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A generalization of Kazhdan and Lusztig's R-polynomials

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Contents

Abstract

Several families of polynomials arise naturally as entries of transition matrices in many parameterized spaces. The Kazhdan-Lusztig polynomials and R-polynomials are well known examples of such families of polynomials. In this paper, we introduce other families which arise in analogous ways and show that the modified R-polynomials are in fact a subset of these new polynomials. We combinatorially describe the coefficients of these new polynomials in a way which generalize previous combinatorial descriptions of the modified R-polynomials. Next we state new symmetry results as well as give alternate proofs of known symmetries, providing a bijective proof of one such result. We then apply these results to give a new formulation of the dual canonical basis of the quantum polynomial ring in terms of Kazhdan-Lusztig immanants. Finally, we look at a two parameter version of the Hecke algebra and quantum polynomial ring, introducing two parameter analogs of the modified R-polynomials and show these satisfy recursive formulas and identities which resemble and generalize known recursive formulas and identities.

Introduction

Representations of quantum groups provide solutions to the Quantum Yang-Baxter equation and have applications in quantum field theory and statistical mechanics [32]. Quantum groups also have connections to other fields of mathematics such as algebraic geometry [33], category theory [29], knot theory [21], as well as having applications in nuclear physics [18], astrophysics [30], and other areas of science.

Kashiwara [20] and Lusztig [27] made contributions to the representation theory of quantum groups by introducing *canonical* (or *crystal*) bases known as enveloping algebras. Quantum coordinate rings such as $\mathcal{O}_q(SL(n,\mathbb{C}))$ are dual to the enveloping algebras and contain *dual canonical* bases. The quantum polynomial ring $\mathcal{A}(n;q)$, while not a quantum group, is closely related to $\mathcal{O}_q(SL(n,\mathbb{C}))$ and contains a *dual canonical basis*, which is related by Hopf algebra duality to Kashiwara's [20] and Lusztig's [27] canonical basis of $\mathfrak{sl}(n,\mathbb{C})$. Understanding these bases will allow the construction of representations of the algebras and lead to representations of the *quantum groups*. However, these bases are defined recursively and are not easy to construct or understand.

The Iwahori-Hecke algebra, $H_n(q)$, is a single parameter deformation of the group algebra $\mathbb{C}[\mathfrak{S}_n]$, where $\mathfrak{S}_n$ is the *symmetric group*. In particular, $H_n(q)$ specializes to $\mathbb{C}[\mathfrak{S}_n]$ when $q = 1$. Kazhdan and Lusztig [22] introduced a basis and irreducible modules for $H_n(q)$, which make use of polynomials defined recursively known as *Kazhdan-Lusztig polynomials*. An important ingredient in the definition of their basis is known as the *bar involution*. Applying this involution to a natural basis element of $H_n(q)$, one obtains a new basis, related to the first by polynomials in $\mathbb{Z}[q]$ known as *R-polynomials*. Alternatively, one may use *modified R-polynomials* in $\mathbb{N}[q]$.

Coefficients of the modified R-polynomials and their combinatorial interpretations were studied by Brenti [3], [4], [5], [6], [7], Deodhar [13], and Dyer [17].

Certain $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-submodules of $H_n(q)$ called *double parabolic modules* inherit a bar involution from $H_n(q)$, and therefore inherit analogs of R-polynomials called *parabolic R-polynomials*. Also belonging to $\mathbb{Z}[q]$, these parabolic R-polynomials appear in numerous papers, yet somehow have not received the modification and combinatorial interpretation granted to their non parabolic siblings.

Related to the bar involutions on $H_n(q)$ and its parabolic modules is another involution on $\mathcal{A}(n;q)$. This last involution, also called the *bar involution*, is an important ingredient in the definition of the dual canonical basis of the quantum polynomial ring. Again, applying this involution to a natural basis of $\mathcal{A}(n;q)$, one obtains a second basis, related to the first by *inverse R-polynomials* and *inverse parabolic R-polynomials* (equivalently, by modifications of these).

To summarize, we have several algebras with the property that a natural basis and its bar image are related by a transition matrix whose entries are variations of R-polynomials. Using an elementary family of bases of $\mathcal{A}(n;q)$, we show that in all cases, the above entries have simple combinatorial interpretations in terms of walks in the Bruhat order.

In Chapter 1 we first review some properties of the symmetric group, as it plays a very important role in the Hecke algebra and quantum polynomial ring. We then define the Hecke algebra $H_n(q)$ and it's submodules $H_{I,J}$ and $H'_{I,J}$, as well as the quantum polynomial ring $\mathcal{A}(n;q)$. Next we introduce a new family of polynomials, which we define combinatorially. This new family of polynomials will be shown to be a generalization of the modified R-polynomials in Chapter 3. The final section of the chapter is spent showing these polynomials appear naturally in $\mathcal{A}(n;q)$, by proving a result connecting them to the basis expansion of the monomials in $\mathcal{A}(n;q)$.

Chapter 2 begins by establishing actions of $H_n(q)$ on various components of $\mathcal{A}(n;q)$, including the immanant space. We then use the actions to show that our polynomials also appear naturally in the multiplicative structure of $H_n(q)$. The second part of the chapter is then spent establishing different symmetries satisfied by our polynomials. These symmetries are found using the algebraic structure of

$H_n(q)$ and $\mathcal{A}(n; q)$, yet suggest interesting combinatorial results. We provide a bijective proof of one such symmetry.

We then shift our focus to bar involutions and the R-polynomials in Chapter 3. We define inverse R-polynomials and show that they are equal to the R-polynomials in the nonparabolic case. Furthermore, the modified R-polynomials are shown to be a special subclass of the polynomials defined in Section 1.4. We also introduce double parabolic R-polynomials, an extension of the single parabolic R-polynomials appearing in the literature, and connect these to our new family of polynomials as well. The second half of the chapter focuses on the dual canonical basis. We give a new formulation in terms of the immanant space and row and column repetition. To conclude the chapter we examine the different variations of the dual canonical basis and bar involution appearing in the literature.

The final chapter introduces two-parameter generalizations of the Hecke algebra and quantum polynomial ring. We focus our attention on the immanant space at first, establishing results analogous to the first three chapters. We introduce definitions of two parameter analogs of the bar involutions and the modified R-polynomials, as well as the family of polynomials introduced in Chapter 1.

Chapter 1

The Hecke algebra and quantum polynomial ring

The symmetric group plays an important role in the behavior of the Hecke algebra and quantum polynomials ring. We will summarize some of the properties of the symmetric group, define the Hecke algebra and certain submodules of the Hecke algebra, following the treatment in [31]. Next we will review the definition of the quantum polynomial ring and generalize previous results about the immanant space in [31] to a general multi-graded component. Finally, we combinatorially define a new family of polynomials and show that these are the elements of transition matrices between bases in the quantum polynomial ring.

1.1 The Symmetric group $\mathfrak{S}_n$

The symmetric group $\mathfrak{S}_n$ has a standard presentation given by the generators $s_1, \ldots, s_{n-1}$ and the relations

$$\begin{aligned} s_i^2 &= 1, && \text{for } i = 1, \ldots, n-1, \\ s_i s_j s_i &= s_j s_i s_j, && \text{if } |i-j| = 1, \\ s_i s_j &= s_j s_i, && \text{if } |i-j| \geq 2. \end{aligned} \tag{1.1.1}$$

Let $[n]$ denote the set $\{1, \dots, n\}$. Let $\mathfrak{S}_n$ act on rearrangements of the letters $[n]$ by

$$s_i \circ v_1 \cdots v_n = v_1 \cdots v_{i-1} v_{i+1} v_i v_{i+2} \cdots v_n. \tag{1.1.2}$$

For each permutation $w = s_{i_1} \cdots s_{i_\ell} \in \mathfrak{S}_n$ we define the *one-line notation* of w to be the word

$$w_1 \cdots w_n = s_{i_1} \circ (\cdots (s_{i_\ell} \circ 1 \cdots n)) \cdots). \tag{1.1.3}$$

The one-line notation does not depend on the expression $s_{i_1} \cdots s_{i_\ell}$ for w. When an expression for w is as short as possible, we say that expression is *reduced*. Furthermore, we call $\ell = \ell(w)$ the *length* of w.

The *Bruhat order* on $\mathfrak{S}_n$ is defined by $v \leq w$ if some (equivalently every) reduced expression for w contains a reduced expression for v as a subword (See [2] for more information). A generator is called a *left ascent* for w if $sv > v$, and a *left descent* otherwise. Right ascents and descents are defined analogously. The unique maximal element in the Bruhat order will be denoted by w_0. This permutation has one line notation $n(n-1)\cdots 2\,1$.

It is known that left or right multiplication by w_0 induces an antiautomorphism of the Bruhat order. Thus if $u < v$, then we have

$$w_0 v < w_0 u, \qquad v w_0 < u w_0. \tag{1.1.4}$$

We also have that

$$\ell(w_0 v) = \ell(v w_0) = \ell(w_0) - \ell(v) \tag{1.1.5}$$

For all $v \in \mathfrak{S}_n$.

Let I and J be subsets of the standard generators (adjacent transpositions) of $W = \mathfrak{S}_n$ and let W_I, W_J be the corresponding parabolic subgroups of W. Define $W_\emptyset = e$. Let w_0^I and w_0^J be the longest elements in W_I and W_J, respectively. It is easy to see that $(w_0^I)^{-1} = w_0^I$ and $(w_0^J)^{-1} = w_0^J$.

Let $W_I \backslash W / W_J$ be the set of double cosets of the form $W_I w W_J$. If $J = \emptyset$, this is the set of single cosets $W_I \backslash W$ of the form $W_I w$. If $I = \emptyset$, this is the set of single cosets W/W_J of the form wW_J. If $I = J = \emptyset$, this is just $W = \mathfrak{S}_n$. Each double coset is an interval in the Bruhat order and has a unique maximal and minimal element [10], [14]. Moreover, each double coset in $W_I\backslash W/W_J$ is equal to a union of single cosets in $W_I\backslash W$ and is equal to a union of single cosets in W/W_J,

$$W_IwW_J = \bigcup_{u\in W_Iw} uW_J = \bigcup_{u\in wW_J} W_Iu. \tag{1.1.6}$$

While the double cosets partition W, unlike single cosets, double cosets need not have the same cardinality, i.e., for $u \neq v$ we may have $|W_IuW_J| \neq |W_IvW_J|$.

Let $W_+^{I,J}$ be the set of maximal representatives of cosets in $W_I\backslash W/W_J$ and let $W_-^{I,J}$ be the set of minimal coset representatives. These sets may be characterized by

$$\begin{aligned}
&W_+^{I,J}\\
&= \{w \mid sw < w \text{ for all } s \in I, ws' < w \text{ for all } s' \in J\}\\
&= \{w \mid w_i > w_{i+1} \text{ for all } s_i \in I, j+1 \text{ appears before } j \text{ in } w_1 \cdots w_r \text{ for all } s_j \in J\}\\
&= W_+^{I,\emptyset} \cap W_+^{\emptyset,J},\\
&W_-^{I,J}\\
&= \{w \mid sw > w \text{ for all } s \in I, ws' > w \text{ for all } s' \in J\}\\
&= \{w \mid w_i < w_{i+1} \text{ for all } s_i \in I, j \text{ appears before } j+1 \text{ in } w_1 \cdots w_r \text{ for all } s_j \in J\}\\
&= W_-^{I,\emptyset} \cap W_-^{\emptyset,J},
\end{aligned} \tag{1.1.7}$$

Following Douglass [14, Lem 2.2], we define the following Bruhat order on double cosets. Let D_1 and D_2 be double cosets with corresponding minimum length representatives $u^{(1)}, u^{(2)}$, and corresponding maximum length representatives $v^{(1)}, v^{(2)}$. We define $D_1 \leq D_2$ if any of the following equivalent conditions hold,

1. $u^{(1)} \leq u^{(2)}$,

2. $v^{(1)} \leq v^{(2)}$,

3. There exist elements $w^{(1)} \in D_1, w^{(2)} \in D_2$ which satisfy $w^{(1)} \leq w^{(2)}$.

Suppose that w belongs to $W_+^{I,J}$. Then [14, Lem 2.2] implies that for each left descent s of w we have $sw \in W_+^{I,J}$ or $sw \in W_IwW_J$, and that for each right descent

s' of w we have $ws' \in W_+^{I,J}$ or $ws' \in W_I w W_J$. Similarly, that w belongs to $W_-^{I,J}$. Then [14, Lem 2.2] implies that for each left ascent s of w we have $sw \in W_-^{I,J}$ or $sw \in W_I w W_J$, and that for each right ascent s' of w we have $ws' \in W_-^{I,J}$ or $ws' \in W_I w W_J$.

For each minimal coset representative $u \in W_-^{I,J}$, we define the subsets

$$\begin{aligned} K = K(I,J,u) &\underset{\text{def}}{=} u^{-1}Iu \cap J \\ &= \{s \in J \mid usu^{-1} \in I\} \\ &= \{s_j \in J \mid s_i u = u s_j \text{ for some } s_i \in I\} \\ &= \{s_j \in J \mid u_i = j, u_{i+1} = j+1 \text{ for some } s_i \in I\}, \end{aligned} \tag{1.1.8}$$

$$\begin{aligned} K' \underset{\text{def}}{=} uKu^{-1} = K(J,I,u^{-1}) &= I \cap uJu^{-1} \\ &= \{s \in I \mid u^{-1}su \in J\} \\ &= \{s_i \in I \mid s_i u = u s_j \text{ for some } s_j \in J\} \\ &= \{s_i \in I \mid u_i = j, u_{i+1} = j+1 \text{ for some } s_j \in J\}, \end{aligned} \tag{1.1.9}$$

of generators. The parabolic subgroups $W_K, W_{K'}$ have longest elements $w_0^K = (w_0^K)^{-1}, w_0^{K'} = (w_0^{K'})^{-1} = u w_0^K u^{-1}$, and satisfy

$$\begin{aligned} W_K = u^{-1}W_I u \cap W_J &= \{v \in W_J \mid uvu^{-1} \in W_I\} \\ &= \{v \in W_J \mid uv = wu \text{ for some } w \in W_I\},, \\ W_{K'} = uW_K u^{-1} = W_I \cap uW_J u^{-1} &= \{w \in W_I \mid u^{-1}wu \in W_J\} \\ &= \{w \in W_I \mid uv = wu \text{ for some } v \in W_J\}. \end{aligned} \tag{1.1.10}$$

The unique minimal representative u and maximal representative v of the double coset $W_I u W_J = W_I v W_J$ are related by

$$v = w_0^I u w_0^K w_0^J = w_0^I w_0^{K'} u w_0^J. \tag{1.1.11}$$

(See [34, Sec. 2].) By [16, (1.3.b)] we also have

$$\begin{aligned} \ell(v) &= \ell(u) + \ell(w_0^I) + \ell(w_0^J) - \ell(w_0^K) = \ell(w_0^I) + \ell(u) + \ell(w_0^K w_0^J) \\ &= \ell(u) + \ell(w_0^I) + \ell(w_0^J) - \ell(w_0^{K'}) = \ell(w_0^I w_0^{K'}) + \ell(u) + \ell(w_0^J). \end{aligned} \tag{1.1.12}$$

Since $W_K \subset W_J$ and $W_{K'} \subset W_I$, we define

$$(W_I)_+^{\emptyset,K'} \underset{\text{def}}{=} W_I \cap W_+^{\emptyset,K'}, \qquad (W_J)_+^{K,\emptyset} \underset{\text{def}}{=} W_J \cap W_+^{K,\emptyset} \tag{1.1.13}$$

to be the collections of maximal representatives of the cosets

$$W_I/W_{K'} = \{vW_{K'} \mid v \in W_I\}, \qquad W_K \backslash W_J = \{W_K v \mid v \in W_J\} \tag{1.1.14}$$

and

$$(W_I)_-^{\emptyset,K'} \underset{\text{def}}{=} W_I \cap W_-^{\emptyset,K'}, \qquad (W_J)_-^{K,\emptyset} \underset{\text{def}}{=} W_J \cap W_-^{K,\emptyset} \tag{1.1.15}$$

to be the collections of minimal representatives. Thus we have

$$W_I = (W_I)_+^{\emptyset,K'} W_I = (W_I)_-^{\emptyset,K'} W_I \text{ and } W_J = W_K (W_J)_+^{K,\emptyset} = W_K (W_J)_-^{K,\emptyset}. \tag{1.1.16}$$

Furthermore, for $u \in W_-^{I,J}$ we have

$$\begin{aligned} W_I u W_J &= W_I u W_K (W_J)_-^{K,\emptyset} = W_I W_{K'} u (W_J)_-^{K,\emptyset} = W_I u (W_J)_-^{K,\emptyset} \\ &= (W_I)_-^{\emptyset,K'} W_{K'} u W_J = (W_I)_-^{\emptyset,K'} u W_K W_J = (W_I)_-^{\emptyset,K'} u W_J \\ &= (W_I)_-^{\emptyset,K'} W_{K'} u (W_J)_-^{K,\emptyset} = (W_I)_-^{\emptyset,K'} u W_K (W_J)_-^{K,\emptyset}. \end{aligned} \tag{1.1.17}$$

In other words given generator sets I, J, each element v of the double coset $W_I u W_J$ of $W = \mathfrak{S}_n$ has unique factorizations $v^I u v_-^J, v_-^I u v^J, v_-^I v^{K'} u v_-^J, v_-^I u v^K v_-^J$ satisfying

$$\begin{aligned} v^I \in W_I, \quad v_-^I \in (W_I)_-^{\emptyset,K'}, \quad v^{K'} \in W_{K'} \\ v^J \in W_J, \quad v_-^J \in (W_J)_-^{K,\emptyset}, \quad v^K \in W_K. \end{aligned} \tag{1.1.18}$$

Furthermore, these factorizations satisfy

$$\begin{aligned} \ell(v) &= \ell(v_-^I) + \ell(u) + \ell(v^J) = \ell(v^I) + \ell(u) + \ell(v_-^J) \\ &= \ell(v_-^I) + \ell(u) + \ell(v^K) + \ell(v_-^J) = +\ell(v_-^I) + \ell(v^{K'}) + \ell(u) + \ell(v_-^J) \end{aligned} \tag{1.1.19}$$

and

$$\begin{gathered} v^{K'} u = u v^K, \quad v^I = v_-^I v^{K'}, \quad v^J = v^K v_-^J, \\ \ell(v^{K'}) = \ell(v^K), \ell(v^I) = \ell(v_-^I) + \ell(v^{K'}), \quad \ell(v^J) = \ell(v^K) + \ell(v_-^J). \end{gathered} \tag{1.1.20}$$

Notice that for $w \in W_+^{I,J}$, we have $w_-^I = w_0^I w_0^{K'}$, $w^I = w_0^I$, $w_-^J = w_0^K w_0^J$, and $w^J = w_0^J$. For more information see [11, Thm. 1.2].

1.2 The Hecke algebra $H_n(q)$

The Hecke algebra $H_n(q)$ is the $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-algebra with multiplicative identity $\widetilde{T}_e = 1$ generated by elements $\{\widetilde{T}_{s_i} \mid 1 \leq i \leq n-1\}$ subject to the relations

$$\begin{aligned} \widetilde{T}_{s_i}^2 &= (q^{\frac{1}{2}} - q^{\frac{-1}{2}})\widetilde{T}_{s_i} + \widetilde{T}_e, && \text{for } i = 1, \ldots, n-1, \\ \widetilde{T}_{s_i}\widetilde{T}_{s_j}\widetilde{T}_{s_i} &= \widetilde{T}_{s_j}\widetilde{T}_{s_i}\widetilde{T}_{s_j}, && \text{if } |i-j| = 1, \\ \widetilde{T}_{s_i}\widetilde{T}_{s_j} &= \widetilde{T}_{s_j}\widetilde{T}_{s_i}, && \text{if } |i-j| \geq 2. \end{aligned} \tag{1.2.1}$$

Note that when $q = 1$ this reduces to the group algebra of the symmetric group $\mathbb{C}[\mathfrak{S}_n]$. If $s_{i_1} \cdots s_{i_\ell}$ is a reduced expression for w we define

$$\widetilde{T}_w = \widetilde{T}_{s_{i_1}} \cdots \widetilde{T}_{s_{i_\ell}}. \tag{1.2.2}$$

We shall call the elements $\{\widetilde{T}_w \mid w \in \mathfrak{S}_n\}$ the *natural basis* of $H_n(q)$ as a $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-module. For $u, v \in \mathfrak{S}_n$ we define $q_{u,v} = (q^{\frac{1}{2}})^{\ell(v)-\ell(u)}$. We remark that basis elements often denoted in the literature by $\{T_w \mid w \in \mathfrak{S}_n\}$ are related to our basis elements by $T_w = q_{e,w}\widetilde{T}_w$; however, we choose to follow the notation in [26].

Note that the first relation may be written as

$$(\widetilde{T}_{s_i} - q^{\frac{1}{2}})(\widetilde{T}_{s_i} - (-q^{\frac{-1}{2}})) = 0. \tag{1.2.3}$$

Inverses of the generators are given by

$$\widetilde{T}_{s_i}^{-1} = \widetilde{T}_{s_i} - (q^{\frac{1}{2}} - q^{\frac{-1}{2}})\widetilde{T}_e. \tag{1.2.4}$$

and a multiplication rule is given by

$$\widetilde{T}_{s_i}\widetilde{T}_w = \begin{cases} \widetilde{T}_{s_i w} & \text{if } s_i w > w, \\ \widetilde{T}_{s_i w} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})\widetilde{T}_w & \text{if } s_i w < w. \end{cases} \tag{1.2.5}$$

Similarly, we have

$$\widetilde{T}_w\widetilde{T}_{s_i} = \begin{cases} \widetilde{T}_{w s_i} & \text{if } w s_i > w, \\ \widetilde{T}_{w s_i} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})\widetilde{T}_w & \text{if } w s_i < w. \end{cases} \tag{1.2.6}$$

More generally, we have

$$\widetilde{T}_{u^{-1}}\widetilde{T}_v = \begin{cases} \widetilde{T}_{u^{-1}s_i}\widetilde{T}_{s_iv} & \text{if } s_iu > u \text{ and } s_iv < v, \\ & \quad\text{or if } s_iu < u \text{ and } s_iv > v, \\ \widetilde{T}_{u^{-1}s_i}\widetilde{T}_{s_iv} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})\widetilde{T}_{u^{-1}s_i}\widetilde{T}_v & \text{if } s_iu < u \text{ and } s_iv < v, \\ \widetilde{T}_{u^{-1}s_i}\widetilde{T}_{s_iv} - (q^{\frac{1}{2}} - q^{\frac{-1}{2}})\widetilde{T}_{u^{-1}s_i}\widetilde{T}_v & \text{if } s_iu > u \text{ and } s_iv > v. \end{cases} \tag{1.2.7}$$

For each coset W_IwW_J in $W_I\backslash W/W_J$, where $w \in W_+^{I,J}$, define the elements

$$\begin{aligned} \widetilde{T}_{W_IwW_J} &= \sum_{v\in W_IwW_J} \epsilon_{v,w}q_{v,w}\widetilde{T}_v, \\ \widetilde{T}'_{W_IwW_J} &= \sum_{v\in W_IwW_J} q_{v,w}^{-1}\widetilde{T}_v. \end{aligned} \tag{1.2.8}$$

Denote by $H_{I,J}$, $H'_{I,J}$, the submodules of $H_n(q)$ spanned by the double coset sums,

$$\begin{aligned} H_{I,J} &= \text{span}_{\mathbb{C}[q^{\frac{1}{2}},q^{\frac{-1}{2}}]}\{\widetilde{T}_{W_IwW_J} \mid w \in W_+^{I,J}\}, \\ H'_{I,J} &= \text{span}_{\mathbb{C}[q^{\frac{1}{2}},q^{\frac{-1}{2}}]}\{\widetilde{T}'_{W_IwW_J} \mid w \in W_+^{I,J}\}. \end{aligned} \tag{1.2.9}$$

For example when $n = 3$, $I = \{s_1\}$, $J = \{s_2\}$, we have

$$\begin{aligned} H_{I,J} &= \text{span}_{\mathbb{C}[q^{\frac{1}{2}},q^{\frac{-1}{2}}]}\{\widetilde{T}_{W_Is_1s_2s_1W_J}, \widetilde{T}_{W_Is_1,s_2W_J}\}, \\ H'_{I,J} &= \text{span}_{\mathbb{C}[q^{\frac{1}{2}},q^{\frac{-1}{2}}]}\{\widetilde{T}'_{W_Is_1s_2s_1W_J}, \widetilde{T}'_{W_Is_1,s_2W_J}\}, \end{aligned} \tag{1.2.10}$$

where

$$\begin{aligned} \widetilde{T}_{W_Is_1s_2s_1W_J} &= \widetilde{T}_{s_1s_2s_1} - q^{\frac{1}{2}}\widetilde{T}_{s_2s_1}, \\ \widetilde{T}_{W_Is_1,s_2W_J} &= \widetilde{T}_{s_1s_2} - q^{\frac{1}{2}}(\widetilde{T}_{s_1} + \widetilde{T}_{s_2}) + q\widetilde{T}_e, \end{aligned} \tag{1.2.11}$$

and

$$\begin{aligned} \widetilde{T}'_{W_Is_1s_2s_1W_J} &= \widetilde{T}_{s_1s_2s_1} + q^{\frac{-1}{2}}\widetilde{T}_{s_2s_1}, \\ \widetilde{T}'_{W_Is_1,s_2W_J} &= \widetilde{T}_{s_1s_2} + q^{\frac{-1}{2}}(\widetilde{T}_{s_1} + \widetilde{T}_{s_2}) + q^{-1}\widetilde{T}_e. \end{aligned} \tag{1.2.12}$$

We can let $H_n(q)$ act on the submodules $H_{I,\emptyset}$ and $H'_{I,\emptyset}$ by right multiplication

$$\begin{aligned} \widetilde{T}_{W_Iw}\widetilde{T}_{s_i} &= \sum_{v\in W_Iw} \epsilon_{v,w}q_{v,w}\widetilde{T}_v\widetilde{T}_{s_i}, \\ \widetilde{T}'_{W_Iw}\widetilde{T}_{s_i} &= \sum_{v\in W_Iw} q_{v,w}^{-1}\widetilde{T}_v\widetilde{T}_{s_i}, \end{aligned} \tag{1.2.13}$$

and on $H_{\emptyset,J}$ and $H'_{\emptyset,J}$ by left multiplication,

$$\begin{aligned}\widetilde{T}_{s_i}\widetilde{T}_{wW_J} &= \sum_{v\in wW_J} \epsilon_{v,w} q_{v,w} \widetilde{T}_{s_i}\widetilde{T}_v,\\ \widetilde{T}_{s_i}\widetilde{T}'_{W_I w} &= \sum_{v\in wW_J} q_{v,w}^{-1} \widetilde{T}_{s_i}\widetilde{T}_v.\end{aligned} \tag{1.2.14}$$

Douglass [14, Prop. 2.3] states formulas for these actions without assuming the representative w of the coset is maximal. In particular, we may write

$$\widetilde{T}_{W_I w}\widetilde{T}_{s_i} = \begin{cases} \widetilde{T}_{W_I w s_i} & \text{if } W_I w s_i > W_I w,\\ -q^{-\frac{1}{2}}\widetilde{T}_{W_I w} & \text{if } W_I w s_i = W_I w,\\ \widetilde{T}_{W_I w s_i} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\widetilde{T}_{W_I w} & \text{if } W_I w s_i < W_I w,\end{cases} \tag{1.2.15}$$

$$\widetilde{T}'_{W_I w}\widetilde{T}_{s_i} = \begin{cases} \widetilde{T}'_{W_I w s_i} & \text{if } W_I w s_i > W_I w,\\ q^{\frac{1}{2}}\widetilde{T}'_{W_I w} & \text{if } W_I w s_i = W_I w,\\ \widetilde{T}'_{W_I w s_i} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\widetilde{T}'_{W_I w} & \text{if } W_I w s_i < W_I w,\end{cases} \tag{1.2.16}$$

$$\widetilde{T}_{s_i}\widetilde{T}_{wW_J} = \begin{cases} \widetilde{T}_{s_i wW_J} & \text{if } s_i wW_J > wW_J,\\ -q^{-\frac{1}{2}}\widetilde{T}_{wW_J} & \text{if } s_i wW_J = wW_J,\\ \widetilde{T}_{s_i wW_J} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\widetilde{T}_{wW_J} & \text{if } s_i wW_J < wW_J,\end{cases} \tag{1.2.17}$$

and

$$\widetilde{T}_{s_i}\widetilde{T}'_{wW_J} = \begin{cases} \widetilde{T}'_{s_i wW_J} & \text{if } s_i wW_J > wW_J,\\ q^{\frac{1}{2}}\widetilde{T}'_{wW_J} & \text{if } s_i wW_J = wW_J,\\ \widetilde{T}'_{s_i wW_J} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\widetilde{T}'_{wW_J} & \text{if } s_i wW_J < wW_J.\end{cases} \tag{1.2.18}$$

From the definitions it is easy to see that for $u \in W_-^{I,\emptyset}$ and $v \in W_-^{\emptyset,J}$,

$$\widetilde{T}'_{W_I u} = \widetilde{T}'_{W_I}\widetilde{T}_u, \qquad \widetilde{T}'_{vW_J} = \widetilde{T}_v\widetilde{T}'_{W_J}. \tag{1.2.19}$$

Furthermore, following [16], define elements d_I and $\widetilde{d}_I$ of $\mathbb{C}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ which serve as generating functions for elements of W_I by the number of inversions,

$$\begin{aligned} d_I &\underset{\text{def}}{=} \sum_{v\in W_I} q^{\ell(v)},\\ \widetilde{d}_I &\underset{\text{def}}{=} q_{e,w_0^I}^{-1} \sum_{v\in W_I} q^{\ell(v)}.\end{aligned} \tag{1.2.20}$$

For example, when $I = \{s_1, s_2\}$ and $J = \{s_1, s_3\}$ we have

$$\begin{gathered} d_I = 1 + 2q + 2q^2 + q^3, \qquad \widetilde{d}_I = q^{\frac{-3}{2}} + 2q^{\frac{-1}{2}} + 2q^{\frac{1}{2}} + q^{\frac{3}{2}}, \\ d_J = 1 + 2q + q^2, \quad \text{and} \quad \widetilde{d}_J = q^{-1} + 2 + q. \end{gathered} \tag{1.2.21}$$

Note that d_I and $\widetilde{d}_I$ are not in general invertible in $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$. In [16] there is a calculation which implies for $w \in W_+^{I,J}$ we have

$$\widetilde{T}'_{W_I} \widetilde{T}_{w_-} \widetilde{T}'_{W_J} = \widetilde{d}_{K(w)} \widetilde{T}'_{W_I w W_J}, \tag{1.2.22}$$

where by w_- we are referring to the minimal coset representative for the double coset $W_I w W_J$ and $K(w) = K(w_-) = K(I, J, w_-)$ is defined by (1.1.8). We omit the details; however, a detailed proof of (1.2.22), which involves using the definitions to factor $\widetilde{T}'_{W_J}$ and the multiplication rules (1.2.16) and (1.2.18), appears in [31]. In a similar manner, using (1.2.16) and (1.2.18) we can see that for $v \in W_I w W_J$,

$$\widetilde{T}'_{W_I} \widetilde{T}_v \widetilde{T}'_{W_J} = q_{v,w}^{-1} \widetilde{T}'_{W_I} \widetilde{T}_w \widetilde{T}'_{W_J}. \tag{1.2.23}$$

Thus for $w \in W_+^{I,J}$, we have

$$(q^{\frac{1}{2}})^{\ell(W_I w W_J)} \widetilde{d}_{K(w)} \widetilde{T}'_{W_I w W_J} = \widetilde{T}'_{W_I} \widetilde{T}_w \widetilde{T}'_{W_J}, \tag{1.2.24}$$

where we define $\ell(W_I w W_J) = \ell(w_0^I) + \ell(w_0^J) - \ell(w_0^{K(w)})$ to be the distance between the minimal and maximal elements of the double coset $W_I w W_J$.

1.3 The quantum polynomial ring $\mathcal{A}(n; q)$

For each $n > 0$, let the *quantum polynomial ring* $\mathcal{A}(n; q)$ be the noncommutative $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-algebra generated by n^2 variables $x = (x_{1,1}, \ldots, x_{n,n})$ representing matrix entries, subject to the relations

$$\begin{aligned} x_{i,\ell} x_{i,k} &= q^{\frac{1}{2}} x_{i,k} x_{i,\ell}, \\ x_{j,k} x_{i,k} &= q^{\frac{1}{2}} x_{i,k} x_{j,k}, \\ x_{j,k} x_{i,\ell} &= x_{i,\ell} x_{j,k}, \\ x_{j,\ell} x_{i,k} &= x_{i,k} x_{j,\ell} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}}) x_{i,\ell} x_{j,k}, \end{aligned} \tag{1.3.1}$$

for all indices $1 \leq i < j \leq n$ and $1 \leq k < \ell \leq n$. Notice that $\mathcal{A}(n;1)$ is the commutative polynomial ring $\mathbb{C}[x_{1,1}, \ldots, x_{n,n}]$.

We can use the relations above to convert any monomial into a linear combination of monomials in lexicographic order, which we shall call *standard*. Thus as a $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-module, $\mathcal{A}(n;q)$ is spanned by monomials in lexicographic order. $\mathcal{A}(n;q)$ has a natural grading by degree,

$$\mathcal{A}(n;q) = \bigoplus_{r \geq 0} \mathcal{A}_r(n;q), \tag{1.3.2}$$

where $\mathcal{A}_r(n;q)$ consists of the homogeneous degree r polynomials within $\mathcal{A}(n;q)$. Furthermore, we may decompose each homogeneous component $\mathcal{A}_r(n;q)$ by considering pairs (L, M) of multisets of r integers, written as weakly increasing sequences $1 \leq \ell_1 \leq \cdots \leq \ell_r \leq n$, and $1 \leq m_1 \leq \cdots \leq m_r \leq n$. Let $\mathcal{A}_{L,M}(n;q)$ be the $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-span of monomials whose row indices and column indices (with multiplicity) are equal to the multisets L and M, respectively. This leads to the multigrading

$$\mathcal{A}(n;q) = \bigoplus_{r \geq 0} \bigoplus_{L,M} \mathcal{A}_{L,M}(n;q). \tag{1.3.3}$$

The graded component $\mathcal{A}_{[n],[n]}(n;q)$ is spanned by the monomials

$$\{x_{1,w_1} \cdots x_{n,w_n} \mid w \in \mathfrak{S}_n\}. \tag{1.3.4}$$

Defining $x^{u,v} = x_{u_1,v_1} \cdots x_{u_n,v_n}$ for any $u, v \in \mathfrak{S}_n$, we may express the above basis as $\{x^{e,w} \mid w \in \mathfrak{S}_n\}$. We will call elements of this submodule *(quantum) immanants* and we will call the module itself the *immanant space* of $\mathcal{A}(n;q)$.

In general, $\mathcal{A}_{L,M}(n;q)$ is the $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-submodule of $\mathcal{A}(n;q)$ spanned by the monomials

$$\{x_{\ell_1,m_{w_1}} \cdots x_{\ell_r,m_{w_r}} \mid w \in \mathfrak{S}_r\} = \{(x_{L,M})^{e,w} \mid w \in \mathfrak{S}_r\}, \tag{1.3.5}$$

where the generalized submatrix $x_{L,M}$ of x is defined by

$$x_{L,M} = \begin{bmatrix} x_{\ell_1,m_1} & x_{\ell_1,m_2} & \cdots & x_{\ell_1,m_r} \\ x_{\ell_2,m_1} & x_{\ell_2,m_2} & \cdots & x_{\ell_2,m_r} \\ \vdots & \vdots & & \vdots \\ x_{\ell_r,m_1} & x_{\ell_r,m_2} & \cdots & x_{\ell_n,m_r} \end{bmatrix}. \tag{1.3.6}$$

Note that the variables in monomials appearing in (1.3.5) do not necessarily appear in lexicographic order, e.g., $(x_{112,123})^{e,s_1 s_2 s_1} = x_{1,3}x_{1,2}x_{2,1}$.

For any r, an r-element multiset $M = m_1 \cdots m_r$ on $[n]$ determines a subset

$$\iota(M) = \{s_i \mid m_i = m_{i+1}\} \tag{1.3.7}$$

of generators of $\mathfrak{S}_r$, satisfying $|\iota(M)| \leq r-1$. For example

$$\iota(11224555) = \{s_1, s_3, s_6, s_7\}. \tag{1.3.8}$$

Using the map ι, one may characterize several subsets of $W = \mathfrak{S}_r$ as follows. Let $I = \iota(L), J = \iota(M)$. Then we have

$$\begin{gathered} I = \{s_i \mid s_i L = L\}, \\ W_I = \{w \in W \mid w_i < w_k \text{ for all } i \leq j < k, s_j \notin I\} = \{w \in W \mid wL = L\}, \end{gathered} \tag{1.3.9}$$

$$\begin{aligned} W_-^{I,\emptyset} &= \{w \in W_I \mid w_i < w_{i+1} \text{ whenever } \ell_i = \ell_{i+1}\} \\ &= \{w \in W \mid \textsc{inv}(w^{-1}L) = \textsc{inv}(w^{-1}) = \textsc{inv}(w)\}, \\ W_+^{I,\emptyset} &= \{w \in W_I \mid w_i > w_{i+1} \text{ whenever } \ell_i = \ell_{i+1}\} \\ &= \{w \in W \mid \textsc{inv}(w^{-1}L) = \textsc{inv}((w_0^I w)^{-1}) = \textsc{inv}(w) - \textsc{inv}(w_0^I)\}, \end{aligned} \tag{1.3.10}$$

$$\begin{gathered} J = \{s_j \mid s_j M = M\}, \\ W_J = \{w \in W \mid w_i < w_k \text{ for all } i \leq j < k, s_j \notin J\} = \{w \in W \mid wM = M\}, \end{gathered} \tag{1.3.11}$$

$$\begin{aligned} W_-^{\emptyset,J} &= \{w \in W_J \mid i \text{ appears before } i+1 \text{ in } w_1 \cdots w_r \text{ whenever } m_i = m_{i+1}\} \\ &= \{w \in W \mid \textsc{inv}(wM) = \textsc{inv}(w)\}, \\ W_+^{\emptyset,J} &= \{w \in W_J \mid\mid i \text{ appears after } i+1 \text{ in } w_1 \cdots w_r \text{ whenever } m_i = m_{i+1}\} \\ &= \{w \in W \mid \textsc{inv}(wM) = \textsc{inv}(w w_0^J) = \textsc{inv}(w) - \textsc{inv}(w_0^J)\}, \end{aligned} \tag{1.3.12}$$

where $\textsc{inv}(w)$ is the number of inversions, or pairs $i < j$ such that $w_i > w_j$, of the permutation w. Note that for $w \in W_I u$ we have $\textsc{inv}(w^{-1}L) = \textsc{inv}(u^{-1}L)$ and that for $w \in uW_J$ we have $\textsc{inv}(wM) = \textsc{inv}(uM)$.

