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WELDED CONTINUOUS FRAMES AND THEIR COMPONENTS

ON THE STABILITY OF FRAMES UNDER INITIAL BENDING MOMENTS

By Le-Wu Lu

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SYNOPSIS

The stability of a portal frame subjected to loads causing initial bending moments in the members is examined. The results indicate that the critical load associated with a symmetrical mode of instability is appreciably reduced by initial moment effects, while the critical load for anti-symmetrical buckling is only affected slightly. The presence of axial thrust in the cross beam is found to be responsible for the major part of the reductions.

The theoretical buckling loads are checked by experiments conducted on model steel frames. It is observed that the elastic buckling strength of portal frames can be closely predicted by the existing methods.

Also presented herein is a method for analyzing the stability of frames with partial base fixity.
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1. **INTRODUCTION**

In recent years there has been an increased interest in problems of the overall stability of structural frames. This has been caused by the need for more precise design of columns in modern steel buildings which generally have no substantial walls or bracing systems to provide the skeleton with additional stiffness. Bleich\(^1\) in 1952 presented a systematical survey of the various stability theories for rigid frames of single and multiple stories. This survey has aroused further interest among research workers in this field. Merchant\(^2\) and his associates have developed some convenient numerical methods for analyzing the stability of tall building frames. The concept of stiffness matrix was first applied to frame buckling problems by Masur\(^3\). He also developed lower and upper bound theorems for determining the critical load of plane frames and trusses\(^4\). More recently Johnson\(^5\) has extended the energy method to multi-story, multi-bay frames, and McMinn\(^6\) has derived a matrix criterion for the in-plane buckling of trusses and frames.

In almost all the previous investigations, it has been assumed that the frames are loaded in such a manner that before the attainment of the critical load all the members are free of initial bending moments. Consequently, there is no bending deformation in any part of the structure and the
members remain straight up to the limit of stability. Obviously these conditions cannot be completely satisfied in practical building frames which are designed to carry loads primarily by bending action. Therefore, it is necessary to study the effect of initial bending moments on the stability of structural frames.

In 1938 Chwalla\textsuperscript{7} presented a study on the sidesway instability of a simple portal frame subjected to vertical loads placed symmetrically on the cross beam. He found that the presence of primary bending moment in the members does not alter the buckling characteristics of the frame. However, the critical value of the applied loads may be slightly less than the critical value of the loads when applied to the tops of the columns. (In the latter case all the members are subjected to only axial force.) A fairly complete review of Chwalla’s work is contained in the book by Bleich\textsuperscript{1}. Due to the mathematical complexities involved, this area of stability analysis had not been further explored until very recently. In 1961 Masur, Chang and Donnell\textsuperscript{8} developed some systematical methods for analyzing frame instability problems, taking into account the effect of initial bending moments. Their methods are generalizations of the standard techniques of buckling analysis, such as the four-moment equation method, the slope-deflection method, and the moment distribution method\textsuperscript{8,9}. These authors have
re-examined the stability problem previously considered by Chwalla and obtained identical results. The theoretical investigations mentioned here are the only ones of which the writer is aware; furthermore, they are concerned mainly with results obtained for frame instability of the sidesway type.

In the following sections, the elastic instability of the frame shown in Fig. 1 is analyzed. Theoretical solutions for both the symmetrical and antisymmetrical (sidesway) types of instability are obtained, and numerical results are given for a variety of frames. The solution for sidesway buckling is then checked by experiments conducted on small scale model frames. Also presented is a method of analyzing frames with partial base fixity.
2. THEORETICAL SOLUTIONS

The slope-deflection approach developed by Masur et al. is adopted here to obtain theoretical solutions for the two types of instability shown in Fig. 2. The frame is assumed to carry simultaneously a uniformly distributed load of intensity \( w \) on the beam and concentrated loads \( P \) applied at the top of the columns. The load \( P \) is related to the uniform load by the parameter \( N \) in the form

\[
P = N \left( \frac{wL^2}{2} \right)
\]

(1)

To achieve proportional loading, \( N \) will be held constant. This loading system is intended to simulate approximately the axial loads and moments which occur in the lower stories of a single bay, multi-story frame. For instance, the total axial force in the columns.

\[
\bar{P} = (1+N) \frac{wL^2}{2}
\]

(2)

with \( N = 3 \) would simulate those for a four-story frame.

2.1 Symmetrical Deformation

The bending moments at the ends of a prismatic member \( AB \) loaded as shown in Fig. 3 are given by the following expressions due to Winter:

\[
M_{AB} = KS \left( \theta_A + C \theta_B - (1+C)P \right) + M_{FAB}
\]

(3a)
and

\[ M_{BA} = KS\left( \theta_B + C\theta_A - (1 + C)r \right) + M_{FBA} \]  \hspace{1cm} (3b)

in which \( \theta_A \) and \( \theta_B \) are the end rotations at A and B, respectively, \( r \) is the rotation of member AB with respect to the undeformed position, and \( K \) denotes \( EI/L \). The coefficients \( S \) and \( C \) are functions of the axial forces \( p \) and are defined as

\[ S = \frac{\lambda L (\sin \lambda L - \lambda L \cos \lambda L)}{2 - 2 \cos \lambda L - \lambda L \sin \lambda L} \]  \hspace{1cm} (4a)

and

\[ C = \frac{\lambda L - \lambda L \sin \lambda L}{\lambda L \sin \lambda L - \lambda L \cos \lambda L} \]  \hspace{1cm} (4b)

in which

\[ \lambda = \sqrt{\frac{p}{E I}} \]  \hspace{1cm} (4c)

