Stability of beam-columns above the elastic limit, Proc. ASCE, Separate 692, 81, (1954), Reprint No. 103 (55-3)

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Welded Continuous Frames and Their Components

PROGRESS REPORT II

A VIRTUAL DISPLACEMENT METHOD FOR DETERMINING STABILITY OF BEAM COLUMNS ABOVE THE ELASTIC LIMIT

by

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March, 1954
(Revised December 1954)

Fritz Laboratory Report No. 205A.14
I. SYNOPSIS

This paper, which discusses the behavior of pin-ended, steel, wide-flange beam-columns, is the development of a virtual displacement method of solution to the inelastic instability problem. The method is equally applicable to the solution of elastic or inelastic beam-column problems but its greatest application seems to be in the inelastic field. Using the proposed method of solution, the critical value of loading for members subjected to concentric, eccentric and/or lateral loads, can be determined in a fraction of the time required by existing solutions.
II. INTRODUCTION

The problem under consideration is the determination of the maximum loading a beam-column type member can carry. The importance of the problem is evident, since if it is possible to predict the "collapse" loading of a member, it is also possible to evaluate for a member with a given loading the true factor of safety.

A large number of variables enter in the solution of problems of this type, and to reduce these in the development that will follow, it will be assumed that:

a. The mode of failure is that of bending in the plane of the applied moments. Furthermore, this plane is coincident with a principal axis of the section. (Lateral-torsional and/or local buckling are not considered.)

b. The material under study possesses stress-strain properties as shown in Fig. 1.

c. Members are originally straight, free from accidental eccentricities, and of uniform cross-section along their lengths.

To illustrate the phenomena being studied, consider the typical load-deflection curves of Figs. (2) and (3).

For the axially loaded member, Fig. 2 (elastic behavior), axial thrust, $P$, can be increased from zero to a certain critical value, $P_c$, with the member remaining in the straight equilibrium position. However, on application of the slightest additional increment of thrust above the critical value, lateral
deformation occurs. Thus at the instant of bending there exist two adjacent positions of equilibrium of the same load; that is to say, the member is indifferent with regard to either a straight or a deflected configuration. This condition, known as bifurcation of the equilibrium position, is called a "condition of instability".

For the eccentrically loaded column problem of Fig. 3 a different type of behavior is observed. First of all, the member starts to deflect on the slightest application of thrust. Elastically this deformation increases with the maximum strength approaching the Euler load. However, as deflections increase with load, the outermost compressive fiber stress at the center of the column increases as the product of P and y_0. Because of this rapidly increasing stress situation, there is reached a load at which yielding of the member takes place, (e.g., point "a" in Fig. 3). As the load is further increased, yielding progresses across and along the member, thereby reducing its resistance to further loading. Finally, there is reached a point at which an increase in load is impossible, the member progressively deflecting as load remains constant. At this instant, the member is indifferent with regard to two possible adjacent equilibrium configurations and it becomes unstable.

Herein then, "instability" is considered as that phenomenon where under a given constant load there exist two possible adjacent equilibrium positions.

To determine this condition of instability, several basic methods of solution are available. Three of these are as follows:
a. If the force required to displace the loaded column into a different, deflected position equals zero, then the member is said to be indifferent with regard to which of the positions it will assume, and a condition of instability exists.

b. If the total work associated with a loaded column remains the same as the member is deflected, that is, the rate of change with respect to deflection equals zero, indifference again has taken place.

c. If it is possible to determine the axial load-lateral deflection relation, the member will become unstable when

\[ \frac{\Delta P}{\Delta y} = 0 \]

since for that value of axial load there exist two adjacent equilibrium positions.
III. EXISTING SOLUTIONS

In the main, previous solutions of the inelastic in-stability, beam-column problem have been based on the criterion of instability that involves the determination of the maximum point of a load-deflection curve. In general each of those solutions has been based on the work of Karman, who, starting from an assumed strain distribution pattern, (and thus a stress distribution pattern), calculated the corresponding moment, thrust and curvature values which would have produced such a stress configuration. Since the three equations defining these variables, \( M, P \) and \( \phi \), contain five unknowns (see Fig. 4), one expression of the form

\[
M_{\text{int}} = f(\phi, P)
\]  

(1)

can be obtained by eliminating the two cross-section variables, \( \alpha \) and \( \sigma_a \), which define the generalized yield pattern.

Consider now a typical beam-column (Fig. 5). Since equilibrium must be satisfied at all times, i.e.

\[
M_{\text{ext}} = M_{\text{int}}
\]  

(2)

and since the equation for external moment, \( M_{\text{ext}} \), at any point along the column can be expressed as a function of \( P, \gamma, \) lateral loads, etc., i.e.

\[
M_{\text{ext}} = g(P, \gamma, \text{lateral loads, etc.})
\]

solution to equation (1) will be of the form

\[
P = h(\phi, \gamma, \text{lateral loads, etc.})
\]  

(3)
At this point the various methods of solutions differ. Karman and Chwalla (3) used numerical or graphical integration to solve equation (3) for direct values of $P$ versus $y$. Jezek (7) and others assumed a known deflection curve such that $\phi$ could be obtained directly in terms of $y$ by differentiation. Both of these methods of solution have as their objective the determination of a means of describing thrust as a function of deflection,

$$P = f(y), \quad (4)$$

for which, according to criterion (c), instability occurs when

$$\frac{\partial P}{\partial y} = 0$$

(See Fig. 3.)

