Limiting deflections in plastic design by limiting l/d ratios, C.E. 406 Report, Lehigh University, (February 1960) M.S. thesis

D. N. C. Pai.
LIMITING DEFLECTIONS IN PLASTIC DESIGN

BY

LIMITING L/d RATIOS

By

David H. C. Pai

Submitted to Professor G. C. Driscoll as fulfillment of the course requirement of C.E. 406 "Special Problems in Civil Engineering"

DEPARTMENT OF CIVIL ENGINEERING
FRITZ ENGINEERING LABORATORY
LEHIGH UNIVERSITY
BETHLEHEM, PENNSYLVANIA

February, 1960
268.6
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT.</td>
<td>11</td>
</tr>
<tr>
<td>1. INTRODUCTION.</td>
<td>1</td>
</tr>
<tr>
<td>2. ASSUMPTIONS AND FUNDAMENTAL CONCEPTS</td>
<td>2</td>
</tr>
<tr>
<td>3. METHODS OF CALCULATING DEFLECTIONS</td>
<td>3</td>
</tr>
<tr>
<td>4. CASE I: BEAM FIXED AT ONE END, SUPPORTED AT THE OTHER - UNIFORMLY DISTRIBUTED LOAD.</td>
<td>5</td>
</tr>
<tr>
<td>5. CASE II: BEAM FIXED AT ONE END SUPPORTED AT THE OTHER - CONCENTRATED LOAD AT MID-SPAN.</td>
<td>8</td>
</tr>
<tr>
<td>6. CASE III: BEAM FIXED AT BOTH ENDS - UNIFORMLY DISTRIBUTED LOADS.</td>
<td>10</td>
</tr>
<tr>
<td>7. CASE IV: CONTINUOUS BEAM - TWO EQUAL SPANS - UNIFORMLY DISTRIBUTED LOADS.</td>
<td>12</td>
</tr>
<tr>
<td>8. CASE V: PIN-BASED RECTANGULAR FRAME, UNIFORMLY DISTRIBUTED VERTICAL LOADS</td>
<td>14</td>
</tr>
<tr>
<td>9. DISCUSSION.</td>
<td>16</td>
</tr>
<tr>
<td>10. ACKNOWLEDGEMENTS</td>
<td>17</td>
</tr>
<tr>
<td>11. NOMENCLATURE</td>
<td>18</td>
</tr>
<tr>
<td>12. REFERENCES</td>
<td>19</td>
</tr>
<tr>
<td>13. FIGURES.</td>
<td>20</td>
</tr>
</tbody>
</table>
Abstract

Deflection calculations at working load are time consuming in elastic design. Specifications of the American Institute of Steel Construction and other specification writing bodies have made it more or less unnecessary to calculate deflections by limiting length of span to depth of beam (L/d) ratios to 24. And when beams are subject to shock or dynamic loading, the L/d ratio is reduced to 20.

It is proposed, therefore, that a study could show that by limiting L/d ratios, beams designed plastically would not have a maximum deflection at working load more than a certain specified amount. These studies would help write specifications limiting L/d ratios in plastic design.
1. **INTRODUCTION**

Plastic analysis and design of steel structures is a good tool for steel construction. It has not, however, eliminated the necessity of deflection calculations. In elastic design, deflection calculations are tedious and are much to be avoided if possible. Therefore, the AISC partially circumvented the necessity of deflection calculations by specifying length of span to depth of beam (L/d) ratios. Deflection calculations in plastic analysis is just as necessary as that in elastic design, although there are approximations which do simplify the calculations to a limited extent. But still, the amount of work involved remains objectionable. It is the purpose of this report to investigate into the feasibility of determining appropriate L/d ratios for plastic design. In this report, five cases will be investigated:

(I) A beam fixed at one end and simply supported at the other with uniformly distributed loads.

(II) Same beam as (I), but loaded with a concentrated load at mid-span.

(III) A fixed-ended beam with uniformly distributed load.

(IV) A two-equal-spaned continuous beam with uniformly distributed load throughout.

(V) A pin-based rectangular frame with uniformly distributed vertical loads.
2. ASSUMPTIONS AND FUNDAMENTAL CONCEPTS

Aside from assumptions made in simple plastic theory, this report is based on the following concepts:

"(1) The $EI\phi$ relationship is idealized as shown in Fig. 1.

