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WELDED CONTINUOUS FRAMES AND THEIR COMPONENTS

INELASTIC BUCKLING OF STEEL FRAMES

By

Le-Wu Lu

FRITZ ENGINEERING LABORATORY

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It is well-known that a symmetrical frame carrying symmetrical load buckles into an asymmetrical configuration when the load has reached certain critical value. This phenomenon, often referred to as "Frame Instability", may occur at a load level below the yield (elastic buckling), but more frequently would take place when the applied load has caused yielding in some portion of the frame (inelastic buckling). In this paper a numerical method for the determination of the buckling strength of partially yielded frames is presented.

The proposed method is an adaptation of the modified moment distribution procedure developed previously for analyzing the elastic stability of plane frames. In the present method the stiffness and carry-over factors of the various members are modified for the combined influence of axial force and nonuniform yielding. The effects of initial residual stresses and the secondary bending moments resulting from deformations are included in the analysis. Two examples are given to illustrate the application of the method in constructing frame buckling curves.

Experiments on three sets of steel frames fabricated from a small wide-flange shape were conducted. Satisfactory agreement between the test results and the theoretical solution has been observed.

On the basis of the results of the theoretical and experimental studies presented herein, the validity of a currently used column design rule is discussed and a new method is suggested.
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1. INTRODUCTION

When a symmetrical frame, unrestrained from sidesway, is subjected to symmetrically applied loads, its deformation configuration will also be symmetrical as long as the loads are below a certain critical value. However, as the critical loading is reached, the frame may buckle suddenly into an antisymmetrical configuration, and consequently a large displacement develops in the lateral direction. At this instant the frame has lost completely its resistance to any imposed lateral force or deformation, and failure by buckling has thus terminated the load-carrying capacity.¹,²

In Fig. 1 schematic load-deflection curves of a portal frame corresponding to various modes of failure are shown. If the frame is prevented from swaying sidewise, the symmetrical deflection form shown in inset (a) will be maintained at all stages of loading. The ultimate load of the frame, \( w_u \), is reached when the bending moment at the top of columns has attained the limiting capacity. This load is indicated by point A on the load-deflection curve. The type of failure which is typical for braced frames will be referred to as "beam-column instability" in the subsequent discussions. In a previous paper a method has been developed for determining the ultimate strength for this type of failure.³

If no external bracing is provided for the frame, sidesway buckling (as shown in inset (b) of Fig. 1) may take place at point B (inelastic buckling) or at point C (elastic buckling), depending on the slenderness ratio of the columns. In the case of buckling below the yield limit, all the members are elastic, and the critical load can be readily determined by the methods developed by Masur, Chang and Donnell⁴ or using the solution presented by the author. For inelastic buckling, however, precise deter-
mination of the critical load becomes extremely laborious if not impossible. This is mainly due to the nonuniform yielding present in the various parts of the structure. The problem is further complicated in that the effect of residual stresses must also be considered.

It is the purpose of this paper to present an engineering solution for the inelastic sidesway buckling of single-story single-bay portal frames. The combined influence of nonuniform yielding and initial stresses are taken into account through a numerical procedure previously developed for analyzing braced frames.

1.1 PREVIOUS RESEARCH ON INELASTIC BUCKLING

Since inelastic buckling strength may be regarded as the limiting strength for unbraced frames, it is important to consider this type of failure in proportioning the columns in such structures, especially when the plastic method is adopted in the design. In recent years several attempts have been made to develop methods for estimating the inelastic buckling load of frames. Merchant in 1954 suggested that for practical calculations it might be reasonable to consider the inelastic buckling strength of an elastic-plastic structure as some function of the elastic buckling load and the simple plastic load. These loads represent two extreme idealizations of the carrying capacity of a steel structure. Bolton, Salem, and Low have tested several series of model steel frames to observe the magnitude of the frame buckling effect. Their results have shown some degree of correlation with the empirical approach proposed by Merchant. Wood in his studies on plastic instability of multi-story frames introduced the concept of "deteriorated critical load" as a theoretical test for the stability of partially yielded structures. In
a recent survey prepared by Horne the importance of considering the deformation effects in instability analysis was stressed. The author in an unpublished report has presented an analytical solution to the sidesway buckling of portal frames in the plastic range. This solution takes into account the influence of residual stresses and inelastic deformations. The present paper is a summary and an extension of the work contained in that report.

1.2 LIMITING STRENGTH OF FRAMES

In studying the buckling problems associated with frames, it is often convenient to present the solution in the form of a frame buckling curve. This curve gives the relationship between the height of the columns and the critical load of the structure. A typical plot of such a curve is shown in Fig. 2. In this plot all the frames are assumed to have a constant span length and acted upon by the same type of loading. The complete frame buckling curve consists essentially of two parts: (1) Portion AB defines the elastic buckling strength, and (2) Portion BC corresponds to buckling in the inelastic range. Point B marks the transition between elastic and inelastic buckling and represents the column height for which buckling and yielding occur simultaneously.

Also shown in Fig. 2 are two curves representing the strength of the frames when they are properly braced to prevent sidesway movement. Line EF gives the maximum load according to simple plastic theory. This theory assumes failure by symmetrical bending and ignores any reduction in the moment capacity of the columns due to axial load and due to the secondary moments in the columns resulting from their deformations. Curve DG represents the ultimate strength corresponding to failure by beam-column instability. This strength
can be determined by considering the reduction of plastic moment capacity at the top of the columns due to beam-column action. It is interesting to note in Fig. 2 that curve BC becomes coincident with curve DG after passing through point C. Therefore, for frames with height less than that indicated by point C, the reduction is strength due to sidesway buckling would be negligibly small.

1.3 SCOPE OF INVESTIGATION

This paper contains the results of an investigation of the following phases:

1. Theoretical development of a method for the determination of the inelastic buckling strength of partially yielded frames. The method will be explained with reference to the simple portal frame shown in Fig. 3. The frame carried simultaneously a uniformly distributed load of intensity \( w \) on the beam and two concentrated loads \( P \) applied along the centerlines of the columns. The load \( P \) is related to the uniform load by the parameter \( N \) in the form \( P = N\left(\frac{wL}{2}\right) \). To achieve proportional loading, \( N \) will be held constant. This condition of loading was originally suggested for investigation by Bleich \(^1\) and is intended to simulate approximately the axial loads and moments occurring in the lower stories of a multi-story building.

2. Experimental verification of the analytical solution,

3. Comparison of the theoretical and experimental results with an existing design rule, and


Throughout this investigation, the frames are assumed to be sufficiently braced in the perpendicular direction so that buckling can occur only in the plane of the applied loads.
2. DEVELOPMENT OF THE THEORY

2.1 ANALYSIS OF PARTIALLY YIELDED FRAMES

In order to obtain solutions to the inelastic buckling problem described above, it is first necessary to develop a method by means of which the distribution of bending moment and the variation of yielding in various members can be determined. Since the frame considered in this investigation is an indeterminate structure, statical conditions alone are not sufficient to obtain all the reaction components. An additional condition based on geometrical compatibility has to be incorporated in the analysis. For the frame shown in Fig. 3 a commonly used compatibility condition is that at joint B or D the slope of the beam should be equal to that of the column. With the aid of this condition the structure can then be analyzed by the classical slope-deflection method modified to take into account the effects of axial forces. This method of solution, however, is applicable only when the frame is loaded within the elastic range. If the applied load has exceeded the elastic limit, some portions of the structure are yielded; and the effect of yielding will have to be considered in the analysis. Obviously the analysis of a partially yielded structure is far more involved than that of an elastic structure.