Considering the representatives of double cosets, we have

$$\begin{aligned}
W_-^{I,J} &= \{u \in W_-^{\emptyset,J} \mid u_i < u_{i+1} \text{ whenever } \ell_i = \ell_{i+1}\} \\
&= \{u \in W_-^{\emptyset,J} \mid \textsc{inv}(u^{-1}L) = \textsc{inv}(u^{-1}) = \textsc{inv}(u)\} \\
&= \{u \in W_-^{I,\emptyset} \mid i \text{ appears before } i+1 \text{ in } u_1 \cdots u_r \text{ whenever } m_i = m_{i+1}\} \\
&= \{u \in W_-^{I,\emptyset} \mid \textsc{inv}(uM) = \textsc{inv}(u)\}, \\
W_+^{I,J} &= \{v \in W_+^{\emptyset,J} \mid v_i > v_{i+1} \text{ whenever } \ell_i = \ell_{i+1}\} \\
&= \{v \in W_+^{\emptyset,J} \mid \textsc{inv}(v^{-1}L) = \textsc{inv}((w_0^I v)^{-1}) = \textsc{inv}(v) - \textsc{inv}(w_0^I)\} \\
&= \{v \in W_+^{I,\emptyset} \mid i \text{ appears after } i+1 \text{ in } v_1 \cdots v_r \text{ whenever } m_i = m_{i+1}\} \\
&= \{v \in W_+^{I,\emptyset} \mid \textsc{inv}(vM) = \textsc{inv}(vw_0^J) = \textsc{inv}(v) = \textsc{inv}(w_0^J)\}.
\end{aligned} \tag{1.3.13}$$

Now fix $u \in W_-^{I,J}$. Defining K, K' as in (1.1.8) and (1.1.9), we have the following.

$$\begin{aligned}
K' &= \{s_i \in I \mid s_i uM = uM\}, \\
W_{K'} &= \{w \in W_I \mid wuM = uM\}, \\
(W_I)_-^{\emptyset,K'} &= \{w \in W_I \mid \textsc{inv}(wuM) = \textsc{inv}(w) + \textsc{inv}(uM)\} \\
&= \{w \in W_I \mid \textsc{inv}(wuM) = \textsc{inv}(wu) = \textsc{inv}(w) + \textsc{inv}(u)\},
\end{aligned} \tag{1.3.14}$$

$$\begin{aligned}
K &= \{s_j \in J \mid (us_j)^{-1}L = s_j u^{-1}L = u^{-1}L\}, \\
W_K &= \{w \in W_J \mid (uw)^{-1}L = w^{-1}u^{-1}L = u^{-1}L\}, \\
(W_J)_-^{K,\emptyset} &= \{w \in W_J \mid \textsc{inv}((uw)^{-1}L) = \textsc{inv}(u^{-1}L) + \textsc{inv}(w)\} \\
&= \{w \in W_J \mid \textsc{inv}((uw)^{-1}L) = \textsc{inv}((uw)^{-1}) = \textsc{inv}(w) + \textsc{inv}(u)\},
\end{aligned} \tag{1.3.15}$$

It is easy to see that the monomials $\{(x_{L,M})^{u,v} \mid u, v \in \mathfrak{S}_r\}$ satisfy

$$(x_{L,M})^{u,v} = \begin{cases} (x_{L,M})^{s_iu,s_iv} & \text{if } \ell_{u_i} < \ell_{u_{i+1}} \text{ and } m_{v_i} > m_{v_{i+1}}, \\ & \quad \text{or if } \ell_{u_i} > \ell_{u_{i+1}} \text{ and } m_{v_i} < m_{v_{i+1}}, \\ & \quad \text{or if } \ell_{u_i} = \ell_{u_{i+1}} \text{ and } m_{v_i} = m_{v_{i+1}}, \\ q^{\frac{1}{2}}(x_{L,M})^{s_iu,s_iv} = (x_{L,M})^{s_iu,v} & \text{if } \ell_{u_i} = \ell_{u_{i+1}} \text{ and } m_{v_i} > m_{v_{i+1}}, \\ q^{\frac{1}{2}}(x_{L,M})^{s_iu,s_iv} = q^{\frac{1}{2}}(x_{L,M})^{s_iu,v} & \text{if } \ell_{u_i} > \ell_{u_{i+1}} \text{ and } m_{v_i} = m_{v_{i+1}}, \\ q^{\frac{-1}{2}}(x_{L,M})^{s_iu,s_iv} = (x_{L,M})^{s_iu,v} & \text{if } \ell_{u_i} = \ell_{u_{i+1}} \text{ and } m_{v_i} < m_{v_{i+1}}, \\ q^{\frac{-1}{2}}(x_{L,M})^{s_iu,s_iv} = q^{\frac{-1}{2}}(x_{L,M})^{s_iu,v} & \text{if } \ell_{u_i} < \ell_{u_{i+1}} \text{ and } m_{v_i} = m_{v_{i+1}}, \\ (x_{L,M})^{s_iu,s_iv} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})(x_{L,M})^{s_iu,v} & \\ \ = (x_{L,M})^{s_iu,s_iv} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})(x_{L,M})^{u,s_iv} & \text{if } \ell_{u_i} > \ell_{u_{i+1}} \text{ and } m_{v_i} > m_{v_{i+1}}, \\ (x_{L,M})^{s_iu,s_iv} - (q^{\frac{1}{2}} - q^{\frac{-1}{2}})(x_{L,M})^{s_iu,v} & \\ \ = (x_{L,M})^{s_iu,s_iv} - (q^{\frac{1}{2}} - q^{\frac{-1}{2}})(x_{L,M})^{u,s_iv} & \text{if } \ell_{u_i} < \ell_{u_{i+1}} \text{ and } m_{v_i} < m_{v_{i+1}}, \end{cases} \tag{1.3.16}$$

Since the dimension of $\mathcal{A}_{L,M}(n; q)$ is typically less than $r!$, the spanning set (1.3.5) is not in general linearly independent. In [31], Skandera states formulas describing the dependency for the cases where $L = [r]$ or $M = [r]$, which we shall call the *single parabolic cases*. Here we extend this result to the *double parabolic case*, where neither L nor M is $[r]$.

Proposition 1.3.1. *Let L, M be r-element multisets of $[n]$ and define subsets $I = \iota(L), J = \iota(M)$ of generators of $\mathfrak{S}_r$. Fix $u, v \in \mathfrak{S}_r$ such that $u \in W_I v W_J$. Then we have*

$$\begin{aligned} (x_{L,M})^{e,u} &= q_{v^I_-,u^I_-}(x_{L,M})^{e,v}, \\ (x_{L,M})^{u^{-1},e} &= q_{v^J_-,u^J_-}(x_{L,M})^{v^{-1},e}. \end{aligned} \tag{1.3.17}$$

Proof. Let w be the minimal element of the double coset $W_I v W_J$, in other words $w \in W^{I,J}_- \cap W_I v W_J$. Recall by (1.3.13) that the variables in $(x_{L,M})^{e,w}$ appear in lexicographic order, and by (1.3.11) we have

$$\begin{aligned}(x_{L,M})^{e,u} &= x_{\ell_1,(u^I_- w u^J M)_1} \cdots x_{\ell_r,(u^I_- w u^J M)_r} \\ &= x_{\ell_1,(u^I_- w M)_1} \cdots x_{\ell_r,(u^I_- w M)_r} \\ &= (x_{L,M})^{e,u^I_- w}.\end{aligned} \tag{1.3.18}$$

Since u^I_- belongs to W_I, (1.3.9) implies that each pair of variables appearing out of lexicographic order with respect to one another in $(x_{L,M})^{e,u^I_- w}$ has the form

$$(x_{k,(u^I_- wM)_i}, x_{k,(u^I_- wM)_j}) \tag{1.3.19}$$

for some indices $i < j$ and k. Since u^I_- belongs more specifically to $(W_I)^{\emptyset,K'}_-$, (1.3.14) implies that the number of such pairs is equal to $\ell(u^I_-)$. Thus when we sort variables in $(x_{L,M})^{e,u} = (x_{L,M})^{e,u^I_- w}$ into lexicographic order we obtain

$$(x_{L,M})^{e,u} = q_{e,u^I_-}(x_{L,M})^{e,w}. \tag{1.3.20}$$

Combining this equality with $(x_{L,M})^{e,v} = q_{e,v^I_-}(x_{L,M})^{e,w}$, we obtain the first identity in (1.3.17).

Similarly, recall by (1.3.13) that the variables in the monomial $(x_{L,M})^{w^{-1},e}$ appear in right-to-left lexicographic order, and by (1.3.9) we have

$$\begin{aligned}(x_{L,M})^{u^{-1},e} &= x_{((u^J_-)^{-1} w^{-1} (u^I)^{-1} L)_1, m_1} \cdots x_{((u^J_-)^{-1} w^{-1} (u^I)^{-1} L)_r, m_r} \\ &= x_{((u^J_-)^{-1} w^{-1} L)_1, m_1} \cdots x_{((u^J_-)^{-1} w^{-1} L)_r, m_r} \\ &= (x_{L,M})^{(u^J_-)^{-1} w^{-1}, e}.\end{aligned} \tag{1.3.21}$$

Since $(u^J_-)^{-1}$ belongs to W_J, (1.3.11) implies that each pair of variables appearing out of right-to-left lexicographic order with respect to one another in $(x_{L,M})^{(u^J_-)^{-1} w^{-1},e}$ has the form

$$(x_{((u^J_-)^{-1} w^{-1} L)_i, k}, x_{((u^J_-)^{-1} w^{-1} L)_j, k}) \tag{1.3.22}$$

for some indices $i < j$ and k. Since $(u^J_-)^{-1}$ belongs more specifically to $(W_J)^{\emptyset,K}_-$, (1.3.15) implies that the number of such pairs is equal to $\ell(u^J_-)$. Thus when we sort variables in $(x_{L,M})^{u^{-1},e} = (x_{L,M})^{(u^J_-)^{-1} w^{-1},e}$ into lexicographic order we obtain

$$(x_{L,M})^{u^{-1},e} = q_{e,u^J_-}(x_{L,M})^{w^{-1},e}. \tag{1.3.23}$$

Combining this equality with $(x_{L,M})^{v^{-1},e} = q_{e,v_-^J}(x_{L,M})^{w^{-1},e}$, we obtain the second identity in (1.3.17). $\square$

It follows that we just need one permutation w from each double coset in $W_I \backslash W / W_J$ to form a basis of $\mathcal{A}_{L,M}(n;q)$. A natural choice of representatives is the set $W_+^{I,J}$, thus we will call the set $\{(x_{L,M})^{e,w} \mid w \in W_+^{I,J}\}$ the *natural basis*. For example in $\mathcal{A}_{[3],[3]}(3;q)$, the immanant space, we have

$$\begin{aligned} x^{s_1,w_0} &= x^{e,s_2s_1} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{e,w_0}, \\ x^{s_1s_2,w_0} &= x^{e,s_1} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})(x^{e,s_1s_2} + x^{e,s_2s_1}) + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})^2 x^{e,w_0}, \\ x^{w_0,w_0} &= x^{e,e} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})(x^{e,s_1} + x^{e,s_2}) + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})^2(x^{e,s_1s_2} + x^{e,s_2s_1}) \\ &\quad + \left((q^{\frac{1}{2}} - q^{\frac{-1}{2}})^3 + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})\right) x^{e,w_0}, \end{aligned} \tag{1.3.24}$$

and in $\mathcal{A}_{112,122}(3,q)$ we have

$$\begin{aligned} (x_{112,122})^{w_0,w_0} = &\left((q^{\frac{1}{2}} - q^{\frac{-1}{2}})^2 q^{\frac{-1}{2}} + 2(q^{\frac{1}{2}} - q^{\frac{-1}{2}})q^{-1} + q^{\frac{-3}{2}}\right)(x_{112,122})^{e,s_1s_2} \\ &+ \left((q^{\frac{1}{2}} - q^{\frac{-1}{2}})^3 q^{\frac{-1}{2}} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})^2 q^{-1} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})q^{\frac{-1}{2}}\right)(x_{112,122})^{e,w_0}. \end{aligned} \tag{1.3.25}$$

In [31], Skandera shows that a second basis, which we will call the *inverse transpose basis*, of $\mathcal{A}_{L,M}(n;q)$ is given by $\{(x_{L,M})^{w^{-1},e} \mid w \in W_+^{I,J}\}$. We state this with proof below.

Proposition 1.3.2. *For each element w of $W_-^{I,J}$, we have*

$$(x_{L,M})^{e,w} = (x_{L,M})^{w^{-1},e}. \tag{1.3.26}$$

Proof. Recall that the variables in the monomial $(x_{L,M})^{e,w} = x_{\ell_1,(wM)_1} \cdots x_{\ell_r,(wM)_r}$ appear in lexicographic order. Writing M in multiplicity notation as $M = 1^{b_1} \cdots n^{b_n}$, we see that for each letter k, the b_k variables having column index k are those having row indices

$$(w^{-1}L)_{b_1+\cdots+b_{k-1}+1} \leq \cdots \leq (w^{-1}L)_{b_1+\cdots+b_k}. \tag{1.3.27}$$

Now consider the monomial $(x_{L,M})^{w^{-1},e} = x_{(w^{-1}L)_1,m_1} \cdots x_{(w^{-1}L)_r,m_r}$. Again, the b_k variables having column index k have row indices (1.3.27). Thus the sequence of index pairs

$$(((w^{-1}L)_1, m_1), \dots, ((w^{-1}L)_r, m_r)) \tag{1.3.28}$$

is a shuffle of disjoint subsequences of the form

$$(((w^{-1}L)_{b_1+\cdots+b_{k-1}+1}, k), \dots, ((w^{-1}L)_{b_1+\cdots+b_k}, k)). \tag{1.3.29}$$

Now let us use the relations (1.3.1) to express $(x_{L,M})^{w^{-1},e}$ as a linear combination of monomials with variables appearing in lexicographic order. Note that for any indices $j < j'$, we must have $m_j \leq m_{j'}$. If $m_j = m_{j'}$, then by (1.3.27) we have $(w^{-1}L)_j \leq (w^{-1}L)_{j'}$, and the two variables $x_{(w^{-1}L)_j, m_j}, x_{(w^{-1}L)_{j'}, m_{j'}}$ are already in lexicographic order. Now suppose that we have $m_j < m_{j'}$. If $(w^{-1}L)_j \leq (w^{-1}L)_{j'}$, then again the variables are in lexicographic order. Otherwise they commute. Thus we only use the third relation in (1.3.1) to sort $(x_{L,M})^{w^{-1},e}$, and we see that it is equal to $(x_{L,M})^{e,w}$. □

It follows that for $w \in W_-^{I,J}$ and $v \in W_I w W_J$, we have

$$\begin{aligned}(x_{L,M})^{e,v} &= q_{e,v_-^I}(x_{L,M})^{e,w} = q_{e,v_-^I}(x_{L,M})^{w^{-1},e} = q_{e,v_-^I} q_{e,v_-^J}^{-1}(x_{L,M})^{v^{-1},e} \\ &= q_{v_-^J, v_-^I}(x_{L,M})^{v^{-1},e}.\end{aligned} \tag{1.3.30}$$

More generally, for $u, v \in \mathfrak{S}_r$ and $y \in uW_I$, $z \in vW_J$ we have

$$(x_{L,M})^{u,v} = (x_{L,M})^{y,z}. \tag{1.3.31}$$

In the immanant space the double cosets are the individual permutations themselves and thus each w is a minimal coset representative and Proposition 1.3.2 says

$$x^{e,w} = x^{w^{-1},e}. \tag{1.3.32}$$

Thus, in the immanant space the natural basis and the inverse transpose basis are identical.

A natural $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-linear involution on $\mathcal{A}(n;q)$ is matrix transposition $x_{L,M} \mapsto (x_{L,M})^\top$, defined by

$$\begin{aligned}(x_{L,M})^{e,v} &\mapsto ((x_{L,M})^{e,v})^\top \underset{\text{def}}{=} (x_{L,M})^{v,e}, \\ f(x_{L,M}) &\mapsto f(x_{L,M})^\top \underset{\text{def}}{=} f((x_{L,M})^\top),\end{aligned} \tag{1.3.33}$$

assuming that we have $v \in W_+^{I,J}$ and that f is expressed as a linear combination of monomials $\{(x_{L,M})^{e,w} \mid w \in W_+^{I,J}\}$. This will be useful in a subsequent chapter.

1.4 Natural and inverse transpose basis expansions

In the previous section we defined two bases, the natural basis and the inverse transpose basis. Therefore we can express any monomial $(x_{L,M})^{u,v}$ as a linear combination of natural basis elements and as a linear combination of inverse transpose basis elements. Looking at the defining relations of $\mathcal{A}(n;q)$ we see that the coefficients of the basis elements in both cases are polynomials in $q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{1}{2}}, q^{\frac{-1}{2}}$ with nonnegative integer coefficients. However, since

$$q^{\frac{-1}{2}}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) + (q^{\frac{-1}{2}})^2 = 1, \tag{1.4.1}$$

these polynomials can be expressed in several ways, making a canonical description of the coefficients difficult. Brenti studied polynomials with nonnegative integer coefficients which appear in a similar manner as entries of transition matrices in $H_n(q)$. In [1], two combinatorial interpretations for such matrices are given. In order to handle the lack of canonical description mentioned earlier, we will define a family of polynomials in a manner analogous to Brenti's combinatorial interpretations and extend his results to a larger family of Laurent polynomials which will evaluate to transition matrix entries within $\mathcal{A}(n;q)$.

For all $u, v \in \mathfrak{S}_r$ and $w \in W_+^{I,J}$, given any reduced expression $s_{i_1} \cdots s_{i_k}$ for u, define the (*Laurent*) polynomials $\{p_{u,v,w}^{I,J}(q_1, q_2) \in \mathbb{N}[q_1, q_2, q_2^{-1}] \mid u, v \in \mathfrak{S}_r, w \in W_+^{I,J}\}$ to be the polynomials whose coefficient of $q_1^a q_2^b$ is equal to the number of sequences $(\pi^{(0)}, \ldots, \pi^{(k)})$ of permutations satisfying

1. $\pi^{(0)} = v$, $\pi^{(k)} \in W_I w W_J$,

2. $\pi^{(j)} \in \{s_{i_j}\pi^{(j-1)}, \pi^{(j-1)}\}$ for $j = 1, \ldots, k$,

3. $\pi^{(j)} = s_{i_j}\pi^{(j-1)}$ if $s_{i_j}\pi^{(j-1)} > \pi^{(j-1)}$,

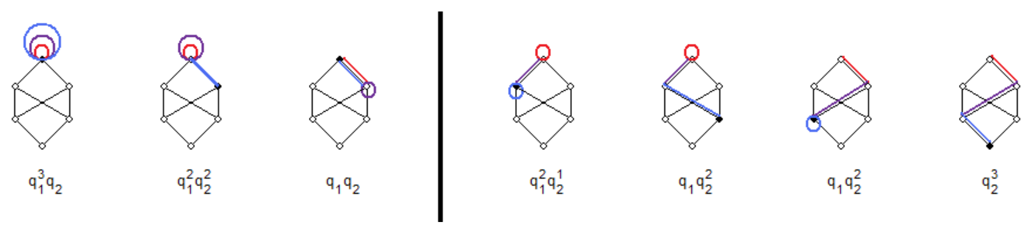
4. $\pi^{(j)} = \pi^{(j-1)}$ for exactly a values of j,

5. $\ell(w) - \ell(\pi^{(k)}) + \ell(u^I) + \ell(v^J) - \ell(w_0^J) = b$.

The sequences begin counted by the coefficients of $p^{I,J}_{u,v,w}(q_1, q_2)$ can be thought of as walks in the Bruhat order, where we allow vertices to be repeated. Alternatively, we could consider the Bruhat order with loops added at each vertex. For example, if $n = 3$, $I = \{s_1\}$, $J = \{s_2\}$, $u = v = w_0$ and we choose the reduced expression $s_1 s_2 s_1$ for u, then we have

$$\ell(u^I) + \ell(v^J) - \ell(w_0^J) = 1 + 1 - 1 = 1. \tag{1.4.2}$$

Therefore, there are seven walks beginning at $v = w_0$, which satisfy the conditions in the definition for some $w \in W_+^{I,J}$. Thus we have



$$\begin{aligned} p^{I,J}_{w_0,w_0,w_0}(q_1, q_2) &= q_1^3 q_2 + q_1^2 q_2^2 + q_1 q_2, \\ p^{I,J}_{w_0,w_0,s_1 s_2}(q_1, q_2) &= q_1^2 q_2 + 2 q_1 q_2^2 + q_2^3. \end{aligned} \tag{1.4.3}$$

Similarly, for all $u, v \in \mathfrak{S}_r$ and $w \in W_+^{I,J}$, given any reduced expression $s_{i_1} \cdots s_{i_k}$ for v, define the (*Laurent*) polynomials $\{r^{I,J}_{u,v,w}(q_1, q_2) \in \mathbb{N}[q_1, q_2, q_2^{-1}] \mid u, v \in \mathfrak{S}_r, w \in W_+^{I,J}\}$ to be the polynomials whose coefficient of $q_1^a q_2^b$ is equal to the number of sequences $(\pi^{(0)}, \ldots, \pi^{(k)})$ of permutations satisfying

1. $\pi^{(0)} = u$, $(\pi^{(k)})^{-1} \in W_I w W_J$,

2. $\pi^{(j)} \in \{s_{i_j} \pi^{(j-1)}, \pi^{(j-1)}\}$ for $j = 1, \ldots, k$,

3. $\pi^{(j)} = s_{i_j} \pi^{(j-1)}$ if $s_{i_j} \pi^{(j-1)} > \pi^{(j-1)}$,

4. $\pi^{(j)} = \pi^{(j-1)}$ for exactly a values of j,

5. $\ell(w) - \ell(\pi^{(k)}) + \ell(u^I) + \ell(v^J) - \ell(w_0^I) = b$.

Looking at these definitions it is easy to see that

$$r^{I,J}_{u,v,w}(q_1, q_2) = p^{J,I}_{v,u,w^{-1}}(q_1, q_2) \tag{1.4.4}$$

for all subsets of generators I, J and permutations $u, v \in \mathfrak{S}_r$ and $w \in W^{I,J}_+$.

In the case where $I = J = \emptyset$, we write

$$\begin{aligned} p_{u,v,w}(q_1) &= p^{\emptyset,\emptyset}_{u,v,w}(q_1, q_2), \\ r_{u,v,w}(q_1) &= r^{\emptyset,\emptyset}_{u,v,w}(q_1, q_2), \end{aligned} \tag{1.4.5}$$

since the power of q_2 is always zero and we have polynomials in one variable. Using the previous example, except with $I = J = \emptyset$, we have the same seven paths as before. This time

$$\ell(w) - \ell(\pi^{(k)}) + \ell(u^I) + \ell(v^J) - \ell(w_0^J) = 0 \tag{1.4.6}$$

for all paths and $w \in \mathfrak{S}_3$, thus we have

$$p_{w_0,w_0,w}(q_1) = \begin{cases} 1 & \text{if } \ell(w) = 0, \\ q_1 & \text{if } \ell(w) = 1, \\ q_1^2 & \text{if } \ell(w) = 2, \\ q_1^3 + q_1 & \text{if } \ell(w) = 3. \end{cases} \tag{1.4.7}$$

As a special case of (1.4.4) we have

$$r_{u,v,w}(q) = p_{v,u,w^{-1}}(q), \tag{1.4.8}$$

for all $u, v, w \in \mathfrak{S}_r$.

Notice that the definitions depend upon a chosen reduced expression for u or v. On the other hand, with a little work we will show that all reduced expressions for u yield the same polynomials $\{p^{I,J}_{u,v,w}(q_1, q_2) \mid v \in \mathfrak{S}_r, w \in W^{I,J}_+\}$, and consequently all reduced expressions for v yield the same polynomials $\{r^{I,J}_{u,v,w}(q_1, q_2) \mid u \in \mathfrak{S}_r, w \in W^{I,J}_+\}$.

Proposition 1.4.1. *Fix $u, v \in \mathfrak{S}_r$ and $w \in W_+^{I,J}$. Then we have*

$$p_{u,v,w}^{I,J}(q_1, q_2) = \sum_{z \in W_I w W_J} q_2^{\ell(w)-\ell(z)+\ell(u^I)+\ell(v^J)-\ell(w_0^J)} p_{u,v,z}(q_1). \tag{1.4.9}$$

Proof. Choose $z \in W_I w W_J$, then by definition the coefficient of $q_1^a q_2^b$ in the polynomial $q_2^{\ell(w)-\ell(z)+\ell(u^I)+\ell(v^J)-\ell(w_0^J)} p_{u,v,z}(q_1)$ is the number of sequences $(\pi^{(0)}, \ldots, \pi^{(k)})$ satisfying conditions (1)-(5) of the definition of $p_{u,v,w}^{I,J}(q_1, q_2)$ which end specifically at $z \in W_I w W_J$. Therefore by summing over all $z \in W_I w W_J$, we get all the sequences satisfying the conditions of the definition of $p_{u,v,w}^{I,J}(q_1, q_2)$ and equality. □

Therefore, the question of whether or not $p_{u,v,w}^{I,J}(q_1, q_2)$ depends on the choice of reduced expression for u, reduces to the question of whether $p_{u,v,z}(q_1)$ depends on the choice of of reduced expression for u. The examples (1.4.7) and (1.3.24) suggest the polynomials $\{p_{u,v,w}(q) \mid u, v, w \in \mathfrak{S}_n\}$ might appear naturally in $\mathcal{A}_{[r],[r]}(r; q)$, the immanant space for $\mathcal{A}(r; q)$. In order to resolve the question of whether the choice of reduced expression changes the value of these polynomials, we prove the following result connecting them to the immanant space precisely as the examples imply.

Proposition 1.4.2. *For all $u, v \in \mathfrak{S}_r$,*

$$x^{u,v} = \sum_{w \geq u^{-1}v} p_{u,v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) x^{e,w}. \tag{1.4.10}$$

Proof. By definition $p_{e,v,w} = \delta_{v,w}$, thus the claim is true for $u = e$. Assume the claim to be true for u having length at most $\ell - 1$, fix a permutation u of length ℓ, and fix a reduced expression $s_{i_1} \cdots s_{i_\ell}$ for u. Then we have

$$x^{u,v} = \begin{cases} x^{s_{i_2}\cdots s_{i_\ell}, s_{i_1}v} & \text{if } s_{i_1}v > v, \\ x^{s_{i_2}\cdots s_{i_\ell}, s_{i_1}v} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}}) x^{s_{i_2}\cdots s_{i_\ell}, v} & \text{if } s_{i_1}v < v. \end{cases} \tag{1.4.11}$$

By induction, we see that if

$$p_{u,v,w}(q) = \begin{cases} p_{s_{i_2}\cdots s_{i_\ell}, s_{i_1}v, w}(q) & \text{if } s_{i_1}v > v, \\ p_{s_{i_2}\cdots s_{i_\ell}, s_{i_1}v, w}(q) + q \cdot p_{s_{i_2}\cdots s_{i_\ell}, v, w}(q) & \text{if } s_{i_1}v < v \end{cases} \tag{1.4.12}$$

then the claim holds.

Suppose first that we have $s_{i_1}v > v$. Then by induction, the coefficient of q^k in $p_{u,v,w}(q)$ is equal to the number of sequences $(\pi^{(1)} = s_{i_1}v, \pi^{(2)}, \ldots, \pi^{(\ell)} = w)$ satisfying Conditions (2)-(4) of the definition. Prepending the permutation $\pi^{(0)}$ to any such sequence and considering the inequality $s_{i_1}v > v$, we see that the new sequence

$$\pi \underset{\text{def}}{=} (\pi^{(0)}, \pi^{(1)}, \ldots, \pi^{(\ell)}) \tag{1.4.13}$$

satisfies all four conditions of the definition if and only if $\pi^{(0)} = v$. Thus the coefficient of q^k in $p_{u,v,w}(q)$ is equal to the number of sequences $(\pi^{(0)}, \ldots, \pi^{(\ell)})$ satisfying Conditions (1)-(4) of the definition.

Now suppose that we have $s_{i_1}v < v$. Then by induction, the coefficient of q^k in $p_{u,v,w}(q)$ is equal to the number of sequences $(\pi^{(1)} = s_{i_1}v, \pi^{(2)}, \ldots, \pi^{(\ell)}) = w)$ satisfying Conditions (2)-(4) of the definition, plus the number of sequences $(\pi^{(1)} = v, \pi^{(2)}, \ldots, \pi^{(\ell)} = w)$ satisfying Conditions (2)-(3) of the definition and in which $\pi^{(j)} = \pi^{(j-1)}$ for exactly $k-1$ values of j. Prepending a permutation $\pi^{(0)}$ to any such sequence, we see that the new sequence

$$\pi \underset{\text{def}}{=} (\pi^{(0)}, \pi^{(1)}, \ldots, \pi^{(\ell)}) \tag{1.4.14}$$

satisfies all four conditions of the definition if and only if $\pi^{(0)} = v$. (No new equality $\pi^{(1)} = \pi^{(0)}$ is introduced for a sequence of the first form; one new such equality is introduced for a sequence of the second form.) Thus we see that the recursion (1.4.12) holds. □

Since the expansion of $x^{u,v}$ in terms of the natural basis does not depend on the choice of reduced expression for u, neither does $p_{u,v,w}(q)$. Therefore, Proposition 1.4.1 implies that all reduced expressions for u yield the same polynomials $\{p^{I,J}_{u,v,w}(q_1, q_2) \mid v \in \mathfrak{S}_r, w \in W^{I,J}_+\}$, as claimed. This also implies the polynomials $\{r^{I,J}_{u,v,w}(q_1, q_2) \mid u \in \mathfrak{S}_r, w \in W^{I,J}_+\}$ are independent of the choice of reduced expression for v, by (1.4.4).

Corollary 1.4.3. *Fix $u, v \in \mathfrak{S}_r$ and $w \in W^{I,J}_+$. Then we have*

$$r^{I,J}_{u,v,w}(q_1, q_2) = \sum_{z \in W_I w W_J} q_2^{\ell(w)-\ell(z)+\ell(u^I)+\ell(v^J)-\ell(w_0^I)} r_{u,v,z}(q_1). \tag{1.4.15}$$

Proof. By (1.4.4), Proposition 1.4.1, and the special case (1.4.8), we have

$$\begin{aligned} r^{I,J}_{u,v,w}(q_1,q_2) &= p^{J,I}_{v,u,w^{-1}}(q_1,q_2) \\ &= \sum_{z^{-1}\in W_J w^{-1} W_I} q_2^{\ell(w^{-1})-\ell(z^{-1})+\ell(v^J)+\ell(u^I)-\ell(w_0^I)} p_{v,u,z^{-1}}(q_1) \\ &= \sum_{z\in W_I w W_J} q_2^{\ell(w)-\ell(z)+\ell(u^I)+\ell(v^J)-\ell(w_0^I)} r_{u,v,z}(q_1). \end{aligned} \tag{1.4.16}$$

□

We have seen a relationship between $p^{I,J}_{u,v,w}(q_1,q_2)$ and $r^{I,J}_{u,v,w}(q_1,q_2)$. However, we can state another, more surprising, relationship between these polynomials by taking advantage of Proposition 1.4.2 and the fact that $x^{e,w} = x^{w^{-1},e}$. Putting these two facts together gives us

$$x^{u,v} = \sum_{w\geq u^{-1}v} p_{u,v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{w^{-1},e}. \tag{1.4.17}$$

Applying the transpose involution to both sides of (1.4.17) gives us

$$x^{v,u} = \sum_{w\geq u^{-1}v} p_{u,v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{e,w^{-1}}. \tag{1.4.18}$$

Proposition 1.4.2 tells us that

$$x^{v,u} = \sum_{w\geq u^{-1}v} p_{v,u,w^{-1}}(q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{e,w^{-1}}. \tag{1.4.19}$$

Comparing terms in the the two expressions for $x^{v,u}$ and recalling (1.4.8) we see that

$$p_{u,v,w}(q) = p_{v,u,w^{-1}}(q) = r_{u,v,w}(q). \tag{1.4.20}$$

This is an interesting and unexpected relationship, which can be generalized to any multi-graded component $\mathcal{A}_{L,M}(n;q)$ as follows.

Proposition 1.4.4. *For all $u, v \in \mathfrak{S}_r$ and $w \in W^{I,J}_+$,*

$$p^{I,J}_{u,v,w}(q_1,q_2) = q_2^{\ell(w_0^I)-\ell(w_0^J)} r^{I,J}_{u,v,w}(q_1,q_2). \tag{1.4.21}$$

Proof. Applying (1.4.20) to (1.4.9) gives

$$\begin{aligned} p^{I,J}_{u,v,w}(q_1,q_2) &= \sum_{z\in W_I w W_J} (q_2)^{\ell(w)-\ell(z)+\ell(u^I)+\ell(v^J)-\ell(w_0^J)} p_{u,v,w}(q_1) \\ &= q_2^{\ell(w_0^I)-\ell(w_0^J)} \sum_{z\in W_I w W_J} (q_2)^{\ell(w)-\ell(z)+\ell(u^I)+\ell(v^J)-\ell(w_0^I)} r_{u,v,w}(q_1) \qquad (1.4.22) \\ &= q_2^{\ell(w_0^I)-\ell(w_0^J)} r^{I,J}_{u,v,w}(q_1,q_2), \end{aligned}$$

where equality comes from (1.4.15). □

This relationship is not the only thing which generalizes nicely from the immanant space to a general component of $\mathcal{A}(n;q)$. In fact, the combinatorial definition is precisely the definition we need in order to generalize Proposition 1.4.2. In order to prove the generalization of Proposition 1.4.2 we first need a few lemmas and corollaries. First, we will state alternate summation formulas, this time in terms of *single parabolic* polynomials (when exactly one of L or M is equal to $[r]$).

Corollary 1.4.5. *Fix $u, v \in \mathfrak{S}_r$ and $w \in W^{I,J}_+$. Then we have*

$$\begin{aligned} p^{I,J}_{u,v,w}(q_1,q_2) &= \sum_{\substack{y\in W^{I,\emptyset}_+ \\ y\in w(W_J)^{K,\emptyset}_-}} q_2^{\ell(w)-\ell(y)+\ell(v^J)-\ell(w_0^J)} p^{I,\emptyset}_{u,v,y}(q_1,q_2) \\ &= \sum_{\substack{y\in W^{\emptyset,J}_+ \\ y\in (W_I)^{\emptyset,K'}_- w}} q_2^{\ell(w)-\ell(y)+\ell(u^I)} p^{\emptyset,J}_{u,v,y}(q_1,q_2) \end{aligned} \qquad (1.4.23)$$

and

$$\begin{aligned} r^{I,J}_{u,v,w}(q_1,q_2) &= \sum_{\substack{y\in W^{I,\emptyset}_+ \\ y\in w(W_J)^{K,\emptyset}_-}} q_2^{\ell(w)-\ell(y)+\ell(v^J)} r^{I,\emptyset}_{u,v,y}(q_1,q_2) \\ &= \sum_{\substack{y\in W^{\emptyset,J}_+ \\ y\in (W_I)^{\emptyset,K'}_- w}} q_2^{\ell(w)-\ell(y)+\ell(u^I)-\ell(w_0^I)} r^{\emptyset,J}_{u,v,y}(q_1,q_2). \end{aligned} \qquad (1.4.24)$$

Proof. Each $z \in W_I w W_J$ is in a coset $W_I y$ for some $y \in W^{I,\emptyset}_+$ and $y \in w(W_J)^{K,\emptyset}_-$.

Therefore (1.4.9) can be written as a double sum in the following manner.

$$\begin{aligned}
p^{I,J}_{u,v,w}(q_1,q_2) &= \sum_{z\in W_I w W_J} q_2^{\ell(w)-\ell(z)+\ell(u^I)+\ell(v^J)-\ell(w_0^J)} p_{u,v,z}(q_1) \\
&= \sum_{\substack{y\in W_+^{I,\emptyset} \\ y\in w(W_J)_-^{K,\emptyset}}} \sum_{z\in W_I y} q_2^{\ell(w)-\ell(y)+\ell(y)-\ell(z)+\ell(u^I)+\ell(v^J)-\ell(w_0^J)} p_{u,v,z}(q_1) \\
&= \sum_{\substack{y\in W_+^{I,\emptyset} \\ y\in w(W_J)_-^{K,\emptyset}}} q_2^{\ell(w)-\ell(y)+\ell(v^J)-\ell(w_0^J)} \sum_{z\in W_I y} q_2^{\ell(y)-\ell(z)+\ell(u^I)} p_{u,v,z}(q_1) \qquad (1.4.25)\\
&= \sum_{\substack{y\in W_+^{I,\emptyset} \\ y\in w(W_J)_-^{K,\emptyset}}} q_2^{\ell(w)-\ell(y)+\ell(v^J)-\ell(w_0^J)} p^{I,\emptyset}_{u,v,y}(q_1,q_2).
\end{aligned}$$

The second equality comes from considering that each $z \in W_I w W_J$ is in a coset yW_J for some $y \in W_+^{\emptyset,J}$ and $y \in (W_I)_-^{\emptyset,K'} w$. Thus we have

$$\begin{aligned}
p^{I,J}_{u,v,w}(q_1,q_2) &= \sum_{z\in W_I w W_J} q_2^{\ell(w)-\ell(z)+\ell(u^I)+\ell(v^J)-\ell(w_0^J)} p_{u,v,z}(q_1) \\
&= \sum_{\substack{y\in W_+^{\emptyset,J} \\ y\in (W_I)_-^{\emptyset,K'} w}} \sum_{z\in y W_J} q_2^{\ell(w)-\ell(y)+\ell(y)-\ell(z)+\ell(u^I)+\ell(v^J)-\ell(w_0^J)} p_{u,v,z}(q_1) \\
&= \sum_{\substack{y\in W_+^{\emptyset,J} \\ y\in (W_I)_-^{\emptyset,K'} w}} q_2^{\ell(w)-\ell(y)+\ell(u^I)} \sum_{z\in y W_J} q_2^{\ell(y)-\ell(z)+\ell(v^J)-\ell(w_0^J)} p_{u,v,z}(q_1) \qquad (1.4.26)\\
&= \sum_{\substack{y\in W_+^{\emptyset,J} \\ y\in (W_I)_-^{\emptyset,K'} w}} q_2^{\ell(w)-\ell(y)+\ell(u^I)} p^{\emptyset,J}_{u,v,y}(q_1,q_2).
\end{aligned}$$

We can express $r^{I,J}_{u,v,w}(q_1,q_2)$ in terms of $p^{I,J}_{u,v,w}(q_1,q_2)$ using Proposition 1.4.4, then express this as either a sum of $p^{I,\emptyset}$ or $p^{\emptyset,J}$-polynomials. Using the relationship with $r^{I,\emptyset}$ and $r^{\emptyset,J}$-polynomials, we get the formulas as claimed. □

The next lemma takes advantage of the quotient ring $\mathbb{Z}[q_1,q_2]/(1-q_1q_2-q_2^2)$, which is the setting in which the polynomials will reside in the natural basis and inverse transpose basis expansions.

Lemma 1.4.6. *Fix $y \in W_-^{\emptyset,J}$, $v \in yW_J$, and $w \in W_+^{I,J}$. In the ring $\mathbb{Z}[q_1,q_2]/(1-q_1q_2-q_2^2)$ we have*

$$r_{u,v,w}^{I,J}(q_1,q_2) = r_{u,y,w}^{I,J}(q_1,q_2). \tag{1.4.27}$$

Proof. Each element v of yW_J factors uniquely as $v = yt$ with $t \in W_J$. If $\ell(t) = 0$, then $v = y$ and the claim is trivial. Now fix $k > 1$, assume the claim to hold for v factoring with $\ell(t) \le k-1$, and choose v factoring with $\ell(t) = k$. Fix a reduced expression $s_{i_1} \cdots s_{i_r}$ for v with $s_{i_{r-k+1}} \cdots s_{i_r}$ a reduced expression for t. Then we have $s_{i_r} \in W_J$, $vs_{i_r} = yts_{i_r}$, $\ell(ts_{i_r}) = k-1$, and by induction, $r_{u,vs_{i_r},w}^{I,J}(q_1,q_2) = r_{u,y,w}^{I,J}(q_1,q_2)$ in the quotient ring.