(2) As a consequence of assumption 1, each span retains its flexural rigidity $EI$ for the whole length between hinge sections.

(3) Unlimited rotation is possible at hinge sections at a moment value of $M = M_p$.*"  

*See pp. 25, 60, 98 of Reference 1.
3. METHODS OF CALCULATING DEFLECTIONS

Deflection calculations in plastic design fall into two types:

(1) The magnitude of deflection at ultimate load: this is sometimes desirable because the load factor of safety does not guarantee absolutely against overload on rare occasions.

(2) The magnitude of deflection at working load: this is of value since a great majority of structures will function at working load most of the time.

In this report interest is centered exclusively around the second type of deflection calculations, i.e. magnitude of deflections at working load. A sufficiently accurate approximation will be used to calculate deflections at working load. This consists of a plastic analysis to obtain the ultimate load, which is then divided by the load factor (1.85) to reduce the loading down to working load. For the types of structures to be investigated, the structures are all in the elastic range at working load. Therefore, elastic deflection equations will be used to calculate deflections.

The scheme for the investigation here is to express L/d as a function of \( \delta /L \) (the ratio of deflection to span length). In general:
\[ \sigma = K \frac{PL^3}{EIX} \quad (P \text{ & } W \text{ are interchangeable}) \]

\[ PL = X M_p \]

\[ \frac{\sigma}{L} = K (X M_p) \cdot \frac{L}{EI} \quad M_p = \sigma_y Z \]

\[ \frac{\sigma}{L} = K X \frac{\sigma_y}{E} \cdot \frac{2d}{I} \left( \frac{L}{d} \right) \]

or \[ \frac{\sigma}{L} = K X \frac{\sigma_y}{E} \cdot 2f \left( \frac{L}{d} \right) \quad \text{......(1)} \]

where \( K \) and \( X \) are the factors which depend on the loading and the geometry of the structure, \[ \frac{\sigma_y}{E} \] is the property of ASTM A-7 Steel the material assumed in this report; \[ \frac{2d}{I} = 2f \], \( f \) is the shape factor which governs how severely the beam will deflect.
4. CASE I

BEAM FIXED AT ONE END, SUPPORTED AT OTHER - UNIFORMLY DISTRIBUTED LOADS

Following the procedure outlined in the previous chapter, we have

\[ W_e = W_1 \]

\[ \frac{W \theta \times L}{2} = M_p \theta \left[ 1 + 1 + \frac{X}{L-X} \right] \]

\[ W_u = \left[ \frac{h}{X} + \frac{2}{L-X} \right] M_p \]

\[ W_u = 11.73 \frac{M_p}{L} \quad \text{(by trial and error)} \]

\[ W_u = \frac{W_u}{F} = 6.35 \frac{M_p}{L} \quad , \quad F = 1.85 \]
Since the structure is elastic at working load, the deflection is then given as

\[
\delta = \frac{WL^3}{185 EI} \quad \text{(Ref. 2)} \quad \ldots (4)
\]

Substituting the value of \( W_w \) in Eq. 3 into Eq. 4, we have

\[
\delta = 6.35 \frac{M_p}{L} \frac{L^3}{185 EI}
\]

But \( M_p = \delta y Z \)

\[
\therefore \delta = 6.35 \frac{\delta y Z}{L} (L^3/185 EI)
\]

Since \( \delta_y = 33 \text{ ksi} \) and \( E = 30 \times 10^3 \text{ ksi} \), we have:

\[
\delta = 3.78 \times 10^{-5} \frac{2L^2}{I}
\]

Divide through by \( L \) and multiply the right hand side of the equation by \( \frac{d}{d} \) we have

\[
\frac{\delta}{L} = 3.78 \times 10^{-5} \frac{Zd}{I} (L/d)
\]

or

\[
\frac{\delta}{L} = 3.78 \times 10^{-5} (2f) (L/d) \quad \ldots (5)
\]