In a recent paper Ojalvo and the author have presented a method for analyzing symmetrical frames stressed into the inelastic range. The method, as applied to the frame considered in this paper, may be summarized as follows: For a given load \( w \), construct the end-moment versus end-rotation curves for the beams and the columns (\( M_{BD} - \Theta_{BD} \) and \( M_{BA} - \Theta_{BA} \) curves), using the numerical integration procedure developed by von Kármán which is based on the moment-curvature relationship of the members. In computing these
curves it is possible to take into account not only the inelastic action, but also the effects of axial force and the deformation due to bending. When the resulting curves are plotted, the point of intersection gives the moment and rotation at joint B for the equilibrium configuration of the structure. By knowing the moment at the top of the column and the deformed shape of the frame, the distribution of moment can then be determined from statical conditions.

In the development of the method several assumptions were made in order to reduce the amount of numerical computations. The assumptions are:

(1) All members are prismatic,
(2) The shear force present at any section of the frames is small and its effect on yielding may be neglected,
(3) Only deformations (elastic or inelastic) due to bending are considered,
(4) The axial force in the beam is small compared with the thrust in the columns and its effect may be ignored,
(5) No lateral (sidesway) displacements at column ends are considered,
(6) No transverse load is applied to the column except at the ends, and
(7) The method is non-historic. It is necessary to specify that during loading there is no strain reversal of material stressed beyond the elastic limit.3

As will be seen in the later discussions, this method of elastic-plastic analysis leads to convenient ways of determining the stiffnesses of the beam and columns. By knowing the stiffnesses of the various members at all stages of loading, it is then possible to determine the buckling strength of the frame by any one of the existing techniques of buckling analysis. In the
present investigation, the moment distribution procedure due to Winter, Hsu, Koo and Loh\(^{15}\) is adopted because of its simplicity.

2.2 ASSUMPTIONS FOR BUCKLING ANALYSIS

Since the method of buckling analysis developed in this paper utilizes directly the results of elastic-plastic analysis as described above, it is also subject to the assumptions stated in Section 2.1. In addition, the following two assumptions are made:

1. The axial force in the beam is small and its effect on the bending stiffness may be ignored in the buckling analysis. The justification of this assumption has been discussed in a previous paper\(^5\) in connection with elastic buckling problems.

2. The frame deforms in a perfectly symmetrical form up to the instant of buckling. This implies that the method of inelastic analysis can be applied to determine the yield configuration at any load level below that which causes sidesway buckling.

2.3 STIFFNESS OF MEMBERS AFTER YIELDING

Analogous to the method commonly used in determining the inelastic buckling strength of centrally loaded columns, the procedure here developed also requires proper evaluation of the reduction of bending stiffness (buckling constant) of all the members due to yielding. By using these reduced stiffnesses in the analysis, the problem of inelastic buckling may be treated in a manner similar to that of the elastic case. Since the modified moment distribution method developed by Winter, et al.\(^{15}\) is adopted in the analysis, it is necessary to obtain the following buckling constants:
(1) For beams: stiffness factor $K_b$ (assuming far end fixed) and carry-over factor $C_b$.

(2) For columns: stiffness factor $K'_c$ (assuming far end hinged)

**Stiffness and Carry-Over Factors of the Beam**

For a given set of loads $w$ and $P$, the bending moment at $B$ (or $D$) is first determined by the method of elastic-plastic analysis. By knowing the two end moments, the moment diagram of the beam can be easily constructed by statics. Figure 4a shows a typical example of such a diagram. According to the elementary theory of strength of materials, the flexural behavior (in the elastic and inelastic range) of any section of the beam is governed completely by the moment-curvature relationship of the member. In the elastic range the slope of the moment-curvature diagram is constant and equal to the flexural rigidity of the section $EI_b$. When the applied moment exceeds the elastic limit, the slope (or rigidity) starts to decrease and approaches zero when the moment is near the plastic moment $M_p$. The effective flexural rigidity $(EI_b)_{eff}$ of the section can thus be determined as the instantaneous slope on the $M$-$\varphi$ diagram corresponding to the applied moment.

In this paper, the moment-curvature ($M$-$\varphi$) curve of a typical beam section 27 WF 94 as shown in Fig. 5a is adopted. The curve was constructed according to the procedure developed by Ketter, Kaminsky and Beedle and is based on an idealized elastic-fully plastic stress-strain relationship and a linearly varying symmetrical residual stress pattern with a maximum compressive residual stress at the flange tips equal to 0.3 times the yield stress of the material. It has been observed from the results of extensive computations, that the $M$-$\varphi$ curves, in their nondimensional form, are approximately the same for most of the WF sections that are commonly used as beams. Therefore the
M-θ curve constructed for this particular section, after being properly nondimensionalized, can be applied to other sections as well. The moment-curvature relationship given in Fig. 5a shows that the actual yield moment is only 70 percent of the nominal yield moment My. Thus, yielding occurs at sections where the moment has exceeded 0.7 My. This is indicated in Fig. 4b for the beam under consideration.

If no strain reversal is assumed to take place at the moment of buckling, the stiffness of the beam can be determined by considering a beam of variable EI. For the elastic part the flexural rigidity of the beam is EIb, while for the plastic part the effective flexural rigidity is reduced as if the yielded portions of the beam were removed. The effective flexural rigidity (EIb)eff, determined from the M-θ curve of Fig. 5a, is plotted as a function of the applied moment in Fig. 5b. Figure 6a shows a symbolic representation of a plasticified beam corresponding to the yield configuration indicated in Fig. 4b. It is called the "reduced beam" in this paper. The stiffness of this beam can be evaluated by the method of column analogy which is commonly used in indeterminate analysis. The analogous column of the reduced beam is shown in Fig. 6b. The width of the column at each section is inversely proportional to the flexural rigidity of that section. To determine the bending moment at B induced by a unit rotation at B, a unit load of one radian is then applied to the analogous column at the end B. The stiffness of the beam B is equivalent to the stress on the analogous column at that point, that is

\[ K_B = m_B = \frac{1}{A} + \frac{L}{I} \cdot \frac{L}{I} \]

(1)

in which A is the area of the analogous column and I is the moment of inertia
about the centroidal axis G-G. Similarly the moment at D is equal to:

\[ m_D = \frac{1}{A} \cdot \frac{1 \cdot \frac{L}{2} \cdot \frac{L}{2}}{I} \]  

(2)

The carry-over factor is simply the ratio of the moment at D to that at B, or

\[ c_b = \frac{m_D}{m_B} \]  

(3)

If it is known that both ends of the beam would rotate through the same angle and in the same direction at the instant of sidesway, the stiffness may be computed by using the analogous column shown in Fig. 7b. The centroidal axis G-G is now at the right end of the column and the area is assumed to be infinity. The stiffness \( K_b'' \) of the beam at the left end is

\[ K_b'' = \frac{1 \cdot \frac{L}{2} \cdot \frac{L}{2}}{I} \]  

(4)

in which \( I \) is the moment of inertia about axis G-G. The carry-over factor is not needed in this case.

**Stiffness of the Columns**

The stiffness factor \( K_c' \) of a column with hinged ends can be determined as the slope of the moment-rotation curve of a beam-column as shown in Fig. 8. Within the elastic range the stiffness is given by

\[ K_c' = \frac{\lambda^2 h^2}{1 - \lambda h \cot \lambda h} \frac{E I_c}{h} \]  

(5)

in which

\[ \lambda = \frac{F}{E I_c} \]
As the applied moment increases beyond the elastic limit, the stiffness of the column decreases and becomes zero when the moment reaches the maximum value. At this instant the column has lost completely its resistance to any further increase of bending moment. However, if a moment of opposite sense is applied, the column will behave elastically again and its stiffness is equal to that given by Eq. 5. This is indicated as "unloading" in Fig. 8.