Now fix a,b and let $\pi = (\pi^{(0)},\ldots,\pi^{(r)})$ be a sequence satisfying the definition of the coefficient of $q_1^a q_2^b$ in $r_{u,v,w}^{I,J}(q_1,q_2)$. Since we have

$$(\pi^{(r-1)})^{-1} \in \{(\pi^{(r)})^{-1}, (s_{i_r}\pi^{(r)})^{-1}\} \subset W_I w W_J, \tag{1.4.28}$$

it is clear that the subsequence $\hat{\pi} = (\pi^{(0)},\ldots,\pi^{(r-1)})$ of π satisfies

1. $\pi^{(0)} = u$, $(\pi^{(r-1)})^{-1} \in W_I w W_J$,
2. $\pi^{(j)} \in \{s_{i_j}\pi^{(j-1)}, \pi^{(j-1)}\}$,
3. $\pi^{(j)} = s_{i_j}\pi^{(j-1)}$ if $s_{i_j}\pi^{(j-1)} > \pi^{(j-1)}$,

and therefore contributes to some coefficient of $r_{u,vs_{i_r},w}^{I,J}(q_1,q_2)$.

Let us call the coefficients of $r_{u,v,w}^{I,J}(q_1,q_2)$ and $r_{u,vs_{i_r},w}^{I,J}(q_1,q_2)$, $c_{a,b}$ and $d_{a,b}$, respectively, so that we have

$$\begin{aligned} r_{u,v,w}^{I,J}(q_1,q_2) &= \sum_{a=0}^{n}\sum_{b=0}^{n} c_{a,b} q_1^a q_2^b, \\ r_{u,vs_{i_r},w}^{I,J}(q_1,q_2) &= \sum_{a=0}^{n}\sum_{b=0}^{n} d_{a,b} q_1^a q_2^b. \end{aligned} \tag{1.4.29}$$

Furthermore let us define $c_{a,b}^{<}$, $c_{a,b}^{=}$, and $c_{a,b}^{>}$ to be the number of sequences satisfying the definition of the coefficient of $q_1^a q_2^b$ in $r_{u,v,w}^{I,J}(q_1,q_2)$ such that $\pi^{(r-1)} < \pi^{(r)}$, $\pi^{(r-1)} = \pi^{(r)}$, and $\pi^{(r-1)} > \pi^{(r)}$, respectively. Similarly, define $d_{a,b}^{<}$ and $d_{a,b}^{>}$ to be the number of sequences $\rho = (\rho^{(0)}, \ldots, \rho^{(r-1)})$ satisfying the definition of the coefficient of $q_1^a q_2^b$ in $r^{I,J}_{u,vs_{i_r},w}(q_1,q_2)$ such that $\rho^{(r-1)} < s_{i_r}\rho^{(r-1)}$ and $\rho^{(r-1)} > s_{i_r}\rho^{(r-1)}$, respectively. Thus we have

$$\begin{aligned} c_{a,b} &= c^<_{a,b} + c^=_{a,b} + c^>_{a,b}, \\ d_{a,b} &= d^<_{a,b} + d^>_{a,b}. \end{aligned} \tag{1.4.30}$$

Note we will define $c_{a,b} = d_{a,b} = 0$ whenever a or b is negative.

To establish the equality of $r^{I,J}_{u,vs_{i_r},w}(q_1,q_2)$ and $r^{I,J}_{u,v,w}(q_1,q_2)$ in the quotient ring, we consider the subsequence $\hat{\pi}$ and the three possible relationships between $\pi^{(r-1)}$ and $\pi^{(r)}$.

First, suppose that $\pi^{(r-1)} < \pi^{(r)}$. Then $\hat{\pi}$ satisfies the conditions defining the coefficient of $q_1^a q_2^b$ in $r^{I,J}_{u,vs_{i_r},w}(q_1,q_2)$. Moreover, sequences $(\rho^{(0)}, \ldots, \rho^{(r-1)})$ contributing to the coefficient of $q_1^a q_2^b$ in $r^{I,J}_{u,vs_{i_r},w}(q_1,q_2)$ and satisfying $\rho^{(r-1)} < s_{i_r}\rho^{(r-1)}$ correspond bijectively to sequences $(\rho^{(0)}, \ldots, \rho^{(r)})$ contributing to the coefficient of $q_1^a q_2^b$ in $r^{I,J}_{u,v,w}(q_1,q_2)$ and satisfying $\rho^{(r-1)} < \rho^{(r)}$. Therefore we have

$$c^<_{a,b} = d^<_{a,b}. \tag{1.4.31}$$

Now, suppose that $\pi^{(r-1)} = \pi^{(r)}$. Then $\hat{\pi}$ satisfies the conditions defining the coefficient of $q_1^{a-1} q_2^{b-1}$ in $r^{I,J}_{u,vs_{i_r},w}(q_1,q_2)$. Moreover, sequences $(\rho^{(0)}, \ldots, \rho^{(r-1)})$ contributing to the coefficient of $q_1^{a-1} q_2^{b-1}$ in $r^{I,J}_{u,vs_{i_r},w}(q_1,q_2)$ and satisfying $\rho^{(r-1)} > s_{i_r}\rho^{(r-1)}$ correspond bijectively to sequences $(\rho^{(0)}, \ldots, \rho^{(r)})$ contributing to the coefficient of $q_1^a q_2^b$ in $r^{I,J}_{u,v,w}(q_1,q_2)$ and satisfying $\rho^{(r-1)} = \rho^{(r)}$. Therefore we have

$$c^=_{a,b} = d^>_{a-1,b-1}. \tag{1.4.32}$$

Finally, suppose that $\pi^{(r-1)} > \pi^{(r)}$. Then $\hat{\pi}$ satisfies the conditions defining the coefficient of $q_1^a q_2^{b-2}$ in $r^{I,J}_{u,vs_{i_r},w}(q_1,q_2)$. Moreover, sequences $(\rho^{(0)}, \ldots, \rho^{(r-1)})$ contributing to the coefficient of $q_1^a q_2^{b-2}$ in $r^{I,J}_{u,vs_{i_r},w}(q_1,q_2)$ and satisfying $\rho^{(r-1)} > s_{i_r}\rho^{(r-1)}$ correspond bijectively to sequences $(\rho^{(0)}, \ldots, \rho^{(r)})$ contributing to the coefficient of $q_1^a q_2^b$ in $r^{I,J}_{u,v,w}(q_1,q_2)$ and satisfying $\rho^{(r-1)} > \rho^{(r)}$. Therefore we have

$$c^>_{a,b} = d^>_{a,b-2}. \tag{1.4.33}$$

Thus we have

$$\begin{aligned}
r^{I,J}_{u,v,w}(q_1,q_2) &= \sum_{a=0}^{n}\sum_{b=0}^{n} c_{a,b}q_1^a q_2^b \\
&= \sum_{a=0}^{n}\sum_{b=0}^{n} \left(c^{<}_{a,b} + c^{=}_{a,b} + c^{>}_{a,b}\right) q_1^a q_2^b \\
&= \sum_{a=0}^{n}\sum_{b=0}^{n} \left(d^{<}_{a,b} + d^{>}_{a-1,b-1} + d^{>}_{a,b-2}\right) q_1^a q_2^b \qquad (1.4.34) \\
&= \sum_{a=0}^{n}\sum_{b=0}^{n} d^{<}_{a,b}q_1^a q_2^b + \sum_{a=0}^{n}\sum_{b=0}^{n} d^{>}_{a,b}q_1^{a+1} q_2^{b+1} + \sum_{a=0}^{n}\sum_{b=0}^{n} d^{>}_{a,b}q_1^a q_2^{b+2} \\
&= \sum_{a=0}^{n}\sum_{b=0}^{n} \left(d^{<}_{a,b} + d^{>}_{a,b}\left(q_1q_2 + q_2^2\right)\right) q_1^a q_2^b.
\end{aligned}$$

In the quotient ring we can make the substitution $q_1q_2 + q_2^2 = 1$ and get

$$\begin{aligned}
r^{I,J}_{u,v,w}(q_1,q_2) &= \sum_{a=0}^{n}\sum_{b=0}^{n} \left(d^{<}_{a,b} + d^{>}_{a,b}\right) q_1^a q_2^b \\
&= \sum_{a=0}^{n}\sum_{b=0}^{n} d_{a,b}q_1^a q_2^b \qquad (1.4.35) \\
&= r^{I,J}_{u,vs_{i_r},w}(q_1,q_2).
\end{aligned}$$

□

Corollary 1.4.7. *Fix $y \in W_-^{\emptyset,I}$, $u \in yW_I$, and $w \in W_+^{I,J}$. In the ring $\mathbb{Z}[q_1,q_2]/(1 - q_1q_2 - q_2^2)$ we have*

$$p^{I,J}_{u,v,w}(q_1,q_2) = p^{I,J}_{y,v,w}(q_1,q_2). \qquad (1.4.36)$$

Proof. Recall that $p^{I,J}_{u,v,w}(q_1,q_2) = r^{I,J}_{v,u,w^{-1}}(q_1,q_2)$ by (1.4.4). Therefore Lemma 1.4.6 implies that $p^{I,J}_{u,v,w}(q_1,q_2) = r^{I,J}_{v,y,w^{-1}}(q_1,q_2)$. Applying (1.4.4) once more gives us (1.4.36). □

The next few lemmas will establish partial natural and inverse transpose expansion results which lead to the general case.

Lemma 1.4.8. *For $u \in W_{-}^{\emptyset,I}$ and $v \in \mathfrak{S}_r$,*

$$(x_{L,[r]})^{u,v} = \sum_{w \in W_{+}^{I,\emptyset}} p_{u,v,w}^{I,\emptyset}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}})(x_{L,[r]})^{e,w}. \tag{1.4.37}$$

Proof. First let us consider the case $u = e$. Letting w be the maximal coset representative of $W_I v$, by 1.3.17, we have $(x_{L,[r]})^{e,v} = q_{v,w}^{-1}(x_{L,[r]})^{e,w}$. Furthermore by definition, $p_{e,v,w}^{I,\emptyset}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) = (q^{\frac{-1}{2}})^{\ell(w)-\ell(v)} = q_{v,w}^{-1}$.

Now assume the claim holds for all u of length at most $k-1$. Choose u of length k and a reduced expression $s_{i_1} \dots s_{i_k}$ for u. Since $s_{i_1}u < u$, we have $u_{i_1} > u_{i_1+1}$ and consequently $\ell_{u_{i_1}} > \ell_{u_{i_1+1}}$. Thus, (1.3.16) reduces to

$$\begin{aligned}
(x_{L,[r]})^{u,v} &= \begin{cases} (x_{L,[r]})^{s_{i_1}u, s_{i_1}v} & \text{if } s_{i_1}v > v, \\ (x_{L,[r]})^{s_{i_1}u, s_{i_1}v} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})(x_{L,[r]})^{s_{i_1}u, v} & \text{if } s_{i_1}v < v, \end{cases} \\
&= \begin{cases} \displaystyle\sum_{w \in W_{+}^{I,\emptyset}} p_{s_{i_1}u, s_{i_1}v, w}^{I,\emptyset}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}})(x_{L,[r]})^{e,w} & \text{if } s_{i_1}v > v, \\ \displaystyle\sum_{w \in W_{+}^{I,\emptyset}} \Big(p_{s_{i_1}u, s_{i_1}v, w}^{I,\emptyset}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \\ \qquad + (q^{\frac{1}{2}} - q^{\frac{-1}{2}}) p_{s_{i_1}u, v, w}^{I,\emptyset}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) \Big)(x_{L,[r]})^{e,w} & \text{if } s_{i_1}v < v, \end{cases}
\end{aligned} \tag{1.4.38}$$

where, since $\ell(s_{i_1}u) = k-1$, induction gives equality. Apparently, it suffices to show that for $\ell_{u_{i_1}} > \ell_{u_{i_1+1}}$, we have

$$p_{u,v,w}^{I,\emptyset}(q_1, q_2) = \begin{cases} p_{s_{i_1}u, s_{i_1}v, w}^{I,\emptyset}(q_1, q_2) & \text{if } s_{i_1}v > v, \\ p_{s_{i_1}u, s_{i_1}v, w}^{I,\emptyset}(q_1, q_2) + q_1 p_{s_{i_1}u, v, w}^{I,\emptyset}(q_1, q_2) & \text{if } s_{i_1}v < v. \end{cases} \tag{1.4.39}$$

Suppose first that $s_{i_1}v > v$. Then, the coefficient of $q_1^a q_2^b$ in $p_{s_{i_1}u, s_{i_1}v, w}^{I,\emptyset}(q_1, q_2)$ is equal to the number of sequences $(\pi^{(1)}, \dots, \pi^{(k)})$ satisfying conditions (2)-(5) of the definition and $\pi^{(1)} = s_{i_1}v$, $\pi^{(k)} \in W_I w$. Prepending the permutation $\pi^{(0)} = v$ to each of these sequences defines a bijection with those satisfying the conditions in the definition of $p_{u,v,w}^{I,\emptyset}(q_1, q_2)$. Therefore $p_{s_{i_1}u, s_{i_1}v, w}^{I,\emptyset}(q_1, q_2) = p_{u,v,w}^{I,\emptyset}(q_1, q_2)$ as desired.

Now suppose that $s_{i_1}v < v$. For all a, b, the coefficient of $q_1^a q_2^b$ in $p_{s_{i_1}u, s_{i_1}v, w}^{I,\emptyset}(q_1, q_2)$ is equal to the number of sequences $(\pi^{(1)}, \dots, \pi^{(k)})$ satisfying conditions (2)-(5)

of the definition and $\pi^{(1)} = s_{i_1}v$, $\pi^{(k)} \in W_I w$, while the coefficient of $q_1^a q_2^b$ in $q_1 p^{I,\emptyset}_{s_{i_1}u,v,w}(q_1,q_2)$ is equal to the number of sequences $(\pi^{(1)},\dots,\pi^{(k)})$ satisfying conditions (2), (3), and (5) of the definition and $\pi^{(j)} = \pi^{(j-1)}$ for exactly $a-1$ values of j, as well as $\pi^{(1)} = v$, $\pi^{(k)} \in W_I w$. Prepending the permutation $\pi^{(0)} = v$ to each of these sequences defines a bijection with those satisfying the conditions in the definition of $p^{I,\emptyset}_{u,v,w}(q_1,q_2)$. Therefore $p^{I,\emptyset}_{s_{i_1}u,s_{i_1}v,w}(q_1,q_2) + q_1 p^{I,\emptyset}_{s_{i_1}u,v,w}(q_1,q_2) = p^{I,\emptyset}_{u,v,w}(q_1,q_2)$ as desired. □

Corollary 1.4.7 allows us to remove the condition on u and strengthen the previous lemma.

Corollary 1.4.9. *For all $u, v \in \mathfrak{S}_r$,*

$$(x_{L,[r]})^{u,v} = \sum_{w \in W_+^{I,\emptyset}} p^{I,\emptyset}_{u,v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}})(x_{L,[r]})^{e,w} \tag{1.4.40}$$

and

$$(x_{L,[r]})^{u,v} = \sum_{w \in W_+^{I,\emptyset}} r^{I,\emptyset}_{u,v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}})(x_{L,[r]})^{w^{-1},e}. \tag{1.4.41}$$

Proof. Let $y \in W_-^{\emptyset,I}$ such that $u \in yW_I$ and $t \in W_I$ where $u = yt$. Then (1.3.31) and Lemma 1.4.8 give

$$(x_{L,[r]})^{u,v} = (x_{L,[r]})^{y,v} = \sum_{w \in W_+^{I,\emptyset}} p^{I,\emptyset}_{y,v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}})(x_{L,[r]})^{e,w}. \tag{1.4.42}$$

Since $(q^{\frac{1}{2}} - q^{\frac{-1}{2}})q^{\frac{-1}{2}} + (q^{\frac{-1}{2}})^2 = 1$, we can apply Corollary 1.4.7 to get the first equation. The second comes from applying (1.3.30) and (1.4.21) to the first equation.

$$\begin{aligned}(x_{L,[r]})^{u,v} &= \sum_{w \in W_+^{I,\emptyset}} p^{I,\emptyset}_{u,v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}})(x_{L,[r]})^{e,w} \\ &= \sum_{w \in W_+^{I,\emptyset}} p^{I,\emptyset}_{u,v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}})(q^{\frac{-1}{2}})^{-\ell(w_0^I)}(x_{L,[r]})^{w^{-1},e} \\ &= \sum_{w \in W_+^{I,\emptyset}} r^{I,\emptyset}_{u,v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}})(x_{L,[r]})^{w^{-1},e}.\end{aligned} \tag{1.4.43}$$

□

We have established the expansion results for the special case $M = [r]$. A few more lemmas are needed in order to get the most general case.

Lemma 1.4.10. *For all $u \in \mathfrak{S}_r$ and $v \in W_-^{\emptyset,J}$ we have*

$$(x_{L,M})^{u,v} = \sum_{y \in W_+^{I,\emptyset}} r_{u,v,y}^{I,\emptyset}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}})(x_{L,M})^{y^{-1},e}. \tag{1.4.44}$$

Proof. Let us first consider the case where $v = e$. If $u^{-1} \in W_I y$ with $y \in W_+^{I,\emptyset}$, then by (1.3.17), we have that $(x_{L,M})^{u,e} = (x_{L,M})^{y^{-1},e}$. Since there is only one sequence, namely the sequence (u), which satisfies the conditions in the definition of $r_{u,e,y}^{I,\emptyset}(q_1, q_2)$, we have by definition that

$$r_{u,e,y}^{I,\emptyset}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) = (q^{\frac{-1}{2}})^{\ell(y)-\ell(u)+\ell(u^I)-\ell(w_0^I)} = 1, \tag{1.4.45}$$

as desired, since $\ell(y) - \ell(u) = \ell(w_0^I) - \ell(u^I)$.

Assume the claim to be true for v having length at most $k-1$. Fix a permutation v having length k. Let s_i be such that $s_i v < v$ and thus $m_{v_i} > m_{v_i+1}$. Then (1.3.16) gives

$$\begin{aligned}
(x_{L,M})^{u,v} &= \begin{cases} (x_{L,M})^{s_iu,s_iv} & \text{if } \ell_{u_i} < \ell_{u_i+1}, \\ q^{\frac{1}{2}}(x_{L,M})^{s_iu,s_iv} & \text{if } \ell_{u_i} = \ell_{u_i+1}, \\ (x_{L,M})^{s_iu,s_iv} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})(x_{L,M})^{u,s_iv} & \text{if } \ell_{u_i} > \ell_{u_i+1}, \end{cases} \\
&= \begin{cases} \displaystyle\sum_{y \in W_+^{I,\emptyset}} r_{s_iu,s_iv,y}^{I,\emptyset}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}})(x_{L,M})^{y^{-1},e}, & \text{if } \ell_{u_i} < \ell_{u_i+1}, \\ \displaystyle\sum_{y \in W_+^{I,\emptyset}} q^{\frac{1}{2}} r_{s_iu,s_iv,y}^{I,\emptyset}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}})(x_{L,M})^{y^{-1},e}, & \text{if } \ell_{u_i} = \ell_{u_i+1}, \\ \displaystyle\sum_{y \in W_+^{I,\emptyset}} \Big(r_{s_iu,s_iv,y}^{I,\emptyset}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \\ \qquad + (q^{\frac{1}{2}} - q^{\frac{-1}{2}}) r_{u,s_iv,y}^{I,\emptyset}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) \Big)(x_{L,M})^{y^{-1},e}, & \text{if } \ell_{u_i} > \ell_{u_i+1}, \end{cases}
\end{aligned} \tag{1.4.46}$$

where induction gives equality. Thus it suffices to show that

$$r^{I,\emptyset}_{u,v,y}(q^{\frac{1}{2}}-q^{\frac{-1}{2}},q^{\frac{-1}{2}})=\begin{cases} r^{I,\emptyset}_{s_iu,s_iv,y}(q^{\frac{1}{2}}-q^{\frac{-1}{2}},q^{\frac{-1}{2}}) & \text{if } \ell_{u_i}<\ell_{u_i+1},\\ q^{\frac{1}{2}}r^{I,\emptyset}_{s_iu,s_iv,y}(q^{\frac{1}{2}}-q^{\frac{-1}{2}},q^{\frac{-1}{2}}) & \text{if } \ell_{u_i}=\ell_{u_i+1},\\ r^{I,\emptyset}_{s_iu,s_iv,y}(q^{\frac{1}{2}}-q^{\frac{-1}{2}},q^{\frac{-1}{2}}) & \\ \quad +(q^{\frac{1}{2}}-q^{\frac{-1}{2}})r^{I,\emptyset}_{u,s_iv,y}(q^{\frac{1}{2}}-q^{\frac{-1}{2}},q^{\frac{-1}{2}}) & \text{if } \ell_{u_i}>\ell_{u_i+1}. \end{cases} \tag{1.4.47}$$

This can be seen by applying Corollary 1.4.9 to both sides of (1.3.16) and comparing like terms. □

Lemma 1.4.11. *For $u\in\mathfrak{S}_r$ and $v\in W^{\emptyset,J}_{-}$,*

$$(x_{L,M})^{u,v}=\sum_{w\in W^{I,J}_{+}} r^{I,J}_{u,v,w}(q^{\frac{1}{2}}-q^{\frac{-1}{2}},q^{\frac{-1}{2}})(x_{L,M})^{w^{-1},e}. \tag{1.4.48}$$

Proof. Applying (1.3.17) to the result of Lemma 1.4.10 gives

$$(x_{L,M})^{u,v}=\sum_{w\in W^{I,J}_{+}}\left(\sum_{\substack{y\in W^{I,\emptyset}_{+}\\ y\in W_IwW_J}} q_{w^J_-,y^J_-}r^{I,\emptyset}_{u,v,y}(q^{\frac{1}{2}}-q^{\frac{-1}{2}},q^{\frac{-1}{2}})\right)(x_{L,M})^{w^{-1},e}. \tag{1.4.49}$$

Since $v\in W^{\emptyset,J}_{-}$ we have that $v^J=e$ and we see this sum of single parabolic r-polynomials is one of the sums in Corollary 1.4.5. Thus we have

$$(x_{L,M})^{u,v}=\sum_{w\in W^{I,J}_{+}} r^{I,J}_{u,v,w}(q^{\frac{1}{2}}-q^{\frac{-1}{2}},q^{\frac{-1}{2}})(x_{L,M})^{w^{-1},e}, \tag{1.4.50}$$

as claimed. □

The preceding lemma allows us to finally prove the result we were seeking, that the polynomials $\{p^{I,J}_{u,v,w}(q_1,q_2)\mid u,v\in\mathfrak{S}_r, w\in W^{I,J}_{+}\}$ are the coefficients in the natural basis expansion of monomials in $\mathcal{A}(n;q)$.

Theorem 1.4.12. *For all $u,v\in\mathfrak{S}_r$*

$$\begin{aligned}(x_{L,M})^{u,v}&=\sum_{w\in W^{I,J}_{+}} p^{I,J}_{u,v,w}(q^{\frac{1}{2}}-q^{\frac{-1}{2}},q^{\frac{-1}{2}})(x_{L,M})^{e,w}\\ &=\sum_{w\in W^{I,J}_{+}} r^{I,J}_{u,v,w}(q^{\frac{1}{2}}-q^{\frac{-1}{2}},q^{\frac{-1}{2}})(x_{L,M})^{w^{-1},e}.\end{aligned} \tag{1.4.51}$$

Proof. Fix $u, v \in \mathfrak{S}_r$ and let $y \in W_-^{\emptyset,J}$ such that $v \in yW_J$. Then (1.3.16), Lemma 1.4.11, and Lemma 1.4.6 give

$$\begin{aligned}(x_{L,M})^{u,v} = (x_{L,M})^{u,y} &= \sum_{w \in W_I w W_J} r^{I,J}_{u,y,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}})(x_{L,M})^{w^{-1},e} \\ &= \sum_{w \in W_I w W_J} r^{I,J}_{u,v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}})(x_{L,M})^{w^{-1},e}.\end{aligned} \tag{1.4.52}$$

Applying (1.3.31) and Proposition 1.4.4 completes the proof. □

Chapter 2

The polynomials $p_{u,v,w}^{I,J}$ and $r_{u,v,w}^{I,J}$

The polynomials $\{p_{u,v,w}^{I,J}(q_1, q_2) \mid u, v \in \mathfrak{S}_r, w \in W_+^{I,J}\}$ and $\{r_{u,v,w}^{I,J}(q_1, q_2) \mid u, v \in \mathfrak{S}_r, w \in W_+^{I,J}\}$ introduced in Section 1.4 appear in many more places than just transition matrices in $\mathcal{A}(n; q)$. They also appear in $H_n(q)$ and its associated submodules $H'_{I,J}$. Furthermore, these families of polynomials have symmetries which, due to the combinatorial definition, imply equinumerocity between several sets of paths in the Bruhat order. In this chapter, following [31], we will define actions of $H_n(q)$ on $\mathcal{A}(n; q)$, which allow us to describe the multiplicative structure of $H_n(q)$ using our new polynomials. Then we will use their appearance in $H_n(q)$ and $\mathcal{A}(n; q)$ to establish symmetries, or identities, which the polynomials satisfy.

2.1 Connections between $H_n(q)$ and $\mathcal{A}(n; q)$

The two spaces $H_n(q)$ and $\mathcal{A}(n; q)$ may not appear to have much in common at first glance; however, it turns out that the polynomials $\{p_{u,v,w}^{I,J}(q_1, q_2) \mid u, v \in \mathfrak{S}_r, w \in W_+^{I,J}\}$ are connected to the multiplicative structure of $H_n(q)$.

Define a left action of $H_n(q)$ on the immanant space $\mathcal{A}_{[n],[n]}(n; q)$ by

$$\widetilde{T}_{s_i} \circ f(x) \underset{\text{def}}{=} f(s_i x), \tag{2.1.1}$$

where s_i is the $n \times n$ defining matrix for s_i, and where we assumed $f(x)$ to be expressed in terms of the natural basis. Similarly, define a right action of $H_n(q)$ on

$\mathcal{A}_{[n],[n]}(n;q)$ by

$$f(x) \circ \widetilde{T}_{s_i} \underset{\text{def}}{=} f(xs_i), \tag{2.1.2}$$

where we assume $f(x)$ to be expressed in terms of the inverse transpose basis. Thus we have the identity

$$f(x) \circ \widetilde{T}_{s_i} = (\widetilde{T}_{s_i} \circ f(x^\top))^\top. \tag{2.1.3}$$

Formulas for these actions on natural basis elements are

$$\begin{gathered}\widetilde{T}_{s_i} \circ x^{e,v} = x^{s_i,v} = \begin{cases} x^{e,s_iv} & \text{if } s_iv > v, \\ x^{e,s_iv} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{e,v} & \text{if } s_iv < v, \end{cases} \\ x^{e,v} \circ \widetilde{T}_{s_i} = x^{v^{-1},e} \circ \widetilde{T}_{s_i} = x^{v^{-1},s_i} = \begin{cases} x^{e,vs_i} & \text{if } vs_i > v, \\ x^{e,vs_i} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{e,v} & \text{if } vs_i < v. \end{cases}\end{gathered} \tag{2.1.4}$$

The following formulas describing the action on monomials of the form $x^{u,v}$ appear in [31].

Proposition 2.1.1. *We have*

$$\widetilde{T}_{s_j} \circ x^{u,v} = \begin{cases} x^{us_j,v} & \text{if } us_j > u, \\ x^{us_j,v} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{u,v} & \text{if } us_j < u, \end{cases} \tag{2.1.5}$$

$$x^{u,v} \circ \widetilde{T}_{s_j} = \begin{cases} x^{u,vs_j} & \text{if } vs_j > v, \\ x^{u,vs_j} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{u,v} & \text{if } vs_j < v. \end{cases} \tag{2.1.6}$$

Proof. Assume the formula (2.1.5) to hold for all monomials $x^{u,v}$ with $\ell(u) < k$. Certainly this is true if $\ell(u) = 0$. Now fix one permutation u of length k, and let s_i be a left descent for u. By (1.3.16) we have

$$\widetilde{T}_{s_j} \circ x^{u,v} \begin{cases} \widetilde{T}_{s_j} \circ x^{s_iu,s_iv} & \text{if } s_iv > v, \\ \widetilde{T}_{s_j} \circ x^{s_iu,s_iv} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})\widetilde{T}_{s_j} \circ x^{s_iu,v} & \text{if } s_iv < v, \end{cases} \tag{2.1.7}$$

which by induction is equal to

$$\begin{cases} x^{s_ius_j,s_iv} & \text{if } s_iv > v \text{ and } s_ius_j > s_iu, \\ x^{s_ius_j,s_iv} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{s_iu,s_iv} & \text{if } s_iv > v \text{ and } s_ius_j < s_iu, \\ x^{s_ius_j,s_iv} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{s_ius_j,v} & \text{if } s_iv < v \text{ and } s_ius_j > s_iu, \\ x^{s_ius_j,s_iv} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})(x^{s_iu,s_iv} + x^{s_ius_j,v}) & \\ \qquad +(q^{\frac{1}{2}} - q^{\frac{-1}{2}})^2 x^{s_iu,v} & \text{if } s_iv < v \text{ and } s_ius_j < s_iu. \end{cases} \tag{2.1.8}$$

Now we return to the right-hand side of the claimed formula. Suppose first that $us_j > u$. This implies that $s_iu < s_ius_j < us_j$. By (1.3.16) we then have

$$x^{us_j,v} = \begin{cases} x^{s_ius_j,s_iv} & \text{if } s_iv > v, \\ x^{s_ius_j,s_iv} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{s_ius_j,v} & \text{if } s_iv < v, \end{cases} \tag{2.1.9}$$

which is equal to $\widetilde{T}_{s_j} \circ x^{u,v}$ by cases 1 and 3 of (2.1.8). Now suppose that $us_j < u$. Then we have $u > s_ius_j$ or $u = s_ius_j$. If $u = s_ius_j$, then $us_j = s_iu < u = s_ius_j$. Applying (1.3.16) to (just the first monomial in)

$$x^{us_j,v} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{u,v} = x^{us_j,v} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{s_ius_j,v}, \tag{2.1.10}$$

we again obtain the expressions on the right-hand side of (2.1.9). If $u > s_ius_j$, then $s_iu < u$ and $s_ius_j < us_j$. By (1.3.16) we then have

$$x^{us_j,v} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{u,v} = \begin{cases} x^{s_ius_j,s_iv} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})(x^{s_ius_j,v} + x^{s_iu,s_iv}) & \\ \qquad +(q^{\frac{1}{2}} - q^{\frac{-1}{2}})^2 x^{s_iu,v} & \text{if } s_iv < v, \\ x^{s_ius_j,s_iv} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{s_iu,s_iv} & \text{if } s_iv > v, \end{cases} \tag{2.1.11}$$

which is equal to $\widetilde{T}_{s_j} \circ x^{u,v}$ by cases 2 and 4 of (2.1.8).

By (2.1.3), we may apply the transpose map to the formula (2.1.5) to obtain the formula (2.1.6). □

It follows that the monomial $x^{e,v}$ may be written as

$$x^{e,v} = \widetilde{T}_v \circ x^{e,e} = x^{e,e} \circ \widetilde{T}_v = x^{v^{-1},e}. \tag{2.1.12}$$

More generally, the monomial $x^{u,v}$ may be written as

$$\begin{aligned} x^{u,v} &= \widetilde{T}_{u^{-1}} \circ x^{e,v} = x^{u,e} \circ \widetilde{T}_v = \widetilde{T}_{u^{-1}} \widetilde{T}_v \circ x^{e,e} = x^{e,e} \circ \widetilde{T}_{u^{-1}} \widetilde{T}_v \\ &= \widetilde{T}_{u^{-1}} \circ x^{e,e} \circ \widetilde{T}_v. \end{aligned} \tag{2.1.13}$$

This allows us to connect the polynomials $\{p_{u,v,w} \mid u, v, w \in \mathfrak{S}_n\}$ to the multiplicative structure of $H_n(q)$.

Proposition 2.1.2. *For all $u, v \in \mathfrak{S}_n$*

$$\widetilde{T}_u \widetilde{T}_v = \sum_{w \in \mathfrak{S}_n} p_{u^{-1},v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) \widetilde{T}_w. \tag{2.1.14}$$

Proof. Define elements $\{a_{u,v,w} \mid u, v, w \in \mathfrak{S}_n\}$ in $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$ by the equation

$$\widetilde{T}_u \widetilde{T}_v = \sum_{w \in \mathfrak{S}_n} a_{u,v,w} \widetilde{T}_w. \tag{2.1.15}$$

Then the element $\widetilde{T}_u \widetilde{T}_v \circ x^{e,e}$ of $\mathcal{A}_{[n],[n]}(n;q)$ expands in the natural basis as

$$\sum_{w \in \mathfrak{S}_n} a_{u,v,w} \widetilde{T}_w \circ x^{e,e} = \sum_{w \in \mathfrak{S}_n} a_{u,v,w} x^{e,w}. \tag{2.1.16}$$

On the other hand, by (2.1.13) and (1.4.10) this element is equal to

$$x^{u^{-1},v} = \sum_{w \in \mathfrak{S}_n} p_{u^{-1},v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) x^{e,w}. \tag{2.1.17}$$

Thus we have $a_{u,v,w} = p_{u^{-1},v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}})$ for all $u, v, w \in \mathfrak{S}_n$. □

Similarly, define a left action of $H_n(q)$ on $\mathcal{A}_{[n],M}(n;q)$ by

$$\widetilde{T}_{s_i} \circ f(x_{[n],M}) \underset{\text{def}}{=} f(s_i x_{[n],M}), \tag{2.1.18}$$

assuming $f(x_{[n],M})$ is expressed in terms of the natural basis, and define a right action of $H_n(q)$ on $\mathcal{A}_{L,[n]}(n:q)$ by

$$f(x_{L,[n]}) \circ \widetilde{T}_{s_i} \underset{\text{def}}{=} f(x_{L,[n]} s_i), \tag{2.1.19}$$

assuming $f(x_{L,[n]})$ is expressed in terms of the inverse transpose basis. Thus we have the identity

$$f(x_{L,[n]}) \circ \widetilde{T}_{s_i} = (\widetilde{T}_{s_i} \circ f(x^{\top}_{L,[n]}))^{\top}. \tag{2.1.20}$$

Formulas for these actions are

$$\widetilde{T}_{s_i} \circ (x_{[n],M})^{e,w} = \begin{cases} (x_{[n],M})^{e,s_iw} & \text{if } s_iwW_J > wW_J, \\ q^{\frac{1}{2}}(x_{[n],M})^{e,w} & \text{if } s_iwW_J = wW_J, \\ (x_{[n],M})^{e,s_iw} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})(x_{[n],M})^{e,w} & \text{if } s_iwW_J < wW_J. \end{cases} \tag{2.1.21}$$

$$(x_{L,[n]})^{e,w} \circ \widetilde{T}_{s_i} = \begin{cases} (x_{L,[n]})^{e,ws_i} & \text{if } W_Iws_i > W_Iw, \\ q^{\frac{1}{2}}(x_{L,[n]})^{e,w} & \text{if } W_Iws_i = W_Iw, \\ (x_{L,[n]})^{e,ws_i} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})(x_{L,[n]})^{e,w} & \text{if } W_Iws_i < W_Iw. \end{cases} \tag{2.1.22}$$

More generally, we have the following formulas which appear in [31].

Proposition 2.1.3. *We have*

$$\widetilde{T}_{s_j} \circ (x_{[n],M})^{u,v} = \begin{cases} (x_{[n],M})^{us_j,v} & \textit{if } us_j > u, \\ (x_{[n],M})^{us_j,v} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})(x_{[n],M})^{u,v} & \textit{if } us_j < u, \end{cases} \tag{2.1.23}$$

$$(x_{L,[n]})^{u,v} \circ \widetilde{T}_{s_j} = \begin{cases} (x_{L,[n]})^{u,vs_j} & \textit{if } vs_j > v, \\ (x_{L,[n]})^{u,vs_j} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})(x_{L,[n]})^{u,v} & \textit{if } vs_j < v. \end{cases} \tag{2.1.24}$$

Proof. Fix $k \geq 1$. Assume the formula (2.1.23) to hold for all monomials $(x_{[n],M})^{u,v}$ with $\ell(u) < k$. Certainly this is true for $k = 1$. Now fix one permutation u of length k and let s_i be a left descent for u. By (1.3.16) we have

$$\widetilde{T}_{s_j} \circ (x_{[n],M})^{u,v} = \begin{cases} \widetilde{T}_{s_j} \circ (x_{[n],M})^{s_iu,s_iv} & \text{if } m_{v_i} < m_{v_{i+1}}, \\ q^{\frac{-1}{2}}\widetilde{T}_{s_j} \circ (x_{[n],M})^{s_iu,s_iv} & \text{if } m_{v_i} = m_{v_{i+1}}, \\ \widetilde{T}_{s_j} \circ (x_{[n],M})^{s_iu,s_iv} & \\ \quad +(q^{\frac{1}{2}} - q^{\frac{-1}{2}})\widetilde{T}_{s_j} \circ (x_{[n],M})^{s_iu,v} & \text{if } m_{v_i} > m_{v_{i+1}}, \end{cases} \tag{2.1.25}$$

which by induction is equal to

$$\begin{cases} (x_{[n],M})^{s_ius_j,s_iv} & \text{if } m_{v_i} < m_{v_{i+1}} \text{ and } s_ius_j > s_iu, \\ (x_{[n],M})^{s_ius_j,s_iv} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})(x_{[n],M})^{s_iu,s_iv} & \text{if } m_{v_i} < m_{v_{i+1}} \text{ and } s_ius_j < s_iu, \\ q^{\frac{1}{2}}(x_{[n],M})^{s_ius_j,s_iv} & \text{if } m_{v_i} = m_{v_{i+1}} \text{ and } s_ius_j > s_iu, \\ q^{\frac{1}{2}}(x_{[n],M})^{s_ius_j,s_iv} + q^{\frac{1}{2}}(q^{\frac{1}{2}} - q^{\frac{-1}{2}})(x_{[n],M})^{s_iu,s_iv} & \text{if } m_{v_i} = m_{v_{i+1}} \text{ and } s_ius_j < s_iu, \\ (x_{[n],M})^{s_ius_j,s_iv} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})(x_{[n],M})^{s_ius_j,v} & \text{if } m_{v_i} > m_{v_{i+1}} \text{ and } s_ius_j > s_iu, \\ (x_{[n],M})^{s_ius_j,s_iv} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})((x_{[n],M})^{s_iu,s_iv} & \\ \quad (x_{[n],M})^{s_ius_j,v}) + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})^2(x_{[n],M})^{s_iu,v} & \text{if } m_{v_i} > m_{v_{i+1}} \text{ and } x_ius_j < s_iu. \end{cases} \tag{2.1.26}$$

Now we return to the right-hand side of the claimed formula. Suppose first that $us_j > u$. This implies that $s_iu < s_ius_j < us_j$. By (1.3.16) we then have

$$(x_{[n],M})^{us_j,v} = \begin{cases} (x_{[n],M})^{s_ius_j,s_iv} & \text{if } m_{v_i} > m_{v_{i+1}}, \\ q^{\frac{1}{2}}(x_{[n],M})^{s_ius_j,s_iv} & \text{if } m_{v_i} = m_{v_{i+1}}, \\ (x_{[n],M})^{s_ius_j,s_iv} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})(x_{[n],M})^{s_ius_j,v} & \text{if } m_{v_i} < m_{v_{i+1}}, \end{cases} \tag{2.1.27}$$

which is equal to $\widetilde{T}_{s_j} \circ (x_{[n],M})^{u,v}$ by cases 1, 3, and 5 of (2.1.26). Now suppose that $us_j < u$. Then we have $u = s_ius_j$ or $u > s_ius_j$. If $u = s_ius_j$, then $us_j = s_iu < u = s_ius_j$. Applying (1.3.16) to (just the first monomial in)

$$(x_{[n],M})^{us_j,v} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})(x_{[n],M})^{u,v} = (x_{[n],M})^{us_j,v} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})(x_{[n],M})^{s_ius_j,v}, \tag{2.1.28}$$

we again obtain the expressions on the right-hand side of (2.1.27). If $u > s_ius_j$, then $s_iu < u$ and $s_ius_j < us_j$. By (1.3.16) we then have

$$(x_{[n],M})^{us_j,v} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})(x_{[n],M})^{u,v} =$$

$$\begin{cases} (x_{[n],M})^{s_ius_j,s_iv} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})(x_{[n],M})^{s_iu,s_iv} & \text{if } m_{v_i} < m_{v_{i+1}}, \\ q^{\frac{1}{2}}(x_{[n],M})^{s_ius_j,s_iv} + q^{\frac{1}{2}}(q^{\frac{1}{2}} - q^{\frac{-1}{2}})(x_{[n],M})^{s_iu,s_iv} & \text{if } m_{v_i} = m_{v_{i+1}}, \\ (x_{[n],M})^{s_ius_j,s_iv} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})((x_{[n],M})^{s_ius_j,v} & \\ \quad +(x_{[n],M})^{s_iu,s_v}) + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})^2(x_{[n],M})^{s_iu,v} & \text{if } m_{v_i} > m_{v_{i+1}} \end{cases} \tag{2.1.29}$$

which is equal to $\widetilde{T}_{s_j} \circ (x_{[n],M})^{u,v}$ by cases 2, 4, and 6 of (2.1.26).