In Eqs. 3 the coefficients \( S \) and \( C \) represent, respectively, the "non-dimensional stiffness" and the "carry-over factor" of the member. If the far end of the member is pinned, then the carry-over factor becomes zero and the stiffness coefficient is expressed as

\[ \tilde{S} = S(1 - C') \]  \hspace{1cm} (5)

Values of \( S, C \) and \( \tilde{S} \) have been tabulated by Lundquist and Knoll, and by Livesley and Chandler.
The terms \( M_F \) in Eqs. 3, which are known as the "fixed-end moment", depend not only on the lateral load \( q \) carried by the member, but also on the axial force \( p \). The fixed end moment at A of member AB can be expressed in the form

\[
M_{FA} = -K_S(\psi_A + C \psi_B)
\]

in which \( \psi_A \) and \( \psi_B \) represent the end rotations of the member when it is simply supported and subjected to the same lateral load. Expressions for \( \psi \) for a number of loading cases have been derived explicitly by Timoshenko and Gere. A case of special interest is that when the lateral load is uniformly distributed throughout the entire length of the member. The value of \( \psi \) for this case is given by

\[
\psi_A = -\psi_B = \frac{wL}{p} (\frac{1}{\lambda L} \tan \frac{\lambda L}{2} - \frac{1}{2})
\]

in which \( w \) represents the intensity of the distributed load.

The sign convention used in Eqs. 3, 6, and 7 is the following: Joint rotation \( \theta \), bar rotation \( \rho \), and end moments \( M \) are considered positive when clockwise. Thus the quantities \( \theta \), \( \rho \) and \( M \) shown in Fig. 3 are all positive.

The frame shown in Fig. 1 is now analyzed for its symmetrical mode of instability. From Eqs. 3a and 5 the moment at the top of column AB is given by

\[
M_{ba} = K_i S_i (1 - C_i^2) \theta_b
\]
Because of the symmetry of the deformation configuration, the rotation at b must be equal to that at d, but of opposite sign, that is \( \theta_b = -\theta_d \). The moment at the left end of beam bd is therefore equal to

\[
M_{bd} = K_2 S_z (\theta_b + C_2 \theta_d) + M_{Fbd} = K_2 S_z (1 - C_2) \theta_b + M_{Fbd} \tag{9}
\]

Joint equilibrium at b requires that

\[
M_{ba} + M_{bd} = 0 \tag{10}
\]

Substitution of \( M_{ba} \) and \( M_{bd} \) from Eqs. 8 and 9 into Eq. 10 leads to

\[
[K_1 S_1 (1 - C_1) + K_2 S_z (1 - C_2)] \theta_b + M_{Fbd} = 0 \tag{11}
\]

The equilibrium of column ab requires that \( M_{ba} = H L_1 \) or

\[
K_1 S_1 (1 - C_1) \theta_b = H L_1 \tag{12}
\]

By eliminating \( \theta_b \) from Eqs. 11 and 12, and substituting the appropriate expressions for \( S, C \) and \( M_F \) from Eqs. 4, 6 and 7 into the resulting equation, the following nondimensional equation relating the horizontal reaction \( H \) to the applied load \( P \) is obtained:

\[
\frac{H}{P} (1 - \lambda_1 L_1 \cot \lambda_1 L_1) + \lambda_2 L_1 \tan \frac{\lambda_2 L_1}{2} - \frac{w L_1}{H} \left( \frac{1}{\lambda_2 L_2 \tan \frac{\lambda_2 L_1}{2}} - \frac{1}{2} \right) = 0 \tag{13}
\]

in which

\[
\lambda_1 = \sqrt{\frac{E}{F_1}} \tag{14a}
\]

and

\[
\lambda_2 = \sqrt{\frac{H}{E I_2}} \tag{14b}
\]

\[\text{Equation} \tag{14c}\]
When the dimensions of the frame and its loading condition are specified, Eq. 13 can be solved numerically to yield a relationship between $H$ and $P$. The maximum value of $P$ thus obtained determines the critical load of the frame. Detailed discussions of the numerical procedure used for solving Eq. 13 are presented in the section titled "NUMERICAL RESULTS".

2.2 Antisymmetrical Mode of Buckling

If the frame under consideration is not braced against sidesway movement at the top of the columns, antisymmetrical buckling will take place at a load level lower than the critical load computed for the symmetrical case. The initiation of antisymmetrical buckling therefore represents a bifurcation of the equilibrium configuration. This phenomenon is analogous to the buckling of a centrally loaded column. The existence of such a bifurcation point on the load-deflection relationship has been proven, for a simple portal, by Chwalla.

In order to establish the condition under which the structure first becomes laterally unstable, it is necessary to consider the equilibrium of the frame in its slightly buckled state as shown in Fig. 4c. This state of equilibrium can be obtained by superimposing on the symmetrical deflection form (Fig. 4a) an infinitely small antisymmetrical deformation associated with a lateral displacement $\Delta R$ of the joints b
and d as shown in Fig. 4b. The assumed antisymmetrical configuration corresponds to a set of small variations in end rotation \( \Delta \theta \) and bar rotation \( \Delta \phi \) of all the members. Associated with these variations in deflection form there are changes in axial force \( \Delta P \) in the members, which in turn cause changes in the stiffness coefficient \( \Delta S \) and carryover factor \( \Delta C \). For the member shown in Fig. 3 the variation in moment at end A due to these changes may be expressed as \(^8\)

\[
\Delta M_{AB} = K \left\{ S \left( \Delta \theta_A + C \Delta \theta_B + \Delta C \theta_B - (1 + C) \Delta \rho - \Delta C \rho \right) + \Delta S \left( \theta_A + \theta_B - (1 + C) \rho \right) \right\} + \Delta M_{FAB} \tag{15}
\]

in which

\[
\Delta S = \frac{dS}{d\rho} \Delta \rho \tag{16a}
\]

\[
\Delta C = \frac{dC}{d\rho} \Delta \rho \tag{16b}
\]

and

\[
\Delta M_{FAB} = \frac{dM_{FAB}}{d\rho} \Delta \rho \tag{16c}
\]