The variation of the stiffness $K_c'$ with the applied end moment for wide-flange columns with slenderness ratios ranging from 40 to 120 and subjected to axial forces of 0.12, 0.2, 0.3 and 0.4 $P_y$ is shown in Fig. 9. The values were obtained by measuring slopes on the moment-rotation curves presented by Ojalvo and Fukumoto. The residual stress pattern used in the construction of these moment-rotation curves is the same as that previously adopted for analyzing the beam members. When the axial force $P$ and the end moment $M$ of a column are specified, its stiffness can be determined from Fig. 9 by interpolation.

2.4 METHOD OF SOLUTION

The proposed method of computing the inelastic buckling strength of frames may be summarized as follows:

(1) Perform a complete elastic-plastic analysis of the frame by assuming that no sidesway instability occurs at all stages of loading.

(2) Select a suitable load level $w_1$ and determine the moment at the column tops. The values of $K_b$, $C_b$ and $K_c'$ can then be obtained by the procedures described above.

(3) Introduce an arbitrary lateral (sidesway) displacement and perform a moment distribution computation for the frame, according to the procedure suggested by Winter, et al. The introduced fixed-end
moment can be taken to be proportional to the stiffness $K_c$ of each column. Using the end moment values resulting from the distribution process, the horizontal shear $Q$ of each column may be determined. The sum of these shears, $\Sigma Q$, should be positive if the selected load $w_1$ is below the critical value. This means that a lateral force is required to produce a sidesway displacement. In the moment distribution procedure, it is required that assumption (1) of Section 2.2 be valid. Thus, the stability of the frame may be examined by considering the simplified loading system shown in Fig. 10b. As explained in a previous report, the buckling load thus determined will be very close to the exact value. Although it is possible to obtain more precise results, with the same procedure, by taking the thrust in the horizontal beam into account (Fig. 10c), the work involved would be prohibitive.

(4) Repeat steps (2) and (3) for several values of $w$ that are in the range between the yield load and the ultimate load. By plotting the total shear $\Sigma Q$ against the load $w$ for each case, a curve such as that shown in Fig. 14 is obtained. The intersection of this curve with the load axis gives the critical load of the frame which will cause it to sway without the application of any lateral load.

In determining the stiffness of the members, the following assumptions are adopted with regard to unloading of the yielded portion:

(1) No strain reversal is assumed to take place for the plastic portion of the beam at the instant of sidesway buckling.
(2) For the case when the first plastic hinge forms at the center of the beam, no unloading of the columns is assumed. This is the situation that usually occurs for tall frames or frames with slender columns.

Both assumptions (1) and (2) are in agreement with the generally accepted concept of inelastic buckling due to Shanley. 18

(3) When no plastic hinge forms in the beam, one of the columns may be assumed to unload. This assumption was adopted in earlier investigations 19,20 and has been checked with experiments.
3. CONSTRUCTION OF A FRAME BUCKLING CURVE

The procedure outlined above will be illustrated by two complete examples in this section in connection with the development of a frame buckling curve. The dimensions and member size of the example frames are shown in Fig. 11. The span length L is arbitrarily chosen to be $80r_x$ (88.2 ft), in which $r_x$ is the radius of gyration about the strong axis of the 33 WF 130 section. A value of 2.0 is assigned for the loading parameter $N$. The cross sectional properties and the material constants adopted in the computations are as follows:

- $A = 38.26$ in$^2$
- $\bar{I}_x = 6699$ in$^4$
- $S_x = 404.8$ in$^3$
- $r_x = 13.23$ in
- $E = 30 \times 10^3$ ksi
- $\sigma_y = 33$ ksi
- $P_y = 1263$ kips
- $M_y = 1113$ ft-kips
- $M = 1282$ ft-kips

The ultimate load of the structures based on simple plastic theory (corresponding to a beam mechanism) is $P_p = 349$ kips or, equivalently, $w_p = 2.64$ kips per ft.

In the elastic range, the buckling load of the frames can be determined from the solution presented by the author in an earlier report. The results are plotted non-dimensionally as the dot-dashed line (curve AF) in Fig. 18. This curve is valid only for frames with slenderness ratios greater than that corresponding to point B shown on the curve. At this point the elastic buckling load is equal to the load which causes initial yielding at the most highly stressed section. For frames having column slenderness ratios less than that indicated by point B, inelastic buckling will govern their load-carrying capacity.
To obtain the buckling curve applicable in the inelastic range (curve BC in Fig. 18), it is necessary to determine the strength of several frames with various slenderness ratios. In this example two frames having \( h = 60 r_x \) (66.2 ft) and \( 80 r_x \) (88.2 ft) are chosen for illustration.

Case 1. - Frame with \( h = 60 r_x \)

The procedure presented in Section 2.4 is applied here to compute the inelastic buckling strength of this frame. The complete analysis consists of the following steps:

1) Perform a complete elastic-plastic analysis of the frame, using the method summarized in Section 2.1, to determine:

   a) The exact load-carrying capacity of the frame if sidesway buckling is prevented. The ultimate load thus obtained gives one point on curve DG of Fig. 18.

   b) The bending moment at joint B or D for any value of the applied load.

The resulting moment vs. rotation curve of joint B is shown as the solid line in Fig. 12. It is determined by combining three pairs of moment-rotation curves for the beam and the column, each pair being constructed for a given applied load. The values of \( w \) that are selected in constructing these pairs of curves are \( w = 1.14, 1.91 \) and 2.28 kips per ft. The corresponding axial thrusts in the columns are \( \bar{P} = 0.12, 0.20, \) and 0.24 \( P_y \). It can be seen in Fig. 12 that if the frame is prevented from sidesway the maximum attainable moment at the top of the columns is 0.926 \( M_y \). The ultimate strength of the frame determined by considering this reduction in moment
capacity is found to be $w_u = 2.38$ kips per ft. Comparison of this load with
the maximum load based on simple plastic load indicates a reduction of 9.8%
of the load-carrying capacity due to beam-column action.

2) Select a trial load $w_1 = 2.20$ kips per ft and compute the moment vs.
rotation curve for joint B. This curve intersects the moment-rotation curve
of the column at point $O_1$. The moment $M_B$ at the column top for this load is
equal to 0.830 $M_y$. The axial thrust in the columns is $P = \frac{3}{2} \times 2.20 \times 88.2 =
291$ kips, and therefore $P_y = 291/1263 = 0.230$. The stiffness of the columns
subjected to this combination of bending moment and axial force are found
from Fig. 9 to be

$$K_c' = 24.0 \ M_y \ \text{(loading)}$$
$$K_c^t = 48.9 \ M_y \ \text{(unloading)}$$

The computations involved in the evaluation of the stiffness factor
$K_b$ and carry-over $C_b$ of the beam are contained in the appendix. The values
of these factors thus obtained are:

$$K_b = 46.6 \ M_y$$
$$C_b = 0.7125$$

Since there is no plastic hinge forming at the center section of the
beam for this trial load (see the calculations contained in the appendix),
then according to assumption (3) of Section 2.4 one of the columns may be
assumed to unload in the buckling analysis. If the frame is assumed to
sway to the right, the left column will be the unloading column.