By (2.1.20), we may apply the transpose map to the formula (2.1.23) to obtain the formula (2.1.24). $\square$

We can use these formulas to derive recursive formulas for the single parabolic p and r-polynomials.

Lemma 2.1.4. *For all $u, v \in \mathfrak{S}_n$, $w \in W_+^{I,\emptyset}$, and $z \in W_+^{\emptyset,J}$, if $vs_i < v$ then*

$$p^{I,\emptyset}_{u,v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) = \begin{cases} p^{I,\emptyset}_{u,vs_i,ws_i}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \text{if } W_I ws_i > W_I w, \\ q^{\frac{1}{2}} p^{I,\emptyset}_{u,vs_i,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \text{if } W_I ws_i = W_I w, \\ p^{I,\emptyset}_{u,vs_i,ws_i}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \\ \quad +(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) p^{I,\emptyset}_{u,vs_i,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \text{if } W_I ws_i < W_I w, \end{cases}$$

$$r^{I,\emptyset}_{u,v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) = \begin{cases} r^{I,\emptyset}_{u,vs_i,ws_i}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \text{if } W_I ws_i > W_I w, \\ q^{\frac{1}{2}} r^{I,\emptyset}_{u,vs_i,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \text{if } W_I ws_i = W_I w, \\ r^{I,\emptyset}_{u,vs_i,ws_i}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \\ \quad +(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) r^{I,\emptyset}_{u,vs_i,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \text{if } W_I ws_i < W_I w, \end{cases} \tag{2.1.30}$$

and if $s_j u < u$ then

$$p^{\emptyset,J}_{u,v,z}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) = \begin{cases} p^{\emptyset,J}_{us_j,v,s_jz}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \text{if } s_j z W_J > z W_J, \\ q^{\frac{1}{2}} p^{\emptyset,J}_{us_j,v,z}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \text{if } s_j z W_J = z W_J, \\ p^{\emptyset,J}_{us_j,v,s_jz}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \\ \quad +(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) p^{\emptyset,J}_{us_j,v,z}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \text{if } s_j z W_J < z W_J, \end{cases}$$

$$r^{\emptyset,J}_{u,v,z}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) = \begin{cases} r^{\emptyset,J}_{us_j,v,s_jz}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \text{if } s_j z W_J > z W_J, \\ q^{\frac{1}{2}} r^{\emptyset,J}_{us_j,v,z}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \text{if } s_j z W_J = z W_J, \\ r^{\emptyset,J}_{us_j,v,s_jz}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \\ \quad +(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) r^{\emptyset,J}_{us_j,v,z}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \text{if } s_j z W_J < z W_J. \end{cases} \tag{2.1.31}$$

Proof. The previous proposition says that $(x_{L,[n]})^{u,v} = (x_{L,[n]})^{u,vs_i} \circ \widetilde{T}_{s_i}$, where $vs_i < v$. Thus using Theorem 1.4.12 and (2.1.22) we can say

$$\begin{aligned}
(x_{L,[n]})^{u,v} &= (x_{L,[n]})^{u,vs_i} \circ \widetilde{T}_{s_i} \\
&= \sum_{w \in W_+^{I,\emptyset}} p^{I,\emptyset}_{u,vs_i,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}})(x_{L,[n]})^{e,w} \circ \widetilde{T}_{s_i} \\
&= \sum_{\substack{w \in W_+^{I,\emptyset} \\ W_I ws_i > W_I w}} p^{I,\emptyset}_{u,vs_i,ws_i}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}})(x_{L,[n]})^{e,w} \\
&+ \sum_{\substack{w \in W_+^{I,\emptyset} \\ W_I ws_i = W_I w}} q^{\frac{1}{2}} p^{I,\emptyset}_{u,vs_i,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}})(x_{L,[n]})^{e,w} \\
&+ \sum_{\substack{w \in W_+^{I,\emptyset} \\ W_I ws_i < W_I w}} \left(p^{I,\emptyset}_{u,vs_i,ws_i}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) + (q^{\frac{1}{2}} - q^{\frac{-1}{2}}) p^{I,\emptyset}_{u,vs_i,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) \right) (x_{L,[n]})^{e,w}.
\end{aligned} \tag{2.1.32}$$

Expanding $(x_{L,[n]})^{u,v}$ in terms of the natural basis and comparing terms gives

$$p^{I,\emptyset}_{u,v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) = \begin{cases} p^{I,\emptyset}_{u,vs_i,ws_i}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \text{if } W_I ws_i > W_I w, \\ q^{\frac{1}{2}} p^{I,\emptyset}_{u,vs_i,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \text{if } W_I ws_i = W_I w, \\ p^{I,\emptyset}_{u,vs_i,ws_i}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \\ \quad +(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) p^{I,\emptyset}_{u,vs_i,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \text{if } W_I ws_i < W_I w. \end{cases} \tag{2.1.33}$$

Thus we have the first equation in the claim. To get the second equation, apply Proposition 1.4.4 to both sides of the first. The last two equations come from applying (1.4.4) to the first two equations in the claim while relabeling $w^{-1} \in W_+^{J,I}$ as z. □

These recursive formulas allow us to connect our polynomials to the action of $H_n(q)$ on its submodules $H'_{I,\emptyset}$ and $H'_{\emptyset,J}$ in a manner analogous to Proposition 2.1.2.

Proposition 2.1.5. *For all $u, w \in W_+^{I,\emptyset}$ and $v \in \mathfrak{S}_n$,*

$$\widetilde{T}'_{W_I u} \widetilde{T}_v = \sum_{w \in W_+^{I,\emptyset}} r^{I,\emptyset}_{u^{-1},v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) \widetilde{T}_{W_I w} \tag{2.1.34}$$

and for all $u \in \mathfrak{S}_n$ and $v, w \in W_+^{\emptyset,J}$,

$$\widetilde{T}_u \widetilde{T}'_{vW_J} = \sum_{w \in W_+^{\emptyset,J}} p^{\emptyset,J}_{u^{-1},v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) \widetilde{T}'_{wW_J}. \tag{2.1.35}$$

Proof. First let us prove the first equation holds. Let $u, w \in W_+^{I,\emptyset}$. Assume $v = e$, then we have $\widetilde{T}'_{W_I u}\widetilde{T}_e = \widetilde{T}'_{W_I u}$. By definition $r^{I,\emptyset}_{u^{-1},e,w}(q_1, q_2) = 1$ if $u^{-1} = w$ and zero otherwise. Thus the claim holds.

Assume the claim holds for all v with length less than k. Choose v to be length k and s_i such that $vs_i < v$. Then by induction and (1.2.16) we have

$$\begin{aligned}
\widetilde{T}'_{W_I u}\widetilde{T}_v &= \widetilde{T}'_{W_I u}\widetilde{T}_{vs_i}\widetilde{T}_{s_i} \\
&= \sum_{w \in W_+^{I,\emptyset}} r^{I,\emptyset}_{u^{-1},vs_i,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) \widetilde{T}'_{W_I w}\widetilde{T}_{s_i} \\
&= \sum_{\substack{w \in W_+^{I,\emptyset} \\ W_I ws_i > W_I w}} r^{I,\emptyset}_{u^{-1},vs_i,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) \widetilde{T}'_{W_I ws_i} \\
&\quad + \sum_{\substack{w \in W_+^{I,\emptyset} \\ W_I ws_i = W_I w}} r^{I,\emptyset}_{u^{-1},vs_i,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) q^{\frac{1}{2}} \widetilde{T}'_{W_I w} \\
&\quad + \sum_{\substack{w \in W_+^{I,\emptyset} \\ W_I ws_i < W_I w}} r^{I,\emptyset}_{u^{-1},vs_i,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) \left(\widetilde{T}'_{W_I ws_i} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})\widetilde{T}'_{W_I w}\right) \\
&= \sum_{\substack{w \in W_+^{I,\emptyset} \\ W_I ws_i > W_I w}} r^{I,\emptyset}_{u^{-1},vs_i,ws_i}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) \widetilde{T}'_{W_I w} \\
&\quad + \sum_{\substack{w \in W_+^{I,\emptyset} \\ W_I ws_i = W_I w}} q^{\frac{1}{2}} r^{I,\emptyset}_{u^{-1},vs_i,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) \widetilde{T}'_{W_I w} \\
&\quad + \sum_{\substack{w \in W_+^{I,\emptyset} \\ W_I ws_i < W_I w}} \left(r^{I,\emptyset}_{u^{-1},vs_i,ws_i}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) + (q^{\frac{1}{2}} - q^{\frac{-1}{2}}) r^{I,\emptyset}_{u^{-1},vs_i,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}})\right) \widetilde{T}'_{W_I w}.
\end{aligned} \tag{2.1.36}$$

Looking at (2.1.30), we see that, in fact, we have

$$\widetilde{T}'_{W_I u}\widetilde{T}_v = \sum_{w \in W_+^{I,\emptyset}} r^{I,\emptyset}_{u^{-1},v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) \widetilde{T}'_{W_I w} \tag{2.1.37}$$

as claimed.

The second equation in the claim is proved in a similar manner, only we induct on the length of u instead of v. Let $v, w \in W_+^{\emptyset,J}$. Assume $u = e$, then we have $\widetilde{T}_e \widetilde{T}'_{vW_J} = \widetilde{T}'_{wW_J}$. By definition $p^{\emptyset,J}_{e,v,w}(q_1, q_2) = 1$ if $v = w$ and zero otherwise. Thus the claim holds.

Assume the claim holds for all u with length less than k. Choose u to be length k and s_i such that $s_i u < u$. Then by induction and (1.2.18) we have

$$\begin{aligned}
\widetilde{T}_u \widetilde{T}'_{vW_J} &= \widetilde{T}_{s_i} \widetilde{T}_{s_i u} \widetilde{T}'_{vW_J} \\
&= \sum_{w \in W_+^{\emptyset,J}} p^{\emptyset,J}_{u^{-1}s_i,v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) \widetilde{T}_{s_i} \widetilde{T}'_{wW_J} \\
&= \sum_{\substack{w \in W_+^{\emptyset,J} \\ s_i w W_J > wW_J}} p^{\emptyset,J}_{u^{-1}s_i,v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) \widetilde{T}'_{s_i wW_J} \\
&\quad + \sum_{\substack{w \in W_+^{\emptyset,J} \\ s_i w W_J = wW_J}} p^{\emptyset,J}_{u^{-1}s_i,v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) q^{\frac{1}{2}} \widetilde{T}'_{wW_J} \\
&\quad + \sum_{\substack{w \in W_+^{\emptyset,J} \\ s_i w W_J < wW_J}} p^{\emptyset,J}_{u^{-1}s_i,v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) \left(\widetilde{T}'_{s_i wW_J} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}}) \widetilde{T}'_{wW_J} \right) \\
&= \sum_{\substack{w \in W_+^{\emptyset,J} \\ s_i w W_J > wW_J}} p^{\emptyset,J}_{u^{-1}s_i,v,s_i w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) \widetilde{T}'_{wW_J} \\
&\quad + \sum_{\substack{w \in W_+^{\emptyset,J} \\ s_i w W_J = wW_J}} q^{\frac{1}{2}} p^{\emptyset,J}_{u^{-1}s_i,v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) \widetilde{T}'_{wW_J} \\
&\quad + \sum_{\substack{w \in W_+^{\emptyset,J} \\ s_i w W_J < wW_J}} \left(p^{\emptyset,J}_{u^{-1}s_i,v,s_i w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) + (q^{\frac{1}{2}} - q^{\frac{-1}{2}}) p^{\emptyset,J}_{u^{-1}s_i,v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) \right) \widetilde{T}'_{wW_J}.
\end{aligned} \tag{2.1.38}$$

Looking at (2.1.31), we see that, in fact, we have

$$\widetilde{T}_u \widetilde{T}'_{vW_J} = \sum_{w \in W_+^{\emptyset,J}} p^{\emptyset,J}_{u^{-1},v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) \widetilde{T}'_{wW_J} \tag{2.1.39}$$

as claimed. □

These methods will not work in the case where neither I, nor J are the empty set, since we do not have nice formulas for an action of $H_n(q)$ on $H'_{I,J}$. If we were to find such formulas, then it should be possible. However, double cosets lack certain nice features of single cosets, presenting difficulties which may not be easily surmounted.

Problem 1. *Find analogous connections between* $H_{I,J}$,$H'_{I,J}$ *and* $\mathcal{A}_{L,M}(n;q)$.

2.2 Symmetries

Shortly after we first defined the polynomials $\{p^{I,J}_{u,v,w}(q_1,q_2)\}$, we saw in (1.4.4) and (1.4.20) that they satisfied certain symmetries,

$$r^{I,J}_{u,v,w}(q_1,q_2) = p^{J,I}_{v,u,w^{-1}}(q_1,q_2) \tag{2.2.1}$$

and

$$p_{u,v,w}(q) = p_{v,u,w^{-1}}(q). \tag{2.2.2}$$

Due to the combinatorial definition of these polynomials, any symmetries suggest bijections between sets of paths in the Bruhat order. In order to explore the symmetries these polynomials satisfy, ideally we would look for and discover bijections; however, another tactic would be to use their presence in algebraic settings. To that end, we can use the multiplicative structure of $H_n(q)$ and the connection to these polynomials.

Proposition 2.2.1. *For all* $u, v, w \in \mathfrak{S}_n$

$$p_{u,v,w}(q) = p_{v^{-1},w^{-1},u}(q). \tag{2.2.3}$$

Proof. Define functions $c^w_{u,v} \in \mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$ by

$$\widetilde{T}_u\widetilde{T}_v = \sum_{w\in\mathfrak{S}_n} c^w_{u,v}\widetilde{T}_w. \tag{2.2.4}$$

[23, Lemma 4.1] shows

$$c^w_{u,v} = c^{u^{-1}}_{v,w^{-1}}. \tag{2.2.5}$$

This fact can also be deduced from the proofs of Lemmas 10.4 and 13.3 in [28]. Proposition 2.1.2 says that $c^w_{u,v} = p_{u^{-1},v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}})$. Putting these two equalities together completes the proof. □

Corollary 2.2.2. *For all $u, v, w \in \mathfrak{S}_n$*

$$\begin{aligned} p_{u,v,w}(q) &= p_{v^{-1},w^{-1},u}(q) = p_{w,u^{-1},v^{-1}}(q) \\ &= p_{v,u,w^{-1}}(q) = p_{w^{-1},v^{-1},u^{-1}}(q) = p_{u^{-1},w,v}(q). \end{aligned} \tag{2.2.6}$$

Proof. This is seen from applying Proposition 2.2.1 repeatedly, then applying (2.2.2) to each of those forms. □

Each of these symmetries implies two sets of paths in the Bruhat order are equinumerous. Furthermore, considering $p_{u,v,w}(q)$ is defined combinatorially, it would be ideal to find bijections between these sets which imply these symmetries, rather than relying on the structure of the immanant space. For example we can see that $p_{u,v,w}(q) = p_{u^{-1},w,v}(q)$ (one of the six symmetries given by (2.2.6)) by considering the map ϕ which reverses the order of a path (or finite sequence),

$$\phi(x_1, \cdots, x_\ell) \underset{\text{def}}{=} (x_\ell, \cdots, x_1). \tag{2.2.7}$$

Proposition 2.2.3. *For $k \in \mathbb{N}$ and $u, v, w \in \mathfrak{S}_n$, the reversal map is a bijection between the set of sequences counted by the coefficient of q^k in $p_{u,v,w}(q)$ and the set of sequences counted by the coefficient of q^k in $p_{u^{-1},w,v}(q)$.*

Proof. Let π be a sequence counted by the coefficient of q^k in $p_{u,v,w}(q)$. We need to show that $\rho = \phi(\pi)$ is a sequence counted by the coefficient of q^k in $p_{u^{-1},w,v}(q)$. Clearly the initial and final points of ρ are w and v, respectively. Furthermore, if $s_{i_1} \cdots s_{i_\ell}$ is a reduced expression for u, then $s_{i_\ell} \cdots s_{i_1}$ is a reduced expression for u^{-1} and the second condition is also satisfied. The fourth condition clearly holds and the fifth condition is trivial in the immanant space. All that remains is to show that the third condition holds.

Let j be an integer between 1 and ℓ for which $t_{i_j}\rho^{(j-1)} > \rho^{(j-1)}$, where $t_{i_1} \cdots t_{i_\ell}$ is the reduced expression for u^{-1} which is the reverse of the reduced expression

$s_{i_1}\cdots s_{i_\ell}$. Assume $\rho^{(j)} \neq t_{i_j}\rho^{(j-1)}$, but rather $\rho^{(j)} = \rho^{(j-1)}$. Since $t_{i_j} = s_{i_{\ell-j+1}}$ and $\rho^{(j)} = \pi^{(\ell-j)}$, this implies $\pi^{(\ell-j)} = \pi^{(\ell-j+1)}$. For this to be true, we must have $s_{i_{\ell-j+1}}\pi^{(\ell-j)} < \pi^{(\ell-j)}$. Since $\pi^{(\ell-j)} = \pi^{(\ell-j+1)}$, we also have $s_{i_{\ell-j+1}}\pi^{(\ell-j+1)} < \pi^{(\ell-j+1)}$. Except that this implies $t_{i_1}\rho^{(j-1)} < \rho^{(j-1)}$, which is a contradiction. Thus ρ satisfies all the conditions in the definition of $p_{u^{-1},w,v}(q)$.

The reversal map is clearly injective and furthermore it is an involution and thus a bijection as claimed. □

Problem 2. *Find bijections between the different sets of sequences, or walks in the Bruhat order, which correspond to the other five symmetries in* (2.2.6).

Some of the symmetries in Corollary 2.2.2 also hold in the single parabolic cases as well. To see this we apply the result of Corollary 1.4.9 to (1.3.16) to get the following recursive formulas

$$
\begin{aligned}
p^{I,\emptyset}_{u,v,w}(q^{\frac12} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) &= \begin{cases} p^{I,\emptyset}_{s_iu,s_iv,w}(q^{\frac12} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \text{if } \ell_{u_i} < \ell_{u_i+1}, \\ q^{\frac12} p^{I,\emptyset}_{u,s_iv,w}(q^{\frac12} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \text{if } \ell_{u_i} = \ell_{u_i+1}, \\ p^{I,\emptyset}_{s_iu,s_iv,w}(q^{\frac12} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \\ \quad +(q^{\frac12} - q^{\frac{-1}{2}}) p^{I,\emptyset}_{u,s_iv,w}(q^{\frac12} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \text{if } \ell_{u_i} > \ell_{u_i+1}, \end{cases} \\
r^{I,\emptyset}_{u,v,w}(q^{\frac12} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) &= \begin{cases} r^{I,\emptyset}_{s_iu,s_iv,w}(q^{\frac12} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \text{if } \ell_{u_i} < \ell_{u_i+1}, \\ q^{\frac12} r^{I,\emptyset}_{u,s_iv,w}(q^{\frac12} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \text{if } \ell_{u_i} = \ell_{u_i+1}, \\ r^{I,\emptyset}_{s_iu,s_iv,w}(q^{\frac12} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \\ \quad +(q^{\frac12} - q^{\frac{-1}{2}}) r^{I,\emptyset}_{u,s_iv,w}(q^{\frac12} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \text{if } \ell_{u_i} > \ell_{u_i+1}, \end{cases}
\end{aligned}
\tag{2.2.8}
$$

where $s_i v < v$, and

$$p^{\emptyset,J}_{u,v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) = \begin{cases} p^{\emptyset,J}_{s_iu,s_iv,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \text{if } m_{v_i} < m_{v_i+1}, \\ q^{\frac{1}{2}} p^{\emptyset,J}_{s_iu,v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \text{if } m_{v_i} = m_{v_i+1}, \\ p^{\emptyset,J}_{s_iu,s_iv,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \\ \quad +(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) p^{\emptyset,J}_{s_iu,v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \text{if } m_{v_i} > m_{v_i+1}, \end{cases}$$

$$r^{\emptyset,J}_{u,v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) = \begin{cases} r^{\emptyset,J}_{s_iu,s_iv,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \text{if } m_{v_i} < m_{v_i+1}, \\ q^{\frac{1}{2}} r^{\emptyset,J}_{s_iu,v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \text{if } m_{v_i} = m_{v_i+1}, \\ r^{\emptyset,J}_{s_iu,s_iv,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \\ \quad +(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) r^{\emptyset,J}_{s_iu,v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \text{if } m_{v_i} > m_{v_i+1}, \end{cases} \tag{2.2.9}$$

where $s_i u < u$. These recursive formulas provide us with the means to establish some symmetry results for the single parabolic cases.

Proposition 2.2.4. *For all $u \in W^{\emptyset,I}_+$, $v \in \mathfrak{S}_n$, and $w \in W^{I,\emptyset}_+$ we have*

$$p^{I,\emptyset}_{u,v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) = p^{I,\emptyset}_{w^{-1},v^{-1},u^{-1}}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}), \tag{2.2.10}$$

and for all $u \in \mathfrak{S}_n$, $v, w \in W^{\emptyset,J}_+$ we have

$$p^{\emptyset,J}_{u,v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) = p^{\emptyset,J}_{u^{-1},w,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}). \tag{2.2.11}$$

Proof. Comparing Lemma 2.1.4 and (2.2.8) we notice that the two polynomials in (2.2.10) satisfy the same recursive formulas. All that is left is to check to see that they have the same initial condition. When $v = e$ we have $p^{I,\emptyset}_{u,e,w}(q_1, q_2) = \ell(w^I_0)$ if $u^{-1} = w$ and zero otherwise. Therefore the formula holds as claimed.

Similarly, comparing Lemma 2.1.4 and (2.2.9), we see that the polynomials in (2.2.11) satisfy the same recursive formulas. Once again they satisfy the same initial condition, $p^{\emptyset,J}_{e,v,w}(q_1, q_2) = 1$ if $v = w$ and zero otherwise. □

The symmetries stated in Proposition 2.2.4 have only been shown to hold when $q_1 = q^{\frac{1}{2}} - q^{\frac{-1}{2}}$ and $q_2 = q^{\frac{-1}{2}}$. We would like to understand whether this is a necessary condition or just sufficient. One possibility that comes to mind is the quotient ring

$\mathbb{Z}[q_1, q_2]/(1-q_1q_2-q_2^2)$. In Lemma 1.4.6 we saw that this was the condition required for the symmetry

$$r^{I,J}_{u,v,w}(q_1, q_2) = r^{I,J}_{u,y,w}(q_1, q_2) \tag{2.2.12}$$

to hold, where $y \in W^{\emptyset,J}_-$, $v \in yW_J$, and $w \in W^{I,J}_+$. Combining Lemma 1.4.6 and Corollary 1.4.7, we get the following result.

Corollary 2.2.5. *Fix $y \in W^{\emptyset,I}_-$, $u \in yW_I$, $z \in W^{\emptyset,J}_-$, $v \in zW_J$, and $w \in W^{I,J}_+$. In the ring $\mathbb{Z}[q_1, q_2]/(1-q_1q_2-q_2^2)$ we have*

$$\begin{aligned} p^{I,J}_{u,v,w}(q_1, q_2) &= p^{I,J}_{y,z,w}(q_1, q_2), \\ r^{I,J}_{u,v,w}(q_1, q_2) &= r^{I,J}_{y,z,w}(q_1, q_2). \end{aligned} \tag{2.2.13}$$

Proof. By Corollary 1.4.7, we have $p^{I,J}_{u,v,w}(q_1, q_2) = p^{I,J}_{y,v,w}(q_1, q_2)$. We also have by (1.4.21) that $p^{I,J}_{y,v,w}(q_1, q_2) = q_2^{\ell(w_0^I)-\ell(w_0^J)} r^{I,J}_{y,v,w}(q_1, q_2)$. Applying Lemma 1.4.6, we see that $r^{I,J}_{y,v,w}(q_1, q_2) = r^{I,J}_{y,z,w}(q_1, q_2)$. Putting all of this together and applying (1.4.21) once more, we have the first equation. The second equation comes from applying (1.4.21) to the first equation. □

Problem 3. *Identify general conditions on $I, I', J, J', u, u', v, v', w, w'$, as well as the parameters q_1, q_2, q_1', q_2', for which*

$$p^{I,J}_{u,v,w}(q_1, q_1) = p^{I',J'}_{u',v',w'}(q_1', q_2'). \tag{2.2.14}$$

Furthermore, find bijective proofs for the known conditions.

Chapter 3

Bar involutions and invariant bases

As mentioned in the introduction, an important ingredient in the definition of Kazhdan and Lusztig's basis of $H_n(q)$ is known as the *bar involution*. Applying this involution gives rise to *modified R-polynomials* in $\mathbb{N}[q]$. These are the polynomials studied by Brenti mentioned when we defined the polynomials $\{p^{I,J}_{u,v,w}(q_1, q_2) \mid u, v \in \mathfrak{S}_r, w \in W^{I,J}_+\}$. In this chapter we will show that the modified R-polynomials are the subset of these new polynomials when $u = w_0$ and $v \in W^{I,J}_-$. In order to show this, we define the bar involution on $H_n(q)$ and the R-polynomials, as well as introduce parabolic analogues, as in [16]. We then follow [31] in defining a bar involution on $\mathcal{A}(n; q)$, which is compatible with the actions of $H_n(q)$, and introduce *(parabolic) inverse R-polynomials*, as a special case of the $p^{I,J}$-polynomials.

In the final two sections of the chapter we will move our interest to special bar-invariant bases in $\mathcal{A}(n; q)$. Kazhdan and Lusztig defined a bar invariant basis in $H_n(q)$ [22], which has been of great interest. Similarly, a bar-invariant basis for $\mathcal{A}(n; q)$, called the *dual canonical basis*, is important to representation theorists studying quantum groups. In [31] Skandera gives a new formulation for the dual canonical basis in the single parabolic case. We extend that formulation to the double parabolic case, giving formulations for the entire dual canonical basis of

$\mathcal{A}(n;q)$. Finally, we examine the different versions of the bar involution and dual canonical basis appearing in several places, connecting our results to the literature.

3.1 The bar involution on $H_n(q)$

Define an involution on $H_n(q)$, commonly known as the *bar involution*, by

$$\sum_{w \in \mathfrak{S}_n} a_w \widetilde{T}_w \mapsto \overline{\sum_{w \in \mathfrak{S}_n} a_w \widetilde{T}_w} = \sum_{w \in \mathfrak{S}_n} \overline{a_w} \overline{\widetilde{T}_w}, \tag{3.1.1}$$

where

$$\overline{q^{\frac{1}{2}}} = q^{\frac{-1}{2}}, \qquad \overline{\widetilde{T}_w} = (\widetilde{T}_{w^{-1}})^{-1}. \tag{3.1.2}$$

We call an element $g \in H_n(q)$ *bar invariant* if $\overline{g} = g$. By (1.2.4) we see that for a generator $\widetilde{T}_s$ we have the special case

$$\overline{\widetilde{T}_s} = \widetilde{T}_s - (q^{\frac{1}{2}} - q^{\frac{-1}{2}})\widetilde{T}_e. \tag{3.1.3}$$

The bar involution is an automorphism of $H_n(q)$,

$$\overline{\widetilde{T}_u \widetilde{T}_v} = \overline{\widetilde{T}_u} \cdot \overline{\widetilde{T}_v}. \tag{3.1.4}$$

It is not too hard to see that the elements $\{\overline{\widetilde{T}_{s_i}} \mid 1 \leq i \leq n-1\}$ generate $H_n(q)$ and satisfy the braid and commutation relations as well as

$$\left(\overline{\widetilde{T}_{s_i}}\right)^2 = \overline{\widetilde{T}_e} - (q^{\frac{1}{2}} - q^{\frac{-1}{2}})\overline{\widetilde{T}_s}. \tag{3.1.5}$$

Writing the basis $\{\overline{\widetilde{T}_v} \mid v \in \mathfrak{S}_n\}$, which we will call the *barred natural basis*, in terms of the natural basis using (3.1.3), (3.1.4), and induction, we see that we have

$$\overline{\widetilde{T}_v} \in \sum_{u \leq v} \mathbb{N}[\overline{q^{\frac{1}{2}} - q^{\frac{-1}{2}}}]\widetilde{T}_u \cap \sum_{u \leq v} \overline{q_{u,v}^{-1}\mathbb{Z}[q]}\widetilde{T}_u. \tag{3.1.6}$$

Specifically, we may write

$$\overline{\widetilde{T}_v} = \sum_{u \leq v} \overline{q_{u,v}^{-1} R_{u,v}(q)}\widetilde{T}_u = \sum_{u \leq v} \overline{\widetilde{R}_{u,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}})}\widetilde{T}_u, \tag{3.1.7}$$

where $\{R_{u,v}(q) \mid u,v \in \mathfrak{S}_n\}$ belong to $\mathbb{Z}[q]$, and $\{\widetilde{R}_{u,v}(q_1) \mid u,v \in \mathfrak{S}_n\}$ belong to $\mathbb{N}[q_1]$. Call these the *R-polynomials* and *modified R-polynomials*, respectively.

The R-polynomials and modified R-polynomials are related by

$$R_{u,v}(q) = q_{u,v}\widetilde{R}_{u,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}). \tag{3.1.8}$$

We will focus our attention on the modified R-polynomials; however, any formulas or symmetries will also hold for the R-polynomials thanks to this relationship.

The combinatorial properties of the R- and modified R-polynomials have been studied by Brenti [3], Deodhar [13], and Dyer [17]. Brenti gives a combinatorial interpretation using what he calls R-chains, where Deodhar uses what he calls distinguished subexpressions. Several of the main results are collected in [2] and [19]. We will review some interesting facts about the modified R-polynomials, which in Section 3.3 will be used, along with the polynomials $\{p_{u,v,w}(q_1) \mid u,v,w \in \mathfrak{S}_n\}$, to state a new combinatorial interpretation for modified R-polynomials.

Using (3.1.4) to write $\overline{\widetilde{T}_v} = \overline{\widetilde{T}_{vs}} \cdot \overline{\widetilde{T}_s}$ for $vs < v$, and using induction on $\ell(v)$, we see that the modified R-polynomials are the unique family $\{\widetilde{R}_{u,v}(q_1) \mid u,v \in \mathfrak{S}_n\}$ of polynomials in $\mathbb{N}[q_1]$ satisfying

1. $\widetilde{R}_{u,v}(q_1) = 0$ if $u \not\leq v$.

2. $\widetilde{R}_{v,v}(q_1) = 1$ for all u.

3. For each right descent s of v we have

$$\widetilde{R}_{u,v}(q_1) = \begin{cases} \widetilde{R}_{us,vs}(q_1) & \text{if } us < u, \\ \widetilde{R}_{us,vs}(q_1) + q_1\widetilde{R}_{u,vs}(q_1) & \text{otherwise.} \end{cases} \tag{3.1.9}$$

(See [19] for more details.) Stating condition (3) in terms of *left* descents s of v, we have

$$\widetilde{R}_{u,v}(q_1) = \begin{cases} \widetilde{R}_{su,sv}(q_1) & \text{if } su < u, \\ \widetilde{R}_{su,sv}(q_1) + q_1\widetilde{R}_{u,sv}(q_1) & \text{otherwise.} \end{cases} \tag{3.1.10}$$

On the other hand, we may fix a right ascent s of u and use (3.1.9) to obtain

$$\widetilde{R}_{u,v}(q_1) = \begin{cases} \widetilde{R}_{us,vs}(q_1) & \text{if } vs > v, \\ \widetilde{R}_{us,vs}(q_1) + q_1 \widetilde{R}_{us,v}(q_1) & \text{otherwise,} \end{cases} \tag{3.1.11}$$

or we may fix a left ascent s of u and use (3.1.10) to obtain

$$\widetilde{R}_{u,v}(q_1) = \begin{cases} \widetilde{R}_{su,sv}(q_1) & \text{if } sv > v, \\ \widetilde{R}_{su,sv}(q_1) + q \widetilde{R}_{su,v}(q_1) & \text{otherwise.} \end{cases} \tag{3.1.12}$$

From the recursive formulas (3.1.9) - (3.1.12), one can verify that for $u \leq v$, $\widetilde{R}_{u,v}(q_1)$ is a monic polynomial of degree $\ell(v) - \ell(u)$ with constant term equal to zero, unless $u = v$. For example, for $u \geq v$ in $\mathfrak{S}_3$, we have

$$\widetilde{R}_{u,v}(q_1) = \begin{cases} 1 & \text{if } \ell(v) - \ell(u) = 0, \\ q_1 & \text{if } \ell(v) - \ell(u) = 1, \\ q_1^2 & \text{if } \ell(v) - \ell(u) = 2, \\ q_1^3 + q_1 & \text{if } \ell(v) - \ell(u) = 3, \end{cases} \tag{3.1.13}$$

and similarly,

$$R_{u,v}(q) = \begin{cases} 1 & \text{if } \ell(v) - \ell(u) = 0, \\ q - 1 & \text{if } \ell(v) - \ell(u) = 1, \\ q - 2q + 1 & \text{if } \ell(v) - \ell(u) = 2, \\ q^3 - 2q^2 + 2q - 1 & \text{if } \ell(v) - \ell(u) = 3. \end{cases} \tag{3.1.14}$$

Furthermore, by (3.1.7), for all $u \leq v$ we have

$$\widetilde{R}_{u,v}(q) = \widetilde{R}_{w_0 v, w_0 u}(q) = \widetilde{R}_{v w_0, u w_0}(q) = \widetilde{R}_{u^{-1}, v^{-1}}(q). \tag{3.1.15}$$

Similarly, we have

$$R_{u,v}(q) = R_{w_0 v, w_0 u}(q) = R_{v w_0, u w_0}(q) = R_{u^{-1}, v^{-1}}(q). \tag{3.1.16}$$

It is not difficult to see that

$$\overline{\widetilde{R}_{u,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}})} = \epsilon_{u,v}\widetilde{R}_{u,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}), \tag{3.1.17}$$

by applying the bar involution to one of the recursive formulas and using induction. Similarly, we have

$$\overline{R_{u,v}(q)} = R_{u,v}(q^{-1}) = \epsilon_{u,v}q_{u,v}^{-2}R_{u,v}(q). \tag{3.1.18}$$

Thus we could have alternatively defined the modified R-polynomials by

$$\overline{\widetilde{T}_v} = \sum_{u \leq v} \epsilon_{u,v}\widetilde{R}_{u,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}})\widetilde{T}_u. \tag{3.1.19}$$

We defined the modified R-polynomials in order to express the barred natural basis in terms of the natural basis. However, we can also express the natural basis in terms of the barred natural basis using the same polynomials. We see this by applying the bar involution to (3.1.7) to get

$$\widetilde{T}_v = \sum_{u \leq v} \widetilde{R}_{u,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}})\overline{\widetilde{T}_u}. \tag{3.1.20}$$

Expanding the right-hand side using (3.1.7) and recognizing that nonnegative powers of $q^{\frac{1}{2}} - q^{\frac{-1}{2}}$ are linearly independent in $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$ gives us

$$\sum_{u \leq v \leq w} \epsilon_{u,v}\widetilde{R}_{u,v}(q_1)\widetilde{R}_{v,w}(q_1) = \widetilde{R}_{u,v}(q_1)\epsilon_{v,w}\widetilde{R}_{v,w}(q_1) = \delta_{u,w}. \tag{3.1.21}$$

In [16], Du shows the bar involution on $H_n(q)$ induces a bar involution on the submodule and $H'_{I,J}$. Thus we would like to define *(double) parabolic R-polynomials* and *modified R-polynomials* in an analogous manner.

First notice that it is easy to see the elements d_I and $\widetilde{d}_I$ satisfy

$$\overline{d_I} = q_{e,w_0^I}^{-2}d_I, \qquad \overline{\widetilde{d}_I} = \widetilde{d}_I. \tag{3.1.22}$$

Another interesting useful fact is that the element $\widetilde{T}'_{W_I}$ is bar-invariant, i.e.

$$\overline{\widetilde{T}'_{W_I}} = \widetilde{T}'_{W_I}, \tag{3.1.23}$$

for all subsets I of generators of $\mathfrak{S}_n$. In fact $\widetilde{T}'_{W_I}$ is a Kazdhan-Lusztig basis element of $H_n(q)$ (see [22] and [24]).

For $w \in W_+^{I,J}$, applying the bar involution to (1.2.22) and using (1.2.23) gives us

$$\begin{aligned}
\overline{\widetilde{d}_{K(w)}\widetilde{T}'_{W_IwW_J}} &= \widetilde{T}'_{W_I}\overline{\widetilde{T}'_{w_-}}\widetilde{T}'_{W_J} \\
&= \sum_{u \le w_-} \overline{q_{u,w_-}^{-1}R_{u,w_-}(q)}\widetilde{T}'_{W_I}\widetilde{T}_u\widetilde{T}'_{W_J} \\
&= \sum_{\substack{v \in W_+^{I,J} \\ v \le w}} \sum_{u \in W_IvW_J} \overline{q_{u,w_-}^{-1}R_{u,w_-}(q)}q_{u,v_-}^{-1}\widetilde{T}'_{W_I}\widetilde{T}_{v_-}\widetilde{T}'_{W_J} \\
&= \sum_{\substack{v \in W_+^{I,J} \\ v \le w}} \sum_{u \in W_IvW_J} \overline{q_{v_-,w_-}^{-1}R_{u,w_-}(q)}\widetilde{d}_{K(v)}\widetilde{T}'_{W_IvW_J} \\
&= \sum_{\substack{v \in W_+^{I,J} \\ v \le w}} \overline{q_{v,w}^{-1}q_{w_0^{K(v)},w_0^{K(w)}}^{-1}\sum_{u \in W_IvW_J} R_{u,w_-}(q)}\widetilde{d}_{K(v)}\widetilde{T}'_{W_IvW_J}.
\end{aligned} \tag{3.1.24}$$

In light of (3.1.24), we define the parabolic R-polynomials in $\mathbb{Z}[q]$ by

$$\overline{\widetilde{d}_{K(w)}\widetilde{T}'_{W_IwW_J}} = \sum_{\substack{v \in W_+^{I,J} \\ v \le w}} \overline{q_{v,w}^{-1}R_{v,w}^{I,J}(q)}\widetilde{d}_{K(v)}\widetilde{T}'_{W_IvW_J}. \tag{3.1.25}$$

For example, when $n = 3$, $I = \{s_1\}$, and $J = \{s_2\}$, we have

$$\begin{aligned}
R_{s_1,s_1}^{I,\emptyset}(q) &= 1, \\
R_{s_1,s_1s_2}^{I,\emptyset}(q) &= q - 1, \\
R_{s_1,w_0}^{I,\emptyset}(q) &= q^2 - q,
\end{aligned} \tag{3.1.26}$$

and

$$\begin{aligned}
R_{s_1s_2,s_1s_2}^{I,J}(q) &= 1, \\
R_{s_1s_2,w_0}^{I,J}(q) &= q^2 - 1.
\end{aligned} \tag{3.1.27}$$

These examples support the claim that the parabolic R-polynomials are in $\mathbb{Z}[q]$, which we will prove shortly.