Using \( \delta/L = 1/360 \) as an arbitrary value, computations are made for the lightest wide-flange of each group in the AISC handbook starting with the 12" members through the 36" shapes. The result is plotted on a \( L/d \) vs \( \delta/L \) curve. (See Figure 2) It is observed that all
curves fall within a narrow boundary. The critical member having a limiting $L/d = 33.0$ for a $\delta/L = 1/360$. Incidentally, the ratio $\delta/L = 1/360$ is specified by the AISC as the maximum recommended live load deflection for members supporting plastered ceilings. Therefore, for this particular type of beam and loading plastically designed, the $L/d$ ratio will be 33 which is considerably greater than a recommended 24 in elastic design.
5. CASE II

BEAM FIXED AT ONE END, SUPPORTED AT THE OTHER

CONCENTRATED LOAD AT MID-SPAN

Procedure here is similar to that of Case I:

\[ M_0 = M_4 \]

\[ \frac{FL\theta}{2} = 3 M_p \theta \]

\[ P_u = \frac{6 M_p}{L} \]

\[ P_u = \frac{6 M_p}{FL} = \frac{6 M_p}{1.85L} = 3.24 \frac{M_p}{L} \]

\[ \cdots (6) \]
Since the structure is still elastic,

\[ \delta = 0.009371 \frac{FL^3}{EI} \] ....(7)

(From p. 370, Ref. 2)

substituting Eq. 6 into Eq. 7,

\[ \delta = 9.317 \times 10^{-5} \times 3.24 \frac{M_pL^2}{EI} \]

\[ \delta = 3.02 \times 10^{-2} \frac{\sigma_y}{E} \frac{2L^2}{I} \]

\[ \delta = 3.33 \times 10^{-5} \frac{2L^2}{I} \]

\[ \frac{\delta}{L} = 3.33 \times 10^{-5} \cdot 2f (L/d) \] ....(8)

After the L/d vs \( \frac{\delta}{L} \) is plotted, (See figure 3) it is observed that the critical L/d ratio for a \( \frac{\delta}{L} = 1/360 \) is 40 which is even better than Case I.
6. **CASE III**

**BEAM FIXED AT BOTH ENDS - DISTRIBUTED LOADS**

**STRUCTURE AND LOADING**

**MECHANISM**

**MOMENT DIAGRAM**

Similarly

\[ W_e = W_1 \]

\[ W \frac{L}{2} \theta \frac{L}{2} = M_p \theta (1+2+1) \]

\[ W_0 = 16 \frac{M_p}{L} \]

\[ W_0 = \frac{W_0}{f} = 8.65 \frac{M_p}{L} \quad ... (9) \]

The structure is elastic at working load,
substituting Eq. 9 into Eq. 10 we have:

\[ \delta = 8.65 \frac{M_pL^2}{384EI} \quad \text{or} \]

\[ \delta = 2.48 \times 10^{-5} \frac{2L^2}{I} \]

\[ \frac{\delta}{L} = 2.48 \times 10^{-5} \cdot 2f \cdot (L/d) \]

The critical L/d ratio for a \( \frac{\delta}{L} = 1/360 \) is 48.3 (See Figure 4) which is greater yet. This is logical since there is more continuity in this beam than in previous one.
7. CASE IV
CONTINUOUS BEAM - TWO EQUAL SPANS - UNIFORMLY DISTRIBUTED LOADS

STRUCTURE AND LOADING

MECHANISM

MOMENT DIAGRAM

By an elastic analysis, the expression for deflection is found to be
\[ \delta = \frac{wx}{48EI} (L^3 + 3Lx^2 + 2x^3) \]
Maximum deflection occurs at
\( x = 0.4215L \), thus giving the expression
\[ \delta_{\text{max}} = \frac{.005416 \text{ } Wl^3}{EI} \] ...

Advantage will be taken of the symmetry of the structure in calculating for the ultimate load. Consider the left span:

\[ W_e = \frac{W_x}{2} \]
\[ \frac{W_x L}{2} (L - x) \theta = M_p \theta \left( \frac{2L}{x} - 1 \right) \]
with \( x = 0.4215L \)
\[ W_n = 11.65 \frac{M_p}{L} \]
\[ W_f = \frac{W_n}{F} = 6.30 \frac{M_p}{L} \] ...