3) A fixed end moment due to a lateral displacement at the column tops
of $M_{FL} = 100$ ft-kips is arbitrarily assigned to act on the unloading column.
The resulting shear force may be computed from expressions derived by Winter. For the left column

\[ Q_L = \frac{1}{h} \left( \frac{M'_L - \frac{P}{K'_L}}{P} \times M_{FL} \times h \right) \]

\[ = \frac{1}{h} \left( 53.10 - \frac{291}{48.9 \times 1113} \times 100 \times 66.2 \right) = \frac{17.7}{h} \]  \hspace{1cm} (6)

and for the right column

\[ Q_R = \frac{1}{h} \left( \frac{M'_R - \frac{P}{K'_R}}{P} \times M_{FR} \times h \right) \]

\[ = \frac{1}{h} \left( 43.25 - \frac{291}{24.0 \times 1113} \times 49 \times 662 \right) = 7.92 \]  \hspace{1cm} (7)

The total shear force is

\[ \Sigma Q = \frac{25.62}{h} > 0 \]  \hspace{1cm} (8)

This shows that the frame is laterally stable at this trial load.

4) Select \( w_2 = 2.28 \) kips per ft as the second trial load and repeat Steps 2 and 3. The total resulting shear force for this trial load is \( \Sigma Q = 0.61/h \). This indicates that the selected load is very close to the true buckling load. In Fig. 14 the total shear \( \Sigma Q \) is plotted against the load \( w \) for these two trials, the critical load is determined as the intersection of this curve with the \( w \)-axis, that is \( w_{cr} = 2.283 \) kips per ft. The total load corresponding to this value of \( w \) is \( P_{cr} = 3/2 \times 2.283 \times 88.2 = 302 \) kips, therefore the ratio \( \frac{P_{cr}}{P_p} = 302/340 = 0.865 \). This furnishes one point on the inelastic buckling curve.
Case 2 - Frame with \( h = 80r_x \)

1) Carry out a complete elastic-plastic analysis in a manner similar to that described in Case 1. The resulting moment-rotation relationship of the column is shown in Fig. 15.

2) As a first try, a load of \( w_1 = 2.20 \) kips per ft is selected. The point of intersection of the moment-rotation curve of the beam for this value of \( w \) with that of the column is marked as \( O_1 \) in the figure. The corresponding moment at joint \( B \) is \( 0.770 \text{ M}_y \). The axial force in the column is \( P = 291 \) kips and the ratio \( P/P_y = 0.230 \). For this combination of axial force and end moment, the column stiffness is

\[
K'_c = 21.0 \text{ M}_y \text{ (loading)}
\]

It may be seen from Fig. 15 that the bending moment at the center of the beam is equal to \( M_p \) for this trial load. Then according to assumption (2) of Section 2.4, neither of the columns should be assumed to unload. Therefore, the beam will be bent in an antisymmetrical form at the instant of buckling. The stiffness factor of the beam may be determined by the simplified procedure shown in Fig. 7. Numerical computations involved are similar to those of the first case. They are also included in the appendix. The stiffness of the beam thus determined is \( K'_b = 70.9 \text{ M}_y \).

3) Introduce a fixed end moment due to lateral displacement of \( M_F = 100 \) ft-kips for each column. These moments can be distributed and balanced in one cycle as indicated on Fig. 16. The resulting shear force is:

\[
\sum Q = \frac{2}{h} \left( M'_L - \frac{P}{K'_L} \times M_{FL} \times h \right)
\]

\[
= \frac{2}{h} \left( 77.15 - \frac{291}{21 \times 1113} \times 100 \times 88.2 \right) = - \frac{65.3}{h}
\]

(9)
This indicates that the trial load is higher than the critical load and that a smaller value of \( w \) should be assumed for the next try.

4) Use \( w = 2.06 \) kips per ft as the second trial load and repeat Steps 2 and 3. The resulting shear force at the column tops is \( \sum Q = -4.90/h \), indicating that the selected \( w \) is still higher than the actual critical load. By using these results, the inelastic buckling load of the frame can be determined graphically as shown in Fig. 17. The value of \( w_{cr} \) is equal to 2.05 kips per ft and the total load \( P = 271 \) kips, thus the ratio \( \frac{P_{cr}}{P} = \frac{271}{349} = 0.776 \). This gives another point on the inelastic buckling curve shown in Fig. 18.

Similar analyses may be performed for frames with different values of \( h/r \). These analyses will result in a series of points in Fig. 18, each of which gives the buckling load of a particular frame. By passing a curve from point B through these points an inelastic buckling curve is obtained. At point C this curve becomes tangent to curve DG which defines the strength of the frames if they are braced to prevent buckling. For any frame with a slenderness ratio less than that corresponding to point C, its load-carrying capacity will not be affected by lateral instability. Therefore within this region the problem of frame stability may be safely ignored and the design can be based on the plastic strength.
In the course of this investigation experiments on model steel frames were conducted to check the validity of the proposed theory. The test program included three sets of welded rectangular frames fabricated from a small wide-flange shape (2 5/8 WF 3.725). Figure 19 shows the dimensions of the test frames and the section properties of the WF shape. The span length L was kept constant for all the frames and was equal to 80 times the radius of gyration \( r_x \) of the section; and the heights \( h \) of the three frames were so chosen that the corresponding slenderness ratios of the columns were equal to 40, 60 and 80. The three sets of test frames are designated as W-1, W-2, and W-3, in order of their column height in Fig. 19.

Since the members of the frames were subjected to bending moments about their major axis, it was necessary to brace the frames in the direction perpendicular to the plane of loading. Past experience in conducting frame buckling tests had indicated that the bracing should be attached in such a manner that no sidesway restraint is offered to the structures at the initiation of buckling. For this reason it was decided to use a two-frame system in all the tests. For each test, two identical frames were fabricated and purlins and cross braces were attached between them to act as the bracing members. The purlins were spaced at a distance equal to \( 45r_y \), where \( r_y \) is the radius of gyration about the minor axis of the WF section. This spacing was based on the recommendation made by Lee and Galambos.

The uniform beam load \( w \) assumed in the theoretical development was replaced by three concentrated loads \( P \) applied as shown in Fig. 19. It was observed that the distribution of bending moment around the frames due to
these concentrated loads is approximately the same as that produced by the uniform load. All the loads were applied to the frames by dead weights magnified by five lever systems. Figure 20 shows a general view of the test setup and the fixtures used for transmitting the loads to their points of application. All the levers and loading fixtures were so arranged that they could sway freely with the frames at any stage of test. The loads were applied in successive increments, and the deflections of the beam and columns were measured after each load application. The increment of load was gradually reduced as the applied load neared the predicted load. Figure 21 shows the deformed shape of the test frame W-3 after all the loads were removed. Typical sidesway buckling may be seen. Details of the test procedure and the experimental techniques employed can be found in a separate report.  

Information pertaining to these model experiments, including the frame dimensions, loading parameter $N$, theoretical predictions and the test results, is summarized in Table 1. The "test load" reported in the table is not the buckling load, but the maximum load observed from each test. Due to the unavoidable imperfection of the test specimens, it was difficult to detect precisely when the test frames started to buckle. However, in general, little increase in load can be expected after the initiation of sidesway movement, so the ultimate load observed from the tests should be very close to the actual buckling load.

It may be seen from the comparisons given in the last column of Table 1 that satisfactory correlation between the theory and the experiments has been obtained. For frames W-1 and W-3, the observed loads are a few percent higher than the predictions, while the experimental and predicted loads are
approximately equal for frame W-2. The average discrepancy of the three tests is about 3.6%. This shows that the procedure developed in this paper is capable of predicting the inelastic buckling strength of frames with a reasonable degree of accuracy.
5. COLUMN DESIGN IN UNBRACED FRAMES

As pointed out earlier in the Introduction, it is important to consider the possibility of overall buckling in proportioning columns in building frames which are not braced to prevent sidesway. This has been recognized by the 1961 Specification of the American Institute of Steel Construction. In allowable-stress design, the Specification requires that the compression members in unbraced frames be designed for their "effective lengths" corresponding to the sidesway buckling mode. (See Section 1.8 of the Specification). Various types of design charts, tables and approximate formulas are available for estimating the effective length of the columns in a variety of frames. A convenient alignment chart, recommended for use by the Column Research Council, is included in the Commentary on the Specification. The chart also provides a rapid means of computing the approximate elastic buckling load of structural frames.