In [12], Deodhar defines single parabolic R-polynomials which are in $\mathbb{Z}[q]$ and specialize to the R-polynomials in the nonparabolic case. This new definition of double parabolic R-polynomials is consistent with Deodhar's definition of single parabolic R-polynomials, except he uses minimal coset representatives to index the polynomials. To see this is an extension of Deodhar's definition we note that (3.1.24) and (3.1.25) imply for $v, w \in W_+^{I,J}$,

$$R_{v,w}^{I,J}(q) = q^{-1}_{w_0^{K(v)},w_0^{K(w)}} \sum_{u \in W_I v W_J} R_{u,w_-}(q). \tag{3.1.28}$$

In the single parabolic case, since $K(u) = \emptyset$ for all $u \in W_-^{I,\emptyset}$, this becomes

$$R_{v,w}^{I,\emptyset}(q) = \sum_{u \in W_I v} R_{u,w_-}(q). \tag{3.1.29}$$

In [7], Brenti states that the right-hand side is equal to the single parabolic R-polynomials defined by Deohdar, as a result of a formula which appears in [12]. Thus our definition of double parabolic R-polynomials is consistent with Deodhar's.

In order to see that the double parabolic R-polynomials are consistent with the literature we needed to express them as sums of nonparabolic R-polynomials where the second index was the minimal representative rather than the maximal representative. We can get a similar formula where the indices match by using (1.2.24). Let $w \in W_+^{I,J}$, then

$$\begin{aligned}
\overline{\widetilde{d}_{K(w)} \widetilde{T}'_{W_I w W_J}} &= (q^{\frac{1}{2}})^{\ell(W_I w W_J)} \widetilde{T}'_{W_I} \overline{\widetilde{T}'_w} \widetilde{T}'_{W_J} \\
&= (q^{\frac{1}{2}})^{\ell(W_I w W_J)} \sum_{u \leq w} \overline{q^{-1}_{u,w} R_{u,w}(q)} \widetilde{T}'_{W_I} \widetilde{T}_u \widetilde{T}'_{W_J} \\
&= (q^{\frac{1}{2}})^{\ell(W_I w W_J)} \sum_{\substack{v \in W_+^{I,J} \\ v \leq w}} \sum_{u \in W_I v W_J} \overline{q^{-1}_{u,w} R_{u,w}(q)} q^{-1}_{u,v} \widetilde{T}'_{W_I} \widetilde{T}_v \widetilde{T}'_{W_J} \\
&= (q^{\frac{1}{2}})^{\ell(W_I w W_J)} \sum_{\substack{v \in W_+^{I,J} \\ v \leq w}} \sum_{u \in W_I v W_J} \overline{q^{-1}_{v,w} R_{u,w}(q)} (q^{\frac{1}{2}})^{\ell(W_I v W_J)} \widetilde{d}_{K(v)} \widetilde{T}'_{W_I v W_J} \\
&= \sum_{\substack{v \in W_+^{I,J} \\ v \leq w}} \overline{q^{-1}_{v,w} (q^{\frac{1}{2}})^{-\ell(W_I v W_J)} (q^{\frac{1}{2}})^{-\ell(W_I w W_J)} \sum_{u \in W_I v W_J} R_{u,w}(q)} \widetilde{d}_{K(v)} \widetilde{T}'_{W_I v W_J}.
\end{aligned} \tag{3.1.30}$$

Looking at the definition of the parabolic R-polynomials we see that for $v, w \in W_+^{I,J}$, we have

$$R_{v,w}^{I,J} = (q^{\frac{-1}{2}})^{\ell(W_I v W_J)+\ell(W_I w W_J)} \sum_{u \in W_I v W_J} R_{u,w}(q). \tag{3.1.31}$$

Recalling that $\ell(W_I v W_J) = \ell(w_0^I) + \ell(w_0^J) - \ell(w_0^{K(v)})$ for $v \in W_+^{I,J}$, we see that (3.1.28) and (3.1.31) imply

$$\sum_{u \in W_I v W_J} R_{u,w}(q) = q^{\ell(W_I w W_J)} \sum_{u \in W_I v W_J} R_{u,w_-}(q), \tag{3.1.32}$$

which shows us that the double parabolic R-polynomials are in fact polynomials in $\mathbb{Z}[q]$, as claimed.

Next we would like to define double parabolic modified R-polynomials. Due to technical details we will define, for $v, w \in W_+^{I,J}$, the double parabolic R-polynomials in $\mathbb{N}[q_1, q_2]$ by

$$\widetilde{R}_{v,w}^{I,J}(q_1, q_2) = q_2^{(\ell(W_I v W_J)+\ell(W_I w W_J))} \sum_{u \in W_I v W_J} q_2^{\ell(u)-\ell(v)} \widetilde{R}_{u,w}(q_1), \tag{3.1.33}$$

rather than using the bar involution in a manner analogous to the definition of the double parabolic R-polynomials.

To see that this new definition is a good double parabolic generalization of the modified R-polynomials, recall (3.1.8). This implies we have

$$\begin{aligned}
\widetilde{R}_{v,w}^{I,J}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) &= (q^{\frac{-1}{2}})^{(\ell(W_I v W_J)+\ell(W_I w W_J))} \sum_{u \in W_I v W_J} q_{u,v} \widetilde{R}_{u,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) \\
&= (q^{\frac{-1}{2}})^{(\ell(W_I v W_J)+\ell(W_I w W_J))} \sum_{u \in W_I v W_J} q_{u,v} q_{u,w}^{-1} R_{u,w}(q) \\
&= q_{v,w}^{-1} (q^{\frac{-1}{2}})^{(\ell(W_I v W_J)+\ell(W_I w W_J))} \sum_{u \in W_I v W_J} R_{u,w}(q) \\
&= q_{v,w}^{-1} R_{v,w}^{I,J}(q).
\end{aligned} \tag{3.1.34}$$

Thus, similar to the nonparabolic case, the parabolic R-polynomials are related to the modified R-polynomials by

$$R_{v,w}^{I,J}(q) = q_{v,w} \widetilde{R}_{v,w}^{I,J}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}). \tag{3.1.35}$$

Therefore, this definition leads to the following result, which is analogous to the definition of the nonparabolic modified R-polynomials,

$$\overline{\widetilde{d}_{K(w)}\widetilde{T}'_{W_IwW_J}} = \sum_{\substack{v \in W_+^{I,J} \\ v \leq w}} \overline{\widetilde{R}^{I,J}_{v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{1}{2}})} \widetilde{d}_{K(v)}\widetilde{T}'_{W_IvW_J}. \tag{3.1.36}$$

3.2 The bar involution on $\mathcal{A}(n;q)$

Analogous to the bar involution on $H_n(q)$ define an involution on $\mathcal{A}(n;q)$ following Brundan [8], which we will also call the *bar involution*, by $\overline{q^{\frac{1}{2}}} = q^{\frac{-1}{2}}$, $\overline{x_{i,j}} = x_{i,j}$, and

$$\overline{x_{a_1,b_1} \cdots x_{a_r,b_r}} = (q^{\frac{1}{2}})^{\alpha(a)-\alpha(b)} x_{a_r,b_r} \cdots x_{a_1,b_1}, \tag{3.2.1}$$

where $\alpha(a)$ is the number of pairs $i < j$ for which $a_i = a_j$. We remark that in [8], Brundan uses q where we have been using $q^{\frac{-1}{2}}$.

Equivalently, if L and M are r-element multisets of $[n]$ and we define generator subsets $I = \iota(L)$, $J = \iota(M)$ of $W = \mathfrak{S}_r$ as before, then we have the definition

$$\overline{(x_{L,M})^{u,v}} \underset{\text{def}}{=} q_{w_0^J,w_0^I}(x_{L,M})^{w_0u,w_0v}. \tag{3.2.2}$$

In the immanant space this reduces to

$$\overline{x^{u,v}} = x^{w_0u,w_0v}. \tag{3.2.3}$$

At first it may seem odd and potentially confusing to have involutions on two different objects denoted in the same way; however, the next proposition, which appears in the literature, e.g. [9], will justify this choice of notation.

Proposition 3.2.1. *The two bar involutions are compatible with the left and right actions of $H_n(q)$ on $\mathcal{A}_{[n],[n]}(n;q)$ in the sense that*

$$\overline{\widetilde{T}_{s_i} \circ x^{e,v}} = \overline{\widetilde{T}_{s_i}} \circ \overline{x^{e,v}}, \qquad \overline{x^{e,v} \circ \widetilde{T}_{s_i}} = \overline{x^{e,v}} \circ \overline{\widetilde{T}_{s_i}}. \tag{3.2.4}$$

Proof. By the definitions we have

$$\overline{\widetilde{T}_{s_i} \circ x^{e,v}} = \overline{x^{s_i,v}} = x^{w_0s_i,w_0v}. \tag{3.2.5}$$

On the other hand, we have

$$
\begin{aligned}
\overline{\widetilde{T}_{s_i}} \circ \overline{x^{e,v}} &= \left(\widetilde{T}_{s_i} - (q^{\frac{1}{2}} - q^{\frac{-1}{2}})\widetilde{T}_e\right) \circ x^{w_0,w_0v} \\
&= \widetilde{T}_{s_i} \circ x^{w_0,w_0v} - (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{w_0,w_0v} \\
&= x^{w_0s_i,w_0v} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{w_0,w_0v} - (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{w_0,w_0v}
\end{aligned}
\tag{3.2.6}
$$

by Proposition 2.1.1. The proof of the second identity is similar. □

Following [31], we can state the following facts concerning the interplay between the two bar involutions. First, notice that (1.3.30) and (3.2.2) allow us to show that the element $(x_{L,M})^{e,w_0}$ is bar invariant. In other words we have

$$
\begin{aligned}
\overline{(x_{L,M})^{e,w_0}} &= q_{w_0^J,w_0^I}(x_{L,M})^{w_0,e} = q_{w_0^J,w_0^I}q^{-1}_{w_0^J,w_0^I}(x_{L,M})^{e,w_0} \\
&= (x_{L,M})^{e,w_0}.
\end{aligned}
\tag{3.2.7}
$$

Recalling (3.1.3) and Proposition 2.1.3, we can see we have

$$
(x_{L,[n]})^{e,w} \circ \overline{\widetilde{T}_{s_i}} = \begin{cases} (x_{L,[n]})^{e,ws_i} - (q^{\frac{1}{2}} - q^{\frac{-1}{2}})(x_{L,[n]})^{e,w} & \text{if } W_Iws_i > W_Iw, \\ q^{\frac{-1}{2}}(x_{L,[n]})^{e,w} & \text{if } W_Iws_i = W_Iw, \\ (x_{L,[n]})^{e,ws_i} & \text{if } W_Iws_i < W_Iw, \end{cases} \tag{3.2.8}
$$

$$
\overline{\widetilde{T}_{s_i}} \circ (x_{[n],M})^{e,w} = \begin{cases} (x_{[n],M})^{e,s_iw} - (q^{\frac{1}{2}} - q^{\frac{-1}{2}})(x_{[n],M})^{e,w} & \text{if } s_iwW_J > wW_J, \\ q^{\frac{-1}{2}}(x_{[n],M})^{e,w} & \text{if } s_IwW_J = wW_J, \\ (x_{[n],M})^{e,s_iw} & \text{if } s_IwW_J < wW_J, \end{cases} \tag{3.2.9}
$$

From the above formulas, one can see that for $w \in W_+^{I,\emptyset}$ we have

$$
(x_{L,[n]})^{e,w} \circ \overline{\widetilde{T}_u} = (x_{L,[n]})^{e,wu}, \qquad \text{if } \ell(wu) = \ell(w) - \ell(u) \text{ and } wu \in W_+^{I,\emptyset}, \tag{3.2.10}
$$

and for $w \in W_+^{\emptyset,J}$ we have

$$
\overline{\widetilde{T}_u} \circ (x_{[n],M})^{e,w} = (x_{[n],M})^{e,uw}, \qquad \text{if } \ell(uw) = \ell(w) - \ell(u) \text{ and } uw \in W_+^{\emptyset,J}. \tag{3.2.11}
$$

In particular, when $w = w_0$, we have

$$
\begin{aligned}
(x_{L,[n]})^{e,v} &= (x_{L,[n]})^{e,w_0} \circ \overline{\widetilde{T}_{w_0v}}, \qquad \text{for } v \in W_+^{I,\emptyset}, \\
(x_{[n],M})^{e,v} &= \overline{\widetilde{T}_{vw_0}} \circ (x_{[n],M})^{e,w_0}, \qquad \text{for } v \in W_+^{\emptyset,J}.
\end{aligned}
\tag{3.2.12}
$$

Furthermore, for $s_i \in I$ and $y \in W_I$, we have

$$(x_{L,[n]})^{e,w_0} \circ \overline{\widetilde{T}_{w_0 s_i w_0}} = q^{-\frac{1}{2}} (x_{L,[n]})^{e,w_0}, \qquad (x_{L,[n]})^{e,w_0} \circ \overline{\widetilde{T}_{w_0 y w_0}} = q_{e,t}^{-1} (x_{L,[n]})^{e,w_0}. \tag{3.2.13}$$

For $s_i \in J$ and $z \in W_J$, we have

$$\overline{\widetilde{T}_{w_0 s_i w_0}} \circ (x_{[n],M})^{e,w_0} = q^{-\frac{1}{2}} (x_{[n],M})^{e,w_0}, \qquad \overline{\widetilde{T}_{w_0 z w_0}} \circ (x_{M,[n]})^{e,w_0} = q_{e,t}^{-1} (x_{[n],M})^{e,w_0}. \tag{3.2.14}$$

It follows that if t factors as $t = yu$ with $u \in W_+^{I,\emptyset}$, $y \in W_I$, we have

$$\begin{aligned} (x_{L,[n]})^{e,w_0} \circ \overline{\widetilde{T}_{w_0 t}} &= (x_{L,[n]})^{e,w_0} \circ \overline{\widetilde{T}_{(w_0 y w_0)(w_0 u)}} \\ &= (x_{L,[n]})^{e,w_0} \circ \overline{\widetilde{T}_{w_0 y w_0} \widetilde{T}_{w_0 u}} \\ &= q_{e,t}^{-1} (x_{L,[n]})^{e,u}. \end{aligned} \tag{3.2.15}$$

Similarly, if t factors as $t = vz$ with $v \in W_+^{\emptyset,J}$, $z \in W_J$, we have

$$\begin{aligned} \overline{\widetilde{T}_{t w_0}} \circ (x_{[n],M})^{e,w_0} &= \overline{\widetilde{T}_{(v w_0)(w_0 z w_0)}} \circ (x_{[n],M})^{e,w_0} \\ &= \overline{\widetilde{T}_{v w_0} \widetilde{T}_{w_0 z w_0}} \circ (x_{[n],M})^{e,w_0} \\ &= q_{e,t}^{-1} (x_{[n],M})^{e,v}. \end{aligned} \tag{3.2.16}$$

Lemma 3.2.2. *For $u \in W_+^{I,\emptyset}$, $t = yu$, with $y \in W_I$, and $v \in W_+^{\emptyset,J}$, $t = zv$, with $z \in W_J$,*

$$\begin{aligned} \overline{(x_{L,[n]})^{e,u}} &= q_{e,y}^{-1} (x_{L,[n]})^{e,w_0} \circ \widetilde{T}_{w_0 t}, \\ \overline{(x_{[n],M})^{e,v}} &= q_{e,z}^{-1} \widetilde{T}_{t w_0} \circ (x_{[n],M})^{e,w_0}. \end{aligned} \tag{3.2.17}$$

Proof. We can write (3.2.15) as

$$(x_{L,[n]})^{e,u} = q_{e,t} (x_{L,[n]})^{e,w_0} \circ \overline{\widetilde{T}_{w_0 u}} \tag{3.2.18}$$

and then apply the bar involution. Recalling that $(x_{L,[n]})^{e,w_0}$ is bar invariant, we get the first formula. The second formula follows a similar argument using (3.2.16). □

3.3 The modified S-polynomials

Writing the basis $\{\overline{(x_{L,M})^{e,v}} \mid v \in W_+^{I,J}\}$ of $\mathcal{A}_{L,M}(n;q)$ in terms of the natural basis, we have

$$\overline{(x_{L,M})^{e,v}} = \sum_{w \in W_+^{I,J}} \overline{\epsilon_{v,w} q_{v,w}^{-1} S_{v,w}^{I,J}(q)} (x_{L,M})^{e,w}. \tag{3.3.1}$$

where $\{S_{v,w}^{I,J}(q) \mid v, w \in W_+^{I,J}\}$ are polynomials in $\mathbb{Z}[q]$ which we shall call *(parabolic) S-polynomials* (or *(parabolic) inverse R-polynomials*). For example, when $n = 3$, $I = \{s_1\}$, and $J = \{s_2\}$, we have

$$\begin{aligned} S_{s_1,s_1}^{I,\emptyset}(q) &= 1, \\ S_{s_1,s_1s_2}^{I,\emptyset}(q) &= q - 1, \\ S_{s_1,w_0}^{I,\emptyset}(q) &= 1 - q, \end{aligned} \tag{3.3.2}$$

and

$$\begin{aligned} S_{s_1s_2,s_1s_2}^{I,J}(q) &= 1, \\ S_{s_1s_2,w_0}^{I,J}(q) &= q - 1. \end{aligned} \tag{3.3.3}$$

As with the parabolic R-polynomials, these examples support the claim that the S-polynomials are in $\mathbb{Z}[q]$. Once again we will see the validity of this claim after establishing a few summation results.

Looking at (3.2.2) and recalling the result of Theorem 1.4.12, we see that

$$\begin{aligned} \overline{(x_{L,M})^{e,v}} &= q_{w_0^J,w_0^I} (x_{L,M})^{w_0,w_0v} \\ &= q_{w_0^J,w_0^I} \sum_{w \in W_+^{I,J}} p_{w_0,w_0v,w}^{I,J}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) (x_{L,M})^{e,w}. \end{aligned} \tag{3.3.4}$$

Therefore we have that the S-polynomials are just special cases of the family of polynomials introduced in Chapter 1. If we define *(parabolic) modified S-polynomials* (or *(parabolic) inverse modified R-polynomials*) by

$$\widetilde{S}_{v,w}^{I,J}(q_1, q_2) = q_2^{\ell(w_0^J) - \ell(w_0^I)} p_{w_0,w_0v,w}^{I,J}(q_1, q_2), \tag{3.3.5}$$

then we have

$$\overline{(x_{L,M})^{e,v}} = \sum_{w \in W_+^{I,J}} \widetilde{S}_{v,w}^{I,J}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) (x_{L,M})^{e,w}, \tag{3.3.6}$$

which is analogous to the definition of the modified R-polynomials, just as the S-polynomials are analogous to the R-polynomials. Notice that we have that

$$\widetilde{S}^{I,J}_{v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) = \epsilon_{v,w} q_{v,w} S^{I,J}_{v,w}(q^{-1}) \tag{3.3.7}$$

and by (1.4.4)

$$\widetilde{S}^{I,J}_{v,w}(q_1, q_2) = r^{I,J}_{w_0, w_0 v, w}(q_1, q_2). \tag{3.3.8}$$

Looking at the definition of $p^{I,J}_{w_0,w_0v,w}(q_1,q_2)$, we see that we have for $v, w \in W^{I,J}_+$, since $w_0 v \in W^{I,J}_-$ and $(w_0v)^J = e$ and $(w_0)^I = w_0^I$, given any reduced expression $s_{i_1} \cdots s_{i_k}$ for w_0, the polynomial $\widetilde{S}^{I,J}_{v,w}(q_1,q_2) \in \mathbb{N}[q_1,q_2]$ is the polynomial whose coefficient of $q_1^a q_2^b$ is equal to the number of sequences $(\pi^{(0)}, \ldots, \pi^{(k)})$ of permutations satisfying

1. $\pi^{(0)} = w_0 v, \pi^{(k)} \in W_I w W_J$,

2. $\pi^{(j)} \in \{\pi^{(j-1)}, s_{i_j}\pi^{(j-1)}\}$ for $j = 1, \ldots, k$,

3. $\pi^{(j)} = s_{i_j}\pi^{(j-1)}$ if $s_{i_j}\pi^{(j-1)} > \pi^{(j-1)}$,

4. $\pi^{(j)} = \pi^{(j-1)}$ for exactly a indices j,

5. $\ell(w) - \ell(\pi^{(k)}) = b$.

Alternatively, by (3.3.8), we have for $v, w \in W^{I,J}_+$, given any reduced expression $s_{i_1} \cdots s_{i_k}$ for $w_0 v$, the polynomial $\widetilde{S}^{I,J}_{v,w}(q_1,q_2) \in \mathbb{N}[q_1,q_2]$ is the polynomial whose coefficient of $q_1^a q_2^b$ is equal to the number of sequences $(\pi^{(0)}, \ldots, \pi^{(k)})$ of permutations satisfying

1. $\pi^{(0)} = w_0, (\pi^{(k)})^{-1} \in W_I w W_J$,

2. $\pi^{(j)} \in \{\pi^{(j-1)}, s_{i_j}\pi^{(j-1)}\}$ for $j = 1, \ldots, k$,

3. $\pi^{(j)} = s_{i_j}\pi^{(j-1)}$ if $s_{i_j}\pi^{(j-1)} > \pi^{(j-1)}$,

4. $\pi^{(j)} = \pi^{(j-1)}$ for exactly a indices j,

5. $\ell(w) - \ell(\pi^{(k)}) = b$.

Just as before we will use the notation $\widetilde{S}_{v,w}(q_1) = \widetilde{S}^{\emptyset,\emptyset}_{v,w}(q_1, q_2)$ for the special case when $I = J = \emptyset$. In the immanant space we can use Proposition 3.2.1 to find recursive formulas for the modified S-polynomials. Using (2.1.6) to write $\overline{x^{e,v}} = \overline{x^{e,vs}} \circ \overline{\widetilde{T}_s^{-1}} = \overline{x^{e,vs}} \circ \widetilde{T}_s$ for $vs > v$, and using induction, we see that the modified S-polynomials are the unique family $\{\widetilde{S}_{v,w}(q_1) \mid v, w \in \mathfrak{S}_r\}$ of polynomials in $\mathbb{N}[q_1]$ satisfying

1. $\widetilde{S}_{v,w}(q_1) = 0$ if $w \not\geq v$.
2. $\widetilde{S}_{v,v}(q_1) = 1$ for all v.
3. For each right ascent s of v we have

$$\widetilde{S}_{v,w}(q_1) = \begin{cases} \widetilde{S}_{vs,ws}(q_1) & \text{if } ws > w, \\ \widetilde{S}_{vs,ws}(q_1) + q_1 \widetilde{S}_{vs,w}(q_1) & \text{otherwise.} \end{cases} \tag{3.3.9}$$

Stating condition (3) in terms of *left* ascents s of v, this is

$$\widetilde{S}_{v,w}(q_1) = \begin{cases} \widetilde{S}_{sv,sw}(q_1) & \text{if } sw > w, \\ \widetilde{S}_{sv,sw}(q_1) + q_1 \widetilde{S}_{sv,w}(q_1) & \text{otherwise.} \end{cases} \tag{3.3.10}$$

On the other hand, we may fix a right descent s of w to obtain

$$\widetilde{S}_{v,w}(q_1) = \begin{cases} \widetilde{S}_{vs,ws}(q_1) & \text{if } vs > v, \\ \widetilde{S}_{vs,ws}(q_1) + q_1 \widetilde{S}_{v,ws}(q_1) & \text{otherwise,} \end{cases} \tag{3.3.11}$$

or we may fix a left descent s of w to obtain

$$\widetilde{S}_{v,w}(q_1) = \begin{cases} \widetilde{S}_{sv,sw}(q_1) & \text{if } sv > v, \\ \widetilde{S}_{sv,sw}(q_1) + q_1 \widetilde{S}_{v,sw}(q_1) & \text{otherwise.} \end{cases} \tag{3.3.12}$$

The various recurrence formulas above show modified S-polynomials and modified R-polynomials are equal.

Proposition 3.3.1. *For all $u, v \in \mathfrak{S}_r$,*

$$\widetilde{S}_{v,w}(q_1) = \widetilde{R}_{v,w}(q_1). \tag{3.3.13}$$

Proof. Comparing the recursive formulas and initial conditions for the modified S- and R-polynomials, we see that

$$\widetilde{S}_{v,w}(q_1) = \widetilde{R}_{w_0w,w_0v}(q_1). \tag{3.3.14}$$

Furthermore, by (3.1.15) we have

$$\widetilde{S}_{v,w}(q_1) = \widetilde{R}_{v,w}(q_1), \tag{3.3.15}$$

as claimed. □

As a consequence of the above proposition we have a new combinatorial interpretation for the modified R-polynomials, which is reminiscent of Dyer's result in [17].

Corollary 3.3.2. *The coefficient of q_1^a in the polynomial $\widetilde{R}_{u,v}(q_1)$ is equal to the number of sequences $(\pi^{(0)}, \dots, \pi^{(k)})$ of permutations satisfying*

1. $\pi^{(0)} = w_0v, \pi^{(k)} \in W_IwW_J$,
2. $\pi^{(j)} \in \{\pi^{(j-1)}, s_{i_j}\pi^{(j-1)}\}$ *for* $j = 1, \dots, k$,
3. $\pi^{(j)} = s_{i_j}\pi^{(j-1)}$ *if* $s_{i_j}\pi^{(j-1)} > \pi^{(j-1)}$,
4. $\pi^{(j)} = \pi^{(j-1)}$ *for exactly a indices j,*

where $s_{i_1} \cdots s_{i_k}$ is any reduced expression for w_0.

Another consequence is that

$$S_{v,w}(q) = R_{v,w}(q). \tag{3.3.16}$$

Furthermore by (3.1.16) and (3.1.15) we have

$$S_{u,v}(q) = S_{w_0v,w_0u}(q) = S_{vw_0,uw_0}(q) = S_{u^{-1},v^{-1}}(q) \tag{3.3.17}$$

and

$$\widetilde{S}_{u,v}(q) = \widetilde{S}_{w_0v,w_0u}(q) = \widetilde{S}_{vw_0,uw_0}(q) = \widetilde{S}_{u^{-1},v^{-1}}(q). \tag{3.3.18}$$

The equations (3.1.17) and (3.3.13) imply

$$\overline{\widetilde{S}_{v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}})} = \epsilon_{v,w}\widetilde{S}_{v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}). \tag{3.3.19}$$

Similarly, (3.1.18) and (3.3.16), we see that

$$\overline{S_{u,v}(q)} = S_{u,v}(q^{-1}) = \epsilon_{u,v}q_{u,v}^{-2}S_{u,v}(q). \tag{3.3.20}$$

Since by definition the polynomial $\widetilde{S}_{v,w}(q) = r_{w_0,w_0v,w}(q) = p_{w_0,w_0v,w}$, we can use (3.1.15) to state more symmetries for the polynomials $\{p_{w_0,v,w}(q) \mid v, w \in \mathfrak{S}_r\}$,

$$p_{w_0,w_0v,w}(q) = p_{w_0,w,w_0v}(q) = p_{w_0,w_0ww_0,vw_0}(q) = p_{w_0,w_0v^{-1},w^{-1}}(q), \tag{3.3.21}$$

which we had not previously found by other methods.

Next we establish some summation results, connecting the parabolic and non-parabolic polynomials. By (1.4.15) and (1.4.24) we have

$$\widetilde{S}^{I,J}_{v,w}(q_1,q_2) = \sum_{z\in W_IwW_J} q_2^{\ell(w)-\ell(z)}\widetilde{S}_{v,z}(q_1), \tag{3.3.22}$$

and

$$\begin{aligned}\widetilde{S}^{I,J}_{v,w}(q_1,q_2) &= \sum_{\substack{y\in W_+^{I,\emptyset}\\ y\in w(W_J)_-^{K,\emptyset}}} q_2^{\ell(w)-\ell(y)}\widetilde{S}^{I,\emptyset}_{v,y}(q_1,q_2)\\ &= \sum_{\substack{z\in W_+^{\emptyset,J}\\ z\in (W_I)_-^{\emptyset,K'}w}} q_2^{\ell(w)-\ell(z)}\widetilde{S}^{\emptyset,J}_{v,z}(q_1,q_2).\end{aligned} \tag{3.3.23}$$

We can use the definition of the modified S-polynomials and (2.2.13) to give us an alternate summation formula, which holds in the ring $\mathbb{Z}[q_1,q_2]/(1-q_1q_2-q_2^2)$.

Lemma 3.3.3. *In the ring $\mathbb{Z}[q_1,q_2]/(1-q_1q_2-q_2^2)$, for $v \in W_+^{I,\emptyset}$ such that $v \in uW_J$, we have*

$$\widetilde{S}^{I,J}_{u,w}(q_1,q_2) = q_2^{\ell(u)-\ell(v)} \sum_{\substack{y\in W_+^{I,\emptyset}\\ y\in w(W_J)_-^{K,\emptyset}}} q_2^{\ell(w)-\ell(y)}\widetilde{S}^{I,\emptyset}_{v,y}(q_1,q_2). \tag{3.3.24}$$

Proof. Since $v \in uW_J$, we have that $w_0v \in w_0uW_J$, and in $\mathbb{Z}[q_1, q_2]/(1 - q_1q_2 - q_2^2)$ (2.2.13) and (1.4.24) tell us

$$\begin{aligned}\widetilde{S}^{I,J}_{u,w}(q_1, q_2) &= r^{I,J}_{w_0,w_0u,w}(q_1, q_2) = r^{I,J}_{w_0,w_0v,w}(q_1, q_2) \\ &= \sum_{\substack{y \in W^{I,\emptyset}_+ \\ y \in w(W_J)^{K,\emptyset}_-}} q_2^{\ell(w)-\ell(y)+\ell((w_0v)^J)} r^{I,\emptyset}_{w_0,w_0v,y}(q_1, q_2).\end{aligned} \tag{3.3.25}$$

Recall that left multiplication by w_0 is an antiautomorphism of the Bruhat order. This implies that it maps the coset uW_J to the coset w_0uW_J, in such a way that if $v < u$ then $w_0u < w_0v$. Thus the length of $(w_0v)^J$, the distance between w_0v and the minimal element in the coset w_0uW_J, is equal to the distance between v and the maximal element in the coset uW_J, or u itself. This distance is just $\ell((w_0v)^J) = \ell(u) - \ell(v)$. Substituting this into the above equation gives us formula as claimed. □

It will be useful to generalize (3.3.22) to linear combinations of the polynomials $\{\widetilde{S}_{t,z}(q_1) \mid t \in W_IvW_J, z \in W_IwW_J\}$. A partial result is stated in [31], when we restrict ourselves to the case where $q_1 = q^{\frac{1}{2}} - q^{\frac{-1}{2}}$ and $q_2 = q^{\frac{-1}{2}}$.

Lemma 3.3.4. *For $u, w \in W^{I,\emptyset}_+$ and $t \in W_Iu$, we have*

$$\widetilde{S}^{I,\emptyset}_{u,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) = q^{-1}_{t,u} \sum_{v \in W_Iw} q^{-1}_{v,w} \widetilde{S}_{t,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}). \tag{3.3.26}$$

Similarly, for $u, w \in W^{\emptyset,J}_+$ and $t \in uW_J$, we have

$$\widetilde{S}^{\emptyset,J}_{u,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) = q^{-1}_{t,u} \sum_{v \in wW_J} q^{-1}_{v,w} \widetilde{S}_{t,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}). \tag{3.3.27}$$

Proof. Expanding (3.2.17) in terms of the modified S- and R-polynomials using (3.1.20) gives us

$$\begin{aligned}\sum_{\substack{w \in W^{I,\emptyset}_+ \\ u \leq w}} \widetilde{S}^{I,\emptyset}_{u,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}})(x_{L,[r]})^{e,w} &= q^{-1}_{e,y}(x_{L,[r]})^{e,w_0} \circ \sum_{w_0v \leq w_0t} \widetilde{R}_{w_0v,w_0t}(q^{\frac{1}{2}} - q^{\frac{-1}{2}})\overline{\widetilde{T}_{w_0v}} \\ &= q^{-1}_{t,u} \sum_{w \in W^{I,\emptyset}_+} \sum_{\substack{v \in W_Iw \\ v = zw}} \widetilde{S}_{t,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}})(x_{L,[r]})^{e,w_0} \circ \overline{\widetilde{T}_{w_0zw_0}\widetilde{T}_{w_0w}}.\end{aligned} \tag{3.3.28}$$

Next we apply (3.2.13) and (3.2.12) to the right-hand side to get

$$\sum_{\substack{w\in W_+^{I,\emptyset}\\ u\leq w}} \widetilde{S}^{I,\emptyset}_{u,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}})(x_{L,[r]})^{e,w} = q_{t,u}^{-1} \sum_{w\in W_+^{I,\emptyset}} \sum_{\substack{v\in W_I w\\ v=zw}} \widetilde{S}_{t,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) q_{e,z}^{-1}(x_{L,[r]})^{e,w}$$
$$= q_{t,u}^{-1} \sum_{w\in W_+^{I,\emptyset}} \sum_{v\in W_I w} q_{v,w}^{-1}\widetilde{S}_{t,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}})(x_{L,[r]})^{e,w}. \tag{3.3.29}$$

Comparing coefficients of $(x_{L,[r]})^{e,w}$ in the equation above, gives us the first result. The second result can be found in a similar manner using (3.2.17), (3.1.20), (3.2.14), and (3.2.12). □

Corollary 3.3.5. *For $u, w \in W_+^{I,J}$ and $t \in W_I u W_J$,*

$$\widetilde{S}^{I,J}_{u,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) = q_{t,u}^{-1} \sum_{v\in W_I w W_J} q_{v,w}^{-1}\widetilde{S}_{t,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}). \tag{3.3.30}$$

Proof. Fix $u, w \in W_+^{I,J}$ and $t \in W_I u W_J$. Then there exists $y \in W_+^{I,\emptyset}$, such that $y \in u(W_J)_-^{K,\emptyset}$ and $t \in W_I y$. Lemmas 3.3.3 and 3.3.4 imply

$$\widetilde{S}^{I,J}_{u,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) = q_{y,u}^{-1} \sum_{\substack{z\in W_+^{I,\emptyset}\\ z\in w(W_J)_-^{K,\emptyset}}} q_{z,w}^{-1}\widetilde{S}^{I,\emptyset}_{y,z}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}})$$
$$= q_{y,u}^{-1} \sum_{\substack{z\in W_+^{I,\emptyset}\\ z\in w(W_J)_-^{K,\emptyset}}} q_{z,w}^{-1} q_{t,y}^{-1} \sum_{v\in W_I z} q_{v,z}^{-1}\widetilde{S}_{t,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) \tag{3.3.31}$$
$$= q_{t,u}^{-1} \sum_{v\in W_I w W_J} q_{v,w}^{-1}\widetilde{S}_{t,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}).$$

□

The relationship between the S- and modified S-polynomials and this corollary imply that for $u, w \in W_+^{I,J}$ and $t \in W_I u W_J$,

$$\epsilon_{u,w} S^{I,J}_{u,w}(q) = \sum_{v\in W_I w W_J} q_{v,w}^2 \epsilon_{t,v} S_{t,v}(q), \tag{3.3.32}$$

In particular if $t = u$,

$$S^{I,J}_{u,w}(q) = \sum_{v\in W_I w W_J} \epsilon_{v,w} q_{v,w}^2 S_{u,v}(q). \tag{3.3.33}$$

This expression finally justifies the claim that the parabolic S-polynomials are in $\mathbb{Z}[q]$.

A generalization of (3.1.21) for the parabolic inverse R-polynomials is given below.

Proposition 3.3.6. *For* $u, w \in W_+^{I,J}$,

$$\sum_{\substack{v \in W_+^{I,J} \\ u \geq v \geq w}} q_{u,v}^{-1} S_{u,v}^{I,J}(q) q_{v,w} S_{v,w}^{I,J}(q^{-1}) = \delta_{u,w}. \tag{3.3.34}$$

Proof. Fix $u \in W_+^{I,J}$, then applying the bar involution to (3.3.1), we have

$$\begin{aligned}
(x_{L,M})^{e,u} &= \sum_{v \in W_+^{I,J}} \epsilon_{u,v} q_{u,v}^{-1} S_{u,v}^{I,J}(q) \overline{(x_{L,M})^{e,v}} \\
&= \sum_{v \in W_+^{I,J}} \epsilon_{u,v} q_{u,v}^{-1} S_{u,v}^{I,J}(q) \sum_{w \in W_+^{I,J}} \overline{\epsilon_{v,w} q_{v,w}^{-1} S_{v,w}^{I,J}(q)} (x_{L,M})^{e,w} \\
&= \sum_{v \in W_+^{I,J}} \sum_{w \in W_+^{I,J}} \epsilon_{u,w} q_{u,v}^{-1} S_{u,v}^{I,J}(q) q_{v,w} S_{v,w}^{I,J}(q^{-1}) (x_{L,M})^{e,w}.
\end{aligned} \tag{3.3.35}$$

Comparing terms, we see that (3.3.34) holds as claimed. □

We mentioned that the definition of the double parabolic R-polynomials is consistent with the literature, since in the single parabolic case they equaled the R-polynomials defined by Deodhar. Brenti in [7] mentions that [12] Deodhar defines two families of R-polynomials, depending on the parameter $x \in \{-1, q\}$. The R-polynomials we have defined are equal to Deodhar's when $x = -1$. It turns out that the single parabolic S-polynomials are equal to Deodhar's family of polynomials when $x = q$. In order to see this we recall Lemma 2.1.4 and recognize by (3.3.8) we have for $vs_i > v$,

$$\widetilde{S}_{v,w}^{I,\emptyset}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) = \begin{cases} \widetilde{S}_{vs_i,ws_i}^{I,\emptyset}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \text{if } W_I ws_i > W_I w, \\ q^{\frac{1}{2}} \widetilde{S}_{vs_i,w}^{I,\emptyset}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \text{if } W_I ws_i = W_I w, \\ \widetilde{S}_{vs_i,ws_i}^{I,\emptyset}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \\ \quad + (q^{\frac{1}{2}} - q^{\frac{-1}{2}}) \widetilde{S}_{vs_i,w}^{I,\emptyset}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}, q^{\frac{-1}{2}}) & \text{if } W_I ws_i < W_I w. \end{cases} \tag{3.3.36}$$

Then (3.3.7) tells us

$$S^{I,\emptyset}_{v,w}(q) = \begin{cases} S^{I,\emptyset}_{vs_i,ws_i}(q) & \text{if } W_Iws_i > W_Iw, \\ -S^{I,\emptyset}_{vs_i,w}(q) & \text{if } W_Iws_i = W_Iw, \\ qS^{I,\emptyset}_{vs_i,ws_i}(q) + (q-1)S^{I,\emptyset}_{vs_i,w}(q) & \text{if } W_Iws_i < W_Iw. \end{cases} \tag{3.3.37}$$

If we define $\hat{I} = w_0Iw_0$, then we see that the polynomials $\{S^{\hat{I},\emptyset}_{w_0w_-,w_0v_-}(q) \mid v,w \in W^{I,\emptyset}_+\}$, for $w_0v_-s_i > w_0v_-$, satisfy

$$S^{\hat{I},\emptyset}_{w_0w_-,w_0v_-}(q) = \begin{cases} S^{\hat{I},\emptyset}_{w_0w_-s_i,w_0v_-s_i}(q) & \text{if } W_{\hat{I}}w_0w_-s_i > W_{\hat{I}}w_0w_-, \\ -S^{\hat{I},\emptyset}_{w_0w_-,w_0v_-s_i}(q) & \text{if } W_{\hat{I}}w_0w_-s_i = W_{\hat{I}}w_0w_-, \\ qS^{\hat{I},\emptyset}_{w_0w_-s_i,w_0v_-s_i}(q) & \\ \quad +(q-1)S^{\hat{I},\emptyset}_{w_0w_-,w_0v_-s_i}(q) & \text{if } W_{\hat{I}}w_0w_-s_i < W_{\hat{I}}w_0w_-, \end{cases} \tag{3.3.38}$$

which is the same recursive formula Brenti asserts Deodhar's single parabolic R-polynomials satisfy in [7]. Furthermore, he states there is a unique family of polynomials satisfying this recursion (and some initial conditions which can easily be seen are also satisfied). Since the polynomials $\{S^{\hat{I},\emptyset}_{w_0w_-,w_0v_-}(q) \mid v,w \in W^{I,\emptyset}_+\}$ are Deodhar's polynomials we can express them as a sum using another result in [7],

$$S^{\hat{I},\emptyset}_{w_0w_-,w_0v_-}(q) = \sum_{w_0u \in W_{\hat{I}w_0w_-}} \epsilon_{w_0u,w_0w} q^{-2}_{w_0u,w_0w} S_{w_0u,w_0v}(q). \tag{3.3.39}$$

Using (3.3.17) and (3.3.33) we have

$$\begin{aligned} S^{\hat{I},\emptyset}_{w_0w_-,w_0v_-}(q) &= \sum_{w_0u \in W_{\hat{I}w_0w_-}} \epsilon_{w_0u,w_0w} q^{-2}_{w_0u,w_0w} S_{v,u}(q) \\ &= \sum_{u \in W_Iw} \epsilon_{u,w} q^2_{u,w} S_{v,u}(q) \\ &= S^{I,\emptyset}_{v,w}(q). \end{aligned} \tag{3.3.40}$$

Thus we conclude the polynomials $\{S^{I,\emptyset}_{v,w}(q) \mid v, wW^{I,\emptyset}_+\}$ are equal to Deodhar's family of polynomials for $x = q$. Brenti relates the two families in [7], which leads us to conclude that

$$\overline{S^{I,\emptyset}_{v,w}(q)} = \epsilon_{v,w} q^{-2}_{v,w} R^{I,\emptyset}_{v,w}(q). \tag{3.3.41}$$

In terms of modified polynomials this relationship is

$$\widetilde{S}^{I,\emptyset}_{v,w}(q) = \widetilde{R}^{I,\emptyset}_{v,w}(q). \tag{3.3.42}$$

Problem 4. *Is there a similar relationship for the double parabolic R- and S-polynomials.*

3.4 Bar invariant bases

There exists a unique basis for $\mathcal{A}(n;q)$ which is invariant under the bar involution, called the *dual canonical basis*. For the immanant space, this basis is somewhat well understood and can be described using *inverse Kazhdan-Lusztig polynomials*. Similarly, we can define the dual canonical basis for an arbitrary multi-graded component $\mathcal{A}_{L,M}(n;q)$ using *parabolic inverse Kazhdan-Lusztig polynomials*. However, this formulation is somewhat cumbersome and not instructive. In this section we first prove the existence of the bar-invariant basis of $\mathcal{A}_{L,M}(n;q)$, by defining parabolic inverse Kazhdan-Lusztig polynomials. Finally, we show the dual canonical basis elements can be described using the immanant space and generalized submatrices, producing a new formulation for the dual canoncial basis.

