used to solve the buckling problems of beam-columns of any number of spans. In the present study, only beam-columns with 3 spans are examined. The variables in this study are the slenderness ratio, the end moment ratio, and the load ratio $P/P_y$.

Four types of joint conditions are studied:

1. No restraints ($K_o(J) = K_w(J) = 0$);
2. Only full warping restraint ($K_o(J) = 0$, $K_w(J) = \infty$);
3. Only full weak-axis bending restraint ($K_o(J) = \infty$, $K_w(J) = 0$); and
4. Full warping and weak-axis bending restraints ($K_o(J) = \infty$, $K_w(J) = \infty$).

A summary of the analytical results for five types of continuous beam-columns are given in Table 5.1. The upper and the lower joints referred to in this table are the upper interior and the lower interior joints respectively.

#### 5.5.1 Continuous Beam-Columns with Equal Spans and Under Symmetrical Loading (Types A and B)

The three-span continuous beam-column under consideration is of $\text{BW}31$ shape subjected to a constant axial load and equal but opposite strong axis bending moments at the interior joints as shown in the sketches in Figs. 5.3-5.10. Thus the end moment ratio is zero for the upper and the lower columns, and -1 for the middle column. The slenderness ratios chosen for this study are 30 and 50.
Shown in Figs. 5.3-5.10 are the in-plane curves of the individual beam-columns and of the joints, and the points of inception of lateral-torsional buckling in individual beam-columns. The buckling moments shown here are the tangent modulus values previously obtained in Chapter 3. The points of inception of lateral-torsional buckling are extrapolated upward to cut the joint curve at two locations representing thus the upper and the lower limits for the occurrence of lateral-torsional buckling in a continuous beam-column. Because of the symmetry of the loading conditions, the joint curve shown in any of these figures is for either the upper or the lower joint. The joint conditions have been stated earlier.

In all instances, the inception of lateral-torsional buckling for the continuous system lies between the two limits. In the individual beam-column analysis, the upper and the lower spans should buckle before the middle column. Thus the results demonstrate the restraining effect from the middle column in delaying the buckling of the upper and the lower columns until the whole system buckles simultaneously.

In the plastic method of design of unbraced beam-columns, the lower limit has been recommended as the critical moment for the entire system. Thus when applied to the examples shown here, the design provision is conservative in estimating the buckling strength of a continuous beam-column.

A comparative study of the critical moments for a continuous beam-column of span length $30 \, r_x$, or of span length $50 \, r_x$ as given in Table 5.1 indicates that the critical moment value is
improved by the presence of restraints at the joints.

5.5.2 Continuous Beam-Columns with Equal Spans and Under Unsymmetrical Loading

a) Constant Axial Load (Type C)

The case study is an 8/31 continuous beam-column with an end-moment ratio equal to zero for the upper and lower columns, and -0.5 for the middle column. The column span is chosen as 50 r_x and the load ratio P/P_y is 0.4 for all spans. Because of the unsymmetrical loading, there are now two joint curves, one for the upper joint and the other for the lower joint. There are an upper limit and a lower limit to each joint curve.

The results are shown in Figs. 5.11-5.14 for the four different joint restraints. In all cases the critical moment for the upper joint occurs between its upper and lower limits. For the lower joint, the buckling moment occurs below the lower limit. This is because under such loading conditions the column that tends to buckle first is the upper column. The middle column merely acts as a restraint to this column. Thus when the entire system buckles simultaneously, the original critical moment for the middle span will be reduced. Since the end moment ratio for the middle column is held constant, a drop in the original critical moment for the middle column will automatically cause a drop in the lower limit of the lower joint.

The presence of joint restraints slightly improves the buckling moment as shown in Table 5.1.
b) **Variable Axial Load (Type D)**

In the case study reported herein, the continuous beam-column is of \(8\times3\) shape with an axial load ratio \(P/P_y\) equal to 0.3 for the upper column, 0.4 for the middle column, and 0.5 for the lower column. The end moment ratios are 0, -1, and 0 respectively. The column length is 50 \(x\) for all spans. The resultant joint in-plane \(M=0\) curves are shown in Figs. 5.15-5.18.

The critical span for this continuous beam-column is the lower column. Thus when buckling occurs for the entire system, the original critical moment for lower column should be improved, and those of the middle and upper columns would be reduced. The theoretical solutions shown in Figs. 5.15-5.18 demonstrate that this phenomenon does happen: the critical moment occurs above the lower limit of the lower joint, but below the lower limit of the upper joint.

The critical moments for the upper and the lower joints are listed in Table 5.1 for the four restraining conditions. It can be seen that the presence of restraints at the joints does improve the critical moment slightly.

The results of the last two examples show that a lower limit may not necessarily be conservative for predicting the lateral-torsional buckling strength of a continuous beam-column. The point to note is that the buckling strength of an assembly of two members is very much influenced by the strength of the members adjacent to this assembly.
5.5.3 Continuous Beam-Columns with Unequal Spans and Under Symmetrical Loading (Type E)

The example chosen for this study is an 8x31 continuous beam-column with constant axial load ratio for all the three spans, and with equal but opposite end moments applied at the interior joints (see sketches in Figs. 5.19-5.22). The slenderness ratio is 30 for the upper and the lower columns, and 50 for the middle column.

In all instances, the critical moment occurs between the two limits. It is closer to the lower limit than the upper limit. The values of the critical moments for the four restraining conditions are also given in Table 5.1. Again the results show the beneficial effects of the presence of restraints at the joints.

5.6 Summary

This chapter has presented a method for predicting the inception of inelastic lateral-torsional buckling in a continuous beam-column with restraints present at every joint. It has been shown that the method is applicable to beam-columns under any combinations of loads and span length.

In the plastic method of design of unbraced beam-columns, the buckling strength of a joint is assumed to be that corresponding to the lower limit. It has been shown in this chapter that this assumption may be unconservative as the buckling strength is very much influenced by the buckling of the members adjacent to the two columns that form the joint.

The solutions presented herein are new.
6. BEHAVIOR OF BEAM-COLUMNS AFTER LATERAL-TORSIONAL BUCKLING

6.1 Introduction

One of the complexities in the study of structural stability of a column or a beam-column for that matter, is to determine its behavior after the occurrence of buckling. Ever since the day of the elastica first investigated by Euler and Lagrange\(^\text{(6.1)}\) in 1770, relatively little work has been done on this important aspect of structural mechanics.

In classical buckling theory, a column is assumed to deform indefinitely at buckling load. Yet common sense tells us that no structural member can deform indefinitely in the real sense. The fallacy lies in the fact that all buckling theories invariably assume small deflections which are invalid when deformations become large. The behavior after buckling requires a second-order analysis which can be very complex. As an example, in the relatively simple problem of the elastica, the ordinary buckling equation for a column becomes a non-linear differential equation when the second-order analysis is performed. It requires for a solution the use of elliptical integral functions.\(^\text{(2.1,6.2)}\) It should be realized that elastica is only applicable to elastic columns.

For inelastic columns, the theoretical solutions were not available until 1947 when Shanley forwarded his famous inelastic column theory.\(^\text{(2.9)}\) His contribution is very significant. For the first
time this new theory shed light on the inelastic behavior of a column after buckling had occurred. It resolved the paradox that had confronted engineers for more than a half a century as to which of the two different theories was correct in predicting the strength of an unbraced column-- the reduced modulus or the tangent modulus concept. Shanley's theory was soon after refined and expanded by many recent investigators. (2.10,6.3-6.6)

Thus far, the post-buckling analysis considers a buckled column to deform in one direction only. This is true for a concentrically loaded column which usually buckles laterally if it is not braced against movement in the lateral direction. The situation is vastly different for a beam-column. After the occurrence of lateral-torsional buckling, the beam-column will deform in the transverse and lateral directions, and at the same time twist about its shear center. The beam-column is biaxially loaded. The problem is therefore seemingly more complex and this possibly explains the lack of research work done in this field.

6.2 Tool of Analysis

If a column, initially out of straightness, is concentrically loaded by an axial load, there is no buckling. The column will continue to deform under increasing load until instability occurs. Thereafter unloading in the column begins with further deformation. (2.4)

The different load-deflection characteristics for either straight or crooked columns are illustrated in Fig. 6.1. The smaller the initial imperfection, the closer the maximum load is to the one that corresponds to zero initial imperfection. Tests on crooked H-shape columns show that the load-deformation characteristics of these columns are affected
by the magnitude of initial imperfections in a manner similar to those sketched in Fig. 6.1(6.7)

The tool of analysis for the post-buckling behavior of a beam-column is now apparent: an utilization of the concept of initial imperfection. A beam-column with in-plane and out-of-plane initial imperfections when loaded by in-plane forces alone would behave like a biaxially loaded beam-column. The problem of beam-columns under biaxial loading has been extensively studied by Santathadaporn at Lehigh University. Much of the derivations of the governing biaxial bending equations to be presented in this chapter has been taken from his analytical work given in Ref. 6.8.