When the plastic method is used in the design of an unbraced frame, the usual approach is to proportion the members first on the basis of their plastic strength (this yields the member sizes for the trial frame), and then modify the columns to take account of the possible reduction in strength due to instability. The second step requires a close estimate of the inelastic buckling strength of the trial frame. As seen in the previous discussion, the procedure for determining the inelastic buckling strength is usually very tedious. It would be impractical to perform such an analysis in an actual design. For this reason, the AISC Specification places limitations on the slenderness ratio and the intensity of axial thrust in columns to safeguard against possible failure due to buckling. It is understood that if the
columns in an unbraced frame are designed to meet the specified limits, the reduction in load-carrying capacity due to sidesway buckling would be very small and can be ignored for practical purposes. In the following the validity of the design limitations will be discussed in the light of the results obtained from this investigation.

5.1 COMPARISON OF RESULTS WITH THE AISC DESIGN RULE

In designing an unbraced frame by the plastic method, the Specification stipulates that its columns be proportioned to satisfy the following rule

\[
\frac{2P}{P_y} + \frac{h}{70r_x} = 1.0
\]

in which \( P \) is the axial force in the column when the frame carries its computed ultimate load; it is equal to \( P_u \) for the symmetrical frame considered in this study. This rule is applicable to columns in continuous frames, where sidesway is not prevented 1) by diagonal bracing, 2) by attachment to an adjacent structure having ample lateral stability or 3) by floor slabs or roof decks secured horizontally by walls or bracing systems parallel to the plane of the frames.

The above rule was derived from an approximate solution of the inelastic frame buckling problem. The analytical procedure presented herein has been used to check the validity of this rule. Two groups of portal frames were selected for this check. In all the frames, the beam and the columns were assumed to be of the same size. The frames in the first group have a constant span length of \( L = 70 \) ft and variable heights ranging from 0.2L to 1.2L. Frame buckling curves, similar to the one shown in Fig. 18, were constructed for these frames. The structural shapes used in the computations
were: 33 WF 130, 27 WF 102, 21 WF 73 and 18 I 54.7. For the frames in the second group a span length of 90 ft was chosen, and computations were carried out for two shapes: 36 WF 260 and 33 WF 130.

A loading parameter of \( N = 2.0 \) was used for all the frames. (See Fig. 3) This choice of \( N \) would produce approximately the loading condition that may occur in the bottom story of a three-story building. When applied to a two-story frame, which is permitted to be designed plastically by the Specification, the value \( N = 2.0 \) provides for additional sources of column loads such as cranes, pipe supports and miscellaneous loading.

Figure 22 shows a comparison between the theoretically computed sidesway buckling loads and the design rule. Also shown are the results obtained from the model frame experiments previously described. All the buckling loads are expressed as percentages of the ultimate load \( P_u \), which was determined by assuming that the frames were restrained from sidesway and that failure was due to the instability of the column members. The line defined by the rule therefore should represent the limit within which the sidesway buckling load should be equal to 100% of the computed ultimate load. It may be seen from Fig. 22 that, within the range of \( \bar{P}/P_y \) and \( h/r_x \) that have been covered by the results presented in this study, the rule is somewhat conservative. The buckling strengths of some frames having combinations of \( \bar{P}/P_y \) and \( h/r_x \) considerably outside of the safe region defined by the rule are more than 95% of the ultimate strength. This indicates that some liberalization of the limitations on slenderness ratio and axial force may be possible.
5.2 DEVELOPMENT OF A NEW DESIGN APPROXIMATION

The above discussion has shown that the AISC rule is adequate for use in proportioning columns to avoid possible reduction in strength due to sidesway buckling. It is felt, however, that a different type of design approximation, developed by considering the overall behavior of frames, may prove more useful in future applications to multi-story buildings. Furthermore, as will be seen in the subsequent discussion, the proposed design approximation leads to a convenient way of estimating the inelastic buckling strength; and thus makes possible the use of this strength as a basis of designing unbraced frames.

In studying the stability of a centrally-loaded column, it is sometimes useful to consider inelastic buckling as a type of failure resulting from the combined effects of yielding and buckling. Therefore the inelastic buckling strength may be expressed as a function of the axial yield load of the column section and the Euler load. Following the same reasoning, Merchant suggested that for practical purposes the inelastic buckling load of a structure may also be empirically expressed as a function of the ultimate load according to the plastic theory and the elastic buckling load. This function depends on the type of structure and on the loading condition under consideration. A great deal of experimental work has been conducted on various types of model frames in an attempt to establish some simple relationships that are useful in practical design. Unfortunately the results obtained so far are rather inconclusive. A similar attempt is made here using the data given in Fig. 22.

A nondimensional plot of the inelastic buckling loads of the frames described in the previous section is shown in Fig. 23. Two independent
parameters are used to nondimensionalize the inelastic buckling loads \( \overline{P}_{cr} \); namely the plastic failure load \( \overline{P}_u \) and the elastic buckling load \( \overline{P}_{cr} \).

It should be pointed out that the load \( \overline{P}_u \) used here is the ultimate load corresponding to failure by beam-column instability, and is not the simple plastic load \( \overline{P}_p \) as was originally suggested by Merchant. A wider scatter of the points had been observed when \( \overline{P}_p \) was used in the plot.

The following may be observed from Fig. 23:

1. For frames with \( \frac{\overline{P}_{cr}}{\overline{P}_u} \) less than about 0.4, the inelastic buckling load may be expected to be close to the ultimate load \( \overline{P}_u \). It is therefore not necessary to consider sideways buckling in the design of such frames.

2. Frames with \( \frac{\overline{P}_{cr}}{\overline{P}_u} \leq 0.8 \) are likely to buckle in the elastic range; and hence their design should be based on a limiting allowable stress.

3. A straight line passing through the points \( G_1 \) (0.4, 1.0) and \( G_2 \) (1.0, 0.8) can be used to approximate the limiting strength of the frames which may fail by inelastic buckling. This straight line is given by

\[
\frac{\overline{P}_{cr}}{\overline{P}_u} + 3 \frac{\overline{P}_{cr}}{\overline{P}_u} = 3.4
\]  

(11)

which is applicable in the region where \( \frac{\overline{P}_{cr}}{\overline{P}_u} \geq 0.4 \) and

\[
\frac{\overline{P}_{cr}}{\overline{P}_u} \geq 0.8.
\]
The above observations were made on the basis of the theoretical and experimental results obtained for the simple frame considered in this study and for a constant loading parameter \( N = 2.0 \). Obviously, for other structures and loading conditions, design approximations different from that given by Eq. (11) may have to be developed. It is believed, however, that the proposed approximation shown in Fig. 23 can be applied to the various types of frames which are permitted to be designed plastically by the present Specification.

The procedure of using the new design approximation is as follows:

1. Perform a preliminary design using simple plastic strength as the design basis.

2. Revise the column sizes to allow for the reduction in strength due to beam-column action. This can be done conveniently by using the beam-column tables contained in the AISC Specification. The ultimate load \( P_u \) of the trial frame is thus equal to the design ultimate load (design load multiplied by a load factor).

3. Compute the elastic buckling load of the trial frame \( (P_{cr})_e \).

   For this purpose the alignment charts contained in the Commentary on the AISC Specification may be used.