Theorem 3.4.1. *For any $v \in W^{I,J}_+$ there exists a unique bar-invariant element $\mathrm{Imm}^{L,M}_v(x) \in \mathcal{A}_{L,M}(n;q)$ such that*

$$\mathrm{Imm}^{L,M}_v(x) = \sum_{\substack{w \in W^{I,J}_+ \\ w \geq v}} \epsilon_{v,w} q^{-1}_{v,w} Q^{I,J}_{v,w}(q)(x_{L,M})^{e,w}, \tag{3.4.1}$$

where $Q^{I,J}_{v,w}(q)$ are polynomials in $\mathbb{Z}[q]$ of degree at most $\frac{1}{2}(\ell(w) - \ell(v) - 1)$ if $v < w$ and $Q^{I,J}_{v,v}(q) = 1$.

Proof. To prove the uniqueness we can rewrite the condition

$$\mathrm{Imm}^{L,M}_u(x) = \overline{\mathrm{Imm}^{L,M}_u(x)} \tag{3.4.2}$$

as

$$q_{u,w}Q^{I,J}_{u,w}(q^{-1}) - q^{-1}_{u,w}Q^{I,J}_{u,w}(q) = \sum_{\substack{v \in W^{I,J}_+ \\ u \leq v < w}} q^{-1}_{u,v}Q^{I,J}_{u,v}(q)q^{-1}_{v,w}S^{I,J}_{v,w}(q) \tag{3.4.3}$$

for all $w > u$. In particular, there is a unique solution $Q^{I,J}_{u,w}(q)$ to the above equation when all other polynomials which appear are known, if we constrain this solution to satisfy the degree condition.

The existence comes from observing that the substitution

$$Q^{I,J}_{v,w}(q) = \sum_{z \in W_I w W_J} \epsilon_{w,z} Q_{v,z}(q), \tag{3.4.4}$$

where the polynomials $Q_{v,z}(q)$ are the *inverse Kazhdan-Lusztig polynomials* related to the Kazhdan-Lusztig polynomials by

$$Q_{v,w}(q) = P_{w_0 w, w_0 v}(q) = P_{w w_0, v w_0}(q), \tag{3.4.5}$$

satisfies (3.4.2). In order to see that this substitution satisfies (3.4.2), we first notice that equation (2.2.a) in [22] can be rewritten as

$$P_{u,w}(q) = \sum_{u \leq v \leq w} \epsilon_{u,v} q^2_{v,w} R_{u,v}(q) \overline{P_{v,w}(q)} \tag{3.4.6}$$

for $u \leq w$ in $W = \mathfrak{S}_r$. Now recalling (3.3.16) and (3.3.17), we have $S_{v,w}(q) = R_{w_0 w, w_0 v}(q)$. Thus (3.4.5) tells us that we have

$$Q_{w_0 w, w_0 u}(q) = \sum_{u \leq v \leq w} \epsilon_{u,v} q^2_{v,w} S_{w_0 v, w_0 u}(q) \overline{Q_{w_0 w, w_0 v}(q)}. \tag{3.4.7}$$

Reindexing the above equation and applying the bar involution gives us

$$\begin{aligned} \overline{Q_{u,w}(q)} &= \sum_{u \leq v \leq w} \epsilon_{u,v} q^{-2}_{v,w} \overline{S_{v,w}(q)} Q_{u,v}(q) \\ &= \sum_{u \leq v \leq w} \epsilon_{u,v} q^{-2}_{v,w} \epsilon_{u,v} q^{-2}_{u,v} S_{v,w}(q) Q_{u,v}(q) \\ &= q^{-2}_{u,w} \sum_{u \leq v \leq w} Q_{u,v}(q) S_{v,w}(q), \end{aligned} \tag{3.4.8}$$

where the second line comes from applying (3.3.20). Finally, we can use this to see that the expression

$$\begin{aligned}
F_v &= \sum_{\substack{w \in W_+^{I,J} \\ w \geq v}} \epsilon_{v,w} q_{v,w}^{-1} \sum_{z \in W_I w W_J} \epsilon_{w,z} Q_{v,z}(q) (x_{L,M})^{e,w} \\
&= \sum_{\substack{w \in W_+^{I,J} \\ w \geq v}} \sum_{z \in W_I w W_J} \epsilon_{v,z} q_{v,w}^{-1} Q_{v,z}(q) (x_{L,M})^{e,w}
\end{aligned} \tag{3.4.9}$$

is bar-invariant. Applying the bar involution to F_v gives

$$\begin{aligned}
\overline{F_v} &= \overline{\sum_{\substack{w \in W_+^{I,J} \\ w \geq v}} \epsilon_{v,w} q_{v,w}^{-1} \sum_{z \in W_I w W_J} \epsilon_{w,z} Q_{v,z}(q) (x_{L,M})^{e,w}} \\
&= \overline{\sum_{\substack{w \in W_+^{I,J} \\ w \geq v}} \sum_{z \in W_I w W_J} \epsilon_{v,z} q_{v,w}^{-1} Q_{v,z}(q) (x_{L,M})^{e,w}} \\
&= \sum_{\substack{w \in W_+^{I,J} \\ w \geq v}} \sum_{z \in W_I w W_J} \epsilon_{v,z} q_{v,w} Q_{v,z}(q^{-1}) \overline{(x_{L,M})^{e,w}} \\
&= \sum_{\substack{w \in W_+^{I,J} \\ w \geq v}} \sum_{z \in W_I w W_J} \epsilon_{v,z} q_{v,w} Q_{v,z}(q^{-1}) \sum_{y \in W_+^{I,J}} \epsilon_{w,y} q_{w,y} S_{w,y}^{I,J}(q^{-1}) (x_{L,M})^{e,y} \\
&= \sum_{\substack{w \in W_+^{I,J} \\ w \geq v}} \sum_{y \in W_+^{I,J}} \sum_{z \in W_I w W_J} \epsilon_{v,z} q_{v,y} Q_{v,z}(q^{-1}) \epsilon_{w,y} S_{w,y}^{I,J}(q^{-1}) (x_{L,M})^{e,y}.
\end{aligned} \tag{3.4.10}$$

Since $z \in W_I w W_J$, we can use (3.3.32) to expand the parabolic S-polynomial as a sum over appropriate nonparabolic S-polynomials and then apply (3.4.8) to get

$$\begin{aligned}\overline{F_v} &= \sum_{\substack{w\in W_+^{I,J}\\ w\geq v}} \sum_{y\in W_+^{I,J}} \sum_{z\in W_I w W_J} \epsilon_{v,z} q_{v,y} Q_{v,z}(q^{-1}) \sum_{u\in W_I y W_J} q_{u,y}^{-2} \epsilon_{z,u} S_{z,u}(q^{-1})(x_{L,M})^{e,y} \\ &= \sum_{y\in W_+^{I,J}} \sum_{u\in W_I y W_J} \epsilon_{v,u} q_{v,u} q_{u,y}^{-1} \sum_{v\leq z\leq u} S_{z,u}(q^{-1}) Q_{v,z}(q^{-1})(x_{L,M})^{e,y} \\ &= \sum_{y\in W_+^{I,J}} \sum_{u\in W_I y W_J} \epsilon_{v,u} q_{v,u} q_{u,y}^{-1} q_{v,u}^{-2} Q_{v,u}(q)(x_{L,M})^{e,y} \\ &= \sum_{y\in W_+^{I,J}} \sum_{u\in W_I y W_J} \epsilon_{v,u} q_{v,y}^{-1} Q_{v,u}(q)(x_{L,M})^{e,y}.\end{aligned} \tag{3.4.11}$$

Thus the expression is bar-invariant as claimed.

The fact that the sum of inverse Kazhdan-Lusztig polynomials satisfies the degree equation comes from the fact that the Kazhdan-Lusztig polynomials satisfy a similar degree equation (see [22]). □

Once again, we will use the notation

$$\mathrm{Imm}_v(x) = \mathrm{Imm}_v^{[r],[r]}(x). \tag{3.4.12}$$

For example, when $r = 3$, we have

$$\begin{aligned}\mathrm{Imm}_{w_0}(x) &= x^{e,w_0}, \\ \mathrm{Imm}_{s_1 s_2}(x) &= x^{e,s_1 s_2} - q^{\frac{-1}{2}} x^{e,w_0}, \\ \mathrm{Imm}_{s_2 s_1}(x) &= x^{e,s_2 s_1} - q^{\frac{-1}{2}} x^{e,w_0}, \\ \mathrm{Imm}_{s_1}(x) &= x^{e,s_1} - q^{\frac{-1}{2}}(x^{e,s_1 s_2} + x^{e,s_2 s_1}) + q^{-1} x^{e,w_0}, \\ \mathrm{Imm}_{s_2}(x) &= x^{e,s_2} - q^{\frac{-1}{2}}(x^{e,s_1 s_2} + x^{e,s_2 s_1}) + q^{-1} x^{e,w_0}, \\ \mathrm{Imm}_{e}(x) &= x^{e,e} - q^{\frac{-1}{2}}(x^{e,s_1} + x^{e,s_2}) + q^{-1}(x^{e,s_1 s_2} + x^{e,s_2 s_1}) - q^{\frac{-3}{2}} x^{e,w_0}.\end{aligned} \tag{3.4.13}$$

Similarly, we have

$$\begin{aligned}\mathrm{Imm}_{s_1 s_2}^{112,122}(x) &= (x_{112,122})^{e,s_1 s_2} - q^{\frac{-1}{2}}(x_{112,122})^{e,w_0}, \\ \mathrm{Imm}_{w_0}^{112,122}(x) &= (x_{112,122})^{e,w_0}.\end{aligned} \tag{3.4.14}$$

An interesting symmetry results as a consequence of the previous theorem.

Lemma 3.4.2. *For $v, w \in W_+^{I,J}$,*

$$Q^{J,I}_{v^{-1},w^{-1}}(q) = Q^{I,J}_{v,w}(q). \tag{3.4.15}$$

Proof. First notice that the left-hand side is defined since we have $v^{-1}, w^{-1} \in W_+^{J,I}$. Next taking advantage of (3.4.4), we can say

$$Q^{J,I}_{v^{-1},w^{-1}}(q) = \sum_{z^{-1}\in W_J w^{-1} W_I} \epsilon_{w^{-1},z^{-1}} Q_{v^{-1},z^{-1}}(q) = \sum_{z\in W_I w W_J} \epsilon_{w,z} Q_{v,z}(q) = Q^{I,J}_{v,w}(q). \tag{3.4.16}$$

□

Another interesting consequence follows.

Lemma 3.4.3. *For $v, w \in W_+^{I,J}$*

$$Q^{I,J}_{v,w}(q) = \sum_{\substack{z\in W_+^{\emptyset,J} \\ z\in W_I w W_J}} \epsilon_{z,w} Q^{\emptyset,J}_{v,z}(q) = \sum_{\substack{z\in W_+^{I,\emptyset} \\ z\in W_I w W_J}} \epsilon_{z,w} Q^{I,\emptyset}_{v,z}(q). \tag{3.4.17}$$

Proof. This is an easy consequence of (3.4.4), recognizing that every element of $W_I w W_J$ is in a right coset and a left coset within the double coset,

$$\begin{aligned} \sum_{\substack{z\in W_+^{\emptyset,J} \\ z\in W_I w W_J}} \epsilon_{z,w} Q^{\emptyset,J}_{v,z}(q) &= \sum_{\substack{z\in W_+^{\emptyset,J} \\ z\in W_I w W_J}} \epsilon_{z,w} \sum_{y\in zW_J} \epsilon_{z,y} Q_{v,y}(q) \\ &= \sum_{y\in W_I w W_J} \epsilon_{w,y} Q_{v,y}(q) = Q^{I,J}_{v,w}(q). \end{aligned} \tag{3.4.18}$$

The last equality in the claim follows similarly. □

Skandera, in [31] expresses the single parabolic Kazhdan-Lusztig immanants in the following manner.

Theorem 3.4.4. *Fix multisets L, M and let $I = \iota(L), J = \iota(M)$. For $u \in W_+^{I,\emptyset}$ and $v \in W_+^{\emptyset,J}$ we have*

$$\begin{aligned} \mathrm{Imm}_u^{L,[r]}(x) &= \mathrm{Imm}_u(x_{L,[r]}), \\ \mathrm{Imm}_v^{[r],M}(x) &= q^{-1}_{e,w_0^J} \mathrm{Imm}_{v^{-1}}((x_{[r],M})^\top). \end{aligned} \tag{3.4.19}$$

Proof. Fix $u \in W_+^{I,\emptyset}$. By definition we have

$$\begin{aligned}
\mathrm{Imm}_u(x_{L,[r]}) &= \sum_{v \geq u} \epsilon_{u,v} q_{u,v}^{-1} Q_{u,v}(q) (x_{L,[r]})^{e,v} \\
&= \sum_{\substack{w \in W_+^{I,\emptyset} \\ w \geq u}} \sum_{v \in W_I w} \epsilon_{u,v} q_{u,v}^{-1} Q_{u,v}(q) q_{v,w}^{-1} (x_{L,[r]})^{e,w} \\
&= \sum_{\substack{w \in W_+^{I,\emptyset} \\ w \geq u}} \Big(\epsilon_{u,w} q_{u,w}^{-1} \sum_{v \in W_I w} \epsilon_{v,w} Q_{u,v}(q) \Big) (x_{L,[r]})^{e,w}.
\end{aligned} \tag{3.4.20}$$

Applying the bar involution to the penultimate expression above, we have

$$\begin{aligned}
\overline{\mathrm{Imm}_u(x_{L,[r]})} &= \sum_{\substack{w \in W_+^{I,\emptyset} \\ w \geq u}} \epsilon_{u,w} q_{u,w} \sum_{v \in W_I w} \epsilon_{v,w} Q_{u,v}(q^{-1}) \overline{(x_{L,[r]})^{e,w}} \\
&= \sum_{\substack{w \in W_+^{I,\emptyset} \\ w \geq u}} \epsilon_{u,w} q_{u,w} \sum_{v \in W_I w} \epsilon_{v,w} Q_{u,v}(q^{-1}) \sum_{\substack{z \in W_+^{I,\emptyset} \\ z \geq w}} \epsilon_{w,z} q_{w,z} S_{w,z}^{I,\emptyset}(q^{-1}) (x_{L,[r]})^{e,z}.
\end{aligned} \tag{3.4.21}$$

By (3.3.32), this is

$$\begin{aligned}
&\sum_{\substack{w \in W_+^{I,\emptyset} \\ w \geq u}} \epsilon_{u,w} q_{u,w} \sum_{v \in W_I w} \epsilon_{v,w} Q_{u,v}(q^{-1}) \sum_{\substack{z \in W_+^{I,\emptyset} \\ z \geq w}} q_{v,w}^{-1} \sum_{y \in W_I z} q_{y,z}^{-1} (\epsilon_{v,y} q_{v,y} S_{v,y}(q^{-1})) (x_{L,[r]})^{e,z} \\
&= \sum_{\substack{z \in W_+^{I,\emptyset} \\ z \geq u}} \sum_{y \in W_I z} \epsilon_{u,y} q_{y,z}^{-1} \sum_{\substack{w \in W_+^{I,\emptyset} \\ u \leq w \leq z}} q_{u,y} \sum_{v \in W_I w} Q_{u,v}(q^{-1}) S_{v,y}(q^{-1}) (x_{L,[r]})^{e,z} \\
&= \sum_{\substack{z \in W_+^{I,\emptyset} \\ z \geq u}} \epsilon_{u,z} q_{u,z}^{-1} \sum_{y \in W_I z} \epsilon_{y,z} q_{u,y}^{2} \sum_{u \leq v \leq y} Q_{u,v}(q^{-1}) S_{v,y}(q^{-1}) (x_{L,[r]})^{e,z} \\
&= \sum_{\substack{z \in W_+^{I,\emptyset} \\ z \geq u}} \epsilon_{u,z} q_{u,z}^{-1} \sum_{y \in W_I z} \epsilon_{y,z} Q_{u,y}(q) (x_{L,[r]})^{e,z} \\
&= \mathrm{Imm}_u(x_{L,[r]}).
\end{aligned} \tag{3.4.22}$$

By the uniqueness of the basis $\{\mathrm{Imm}_u^{L,[r]}(x) \,|\, u \in W_+^{I,\emptyset}\}$, we have the first desired result.

The proof of the second result is similar. Fix $u \in W_+^{\emptyset,J}$. By definition we have

$$\begin{aligned}
q_{e,w_0^J}^{-1}\mathrm{Imm}_{u^{-1}}((x_{[r],M})^\top) &= q_{e,w_0^J}^{-1} \sum_{v^{-1}\geq u^{-1}} \epsilon_{u,v} q_{u,v}^{-1} Q_{u,v}(q)((x_{[r],M})^\top)^{e,v^{-1}} \\
&= q_{e,w_0^J}^{-1} \sum_{v\geq u} \epsilon_{u,v} q_{u,v}^{-1} Q_{u,v}(q)(x_{[r],M})^{v^{-1},e} \\
&= q_{e,w_0^J}^{-1} \sum_{\substack{w\in W_+^{\emptyset,J}\\ w\geq u}} \sum_{v\in wW_J} \epsilon_{u,v} q_{u,v}^{-1} Q_{u,v}(q) q_{v,w}^{-1}(x_{[r],M})^{w^{-1},e} \\
&= q_{e,w_0^J}^{-1} \sum_{\substack{w\in W_+^{\emptyset,J}\\ w\geq u}} \Big(\epsilon_{u,w} q_{u,w}^{-1} \sum_{v\in wW_J} \epsilon_{v,w} Q_{u,v}(q)\Big)(x_{[r],M})^{w^{-1},e} \\
&= \sum_{\substack{w\in W_+^{\emptyset,J}\\ w\geq u}} \Big(\epsilon_{u,w} q_{u,w}^{-1} \sum_{v\in wW_J} \epsilon_{v,w} Q_{u,v}(q)\Big)(x_{[r],M})^{e,w}.
\end{aligned} \tag{3.4.23}$$

Applying the bar involution to the penultimate expression above, we have

$$\begin{aligned}
\overline{q_{e,w_0^J}^{-1}\mathrm{Imm}_{u^{-1}}((x_{[r],M})^\top)} &= \sum_{\substack{w\in W_+^{\emptyset,J}\\ w\geq u}} \epsilon_{u,w} q_{u,w} \sum_{v\in wW_J} \epsilon_{v,w} Q_{u,v}(q^{-1}) \overline{(x_{[r],M})^{e,w}} \\
= \sum_{\substack{w\in W_+^{\emptyset,J}\\ w\geq u}} \epsilon_{u,w} q_{u,w} &\sum_{v\in wW_J} \epsilon_{v,w} Q_{u,v}(q^{-1}) \sum_{\substack{z\in W_+^{\emptyset,J}\\ z\geq w}} \epsilon_{w,z} q_{w,z} S_{w,z}^{\emptyset,J}(q^{-1})(x_{[r],M})^{e,z}.
\end{aligned} \tag{3.4.24}$$

By (3.4.8), this is

$$\begin{aligned}
&\sum_{\substack{w\in W_+^{\emptyset,J}\\ w\geq u}} \epsilon_{u,w} q_{u,w} \sum_{v\in wW_J} \epsilon_{v,w} Q_{u,v}(q^{-1}) \sum_{\substack{z\in W_+^{\emptyset,J}\\ z\geq w}} q_{v,w}^{-1} \sum_{y\in zW_J} q_{y,z}^{-1}(\epsilon_{v,y}q_{v,y}S_{v,y}(q^{-1}))(x_{[r],M})^{e,z}\\
&= \sum_{\substack{z\in W_+^{\emptyset,J}\\ z\geq u}} \sum_{y\in zW_J} \epsilon_{u,y} q_{y,z}^{-1} \sum_{\substack{w\in W_+^{\emptyset,J}\\ u\leq w\leq z}} q_{u,y} \sum_{v\in zW_J} Q_{u,v}(q^{-1})S_{v,y}(q^{-1})(x_{[r],M})^{e,z}\\
&= \sum_{\substack{z\in W_+^{\emptyset,J}\\ z\geq u}} \epsilon_{u,z} q_{u,z}^{-1} \sum_{y\in zW_J} \epsilon_{y,z} q_{u,y}^2 \sum_{u\leq v\leq y} Q_{u,v}(q^{-1})S_{v,y}(q^{-1})(x_{[r],M})^{e,z}\\
&= \sum_{\substack{z\in W_+^{\emptyset,J}\\ z\geq u}} \epsilon_{u,z} q_{u,z}^{-1} \sum_{y\in zW_J} \epsilon_{y,z} Q_{u,y}(q)(x_{[r],M})^{e,z}\\
&= q_{e,w_0^J}^{-1} \mathrm{Imm}_{u^{-1}}((x_{[r],M})^{\top}).
\end{aligned} \tag{3.4.25}$$

By the uniqueness of the basis $\{\mathrm{Imm}_u^{[r],M}(x) \,|\, u \in W_+^{\emptyset,J}\}$, we have the second desired result. □

We are not be able to express the double parabolic immanants simply as a non-parabolic immanant evaluated at a generalized submatrix as in the single parabolic case. However, the following results are generalizations of the previous result, in which we express double parabolic immanants as single parabolic immanants evaluated at a generalized submatrix.

Theorem 3.4.5. *For $v \in W_+^{I,J}$*

$$\begin{aligned}
\mathrm{Imm}_v^{L,M}(x) &= \mathrm{Imm}_v^{[r],M}(x_{L,[r]}),\\
\mathrm{Imm}_v^{L,M}(x) &= q_{w_0^I,w_0^J}^{-1} \mathrm{Imm}_{v^{-1}}^{L,[r]}((x_{[r],M})^{\top}).
\end{aligned} \tag{3.4.26}$$

Proof. By definition we have

$$\begin{aligned}
\mathrm{Imm}_v^{[r],M}(x_{L,M}) &= \sum_{\substack{z\in W_+^{\emptyset,J}\\ z\geq v}} \epsilon_{v,z} q_{v,z}^{-1} Q_{v,z}^{\emptyset,J}(q)(x_{L,M})^{e,z}\\
&= \sum_{\substack{w\in W_+^{I,J}\\ w\geq v}} \sum_{\substack{z\in W_+^{\emptyset,J}\\ z\in W_I w W_J}} \epsilon_{v,z} q_{v,z}^{-1} Q_{v,z}^{\emptyset,J}(q) q_{z,w}^{-1}(x_{L,M})^{e,w},
\end{aligned} \tag{3.4.27}$$

where the second line comes from the fact that the difference in length between z and w is just the difference in z_-^I and w_-^I, since they are both maximal with respect to J. Lemma 3.4.3 allows us to complete the proof,

$$\begin{aligned}
\mathrm{Imm}_v^{[r],M}(x_{L,M}) &= \sum_{\substack{w\in W_+^{I,J}\\ w\geq v}} \epsilon_{v,w} q_{v,w}^{-1} \sum_{\substack{z\in W_+^{\emptyset,J}\\ z\in W_I w W_J}} \epsilon_{z,w} Q_{v,z}^{\emptyset,J}(q)(x_{L,M})^{e,w}, \\
&= \sum_{\substack{w\in W_+^{I,J}\\ w\geq v}} \epsilon_{v,w} q_{v,w}^{-1} Q_{v,w}^{I,J}(q)(x_{L,M})^{e,w}, \\
&= \mathrm{Imm}_v^{L,M}(x).
\end{aligned} \tag{3.4.28}$$

The second result is proved in a similar manner. By definition we have

$$\begin{aligned}
\mathrm{Imm}_{v^{-1}}^{L,[r]}((x_{L,M})^\top) &= \sum_{\substack{z\in W_+^{I,\emptyset}\\ z\geq v}} \epsilon_{v,z} q_{v,z}^{-1} Q_{v,z}^{I,\emptyset}(q)(x_{L,M})^{z^{-1},e} \\
&= \sum_{\substack{w\in W_+^{I,J}\\ w\geq v}} \sum_{\substack{z\in W_+^{I,\emptyset}\\ z\in W_I w W_J}} \epsilon_{v,z} q_{v,z}^{-1} Q_{v,z}^{I,\emptyset}(q) q_{z,w}^{-1}(x_{L,M})^{w^{-1},e},
\end{aligned} \tag{3.4.29}$$

where the second line comes from the fact that the difference in length between z and w is just the difference in z_-^J and w_-^J, since they are both maximal with respect to I. Lemma 3.4.3 allows us to complete the proof once again,

$$\begin{aligned}
\mathrm{Imm}_v^{L,[r]}(x_{L,M}) &= \sum_{\substack{w\in W_+^{I,J}\\ w\geq v}} \epsilon_{v,w} q_{v,w}^{-1} \sum_{\substack{z\in W_+^{I,\emptyset}\\ z\in W_I w W_J}} \epsilon_{z,w} Q_{v,z}^{I,\emptyset}(q)(x_{L,M})^{w^{-1},e}, \\
&= \sum_{\substack{w\in W_+^{I,J}\\ w\geq v}} \epsilon_{v,w} q_{v,w}^{-1} Q_{v,w}^{I,J}(q)(x_{L,M})^{w^{-1},e}, \\
&= \sum_{\substack{w\in W_+^{I,J}\\ w\geq v}} \epsilon_{v,w} q_{v,w}^{-1} Q_{v,w}^{I,J}(q) q_{w_0^I,w_0^J}, \\
&= q_{w_0^I,w_0^J} \mathrm{Imm}_v^{L,M}(x).
\end{aligned} \tag{3.4.30}$$

□

At first, it seems we should be able to use Theorems 3.4.4 and 3.4.5 to express the double parabolic immanants as regular immanants. However this does not work as

$$q_{e,w_0^J}^{-1}\mathrm{Imm}_{v^{-1}}((x_{L,M})^{\top}) = q_{w_0^I,e}\sum_{\substack{w\in W_+^{I,J}\\ w\geq v}}\sum_{z\in W_IwW_J}\epsilon_{v,z}q_{v,z}^{-1}q_{z_-^J,w_-^J}^{-1}Q_{v,z}(q)(x_{L,M})^{e,w} \tag{3.4.31}$$

does not have the complete difference in length between z and w, but rather just the J part. We see that when we plug in $x_{L,M}$ to the nonparabolic immanant we pick up a factor depending on I that we don't pick up when we plug in $x_{[r],M}$. Thus, if we wait until after we have gone to the single parabolic form, then plug in the $x_{L,M}$, the extra factor does not appear. Thus we can still express double parabolic immanants as regular immanants if we use a two step process, first evaluating at one generalized submatrix and simplifying and then evaluating at another generalized submatrix and simplifying.

3.5 Alternative bar involutions

In the literature, the bar involution is sometimes defined without the power of $q^{\alpha(a)-\alpha(b)}$, or with an alternate power. The dual canonical basis is then defined to be invariant under these alternate bar involutions. In an effort to address this let us generalize the definition of the bar involution. Define the *k-bar involution* by $\varphi_k(q) = q^{-1}$, $\varphi_k(x_{i,j}) = x_{i,j}$, and

$$\varphi_k(x_{a_1,b_1}\cdots x_{a_r,b_r}) = q^{\frac{k}{2}}x_{a_r,b_r}\cdots x_{a_1,b_1}. \tag{3.5.1}$$

Looking at the definition of the bar involution on $\mathcal{A}_{L,M}(n;q)$, we see that this is just the k-bar involution where $k = \ell(w_0^I) - \ell(w_0^J)$.

It is straight forward to show that for $j \neq k$ we have

$$\varphi_j((x_{L,M})^{u,v}) = (q^{\frac{1}{2}})^{j-k}\varphi_k((x_{L,M})^{u,v}). \tag{3.5.2}$$

The existence of a unique bar-invariant basis for one of these involutions guarantees the existence and uniqueness of an invariant basis for all choices of k even, as we will see below. Define $\{\mathrm{Imm}_{v,k}^{L,M}(x) \mid v \in W_+^{I,J}\}$ to be the k-bar-invariant basis for $\mathcal{A}_{L,M}(n;q)$ and call them *k-immanants*. Thus we have $\mathrm{Imm}_v^{L,M}(x) = \mathrm{Imm}_{v,\ell(w_0^I)-\ell(w_0^J)}^{L,M}(x)$, for each $v \in W_+^{I,J}$. Furthermore we have

$$\begin{aligned}\varphi_k((q^{\frac{1}{2}})^{\frac{k-(\ell(w_0^I)-\ell(w_0^J))}{2}}\mathrm{Imm}_v^{L,M}(x)) &= (q^{\frac{-1}{2}})^{\frac{k-(\ell(w_0^I)-\ell(w_0^J))}{2}}(q^{\frac{1}{2}})^{k-(\ell(w_0^I)-\ell(w_0^J))}\mathrm{Imm}_v^{L,M}(x)\\ &= (q^{\frac{1}{2}})^{\frac{k-(\ell(w_0^I)-\ell(w_0^J))}{2}}\mathrm{Imm}_v^{L,M}(x).\end{aligned} \tag{3.5.3}$$

Thus we can relate the k-immanants to the Kazhdan-Lusztig immanants by

$$\mathrm{Imm}_{v,k}^{L,M}(x) = q^{\frac{k}{4}} q_{w_0^I,w_0^J}\mathrm{Imm}_v^{L,M}(x). \tag{3.5.4}$$

An interesting thing to note is that if k is not even, we would need to extend our space to allow fourth powers of q in order to have a bar invariant basis.

In [8] Theorem 15 gives a basis for $\mathcal{A}(n;q)$, $\{M_{\alpha,\beta} \mid (\alpha,\beta) \in \bigcup(I_\mu \times I_\nu)^+\}$. For each component $\mathcal{A}_{L,M}(n;q)$, this basis is just $\{\overline{q_{w_0^I,w_0^J}^{-1}(x_{L,M})^{e,v}} \mid v \in W_+^{I,J}\}$. In Theorem 16, Brundan describes a bar involution. This involution turns out to be the $2(\ell(w_0^J)-\ell(w_0^I))$-bar involution. This is as expected since, as mentioned earlier, q is being used where we have been using $q^{\frac{-1}{2}}$. Brundan then defines the dual canonical basis to be invariant under this involution. Therefore, what he calls the dual canonical basis consists of the $\ell(w_0^J)-\ell(w_0^I)$-immanants.

In [36], Zhang uses defines a bar involution which is the 0-bar involution. Thus the dual canonical basis he defines is the set of 0-immanants.

In [15], Du defines the dual canonical basis to be the set $\{Z_{\lambda,v,\mu} \mid v \in W_-^{I,J}\}$. Letting $v \in W_+^{I,J}$, we can express this in our notation as

$$Z_{\lambda,v_-,\mu} = Z_v^{L,M} = q_{e,w_0^J}^{-1}\sum_{\substack{w\in W_+^{I,J}\\ w\geq v}} \epsilon_{v,w} q_{v,w}^{-1} Q_{v,w}^{I,J}(q)(x_{L,M})^{w_-^{-1},e}, \tag{3.5.5}$$

where $v_-, w_- \in W_-^{I,J}$ are the minimal representatives of the cosets $W_I v W_J, W_I w W_J$, respectively. This looks very similar to the Kazhdan-Lusztig immanant basis, but is not equivalent. In fact we have

$$\mathrm{Imm}_v^{L,M}(x) = q_{e,w_0^J}^{-1}\sum_{\substack{w\in W_+^{I,J}\\ w\geq v}} \epsilon_{v,w} q_{v,w_-}^{-1} Q_{v,w}^{I,J}(q)(x_{L,M})^{w_-^{-1},e}. \tag{3.5.6}$$

Alternatively we could express them more naturally as

$$\begin{aligned} Z_v^{L,M} &= q^{\frac{-\left(\ell\left(w_0^I\right)+\ell\left(w_0^J\right)\right)}{2}} \sum_{\substack{w \in W_+^{I,J} \\ w \geq v}} \epsilon_{v,w} q_{v,w}^{-1} Q_{v,w}^{I,J}(q) q_{e,w_0^K} (x_{L,M})^{e,w}, \\ \mathrm{Imm}_v^{L,M}(x) &= \sum_{\substack{w \in W_+^{I,J} \\ w \geq v}} \epsilon_{v,w} q_{v,w}^{-1} Q_{v,w}^{I,J}(q) (x_{L,M})^{e,w}. \end{aligned} \tag{3.5.7}$$

The difference between them is an overall power of q and (more importantly) the q_{e,w_0^K} term in the summation. This implies that the Z-basis is not k-bar invariant for any k in the double parabolic case. In the single and nonparabolic cases $\ell(w_0^K) = 1$ and the Z-basis consists of k-immanants.

Proposition 3.5.1. *For all u, $v \in W_+^{I,\emptyset}$, and $w \in W_+^{\emptyset,J}$, we have*

$$\begin{aligned} Z_v^{[r],[r]} &= Z_v = \mathrm{Imm}_v(x), \\ Z_v^{L,[r]} &= q_{e,w_0^I}^{-1} \mathrm{Imm}_v^{L,[r]}(x) = \mathrm{Imm}_{v,0}^{L,[r]}(x), \\ Z_v^{[r],M} &= q_{e,w_0^J}^{-1} \mathrm{Imm}_v^{[r],M}(x) = \mathrm{Imm}_{v,-2\ell(w_0^J)}^{[n],M}(x). \end{aligned} \tag{3.5.8}$$

Proof. This is an easy consequence of (3.5.7), recognizing what I, J, and K are in each instance. □

Corollary 3.5.2. *The bases $\{Z_v \mid v \in \mathfrak{S}_r\}$, $\{Z_v^{L,[r]} \mid v \in W_+^{I,\emptyset}\}$, and $\{Z_v^{[r],M} \mid v \in W_+^{\emptyset,J}\}$ are bar invariant, 0-bar invariant, and $-2\ell(w_0^J)$-bar invariant, respectively.*

Proof. This follows from the definitions and Proposition 3.5.1. □

The previous results would lead us to believe that Du is using the $\ell(w_0^J)$-bar involution. This works for the single and nonparabolic cases. Since K depends on w and the length of K is not constant for each double coset, we cannot just factor out q_{e,w_0^K} and express the Z-basis as a multiple of the Kazhdan-Lusztig immanants.

Define the $\star$-bar operation by $\varphi_\star(q) = q^{-1}$, $\varphi_\star(x_{i,j}) = x_{i,j}$, and

$$\varphi_\star((x_{L,M})^{e,v}) = q^{-\left(\ell\left(w_0^I\right)+\ell\left(w_0^J\right)\right)} \sum_{\substack{w \in W_+^{I,J} \\ w \geq v}} \epsilon_{v,w} q_{v,w} S_{v,w}^{I,J}(q^{-1}) q_{e,w_0^{K(v)}} q_{e,w_0^{K(w)}} (x_{L,M})^{e,w}. \tag{3.5.9}$$

Lemma 3.5.3. *The $\star$-bar operation is an involution.*

Proof. Let $A = \ell\left(w_0^I\right) + \ell\left(w_0^J\right)$, then applying the $\star$-bar involution to a natural basis element twice gives

$$\begin{aligned}\varphi_\star(\varphi_\star((x_{L,M})^{e,v})) &= \varphi_\star\Big(\sum_{\substack{w\in W_+^{I,J}\\ w\geq v}} q^{-A}\epsilon_{v,w}q_{v,w}S_{v,w}^{I,J}(q^{-1})q_{e,w_0^{K(v)}}q_{e,w_0^{K(w)}}(x_{L,M})^{e,w}\Big)\\ &= q^A q^{-1}_{e,w_0^{K(v)}}\sum_{\substack{w,z\in W_+^{I,J}\\ z\geq w\geq v}} \epsilon_{v,w}q_{v,w}^{-1}S_{v,w}^{I,J}(q)q^{-A}\epsilon_{w,z}q_{w,z}S_{w,z}^{I,J}(q^{-1})q_{e,w_0^{K(z)}}(x_{L,M})^{e,z}\\ &= q^{-1}_{e,w_0^{K(v)}}\sum_{\substack{z\in W_+^{I,J}\\ z\geq v}} \epsilon_{v,z}q_{e,w_0^{K(z)}}\sum_{\substack{w\in W_+^{I,J}\\ z\geq w\geq v}} q_{v,w}^{-1}S_{v,w}^{I,J}(q)q_{w,z}S_{w,z}^{I,J}(q^{-1})(x_{L,M})^{e,z}.\end{aligned} \tag{3.5.10}$$

By (3.3.34), we have

$$\varphi_\star(\varphi_\star((x_{L,M})^{e,v})) = q^{-1}_{e,w_0^{K(v)}}q_{e,w_0^{K(v)}}(x_{L,M})^{e,v} = (x_{L,M})^{e,v}. \tag{3.5.11}$$

□

The $\star$-bar involution is not a k-bar involution for any value of k in the double-parabolic case. Notice whenever either L or M is $[r]$, then the $\star$-bar involution is equal to the $\ell(w_0^J)$-bar involution and we have Du's Z-basis is $\star$-bar invariant.