The difficulties involved in either of these methods of solution arise from the fact that for any conventional type of cross-section other than a simple rectangle or circle, it is impossible to write a continuous function for Eq. (1); therefore curves or graphs must first be devised to supply the $M$-$P$-$\phi$ information corresponding to various stress conditions (9). Proceeding from these plots it is necessary to integrate suitable $\phi$ values and obtain the actual load-deflection curve.

To circumvent this difficulty, Jezek (7) introduced in his approximate solution a "shape factor"* the purpose of which was to relate the relative stiffness of the section in question to that of the rectangle. This type of solution lends itself readily to the formulation of design equations* which are comparatively easy to solve.

* Page 45, Ref. (1).
One notably different type of solution to the eccentric instability problem was presented by Ros (11), (12), who reasoned that if the rate of increase in effective, internal, resistance eccentricity, $M/P$, is the same as the rate of increase in the external applied centerline eccentricity, then instability will occur. Using this condition he obtained a graphical solution for the rectangular section problem by expressing the internal eccentricity as a function of the sum of the outer fiber strains (a measure of curvature).

It has been shown that the presence of residual stress is of definite influence on the strength of columns (5), (9). This makes solution by the above methods even more complicated.

To overcome the difficulties encountered in each of these solutions, the method presented in the next section of this paper was developed. The mechanics are similar to those presented by Ros (12) for the rectangular section.
IV. THE PROPOSED VIRTUAL DISPLACEMENT METHOD OF SOLUTION

Bending instability can be considered as a breakdown in the resistance the member offers to further bending. Since bending is synonymous with moment application, it should be possible, then, to determine for a given loading condition if the section offers sufficient resistance to further bending by displacing the loaded specimen a virtual amount and noting the change in external applied moment at the most critical section and the internal moment of resistance offered at that same section. If the increase in external moment is greater than that which the internal stress pattern can supply, the member fails since resisting moment cannot "keep up" with applied moments. When these two ratios of increase are the same a condition of indifference exists. The necessary condition for instability can then be stated:

When external moment at the most critical section increases at a rate equal to the rate of increase in resistance moment supplied by the internal stress system at that same section, the member becomes unstable.

Writing in the form of an equation

\[ \frac{\Delta M}{\Delta \theta} \bigg|_{\text{ext}} = \frac{\Delta M}{\Delta \theta} \bigg|_{\text{int}} \]  

where \( \Delta M/\Delta \theta \) is the rate of change in moment with respect to curvature, \( \theta \).
For illustration consider the axially loaded column shown in Fig. 2 assumed, for ease of solution, to buckle in the elastic stress range. As the thrust, \( P \), is increased the potential of the external system to subject the various cross-sections along the member to bending moment in addition to applied thrust increase as the thrust, i.e.

\[
\Delta M_{\text{ext}} = P \Delta y
\]

Whereas, internally (as shown in Fig. 6) the resistance to this possible external moment is constant at a value defined by the equation

\[
\Delta M_{\text{int}} = EI \Delta \phi
\]

\( \Delta M_{\text{ext}} \) and \( \Delta M_{\text{int}} \) represent changes in moment when going from the straight (original) to the deflected position (a virtual displacement away).

Assuming the deflection curve is of the form

\[
\Delta y = \Delta y_0 \cos \frac{\pi x}{L}
\]

then

\[
\Delta \phi_0 = \frac{\partial^2}{\partial x^2} (\Delta y_0 \cos \frac{\pi x}{L}) = \Delta y_0 \frac{\pi^2}{L^2}
\]

or

\[
\Delta y_0 = \Delta \phi_0 \frac{L^2}{\pi^2},
\]

where the zero subscript denotes conditions at the center of the column (the most critically deformed section). Substituting Eq. 8 in Eq. 6,
Writing this expression and Eq. (7) as the change in moment divided by the virtual curvature to obtain the rate of change in moment with respect to $\phi_0$ gives

$$\left. \frac{\Delta M_o}{\Delta \phi_o} \right|_{\text{ext}} = P \frac{L^3}{\eta^2} \Delta \phi_0$$

Each of Eqs. 9 can be plotted as function of $P$ and $(\Delta M/\Delta \phi)$, (see Fig. 7). There is reached a point at which the rate of external potentiality of moment application ($\Delta M/\Delta \phi \left|_{\text{ext}} \right.$) exactly equals the rate of internal resistance to lateral deformation ($\Delta M/\Delta \phi \left|_{\text{int}} \right.$). At that point the member can assume either the straight or the deflected position, and therefore a condition of instability exists. Above that critical value of load, $P_{cr}$, it is impossible for the internal change in resistance moment to "keep up" with the potential change in externally applied moments and therefore the member collapses. To evaluate the magnitude of the load corresponding to this critical condition, use the instability condition of Eq. 5,

$$\left. \frac{\Delta M_o}{\Delta \phi_o} \right|_{\text{ext}} = \left. \frac{\Delta M_o}{\Delta \phi_o} \right|_{\text{int}}$$

or

$$P_{cr} \frac{L^3}{\eta^2} = EI,$$
which when solved for $P_{cr}$ gives

$$P_{cr} = \frac{\pi^2 E I}{L^2},$$

the Euler buckling load for a straight, axially loaded, elastic column.