Shown in Fig. 6.2a is an initially crooked beam-column subjected to an axial load P and strong-axis bending moments $M_{Bx}$ and $M_{Tx}$. The restraining moments are $M_{By}$, $M_{Ty}$, and $M_{zo}$. A cross-sectional view of the beam-column in the unloaded state at a distance z from the bottom end is shown in Fig. 6.2b. The imperfections for the column are designated $u_i$, $v_i$, and $\beta_i$. It is assumed that the imperfections may be represented by sine functions as follows:

$$
\begin{align*}
    u_i &= u_o \sin \frac{n\pi z}{L} \\
    v_i &= v_o \sin \frac{n\pi z}{L} \\
    \beta_i &= \beta_o \sin \frac{n\pi z}{L}
\end{align*}
$$

(6.1)

The parameters $u_o$, $v_o$, and $\beta_o$ denote respectively the initial lateral displacement, the in-plane displacement and the twist at the midheight of the column. The displacements of the loaded column measured from the coordinate axes are

$$
\begin{align*}
    u &= u_i + \ddot{u} \\
    v &= v_i + \ddot{v} \\
    \beta &= \beta_i + \ddot{\beta}
\end{align*}
$$

(6.2)
where \( \tilde{u}, \tilde{v}, \) and \( \tilde{\beta} \) denote the actual displacements. It can be found from geometry that the displacements \( u_c, v_c, \) and \( \beta_c \) of the centroid are:

\[
\begin{align*}
    u_c &= u + y_o \beta \\
         &= (u_i + \tilde{u}) + y_o (\beta_i + \tilde{\beta}) \quad (6.3) \\
    v_c &= v - x_o \beta \\
         &= (v_i + \tilde{v}) - x_o (\beta_i + \tilde{\beta}) \quad (6.4) \\
    \beta_c &= \beta = \beta_i + \tilde{\beta} \quad (6.5)
\end{align*}
\]

The internal resisting moments \( M_\xi, M_\eta, \) and \( M_\zeta \) can be found be a procedure similar to the one described in Chapter 2. They are as follows:

\[
\begin{align*}
    M_\xi &= - M_{Bx} + \frac{z}{L} (M_{Bx} + M_{Tx}) + P v_c + \\
           &\quad \beta \left[ - M_{By} + \frac{z}{L} (M_{By} + M_{Ty}) - P u_c \right] - u' M_{zo} \quad (6.6) \\
    M_\eta &= - \beta \left[ - M_{Bx} + \frac{z}{L} (M_{Bx} + M_{Tx}) + P v_c \right] - \\
           &\quad - M_{By} + \frac{z}{L} (M_{By} + M_{Ty}) - P u_c - v' M_{zo} \quad (6.7) \\
    M_\zeta &= u' \left[ - M_{Bx} + \frac{z}{L} (M_{Bx} + M_{Tx}) + P v_c \right] \\
           &\quad + v' \left[ - M_{By} + \frac{z}{L} (M_{By} + M_{Ty}) - P u_c \right] \\
           &\quad + M_{zo} + P y_o u' - P x_o v' - \\
           &\quad \beta' \int_A \sigma a^2 \, da - \frac{1}{L} (M_{Bx} + M_{Tx}) (u + x_o - x_{o,z=0}) \\
           &\quad - \frac{1}{L} (M_{By} + M_{Ty}) (v + y_o - y_{o,z=0}) \quad (6.8)
\end{align*}
\]
In small deflection analysis the internal moments $M_\xi$, $M_\eta$ and $M_\zeta$ are given by:

\begin{align}
M_\xi &\approx -EI_x \dddot{\bar{v}} \\
M_\eta &\approx EI_y \dddot{\bar{u}} \\
M_\zeta &\approx C_T \dddot{\bar{\beta}} - (C_w \bar{\beta})''
\end{align} 

(6.9) (6.10) (6.11)

The above relationships have been derived in Chapter 2 for lateral-torsional buckling problems. They are valid as long as deflections are small. When applying them to large deflection problems, it has been demonstrated in Ref. 6.8 that the error incurred can be substantial. The stiffnesses and the displacements should be referred to the local coordinates. Thus, in place of Eqs. 6.9 and 6.10, the following relationships are introduced:

\begin{align}
M_\xi &= -EI_\xi \dddot{\bar{v}} \\
M_\eta &= EI_\eta \dddot{\bar{u}} \\
M_\zeta &= C_T \dddot{\bar{\beta}} - (C_w \bar{\beta})''
\end{align} 

(6.8)

(6.12) (6.13)

In the above equations, $\dddot{\bar{v}}_\eta$ and $\dddot{\bar{u}}_\xi$ denote the displacements in the $\eta$ and $\xi$ directions respectively. The double primes denote the second derivatives with respect to $\zeta$. Using Eq. 6.2,

\begin{align}
\dddot{\bar{u}}_\xi &= u_\xi - u_i \\
\dddot{\bar{v}}_\eta &= v_\eta - v_i
\end{align} 

(6.14) (6.15)

Taking the second derivative and substituting them into Eqs. 6.12 and 6.13, the following relationships are obtained:
The direction cosine relationships for the x, y, z and $\xi$, $\eta$, $\zeta$ axes have been given in Chapter 2. In matrix notation, the direction cosine relationships are simply

\[
\begin{bmatrix}
\eta \\
\xi \\
\zeta \\
\end{bmatrix} =
\begin{bmatrix}
1 & -\beta & -v' \\
+\beta & 1 & -u' \\
v' & u' & 1 \\
\end{bmatrix}
\begin{bmatrix}
y \\
x \\
z \\
\end{bmatrix}
\]

In terms of displacement, the direction cosine may be written as:

\[
\begin{bmatrix}
v_{\eta} \\
u_{\xi} \\
w_{\zeta} \\
\end{bmatrix} =
\begin{bmatrix}
1 & -\beta & -v' \\
\beta & 1 & -u' \\
v' & u' & 1 \\
\end{bmatrix}
\begin{bmatrix}
v \\
u \\
w \\
\end{bmatrix}
\]

Taking the second derivatives of the displacements $u_{\xi}$ and $v_{\eta}$, the above equation can be written as:

\[
\begin{bmatrix}
v''_{\eta} \\
u''_{\xi} \\
w''_{\zeta} \\
\end{bmatrix} =
\begin{bmatrix}
1 & -\beta & -v' \\
\beta & 1 & -u' \\
v' & u' & 1 \\
\end{bmatrix}
\begin{bmatrix}
v'' \\
u'' \\
w'' \\
\end{bmatrix}
\]

or simply

\[
[\ddot{\delta}] = [T] [\ddot{\delta}] 
\]

where

\[
[\ddot{\delta}] =
\begin{bmatrix}
v''_{\eta} \\
u''_{\xi} \\
w''_{\zeta} \\
\end{bmatrix}
\]

\[
M_{\xi} = -EI_{\xi} [v''_{\eta} - v''_{i}] 
\]

\[
M_{\eta} = EI_{\eta} (u''_{\xi} - u''_{i})
\]
The relationship between axial load $P$ and axial deformation $w$ is simply

$$P = E A w \quad (6.22)$$

In matrix notation, the forces are related to the displacements as:

$$\begin{bmatrix} M_{\xi} \\ M_{\eta} \\ P \end{bmatrix} = \begin{bmatrix} -EI_{\xi} & 0 & 0 \\ 0 & EI_{\eta} & 0 \\ 0 & 0 & EA \end{bmatrix} \begin{bmatrix} v'' \\ u'' \\ w'' \end{bmatrix} + \begin{bmatrix} EI_{\xi} & 0 & 0 \\ 0 & -EI_{\eta} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_i'' \\ u_i'' \\ w_i'' \end{bmatrix}$$

The above equation can be written simply

$$[F] = [Q][\delta] + [Q_o][\delta_o] \quad (6.24)$$

The displacement vector $[\delta]$ has been found in Eq. 6.18. The matrix $[Q_o]$ is simply

$$[Q_o] = \begin{bmatrix} EI_{\xi} & 0 & 0 \\ 0 & -EI_{\eta} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (6.25)$$
The vector $[\delta_o]$ is

$$[\delta_o] = \begin{bmatrix} v''_1 \\ u''_i \\ w \end{bmatrix} \quad (6.26)$$

It can be related to vector $[\delta]$ by the use of a transformation matrix as follows:

$$[\delta_o] = [T_o] [\delta] \quad (6.27)$$

where

$$[T_o] = \begin{bmatrix} 1 & -\beta_i & -v'_i \\ \beta & 1 & -u'_i \\ v'_i & u'_i & 1 \end{bmatrix} \quad (6.28)$$

Equation 6.24 can now be expressed in terms of $[\delta]$ as follows:

$$[F] = [Q] [T] [\delta] + [Q_o] [T_o] [\delta]$$

$$= \left\{ [Q] [T] + [Q_o] [T_o] \right\} [\delta]$$

$$= \left\{ [H] + [H_o] \right\} [\delta] \quad (6.29)$$

The force vector $[F]$ contains the internal moments $M_\eta, M_\eta$ and the axial load $P$. The internal twisting moment $M_\xi$ is given by Eq. 6.11.

The internal moments in terms of the end moments and load are given by Eqs. 6.6, 6.7 and 6.8. Thus by substituting Eqs. 6.6-6.8 into Eqs. 6.29 and 6.11, it is possible to relate the end forces to the deformation parameters $u'', v'', w, \beta'$ and $\beta''$. The resultant set of three differential equations can be solved once the boundary and end equilibrium equations have been established.
A method of solution has been developed in Ref. 6.8 for biaxially loaded as-rolled wide-flange columns. This method, known as Tangent Stiffness Method, is applicable to beam-columns under symmetrical loading, that is, a beam-column subjected to an axial load, equal but opposite end moments about x and y axes, and a pair of twisting moments. A detailed description of the method is given in Ref. 6.8. In brief, a column is assumed to deform in such a way that the displacement functions $u$, $v$ and $\beta$ can be expressed as sine-parabolic curves. The rate of change of the internal forces at the mid-column is then equated to the rate of change of the external forces at the same section. The final equation has the following form:

$$[R] \dot{\Delta} = [W]$$

(6.30)

In the above equation, $[R]$ is the tangent stiffness matrix of size 4x4. It contains the combinations of the stiffness matrices $[H]$ and $[H_0]$, axial load $P$, bending and restraining moments, column length $L$, St. Venant torsional constant $C_T$, and warping rigidity $C_w$. The vector $[\Delta]$ is

$$[\Delta] = \begin{bmatrix}
\dot{u}_m \\
\dot{v}_m \\
\dot{u}_m \\
\dot{w}_m \\
\dot{\beta}_m
\end{bmatrix}
\quad (6.31)$$

The subscript $m$ denotes mid-column and the dot above the displacement parameters represents the rate of change of the quantity concerned.

The vector $[W]$ represents the rate of change of the external forces. It may be briefly written as follows:
\[ [W] = W \left( P, M_x, M_y, \gamma_3, u_m, v_m, v_m', v_m'' \right) \]  

(6.32)

The quantity \( \gamma_3 \) in the above equation is a warping restraint factor. A computer program to solve Eq. 6.30 by an iterative method has been documented in Ref. 6.8.