Equation 15 is referred to as the "slope-deflection equation for neutral equilibrium". As pointed out by Masur\(^8\), the terms containing \( \theta_A \), \( \theta_B \) and \( \rho \) and the term \( \Delta M_{FAB} \) in this equation account for the effect of the prebuckling deformations produced by the initial bending moments.
The terms \( \frac{dS}{dp} \) and \( \frac{dC}{dp} \) in Eqs. 16a and 16b represent, respectively, the rate of change of the coefficients \( S \) and \( C \) with respect to the axial force \( p \). Their values are given in terms of \( S \), \( C \) and \( p \) by the following expressions:

\[
\frac{dS}{dp} = S' = \frac{S}{2P} (1 - C^2 S) \quad (17a)
\]

\[
\frac{dC}{dp} = C' = \frac{1 + C}{2P} [1 - C S (1 + C)] \quad (17b)
\]

In a similar manner the term \( \frac{dM_{FAB}}{dp} \) in Eq. 15c may be considered as the rate of change of the fixed-end moment with respect to \( p \); and its value may be evaluated from the relationship

\[
\frac{dM_{FAB}}{dp} = M'_{FAB} = -K \left[ S (\psi'_A + C \psi'_B + C' \psi'_B) + S' (\psi'_A + C \psi'_B) \right] \quad (17c)
\]

in which \( \psi' \) denotes \( \frac{d\psi}{dp} \).

If the member shown in Fig. 3 is not subjected to any lateral loads and if end \( B \) is actually hinged, then from Eq. 5, the moment at end \( A \) is given by

\[
M_{AB} = KS(1-C^2)(\theta_A - \rho) \quad (18)
\]

The incremental form of the above equation may be seen to be

\[
\Delta M_{AB} = K \left\{ S(1-C^2)(\Delta \theta_A - \Delta \rho) + [\Delta S(1-C^2) - 2SC \Delta C](\theta_A - \rho) \right\} \quad (19)
\]
Equation 19 is useful in expressing the change in moment at the top of the column due to the imposed lateral displacement $\Delta R$ (Fig. 4b).

The procedure for analyzing the sidesway buckling of the frame considered in this investigation follows closely that originally presented by Masur and may be summarized as follows:

1. Introduce an infinitesimal sidesway displacement $\Delta R$ at the top of the columns as shown in Fig. 4b. Associated with this displacement, there are changes in joint rotations at b and d. Due to these changes in deformation configuration, the bending moment at the ends of the members and the reaction at the supports should also change. For the perfectly anti-symmetrical configuration assumed here, it is necessary that the change in rotation at b equals to that at d, or $\triangle \theta_\text{b} = \triangle \theta_\text{d}$, and that the change in horizontal reaction, $\Delta H$, should be zero.\(^ 1,8\)

2. Determine the change in vertical reaction at the two supports ($\Delta V_1$ and $\Delta V_3$) in terms of $\Delta R$ by considering the equilibrium of the whole frame. These changes in reaction are equal to the changes in axial force in the columns, that is, $\Delta V_1 = \Delta P_1$ and $\Delta V_3 = \Delta P_3$.\(^ 1,8\)
(3) Obtain expressions for the change in end moments of columns ab and de (ΔM_{ba} and ΔM_{de}) by using Eq. 19 and taking ρ = 0 and Δρ = ΔR/L₁. Similarly, expressions for the changes in end moment of beam bd (ΔM_{bd} and ΔM_{db}) can be obtained by applying Eq. 15 and assuming ρ = Δρ = 0. In these expressions the changes in end moment are given in terms of the following variables: θ₂ (or θ₃), Δθ₂ (or Δθ₃), and ΔR. (Note that the terms containing the change in axial force can be replaced by terms involving ΔR, using the results of Step 2. Therefore the changes in axial force (Δp₁ and Δp₃) do not appear in the final expressions).

(4) Establish the required statical conditions of the structure by considering the equilibrium of joints b and d, and of columns ab and de. Because of the assumed symmetry in dimensions and loading, only two independent equilibrium equations are obtained for this frame.

(5) Substitute the moment expressions found in Step 3 into the two equilibrium conditions. This results in two linear homogeneous equations in the two unknowns Δθ₂ and ΔR. A nontrivial solution is possible only if the determinant of the coefficients of the unknowns is equal to zero; this represents the condition of neutral equilibrium.

The procedure outlined above results in the following characteristic equation for the antisymmetrical buckling of
the frame shown in Fig. 1

$$\lambda_1 L_1 \cot \lambda_1 L_1 - \frac{P L_1}{H L_2} (2 - \lambda_2 L_2 \cot \alpha \frac{L_1}{2}) + \frac{H L_1}{P L_2} (2 - \lambda_1 L_1 \cot \lambda_1 L_1 - \frac{\lambda_1^2 L_1}{\alpha \lambda_1 L_1}) = 0$$

Equation 20 defines the value of $P$ (as a function of $H$) at which sideways displacement first becomes possible. Simultaneous solution of this equation and Eq. 13 determines the antisymmetric buckling load of the frame. In the following section detailed discussions of the numerical results obtained for a variety of frames will be presented.
3. NUMERICAL RESULTS