Proposition 3.5.4. *For all $v \in W_+^{I,J}$, $Z_v^{L,M}$ is $\star$-bar invariant, or in other words,*

$$\varphi_\star(Z_v^{L,M}) = Z_v^{L,M}. \tag{3.5.12}$$

Proof. Let $A = \ell\left(w_0^I\right) + \ell\left(w_0^J\right)$, then applying the $\star$-bar involution gives

$$\begin{aligned}
\varphi_\star(Z_v^{L,M}) &= q^{\frac{A}{2}} \sum_{\substack{w \in W_+^{I,J} \\ w \geq v}} \epsilon_{v,w} q_{v,w} Q_{v,w}^{I,J}(q^{-1}) q^{-1}_{e,w_0^{K(w)}} \varphi_\star((x_{L,M})^{e,w}) \\
&= q^{\frac{A}{2}} \sum_{\substack{w,z \in W_+^{I,J} \\ z \geq w \geq v}} \epsilon_{v,w} q_{v,w} Q_{v,w}^{I,J}(q^{-1}) q^{-A} \epsilon_{w,z} q_{w,z} S_{w,z}^{I,J}(q^{-1}) q_{e,w_0^{K(z)}} (x_{L,M})^{e,z} \\
&= q^{\frac{-A}{2}} \sum_{\substack{w,z \in W_+^{I,J} \\ z \geq w \geq v}} q_{e,w_0^{K(z)}} \epsilon_{v,w} q_{v,z} \sum_{y \in W_I w W_J} \epsilon_{w,y} Q_{v,y}(q^{-1}) \sum_{x \in W_I z W_J} q_{x,z}^{-2} \epsilon_{y,x} S_{y,x}(q^{-1}) (x_{L,M})^{e,z} \\
&= q^{\frac{-A}{2}} \sum_{\substack{z \in W_+^{I,J} \\ z \geq v}} q_{e,w_0^{K(z)}} \sum_{x \in W_I z W_J} \epsilon_{v,x} q_{v,x} q_{x,z}^{-1} \sum_{z \geq y \geq v} Q_{v,y}(q^{-1}) S_{y,x}(q^{-1}) (x_{L,M})^{e,z} \\
&= q^{\frac{-A}{2}} \sum_{\substack{z \in W_+^{I,J} \\ z \geq v}} q_{e,w_0^{K(z)}} \sum_{x \in W_I z W_J} \epsilon_{v,x} q_{v,x} q_{x,z}^{-1} q_{v,x}^{-2} Q_{v,z}(q) (x_{L,M})^{e,z} \\
&= q^{\frac{-A}{2}} \sum_{\substack{z \in W_+^{I,J} \\ z \geq v}} q_{e,w_0^{K(z)}} \epsilon_{v,z} q_{v,z}^{-1} \sum_{x \in W_I z W_J} \epsilon_{x,z} Q_{v,x}(q) (x_{L,M})^{e,z} \\
&= q^{\frac{-A}{2}} \sum_{\substack{z \in W_+^{I,J} \\ z \geq v}} q_{e,w_0^{K(z)}} \epsilon_{v,z} q_{v,z}^{-1} Q_{v,z}^{I,J}(q) (x_{L,M})^{e,z} \\
&= Z_v^{L,M}.
\end{aligned} \tag{3.5.13}$$

∎

Du's choice to use minimal coset representatives seems to be a minor difference, but ends up being quite significant. Since, Du's basis agrees with a k-immanant basis in the single parabolic cases, one might think the $\star$-bar involution may fix $\mathrm{Imm}_v(x_{L,M})$ when the bar involution did not. However, this is not the case and we are left wondering if there is an involution which would allow us to express the double parabolic immanants as nonparabolic immanants expressed at a generalized submatrix directly, rather than in a two step process.

Problem 5. *Find an involution $f : \mathcal{A}(n;q) \to \mathcal{A}(n;q)$ which fixes* $\mathrm{Imm}_v(x_{L,M})$.

Chapter 4

Two parameter generalizations of $H_n(q)$ and $\mathcal{A}(n;q)$

In the previous chapters we have seen that the various families of polynomials have many properties when restricted to the quotient ring $\mathbb{Z}[q_1, q_2]/(1 - q_1q_2 - q_2^2)$. This ring arises as we are often using $q_1 = q^{\frac{1}{2}} - q^{\frac{-1}{2}}$ and $q_2 = q^{\frac{-1}{2}}$. In [25] and [35], two-parameter generalizations of $H_n(q)$ and $\mathcal{A}(n;q)$ are defined. In this chapter we make analogous definitions to the single parameter case and seek to better understand the role of $\mathbb{Z}[q_1, q_2]/(1 - q_1q_2 - q_2^2)$. We will follow the structure of [31] and focus first on the special immanant space submodule before considering a general multi-graded component and parabolic submodules.

4.1 The two-parameter Hecke algebra.

In [25], Lascoux defines a two-parameter version of the Hecke algebra. Define the *two-parameter Hecke algebra* $H_n(q_2, q_4)$ to be the $\mathbb{C}[q_2, q_2^{-1}, q_4, q_4^{-1}]$-algebra with multiplicative identity $\widetilde{T}_e = 1$, generated by elements $\{\widetilde{T}_{s_i} \mid 1 \leq i \leq n-1\}$ subject to the relations

$$\begin{aligned} (\widetilde{T}_{s_i} - q_2\widetilde{T}_e)(\widetilde{T}_{s_i} - q_4\widetilde{T}_e) = 0, \qquad & \text{for } i = 1, \ldots, n-1, \\ \widetilde{T}_{s_i}\widetilde{T}_{s_j}\widetilde{T}_{s_i} = \widetilde{T}_{s_j}\widetilde{T}_{s_i}\widetilde{T}_{s_j}, \qquad & \text{if } |i-j| = 1, \\ \widetilde{T}_{s_i}\widetilde{T}_{s_j} = \widetilde{T}_{s_j}\widetilde{T}_{s_i}, \qquad & \text{if } |i-j| \geq 2. \end{aligned} \tag{4.1.1}$$

Notice that we have $H_n(q^{\frac{1}{2}}, -q^{\frac{-1}{2}}) = H_n(q)$. Furthermore, inverses of the generators are given by

$$\widetilde{T}_{s_i}^{-1} = \frac{\widetilde{T}_{s_i} - (q_2 + q_4)\widetilde{T}_e}{-q_2q_4}, \tag{4.1.2}$$

and multiplication rules are given by

$$\widetilde{T}_{s_i}\widetilde{T}_w = \begin{cases} \widetilde{T}_{s_iw} & \text{if } s_iw > w, \\ -q_2q_4\widetilde{T}_{s_iw} + (q_2 + q_4)\widetilde{T}_w & \text{if } s_iw < w \end{cases} \tag{4.1.3}$$

and

$$\widetilde{T}_w\widetilde{T}_{s_i} = \begin{cases} \widetilde{T}_{ws_i} & \text{if } ws_i > w, \\ -q_2q_4\widetilde{T}_{ws_i} + (q_2 + q_4)\widetilde{T}_w & \text{if } ws_i < w. \end{cases} \tag{4.1.4}$$

More generally, we have

$$\widetilde{T}_{u^{-1}}\widetilde{T}_v = \begin{cases} \widetilde{T}_{u^{-1}s_i}\widetilde{T}_{s_iv} & \text{if } s_iu < u \text{ and } s_iv > v, \\ & \text{ or if } s_iu > u \text{ and } s_iv < v, \\ -q_2q_4\widetilde{T}_{u^{-1}s_i}\widetilde{T}_{s_iv} + (q_2 + q_4)\widetilde{T}_{u-1s_i}\widetilde{T}_v & \text{if } s_iu < u \text{ and } s_iv < v, \\ (-q_2q_4)^{-1}\widetilde{T}_{u^{-1}s_i}\widetilde{T}_{s_iv} & \\ \quad +(-q_2q_4)^{-1}(q_2 + q_4)\widetilde{T}_{u^{-1}s_i}\widetilde{T}_v & \text{if } s_iu > u \text{ and } s_iv > v. \end{cases} \tag{4.1.5}$$

Thus we have

$$\widetilde{T}_{u^{-1}}\widetilde{T}_v \in (-q_2q_4)^{k/2}\widetilde{T}_{u^{-1}v} + \sum_{w \geq u^{-1}v} \mathbb{N}[-q_2q_4, q_2 + q_4]\widetilde{T}_w. \tag{4.1.6}$$

Notice that in $H_n(q^{\frac{1}{2}}, -q^{\frac{-1}{2}}) = H_n(q)$, we have $-q_2q_4 = 1$ and $q_2 + q_4 = q^{\frac{1}{2}} - q^{\frac{-1}{2}}$. Therefore, these formulas are consistent with the earlier definitions and we see why the $-q_2q_4$ term does not appear in $H_n(q)$.

In [25], Lascoux also mentions that the elements

$$\widetilde{\mathcal{T}}_{s_i} = -q_2q_4\widetilde{T}_{s_i}^{-1} \tag{4.1.7}$$

generate $H_n(q_2, q_4)$ and satisfy the condition

$$(\widetilde{\mathcal{T}}_{s_i} + q_2)(\widetilde{\mathcal{T}}_{s_i} + q_4) = 0. \tag{4.1.8}$$

Therefore we have

$$\widetilde{\mathcal{T}}_{s_i} = \widetilde{T}_{s_i} - (q_2 + q_4)\widetilde{T}_e \tag{4.1.9}$$

and

$$\widetilde{\mathcal{T}}_{s_i}^2 = -q_2q_4\widetilde{\mathcal{T}}_e - (q_2 + q_4)\widetilde{\mathcal{T}}_{s_i}. \tag{4.1.10}$$

Letting $\widetilde{\mathcal{T}}_w = \widetilde{\mathcal{T}}_{s_{i_1}} \cdots \widetilde{\mathcal{T}}_{s_{i_\ell}}$, where $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced expression, we have

$$\widetilde{\mathcal{T}}_w = (-q_2q_4)^{\ell(w)}\widetilde{T}_w^{-1}. \tag{4.1.11}$$

We have that $\{\widetilde{\mathcal{T}}_w \mid w \in \mathfrak{S}_n\}$ is a basis for $H_n(q_2, q_4)$, which we will call the *inverse natural basis*. Multiplication rules are given by

$$\widetilde{\mathcal{T}}_{s_i}\widetilde{\mathcal{T}}_w = \begin{cases} \widetilde{\mathcal{T}}_{s_iw} & \text{if } s_iw > w, \\ -q_2q_4\widetilde{\mathcal{T}}_{s_iw} - (q_2 + q_4)\widetilde{\mathcal{T}}_w & \text{if } s_iw < w, \end{cases} \tag{4.1.12}$$

and

$$\widetilde{\mathcal{T}}_w\widetilde{\mathcal{T}}_{s_i} = \begin{cases} \widetilde{\mathcal{T}}_{ws_i} & \text{if } ws_i > w, \\ -q_2q_4\widetilde{\mathcal{T}}_{ws_i} - (q_2 + q_4)\widetilde{\mathcal{T}}_w & \text{if } ws_i < w. \end{cases} \tag{4.1.13}$$

4.2 The two-parameter quantum polynomial ring

In [35], Takeuchi defines a two-parameter version of GL_n, which leads to a two-parameter version of the quantum polynomial ring. Following [35], for each $n \geq 0$, let the *two-parameter quantum polynomial ring* $\mathcal{A}(n; q_2, q_4)$ be the noncommutative $\mathbb{C}[q_2, q_2^{-1}, q_4, q_4^{-1}]$-algebra generated by n^2 variables $x = (x_{1,1}, \ldots, x_{n,n})$, subject to the relations

$$
\begin{aligned}
x_{i,\ell}x_{i,k} &= -q_4^{-1}x_{i,k}x_{i,\ell} \\
x_{j,k}x_{i,k} &= q_2x_{i,k}x_{j,k} \\
x_{j,k}x_{i,\ell} &= -q_2q_4x_{i,\ell}x_{j,k} \\
x_{j,\ell}x_{i,k} &= x_{i,k}x_{j,\ell} + (q_2+q_4)x_{i,\ell}x_{j,k},
\end{aligned}
\tag{4.2.1}
$$

for all indices $1 \leq i < j \leq n$ and $1 \leq k < l \leq n$. As with the two-parameter Hecke algebra, we have $\mathcal{A}(n; q^{\frac{1}{2}}, -q^{\frac{-1}{2}}) = \mathcal{A}(n; q)$. Just as before, as a $\mathbb{C}[q_2, q_2^{-1}, q_4, q_4^{-1}]$-module, $\mathcal{A}(n; q_2, q_4)$ is spanned by monomials in lexicographic order, or standard form. The natural grading of $\mathcal{A}(n; q_2, q_4)$ by degree,

$$\mathcal{A}(n; q_2, q_4) = \bigoplus_{r\geq 0} \mathcal{A}_r(n; q_2, q_4), \tag{4.2.2}$$

where $\mathcal{A}_r(n; q_2, q_4)$ consists of the homogeneous degree r polynomials, has a finer grading by multidegree, in which for each monomial we keep track of the multiset of row indices and the multiset of column indices,

$$\mathcal{A}_r(n; q_2, q_4) = \bigoplus_{L,M} \mathcal{A}_{L,M}(n; q_2, q_4), \tag{4.2.3}$$

where the sum is over pairs (L, M) of multisets of $[n]$ having cardinality r.

We will focus our attention on $\mathcal{A}_{[n],[n]}(n; q_2, q_4)$, the *two-parameter immanant space*, the $\mathbb{C}[q_2, q_2^{-1}, q_4, q_4^{-1}]$-submodule of $\mathcal{A}(n; q_2, q_4)$ spanned by the monomials

$$\{x_{1,w_1} \cdots x_{n,w_n} \mid w \in \mathfrak{S}_n\}, \tag{4.2.4}$$

which we will once again call the *natural basis* of $\mathcal{A}_{[n],[n]}(n; q_2, q_4)$. Using the same notation as before we can see that the monomials $\{x^{u,v} \mid u, v \in \mathfrak{S}_n\}$ satisfy

$$x^{u,v} = \begin{cases} -q_2q_4x^{s_iu,s_iv} & \text{if } s_iu < u \text{ and } s_iv > v, \\ (-q_2q_4)^{-1}x^{s_iu,s_iv} & \text{if } s_iu > u \text{ and } s_iv < v, \\ x^{s_iu,s_iv} + (q_2+q_4)x^{s_iu,v} & \text{if } s_iu < u \text{ and } s_iv < v, \\ x^{s_iu,s_iv} - (-q_2q_4)^{-1}(q_2+q_4)x^{s_iu,v} & \text{if } s_iu > u \text{ and } s_iv > v. \end{cases} \tag{4.2.5}$$

For example in $\mathcal{A}_{[3],[3]}(3; q_2, q_4)$, we have

$$\begin{aligned} x^{s_1,w_0} &= x^{e,s_2s_1} + (q_2+q_4)x^{e,w_0}, \\ x^{s_1s_2,w_0} &= x^{e,s_1} + (q_2+q_4)(x^{e,s_1s_2} + x^{e,s_2s_1}) + (q_2+q_4)^2 x^{e,w_0}, \\ x^{w_0,w_0} &= x^{e,e} + (q_2+q_4)(x^{e,s_1} + x^{e,s_2}) + (q_2+q_4)^2(x^{e,s_1s_2} + x^{e,s_2s_1}) \\ &\quad + \left((q_2+q_4)^3 - q_2q_4(q_2+q_4)\right) x^{e,w_0}. \end{aligned} \tag{4.2.6}$$

We no longer have (1.3.32), but rather

$$x^{w^{-1},e} = (-q_2q_4)^{\ell(w)} x^{e,w}. \tag{4.2.7}$$

In $\mathcal{A}(n;q)$ we have a transpose involution, $(x^{e,w})^\top = x^{w,e}$. In $\mathcal{A}(n;q_2,q_4)$ this is no longer an involution. Rather, by (4.2.7) we see that

$$((x^{e,w})^\top)^\top = (-q_2q_4)^{2\ell(w)} x^{e,w}. \tag{4.2.8}$$

Define a modified transpose operation by $(x^{e,w})^\dagger = (-q_2q_4)^{-\ell(w)}(x^{e,w})^\top$. This is also an involution specializing to the transpose operation in $\mathcal{A}(n;q)$.

Proposition 4.2.1. *The map $x \to x^\dagger$ defined by $(x^{e,w})^\dagger = (-q_2q_4)^{-\ell(w)} x^{w,e}$, is a well-defined $\mathbb{C}[q_2,q_4]$-linear transposition on $\mathcal{A}(n;q_2,q_4)$. In particular, we have*

$$(x^{u,v})^\dagger = (-q_2q_4)^{\ell(u)-\ell(v)} x^{v,u}. \tag{4.2.9}$$

Proof. Observe that by (4.2.7) we have

$$((x^{e,w})^\dagger)^\dagger = (-q_2q_4)^{-\ell(w)}(x^{w,e})^\dagger = (x^{e,w^{-1}})^\dagger = (-q_2q_4)^{-\ell(w)} x^{w^{-1},e} = x^{e,w}. \tag{4.2.10}$$

Now suppose that we know $(x^{u,v})^\dagger = x^{v,u}$ whenever $\ell(u) \leq k$ and choose a permutation su with length of $k+1$ and $u < su$. Then by (4.2.5) and induction we have

$$\begin{aligned} (x^{su,v})^\dagger &= \begin{cases} -q_2q_4(x^{u,sv})^\dagger & \text{if } sv > v, \\ (x^{u,sv})^\dagger + (q_2+q_4)(x^{u,v})^\dagger & \text{if } sv < v, \end{cases} \\ &= \begin{cases} -q_2q_4(-q_2q_4)^{\ell(u)-\ell(sv)} x^{sv,u} & \text{if } sv > v, \\ (-q_2q_4)^{\ell(u)-\ell(sv)} x^{sv,u} + (q_2+q_4)(-q_2q_4)^{\ell(u)-\ell(v)} x^{v,u} & \text{if } sv < v. \end{cases} \end{aligned} \tag{4.2.11}$$

If $sv > v$ then $\ell(sv) = \ell(v) + 1$ and by (4.2.5) we have

$$-q_2q_4(-q_2q_4)^{\ell(u)-\ell(sv)}x^{sv,u} = (-q_2q_4)^{\ell(u)-\ell(v)}(-q_2q_4)x^{v,su} = (-q_2q_4)^{\ell(su)-\ell(v)}x^{v,su}. \tag{4.2.12}$$

Similarly, if $sv < v$ then $\ell(sv) = \ell(v) - 1$ and by (4.2.5) we have

$$\begin{aligned}(-q_2q_4)^{\ell(u)-\ell(sv)}&x^{sv,u} + (q_2+q_4)(-q_2q_4)^{\ell(u)-\ell(v)}x^{v,u} \\ &= (-q_2q_4)^{\ell(u)-\ell(v)+1}\left(x^{v,su} - (-q_2q_4)^{-1}(q_2+q_4)x^{v,u}\right) \\ &\quad + (q_2+q_4)(-q_2q_4)^{\ell(u)-\ell(v)}x^{v,u} \\ &= (-q_2q_4)^{\ell(su)-\ell(v)}x^{v,su}.\end{aligned} \tag{4.2.13}$$

Therefore, we have that $(x^{su,v})^\dagger = (-q_2q_4)^{\ell(su)-\ell(v)}x^{v,su}$ as claimed. □

In the proof above we see that (4.2.10) implies we have

$$(x^{e,w})^\dagger = x^{e,w^{-1}}. \tag{4.2.14}$$

Using the defining relations we can see that

$$x^{u,v} \in (-q_2q_4)^{\frac{k}{2}}x^{e,u^{-1}v} + \sum_{w>u^{-1}v}\mathbb{N}[-q_2q_4, q_2+q_4]x^{e,w}, \tag{4.2.15}$$

where $k = \ell(u) - \ell(v) + \ell(u^{-1}v)$. The following result also provides a definition for two-parameter p-polynomials, as well as suggesting how we might later define the two-parameter $p^{I,J}$-polynomials in order for them to evaluate to the coefficients in the natural basis expansion as in $\mathcal{A}(n;q)$.

Proposition 4.2.2. *The coefficient of $x^{e,w}$ in the natural expansion of $x^{u,v}$ has the form $p_{u,v,w}(-q_2q_4, q_2+q_4)$ for some polynomial $p_{u,v,w}(q_0,q_1)$ in $\mathbb{N}[q_0,q_1]$,*

$$x^{u,v} = \sum_{w\geq u^{-1}v} p_{u,v,w}(-q_2q_4, q_2+q_4)x^{e,w}. \tag{4.2.16}$$

In particular, given a reduced expression $s_{i_1}\cdots s_{i_\ell}$ for u, the coefficient of $q_0^aq_1^b$ in $p_{u,v,w}(q_0,q_1)$ is equal to the number of sequences $(\pi^{(0)},\ldots,\pi^{(\ell)})$ of permutations satisfying

1. $\pi^{(0)} = v, \pi^{(\ell)} = w$.

2. $\pi^{(j)} \in \{s_{i_j}\pi^{(j-1)}, \pi^{(j-1)}\}$ *for* $j = 1, \ldots, \ell$.

3. $\pi^{(j)} = s_{i_j}\pi^{(j-1)}$ *if* $s_{i_j}\pi^{(j-1)} > \pi(j-1)$.

4. $\pi^{(j)} = \pi^{(j-1)}$ *for exactly* b *values of* j.

5. $\pi^{(j)} > \pi^{(j-1)}$ *for exactly* a *values of* j.

Proof. Clearly the claim is true for $u = e$. Assume the claim to be true for u having length at most $\ell - 1$, fix a permutation u of length ℓ, and fix a reduced expression $s_{i_1} \cdots s_{i_\ell}$ for u. Then we have

$$x^{u,v} = \begin{cases} -q_2q_4x^{s_{i_2}\cdots s_{i_\ell}, s_{i_1}v} & \text{if } s_{i_1}v > v, \\ x^{s_{i_2}\cdots s_{i_\ell}, s_{i_1}v} + (q_2 + q_4)x^{s_{i_2}\cdots s_{i_\ell}, v} & \text{if } s_{i_1}v < v. \end{cases} \tag{4.2.17}$$

By induction, we see immediately that the coefficient of $x^{e,w}$ in the natural expansion of $x^{u,v}$ has the form $p_{u,v,w}(-q_2q_4, q_2+q_4)$ for some polynomial $p_{u,v,w}(q_0, q_1) \in \mathbb{N}[q_0, q_1]$. In particular,

$$p_{u,v,w}(q_0, q_1) = \begin{cases} q_0 \cdot p_{s_{i_2}\cdots s_{i_\ell}, s_{i_1}v, w}(q_0, q_1) & \text{if } s_{i_1}v > v, \\ p_{s_{i_2}\cdots s_{i_\ell}, s_{i_1}v, w}(q) + q_1 \cdot p_{s_{i_2}\cdots s_{i_\ell}, v, w}(q) & \text{if } s_{i_1}v < v. \end{cases} \tag{4.2.18}$$

Suppose first that we have $s_{i_1}v > v$. Then by induction, the coefficient of $q_0^aq_1^b$ in $p_{u,v,w}(q_0, q_1)$ is equal to the number of sequences $(\pi^{(1)} = s_{i_1}v, \pi^{(2)}, \ldots, \pi^{(\ell)} = w)$ satisfying Conditions (2)-(4) of the proposition and in which $\pi^{(j)} > \pi^{(j-1)}$ for exactly $a - 1$ values of j. Prepending the permutation $\pi^{(0)}$ to any such sequence and considering the inequality $s_{i_1}v > v$, we see that the new sequence

$$\pi \underset{\text{def}}{=} (\pi^{(0)}, \pi^{(1)}, \ldots, \pi^{(\ell)}) \tag{4.2.19}$$

satisfies all five conditions of the proposition if and only if $\pi^{(0)} = v$. (One new inequality $\pi^{(1)} > \pi^{(0)}$ is introduced.) Thus the coefficient of $q_0^aq_1^b$ in $p_{u,v,w}(q_0, q_1)$ is equal to the number of sequences $(\pi^{(0)}, \ldots, \pi^{(\ell)})$ satisfying Conditions (1)-(5) of the proposition.

Now suppose that we have $s_{i_1}v < v$. Then by induction, the coefficient of $q_0^a q_1^b$ in $p_{u,v,w}(q_0, q_1)$ is equal to the number of sequences $(\pi^{(1)} = s_{i_1}v, \pi^{(2)}, \ldots, \pi^{(\ell)} = w)$ satisfying Conditions (2)-(5) of the proposition, plus the number of sequences $(\pi^{(1)} = v, \pi^{(2)}, \ldots, \pi^{(\ell)} = w)$ satisfying Conditions (2),(3), and (5) of the proposition and in which $\pi^{(j)} = \pi^{(j-1)}$ for exactly $b-1$ values of j. Prepending a permutation $\pi^{(0)}$ to any such sequence, we see that the new sequence

$$\pi \underset{\text{def}}{=} (\pi^{(0)}, \pi^{(1)}, \ldots, \pi^{(\ell)}) \tag{4.2.20}$$

satisfies all five conditions of the proposition if and only if $\pi^{(0)} = v$. (No new equality $\pi^{(1)} = \pi^{(0)}$ is introduced for a sequence of the first form; one new such equality is introduced for a sequence of the second form.) Thus we see again that the coefficient of $q_0^a q_1^b$ in $p_{u,v,w}(q_0, q_1)$ is equal to the number of sequences $(\pi^{(0)}, \ldots, \pi^{(\ell)})$ satisfying Conditions (1)-(5) of the proposition. $\square$

From (4.2.6) we see that for permutations in $w \in \mathfrak{S}_3$, we have

$$p_{w_0,w_0,w}(q_0, q_1) = \begin{cases} 1 & \text{if } \ell(w) = 0, \\ q_1 & \text{if } \ell(w) = 1, \\ q_1^2 & \text{if } \ell(w) = 2, \\ q_1^3 + q_0 q_1 & \text{if } \ell(w) = 3. \end{cases} \tag{4.2.21}$$

Corollary 4.2.3. *For all $u, v, w \in \mathfrak{S}_n$ we have*

$$p_{v,u,w^{-1}}(-q_2q_4, q_2 + q_4) = (-q_2q_4)^{\ell(v)-\ell(u)} p_{u,v,w}(-q_2q_4, q_2 + q_4). \tag{4.2.22}$$

Proof. Applying the dagger operation to both sides of (4.2.16) gives

$$\begin{aligned} (-q_2q_4)^{\ell(u)-\ell(v)} x^{v,u} &= \sum_{w \geq u^{-1}v} p_{u,v,w}(-q_2q_4, q_2 + q_4)(-q_2q_4)^{-\ell(w)} x^{w,e} \\ &= \sum_{w \geq u^{-1}v} p_{u,v,w}(-q_2q_4, q_2 + q_4) x^{e,w^{-1}}. \end{aligned} \tag{4.2.23}$$

Thus we have

$$x^{v,u} = \sum_{w^{-1} \geq uv^{-1}} (-q_2q_4)^{\ell(v)-\ell(u)} p_{u,v,w}(-q_2q_4, q_2 + q_4) x^{e,w^{-1}}. \tag{4.2.24}$$

Expanding the left-hand side using (4.2.16) and comparing terms, we see the claim is true. □

Previously, we defined two families of polynomials, the $p^{I,J}$- and $r^{I,J}$-polynomials. Corollary 4.2.3 suggests the two-parameter version of the r-polynomials should have the following definition. For all $u, v, w \in \mathfrak{S}_n$, given any reduced expression $s_{i_1} \cdots s_{i_k}$ for v, define the (*Laurent*) polynomials $\{r_{u,v,w}(q_0, q_1) \in \mathbb{N}[q_0, q_0^{-1}, q_1] \mid u, v, w \in \mathfrak{S}_n\}$ to be the polynomials whose coefficient of $q_0^a q_1^b$ is equal to the number of sequences $(\pi^{(0)}, \ldots, \pi^{(k)})$ of permutations satisfying

1. $\pi^{(0)} = u$, $(\pi^{(k)})^{-1} = w$,
2. $\pi^{(j)} \in \{s_{i_j}\pi^{(j-1)}, \pi^{(j-1)}\}$ for $j = 1, \ldots, k$,
3. $\pi^{(j)} = s_{i_j}\pi^{(j-1)}$ if $s_{i_j}\pi^{(j-1)} > \pi^{(j-1)}$,
4. $\pi^{(j)} = \pi^{(j-1)}$ for exactly b values of j,
5. $\pi^{(j)} > \pi^{(j-1)}$ for exactly a values of j.

Thus we see the following equation holds,

$$r_{u,v,w}(q_0, q_1) = p_{v,u,w^{-1}}(q_0, q_1) \tag{4.2.25}$$

and the two families of polynomials satisfy an analogous relation to (1.4.8). Furthermore, by Corollary 4.2.3, we can say

$$r_{u,v,w}(-q_2q_4, q_2 + q_4) = (-q_2q_4)^{\ell(v)-\ell(u)} p_{u,v,w}(-q_2q_4, q_2 + q_4), \tag{4.2.26}$$

as well as

$$x^{u,v} = \sum_{w \geq u^{-1}v} r_{u,v,w}(-q_2q_4, q_2 + q_4) x^{w^{-1},e}. \tag{4.2.27}$$

4.3 Connecting $H_n(q_2, q_4)$ and $\mathcal{A}_{[n],[n]}(n; q_2, q_4)$

In the one parameter case we defined actions of $H_n(q)$ on $\mathcal{A}(n;q)$ and used them to connect the multiplicative structure of $H_n(q)$ to the p-polynomials. Similarly, define a left action of $H_n(q_2, q_4)$ on $\mathcal{A}_{[n],[n]}(n; q_2, q_4)$ by

$$\widetilde{T}_{s_i} \circ f(x) = f(s_i x), \tag{4.3.1}$$

where s_i is the $n \times n$ defining matrix for s_i, and where we assume $f(x)$ to be expressed in terms of the natural basis. Similarly, define a right action of $H_n(q_2, q_4)$ on $\mathcal{A}_{[n],[n]}(n; q_2, q_4)$ by

$$f(x) \circ \widetilde{T}_{s_i} = f(x s_i), \tag{4.3.2}$$

where we assume $f(x)$ to be expressed in terms of the basis $\{x^{v,e} \mid v \in \mathfrak{S}_n\}$.

Formulas for these actions on monomials in standard form are

$$\widetilde{T}_{s_i} \circ x^{e,v} = x^{s_i,v} = \begin{cases} -q_2 q_4 x^{e,s_i v} & \text{if } s_i v > v, \\ x^{e,s_i v} + (q_2 + q_4) x^{e,v} & \text{if } s_i v < v, \end{cases} \tag{4.3.3}$$

$$\begin{aligned} x^{e,v} \circ \widetilde{T}_{s_i} &= (-q_2q_4)^{-\ell(v)} x^{v^{-1},e} \circ \widetilde{T}_{s_i} = (-q_2q_4)^{-\ell(v)} x^{v^{-1},s_i} \\ &= \begin{cases} (-q_2q_4)^{-\ell(v)-1} x^{s_i v^{-1},e} & \text{if } vs_i > v, \\ (-q_2q_4)^{-\ell(v)} x^{s_i v^{-1},e} & \\ \qquad +(-q_2q_4)^{-\ell(v)}(-q_2q_4)^{-1}(q_2+q_4) x^{v^{-1},e} & \text{if } vs_i < v, \end{cases} \\ &= \begin{cases} x^{e,vs_i} & \text{if } vs_i > v, \\ (-q_2q_4)^{-1} x^{e,vs_i} + (-q_2q_4)^{-1}(q_2+q_4) x^{e,v} & \text{if } vs_i < v, \end{cases} \end{aligned} \tag{4.3.4}$$

With a bit more work, we obtain the following formulas describing the action on monomials of the form $x^{u,v}$ not necessarily belonging to the natural basis.

Proposition 4.3.1. *We have*

$$\widetilde{T}_{s_j} \circ x^{u,v} = \begin{cases} x^{us_j,v} & \text{if } us_j > u, \\ -q_2q_4 x^{us_j,v} + (q_2+q_4) x^{u,v} & \text{if } us_j < u. \end{cases} \tag{4.3.5}$$

$$x^{u,v} \circ \widetilde{T}_{s_j} = \begin{cases} x^{u,vs_j} & \textit{if } vs_j > v, \\ (-q_2q_4)^{-1}x^{u,vs_j} + (-q_2q_4)^{-1}(q_2+q_4)x^{u,v} & \textit{if } vs_j < v. \end{cases} \tag{4.3.6}$$

Proof. Assume the formula (4.3.5) to hold for all monomials $x^{u,v}$ with $\ell(u) < k$. Certainly, this is true if $\ell(u) = 0$. If $\ell(u) = 1$, then u is just a single adjacent transposition s_i. We can restate (4.3.5) as

$$\widetilde{T}_{s_j} \circ x^{s_i,v} = \begin{cases} x^{s_is_j,v} & \text{if } s_i \neq s_j, \\ -q_2q_4x^{e,v} + (q_2+q_4)x^{s_i,v} & \text{if } s_i = s_j. \end{cases} \tag{4.3.7}$$

First assume $s_i = s_j$. Then we have by (4.2.5)

$$\begin{aligned} \widetilde{T}_{s_j} \circ x^{s_i,v} &= \begin{cases} \widetilde{T}_{s_i} \circ (-q_2q_4)x^{e,s_iv} & \text{if } s_iv > v, \\ \widetilde{T}_{s_i} \circ (x^{e,s_iv} + (q_2+q_4)x^{e,v}) & \text{if } s_iv < v, \end{cases} \\ &= \begin{cases} -q_2q_4x^{s_i,s_iv} & \text{if } s_iv > v, \\ x^{s_i,s_iv} + (q_2+q_4)x^{s_i,v} & \text{if } s_iv < v. \end{cases} \end{aligned} \tag{4.3.8}$$

By (4.2.5), when $s_iv > v$ we have

$$-q_2q_4x^{s_i,s_iv} = -q_2q_4x^{e,v} + -q_2q_4(q_2+q_4)x^{e,s_iv} = -q_2q_4x^{e,v} + (q_2+q_4)x^{s_i,v}. \tag{4.3.9}$$

Similarly, when $s_iv < v$ we have

$$x^{s_i,s_iv} + (q_2+q_4)x^{s_i,v} = -q_2q_4x^{e,v} + (q_2+q_4)x^{s_i,v}. \tag{4.3.10}$$

Thus when $s_i = s_j$,

$$\widetilde{T}_{s_j} \circ x^{s_i,v} = -q_2q_4x^{e,v} + (q_2+q_4)x^{s_i,v}, \tag{4.3.11}$$

as claimed. Next assume $s_i \neq s_j$, then by similar steps as before we have

$$\begin{aligned} \widetilde{T}_{s_j} \circ x^{s_i,v} &= \begin{cases} \widetilde{T}_{s_j} \circ (-q_2q_4)x^{e,s_iv} & \text{if } s_iv > v, \\ \widetilde{T}_{s_j} \circ (x^{e,s_iv} + (q_2+q_4)x^{e,v}) & \text{if } s_iv < v, \end{cases} \\ &= \begin{cases} -q_2q_4x^{s_j,s_iv} & \text{if } s_iv > v, \\ x^{s_j,s_iv} + (q_2+q_4)x^{s_j,v} & \text{if } s_iv < v. \end{cases} \end{aligned} \tag{4.3.12}$$

Since $s_i s_j > s_i$, (4.2.5) tells us that when $s_i v > v$ we have

$$-q_2 q_4 x^{s_j, s_i v} = -q_2 q_4 (-q_2 q_4)^{-1} x^{s_i s_j, v} = x^{s_i s_j, v}. \tag{4.3.13}$$

Similarly, when $s_i v < v$ we have

$$x^{s_j, s_i v} + (q_2 + q_4) x^{s_j, v} = x^{s_i s_j, v} - (q_2 + q_4) x^{s_j, v} + (q_2 + q_4) x^{s_j, v} = x^{s_i s_j, v}. \tag{4.3.14}$$

Again we have the claimed formula.

Now fix one permutation u of length k, and let s_i be a left descent for u. By (4.2.5) we have

$$\widetilde{T}_{s_j} \circ x^{u,v} = \begin{cases} \widetilde{T}_{s_j} \circ (-q_2 q_4) x^{s_i u, s_i v} & \text{if } s_i v > v, \\ \widetilde{T}_{s_j} \circ x^{s_i u, s_i v} + \widetilde{T}_{s_j} \circ (q_2 + q_4) x^{s_i u, v} & \text{if } s_i v < v, \end{cases} \tag{4.3.15}$$

which by induction is equal to

$$\begin{cases} -q_2 q_4 x^{s_i u s_j, s_i v} & \text{if } s_i u s_j > s_i u \text{ and } s_i v > v, \\ (-q_2 q_4)^2 x^{s_i u s_j, s_i v} + (-q_2 q_4)(q_2 + q_4) x^{s_i u, s_i v} & \text{if } s_i u s_j < s_i u \text{ and } s_i v > v, \\ x^{s_i u s_j, s_i v} + (q_2 + q_4) x^{s_i u s_j, v} & \text{if } s_i u s_j > s_i u \text{ and } s_i v < v, \\ -q_2 q_4 x^{s_i u s_j, s_i v} + (q_2 + q_4) x^{s_i u, s_i v} & \\ \qquad + (-q_2 q_4)(q_2 + q_4) x^{s_i u s_j, v} + (q_2 + q_4)^2 x^{s_i u, v} & \text{if } s_i u s_j < s_i u \text{ and } s_i v < v. \end{cases} \tag{4.3.16}$$

Now we return to the right-hand side of (4.3.5). Suppose first that $u s_j > u$. This implies that $s_i u < s_i u s_j < u s_j$. By (4.2.5) we then have

$$x^{u s_j, v} = \begin{cases} -q_2 q_4 x^{s_i u s_j, s_i v} & \text{if } s_i v > v, \\ x^{s_i u s_j, s_i v} + (q_2 + q_4) x^{s_i u s_j, v} & \text{if } s_i v < v, \end{cases} \tag{4.3.17}$$

which is equal to $\widetilde{T}_{s_j} \circ x^{u,v}$ by cases 1 and 3 of (4.3.16). Now suppose that $u s_j < u$. Then we have $u > s_i u s_j$ or $u = s_i u s_j$. If $u = s_i u s_j$, then $u s_j = s_i u < u = s_i u s_j$. Applying (4.2.5) to the first monomial in

$$-q_2 q_4 x^{u s_j, v} + (q_2 + q_4) x^{u,v} = -q_2 q_4 x^{u s_j, v} + (q_2 + q_4) x^{s_i u s_j, v}, \tag{4.3.18}$$

we again obtain the expressions on the right-hand side of (4.3.17). If $u > s_i u s_j$, then $s_i u < u$ and $s_i u s_j < u s_j$. By (4.2.5) we then have

$$
\begin{aligned}
&-q_2q_4x^{us_j,v} + (q_2+q_4)x^{u,v} \\
&\qquad = \begin{cases} (-q_2q_4)^2x^{s_ius_j,s_iv} + (-q_2q_4)(q_2+q_4)x^{s_iu,s_iv} & \text{if } s_iv > v, \\ -q_2q_4x^{s_ius_j,s_iv} + (-q_2q_4)(q_2+q_4)x^{s_ius_j,v} & \\ \quad +(q_2+q_4)x^{s_iu,s_iv} + (q_2+q_4)^2x^{s_iu,v} & \text{if } s_iv < v, \end{cases}
\end{aligned} \tag{4.3.19}
$$

which is equal to $\widetilde{T}_{s_j} \circ x^{u,v}$ by cases 2 and 4 of (4.3.16).