Since the maximum deflection is \( \delta = 0.005416 \frac{WL^3}{EI} \), the value of \( W \) will be substituted into the above equation giving the maximum deflection at working load as

\[
\delta = 0.005416 \left( 6.30 \frac{M_0}{L} \right) \frac{L^3}{EI} \quad \text{or}
\]

\[
\frac{\delta}{L} = 3.78 \times 10^{-5} \quad 2f (L/d)
\]

...(14)

The \( L/d \) ratio for this case has a critical value of 33, which is identical to that in Case I. (See Figure 5)
An elastic analysis was carried out for a three span continuous beam with side spans $\beta L$ and side span loads $\alpha w$. The deflection at the center line of the main span is given as

$$\delta = \frac{WL^3}{384 (3+2\beta)^3 EI} \left( -24\alpha \beta^4 -36\alpha \beta^3 + 20\beta^2 + 36\beta + 9 \right).$$

By letting $\alpha$, the side span load factor equal zero, a deflection expression for all pin-based rectangular frames with vertical loads is obtained. Thus,
\[ \delta = \frac{ML^3}{384(3+2\beta)} \frac{L}{EI} \left(20\beta^2 + 32\beta + 9\right) \]  
\text{...(15)}

is the deflection expression for a rectangular frame with variable column height \( \beta \).

As for the plastic analysis to obtain the ultimate load, similar procedure will be followed noting that hinges at the knees form simultaneously. Thus the ultimate load is

\[ W_u = 16 \frac{M_p}{L} \]

and \[ W_u = \frac{W}{F} = 8.65 \frac{M_p}{L} \]  
\text{...(16)}

Substituting this value into the expression for deflection, we have

\[ \delta = \frac{8.65}{384(3+2\beta)} \frac{M_p L^2}{EI} \frac{L}{(20\beta^2 + 36\beta + 9)} \]

or \[ \frac{\delta}{L} = 2.48 \times 10^{-5} \left(2f\right) \frac{L}{d} \left(\frac{20\beta^2 + 36\beta + 9}{(3+2\beta)^2}\right) \]  
\text{...(17)}

This equation shows that the severity of deflection is directly proportional to the column height and the magnitude of the shape factor.

In figure 6, graphs of \( S/L \) vs \( L/d \) for several factors of column height \( \beta \) have been plotted. The graphs show that the critical \( L/d \) ratios are 3.1, 2.4, 2.1 and 1.9, for a \( \beta \) value of \( \frac{1}{4} \), \( \frac{1}{2} \), \( 3/4 \) and 1 respectively.
9. DISCUSSION

From this investigation, it is observed that L/d ratios could be specified in plastic design in order to circumvent the necessity of deflection calculations. Before discussing suitable L/d values, it should be borne in mind that the investigation here is limited to the particular cases of frequently occurring structures only. Also, a $\delta /L = 1/360$ represents the deflection limitation in the elastic AISC specifications due to live load. Further, the structural shape assumed in this report is the standard American wide-flange made of ASTM A-7 steel. Keeping the above conditions in mind one may safely say that the critical L/d ratio for a plastically designed beam is 33 as compared with 24 in an elastically designed one. As for a rectangular frame with uniformly distributed vertical loads, the L/d ratios vary according to the column height, i.e., the taller the columns, the more deflection (see Figure 6). Therefore, the general derivation in case V serves as a guide to obtain limiting L/d ratios for any factor of column height to span length.

As was mentioned previously, this report assumes the wide-flange as the structural shape. The values of $2f$ or $2d/I$ varies from 2.13 to 2.32 (see table in back of report) and these values establish the boundaries for the $\delta /L$ vs L/d curves. For other structural shapes such as the I beams, the $2f$ values are higher, being between 2.28 and 2.46. Therefore, there will be a corresponding reduction in limiting L/d ratios. However, the procedure for obtaining critical L/d ratios remains the same regardless of the type of structural shapes used.
The author is deeply indebted to Professor G. C. Driscoll, supervisor of this study. His most valuable suggestions and criticisms are sincerely appreciated.