The columns under study in this chapter are those subjected to an axial load and equal but opposite strong-axis end moments. The buckling strength for this type of beam-column has been investigated in Chapter 3 for the lower bound solution, and in Chapter 4 for the upper bound solution. Based on the earlier discussion, the maximum strength of a laterally unbraced beam-column should lie between the upper and lower bound solutions.

As discussed earlier, an unbraced beam-column with initial imperfection will attain a maximum strength slightly below that of a straight member. A beam-column with a larger imperfection will exhibit a smaller strength than a beam-column with smaller imperfection. In the present study, two values of initial imperfections are investigated for a particular beam-column. The "true" maximum strength is obtained by extrapolation of the maximum strength values corresponding to the two selected values of imperfections. In Fig. 6.3 is demonstrated the technique used in estimating the "true" maximum strength of an unbraced beam-column. The quantities \( \chi_1 \) and \( \chi_2 \) denote the two selected values of imperfections, and \( M_1 \) and \( M_2 \) denote the corresponding maximum moments. The maximum strength \( M_m \) of a straight column is estimated from the following relationship:

\[ M_m = M_2 \left(1 - \frac{X_2}{X_2 - X_1}\right) + M_1 \left(\frac{X_2}{X_2 - X_1}\right) \]

(6.33)
In general, the imperfection function \( \chi_1 \) or \( \chi_2 \) is related to \( u_i \), \( v_i \) and \( \beta_i \), that is,

\[
\chi = F_\alpha (u_i, v_i, \beta_i)
\]  \hspace{1cm} (6.34)

In the present study, it is assumed that the beam-column is initially straight with respect to \( v \) and \( \beta \). In other words,

\[
v_i = \beta_i = 0
\] \hspace{1cm} (6.35)

Thus the imperfection functions \( \chi_1 \) and \( \chi_2 \) are now simply

\[
\chi_1 = u_{i_1} = u_{o_1} \sin \frac{\pi z}{L}
\] \hspace{1cm} (6.36)

\[
\chi_2 = u_{i_2} = u_{o_2} \sin \frac{\pi z}{L}
\] \hspace{1cm} (6.37)

The values for \( u_{o_1} \) and \( u_{o_2} \) have been arbitrarily chosen as 0.02 in. and 0.05 in. respectively.

6.3 Pinned-End Beam-Columns

6.3.1 Maximum Strength

Typical results of the maximum strength curves are given in Figs. 6.4 and 6.5 for beam-columns of \( 8\times31 \) and \( 14\times142 \) shape respectively. The maximum strength values are extrapolated from the two moment values corresponding to two small imperfections as explained in the earlier section. Also plotted in these figures are the in-plane strength curve and the lateral-torsional buckling curves based on the reduced modulus and the tangent modulus concepts.

In both figures, the maximum strength curve is bounded by the reduced modulus and the tangent modulus buckling curves. For longer columns, the maximum strength curve coincides with the tangent modulus buckling strength curve. This indicates that long columns have little
or no post-buckling strength. However, the maximum strength curve rapidly deviates from the tangent modulus buckling curve to follow approximately the same slope as the reduced modulus buckling curve. It intercepts the in-plane strength curve at \( L/r_x = 39 \) for \( S_{31} \) beam-columns and \( L/r_x = 49 \) for \( 14W142 \) beam-columns. Thus it may be stated that for columns of practical length, usually around \( L/r_x = 40 \), the in-plane moment capacity will not be impaired by the occurrence of lateral-torsional buckling. It should be realized that the above generalized statement is valid for column shapes with \( D_y \geq 925 \) and \( \frac{P}{F_y} = 0.4 \).

The various strength curves in terms of the in-plane moment capacity \( M_{\text{max}} \) are given in Fig. 6.6. This figure also shows that the maximum strength curve is very much influenced by the \( D_T \) values.

6.3.2 Behavior

The typical behavior of unbraced columns is given in Figs. 6.7 and 6.8 for beam-columns of \( S_{31} \) and \( 14W142 \) shape respectively. This is in the form of moment-rotation response for columns of several \( L/r_x \) ratios. Also shown in these two figures are the corresponding in-plane characteristics and the points of inception of lateral-torsional buckling. These points of inception of lateral-torsional buckling are the tangent modulus values obtained in Chapter 3. It can be seen that in all instances, the behavior before lateral-torsional buckling is strictly in-plane type. For short columns \( (L/r_x \leq 40) \), the \( M-\theta \) curves after buckling for unbraced columns almost coincide with those for braced columns. There is no noticeable loss in the rotation capacity for the short columns. However, as the column length becomes larger, not only does the beam-column fail to reach its in-plane strength,
but the in-plane rotation capacity after buckling is badly impaired. As an example, for a beam-column of $L/r_x = 60$ for either of the two column shapes, it starts to unload rapidly soon after the attainment of its maximum strength which is slightly above the buckling strength. It may therefore be concluded that the post-buckling behavior of an unbraced beam-column is very much influenced by the slenderness ratio. Fortunately, since most practical columns have $L/r_x$ around 40, the in-plane behavior may be assumed for an unbraced column. This is valid for columns with $D_t$ greater than 925, and $\frac{P}{P_y} = 0.4$.

6.4 Restrained Beam-Columns

In the case study reported herein, only warping restraint is considered. Thus a beam-column is free to bend about its weak axis at the column ends. The warping restraint is assumed infinitely stiff, that is,

$$K_{WB} = K_{WT} = \infty$$

where $K_{WB}$ and $K_{WT}$ are the warping restraint parameters as defined in Chapter 3.

6.4.1 Maximum Strength

The maximum strength of 8W31 beam-columns is examined. Figure 6.9 shows the interaction curves for the column with full warping restraints. As in the case of pinned-end beam-columns, the maximum strength curve lies somewhere between the reduced and the tangent modulus buckling curves. The cut-off point now occurs at $L/r_x = 50$.

A comparison of the maximum strength curves for pinned-end beam-columns and those with full warping restraints is given in Figs.
6.10 and 6.11. Also plotted in these figures are the CRC interaction curves. Two conclusions may be made from these figures:

(1) The effect of warping restraint on the maximum strength of an unbraced beam-column is not very significant. The warping restraint only slightly improves the strength of long columns.

(2) The CRC Interaction curve computed from Eq. 3.10 has been found theoretically to be unconservative in predicting the inception of lateral-torsional buckling in unbraced beam-columns. However as shown in Figs. 6.10 and 6.11, it gives a good approximation of the maximum strength capacity of such columns although it tends to be conservative for shorter beam-columns.

It should be realized that the above conclusions are valid for beam-columns of $D_T$ varying from about 900 to 1500. For columns of other shapes, an analysis similar to the one described in this chapter may be performed.

6.4.2 Behavior

The behavior of unbraced 8W31 and 14W142 beam-columns with full warping restraints is shown in Figs. 6.12 and 6.13 respectively for a number of slenderness ratios. Also shown in these figures are the in-plane $M-\theta$ curves and the points of inception of lateral-torsional buckling. As in the case of pinned-end beam-columns, the post-buckling behavior is very much influenced by the slenderness ratio. Longer columns tend to lose their rotation capacity after buckling has occurred.
A comparison of Fig. 6.12 with Fig. 6.7 and of Fig. 6.13 with Fig. 6.8 indicates that the general moment-rotation behavior for unbraced beam-columns is improved by the presence of restraints at the column ends.

6.5 Summary

This chapter has presented a method to determine the strength and behavior of unbraced beam-columns after the occurrence of lateral-torsional buckling. It has been found analytically that the behavior of unbraced beam-columns is very much influenced by the column length. Long columns have been found to have virtually no post-buckling strength. As soon as lateral-torsional buckling has occurred, the beam-column unloads very rapidly with little rotation.

Beam-columns of practical length \((L/r_x \leq 40)\) with \(P/P_y = 0.4\) are found to have remarkably high post-buckling strength. In fact, for the two column shapes examined herein, the post-buckling behavior for the unbraced beam-columns is, and in some cases almost identical to their in-plane behavior. The beam-columns can, in some cases, still be loaded to their in-plane moment capacity even though inelastic lateral-torsional buckling has already occurred. This is a very significant fact since it means that in designing unbraced beam-columns by the plastic method, the in-plane moment-rotation response may be assumed if the slenderness ratio does not exceed 40. This limitation on the slenderness ratio applies only to unbraced beam-columns with the load ratio \(P/P_y\) equals to 0.4. For beam-columns with other \(P/P_y\) ratios, or with slenderness ratio greater than 40, an independent analysis must be performed to determine their maximum strength capacity.
The analytical solutions to the post-buckling strength and behavior are the first available for inelastic as well as elastic W beam-columns.
7. SUMMARY AND CONCLUSIONS

It has been the purpose of this investigation to develop methods for predicting the strength and behavior of laterally unbraced beam-columns. The beam-columns in this case are commercially rolled W shapes subjected to constant axial load and monotonically increasing strong-axis bending moments. The strength and behavior of this kind of beam-columns are very much affected by lateral-torsional buckling. The behavior before the occurrence of lateral-torsional buckling is entirely in-plane type. Immediately after buckling has occurred, the in-plane behavior is terminated and the beam-column will displace in both the transverse and the lateral directions, and twist about the shear center.

The concept of elastic spring representation has been introduced to account for the warping and weak-axis bending restraints at the column ends for single-span beam-columns or at the joints in continuous beam-columns. The buckling strength of either pinned-end or restrained beam-columns has been determined utilizing both the tangent modulus and the reduced modulus concepts of the unloading of the yielded portions in column sections. The tangent modulus solutions are the lower bound solutions to the maximum strength of unbraced beam-columns. The solutions based on the reduced modulus method are the upper bounds. The method of analysis for the determination of the buckling strength developed in this investigation is applicable to beam-columns under any loading conditions.
Based on the analytical results obtained in the lateral-torsional buckling analysis, the major conclusions are:

1. The presence of warping restraints slightly improves the buckling strength of pinned-end beam-columns.

2. Weak-axis bending restraints have little effect on the buckling strength of short beam-columns. However as the column length becomes larger, the effect becomes significant.