3.1 Symmetrical Mode

The critical load for symmetrical mode of instability can be obtained from the solution of Eq. 13, which expresses implicitly the horizontal reaction \( H \) at the support as a function of the applied load \( P \). The solution of this equation can be facilitated by introducing a non-dimensional parameter \( \alpha \) defined as

\[
\alpha = \sqrt{\frac{H}{P}} \tag{21}
\]

and by rewriting the equation in the following form:

\[
\alpha^2 \left( 1 - \lambda_1 L_1 \cot \lambda_1 L_1 \right) + \lambda_1 L_1 \beta \tan \frac{\lambda_1 L_1 Y}{2} - \frac{2}{\lambda_1 L_1} \alpha^2 (1 + N) \tan \frac{\lambda_1 L_1 Y}{2} + \frac{1}{\alpha^2 (1 + N)} = 0 \tag{22}
\]

in which

\[
\beta = \alpha \sqrt{\frac{I_1}{I_2}} \tag{23a}
\]

and

\[
\gamma = \beta \frac{L_2}{L_1} \tag{23b}
\]

When the dimensions of the frame and its loading condition are specified, Eq. 22 can be solved numerically - with the aid of a digital computer - to obtain the value of \( \alpha \) for an assumed value of \( \lambda_1 L_1 \). The applied load \( P \) and the horizontal
reaction \( H \) corresponding to these values of \( a \) and \( \lambda_1 L_1 \) can then be determined from the relations

\[
\frac{\tilde{P}}{\ell_1} = (\lambda_1 L_1)^2 \quad \text{and} \quad \frac{H}{\ell_1} = (\lambda_1 L_1 a)^2
\]  

(24a, b)

If the process is repeated for several assumed values of \( \lambda_1 L_1 \), and if the results are plotted graphically, a curve relating the horizontal reaction to the applied load is obtained, from which the critical load may be determined.

Figure 5 shows the results obtained for a frame with \( L_2/L_1 = 3 \) and \( I_1 = I_2 \). In this case, Eq. 22 becomes

\[
\alpha^2(1 - \lambda_1 L_1 \cot \lambda_1 L_1) + \lambda_1 L_1 a \tan \frac{3\lambda_1 L_1 a}{\ell} - \frac{\ell}{3\lambda_1 L_1 a^2 (1 + N) \tan \frac{3\lambda_1 L_1 a}{\ell}} + \frac{1}{\alpha^2 (1 + N)} = 0
\]

(25)

For a fixed value of \( N \), repeated solutions of the above equation result in a series of \( a \) values; each corresponding to a given value of \( \lambda_1 L_1 \). For example, the following values of \( a \) were found for the case \( N = 2.0 \): 0.375, 0.383, 0.402, 0.428, 0.528, and 0.583. These values were computed for \( \lambda_1 L_1 = 1.0, 1.5, 2.0, 2.25, 2.30, \) and 2.25. The applied load \( \tilde{P} \) and the reaction \( H \) are then determined by substituting the corresponding values of \( a \) and \( \lambda_1 L_1 \) into Eqs. 24. In Fig. 5 the resulting \( \tilde{P} \) versus \( H \) curves for three selected values of \( N \) are presented. It is seen that the critical loads of the frame occur at the peaks of these curves.
The value $P_{cr}$ given at the top of Fig. 5 represents the critical load of the frame if all the loads are assumed to act along the axes of the columns, in which case there is no initial bending moment present in the members at the instant of buckling. Comparison of this load with the maximum attainable loads as shown by the curves indicates that the presence of initial moments causes a significant reduction in the critical load. For the curve with $N = 0$, the critical load is only $1.91 \frac{EI}{L^2}$, or about 17.2% of the load $P_{cr}$. If the critical load is expressed in the form of the Euler formula for pin-ended column, the effective length factor $k$ for this case is found to be 2.27. This means that the effective length of the column in the frame is 2.27 times longer than the actual length.

The results shown in Fig. 5 also indicate that the critical load $P_{cr}$ increases as the loading parameter $N$ increases. This can be explained by considering the relative magnitude of the axial thrust in the cross beam at the limit of stability. As $N$ increases, the portion of the total load that is applied to the beam becomes less, consequently, the horizontal reaction of the base (which is equal to the axial force in the beam) is smaller. The stiffness of the beam is therefore increased. This results in an increase in critical load.
3.2 **Antisymmetrical Mode**

In the analysis of symmetrical deformation discussed above, the relationship between the horizontal reaction $H$ and the applied load $P$ has been established for the entire range of loading. Antisymmetrical deformation becomes possible when the applied load reaches such a magnitude that Eq. 13 and Eq. 20 are simultaneously satisfied. This implies that both the symmetrical and the antisymmetrical configuration are equally possible for the frame under this load.

A numerical solution of Eq. 20 can be performed by the same procedure as that used previously for solving Eq. 13. By substituting the non-dimensional parameters defined by Eqs. 21 and 23 into Eq. 20, the following expression is obtained:

$$
\alpha \gamma \lambda_1 L_1 \cot \lambda_1 L_1 - \beta (z - \lambda_1 L_1) \cot \frac{\lambda_1 L_1}{2} + \frac{4}{\alpha} \beta (z - \lambda_1 L_1) \cot \lambda_1 L_1 - \frac{\lambda_1 L_1^3}{\beta \pi \lambda_1 L_1} = 0
$$

(26)

It may be noticed that the loading parameter $N$ does not appear in this equation; therefore, the manner in which the loads are applied to the frame is immaterial in analyzing the buckling condition of Eq. 20.
In Fig. 6 numerical results obtained for the frame which has been analyzed previously for its symmetrical deformation are shown. The dotted line in this figure represents the solution of Eq. 26. The intersections of this line with the $P - H$ curves reproduced from Fig. 5 (shown as solid lines) determine the buckling loads of the structure. Also, the point at which the dotted line intersects with the vertical axis corresponds to the critical load of the frame when all the loads are applied at the top of the columns. This critical load is designated as $P_{cr}^*$. 