Had the thrust been applied eccentrically and/or a lateral load superimposed thereby introducing bending from the start, the following would have resulted. In the original deflected equilibrium position (see Fig. 8), prior to subjecting the member to the virtual disturbance,

$$M_0 \big|_{\text{ext}} = P (e + y_0) + \frac{RL}{4}$$

whereas, after the $\Delta y_0$ change

$$M'_0 \big|_{\text{ext}} = P (e + y_0 + \Delta y_0) + \frac{RL}{4}$$

The increase in external moment caused by this disturbance is equal to the difference between the final and original moment values, i.e.

$$\Delta M_0 \big|_{\text{ext}} = M'_0 \big|_{\text{ext}} - M_0 \big|_{\text{ext}}$$

or

$$\Delta M_0 \big|_{\text{ext}} = P (\Delta y_0)$$

It should be noted that this is the same as for the axially
loaded column problem shown previously*.

Internally, again considering elastic conditions for simplification, (see Fig. 9), resistance to bending is constant at a value

\[ \frac{\Delta M}{\Delta \theta} = EI \]

Solution to this problem then is the same as for the pure axial load case, instability occurring at the Euler buckling load. However from the discussion of Fig. 3 and from the bending resistance plot of Fig. 10 it is obvious that this condition can never be reached and satisfy the assumption of an elastic internal stress situation. To solve the problem then, it is first necessary to determine internal bending stiffness above the elastic limit.

**Bending Stiffness Above the Elastic Limit**

As was discussed in Section III (Fig. 4) the moment-curvature relation above the elastic limit is a function of axial thrust as well as cross-sectional dimensions and mechanical properties of the material. Thus to be able to use the resulting M-\( \theta \) curves for integration it is necessary to first determine compatible moment and thrust quantities corresponding to given \( \theta \) values. This can become a major problem for anything other than simple shapes; however, by modifying the problem under study

* Generalizing, it can be shown that if the member is pin-ended, regardless of lateral loads, inelastic or elastic action, etc.,

\[ \Delta M = P (A_y) \]

For the case of restrained members, a factor taking into account the end condition will need to be added to the right hand side of the equation.
to one where axial thrust is held constant and moments increased to collapse, a relatively simple, semigraphical solution is available (9) whereby compatible $M$, $P$, $\phi$ values can be determined for any type of cross-section having one axis of symmetry.

Two sets of curves resulting from that previous study are here shown as Fig. 11 and Fig. 12 and are the $M-\phi$ plots (at constant $P$-values) for the 8WF51 (strong axis bending) and for the rectangular section respectively. These plots have been made non-dimensional by dividing the moment value by $M_y$, the moment corresponding to initial yield in the absence of axial thrust, and the curvature value by $\phi_y$, the curvature corresponding to $M_y$.

To have these results in a form suitable for direct use in Eq. 5 it is necessary to further develop them by obtaining tangent values at various points along each curve. These tangent values will be of the form

$$\frac{\Delta M}{\Delta \phi} \cdot \frac{\phi_y}{M_y}$$

which can be considered as a measure of the desired internal stiffness term, $\frac{\Delta M}{\Delta \phi} \bigg|_{\text{int}}$, in Eq. 5. The tangent curves obtained from Figs. 11 and 12 are shown in Figs. 13 and 14. In each case the tangent values are plotted as a function of the two coordinates $M/M_y$ and $\phi/\phi_y$.

If a cosine deflection curve is assumed and the external rate of change substituted in

$$\frac{\Delta M}{\Delta \phi} \cdot \frac{\phi_y}{M_y} \bigg|_{\text{ext}} = \frac{\Delta M}{\Delta \phi} \cdot \frac{\phi_y}{M_y} \bigg|_{\text{int}}$$
then,

$$\left. \frac{\Delta M}{\Delta \phi} \cdot \frac{\phi_y}{M_y} \right|_{\text{int}} = \frac{Pcr}{P_e},$$  

(10)

since

$$\left. \frac{\Delta M}{\Delta \phi} \cdot \frac{\phi_y}{M_y} \right|_{\text{ext}} = \frac{PL^2}{\gamma^2} \cdot \frac{\phi_y}{N_y} = \frac{PL^2}{\gamma^2} \cdot \frac{1}{EI} = \frac{P}{P_e}$$

In these equations, $P_e$ is the Euler buckling load for the member of length $L$. 
V. APPLICATIONS

For any member subjected to both end thrust and bending moments, the total moment at any section is composed of two parts: one is due to the axial thrust times the deflection; the other is independent of deflection (e.g. end moments, lateral loads, etc.). That part of the centerline moment which is independent of deflection is noted throughout the remainder of this discussion as "m_o".