3. The CRC interaction formula over-estimates the buckling strength of pinned-end beam-columns of $L/r_x \leq 60$ and $P/P_y = 0.4$.

4. For 8W31 beam-columns with $P/P_y = 0.4$ and any end moment ratio, lateral-torsional buckling will always be predicted if the analysis is based on the tangent modulus method. However, if the analysis is based on the reduced modulus method, lateral-torsional buckling will not be predicted for columns of length common in practical design, that is, $L/r_x \leq 40$.

5. The warping and weak-axis bending restraints affect the reduced modulus buckling strength solutions in a manner similar to the solutions based on the tangent modulus method.

6. The buckling strength of a structural joint in a continuous beam-column is very much affected by the buckling condition of the joints adjacent to the one under consideration. It has been found that the lower limit to
the buckling strength of a joint as predicted by the current design practice \((1.3)\) can be unconservative when compared to the buckling solution obtained for a continuous beam-column.

The concept of initial imperfections is utilized to predict the strength and behavior of an unbraced and initially straight beam-column. The maximum strength of this beam-column after the occurrence of lateral-torsional buckling is obtained by extrapolation of two maximum load values corresponding to two arbitrarily selected initial imperfections. Major conclusions from this analytical study are as follows:

1. The behavior of unbraced beam-columns is very much affected by the column length.

2. For long columns, unloading occurs immediately after lateral-torsional buckling with relatively little rotation. There is virtually no post-buckling strength.

3. Short beam-columns have large strength and rotation capacities. As a matter of fact, for \(8\sqrt{31}\) and \(14\sqrt{42}\) beam-columns with a load ratio \(P/P_y = 0.4\), the post-buckling behavior resembles remarkably well the in-plane response for \(L/r_x \leq 40\). Their maximum strength is the same as their in-plane strength.

4. The maximum strength curve has been found to be between the tangent modulus and the reduced modulus buckling strength curves. This indirectly provides a check to the tangent modulus and the reduced modulus solutions developed in this investigation.
5. The CRC Interaction formula has been found to be unconservative in predicting the buckling strength of unbraced beam-columns having a slenderness ratio less than 60. However it does give a fair estimation of the maximum strength of unbraced beam-columns of $L/r_x \geq 40$. For shorter beam-columns which have considerably large post-buckling strength, the CRC Interaction formula is conservative with respect to the maximum strength prediction.

6. The presence of warping restraints at the column ends slightly improves the maximum strength capacity of an unbraced beam-column.
8. APPENDICES
APPENDIX 1: DETERMINATION OF TORSIONAL MOMENT FOR INELASTIC BEAM-COLUMNS

The torsional moment $M_z$ of an open section is the sum of the St. Venant contribution and warping contribution:

$$M_z = M_{sv} + M_w$$  \hspace{1cm} (A1.1)

It has been discussed by the previous investigators that the St. Venant torsional moment $M_{sv}$ for an inelastic beam-column does not change appreciably from the value for an elastic beam-column:

$$M_{sv} = G K_T \beta'$$  \hspace{1cm} (A1.2)

where $G$ is the shearing modulus and $K_T$ the torsion constant. For a W shape, $K_T$ is approximately equal to (see discussion in Sect. 3.1.2):

$$K_T = \frac{1}{3} \left[ 2 b t^3 + (d-2t)w^3 \right]$$  \hspace{1cm} (A1.3)

The warping torsional moment for an elastic (prismatic) column is:

$$M_w = -E I_w \beta''$$  \hspace{1cm} (A1.4)

where $I_w$ is the warping moment of inertia. It is a function of the normalized unit warping $w_n$. The exact expression for $I_w$ will be given later in this appendix.

The equation of torsional moment for an inelastic (non-prismatic) member is derived based on the assumptions that (1) deformation is small, and (2) the shearing deformation in the middle surface is zero.
It is given in Ref. 2.4 that for an open section, the unit warping with respect to the shear center \( \omega_o \) is equal to

\[
\omega_o = \int_0^s \rho_o \, ds \quad \text{(A1.5)}
\]

The tangential distance \( \rho_o \) is measured from the shear center as shown in Fig. A1.1. The normal stress \( \sigma_w \) due to warping of the section is:

\[
\sigma_w = E \omega_n \beta'' \quad \text{(A1.6)}
\]

where \( \omega_n \), the unit normalized warping, is given by:

\[
\omega_n = \frac{1}{A} \int_0^s \omega_o \ t \ ds - \omega_o \quad \text{(A1.8)}
\]

It varies along the length of a non-prismatic column.

The shear flow \( \tau_\omega t \) in the section is found by considering the equilibrium of a strip of the cross-section:

\[
\tau_\omega t = -\int_0^s t \left( \frac{d \sigma_w}{dz} \right) ds \quad \text{(A1.9)}
\]

This shear flow results in a torsional moment:

\[
M_w = \int_0^{E_n} \tau_\omega t \rho_o \, ds \quad \text{(A1.10)}
\]

With proper substitution, the warping moment \( M_w \) becomes

\[
M_w = -\int_0^{E_n} \rho_o \left[ \int_0^s t \left( E \omega_n \beta'' \right)' \, ds \right] ds \quad \text{(A1.11)}
\]
The quantity \( (E \omega_n \beta'')' \) is simply:

\[
(E \omega_n \beta'')' = E \omega_n \beta'' + E \omega_n' \beta''
\]

(A1.12)

Thus, Eq. A1.11 may be written as:

\[
M_w = -E \beta'' \int_0^E \rho_o \left[ \int_0^s \omega_n' ds \right] t ds
\]

- \( E \beta'' \int_0^E \rho_o \left[ \int_0^s \omega_n ds \right] t ds \)

(A1.13)

The first term on the right hand side is \((2.4)\)

\[
E \beta'' \int_0^E \rho_o \left[ \int_0^s \omega_n' ds \right] t ds
\]

\[
E \beta'' \int_0^E \omega_n^2 t ds
\]

\[
E I_w \beta''
\]

(A1.14)

The second term on the right hand side of Eq. A1.13 can be found by integration by parts:

\[
E \beta'' \int_0^E \rho_o \left[ \int_0^s \omega_n' ds \right] t ds
\]

\[
E \beta'' \left[ \omega_o \int_0^E \omega_n' t ds - \int_0^E \omega_o \omega_n' t ds \right]
\]

\[
= E \beta'' \left\{ \omega_o A \left( \frac{1}{A} \right)' \right\} \int_0^E \omega_o t ds
\]
\[
\left[ \frac{1}{A} \int_0^{E_n} w_o \ t \ ds \right] \left[ A \left( \frac{1}{A} \right) \int_0^{E_n} w_o \ t \ ds \right] - \\
\int_0^{E_n} w_n w_n' \ t \ ds \right) \right] \\
= E \beta'' \left[ - A \left( \frac{1}{A} \right) \int_0^{E_n} w_n \ t \ ds + \int_0^{E_n} w_n w_n' \ t \ ds \right] \\
\text{(A1.15)}
\]

The total warping moment is then equal to:

\[
M_w = - E I w'' - E \beta'' \left[ - A \left( \frac{1}{A} \right) \int_0^{E_n} w_o \ t \ ds + \int_0^{E_n} w_n w_n' \ t \ ds \right] \\
\text{(A1.16)}
\]

The first term in the parenthesis is highly redundant because of the fact that \( A, w_n \) and \( w_o \) are all functions of \( z \) which are very difficult to determine. The derivative of the quantity \( \left( \frac{1}{A} \right) \) will change the negative sign in front of the first term in the parenthesis to positive. Because this term is highly redundant, it is assumed here that:

\[
- A \left( \frac{1}{A} \right) \int_0^{E_n} w_n \ t \ ds \approx \int_0^{E_n} w_n w_n' \ t \ ds \ \\
\text{(A1.17)}
\]
then Eq. A1.16 becomes

$$M_w = - E I_w \beta'' - E \left[ 2 \int_0^{En} w_n w_n' t \, ds \right] \beta'' \quad (A1.18)$$

It was defined earlier that the warping moment of inertia $I_w$ is:

$$I_w = \int_0^{En} w_n^2 t \, ds$$

Taking derivative on both sides:

$$I_w' = 2 \int_0^{En} w_n w_n' t \, ds \quad (A1.19)$$

Equation A1.18 can now be simplified to:

$$M_w = - E I_w \beta'' - E I_w' \beta''$$

$$= - (E I_w \beta'')'$$

$$= - (C_w \beta'')' \quad (A1.20)$$

where $C_w = E I_w = $ warping rigidity. Thus the total torsional moment (Eq. A1.1) is

$$M_z = G K_T \beta' - (C_w \beta'')'$$

$$= C_T \beta' - (C_w \beta'')' \quad (A1.21)$$

where $C_T = G K_T = $ St. Venant constant.
APPENDIX 2: DETERMINATION OF THE INDUCED MOMENTS

The three differential equations for lateral-torsional buckling of beam-columns have previously been derived in Chapter 2. They are rewritten below:

\[ B_x (v'' + v_o '') - M_{Bx} + \frac{z}{L} (M_{Bx} + M_{Tx}) + P v_c + \]
\[ \beta \left[ - M_{By} + \frac{z}{L} (M_{By} + M_{Ty}) - P u_c \right] - u' M_{z=0} = 0 \tag{2.24} \]

\[ B_y u'' + M_{By} - \frac{z}{L} (M_{By} + M_{Ty}) + P u_c + \]
\[ \beta \left[ - M_{Bx} + \frac{z}{L} (M_{Bx} + M_{Tx}) + P v_c \right] + v' M_{z=0} = 0 \tag{2.25} \]

\[ (C_w \beta'')' - (C_T + \int_A a^2 \, dA) \beta' + u'[ - M_{Bx} + \frac{z}{L} (M_{Bx} + M_{Tx}) + P v_c ] \]
\[ + v' [ - M_{By} + \frac{z}{L} (M_{By} + M_{Ty}) - P u_c ] + \]
\[ M_{z=0} + P y_o u' - P x_o v' - \frac{1}{L} (M_{Bx} + M_{Tx}) (u + x_o - \]
\[ x_o z=0 ) - \frac{1}{L} (M_{By} + M_{Ty}) (v + v_o + y_o - y_o z=0 ) = 0 \tag{2.26} \]

It was also found in Chapter 2 that the displacements \( u_c \) and \( v_c \) are:

\[ u_c = u + y_o \beta \tag{2.2} \]
\[ v_c = v_o + v - x_o \beta \tag{2.3} \]

At the support \( z=0 \); the in-plane displacements \( v z=0 = v o z=0 = 0 \); Eq. 2.25 reduces to:
\[
(B_y u'')_{z=0} + M_{By} + P (u + y_o \beta)_{z=0} + \beta_{z=0} \left[ -M_{Bx} - P \right.
\]
\[
\left. (x_o \beta)_{z=0} + v' M_{zo} = 0 \right]
\]
\[
M_{By} + v'_{z=0} M_{zo} \approx - (B_y u'')_{z=0} - P (u + y_o \beta)_{z=0} + \beta_{z=0} M_{Bx}
\]

(A2.1)

neglecting in above the cross product term of \( P (x_o \beta^2)_{z=0} \).