Consider the particular case $N = 0$; when loads $P$ are gradually applied, the frame is initially deformed into a symmetrical configuration, and the horizontal reaction increases according to the solid line shown in Fig. 6. When $P$ is equal to about $1.08 \frac{EI}{L_1^2}$, any further increase in $P$ may give rise to a lateral displacement at the column tops, hence, this value of $P$ may be taken as the critical load for antisymmetrical buckling. The remaining part of the curve for $N = 0$ in Fig. 5, defining symmetrical deformation of the frame, is only attainable if sideways movement is prevented. This means that the critical load in the case of antisymmetrical buckling is always less than that associated with symmetrical instability.
The results given in Fig. 6 show that the sidesway buckling load is not appreciably affected by the initial bending moments. This is in agreement with the earlier finding by Chwalla. The critical loads of the frame for \( N = 0, 1.0 \) and 2.0 are only 6.7%, 3.4% and 2.3%, respectively, lower than the buckling load \( P_{cr}^{*} \). The increase in the effective length factor \( k \) for the three cases shown is also rather insignificant. It should be noted that for this frame the effective length of the column is approximately three times longer than its actual length; thus, the buckling load is about nine times lower than the Euler load of the individual columns.

Numerical results for antisymmetrical buckling have also been obtained for frames with \( L_2/L_1 = 1 \) and \( L_2/L_1 = 2 \). The moment of inertia is assumed constant for all the members. Table 1 summarizes the critical loads computed for three loading conditions: \( N = 0, 1.0 \) and 2.0. Also included are the critical loads for the cases in which only column loads are applied to the frames (referred to as \( P_{cr}^{*} \) in the previous discussions). In general the percent reduction in critical load due to the presence of initial bending moments becomes smaller as the parameter \( N \) increases and as the span-to-height ratio decreases. From the numerical values given in Table 1, it may be concluded that for portal frames of practical proportions the maximum reduction in critical load should not exceed about 10% of the load \( P_{cr}^{*} \).
4. EXPERIMENTAL INVESTIGATIONS

The theoretical solution for antisymmetrical buckling presented above has been checked by experiments conducted on model steel frames. The cross-sectional shape and the loading arrangement of the test frames are shown in Fig. 7. The uniform load \( w \) assumed in the analysis was replaced by two concentrated loads \( P_1 \) applied at a distance \( 0.3 L_2 \) from the center line of the columns*. Figure 8 shows the test setup and the fixtures used for transmitting the loads to their points of application. A dead weight and lever system was used to produce the downward force in the sling. The loading system as a whole could sway freely with the frame at all stages of the test. Details of the test procedure and the experimental techniques employed will be described in a forthcoming report15.

Information pertaining to these model tests, including the frame dimensions, load ratio \( N \), theoretical predictions, and the test results, is summarized in Table 2. The "test load" reported in the table is not the buckling load, but the maximum load observed in each test. Because of the unavoidable imperfection of the test specimens, it was impossible to detect exactly when the test frame started to buckle. However,

* A separate solution was made for this loading condition for predicting the buckling loads.
in general very little increase of load can be expected after
the initiation of sidesway movement, so the ultimate load
observed from the tests should be very close to the actual
buckling load.

Figure 9 shows the load-deflection curves of the test
frame P-4. The curve shown in Fig. 9b is analogous to the
load-deflection curve usually obtained from a centrally
loaded column test. Figure 10 shows the same frame after
unloading, typical sidesway buckling can be seen.

It may be seen from the comparisons given in Table 2
that satisfactory correlation between the theory and the tests
has been obtained. For both frames P-3 and P-4, the test loads
are a few percent lower than the prediction. These dis-
crepancies were due partly to local yielding at the welded
joints, at which several yield lines were observed.
5. FRAMES WITH PARTIAL BASE FIXITY

In the theoretical and experimental studies described above, the base of the frame was assumed to be perfectly pinned. However, in actual structures this condition usually does not exist. In most of the so-called "pinned" column bases, the rotational restraint at these bases may be rather appreciable. The actual amount of base restraint depends on the details used in construction and on the foundation soil.

In 1960 Galambos showed that the buckling strength of portal frames with small amounts of foundation restraint can be considerably higher than that of pinned-base frames. This has also been observed experimentally in model frame tests.

This section is intended to indicate how the theoretical solution obtained in this paper can be used to determine the buckling load of frames with partial base fixity. As suggested by Galambos, the base restraint may be simulated by inserting a restraining beam between the two column bases as shown in Fig. 11. This beam restrains the column ends in the same way as would be done by an actual base consisting of base plates, the footing, and the soil. With this simplification, the problem can then be solved approximately according to the following steps:
1. Assume a distance \( L_1' \) from the column top to the inflection point in the column, then \( L_1'' = L_1 - L_1' \). As can be seen from the moment diagram shown in Fig. 11b, the given frame may be considered as two separate pinned-base frames with their respective column heights equal to \( L_1' \) and \( L_1'' \).

2. Determine the buckling load \( P_{cr}' \) for the upper frame using the solution presented previously.

3. Compute the buckling load \( P_{cr}'' \) for the lower (inverted) frame using the conventional methods and ignoring the effect of initial bending moments.