4. Determine the ratio \( \frac{(P_{cr})_1}{(P_{cr})_e} \) from the expression

   \[
   \frac{(P_{cr})_1}{(P_{cr})_e} = \frac{3.4}{1 + 3 \frac{(P_{cr})_e}{P_u}} \tag{12}
   \]
If this ratio is less than 0.4, then the member sizes chosen for the trial frame are satisfactory and no further revision of the design will be needed.

5. If the ratio is greater than 0.4, the carrying capacity of the trial frame would be affected by overall buckling and will generally be somewhat less than the design ultimate load. This indicates that a slight increase in the size of the columns (or the beams) of the trial frame is necessary.

6. Repeat Steps 3 and 4 for the second trial frame. The selected member sizes will be satisfactory if the inelastic buckling strength \( (\bar{P}_{cr})_i \) of the frame is equal to or greater than the design ultimate load.

The procedure outlined above has been used in the design of several unbraced building frames. It was found that in most cases no more than two trials are needed in each design.
6. SUMMARY AND CONCLUSIONS

This paper presents a numerical method for the determination of the sidesway buckling strength of partially yielded steel frames. The method has been illustrated with reference to the simple rectangular frame shown in Fig. 3. Extension of the procedure to other types of frames is possible so long as the applied loads and the frame geometry are symmetrical and if failure is characterized by sidesway buckling in the plane of the frames. It is believed that the proposed method may also be adapted to the study of the dynamic response of elastic-plastic structures, when the effect of axial force is of considerable importance.

The inelastic buckling strength of a given frame subjected to a specified system of loads can be determined in the following manner:

(1) A complete elastic-plastic analysis of the frame is made using the graphical method developed in a previous paper. In using this method, it is possible to take into account such effects as axial force, residual stress, and bending moment resulting from elastic as well as inelastic deformations. This analysis assumed that no sidesway buckling occurs at any stage of loading and, therefore, the symmetrical deformation configuration is maintained.

(2) A trial load which is higher than the load causing initial yielding in the frame is selected. Corresponding to this load, the stiffness and carry-over factors of the beam can be determined by applying Eqs. 1, 3 or 4, and the stiffness for the column is obtained from charts given in Fig. 9.
(3) A stability check is made for the frame carrying the trial load.

This is conveniently done by applying the modified moment distribution procedure developed by Winter, et al. A specified amount of lateral displacement is introduced to the frame and the resulting moment of the column top is determined by a moment distribution computation using the stiffness factors obtained previously. Dividing this moment by the column height gives the horizontal shear of each column. If the sum of the shears is positive then the trial load is less than the buckling load.

(4) A higher load is chosen for the next trial and the above steps are repeated. The critical condition is reached when the sum of the resulting shears becomes zero. The load at which this occurs determines the inelastic buckling strength of the frame.

The proposed method has been applied to determine the buckling strength of the frame shown in Fig. 11 and the results are presented in the form of a frame buckling curve in Fig. 18.

Experiments were conducted on three sets of model steel frames fabricated from a small wide-flange shape. The dimensions of the frames are given in Fig. 19 and the test setup is shown in Figs. 20 and 21. It has been noted from the comparison given in Table 1 that the average discrepancy between the theory and the experiments is about 3.6%.

Extensive calculations of the buckling strength of six series of portal frames were made by using the procedure presented herein. Figure 22 shows a comparison of the results with the column design rule (Formula (20))
contained in the 1961 AISC Specification. It has been observed that, within
the range of the two variables \( \frac{F}{P_y} \) and \( \frac{h}{r_x} \) that were covered by the calcu-
lations, the rule is somewhat conservative.

A new design approximation taking into account the overall behavior
of unbraced frames has been developed. It is based on the interaction
relation shown in Fig. 23 and expressed mathematically by Eq. 11. This
method would lead to a more rational design of the columns, especially
when the structure is designed to support heavy gravity loads.
This paper is based on part of a thesis submitted to the Graduate Faculty of Lehigh University in partial fulfillment of the requirements for the Degree of Doctor of Philosophy in Civil Engineering. The thesis was written under the supervision of Professor George C. Driscoll, Jr.

The study leading to this paper is part of the general investigations "WELDED CONTINUOUS FRAMES AND THEIR COMPONENTS" being carried out at the Fritz Engineering Laboratory, Lehigh University, under the general direction of Professor Lynn S. Beedle. The investigation is sponsored jointly by the Welding Research Council and the Department of the Navy, with funds furnished by the American Institute of Steel Construction, American Iron and Steel Institute, Office of Naval Research, Bureau of Ships, and Bureau of Yards and Docks. Technical guidance for the project is provided by the Lehigh Project Subcommittee of the Structural Steel Committee of the Welding Research Council. Dr. T. R. Higgins is Chairman of the Lehigh Project Subcommittee.

The author wishes to express his appreciation to Dr. Morris Ojalvo for fruitful discussions and to Mr. Yu-Chin Yen for his effort in conducting the experiments.
8. TABLES AND FIGURES
### TABLE I COMPARISON OF TEST RESULTS WITH THEORETICAL PREDICTIONS

<table>
<thead>
<tr>
<th>FRAME NO.</th>
<th>COLUMN HEIGHT h(in)</th>
<th>SLENDERNESS RATIO h/rx</th>
<th>LOADING PARAMETER N</th>
<th>SIMPLE PLASTIC LOAD Pp (kips)</th>
<th>BEAM-COLUMN INSTABILITY LOAD Pu (kips)</th>
<th>PREDICTED BUCKLING LOAD Pc (kips)</th>
<th>TEST LOAD Pexp (kips)</th>
<th>P_exp/P_cr</th>
</tr>
</thead>
<tbody>
<tr>
<td>W-1</td>
<td>43.8</td>
<td>40</td>
<td>2.0</td>
<td>12.43</td>
<td>11.53</td>
<td>0.928</td>
<td>10.65</td>
<td>0.857</td>
</tr>
<tr>
<td>W-2</td>
<td>65.7</td>
<td>60</td>
<td>2.0</td>
<td>12.43</td>
<td>11.47</td>
<td>0.923</td>
<td>10.18</td>
<td>0.819</td>
</tr>
<tr>
<td>W-3</td>
<td>87.6</td>
<td>80</td>
<td>1.8</td>
<td>11.43</td>
<td>10.44</td>
<td>0.913</td>
<td>8.61</td>
<td>0.753</td>
</tr>
</tbody>
</table>

Ave. = 1.036
FIG. 1 ILLUSTRATION OF FRAME BUCKLING

FIG. 2 LOAD CARRYING CAPACITY OF FRAMES
\[ P = N \left( \frac{wL}{2} \right), \quad \overline{P} = (1 + N) \frac{wL}{2} \]

**FIG. 3 FRAME DIMENSIONS AND LOADING**

**FIG. 4 MOMENT DIAGRAM AND VARIATION OF YIELDING OF THE BEAM**
FIG. 5

FIG. 6 DETERMINATION OF $K_b$ AND $C_b$ BY COLUMN ANALOGY
Analogous Column

A = \infty

I_{GG} is finite

(a) \[ \frac{L}{2} \quad \frac{L}{2} \theta \]

(b) \[ \text{Figure 7: Determination of } K_b \text{ for member with equal end rotation} \]

(c) \[ 1 \text{ Rad.} \quad \text{Load on Column} \]

FIG. 7 DETERMINATION OF $K_b$ FOR MEMBER WITH EQUAL END ROTATION

FIG. 8 STIFFNESS OF A BEAM-COLUMN WITH CONSTANT AXIAL FORCE
FIG. 9 VARIATIONS OF COLUMN STIFFNESS FACTOR $K'_c$ WITH APPLIED MOMENT (STRONG AXIS BENDING)
Loading Condition for Determining Buckling Constants

\[ P = (1 + N) \frac{WL}{2} \]