Let
\[
A_o = - (B_y u'')_{z=0} - P (u + y_o \beta)_{z=0} + \beta_{z=0} M_{Bx}
\]

(A2.2)

Equation A2.1 becomes:
\[
M_{By} + v'_{z=0} M_{zo} = A_o
\]

(A2.3)

At the other end of the column, the in-plane displacements are also zero, i.e. \( v_{z=L} = v_{o_{z=L}} = 0 \), Eq. 2.25 reduces to:
\[
(B_y u'')_{z=L} - M_{Ty} + P (u + y_o \beta)_{z=L} +
\]
\[
\beta_{z=L} \left[ M_{Tx} - P(x_o \beta)_{z=L} \right] + v'_{z=L} M_{zo} = 0
\]
\[
M_{Ty} - v'_{z=L} M_{zo} \approx (B_y u'')_{z=L} + P (u + y_o \beta)_{z=L} + \beta_{z=L} M_{Tx}
\]

(A2.4)

Again the cross product term \( P (x_o \beta^2)_{z=L} \) has been neglected. Let
\[
C_o = (B_y u'')_{z=L} + P (u + y_o \beta)_{z=L} + \beta_{z=L} M_{Tx}
\]

(A2.5)

Equation A2.4 can be written simply
\[
M_{Ty} - v'_{z=L} M_{zo} = C_o
\]

(A2.6)
Using the same techniques assuming that \( v = v_0 = 0 \) at \( z = 0 \) and \( z = 1 \), and neglecting small terms, it can be found that:

\[
\begin{align*}
M_{zo} - v' z=0 & = B_0 \quad \text{(A.7)} \\
M_{zo} + \left[ v' z=L - \frac{(y_{o z=L} - y_{o z=0})}{L} \right] M_{By} & = D_0 \quad \text{(A.8)}
\end{align*}
\]

where

\[
B_0 = - (C_w \beta'')' z=0 + (C_T + \int_A \sigma a^2 \, dA)_{z=0} \beta' z=0 - u' z=0 (- M_{Bx} + P y_{o z=0}) + v' z=0 P x_{o z=0} \\
+ \frac{1}{L} (M_{Bx} + M_{Tx}) u z=0
\]

\[
D_0 = - (C_w \beta'')' z=L + (C_T + \int_A \sigma a^2 \, dA)_{z=L} \beta' z=L - u' z=L (M_{Tx} + P y_{o z=L}) + v' z=L (M_{Ty} - P x_{o z=L}) \\
- \frac{1}{L} (M_{Bx} + M_{Tx}) (u + x_{o z=L} - x_{o z=0}) - \frac{A_0}{L} (y_{o z=L} - y_{o z=0}) \quad \text{(A2.10)}
\]

Solving simultaneously Eqs. A2.3 and A2.7 to obtain

\[
M_{By} = \frac{\left[ A_0 - v' z=0 \frac{B_0}{1 + (v' z=0)^2} \right]}{1 + (v' z=0)^2} \quad \text{(A2.11)}
\]

\[
M_{zo} = \frac{\left[ B_0 + v' z=0 A_0 \right]}{1 + (v' z=0)^2} \quad \text{(A2.12)}
\]

In a beam-column, the end slope \( v' \) is generally very small. If it is assumed that this quantity can be neglected, then Eqs. A2.11 and
A2.12 reduce to:

\[ M_{By} \approx A_0 \]

\[ = - (B_y u'')_{z=0} - P (u + y_o \beta)_{z=0} + \beta z=0 M_{Bx} \quad (A2.13) \]

\[ M_{zo} \approx B_0 \]

\[ = - (C_w \beta'')_{z=0} + (C_T + \int_A a^2 dA)_{z=0} \beta'_{z=0} + u'_{z=0} (- M_{Bx} + P y_o z=0) + v'_{z=0} P x_o z=0 + \]

\[ \frac{1}{L} (M_{Bx} + M_{Tx}) u_{z=0} \quad (A2.14) \]

Solving simultaneously Eqs. A2.6 and A2.8 to obtain:

\[ M_{Ty} = \frac{C_o + v'_{z=L} D_o}{1 + v'_{z=L} \left[ v'_{z=L} - \frac{1}{L} (y_o z=L - y_o z=0) \right]} \]

\[ \approx C_o \]

\[ = (B_y u'')_{z=L} + P (u + y_o \beta)_{z=L} + \beta z=L M_{Tx} \quad (A2.15) \]

\[ M_{zo} = \frac{D_o - \left[ v'_{z=L} - \frac{1}{L} (y_o z=L - y_o z=0) \right] C_o}{1 + v'_{z=L} \left[ v'_{z=L} - \frac{1}{L} (y_o z=L - y_o z=0) \right]} \]

\[ \approx D_o \]

\[ \approx - (C_w \beta'')_{z=L} + (C_T + \int_A a^2 dA)_{z=L} \beta'_{z=L} - u'_{z=L} (M_{Tx} + P y_o z=L) + v'_{z=L} (M_{Ty} - P x_o z=L) - \frac{1}{L} (M_{Bx} + M_{Tx}) (u + x_o z=0) - \frac{A_o}{L} (y_o z=L - y_o z=0) \quad (A2.16) \]

The twisting moment \( M_{zo} \) is given by Eq. A2.14 or A2.16. In this investigation Eq. A2.14 is adopted because of its simpler form.
APPENDIX 3: DETERMINATION OF BI-MOMENT OF A COLUMN SECTION

A "bi-moment" is a statical quantity which has the dimensions Force x Length. It is given in Ref. 2.3 as:

\[
M_B = \int_A \sigma_w w_n \, dA
\]  
(A3.1)

where \( \sigma_w \), the normal warping stress, and \( w_n \), the unit normalized warping have been defined previously in Appendix 1. It has also been given in Appendix 1 that the unit normalized warping \( w_n \) and the unit warping \( w_o \) are respectively:

\[
w_n = \frac{1}{A} \int_0^E n \omega_t ds - w_o \]  
(A1.8)

\[
w_o = \int_0^s \rho_o ds
\]  
(A1.7)

Substituting for the values of \( w_n \) and \( w_o \) in Eq. A3.1 with those from Eqs. A1.7 and A1.8, the following equation is obtained:

\[
M_B = \int_0^E n \sigma_w \left[ \frac{1}{A} \int_0^E n \omega_t ds - w_o \right] t \, ds
\]

\[
= \int_0^E n \sigma_w \left[ \frac{1}{A} \int_0^E n \left( \int_0^s \rho_o ds \right) t \, ds - \int_0^s \rho_o ds \right] t \, ds
\]

\[
= \int_0^E n \sigma_w \left[ \frac{1}{A} \rho_o t \left[ \frac{s^2}{2} \right] - \rho_o s \right] t \, ds
\]  
(A3.2)
Consider now the top flange of a W shape as shown in Fig. A3.1. In this instance

\[ \rho_o = d_U \]  
(A3.3)

Equation A3.2 becomes

\[ M_{BU} = \int_0^b \sigma_w \left[ \frac{1}{bt} d_U t \left( \frac{b^2}{2} - d_U s \right) \right] t \, ds \]
\[ = d_U \int_0^b \sigma_w \left( \frac{b}{2} - s \right) t \, ds \]
\[ = M_U d_U \]  
(A3.4)

where \( M_U \) is the upper flange moment about the y axis.

By similar reasoning it can be shown that the lower flange contribution is:

\[ M_{BL} = M_L d_L \]  
(A3.5)

Thus, the bi-moment of a W section is

\[ M_B = M_{BU} + M_{BL} \]
\[ = M_U d_U + M_L d_L \]  
(A3.6)
APPENDIX 4: DETERMINATION OF LATERAL DISPLACEMENT AND TWISTING RELATIONSHIPS IN FINITE DIFFERENCE EXPRESSIONS FOR IMAGINARY POINTS ADJACENT TO A JOINT

Lateral Displacement Relationships

The continuity equation is given by Eq. 5.7:

\[ u'_{z=L(n)} = u'_{z=0(n+1)} \]  

(5.7)

Using the designations as shown in Fig. 5.2, the finite difference expression for Eq. 5.7 is

\[ \frac{1}{2\delta(n)} \left[ -u(J-1)(n) + u(I(J+1))(n) \right] = \frac{1}{2\delta(n+1)} \left[ -u_I(J-1)(n+1) + u(J+1)(n+1) \right] \]  

(A4.1)

Let

\[ \gamma = \frac{\delta(n)}{\delta(n+1)} \]  

(A4.2)

Using the above notation and rearranging Eq. A4.1, the lateral displacement relationship for the imaginary point I(J+1) in span n is:

\[ u_{I(J+1)}(n) = u_{(J-1)}(n) - \gamma u_I(J-1)(n+1) + \gamma u_{(J+1)}(n+1) \]  