4. Compare \( P_{cr}' \) with \( P_{cr}'' \). If they are not equal, a new value of \( L_1' \) should be assumed and the process repeated. The correct buckling load is obtained when the assumed \( L_1' \) value gives identical critical loads for both frames, that is, \( P_{cr}' = P_{cr}'' \).

Calculations using the procedure here outlined have also shown that the effective length of columns in a frame can be reduced appreciably when a small amount of rotational restraint at the base was taken into account. Future research in this field should include the development of methods by which the restraints offered by different types of footing can be evaluated.
6. CONCLUSIONS

Based on the results presented herein, the following conclusions may be drawn regarding the stability of portal frames under initial moments:

1. The critical load associated with the symmetrical mode of instability is considerably reduced when the loads are not applied directly on the columns. The decrease in critical load is due mainly to the presence of axial thrust in the beam which causes a reduction in the bending stiffness.

2. For antisymmetrical buckling, the critical load is also influenced by initial bending moments. However, the reduction is much less than that in the symmetrical case. A reduction of 10% may be considered as the maximum for the type of frame studied in this paper.

3. The buckling strength of portal frames can be predicted with a reasonable degree of accuracy by the existing methods. This can be justified by comparing the theoretical and experimental buckling loads given in Table 2.

4. The theoretical solution obtained herein can be used to determine the buckling of frames with base restraint. Preliminary investigation has shown that a small amount of base restraint has a great effect on the total resistance of a frame against buckling.
This paper deals only with the stability of portal frames loaded within the elastic limit. For most practical frames, instability of both the symmetrical and the anti-symmetrical type occurs after the applied load has caused yielding in parts of the members. The conclusions reached in this paper are considered to be useful in solving the more complicated stability problems associated with partially plastic frames.
This study is part of the general investigations "WELDED CONTINUOUS FRAMES AND THEIR COMPONENTS" being carried out at the Fritz Engineering Laboratory, Lehigh University, under the general direction of Professor Lynn S. Beedle. The investigation is sponsored jointly by the Welding Research Council and the Department of the Navy, with funds furnished by the American Institute of Steel Construction, American Iron and Steel Institute, Office of Naval Research, Bureau of Ships, and Bureau of Yards and Docks. Technical guidance for the project is provided by the Lehigh Project Subcommittee of the Structural Steel Committee of the Welding Research Council. Dr. T. R. Higgins is Chairman of the Lehigh Project Subcommittee.

The author wishes to express his appreciation to Dr. George C. Driscoll, Jr., for his constructive suggestions that have been incorporated in this paper, and to Mr. Donald A. Recchio for his assistance in conducting the experiments.
Derivation of the Governing Equation for Antisymmetrical Buckling

To derive the buckling condition of Eq. 20, it is necessary to consider both the symmetrical and the antisymmetrical configuration of the frame as shown in Fig. 4.

For the symmetrical deflection form of Fig. 4a, the vertical reactions at a and e are equal to the axial forces in columns ab and de, respectively, that is

\[ V_1 = \bar{P} \quad \text{and} \quad V_3 = \bar{P} \quad \text{(27)} \]

The corresponding reactions associated with the buckled configuration (Fig. 4c) can be shown to be

\[ V_1 + \Delta V_1 = \bar{P} - 2\bar{P} \frac{\Delta R}{L^2} \quad \text{(28a)} \]

and

\[ V_3 + \Delta V_3 = \bar{P} + 2\bar{P} \frac{\Delta R}{L^2} \quad \text{(28b)} \]

Combination of Eqs. 27 and 28 leads to

\[ \Delta V_1 = \Delta P_1 = -2\bar{P} \frac{\Delta R}{L^2} \quad \text{(29a)} \]

and

\[ \Delta V_3 = \Delta P_3 = 2\bar{P} \frac{\Delta R}{L^2} \quad \text{(29b)} \]
in which $\Delta P_1$ and $\Delta P_3$ represent the change in axial force in the left and the right columns, respectively, due to the imposed antisymmetrical configuration shown in Fig. 4b.

The change in end moment of the left column can be expressed, by applying Eq. 19, as

$$\Delta M_{ba} = K_1 \left[ S_1 (1 - C_1') (\Delta \theta_b - \frac{\Delta R}{L_1}) + (\Delta S_1 (1 - C_1') - 2 S_1 C_1 \Delta C_1) \theta_b \right]$$

in which

$$\Delta S_1 = S_1' \Delta P_1$$

and

$$\Delta C_1 = C_1' \Delta P_1$$

By combining Eq. 29a with Eqs. 31, and substituting the resulting expressions for $\Delta S_1$ and $\Delta C_1$ into Eq. 30, the following expression for $\Delta M_{ba}$ is obtained:

$$\Delta M_{ba} = K_1 \left[ S_1 (1 - C_1') (\Delta \theta_b - \frac{\Delta R}{L_1}) - 2 \bar{P} (S_1' (1 - C_1') - 2 S_1 C_1 C_1') \theta_b \frac{\Delta R}{L_2} \right]$$

Similarly, the change in end moment of column de is given by

$$\Delta M_{de} = K_3 \left[ S_3 (1 - C_3') (\Delta \theta_d - \frac{\Delta R}{L_1}) + 2 \bar{P} (S_3' (1 - C_3') - 2 S_3 C_3 C_3') \theta_d \frac{\Delta R}{L_2} \right]$$
The direction of the moments in Eqs. 32 and 33 is clockwise as shown in Fig. 4b.