Professor W. J. Eney is the director of Fritz Engineering Laboratory and Head of the Department of Civil Engineering.
11. NOMENCLATURE

SYMBOLS

- \( d \) depth of beam
- \( e \) external when used as a subscript
- \( E \) Young's modulus
- \( f \) Shape factor, \( f = z/S \) \( 2f = 2d/l \)
- \( F \) load factor of safety = 1.85
- \( i \) means internal when used as a subscript
- \( L \) length of span
- \( M_p \) plastic hinge moment = \( \sigma_y Z \)
- \( P \) concentrated load
- \( S \) section modulus
- \( u \) means ultimate when used as a subscript
- \( w \) weight per unit length, or "working" when used as a subscript
- \( W \) total uniform load
- \( Z \) fully plastic section modulus
- \( \varepsilon \) strain
- \( \sigma \) stress
- \( \sigma_y \) yield stress of steel
- \( \delta \) vertical deflection
- \( \Theta \) slope of deflection curve
12. REFERENCES

1. Beedle, L. S. 
   PLASTIC DESIGN OF STEEL FRAMES, 
   John Wiley & Sons, New York, 
   1958, pp. 184-204.

2. Steel Construction 
   AMERICAN INSTITUTE OF STEEL 
   CONSTRUCTION, New York, 5th 

3. Driscoll, G. C., Jr. 
   ROTATION CAPACITY OF A THREE 
   SPAN CONTINUOUS BEAM, Fritz 
   Laboratory Report No. 268.2, 
   Lehigh University, 1957.

4. Driscoll, G. C., Jr. 
   ROTATION CAPACITY REQUIREMENTS 
   FOR BEAMS AND FRAMES OF STRUCTURAL 
   STEEL, Dissertation, Lehigh 
   University, 1957, pp. 186, 196.
<table>
<thead>
<tr>
<th>MEMBER</th>
<th>2f</th>
<th>MEMBER</th>
<th>2f</th>
</tr>
</thead>
<tbody>
<tr>
<td>12 WF 27</td>
<td>2.23</td>
<td>18 WF 96</td>
<td>2.21</td>
</tr>
<tr>
<td>12 WF 40</td>
<td>2.23</td>
<td>21 WF 62</td>
<td>2.28</td>
</tr>
<tr>
<td>12 WF 53</td>
<td>2.20</td>
<td>21 WF 82</td>
<td>2.29</td>
</tr>
<tr>
<td>12 WF 65</td>
<td>2.18</td>
<td>21 WF 112</td>
<td>2.31</td>
</tr>
<tr>
<td>14 WF 30</td>
<td>2.28</td>
<td>24 WF 76</td>
<td>2.29</td>
</tr>
<tr>
<td>14 WF 43</td>
<td>2.28</td>
<td>24 WF 100</td>
<td>2.23</td>
</tr>
<tr>
<td>14 WF 61</td>
<td>2.23</td>
<td>24 WF 130</td>
<td>2.21</td>
</tr>
<tr>
<td>14 WF 78</td>
<td>2.20</td>
<td>27 WF 94</td>
<td>2.29</td>
</tr>
<tr>
<td>14 WF 142</td>
<td>2.13</td>
<td>27 WF 145</td>
<td>2.25</td>
</tr>
<tr>
<td>16 WF 36</td>
<td>2.29</td>
<td>30 WF 108</td>
<td>2.32</td>
</tr>
<tr>
<td>16 WF 58</td>
<td>2.27</td>
<td>30 WF 172</td>
<td>2.25</td>
</tr>
<tr>
<td>16 WF 88</td>
<td>2.21</td>
<td>33 WF 130</td>
<td>2.30</td>
</tr>
<tr>
<td>18 WF 50</td>
<td>2.27</td>
<td>33 WF 200</td>
<td>2.25</td>
</tr>
<tr>
<td>18 WF 64</td>
<td>2.27</td>
<td>36 WF 150</td>
<td>2.31</td>
</tr>
</tbody>
</table>
(a) IDEALIZED STRESS-STRAIN CURVE

(b) IDEALIZED MOMENT-CURVATURE CURVE

FIGURE 1
CURVE OF $L/d$ vs $\delta/L$ FOR CASE I

FIGURE 2
CURVE OF L/d vs δ/L FOR CASE II

FIGURE 3
CURVE OF $L/d$ vs $\delta/L$ FOR CASE III

FIGURE 4
CURVE OF L/d vs δ/L FOR CASE IV

Figure 5
CURVES OF L/d vs δ/L FOR CASE V

FIGURE 6