(A4.3)
The equilibrium equation at the joint is:

\[(B \cdot u')_{z=L(n)} - (B \cdot u')_{z=0(n+1)} + K_0(J) u'_{z=0(n+1)} = 0 \quad (5.11)\]

In finite difference expressions, Eq. 5.11 becomes:

\[B_y J(n) \frac{1}{\delta(n)^2} \left[ u_{(J-1)(n)} + u_{I(J+1)(n)} \right] - \]

\[B_y J(n+1) \frac{1}{\delta^2(n+1)} \left[ u_{I(J-1)(n+1)} + u_{(J+1)(n+1)} \right] + \]

\[K_0(J) \frac{1}{\delta^2(n+1)} \left[ - u_{I(J-1)(n+1)} + u_{(J+1)(n+1)} \right] = 0 \quad (A4.4)\]

Substituting for \(u_{I(J+1)(n)}\) in the above equation with that given by Eq. A4.3, and arranging to obtain:

\[u_{I(J-1)(n+1)} = \frac{B_y J(n) - \gamma B_y J(n+1) + K_0(J) \frac{\delta(n)}{2}}{B_y J(n) + \gamma B_y J(n+1) + K_0(J) \frac{\delta(n)}{2}} u_{(J+1)(n+1)} + \]

\[\frac{2B_y J(n)}{\gamma} \left[ B_y J(n) + \gamma B_y J(n+1) + K_0(J) \frac{\delta(n)}{2} \right] u_{(J-1)(n)} \quad (A4.5)\]

Equation A4.3 then becomes

\[u_{I(J+1)(n)} = \frac{2 \gamma^2 B_y J(n+1)}{B_y J(n) + \gamma B_y J(n+1) + K_0(J) \frac{\delta(n)}{2}} u_{(J+1)(n+1)} + \]

\[\frac{2 \gamma^2 B_y J(n)}{B_y J(n) + \gamma B_y J(n+1) + K_0(J) \frac{\delta(n)}{2}} \left[ B_y J(n) + \gamma B_y J(n+1) + K_0(J) \frac{\delta(n)}{2} \right] u_{(J-1)(n)} \quad (A4.6)\]
Twisting Relationships

The continuity and equilibrium equations are respectively:

\[ \beta' z = L(n) = \beta' z = 0(n+1) \quad (5.8) \]

\[ (C_w \beta''') z = L(n) - (C_w \beta''') z = 0(n+1) + K_w(J) \beta' z = 0(n+1) = 0 \quad (5.9) \]

The above equations are respectively similar to Eqs. 5.7 and 5.11 except that \( u \) is now \( \beta \) and \( B_y \) is \( C_w \). Thus it is not necessary to repeat solving the displacement relationships for the two imaginary points \( I(J+1)(n) \) and \( I(J-1)(n+1) \). All that is to be done is to replace Eqs. A4.5 and A4.6 with the appropriate terms. The twisting relationships for the imaginary points are thus:

\[ \beta_{I(J-1)(n+1)} = \frac{\left[ \frac{1}{C_w J(n)} - \gamma C_w J(n+1) + K_w(J) \frac{\delta(n)}{2} \right]}{\left[ C_w J(n) + \gamma C_w J(n+1) + K_w(J) \frac{\delta(n)}{2} \right]} \beta_{(J+1)(n+1)} \]

\[ + \frac{2 C_w J(n)}{C_w J(n) + \gamma C_w J(n+1) + K_w(J) \frac{\delta(n)}{2}} \beta_{(J-1)(n)} \quad (A4.7) \]

\[ \beta_{I(J+1)(n)} = \frac{\left[ \frac{2 \gamma^2 C_w J(n+1)}{C_w J(n) + \gamma C_w J(n+1) + K_w(J) \frac{\delta(n)}{2}} \right]}{\left[ \frac{-C_w J(n) + \gamma C_w J(n+1) + K_w(J) \frac{\delta(n)}{2}}{C_w J(n) + \gamma C_w J(n+1) + K_w(J) \frac{\delta(n)}{2}} \right]} \beta_{(J+1)(n+1)} \]

\[ + \frac{2 \gamma^2 C_w J(n+1)}{C_w J(n) + \gamma C_w J(n+1) + K_w(J) \frac{\delta(n)}{2}} \beta_{(J-1)(n)} \quad (A4.8) \]
9. **SYMBOLS**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>area of cross section;</td>
</tr>
<tr>
<td>a</td>
<td>distance between the shear center and any point in the section;</td>
</tr>
<tr>
<td>B&lt;sub&gt;x&lt;/sub&gt;</td>
<td>bending rigidity about x-axis;</td>
</tr>
<tr>
<td>B&lt;sub&gt;y&lt;/sub&gt;</td>
<td>bending rigidity about y axis (subscript i denotes pivotal point i);</td>
</tr>
<tr>
<td>b</td>
<td>width of flange;</td>
</tr>
<tr>
<td>C&lt;sub&gt;m&lt;/sub&gt;</td>
<td>end moment correct factor;</td>
</tr>
<tr>
<td>C&lt;sub&gt;T&lt;/sub&gt;</td>
<td>St. Venant constant (subscript i denotes pivotal point i);</td>
</tr>
<tr>
<td>C&lt;sub&gt;w&lt;/sub&gt;</td>
<td>warping rigidity (subscript i denotes pivotal point i);</td>
</tr>
<tr>
<td>d</td>
<td>depth of section;</td>
</tr>
<tr>
<td>d&lt;sub&gt;L&lt;/sub&gt;</td>
<td>distance of the lower flange from shear center (subscripts B and T denote section z = 0 and z = L respectively);</td>
</tr>
<tr>
<td>d&lt;sub&gt;U&lt;/sub&gt;</td>
<td>distance of the upper flange from shear center (subscripts B and T denote section z = 0 and z = L respectively);</td>
</tr>
<tr>
<td>d&lt;sub&gt;w&lt;/sub&gt;</td>
<td>depth of web;</td>
</tr>
<tr>
<td>E</td>
<td>elastic modulus;</td>
</tr>
<tr>
<td>En</td>
<td>end of an open cross-section;</td>
</tr>
<tr>
<td>F</td>
<td>force vector;</td>
</tr>
<tr>
<td>G</td>
<td>shearing modulus;</td>
</tr>
<tr>
<td>H</td>
<td>stiffness matrix;</td>
</tr>
<tr>
<td>I</td>
<td>moment of inertia (subscripts x, y, ξ, and η denote axes);</td>
</tr>
<tr>
<td>I&lt;sub&gt;w&lt;/sub&gt;</td>
<td>warping moment of inertia;</td>
</tr>
<tr>
<td>K</td>
<td>effective length factor;</td>
</tr>
<tr>
<td>K&lt;sub&gt;L&lt;/sub&gt;</td>
<td>warping restraint in the lower flange (subscripts B and T denote section z = 0 and z = L respectively);</td>
</tr>
</tbody>
</table>
$K_{oB}$ = weak-axis bending restraint at section $z = 0$;
$K_{oT}$ = weak-axis bending restraint at section $z = L$;
$K_{o(J)}$ = weak-axis bending restraint at joint $J$ in a continuous beam-column;
$K_T$ = torsion constant;
$K_U$ = warping restraint in the upper flange (subscripts $B$ and $T$ denote section $z = 0$ and $z = L$ respectively);
$K_{WB}$ = warping restraint at section $z = 0$;
$K_{WT}$ = warping restraint at section $z = L$;
$K_{W(J)}$ = warping restraint at joint $J$ in a continuous beam-column;
$L$ = length of a beam-column (subscript $n$ denotes span $n$);
$M$ = moment;
$M_B$ = bi-moment;
$M_{Bx}$ = strong axis bending moment at the bottom end of the column;
$M_{By}$ = weak-axis bending moment at the bottom end of the column;
$M_{cr}$ = critical moment;
$M_m$ = maximum strength of an unbraced beam-column;
$M_{max}$ = in-plane moment capacity;
$M_o$ = maximum bending moment a column can sustain in the absence of axial load;
$M_{pc}$ = plastic hinge moment modified to include the effect of axial compression;
$M_{sv}$ = St. Venant torsion;
$M_{Tx}$ = strong axis bending moment at the top end of the column;
$M_{Ty}$ = weak-axis bending moment at the top end of the column;
$M_w$ = warping torsion;
$M_x$ = internal resisting moment about x axis;
$M_y$ = internal resisting moment about y axis;
$M_z$ = internal twisting moment;
$M_{zo}$ = twisting moment at the column end;
\[ M_{\xi} = \text{internal torsional moment; } \]
\[ M_{\eta} = \text{internal moment about } \eta \text{ axis; } \]
\[ M_{\xi} = \text{internal moment about } \xi \text{ axis; } \]
\[ \text{o} = \text{beginning of an open cross-section; } \]
\[ P = \text{axial load; } \]
\[ P_e = \text{elastic buckling load in the plane of bending; } \]
\[ P_o = \text{axial load a column can support in the absence of bending moments; } \]
\[ P_y = \text{axial load corresponding to yield stress level, } P_y = A \sigma_y; \]
\[ q = \text{end moment ratio; } \]
\[ R = \text{tangent stiffness matrix; } \]
\[ r = \text{radius of gyration (subscripts } x \text{ and } y \text{ denote flexural axes); } \]
\[ T = \text{cosine transformation matrix; } \]
\[ T_0 = \text{cosine transformation matrix for initial imperfection functions; } \]
\[ t = \text{thickness of flange; } \]
\[ u = \text{displacement in } x \text{ direction (subscripts } U \text{ and } L \text{ denote upper and lower flange respectively; subscripts } B \text{ and } T \text{ denote section } z = 0 \text{ and } z = L \text{ respectively; subscript } J \text{ denotes joint } J); \]
\[ u_c = \text{lateral displacement of the centroid; } \]
\[ u_i = \text{initial imperfection in } x \text{ axis; } \]
\[ u_o = \text{initial imperfection in } x \text{ axis at mid-column; } \]
\[ u_{\xi} = \text{displacement in } \xi \text{ direction; } \]
\[ v = \text{displacement in } y \text{ direction; } \]
\[ v_c = \text{transverse displacement of the centroid; } \]
\[ v_i = \text{initial imperfection in } y \text{ axis; } \]
\[ v_o = \text{pre-buckling deformation in } y \text{ axis; } \]
\[ \text{initial imperfection in } y \text{ axis at mid-column; } \]
\( \nu \eta \) = displacement in \( \eta \) direction;  
\( W \) = external force vector;  
\( w \) = width of web;  
= axial deformation;  
\( x, y, z \) = coordinate system for a straight member in the undeformed state;  
\( x^o, y^o \) = coordinates of shear center;  
\( \alpha \) = yield length ratio (see Fig. 3.1);  
\( \alpha^o \) = 0 if \( \sigma = \sigma_y \), 1 if \( \sigma < \sigma_y \);  
\( \beta \) = twist about shear center of a section;  
\( \beta^i \) = initial twist;  
\( \beta^o \) = initial twist at mid-column;  
\( \gamma \) = yield length ratio (see Fig. 3.1);  
= ratio of segment length \( \frac{\delta(n)}{\delta(n+1)} \);  
\( \Delta \) = displacement vector;  
\( \delta \) = length of an arbitrary segment in a beam-column (subscript \( n \) denotes span \( n \), etc.);  
\( \Theta \) = rotation;  
\( \xi, \eta, \zeta \) = coordinate system for a member in the deformed state;  
\( \lambda \) = restraint factor (subscripts \( b \) and \( w \) denote weak-axis bending and warping respectively);  
\( \nu \) = yield length ratio (see Fig. 3.1);  
\( \rho^o \) = tangential distance from shear center;  
\( \rho_1, \rho_2 \) = lateral displacement coefficients;  
\( \rho_3, \rho_4 \) = stress;  
\( \sigma_{rc} \) = compressive residual stress in the tip of the flange of a \( W \) shape;  
\( \sigma_{rt} \) = tensile residual stress in the web of a \( W \) shape;  
\( \sigma_w \) = warping normal stress;
\[ \sigma_y = \text{static yield stress;} \]
\[ \tau_w = \text{shear stress due to warping of the section;} \]
\[ \varnothing = \text{curvature;} \]
\[ x_1, x_2 = \text{initial imperfection functions;} \]
\[ \gamma = \text{yield length ratio (see Fig. 3.1);} \]
\[ \Omega_1, \Omega_2 = \text{twisting displacement coefficients;} \]
\[ w_n = \text{normalized unit warping;} \text{ and} \]
\[ w_o = \text{unit warping with respect to shear center.} \]
10. TABLES
TABLE 3.1 LATERAL-TORSIONAL BUCKLING MODES FOR PINNED-END BEAM-COLUMNS OF 8/31 SHAPE SUBJECTED TO ONE END MOMENT