Expressions for the change in moment at the ends of member bd are obtained by using Eq. 15 and assuming $\rho = \Delta \rho = 0$. They are as follows:

$$\Delta M_{bd} = K_1 \left[ S_2 (\Delta \theta_b + C_1 \Delta \theta_d + \Delta C_2 \theta_d) + \Delta S_2 (\theta_b + C_2 \theta_d) \right] + \Delta M_{Fbd} \tag{34}$$

and

$$\Delta M_{db} = K_1 \left[ S_2 (\Delta \theta_d + C_1 \Delta \theta_b + \Delta C_2 \theta_b) + \Delta S_2 (\theta_d + C_2 \theta_b) \right] + \Delta M_{Fdb} \tag{35}$$

in which

$$\Delta S_2 = S_2' \Delta H \tag{36a}$$

$$\Delta C_2 = C_2' \Delta H \tag{36b}$$

and

$$\Delta M_{F} = M_{F}' \Delta H \tag{36c}$$

For the perfectly antisymmetrical deformation assumed in this analysis, the change in horizontal reaction, $\Delta H$ at the support is equal to zero. Therefore the terms containing $\Delta S_2$ and $\Delta C_2$ and the terms $\Delta M_{Fbd}$ and $\Delta M_{Fdb}$ in Eqs. 34 and 35 should vanish. The expressions for $\Delta M_{bd}$ and $\Delta M_{db}$ are thus simplified to
\[ \Delta M_{bd} = K_1 S_2 (\Delta \theta_b + C_1 \Delta \theta_d) \]  
(37)

and

\[ \Delta M_{db} = K_2 S_1 (\Delta \theta_d + C_2 \Delta \theta_b) \]  
(38)

Since the loading and the dimensions are symmetrical, the deformation configuration shown in Fig. 4a requires that \( \theta_b = - \theta_d \) before buckling. Furthermore, \( \Delta \theta_b \) should be equal to \( \Delta \theta_d \) for the antisymmetrical configuration of Fig. 4b. If these conditions are taken into consideration, then Eq. 32 becomes the same as Eq. 33, and Eq. 37 the same as Eq. 38. Thus, there are only two independent equations involved in this problem.

Next the equations of equilibrium are obtained for joint b and column ab.

(1) Joint b

\[ \Delta M_{ba} + \Delta M_{bd} = 0 \]  
(39)

Substituting \( \Delta M_{ba} \) and \( \Delta M_{bd} \) from Eqs. 32 and 37 and rearranging terms, Eq. 39 becomes

\[ \left[ K_1 S_1 (1 - C_i^1) + K_2 S_2 (1 + C_2) \right] \Delta \theta_b - \left[ K_1 S_1 (1 - C_i^1) + 2PK_1 \frac{L_1}{L_2} \right] \Delta \theta_d = 0 \]  
(40)
(2) Column ab

\[ \Delta M_{ba} = -\bar{P} \Delta R \]

or

\[ K_1 S_1 (1 - C_1^2) \Delta \theta_b - \left\{ K_1 S_1 (1 - C_1^2) + 2 \bar{P} K_1 \frac{L_1}{L_2} \left( S_1 (1 - C_1^2) - 2 S_1 C_1 \theta_b \right) - \bar{P} L_1 \right\} \Delta R = 0 \]

(41)

The vanishing of the determinant of the coefficients in Eqs. 40 and 41 furnishes the stability condition

\[ \frac{K_2 S_2 (1 + C_2)}{K_1 S_1 (1 - C_1^2) + K_2 S_2 (1 + C_2)} \left\{ K_1 S_1 (1 - C_1^2) + 2 \bar{P} K_1 \frac{L_1}{L_2} \left( S_1 (1 - C_1^2) - 2 S_1 C_1 \theta_b \right) - \bar{P} L_1 \right\} = 0 \]

(42)

This equation can be further simplified by introducing the appropriate expressions for \( S, C, S', \) and \( C' \) from Eqs. 4 and 17 and by substituting the value of \( \theta_b \) from Eq. 11. The buckling condition is finally obtained in the form

\[ \frac{P L_1 L_2}{H L_2} (2 - \lambda_2 L_2 \cot \lambda_1 L_1) + \frac{H L_1}{P L_2} (2 - \lambda_1 L_1 \cot \lambda_1 L_1 - \frac{\lambda_1 L_1^2}{\lambda_1 \lambda_1 \lambda_1 L_1}) = 0 \]

(43)
9. NOTATIONS

$C = \text{carry-over factor, defined by Eq. 4b}$

$\Delta C = \text{change in carry-over factor}$

$C' = \frac{dC}{dp}, \text{given by Eq. 17b}$

$E = \text{modulus of elasticity}$

$H = \text{horizontal reaction at support}$

$I = \text{moment of inertia}$

$I_1 = \text{moment of inertia of column}$

$I_2 = \text{moment of inertia of beam}$

$I_s = \text{moment of inertia of fictitious base restraint}$

$K = \frac{EI}{L}$

$k = \text{effective length factor}$

$L = \text{length of member}$

$L_1 = \text{height of frame}$

$L_1, L_1'' = \text{heights of fictitious frames}$

$L_2 = \text{span length}$

$M_{AB} = \text{moment at the end A of member AB}$

$M_{FAB} = \text{fixed-end moment at the end A of member AB}$

$\Delta M_{FAB} = \text{Change in } M_{FAB}$

$M'_{FAB} = \frac{dM_{FAB}}{dp}, \text{given by Eq. 17c}$

$N = \text{loading parameter relating the concentrated load P to the uniformly distributed load w, as defined in Eq. 1}$