If it is assumed that the moment producing loads are symmetrically placed on the structure and that the deflection curve can be approximated by the equation

\[ y = y_0 \cos \frac{\eta x}{L} \]  

(11)

the external rate of increase in centerline moment with respect to curvature will be given by the equation

\[ \frac{\Delta M}{\Delta \theta} \bigg|_{\text{ext}} = P \frac{L^2}{\eta^3} \]

However, to have this in the non-dimensional form suitable for use in Figs. 13 and 14, it is necessary to multiply both sides of the equation by 1/\(E I\). This then, as was shown in the last section, gives

\[ \frac{\Delta M}{\Delta \theta} \bigg|_{\text{int}} = \frac{P}{P_e} = \left( \frac{P}{P_e} \right) \left( \frac{P}{P_e} \right) \left( \frac{L}{P} \right)^2 \]  

(12)

where \( P_e \) is the Euler buckling load for the pin-ended member.
Since a cosine deflection curve (Eq. 11) was assumed, the centerline deflection can be expressed as

\[ \gamma_0 = \left( \frac{\phi_0}{\phi_y} \right) \frac{L^2}{\pi^2} \phi_y \]  

(11a)

Substituting Eq. (11a) in the following expression for \( m_0 \)

\[ \frac{m_0}{M_y} = \frac{M_o}{M_y} - \frac{P \gamma_0}{M_y} \]  

(13)

\[ \frac{m_0}{M_y} = \frac{M_o}{M_y} - \frac{P}{F_e} \left( \frac{\phi_0}{\phi_y} \right) \]  

(14)

It should here be noted in the derivation of Eq. 14 it was not necessary to consider the explicit form of the moment producing loads; only the axial thrust, an assumed deflection curve and the length of the member was used. Eq. (14) then is the solution to the general inelastic instability problem for members, which due to applied loads, deform symmetrically according to Eq. (11).

To illustrate the procedure, consider the following numerical example:

Given an SWF31 section, \( L/r = 80 \), subjected to a constant axial thrust value of \( P/P_y = 0.4 \); determine the critical values of various "moment producing, loading components". (See Fig. 15.)

For this problem, according to Eq. (12), \( \Delta M/\Delta \phi \cdot \phi_y/M_y \mid_{\text{int}} = 0.346 \), which from Figs. 13 and 14 give the critical centerline
Substitution of these quantities in Eq. 14 gives the following values for $\frac{m_0}{M_y}$

<table>
<thead>
<tr>
<th>$\frac{\Delta M}{\Delta \theta} \cdot \frac{\phi_y}{M_y}$</th>
<th>Strong Axis Bending</th>
<th>Weak Axis Bending (Rectangle)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.346$</td>
<td>$0.624$</td>
<td>$0.904$</td>
</tr>
<tr>
<td>$\frac{N_o}{M_y}$</td>
<td>$0.653$</td>
<td>$1.105$</td>
</tr>
</tbody>
</table>

Since in Fig. 15 the moment at the center of the column exclusive of thrust times deflection is equal to

$$m_0 = (N_o + \frac{Rl}{4})$$

the critical value of the moment producing loads will be given by the following expressions:

$$\frac{m_0}{M_y} = \frac{\frac{N_o}{M_y}}{cr} = 0.398 \text{ (for strong axis bending)}$$

$$= 0.522 \text{ (for weak axis bending)}$$

* These values have been circled in Figs. 13 and 14.
Had other loading conditions been imposed, a similar procedure would have been used. The expressions for a few typical problems are shown in Table I.

In each of these problems, the quantity of primary importance is \( \frac{m_0}{M_y} \). To facilitate solution of other problems where length and \( P/P_y \) values are different from this one, values of \( \frac{m_0}{M_y} \) have been obtained for both strong axis and weak axis bending in a range of slenderness from \( 0 \leq L/r \leq 120 \). The plots resulting from these calculations are shown in Figs. 16 and 17. They are applicable only to those cases of load which produce single curvature deformation that can be approximated by a "full" cosine curve.

Figs. 16 and 17 can be used to solve three basically different types of beam-column problems.

1. Given a constant axial thrust, determine the maximum lateral load and/or end moment the member can carry.

2. Given the lateral loads and/or end moments, determine the maximum axial thrust to which the member can be subjected.

3. Given an eccentricity, determine the maximum axial thrust.

The first two are solvable directly from the figures. For the third case, however, it is first necessary to "lay-off" a line which described the relationship between \( m_0 \) and \( P \).* The point of intersection of this "e-line" with the given slenderness ratio will determine the critical axial thrust.