(a)

Loading Condition for Analyzing Buckling Load

(b)

(c)

FIG. 10 SIMPLIFICATION OF LOADING CONDITION

\[ P = 2 \left( \frac{WL}{2} \right) , \quad \bar{P} = 3 \left( \frac{WL}{2} \right) \]

FIG. 11 ILLUSTRATIVE EXAMPLE
FIG. 12 ELASTIC-PLASTIC ANALYSIS OF FRAME WITH $h = 60r_x$
<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>-48.80</td>
<td>-32.34</td>
<td>-34.77</td>
<td>+8.01</td>
</tr>
<tr>
<td>-23.04</td>
<td>+11.24</td>
<td>+22.95</td>
<td>-5.29</td>
</tr>
<tr>
<td>+16.35</td>
<td>-7.98</td>
<td>-5.69</td>
<td>+3.76</td>
</tr>
<tr>
<td>-3.77</td>
<td>+1.84</td>
<td>+1.31</td>
<td>-0.86</td>
</tr>
<tr>
<td>+2.68</td>
<td>-1.31</td>
<td>-0.93</td>
<td>+0.61</td>
</tr>
<tr>
<td>-0.61</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>+0.30</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ C_b = 0.7125 \]

\[ M_{FL} = +100.00 \]
\[ +51.20 \]
\[ +11.80 \]
\[ -8.37 \]
\[ +1.83 \]
\[ +0.37 \]
\[ M_L' = +53.10 \]

\[ M_{FR} = 0.488 \]
\[ 0.660 \]
\[ 0.340 \]

\[ 0.512 \]

\[ M_R = +43.25 \]

\[ 0.526 \]

**FIG. 13**

**FIG. 14** DETERMINATION OF CRITICAL LOAD \( (h=60r_h) \)
FIG. 15 ELASTIC-PLASTIC ANALYSIS OF FRAME WITH $h = 80r_x$
FIG. 16

\[ M_{FL} = +100.00 \]
\[ -22.85 \]
\[ M_L' = +77.85 \]

\[ M_{FR} = +100.00 \]
\[ -22.85 \]
\[ M_R = +77.85 \]

\[ \Sigma Q \times h \]

FIG. 17 DETERMINATION OF CRITICAL LOAD \((h = 80r_x)\)
Inelastic Buckling

Simple Plastic Load

Beam-Column Instability

Elastic Buckling

FIG. 18 ILLUSTRATION OF FRAME BUCKLING CURVE

\[ P = N \left( \frac{wL}{2} \right) \]

\[ \bar{P} = (1 + N) \frac{wL}{2} \]

\[ N = 2.0 \]

Slenderness Ratio of Column \( \frac{h}{r_x} \)
\[ P = N \left( \frac{2}{3} P_1 \right) \]
\[ \bar{P} = (1+N) \frac{2}{3} P_1 \]

Dimensions of the Test Frames

\[ h = 40r_x(W-1) \]
\[ 60r_x(W-2) \]
\[ 80r_x(W-3) \]
\[ L = 80r_x = 87.6'' \]

Cross-Sectional Properties

- \( A = 1.043 \text{ in}^2 \)
- \( I_x = 1.251 \text{ in}^4 \)
- \( S_x = 0.953 \text{ in}^3 \)
- \( r_x = 1.095 \text{ in} \)
- \( Z_x = 1.067 \text{ in}^3 \)
- \( r_x = 0.421 \text{ in} \)

FIG. 19
FIG. 20 TESTING OF FRAME W-3

FIG. 21 FRAME W-3 AFTER TESTING
FIG. 22 THEORETICAL AND EXPERIMENTAL RESULTS COMPARED WITH THE AISC RULE
FIG. 23 PROPOSED DESIGN APPROXIMATION

\[
\frac{(P_{cr})_i}{P_u}
\]

\[
G_1(0.4,1.0)
\]

\[
G_2(1.0,0.8)
\]

L = 70'
- 33WF130
- 27WF102
- 21WF73
- 18 I 54.7

L = 90'
- 36WF260
- 33WF130

Model Frame Tests
9. **APPENDIX**

**DETERMINATION OF STIFFNESS OF PARTIALLY PLASTIC BEAMS**

**BY COLUMN ANALOGY**

This appendix contains the numerical computations involved in determining the stiffnesses and carry-over factors for the beams of the two frames considered in Art. 3. The dimensions of these frames and the applied loads (selected trial loads) are as follows:

<table>
<thead>
<tr>
<th>Member size</th>
<th>Case 1</th>
<th>Case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>33 WF 130</td>
<td>33 WF 130</td>
<td></td>
</tr>
<tr>
<td>Radius of gyration, $r_x$</td>
<td>13.23 in.</td>
<td>13.23 in.</td>
</tr>
<tr>
<td>Span length, $L$</td>
<td>80 $r_x$</td>
<td>80 $r_x$</td>
</tr>
<tr>
<td>Column height, $h$</td>
<td>60 $r_x$</td>
<td>80 $r_x$</td>
</tr>
<tr>
<td>Distributed load, $w$</td>
<td>2.20 kips/ft.</td>
<td>2.20 kips/ft.</td>
</tr>
</tbody>
</table>

For each case the moments at the ends of the beam are first determined by the graphical method of elastic-plastic analysis. The moment at all the sections of the beam can then be computed by statics. From the computed moment values the effective flexural rigidity of all the sections can be determined from the flexural rigidity-moment relationship of the 33 WF 130 section (similar to that shown in Fig. 5). Thus, in effect, a beam of variable EI is obtained. In the buckling analysis, it is required to evaluate the stiffness factor and carry-over factor of this beam. This can be done conveniently by the method of column analogy. The procedure of applying this method was discussed in Section 2.3. Detailed computations for the beams of the frames considered here will be explained below.
Case 1 - Frame with $h = 60 \text{ in.}$

The moment at the top of the column corresponding to $w = 2.20 \text{ kips/ft}$ is found from Fig. 12 to be $0.83 M_y = 924 \text{ ft-kips}$. By statics the moment at the center is

$$M_C = \frac{wL^2}{2} - M_B = 2136 - 924 = 1212 \text{ ft-kips}$$

or

$$\frac{M_C}{M_y} = 1.089$$

This indicates that the center section of the beam is not fully plastified at the trial load, since $M_p$ equals $1.15 M_y$. Figure A1 shows the distribution of bending moment of the beam and the corresponding yield configuration. The stiffness of this partially yielded beam will be computed by the method of column analogy. Numerical computations are shown in Table A1. It is convenient (and also accurate enough) to divide the yielded portion into segments, each having a length of one foot. Within each segment the flexural rigidity may be assumed to be constant. In column (1) of Table A1 are listed the "station" numbers or distances from the origin G-G of the beginning and end of each segment. Each segment is one foot long except the 22.7 ft unyielded segment labelled 19.9-42.6. In column (2) the distance from the origin G-G to the center section of each segment is listed.

The moment at the center section of each segment is computed from the known values of $M_B$ and $M_D$ and is listed in column (3) of Table A1 in dimensionless form. The effective flexural rigidity $\frac{(EI_b)_{\text{eff}}}{EI_b}$ of these sections is then determined from the flexural rigidity-moment curve constructed for the 33 WF 130, similar to that shown in Fig. 5. This gives the values shown in column (4).
The area and the moment of inertia of the analogous column can then be calculated numerically as tabulated in Table A1: columns (5), (6), and (7). Since the width of the analogous column at any section is inversely proportional to the flexural rigidity of that section, the reciprocal of the values of column (4) gives directly the width listed in column (5) of the analogous column at the center sections. For example, the width of the first segment is 
\[ \frac{1}{0.134} \frac{1}{E_I_b} = 7.46 \]
where \( E_I_b \) is the flexural rigidity of the section in the elastic range. The area of each segment of the analogous column can be computed by multiplying its width in column (5) by its length. The areas are listed in column (6). The moment of inertia of each segment with respect to axis G-G may be computed by using the parallel-axis theorem. The values obtained for all the segments are listed in column (7). By summing columns (6) and (7) vertically and multiplying by two, the total area and the total moment of inertia thus obtained are \( \frac{187.04}{E_I_b} \) and \( \frac{61,067.08}{E_I_b} \) respectively.