$P = \text{concentrated load applied at column top}$
\( \bar{P} \) = total axial force in column
\( P_{cr} \) = critical value of \( \bar{P} \)
\( P_{cr}^* \) = critical value of \( \bar{P} \) if all the loads are applied at the top of columns
\( p \) = axial force in member
\( \Delta p \) = change in axial force in member
\( q \) = lateral load carried by member
\( \Delta R \) = infinitesimal sidesway displacement
\( S \) = non-dimensional stiffness coefficient, defined by Eq. 4a
\( \Delta S \) = change in stiffness coefficient
\( S' \) = \( \frac{dS}{dp} \), given by Eq. 17a
\( \bar{S} \) = \( S(1-C^2) \)
\( V \) = vertical reaction
\( \Delta V \) = change in vertical reaction
\( w \) = intensity of uniformly distributed load
\( \alpha \) = \( \sqrt{\frac{H}{\bar{P}}} \)
\( \beta \) = \( \alpha \sqrt{\frac{I_2}{I_1}} \)
\( \gamma \) = \( \beta \frac{L_2}{L_1} \)
\( \theta \) = joint rotation
\( \Delta \theta \) = change in joint rotation
\( \lambda \) = \( \sqrt{\frac{p}{EI}} \)
\( \rho \) = bar rotation
\( \Delta \rho \) = change in bar rotation
\( \psi \) = end slope of member when it is simply supported
\( \psi' \) = \( \frac{d\psi}{dp} \)
10. **TABLES AND FIGURES**
### TABLE 1 ANTI-SYMMETRICAL BUCKLING LOADS

<table>
<thead>
<tr>
<th>Loading Conditions</th>
<th>Reductions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) $N = 0$</td>
<td>(5)</td>
</tr>
<tr>
<td>$N = 1.0$</td>
<td>(6)</td>
</tr>
<tr>
<td>$N = 2.0$</td>
<td>(7)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$L_2 = L_1$</th>
<th>$\frac{P}{EI} L_1^2$</th>
<th>$L_2 = L_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>1.787</td>
<td>1.810</td>
</tr>
<tr>
<td></td>
<td>2.35</td>
<td>2.34</td>
</tr>
<tr>
<td>$L_2 = 2L_1$</td>
<td>$\frac{P}{EI} L_1^2$</td>
<td>$L_2 = 2L_1$</td>
</tr>
<tr>
<td>$k$</td>
<td>1.390</td>
<td>1.400</td>
</tr>
<tr>
<td></td>
<td>2.66</td>
<td>2.65</td>
</tr>
<tr>
<td>$L_2 = 3L_1$</td>
<td>$\frac{P}{EI} L_1^2$</td>
<td>$L_2 = 3L_1$</td>
</tr>
<tr>
<td>$k$</td>
<td>1.082</td>
<td>1.120</td>
</tr>
<tr>
<td></td>
<td>3.02</td>
<td>2.97</td>
</tr>
</tbody>
</table>

### TABLE 2 ELASTIC BUCKLING TEST RESULTS

<table>
<thead>
<tr>
<th>Test No.</th>
<th>Column Height $L_1$ (in)</th>
<th>Slenderness Ratio $L_1/r$</th>
<th>$\frac{P}{P_1}$</th>
<th>Yield Load $2\bar{P}_y$ (kips)</th>
<th>Predicted Buckling Load $2\bar{P}_{cr}$ (kips)</th>
<th>Test Load $2P_{exp}$ (kips)</th>
<th>$\frac{\bar{P}<em>{exp}}{\bar{P}</em>{cr}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>P-3</td>
<td>30</td>
<td>83.4</td>
<td>8.5</td>
<td>16.14</td>
<td>8.50</td>
<td>8.15</td>
<td>0.96</td>
</tr>
<tr>
<td>P-4</td>
<td>33</td>
<td>97.2</td>
<td>7.7</td>
<td>14.12</td>
<td>6.61</td>
<td>6.46</td>
<td>0.98</td>
</tr>
</tbody>
</table>
\[
\bar{P} = N \left( \frac{w L_2}{2} \right), \quad \bar{P} = (1 + N) \frac{w L_2}{2}
\]

**FIG. 1** FRAME DIMENSIONS AND LOADING

**FIG. 2** DEFORMATION MODES

Symmetric

Antisymmetric (sidesway)
FIG. 4
\[
\bar{P}_{cr}^* = 11.08 \frac{EI}{L^2} (k=0.945)
\]

\[
L_2 = 3 L_1
\]

\[
\frac{P}{EI} = \frac{wL_2}{2}
\]

\[
\bar{P} = (1 + N) \frac{wL_2}{2}
\]

\[
\bar{P}_{cr} = \frac{\pi^2 EI}{(k L_1)^2}
\]

\[
N = 2.0, k = 1.36
\]

\[
N = 1.0, k = 1.63
\]

\[
N = 0, k = 2.27
\]

FIG. 5 CRITICAL LOADS FOR SYMMETRICAL INSTABILITY
\[ P_{cr}^{*} = 1.160 \frac{EI}{L_1^2} (k = 2.92) \]

\[ P = N \left( \frac{wL_2^2}{2} \right) \]

\[ \bar{P} = (1 + N) \frac{wL_2^2}{2} \]

\[ \bar{P}_{cr} = \frac{\pi^2 EI}{(k L_1)^2} \]

**FIG. 6 CRITICAL LOADS FOR ANTISYMMETRICAL BUCKLING**
\[ P = NP_1 \]
\[ \bar{P} = (1+N)P_1 \]

L_2 = 50"
FIG. 8 SETUP FOR MODEL FRAME TEST

![Setup Image]

FIG. 9 LOAD-DEFLECTION CURVES OF FRAME P-4

(a) Vert. Defl. of Point c (inches)

(b) Hori. Defl. of Point d (inches)

Predicted Buckling Load
**FIG. 10** FRAME P-4 AFTER TESTING

**FIG. 11** THE EFFECT OF PARTIAL BASE FIXITY
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