* This line would correspond to the equation \( m_0 = Pe \) or \( \left( \frac{m_0}{M_y} \right) = \left( \frac{P}{P_y} \right) \left( \frac{ec}{r^2} \right) \).
VI. DISCUSSION AND SUMMARY

In the foregoing presentation a statistical method was used for determining conditions of indifference. This method implies that an equilibrium position is known and it is desired to determine if that position is stable or unstable. Solution was achieved by comparing the rate of change in external moment to the rate of change in internal resistance moment for a given virtual displacement of the loaded specimen. If the internal moment increased at a rate greater than the external moment, the specimen was stable. If the rate of change in external was the larger, the member was unstable. For the case where the rates of change were equal, a condition of indifference (incipient instability) existed.

a. Influence of Assumed Deflected Configuration

For ease of solution, it was assumed throughout this report that the over-all behavior of the member could be predicted with sufficient accuracy by assuming a deflection curve, the magnitude of which would be governed by the strain distribution pattern at the most highly deformed cross-section. This simplification eliminated the need of minimizing integrals and resulted in a direct determination of an end moment, axial thrust and critical length corresponding to a given stress distribution pattern. However, when using such a procedure, a certain amount of error is introduced, the member essentially being forced into an unnatural configuration. To determine the magnitude of this error, the following three comparisons will be made.
a. Comparison of strength results obtained for several different curves.

b. Comparison of the deflection and strength results of each of these curves with individual solutions obtained by numerical integration, and

c. Comparison with previous published solutions; namely, Chwalla's(1), Jezek's(1) and Timoshenko's.(14)

In the first comparison, the three types of configuration shown in Fig. 18 are considered. The first of these, that of a "full-cosine" curve, was used in the development of the general method of solution (Section III) and also in the preceding section on "Applications". The second is a parabola, and the third is a "partial cosine" curve of the form suggested by Westergaard and Osgood (16). In the first two cases the moment boundary condition is violated.

Since each of these assumed curves involves one parameter, \( y_0 \), the previously derived instability condition,

\[
\frac{\Delta M}{\Delta \theta} \bigg|_{\text{ext}} = \frac{\Delta M}{\Delta \theta} \bigg|_{\text{int}},
\]

can be directly applied. For the first two cases,

a. "Full Cosine" curve

\[
\frac{\Delta M}{\Delta \theta} \bigg|_{\text{int}} = P \frac{L^2}{\gamma^2}, \quad \text{and} \quad (15)
\]

b. "Parabolic" curve

\[
\frac{\Delta M}{\Delta \theta} \bigg|_{\text{int}} = P \frac{L^2}{8}. \quad (16)
\]
For the "Partial Cosine" curve (assumption "c"), it is necessary to further develop \( \frac{\Delta M}{\Delta \phi} \bigg|_{\text{ext}} \) so that the ends of the "L" length are held at a given deflection, \( \epsilon \), rather than maintaining a hypothetical "L"-length member. (See Fig. 19.) If such a procedure is carried out a complex equation containing functions of \( \Delta y_0 \) and \( \Delta \phi \) and cosines thereof is obtained which cannot be solved explicitly for \( \Delta y_0 \) in terms of \( \Delta \phi_0 \). However, by considering only the first two terms of a cosine series, the expression can be reduced and results in the following equation:

\[
\frac{\Delta M}{\Delta \phi} \bigg|_{\text{int}} = P \frac{L^2}{\pi^2} \tag{17}
\]

Even though this is the same external rate of moment increase as found for case "a", the solutions for members "a" and "c" are different, the \( M_e \) values being determined by different equations.

a. "Full Cosine" curve

\[
M_e = M_o - P \phi_o \frac{L^2}{\pi^2} \tag{18}
\]

b. "Partial Cosine" curve

\[
M_e = M_o \cos \frac{\pi L}{2} \tag{19}
\]

\( L \) is determined by considering the full value of \( M_o \) and \( \phi_o \) at the center section, i.e.

\[
L^2 = \frac{M_o}{\phi_o} \cdot \frac{\pi^2}{P} \tag{20}
\]

\[
\left( \frac{L}{r} \right)^2 = \frac{M_o/M_y}{P/P_y} \left( \frac{\sigma_y}{\pi^2 E} \right) \tag{20a}
\]
The comparison between these solutions is illustrated in Figs. 20 and 21. Here is presented a series of interaction curves for various length ratios for the SWF31 member bent about both its strong and weak axes. The "full cosine" deflection curve gave the highest strength values. The "parabola" gave the lowest prediction.

A recent paper by John Clark (4) discusses, in part, this same question. Therein he not only investigates the applicability of the "partial cosine" curve but also a much more complicated three parameter deflection equation. Comparison with test results is also shown in that report. In general, however, his results indicate that the increase in accuracy obtained by using the more complicated deflection expression does not warrant the added work required to obtain a solution.

The second means of comparison, numerical integration, was used as a check in a previous report (9) and one figure based on that work is here included to indicate the observed trends. (See Fig. 22.) There is difference between the load deflection curves obtained by the various methods; however, the ultimate carrying capacities of the members are in comparatively good agreement for these examples.

The third comparison is that between previous published solutions and the curves of ultimate strength as predicted by the "partial cosine" deflection curves presented earlier in this discussion. Since previous information regarding WF section behavior is lacking, comparison is made only for the rectangular section; furthermore, only for two slenderness ratios.
Chwalla's solution is based on a numerical integration procedure and should therefore be considered as being "exact". His derivation is based on (a) a typical German steel with a stress-strain curve having a proportional limit of 27 ksi, and a yield stress level of 34 ksi. Jezek's is an approximation based on (a) a "full cosine" deflection curve and (b) a stress-strain curve of the idealized elastic type (see Fig. 1). Timoshenko's solution also considers (a) the "full cosine" curve but (b) the stress-strain curve has a proportional limit 30 ksi, a yield level of 36 ksi and an ultimate strength of 56 ksi.