<table>
<thead>
<tr>
<th>MODE</th>
<th>$\frac{M}{M_{pc}}$</th>
<th>$L/r_x = 30$</th>
<th>$L/r_x = 50$</th>
<th>$L/r_x = 80$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$0.751$</td>
<td>$0.663$</td>
<td>$0.467$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$0.763$</td>
<td>$0.672$</td>
<td>$0.680$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$0.780$</td>
<td>$0.686$</td>
<td>$0.693$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$0.794$</td>
<td>$0.712$</td>
<td>$0.700$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$0.802$</td>
<td>$0.772$</td>
<td>$0.715$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$0.809$</td>
<td>$0.796$</td>
<td>$0.726$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$0.810$</td>
<td>$0.829$</td>
<td>$0.732$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$0.817$</td>
<td>$0.874$</td>
<td>$0.744$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$0.860$</td>
<td>$0.879$</td>
<td>$0.748$</td>
</tr>
</tbody>
</table>

FUKUMOTO'S $0.862$ $0.827$ $0.467$
<table>
<thead>
<tr>
<th>TYPE</th>
<th>JOINT</th>
<th>CRITICAL JOINT MOMENT (kip-in.)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>NO JOINT RERAINTS</td>
</tr>
<tr>
<td>A</td>
<td>UPPER</td>
<td>1018</td>
</tr>
<tr>
<td></td>
<td>LOWER</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>UPPER</td>
<td>885</td>
</tr>
<tr>
<td></td>
<td>LOWER</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>UPPER</td>
<td>990</td>
</tr>
<tr>
<td></td>
<td>LOWER</td>
<td>655</td>
</tr>
<tr>
<td>D</td>
<td>UPPER</td>
<td>755</td>
</tr>
<tr>
<td></td>
<td>LOWER</td>
<td>730</td>
</tr>
<tr>
<td>E</td>
<td>UPPER</td>
<td>800</td>
</tr>
<tr>
<td></td>
<td>LOWER</td>
<td></td>
</tr>
</tbody>
</table>
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\[ R_x = \frac{1}{L} (M_{By} + M_{Ty}) \]
\[ R_y = \frac{1}{L} (M_{Bx} + M_{Tx}) \]

\[ \bar{u} = u + x_0 - x_0 z = 0 \]

\[ \bar{v} = v + v_0 + y_0 - y_0 z = 0 \]

**Fig. 2.9 Twisting due to the End Shears**
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Fig. 2.12 Warping Restraints at Column Ends
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Fig. 2.16 Distorted Shapes of the Upper and Lower Flanges of a Beam-Column
PROGRAM 1

(1) Input: b, t, d, w, \( \sigma_y \), E, \( \frac{M_{tx}}{P_y} \), \( \frac{M_{tx}}{M_{tx}} \), \( \frac{\sigma_{pc}}{\sigma_{pc}} \), \( \frac{L}{r_x} \), \( \Delta \left( \frac{\theta}{\theta_{pc}} \right) \), \( \frac{\theta_{max}}{\theta_{pc}} \), \( \Delta \tau_0 \), \( \tau_{o_{max}} \), JTAN

\( \theta/\theta_{pc} \)

(2) M-\( \theta \)

\( B_y, C_T, C_w, y_o, \int_0^\sigma a^2 \, dA \)

\( \frac{\theta}{\theta_{pc}} \geq \frac{\theta_{max}}{\theta_{pc}} \)

\( \frac{\theta}{\theta_{pc}} = \frac{\theta}{\theta_{pc}} + \Delta \left( \frac{\theta}{\theta_{pc}} \right) \)

Yes

(3) Punch and print data of sectional properties computed in stage (2)

\( \tau_0 \)

(4) CDC

(5) M-\( \theta \), \( v_{o_{i}} \), \( M_{i} \)

(6) Punch and print data of M-\( \theta \), \( v_{o_{i}} \) and \( M_{i} \) computed in Stage (5)

\( \tau_0 \geq \tau_{o_{max}} \)

\( \tau_0 = \tau_0 + \Delta \tau_0 \)

Yes

END

Fig. 2.17 Flow-Chart for Computations of Sectional Properties and Pre-Buckling Deformation
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Fig. 3.3 $\frac{M}{M_y}$ Versus $\frac{C_w}{C_T}$ Curves
Fig. 3.4 \( \frac{M}{M_y} \) Versus \( \frac{y_0}{d} \) Curves

- Present Investigation
- Fukumoto\(^{(3.1)}\)
Fig. 3.5 $\frac{M}{M_y}$ Versus $1 - \frac{\int_A \sigma a^2 \, dA}{C_T}$ Curves
Fig. 3.6 Comparison with Galambos' Solutions
Fig. 3.7 Comparison with Fukumoto's Solution
Fig. 3.8 Effect of Pre-Buckling Deformation

- Pre-Buckling Deformation Neglected
- Pre-Buckling Deformation Considered

$8 WF=31$

$P/P_y = 0.4$
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Fig. 3.10 Comparison of Various Solutions
Fig. 3.11 Effect of Yield Stress Level on Lateral-Torsional Buckling
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Fig. 3.13 Lateral-Torsional Buckling Strength for 8WF31 Beam-Columns
Fig. 3.14 \( \frac{M_{cr}}{M_{max}} \) Vs. \( D_T \) Relationships

- Square Shape
- Rectangular Shape

\( P/P_y = 0.4 \quad L/r_y = 60 \)
Fig. 3.15 Comparison of Lateral-Torsional Buckling Curves for Pinned-End Beam-Columns
Fig. 3.16 Effect of Warping Restraints on Lateral-Torsional Buckling Strength

8 W = 3l
P/Pₙ = 0.4
No Bending Restraints
Fig. 3.17 Effect of Weak-Axis Bending Restraints on Lateral-Torsional Buckling Strength
Fig. 3118 Combined Effect of Weak-Axis Bending and Warping Restraints on Lateral-Torsional Buckling Strength
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Fig. 3.26 Test Results of "RC-10" Beam-Column

8WF 31
\( \sigma_y = 34.1 \text{ ksi} \)
\( P = 0.425 P_y \)
\( L/r_x = 60.5 \)

- In-Plane Strength
- LTB Upper Bound
- LTB Lower Bound
- Present Solution
- Test RC-10
Fig. 3.27 Test Result of "HT-10" Beam-Column (1.10)
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Reduced Modulus

\[ \frac{M}{M_y} \]

\[ \frac{y_0}{d} \]

Fig. 4.4 \( \frac{M}{M_y} \) VS. \( \frac{y_0}{d} \) Curves
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Fig. 4.6 Lateral-Torsional Buckling Strength Curves
Fig. 4.7 Lateral-Torsional Buckling Strength Curves
Fig. 4.8 Lateral-Torsional Buckling Strength Curves

- Reduced Modulus
- Tangent Modulus
- In-Plane Strength

8WF = 31'
P/P_y = 0.4
No Restraints
Fig. 4.9 Lateral-Torsional Buckling Strength Curves

\[
\frac{M}{M_{pc}} \quad \frac{L}{r_x}
\]

- In-Plane Strength
- Reduced Modulus
- Tangent Modulus

8WF 31
\[P/P_y = 0.4\]
No Restraints
Fig. 4.10  Lateral-Torsional Buckling Strength Curves
Fig. 4.11 Reduced Modulus Lateral-Torsional Buckling Strength for 8W31 Beam-Columns
Fig. 4.12 Effect of DT on Lateral-Torsional Buckling Strength
Fig. 4.13 Lateral-Torsional Buckling Strength Curves for Restrained Beam-Columns