The stiffness of the beam or the moment at end B induced by an imposed unit rotation at B is given by Eq. 1:

\[
K_b = m_B = \frac{1}{187.04 \frac{1}{E_I_b}} + \frac{1 \cdot (44.1) \cdot (44.1)}{61,067.08 \frac{1}{E_I_b}}
\]

\[= 0.003719 \frac{E_I_b}{E_I_b}\]

where the unit rotation applied at end B is represented by a unit load applied to the analogous column at point B. The stiffness factor may be expressed in terms of \( M_y \) by substituting \( M_y/\phi_y \) for \( E_I_b \), that is,

\[
K_b = \frac{0.003719}{\phi_y} M_y = 46.63 M_y
\]
The moment at D is

\[ m_D = \frac{1}{187.04} - \frac{1 \cdot (44.1) \cdot (44.1)}{61,067.08} \]

\[ = -0.002650 \frac{EI_b}{EI_b} \]

The carry-over factor is therefore equal to \( C_b = \frac{0.002650}{0.003719} = 0.7125 \).

Case 2 - Frame with \( h = 80 \text{ in} \)

It is required to determine the stiffness of the plastified beam when small anti-symmetrical rotations are imposed at the two ends (see Fig. 7a). The bending moment diagram of the beam and the yield configuration are shown in Fig. A2. In this case only half of the analogous column needs to be considered and its area may be assumed to be infinity. Table A2 contains all the computations involved in determining the moment of inertia of the half column about the axis G-G. The stiffness of the beam, according to Eq. 4, is

\[ K_b'' = m_B' = \frac{1 \cdot (44.1) \cdot (44.1)}{34,405.22} = 0.05653 \frac{EI_b}{EI_b} \]

When expressed in terms of \( M_y \), the stiffness if \( K_b'' = 70.87 M_y \).
### Table A1. Determination of Beam Stiffness by Column Analogy

\((h = 60 \, r_x)\)

<table>
<thead>
<tr>
<th>Segment</th>
<th>Distance from Axis G-G</th>
<th>(\frac{M}{M_y})</th>
<th>(\frac{(E_t b)_{eff}}{E b})</th>
<th>Width from 1.0</th>
<th>Area</th>
<th>Moment of Inertia About Axis G-G</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 - 1</td>
<td>0.5</td>
<td>1.088</td>
<td>0.134</td>
<td>7.46</td>
<td>7.46</td>
<td>2.49</td>
</tr>
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<td>1.5</td>
<td>1.087</td>
<td>0.134</td>
<td>7.46</td>
<td>7.46</td>
<td>17.41</td>
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<td>2 - 3</td>
<td>2.5</td>
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<td>0.140</td>
<td>7.14</td>
<td>7.14</td>
<td>45.22</td>
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<td>3 - 4</td>
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<td>0.160</td>
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<td>6.25</td>
<td>77.08</td>
</tr>
<tr>
<td>4 - 5</td>
<td>4.5</td>
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<td>0.172</td>
<td>5.81</td>
<td>5.81</td>
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<td>1.059</td>
<td>0.190</td>
<td>5.26</td>
<td>5.26</td>
<td>159.56</td>
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<td>6 - 7</td>
<td>6.5</td>
<td>1.047</td>
<td>0.216</td>
<td>4.63</td>
<td>4.63</td>
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<td>7 - 8</td>
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<td>4.00</td>
<td>225.33</td>
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<td>3.45</td>
<td>3.45</td>
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<td>2.96</td>
<td>2.96</td>
<td>267.39</td>
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<tr>
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<td>0.400</td>
<td>2.50</td>
<td>2.50</td>
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<tr>
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<tr>
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<td>1.82</td>
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<tr>
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<td>1.56</td>
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<tr>
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<td>1.39</td>
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<td>0.800</td>
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<td>1.08</td>
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<td>1.03</td>
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<td>1.00</td>
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<tr>
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<td>1.01</td>
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<td>992.93</td>
</tr>
<tr>
<td>43.1 - 44.1</td>
<td>43.6</td>
<td>0.787</td>
<td>0.930</td>
<td>1.08</td>
<td>1.08</td>
<td>2053.13</td>
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\(\sum = 93.52\) \(30,533.54\)
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<th>Section</th>
<th>Distance from Axis G-G</th>
<th>$\frac{M}{M_y}$</th>
<th>$\frac{(EI_b)^{eff}}{EI_b}$</th>
<th>1.0</th>
<th>Moment of Inertia About Axis G-G</th>
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</thead>
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<td>100.00</td>
<td>33.33</td>
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<td>0.010</td>
<td>100.00</td>
<td>233.33</td>
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<td>1.145</td>
<td>0.015</td>
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<tr>
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<td>0.806</td>
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<td>0.884</td>
<td>1.13</td>
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<td>0.700</td>
<td>1.000</td>
<td>1.00</td>
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$\sum = 34,405.22$
FIG. A1 DETERMINATION OF BEAM STIFFNESS FOR THE CASE $h=60r_x$

FIG. A2 DETERMINATION OF BEAM STIFFNESS FOR THE CASE $h=80r_x$
10. **NOTATION**

A = cross sectional area

C = carry-over factor

E = Young's modulus

H = horizontal reaction at base

h = height of frame

I = moment of inertia

K = stiffness of member

K' = stiffness of member with far end hinged

K'' = stiffness of member having equal end rotations

L = span length

M = bending moment

M' = column moment resulting from moment distribution computations

M_F = fixed-end moment

M_P = full plastic moment

M_y = nominal yield moment

m = moment at a point on an analogous column

N = loading parameter relating the concentrated load P to the uniformly distributed load w.

P = concentrated load applied at column top

P_y = axial yield load

F_1 = concentrated beam load

\( \bar{P} \) = total axial force in a column = \((1 + N)wL/2\)

\( \bar{P}_p \) = total axial force in a column when the applied load is equal to the simple plastic load

\( \bar{P}_u \) = total axial force in a column when the applied load is equal to the computed ultimate load taking into account the effect of beam-column instability
\( \bar{P}_{cr} \) = critical value of \( \bar{P} \)

\( (\bar{P}_{cr})_e \) = \( \bar{P}_{cr} \) corresponding to elastic buckling

\( (\bar{P}_{cr})_i \) = \( \bar{P}_{cr} \) corresponding to inelastic buckling

\( \bar{P}_{exp} \) = \( \bar{P}_{cr} \) observed from experiment

\( Q \) = shear force at column top

\( r \) = radius of gyration

\( S \) = section modulus

\( w \) = intensity of uniformly distributed load

\( w_p \) = ultimate value of \( w \) based on simple plastic theory

\( w_u \) = ultimate value of \( w \) computed by considering the effect of beam-column instability

\( w_{cr} \) = critical value of \( w \)

\( Z \) = plastic modulus

\( \delta \) = horizontal deflection of column

\( \Theta \) = end rotation of member

\( \lambda \) = \( \sqrt{E I_c / P} \)

\( \sigma_y \) = yield stress of material

\( \phi \) = curvature

\( \phi_y \) = curvature corresponding to initial yielding
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