Fig. 23 shows the comparison obtained between each of these previous solutions and the "partial cosine" solution of this report. All curves have been adjusted to a yield stress level of 40 ksi. Length adjustment was according to the equation

\[
\left( \frac{L}{T} \right)_{\text{adj}} = \left( \frac{L}{T} \right) \sqrt{\frac{\sigma_y}{40.0}}
\]

In general, Chwalla's solution gives the lowest strength values; Jezek's gives the highest. The partial cosine solution is between these two limits, as is Timoshenko's solution.

b. Elastically Restrained Members

In actual structures, columns are usually restrained at their ends rather than being pin-ended, and a recent paper\(^2\) discusses such a condition. In that study a method was developed for reducing the actual restrained system to one of an equivalent, pin-ended, eccentrically loaded column. Having determined by that
method the equivalent member for a restrained column problem, the solution and curves presented in this paper directly apply. c. Residual Stresses

To include the variable of residual stress, it is necessary to develop a complementary set of curves for the desired pattern and magnitude of residual stress. This development would start from a given M-Φ curve (9), which included the effect of the residual stress pattern, and proceed along the same lines previously described.

d. Load-Deflection History

It should be pointed out that no special consideration has been given to the manner in which a condition of instability is approached; that is, the load-deflection history of the member in question. The presented curves and numerical work assume that an axial load, smaller than that required for failure, is first applied and that bending is then increased from zero to its maximum value. Under this condition there will be no unloading of previously yielded fibers. The resulting critical loads then will be comparable to the reduced modulus load for axially loaded members.

As was clearly demonstrated, first by Shanley (13) and later by Duberg and Wilder (5) and others, if an axially loaded column is considered as an imperfect compression member with the imperfection approaching zero, the column will start to bend at the tangent modulus load. Increased thrust beyond this value will be accompanied by an increasing rate of lateral deflection, the member reaching its maximum strength at some point prior to attainment of the reduced modulus load.
Since one limit of the generalized beam-column problem is the axially loaded column, there will be a certain amount of error involved in the method of solution developed in this report. While it is felt that for the assumed stress strain curve of Fig. 1, the resulting error is quite small, further extension of this work to materials having curved stress-strain relations (either as a basic material property or as a result of residual stress, etc.) should be accompanied by suitable check calculations to indicate the amount of error involved in this type of a procedure. The error will be primarily a function of the initial eccentricity. Other important factors would be the type of cross-section and the stress-strain relation.

A recent NACA report (17) provided one source of information for these check calculations. Therein load-deflection curves are presented (for an idealized WF section column) for various initial eccentricities and stress-strain curves.

Since a majority of the practical beam-column problems encountered in the analysis of Civil Engineering type structures contain moment producing loads greater than that which could be attributed to accidental imperfections, this "tangent modulus influence" will more than likely not be a real problem.

It is interesting to note that the early numerical work on collapse strength of rectangular section, beam-columns carried out by Chwalla was based on a typical "German Steel" type stress-strain relation (linear up to the proportional limit, parabolic from the proportional limit to the yield stress level and then horizontal to the point at which strain hardening
commences). Recently (2) this work has been corrected and it was found that adjustment was needed only when the ratio of the centerline deflection at maximum load to the depth of the section was less than 0.1, a comparatively small deflection value.

**e. Local and Lateral Instability**

Neither local nor lateral types of instability were considered in the development presented in this paper. The considered condition of instability in each case has been that of excessive bending in the plane of the applied moment. However, when bending is forced about the major axis of a section (that condition most frequently encountered in practice); when there is appreciable difference in bending stiffness between the two principal axes of the section; and when there is little if any lateral support along the member, the mode of failure will more than likely be that due to a combined bending and twisting action (8). In practice, however, this will not necessarily be the case since members will be frequently restrained by bracing, walls, floors, etc. Where sufficient lateral restraint is provided, the problem of bending plus twist is non-existent, and collapse can be determined by the methods previously described.

Another entirely different type of problem can develop when the cross-section is composed of connected, relatively thin, projecting elements. For those types of members, the projecting elements may individually or collectively deform in certain local regions and thereby change the original cross-sectional properties of the member.

Assuming the validity of any of the currently available inelastic local buckling theories, the maximum strain, $\varepsilon_{\text{max}}$,
that a flange can sustain and remain in the straight position can be predicted. From this can be determined the critical value of \( \frac{\varepsilon_{\text{max}}}{\varepsilon_y} \). Using the procedure outlined in Ref. 9 and noting that
\[
\frac{\varepsilon_{\text{max}}}{\varepsilon_y} = 1 + 2\alpha \left( \frac{\phi_o}{\phi_y} \right)
\]
(see Fig. 24), the corresponding critical value of \( \phi_o/\phi_y \) for available, rolled, wide-flange shapes can be determined and is found to be comparatively large \( \phi_o/\phi_y \geq 3.0 \). From Fig. 13 it can be seen that a condition will be realized only at extremely low values of thrust, \( P_{\text{cr}}/P_e \leq 0.05 \).

**SUMMARY**

This report presents a virtual displacement solution to the inelastic instability problem, the method being equally adaptable to elastic, inelastic, concentrically loaded, eccentrically loaded and/or laterally loaded columns.