- $P/P_y = 0.4$
- No Bending Restraints

- Tangent Modulus Solutions
- Reduced Modulus Solution

- $K_{WB} = K_{WT} = 0$
- $K_{WB} = K_{WT} = 5\lambda_w$
- $K_{WB} = K_{WT} = \infty$

In-Plane Strength

$8WF 31$

$L/r_x$ vs. $M/M_{pc}$
Fig. 4.14 Lateral-Torsional Buckling Strength Curves for Restrainted Beam-Columns

$P/P_y = 0.4$

- No Warping Restraints
- Tangent Modulus Solution
- Reduced Modulus Solutions

$K_{oB} = K_{oT} = 0$

$K_{oB} = K_{oT} = \lambda_b$

$K_{oB} = K_{oT} = \infty$

$L/r_x$

$M/M_{pc}$

Fig. 4.14 Lateral-Torsional Buckling Strength Curves for Restrained Beam-Columns
Fig. 4.15 Lateral-Torsional Buckling Strength Curves for Restrained Beam-Columns
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- Lateral-Torsional Buckling for the Continuous Beam-Column

Fig. 5.3 Lateral-Torsional Buckling in a Continuous Beam-Column (TYPE A)
Lateral-Torsional Buckling for Individual Beam-Columns

Lateral-Torsional Buckling for the Continuous Beam-Column

Fig. 5.4 Lateral-Torsional Buckling in a Continuous Beam-Column (TYPE A)
- Lateral-Torsional Buckling for Individual Beam-Columns

- Lateral-Torsional Buckling for the Continuous Beam-Column

Fig. 5.5 Lateral-Torsional Buckling in a Continuous Beam-Column (TYPE A)
Lateral-Torsional Buckling for Individual Beam-Columns

Lateral-Torsional Buckling for the Continuous Beam-Column

Fig. 5.6 Lateral-Torsional Buckling in a Continuous Beam-Column (TYPE A)
Lateral-Torsional Buckling for Individual Beam-Columns

Lateral-Torsional Buckling for the Continuous Beam-Column

Fig. 5.7 Lateral-Torsional Buckling in a Continuous Beam-Column (TYPE B)
Lateral-Torsional Buckling for Individual Beam-Columns

Lateral-Torsional Buckling for the Continuous Beam-Column

Fig. 5.8 Lateral-Torsional Buckling in a Continuous Beam-Column (TYPE B)
- Lateral-Torsional Buckling for Individual Beam-Columns

- Lateral-Torsional Buckling for the Continuous Beam-Column

**Fig. 5.9** Lateral-Torsional Buckling in a Continuous Beam-Column (TYPE B)
- Lateral-Torsional Buckling for Individual Beam-Columns

- Lateral-Torsional Buckling for the Continuous Beam-Column

Fig. 5.10 Lateral-Torsional Buckling in a Continuous Beam-Column (TYPE B)
Lateral-Torsional Buckling for Individual Beam-Columns

Lateral-Torsional Buckling for the Continuous Beam-Column

Fig. 5.11 Lateral-Torsional Buckling in a Continuous Beam-Column (TYPE C)
- Lateral-Torsional Buckling for Individual Beam-Columns
- Lateral-Torsional Buckling for the Continuous Beam-Column

Fig. 5.12 Lateral-Torsional Buckling in a Continuous Beam-Column (TYPE C)
- Lateral-Torsional Buckling for Individual Beam-Columns
- Lateral-Torsional Buckling for the Continuous Beam-Column

Fig. 5.13 Lateral-Torsional Buckling in a Continuous Beam-Column (TYPE C)
- Lateral-Torsional Buckling for Individual Beam-Columns

- Lateral-Torsional Buckling for the Continuous Beam-Column

**Fig. 5.14** Lateral-Torsional Buckling in a Continuous Beam-Column (TYPE C)
- Lateral - Torsional Buckling for Individual Beam - Column

- Lateral - Torsional Buckling for the Continuous Beam - Column

Fig. 5.15 Lateral-Torsional Buckling in a Continuous Beam-Column (TYPE D)
Lateral-Torsional Buckling for Individual Beam-Columns

Lateral-Torsional Buckling for the Continuous Beam-Column

Fig. 5.16 Lateral-Torsional Buckling in a Continuous Beam-Column (TYPE D)
- Lateral-Torsional Buckling for Individual Beam-Columns

- Lateral-Torsional Buckling for the Continuous Beam-Column

Fig. 5.17 Lateral-Torsional Buckling in a Continuous Beam-Column (TYPE D)
Lateral-Torsional Buckling for Individual Beam-Columns

Lateral-Torsional Buckling for the Continuous Beam-Column

Fig. 5.18 Lateral-Torsional Buckling in a Continuous Beam-Column (TYPE D)
- Lateral-Torsional Buckling for Individual Beam-Columns

- Lateral-Torsional Buckling for the Continuous Beam Column

Fig. 5.19 Lateral-Torsional Buckling in a Continuous Beam-Column (TYPE E)
Lateral-Torsional Buckling for Individual Beam-Columns

Lateral-Torsional Buckling for the Continuous Beam-Column

Fig. 5.20 Lateral-Torsional Buckling in a Continuous Beam-Column (TYPE E)
Fig. 5.21 Lateral-Torsional Buckling in a Continuous Beam-Column (TYPE E)
Lateral - Torsional Buckling for Individual Beam - Columns

Lateral - Torsional Buckling for the Continuous Beam - Column

Fig. 5.22 Lateral-Torsional Buckling in a Continuous Beam-Column (TYPE E)
Fig. 6.1 Schematic Diagram of Load-Deflection Relationships for Straight and Crooked Columns
Fig. 6.2 End Forces on a Crooked Beam-Column
Fig. 6.3 Extrapolation Technique to Determine the Maximum Strength of an Unbraced Beam-Column
Fig. 6.4 $\frac{M}{M_{pc}}$ vs. $L/r_x$ Curves for Pinned-End 8W31 Beam-Columns
In-Plane Strength

Maximum Strength

P/P_y = 0.4
No Restraints

Fig. 6.5 \(\frac{M}{M_{pc}}\) vs. \(L/r_x\) Curves for Pinned-End 14W142 Beam-Columns
Maximum Strength

Reduced Modulus

Tangent Modulus

Maximum Strength

\[
P/P_y = 0.4
\]

No End Restraints

\[
14\text{WF142 (}D_T=1580)\]

\[
8\text{WF-31 (}D_T=925)\]

\[
\frac{M}{M_{\max}} \text{ vs. } \frac{L}{r_x}
\]

Fig. 6.6 \(\frac{M}{M_{\max}}\) vs. \(\frac{L}{r_x}\) Curves for Pinned-End Beam-Columns
Fig. 6.7 Moment-rotation Relationships for Laterally Unbraced 8W31 Beam-Columns
Fig. 6.8 Moment-Rotation Relationships for Laterally Unbraced 14W142 Beam-Columns
Fig. 6.9 $M/M_{pc}$ vs. $L/r_x$ Curves for Restrained 8WF31 Beam-Columns
Fig. 6.10 Comparison of Interaction Curves for 8W31 Beam-Columns
Fig. 6.11 Comparison of Interaction Curves for 14W142 Beam-Columns
\textbf{Fig. 6.12} M-\(\theta\) Curves for Restrained 8WF31 Beam-Columns

\[ P/P_y = 0.4 \]

- In-Plane Behavior
- Out-of-Plane Deformations Permitted (Warping Fully Restrained at Column Ends)
- Predicted Lateral-Torsional Buckling

\[ \frac{M}{M_{pc}} \]

\[ \frac{L}{r_x} = 20 \]

\[ 8WF31 \]

\[ 40 \]

\[ 50 \]

\[ 60 \]

\[ \theta \text{ (RADIANS)} \]
14 W 142

$P/P_y = 0.4$

- In-Plane Behavior
- Out-of-Plane Deformations Permitted (Warping Fully restrained at Column Ends)

Predicted Lateral-Torsional Buckling

Fig. 6.13 M-θ Curves for Restrained 14W142 Beam-Columns
Fig. A1.1 Coordinates and Tangential Distance in an Open Cross Section
Fig. A3.1 Notations for Bi-Moment Relationships
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VITA

The author was born in Singapore on August 26, 1938, and is the second son of Mrs. Peng Whye Lim and the late Mr. Peng Whye Lim, of Singapore. He was educated at Raffles High School in Singapore from which he graduated with Class I Cambridge University Overseas School Certificate in 1957. He received his undergraduate training at the University of Sydney in New South Wales, Australia. He was conferred the degree of Bachelor of Engineering in Civil Engineering with Class II Honors in 1964.

Immediately after his graduation, the author was appointed Research Assistant in the School of Civil Engineering at University of Sydney. He resigned from that position in 1965 to enroll as a Master's student. He was reappointed in December 1965 to work in the School of Civil Engineering at the same University as Senior Research Assistant, the position he held until March 1966. He subsequently joined Taylor, Thomson, Whitting Consulting Engineers of St. Leonard, New South Wales, Australia, and served them as Design Engineer.

The author was awarded by the University of Sydney the degree of Master of Engineering Science in 1967. A paper entitled "The Behavior of Composite Steel and Lightweight Concrete Beams", detailing the thesis work has been presented and published in the Civil Engineering Transactions of the Institution of Engineers, Australia.
Prior to coming to Lehigh, the author had also served for two years as part-time Lecturer in the School of Civil Engineering at University of New South Wales, Kensington, New South Wales, Australia. He was also Teaching Assistant at the University of Sydney in 1965 while working for his Master's degree.

While at Lehigh University, the author has held the position of Research Assistant. He has served as Secretary on ASCE Subcommittee on Commentary Revision. He is an author and an editor of the second edition of ASCE Manual No. 41 "Plastic Design in Steel".

He is married to the former Molly Zee of Hong Kong.