The essence of the solution is that of applying a virtual curvature change to the loaded member and equating the rate of change in external moment at the most highly stressed section to the internal stiffness resulting from that curvature change. For ease of solution, deflection curves of one parameter were assumed causing the system to reduce to collocation at one point. It was shown that the ultimate strength of such members is relatively unaffected by choice of deflection curve, the "full cosine" giving the highest strength prediction and the parabola giving the lowest prediction. (Figs. 20 and 21.) The "partial cosine" solution compared favorably with previously available solutions. (Fig. 23.)
To facilitate solution of problems other than those discussed herein, curves have been presented which ensure rapid determination of critical loading values (Figs. 20 and 21) for any symmetrically deformed pin-ended member.
VII. ACKNOWLEDGMENTS

The author wishes to express his sincere appreciation to Bruno Thurlimann, who offered many valuable suggestions throughout the development of this approach to the inelastic instability problem. Grateful acknowledgment is also extended to Paul C. Paris for criticism and suggestions of the manuscript. This work has been carried out as part of the project "Welded Continuous Frames and Their Components" being conducted under the general direction of Lynn S. Beedle. It is sponsored jointly by the Welding Research Council and the Navy Department with funds furnished by the following: American Institute of Steel Construction, American Iron and Steel Institute, Column Research Council (Advisory), Office of Naval Research (Contract No. 39303), Bureau of Ships and Bureau of Yards and Docks. The helpful criticisms of members of the Welding Research Council Lehigh Project Sub-committee (T. R. Higgins, chairman), and the Column Research Council Research Committee D (N. M. Newmark, chairman and N. J. Hoff, co-chairman) are sincerely appreciated. This program is being carried out in the Fritz Engineering Laboratory, Lehigh University of which William J. Eney is Director.
### VIII. NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Area of cross-section</td>
</tr>
<tr>
<td>b</td>
<td>Width of cross-section</td>
</tr>
<tr>
<td>d</td>
<td>Depth of cross-section</td>
</tr>
<tr>
<td>e</td>
<td>End eccentricity of axial load application</td>
</tr>
<tr>
<td>E</td>
<td>Young's modulus of elasticity</td>
</tr>
<tr>
<td>I</td>
<td>Moment of inertia of total cross-section about the principle axis of section under consideration</td>
</tr>
<tr>
<td>L</td>
<td>Length of member (distance between pins)</td>
</tr>
<tr>
<td>l</td>
<td>Length of hypothetical member (used in &quot;Partial Cosine&quot; deflection curve assumption)</td>
</tr>
<tr>
<td>Δl</td>
<td>Change in length of hypothetical member due to a virtual change in centerline curvature</td>
</tr>
<tr>
<td>M</td>
<td>Bending moment</td>
</tr>
<tr>
<td>$M_{int}$</td>
<td>Internal bending moment due to stress distribution across section</td>
</tr>
<tr>
<td>$M_{ext}$</td>
<td>External bending moment due to lateral loads, end moment, initial curvature, and axial thrust times deflection</td>
</tr>
<tr>
<td>$M_Y$</td>
<td>Bending moment at which yield point is reached in flexure</td>
</tr>
<tr>
<td>$M_o$</td>
<td>Bending moment at the centerline section of the member</td>
</tr>
<tr>
<td>$m_o$</td>
<td>That part of the centerline moment which is independent of deflection</td>
</tr>
<tr>
<td>$M_e$</td>
<td>End applied bending moment</td>
</tr>
<tr>
<td>$ΔM_o$</td>
<td>Change in value of centerline moment due to a virtual change in curvature at that same section</td>
</tr>
<tr>
<td>P</td>
<td>Axial thrust</td>
</tr>
<tr>
<td>$P_Y$</td>
<td>Axial thrust corresponding to yield stress level over entire section</td>
</tr>
<tr>
<td>$P_e$</td>
<td>Euler buckling load</td>
</tr>
<tr>
<td>$P_{cr}$</td>
<td>Maximum thrust a member can carry</td>
</tr>
</tbody>
</table>
Concentrated lateral load
Flange thickness (wide-flange section)
Deflection parallel to web in undeformed position
Deflection perpendicular to web in undeformed position
Deflection in plane of moment application
Centerline deflection
Change in centerline deflection due to a virtual change in centerline curvature
Ratio of the depth of yield penetration in compression to total depth of section
Unit strain
Unit strain corresponding to initial yield
Maximum average unit strain on compression flange of wide-flange section (strong axis bending)
Unit normal stress
Lower yield point stress
Variable normal stress on tension side of cross-section (positive when in tension)
Curvature
Centerline curvature
Virtual change in centerline curvature
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FIG. 1 IDEALIZED STRESS-STRAIN RELATION

FIG. 2 TYPICAL LOAD-DEFLECTION RELATION FOR AXIALLY LOADED COLUMNS

FIG. 3 LOAD-DEFLECTION RELATION FOR ECCENTRICALLY LOADED COLUMNS
FIG. 4 BASIC INTERNAL STRESS-LOAD RELATIONS

\[ M = \int \sigma_y \, dA \]
\[ P = \int \sigma \, dA \]
\[ \Phi = \left. \frac{d\sigma}{dy} \right|_{\text{neutral axis}} \]

FIG. 5 TYPICAL BEAM-COLUMN

\[ \Delta M_{\text{int}} = \int \Delta \sigma_y \, dA = EI \Delta \Phi \]

FIG. 6 CHANGE IN STRESS DISTRIBUTION AND RESISTING MOMENT DUE TO A VIRTUAL CHANGE IN CURVATURE (ELASTIC-AXIAL LOAD)

FIG. 7 TYPICAL BENDING RESISTANCE RELATION FOR AXIALLY LOADED, ELASTIC COLUMNS
FIG. 8 TYPICAL BEAM-COLUMN SUBJECTED TO VIRTUAL DISPLACEMENT

\[ \Delta M_{int} = \int_A \sigma y dA = EI \Delta \phi \]

FIG. 9 CHANGE IN STRESS DISTRIBUTION AND RESISTING MOMENT DUE TO A VIRTUAL CHANGE IN CURVATURE (ELASTIC-ECCENTRIC LOAD)

FIG. 10 TYPICAL BENDING RESISTANCE RELATION FOR ECCENTRICALLY LOADED COLUMNS
FIG. 11  MOMENT-CURVATURE RELATIONS FOR THE 8 WF 31 SECTION, STRONG AXIS BENDING (9)

FIG. 12  MOMENT-CURVATURE RELATIONS FOR THE RECTANGULAR SECTION (9)
INTERNAL RATE OF CHANGE OF MOMENT VERSUS MOMENT AND CURVATURE FOR CONSTANT AXIAL THRUST VALUES
FIG. 15  TYPICAL BEAM-COLUMN
(used in example problem)

FIG. 16  INTERACTION CURVES FOR 8 WF 3I SECTION
(STRONG AXIS BENDING)

FIG. 17  INTERACTION CURVES FOR 8 WF 3I SECTION
(WEAK AXIS BENDING)
\[ y = y_0 \cos \frac{\pi x}{L} \]

\[ y = y_0 \left[1 - 4x^2/L^2\right] \]

\[ y = y_0 \cos \frac{\pi x}{L} \]

FIG. 18 ASSUMED DEFLECTION CURVES

FIG. 19 VIRTUAL DEFLECTION FOR "PARTIAL COSINE" ASSUMPTION

FIG. 20 INTERACTION CURVES SHOWING INFLUENCE OF DEFLECTION CURVE ASSUMPTION (STRONG AXIS BENDING)
Fig. 21 Interaction curves showing influence of deflection curve assumption (weak axis)

Fig. 22 Load-deflection curves showing influence of deflection curve assumption (weak axis bending)
* Lengths have been adjusted to account for differences in yield level.
Curves correspond to $\sigma_y = 40$ ksi

**FIG. 23 INTERACTION CURVES SHOWING COMPARISON WITH PREVIOUS SOLUTIONS**

**FIG. 24 STRESS DISTRIBUTION PATTERN WHEN STRAINS EXCEED THE ELASTIC LIMIT**
<table>
<thead>
<tr>
<th>LOADING CONDITION</th>
<th>MOMENT DIAGRAM</th>
<th>DETERMINATION OF UNKNOWN</th>
</tr>
</thead>
<tbody>
<tr>
<td>P Me Me P</td>
<td><img src="image1" alt="Moment Diagram" /></td>
<td>( \frac{m_0}{M_y} = \frac{Me}{Me} )</td>
</tr>
<tr>
<td>P Pe P</td>
<td><img src="image2" alt="Moment Diagram" /></td>
<td>( \frac{m_0}{M_y} = \frac{P(e)_{cr}}{M_y} )</td>
</tr>
<tr>
<td>P R R P</td>
<td><img src="image3" alt="Moment Diagram" /></td>
<td>( \frac{m_0}{M_y} = R_{cr} \left[ \frac{L}{4M_y} \right] )</td>
</tr>
<tr>
<td>P W P</td>
<td><img src="image4" alt="Moment Diagram" /></td>
<td>( \frac{m_0}{M_y} = W_{cr} \left[ \frac{L^2}{8M_y} \right] )</td>
</tr>
<tr>
<td>P Me Me P</td>
<td><img src="image5" alt="Moment Diagram" /></td>
<td>( \frac{m_0}{M_y} = \frac{(Me+2RL)_{cr}}{My} )</td>
</tr>
<tr>
<td>P Me Me P</td>
<td><img src="image6" alt="Moment Diagram" /></td>
<td>( \frac{m_0}{M_y} = \frac{(Me+\frac{1}{2}WL^2)_{cr}}{My} )</td>
</tr>
<tr>
<td>P Pao P</td>
<td><img src="image7" alt="Moment Diagram" /></td>
<td>( \frac{m_0}{M_y} = \frac{Pao}{P_{cr}} \left[ \frac{P}{P_y} \right] \left[ \varphi (\alpha)_{cr} \right] )</td>
</tr>
</tbody>
</table>

\[
\left( \frac{ec}{r^2} \right) = \left[ \frac{m_0}{M_y} \right] \]