The formula for the right action is proved in much the same way. Unlike in the one parameter version we cannot just use the transpose operation to simplify the proof.

Assume the formula (4.3.6) to hold for all monomials $x^{u,v}$ with $\ell(v) < k$. Certainly, this is true if $\ell(v) = 0$. If $\ell(v) = 1$, then v is just a single adjacent transposition s_i. We can restate (4.3.6) as

$$
x^{u,s_i} \circ \widetilde{T}_{s_j} = \begin{cases} x^{u,s_is_j} & \text{if } s_i \neq s_j, \\ (-q_2q_4)^{-1}x^{u,e} + (-q_2q_4)^{-1}(q_2+q_4)x^{u,s_i} & \text{if } s_i = s_j. \end{cases} \tag{4.3.20}
$$

First assume $s_i = s_j$. Then we have by (4.2.5)

$$
\begin{aligned}
x^{u,s_i} \circ \widetilde{T}_{s_j} &= \begin{cases} (-q_2q_4)^{-1}x^{s_iu,e} \circ \widetilde{T}_{s_j} & \text{if } s_iu > u, \\ (x^{s_iu,e} + (-q_2q_4)^{-1}(q_2+q_4)x^{u,e}) \circ \widetilde{T}_{s_j} & \text{if } s_iu < u, \end{cases} \\
&= \begin{cases} (-q_2q_4)^{-1}x^{s_iu,s_i} & \text{if } s_iu > u, \\ x^{s_iu,s_i} + (-q_2q_4)^{-1}(q_2+q_4)x^{u,s_i} & \text{if } s_iu < u. \end{cases}
\end{aligned} \tag{4.3.21}
$$

By (4.2.5), we have

$$
x^{u,s_i} \circ \widetilde{T}_{s_j} = (-q_2q_4)^{-1}x^{u,e} + (-q_2q_4)^{-1}(q_2+q_4)x^{u,s_i} \tag{4.3.22}
$$

as claimed. Next assume $s_i \neq s_j$, then by similar steps as before we have

$$
x^{u,s_i} \circ \widetilde{T}_{s_j} = \begin{cases} (-q_2q_4)^{-1}x^{s_iu,s_j} & \text{if } s_iu > u, \\ x^{s_iu,s_j} + (-q_2q_4)^{-1}(q_2+q_4)x^{u,s_j} & \text{if } s_iu < u. \end{cases} \tag{4.3.23}
$$

By (4.2.5), we have

$$x^{u,s_i} \circ \widetilde{T}_{s_j} = x^{u,s_is_j} \tag{4.3.24}$$

as claimed.

Now fix one permutation v of length k, and let s_i be a left descent for v. By (4.2.5) we have

$$x^{u,v} \circ \widetilde{T}_{s_j} = \begin{cases} (-q_2q_4)^{-1} x^{s_iu,s_iv} \circ \widetilde{T}_{s_j} & \text{if } s_iu > u, \\ (x^{s_iu,s_iv} + (q_2+q_4)x^{s_iu,v}) \circ \widetilde{T}_{s_j} & \text{if } s_iu < u, \end{cases} \tag{4.3.25}$$

which by (4.2.5) and induction is equal to

$$\begin{cases} (-q_2q_4)^{-1} x^{s_iu,s_iv} \circ \widetilde{T}_{s_j} & \text{if } s_iu > u, \\ (x^{s_iu,s_iv} + (-q_2q_4)^{-1}(q_2+q_4)x^{u,s_iv}) \circ \widetilde{T}_{s_j} & \text{if } s_iu < u, \end{cases}$$

$$= \begin{cases} (-q_2q_4)^{-1} x^{s_iu,s_ivs_j} & \text{if } s_iu > u \text{ and } s_ivs_j > s_iv, \\ (-q_2q_4)^{-2} \left(x^{s_iu,s_ivs_j} + (q_2+q_4)x^{s_iu,s_iv}\right) & \text{if } s_iu > u \text{ and } s_ivs_j < s_iv, \\ x^{s_iu,s_ivs_j} + (-q_2q_4)^{-1}(q_2+q_4)x^{u,s_ivs_j} & \text{if } s_iu < u \text{ and } s_ivs_j > s_iv, \\ (-q_2q_4)^{-1}x^{s_iu,s_ivs_j} + (-q_2q_4)^{-1}(q_2+q_4)x^{s_iu,s_iv} & \\ \qquad +(-q_2q_4)^{-2}(q_2+q_4)x^{u,s_ivs_j} & \\ \qquad\qquad +(-q_2q_4)^{-2}(q_2+q_4)^2x^{u,s_iv} & \text{if } s_iu < u \text{ and } s_ivs_j < s_iv. \end{cases} \tag{4.3.26}$$

Now we return to the right-hand side of (4.3.6). Suppose first that $vs_j > v$. This implies that $s_iv < s_ivs_j < vs_j$. By (4.2.5) we then have

$$x^{u,vs_j} = \begin{cases} (-q_2q_4)^{-1} x^{s_iu,s_ivs_j} & \text{if } s_iu > u, \\ x^{s_iu,s_ivs_j} + (q_2+q_4)x^{s_iu,vs_j} & \text{if } s_iu < u, \end{cases} \tag{4.3.27}$$

which, by (4.2.5), is equal to $x^{u,v} \circ \widetilde{T}_{s_j}$ by cases 1 and 3 of (4.3.26). Now suppose that $vs_j < v$. Then we have $v > s_ivs_j$ or $v = s_ivs_j$. If $v = s_ivs_j$, then $vs_j = s_iv < v = s_ivs_j$. Applying (4.2.5) to the first monomial in

$$(-q_2q_4)^{-1}x^{u,vs_j} + (-q_2q_4)^{-1}(q_2+q_4)x^{u,v} = (-q_2q_4)^{-1}x^{u,vs_j} + (-q_2q_4)^{-1}(q_2+q_4)x^{u,s_ivs_j} \tag{4.3.28}$$

and using (4.2.5), we again obtain the expressions on the right-hand side of (4.3.27). If $v > s_i v s_j$, then $s_i v < v$ and $s_i v s_j < v s_j$. By (4.2.5) we then have

$$\begin{aligned}
&(-q_2q_4)^{-1}x^{u,vs_j} + (-q_2q_4)^{-1}(q_2+q_4)x^{u,v} \\
&\quad = \begin{cases} (-q_2q_4)^{-2}x^{s_iu,s_ivs_j} + (-q_2q_4)^{-2}(q_2+q_4)x^{s_iu,s_iv} & \text{if } s_iu > u, \\ (-q_2q_4)^{-1}x^{s_iu,s_ivs_j} + (-q_2q_4)^{-1}(q_2+q_4)x^{s_iu,vs_j} & \\ \quad +(-q_2q_4)^{-1}(q_2+q_4)x^{s_iu,s_iv} + (-q_2q_4)^{-1}(q_2+q_4)^2x^{s_iu,v} & \text{if } s_iu < u, \end{cases}
\end{aligned} \tag{4.3.29}$$

which, by (4.2.5), is equal to $x^{u,v} \circ \widetilde{T}_{s_j}$ by cases 2 and 4 of (4.3.26). □

Unlike the one-parameter case, we no longer have a nice connection between the right action and the transpose operation. In fact

$$\begin{aligned}
\left(\widetilde{T}_{s_j} \circ (x^{u,v})^\top\right)^\top &= \left(\widetilde{T}_{s_j} \circ x^{v,u}\right)^\top \\
&= \begin{cases} (x^{vs_j,u})^\top & \text{if } vs_j > v, \\ -q_2q_4(x^{vs_j,u})^\top + (q_2+q_4)(x^{v,u})^\top & \text{if } vs_j < v, \end{cases} \\
&= \begin{cases} x^{u,vs_j} & \text{if } vs_j > v, \\ -q_2q_4x^{u,vs_j} + (q_2+q_4)x^{u,v} & \text{if } vs_j < v. \end{cases}
\end{aligned} \tag{4.3.30}$$

Looking at Proposition 4.3.1 we can clearly see this is not the same as $x^{u,v} \circ \widetilde{T}_{s_j}$.

When we try using the modified transpose (dagger) operation we get

$$\begin{aligned}
\left(\widetilde{T}_{s_j}\circ(x^{u,v})^\dagger\right)^\dagger &= \left(\widetilde{T}_{s_j}\circ(-q_2q_4)^{\ell(u)-\ell(v)}x^{v,u}\right)^\dagger \\
&= \begin{cases} (-q_2q_4)^{\ell(u)-\ell(v)}(x^{vs_j,u})^\dagger & \text{if } vs_j > v, \\ (-q_2q_4)^{\ell(u)-\ell(v)+1}(x^{vs_j,u})^\dagger & \\ \quad +(-q_2q_4)^{\ell(u)-\ell(v)}(q_2+q_4)(x^{v,u})^\dagger & \text{if } vs_j < v, \end{cases} \\
&= \begin{cases} (-q_2q_4)^{\ell(u)-\ell(v)+\ell(vs_j)-\ell(u)}x^{u,vs_j} & \text{if } vs_j > v, \\ (-q_2q_4)^{\ell(u)-\ell(v)+1+\ell(vs_j)-\ell(u)}x^{u,vs_j} & \\ \quad +(-q_2q_4)^{\ell(u)-\ell(v)+\ell(v)-\ell(u)}(q_2+q_4)x^{u,v} & \text{if } vs_j < v, \end{cases} \\
&= \begin{cases} -q_2q_4x^{u,vs_j} & \text{if } vs_j > v, \\ x^{u,vs_j}+(q_2+q_4)x^{u,v} & \text{if } vs_j < v. \end{cases}
\end{aligned} \tag{4.3.31}$$

Again, this is not the same as $x^{u,v}\circ\widetilde{T}_{s_j}$. If you apply (4.2.5) to (4.3.5) in the case where $u = e$, then you get something very similar to this formula.

Using Proposition 4.3.1 we can see that

$$\begin{aligned}
\widetilde{T}_v\circ x^{e,s_i} &= \begin{cases} (-q_2q_4)^{\ell(v)}x^{e,vs_i} & \text{if } vs_i > v, \\ (-q_2q_4)^{\ell(v)}\left((-q_2q_4)^{-1}x^{e,vs_i}+(-q_2q_4)^{-1}(q_2+q_4)x^{e,v}\right) & \text{if } vs_i < v, \end{cases} \\
x^{e,s_i}\circ\widetilde{T}_v &= \begin{cases} x^{e,s_iv} & \text{if } s_iv > v, \\ (-q_2q_4)^{-1}x^{e,s_iv}+(-q_2q_4)^{-1}(q_2+q_4)x^{e,v} & \text{if } s_iv < v. \end{cases}
\end{aligned} \tag{4.3.32}$$

Furthermore we have

$$x^{e,v} = (-q_2q_4)^{-\ell(v)}\widetilde{T}_v\circ x^{e,e} = x^{e,e}\circ\widetilde{T}_v \tag{4.3.33}$$

and

$$x^{u,e} = \widetilde{T}_{u^{-1}}\circ x^{e,e} = (-q_2q_4)^{\ell(u)}x^{e,e,}\circ\widetilde{T}_{u^{-1}}. \tag{4.3.34}$$

More generally, we can express any monomial $x^{u,v}$ as

$$\begin{aligned}
x^{u,v} &= \widetilde{T}_{u^{-1}}\circ x^{e,v} = x^{u,e}\circ\widetilde{T}_v = \widetilde{T}_{u^{-1}}\circ x^{e,e}\circ\widetilde{T}_v \\
&= (-q_2q_4)^{-\ell(v)}\widetilde{T}_{u^{-1}}\widetilde{T}_v\circ x^{e,e} = (-q_2q_4)^{\ell(u)}x^{e,e}\circ\widetilde{T}_{u^{-1}}\widetilde{T}_v.
\end{aligned} \tag{4.3.35}$$

Thus we have that

$$\widetilde{T}_{u^{-1}}\widetilde{T}_v \circ x^{e,e} = (-q_2q_4)^{\ell(u)+\ell(v)} x^{e,e} \circ \widetilde{T}_{u^{-1}}\widetilde{T}_v. \tag{4.3.36}$$

Therefore we have

$$\mathcal{A}_{[n],[n]}(n; q_2, q_4) = H_n(q_2, q_4) \circ x^{e,e} = x^{e,e} \circ H_n(q_2, q_4). \tag{4.3.37}$$

We have shown that the two spaces are connected in a similar manner to the one-parameter case; however, the right action defined as such is no longer nicely connected to the left action. To this end let us define an alternate right action of $H_n(q_2, q_4)$ on $\mathcal{A}_{[n],[n]}(n; q_2, q_4)$ by

$$f(x) \triangleleft \widetilde{T}_{s_i} = -q_2q_4 f(xs_i), \tag{4.3.38}$$

where $f(x)$ is expressed in terms of the basis $\{x^{v,e} \mid v \in \mathfrak{S}_n\}$. This action will reduce to the standard action in the case where $-q_2q_4 = 1$, as in $H_n(q)$ and $\mathcal{A}(n; q)$. Thus it is a generalization of the previous action defined in Chapter 2.

Using similar methods as before we see that we have

$$x^{e,v} \triangleleft \widetilde{T}_{s_i} = \begin{cases} -q_2q_4 x^{e,vs_i} & \text{if } vs_i > v, \\ x^{e,vs_i} + (q_2+q_4)x^{e,v} & \text{if } vs_i < v. \end{cases} \tag{4.3.39}$$

This recursive formula can be generalized in the following manner.

Proposition 4.3.2. *We have*

$$x^{u,v} \triangleleft \widetilde{T}_{s_j} = \begin{cases} -q_2q_4 x^{u,vs_j} & \textit{if } vs_j > v, \\ x^{u,vs_j} + (q_2+q_4)x^{u,v} & \textit{if } vs_j < v. \end{cases} \tag{4.3.40}$$

Proof. Assume the formula (4.3.40) to hold for all monomials $x^{u,v}$ with $\ell(v) < k$. Certainly, this is true if $\ell(v) = 0$. If $\ell(v) = 1$, then v is just a single adjacent transposition s_i. We can restate (4.3.40) as

$$x^{u,s_i} \triangleleft \widetilde{T}_{s_j} = \begin{cases} -q_2q_4 x^{u,s_is_j} & \text{if } s_i \neq s_j, \\ x^{u,e} + (q_2+q_4)x^{u,s_i} & \text{if } s_i = s_j. \end{cases} \tag{4.3.41}$$

First assume $s_i = s_j$. Then we have by (4.2.5)

$$x^{u,s_i} \lhd \widetilde{T}_{s_j} = \begin{cases} (-q_2q_4)^{-1}x^{s_iu,e} \lhd \widetilde{T}_{s_j} & \text{if } s_iu > u, \\ (x^{s_iu,e} + (-q_2q_4)^{-1}(q_2+q_4)x^{u,e}) \lhd \widetilde{T}_{s_j} & \text{if } s_iu < u, \end{cases}$$
$$= \begin{cases} x^{s_iu,s_i} & \text{if } s_iu > u, \\ -q_2q_4x^{s_iu,s_i} + (q_2+q_4)x^{u,s_i} & \text{if } s_iu < u. \end{cases} \tag{4.3.42}$$

By (4.2.5), we have

$$x^{u,s_i} \lhd \widetilde{T}_{s_j} = x^{u,e} + (q_2+q_4)x^{u,s_i} \tag{4.3.43}$$

as claimed. Next assume $s_i \neq s_j$, then by similar steps as before we have

$$x^{u,s_i} \lhd \widetilde{T}_{s_j} = \begin{cases} x^{s_iu,s_j} & \text{if } s_iu > u, \\ -q_2q_4x^{s_iu,s_j} + (q_2+q_4)x^{u,s_j} & \text{if } s_iu < u. \end{cases} \tag{4.3.44}$$

By (4.2.5), we have

$$x^{u,s_i} \lhd \widetilde{T}_{s_j} = -q_2q_4x^{u,s_is_j} \tag{4.3.45}$$

as claimed.

Now fix one permutation v of length k, and let s_i be a left descent for v. By (4.2.5) we have

$$x^{u,v} \lhd \widetilde{T}_{s_j} = \begin{cases} (-q_2q_4)^{-1}x^{s_iu,s_iv} \lhd \widetilde{T}_{s_j} & \text{if } s_iu > u, \\ (x^{s_iu,s_iv} + (q_2+q_4)x^{s_iu,v}) \lhd \widetilde{T}_{s_j} & \text{if } s_iu < u, \end{cases} \tag{4.3.46}$$

which by (4.2.5) and induction is equal to

$$\begin{cases}(-q_2q_4)^{-1}x^{s_iu,s_iv}\lhd\widetilde{T}_{s_j} & \text{if } s_iu>u,\\ (x^{s_iu,s_iv}+(-q_2q_4)^{-1}(q_2+q_4)x^{u,s_iv})\lhd\widetilde{T}_{s_j} & \text{if } s_iu<u,\end{cases}$$

$$=\begin{cases}x^{s_iu,s_ivs_j} & \text{if } s_iu>u \text{ and } s_ivs_j>s_iv,\\ (-q_2q_4)^{-1}\left(x^{s_iu,s_ivs_j}+(-q_2q_4)^{-1}(q_2+q_4)x^{s_iu,s_iv}\right) & \text{if } s_iu>u \text{ and } s_ivs_j<s_iv,\\ -q_2q_4x^{s_iu,s_ivs_j}+(q_2+q_4)x^{u,s_ivs_j} & \text{if } s_iu<u \text{ and } s_ivs_j>s_iv,\\ x^{s_iu,s_ivs_j}+(q_2+q_4)x^{s_iu,s_iv} & \\ \qquad +(-q_2q_4)^{-1}(q_2+q_4)x^{u,s_ivs_j} & \\ \qquad\qquad +(-q_2q_4)^{-1}(q_2+q_4)^2x^{u,s_iv} & \text{if } s_iu<u \text{ and } s_ivs_j<s_iv.\end{cases} \tag{4.3.47}$$

Now we return to the right-hand side of (4.3.6). Suppose first that $vs_j>v$. This implies that $s_iv<s_ivs_j<vs_j$. By (4.2.5) we then have

$$-q_2q_4x^{u,vs_j}=\begin{cases}x^{s_iu,s_ivs_j} & \text{if } s_iu>u,\\ -q_2q_4x^{s_iu,s_ivs_j}+-q_2q_4(q_2+q_4)x^{s_iu,vs_j} & \text{if } s_iu<u,\end{cases} \tag{4.3.48}$$

which, by (4.2.5), is equal to $x^{u,v}\lhd\widetilde{T}_{s_j}$ by cases 1 and 3 of (4.3.47). Now suppose that $vs_j<v$. Then we have $v>s_ivs_j$ or $v=s_ivs_j$. If $v=s_ivs_j$, then $vs_j=s_iv<v=s_ivs_j$. Applying (4.2.5) to the first monomial in

$$x^{u,vs_j}+(q_2+q_4)x^{u,v}=x^{u,vs_j}+(q_2+q_4)x^{u,s_ivs_j} \tag{4.3.49}$$

and using (4.2.5), we again obtain the expressions on the right-hand side of (4.3.48). If $v>s_ivs_j$, then $s_iv<v$ and $s_ivs_j<vs_j$. By (4.2.5) we then have

$$\begin{aligned}&x^{u,vs_j}+(q_2+q_4)x^{u,v}\\ &\quad=\begin{cases}(-q_2q_4)^{-1}x^{s_iu,s_ivs_j}+(-q_2q_4)^{-1}(q_2+q_4)x^{s_iu,s_iv} & \text{if } s_iu>u,\\ x^{s_iu,s_ivs_j}+(q_2+q_4)x^{s_iu,vs_j} & \\ \qquad +(q_2+q_4)x^{s_iu,s_iv}+(q_2+q_4)^2x^{s_iu,v} & \text{if } s_iu<u,\end{cases}\end{aligned} \tag{4.3.50}$$

which, by (4.2.5), is equal to $x^{u,v}\lhd\widetilde{T}_{s_j}$ by cases 2 and 4 of (4.3.26). □

Furthermore, the action $\lhd$ defined in (4.3.38) is related to the dagger operation as follows

$$x^{u,v} \lhd \widetilde{T}_{s_j} = \left(\widetilde{T}_{s_j} \circ (x^{u,v})^{\dagger}\right)^{\dagger}. \tag{4.3.51}$$

Using the dagger operation, or Proposition 4.3.2, we can see that

$$x^{e,s_i} \lhd \widetilde{T}_v = \begin{cases} (-q_2q_4)^{\ell(v)} x^{e,s_iv} & \text{if } s_iv > v, \\ (-q_2q_4)^{\ell(v)} \left((-q_2q_4)^{-1}x^{e,s_iv} + (-q_2q_4)^{-1}(q_2+q_4)x^{e,v}\right) & \text{if } s_iv < v. \end{cases} \tag{4.3.52}$$

Moreover, we have

$$x^{e,v} = (-q_2q_4)^{-\ell(v)}\widetilde{T}_v \circ x^{e,e} = (-q_2q_4)^{-\ell(v)} x^{e,e} \lhd \widetilde{T}_v \tag{4.3.53}$$

and

$$x^{u,e} = \widetilde{T}_{u^{-1}} \circ x^{e,e} = x^{e,e,} \lhd \widetilde{T}_{u^{-1}}. \tag{4.3.54}$$

In general, we can express any monomial $x^{u,v}$ as

$$\begin{aligned} x^{u,v} &= \widetilde{T}_{u^{-1}} \circ x^{e,v} = (-q_2q_4)^{-\ell(v)} x^{u,e} \lhd \widetilde{T}_v = (-q_2q_4)^{-\ell(v)} \widetilde{T}_{u^{-1}} \circ x^{e,e} \lhd \widetilde{T}_v \\ &= (-q_2q_4)^{-\ell(v)} \widetilde{T}_{u^{-1}}\widetilde{T}_v \circ x^{e,e} = (-q_2q_4)^{-\ell(v)} x^{e,e} \lhd \widetilde{T}_{u^{-1}}\widetilde{T}_v. \end{aligned} \tag{4.3.55}$$

Thus we have that

$$\widetilde{T}_{u^{-1}}\widetilde{T}_v \circ x^{e,e} = \widetilde{T}_{u^{-1}} \circ x^{e,e} \lhd \widetilde{T}_v = x^{e,e} \lhd \widetilde{T}_{u^{-1}}\widetilde{T}_v, \tag{4.3.56}$$

and

$$\mathcal{A}_{[n],[n]}(n; q_2, q_4) = H_n(q_2, q_4) \circ x^{e,e} = x^{e,e} \lhd H_n(q_2, q_4). \tag{4.3.57}$$

All of this evidence suggests the $\lhd$ action may in fact be the correct two-parameter generalization of the action of $H_n(q)$ on $\mathcal{A}(n; q)$.

Now that we have a better understanding of $H_n(q_2, q_4)$ and $\mathcal{A}_{[n],[n]}(n; q_2, q_4)$ are related we can state a result analogous to Proposition .

Proposition 4.3.3. *For all $u, v \in \mathfrak{S}_n$ we have*

$$\widetilde{T}_{u^{-1}}\widetilde{T}_v = \sum_{w \in \mathfrak{S}_n} (-q_2q_4)^{\ell(v)-\ell(w)} p_{u,v,w}(-q_2q_4, q_2+q_4)\widetilde{T}_w. \tag{4.3.58}$$

Proof. Define $a_{u,v,w}$ by $\widetilde{T}_{u^{-1}}\widetilde{T}_v = \sum a_{u,v,w}\widetilde{T}_w$. Then

$$\begin{aligned}(-q_2q_4)^{\ell(v)}x^{u,v} &= \widetilde{T}_{u^{-1}}\widetilde{T}_v \circ x^{e,e} = \sum a_{u,v,w}\widetilde{T}_w \circ x^{e,e} \\ &= \sum a_{u,v,w}(-q_2q_4)^{\ell(w)}x^{e,w}.\end{aligned} \tag{4.3.59}$$

Expanding the left-hand side using (4.2.16) and collecting terms completes the proof. □

4.4 The bar involution on $H_n(q_2, q_4)$.

Just as for $H_n(q)$, we may define a bar-involution on $H_n(q_2, q_4)$, which will lead to two parameter versions of the modified R-polynomials. Furthermore, this will lead to generalizations of Kazhdan and Lusztig's work on bar-invariant bases and constructing representations of $H_n(q_2, q_4)$. We will focus our attention on the modified R-polynomials as before.

Define an involution on $H_n(q_2, q_4)$ by

$$\sum_{w\in\mathfrak{S}_n} a_w\widetilde{T}_w \mapsto \overline{\sum_{w\in\mathfrak{S}_n} a_w\widetilde{T}_w} = \sum_{w\in\mathfrak{S}_n} \overline{a_w}\cdot\overline{\widetilde{T}_w}, \tag{4.4.1}$$

where

$$\overline{q_2} = -q_4, \qquad \overline{q_4} = -q_2, \overline{\widetilde{T}_w} = (\widetilde{T}_w)^{-1}. \tag{4.4.2}$$

Once again the bar involution is an automorphism of $H_n(q_2, q_4)$ and we have

$$\overline{\widetilde{T}_{uv}} = \overline{\widetilde{T}_u}\cdot\overline{\widetilde{T}_v}, \qquad \text{if } \ell(uv) = \ell(u) + \ell(v). \tag{4.4.3}$$

Using this fact, (4.1.2), and induction we can express $\overline{\widetilde{T}_v}$ in terms of the natural basis in several ways,

$$\begin{aligned}\overline{\widetilde{T}_v} &\in (-q_2q_4)^{-\ell(v)}\left(\widetilde{T}_v + \sum_{u<v}\overline{\mathbb{N}[-q_2q_4, q_2+q_4]}\widetilde{T}_u\right), \\ \overline{\widetilde{T}_v} &\in (-q_2q_4)^{-\ell(v)}\left(\widetilde{T}_v + \sum_{u<v}\mathbb{Z}[q_2, q_4]\widetilde{T}_u\right).\end{aligned} \tag{4.4.4}$$

Using the first expression above we can define polynomials $\widetilde{R}_{u,v}(q_0, q_1)$ in $\mathbb{N}[q_0, q_1]$, which we'll call *modified R-polynomials*, by

$$\overline{\widetilde{T}_v} = (-q_2q_4)^{-\ell(v)} \sum_{u \le v} \overline{\widetilde{R}_{u,v}(-q_2q_4, q_2 + q_4)} \widetilde{T}_u. \tag{4.4.5}$$

The second expression suggests we can define two-parameter R-polynomials; however, in $H_n(q)$ we factored out a power of q. It is not clear which powers of q_2 and q_4 should be factored out to define the R-polynomials, as we would want them to be related to the bar-invariant basis of $H_n(q_2, q_4)$ in a manner analogous to their original definition. For our immediate purposes, we are interested in the modified R-polynomials and will leave this problem for another paper.

The modified R-polynomials in $\mathbb{N}[q_0, q_1]$ satisfy

1. $\widetilde{R}_{u,v}(q_0, q_1) = 0$ if $u \not\le v$.
2. $\widetilde{R}_{v,v}(q_0, q_1) = 1$ for all v.
3. For each left descent s of v we have

$$\widetilde{R}_{u,v}(q_0, q_1) = \begin{cases} \widetilde{R}_{su,sv}(q_0, q_1) & \text{if } su < u, \\ q_0\widetilde{R}_{su,sv}(q_0, q_1) + q_1\widetilde{R}_{u,sv}(q_0, q_1) & \text{otherwise.} \end{cases} \tag{4.4.6}$$

Using the above conditions we calculate for $u \le v$ in $\mathfrak{S}_3$, we have

$$\widetilde{R}_{u,v}(q_0, q_1) = \begin{cases} 1 & \text{if } \ell(v) - \ell(u) = 0, \\ q_1 & \text{if } \ell(v) - \ell(u) = 1, \\ q_1^2 & \text{if } \ell(v) - \ell(u) = 2, \\ q_1^3 + q_0q_1 & \text{if } \ell(v) - \ell(u) = 3. \end{cases} \tag{4.4.7}$$

Similarly, for each right descent s of v we have

$$\widetilde{R}_{u,v}(q_0, q_1) = \begin{cases} \widetilde{R}_{us,vs}(q_0, q_1) & \text{if } us < u, \\ q_0\widetilde{R}_{us,vs}(q_0, q_1) + q_1\widetilde{R}_{u,vs}(q_0, q_1) & \text{otherwise.} \end{cases} \tag{4.4.8}$$

On the other hand, we may fix a right ascent s of u and obtain

$$\widetilde{R}_{u,v}(q_0,q_1) = \begin{cases} \widetilde{R}_{us,vs}(q_0,q_1) & \text{if } vs > v, \\ q_0\widetilde{R}_{us,vs}(q_0,q_1) + q_1\widetilde{R}_{us,v}(q_0,q_1). \end{cases} \tag{4.4.9}$$

Or we may fix a left ascent s of u and obtain

$$\widetilde{R}_{u,v}(q_0,q_1) = \begin{cases} \widetilde{R}_{su,sv}(q_0,q_1) & \text{if } sv > v, \\ q_0\widetilde{R}_{su,sv}(q_0,q_1) + q_1\widetilde{R}_{su,v}(q_0,q_1). \end{cases} \tag{4.4.10}$$

From the recursive formulas above, one can verify that for $u \leq v$, $\widetilde{R}_{u,v}(q_0,q_1)$ is a monic polynomial of degree $\ell(v) - \ell(u)$ with constant term equal to zero, unless $u = v$. By (4.4.5) we have

$$\widetilde{R}_{u,v}(q_0,q_1) = \widetilde{R}_{w_0v,w_0u}(q_0,q_1) = \widetilde{R}_{vw_0,uw_0}(q_0,q_1) = \widetilde{R}_{u^{-1},v^{-1}}(q_0,q_1). \tag{4.4.11}$$

Furthermore, in a manner similar to the one-parameter case, using induction we can see that

$$\overline{\widetilde{R}_{u,v}(-q_2q_4, q_2+q_4)} = \epsilon_{u,v}\widetilde{R}_{u,v}(-q_2q_4, q_2+q_4). \tag{4.4.12}$$

4.5 The bar involution on $\mathcal{A}_{[n],[n]}(n;q_2,q_4)$.

Just as with the Hecke algebra, we can define a bar involution on $\mathcal{A}_{[n],[n]}(n;q_2,q_4)$, which will be analogous to the one-parameter case. Define the bar involution on $\mathcal{A}_{[n],[n]}(n;q_2,q_4)$ by,

$$\overline{x^{u,v}} = x^{w_0u,w_0v}. \tag{4.5.1}$$

Recall in chapter 3, we saw that the two bar involutions on $H_n(q)$ and $\mathcal{A}(n;q)$ were compatible, and this allowed us to connect the modified R- and inverse R-polynomials. We would like this to be the case in the two-parameter version as well, which we will see is almost true.

Proposition 4.5.1. *The two bar involutions are compatible with the left action of $H_n(q_2,q_4)$ on $\mathcal{A}_{[n],[n]}(n;q_2,q_4)$ in the sense that*

$$\overline{\widetilde{T}_{s_i} \circ x^{e,v}} = \overline{\widetilde{T}_{s_i}} \circ \overline{x^{e,v}}. \tag{4.5.2}$$

Proof. First look at the left-hand side and see

$$\overline{\widetilde{T}_{s_i} \circ x^{e,v}} = \overline{x^{s_i,v}} = x^{w_0 s_i, w_0 v}. \tag{4.5.3}$$

Next we look at the right-hand side and see

$$\begin{aligned}
\overline{\widetilde{T}_{s_i}} \circ \overline{x^{e,v}} &= (-q_2 q_4)^{-1} \left(\widetilde{T}_{s_i} - (q_2+q_4)\widetilde{T}_e\right) \circ x^{w_0, w_0 v} \\
&= (-q_2q_4)^{-1} \left(-q_2 q_4 x^{w_0 s_i, w_0 v} + (q_2+q_4) x^{w_0, w_0 v}\right) \\
&\quad - (-q_2q_4)^{-1}(q_2+q_4)x^{w_0,w_0v} \\
&= x^{w_0 s_i, w_0 v}.
\end{aligned} \tag{4.5.4}$$

□

The compatibility holds for the left action; however the two bar involutions are not compatible with the right actionl $\circ$. We see this in a similar manner. We have that

$$\overline{x^{e,v} \circ \widetilde{T}_{s_i}} = (-q_2q_4)^{-\ell(v)} x^{w_0 v^{-1}, w_0 s_i}, \tag{4.5.5}$$

while

$$\overline{x^{e,v}} \circ \overline{\widetilde{T}_{s_i}} =$$
$$(-q_2q_4)^{-\ell(v)} \left((-q_2q_4)^{-2} x^{w_0v^{-1}, w_0 s_i} - (-q_2q_4)^{-1}(q_2+q_4)((-q_2q_4)^{-1} - 1)x^{w_0 v^{-1}, w_0}\right). \tag{4.5.6}$$

Whenever $-q_2q_4 = 1$, these are the same (as in the one parameter case), but in general they are not.

On the other hand, the two bar involutions are almost compatible, in a sense, with the $\lhd$ right action.

Proposition 4.5.2. *The two bar involutions are compatible with the $\lhd$ right action of $H_n(q_2,q_4)$ on $\mathcal{A}_{[n],[n]}(n;q_2,q_4)$ in the sense that*

$$\overline{x^{e,v} \lhd \widetilde{T}_{s_i}} = (-q_2q_4)^2 \overline{x^{e,v}} \lhd \overline{\widetilde{T}_{s_i}}. \tag{4.5.7}$$

Proof. First look at the left-hand side and see

$$\overline{x^{e,v} \lhd \widetilde{T}_{s_i}} = \overline{(-q_2q_4)^{-\ell(v)+1} x^{v^{-1}, s_i}} = (-q_2q_4)^{-\ell(v)+1} x^{w_0 v^{-1}, w_0 s_i}. \tag{4.5.8}$$

Next we look at the right-hand side and see

$$\begin{aligned}\overline{x^{e,v}} \lhd \overline{\widetilde{T}_{s_i}} &= (-q_2q_4)^{-\ell(v)}x^{w_0v^{-1},w_0} \lhd (-q_2q_4)^{-1}\left(\widetilde{T}_{s_i} - (q_2+q_4)\widetilde{T}_e\right) \\ &= (-q_2q_4)^{-\ell(v)-1}\left(x^{w_0v^{-1},w_0s_i} + (q_2+q_4)x^{w_0v^{-1},w_0} - (q_2+q_4)x^{w_0v^{-1},w_0}\right) \\ &= (-q_2q_4)^{-\ell(v)-1}x^{w_0s_i,w_0v}.\end{aligned} \tag{4.5.9}$$

□

This provides more evidence that the $\lhd$ action is the correct generalized right action.

As before we define the *modified S-polynomials* by

$$\widetilde{S}_{v,w}(-q_2q_4, q_2+q_4) = p_{w_0,w_0v,w}(-q_2q_4, q_2+q_4). \tag{4.5.10}$$

Then by (4.2.16) we once again have

$$\overline{x^{e,v}} = \sum_{w \geq v} \widetilde{S}_{v,w}(-q_2q_4, q_2+q_4)x^{e,w}. \tag{4.5.11}$$

Furthermore, we can see from the combinatorial interpretation that

$$p_{u,v,w}(q_0, -q_1) = \epsilon_{e,u}\epsilon_{v,w}p_{u,v,w}(q_0, q_1), \tag{4.5.12}$$

which implies

$$\begin{aligned}\overline{\widetilde{S}_{v,w}(-q_2q_4, q_2+q_4)} &= \widetilde{S}_{v,w}\left(-q_2q_4, -(q_2+q_4)\right) \\ &= \epsilon_{e,w_0}\epsilon_{w_0v,w}\widetilde{S}_{v,w}(-q_2q_4, q_2+q_4) \\ &= \epsilon_{v,w}\widetilde{S}_{v,w}(-q_2q_4, q_2+q_4).\end{aligned} \tag{4.5.13}$$

In the single-parameter case we were able to eventually conclude that the modified S-polynomials were just the modified R-polynomials. This is not as easy to see in the two-parameter setting.

Problem 6. *Are the two-parameter modified S-polynomials equal to the modified R-polynomials, or are there certain conditions on the variables q_0 and q_1 for which they are equal?*

Chapter 5

Conclusion

In this paper we have introduced a new family of polynomials, defined combinatorially in terms of walks in the Bruhat order. These polynomials were shown to be transition matrix entries for the natural basis and inverse transpose bases within $\mathcal{A}(n;q)$. We then showed that these polynomials turn out to be a superset of the family of modified R-polynomials which have been studied by Brenti, Deodhar, and Dyer. Moreover, we introduced new double parabolic versions of the R- and modified R-polynomials, as wells as inverse R- and modified R-polynomials. These polynomials were shown to be linear combinations of ordinary ones and were used to develop a new formulation for the dual canonical basis of $\mathcal{A}(n;q)$. Due to the lack of consistency in the literature, we surveyed the different definitions which appear for the dual canonical basis and connected them to our results.

The new family of polynomials inherited symmetries from the R-polynomials. We were able to identify many more symmetries by connecting them to the multiplicative structure of $H_n(q)$ and by taking advantage of their role in the natural basis and inverse transpose basis expansions. These symmetries imply many combinatorial results. We were able to provide a bijective proof of one such symmetry.

Finally, we looked at two-parameter versions of $H_n(q)$ and $\mathcal{A}(n;q)$, defining two-parameter versions of the modified R-polynomials as well as our new polynomials for the nonparabolic case. We would like to prove results similar to those for the one-parameter case. One such result would be to find a relationship between the two-parameter inverse modified R-polynomials and the two-parameter modified R-polynomials. We used recursive formulas in the one-parameter situation to establish the link; however, in the case the recursive formulas are not exactly the same, but related. It seems probable there is a relation which is similar, yet different.

Several other open problems were mentioned throughout the paper. We saw that the formulation for the dual canonical basis in the double parabolic case required a two step process. It would be interesting to find an involution which fixed $\mathrm{Imm}_v(x_{L,M})$. Du's involution appeared to be close, but did not succeed. Such an involution may prove to have nice properties and be connected to the dual canonical basis. However, it is hard to imagine such an involution existing, since when we simplify the element $(x_{L,M})^{e,v}$ in terms of maximal elements, the length of the coset comes into play. Seeing how this is not the same for all cosets, it would be difficult to define an involution which accounts for this properly.

Another open problem was to connect the double parabolic R-polynomials and the double parabolic inverse R-polynomials. We named them inverse R-polynomials because of the relationship which holds in the single and nonparabolic cases; however, we do not see this holding in the double parabolic case. Due to the summation results, one would think we would be able to connect the two, possibly with some appropriate factors of q floating around. This may just take some careful consideration and use of symmetries holding in the other cases.

In Chapter 2, we saw that the nonparabolic polynomials satisfied many symmetry identities. Yet as we moved to the single parabolic case the number of symmetries appeared to shrink. In the end we have yet to find a symmetry result for the double parabolic case. This may in fact mirror the lack of connection between the double parabolic R-polynomials and the inverse R-polynomials. Just because we have not found a symmetry, does not mean none exist. Possibly, if we could identify general conditions which give symmetries, we will be able to make progress on the R-polynomial problems.

One possibility would be to find bijections which give the symmetry results we currently know. It seems likely that once we have a better sense of the combinatorics involved in these paths on the Bruhat order, we might gain insight into the symmetries satisfied by the parabolic versions. This may be the most most promising way of approaching the problem. In the future, we hope to find bijective proofs of all the relations in (2.2.6) and use these to make progress toward resolving all of the previously mentioned problems. However, it is worth mentioning one other possible strategy. If we could find a connection between the multiplicative structure of the submodules $H'_{I,J}$ and our new family of polynomials analogous to the connections in the single and nonparabolic cases, this may provide a way of bypassing the need for combinatorial proofs of the symmetry results.

It is easy to see that there are many interesting problems which have sprung up from our results in this paper. While we have managed to generalize the R-polynomials and defined double parabolic versions, much work remains in order to understand all of the combinatorics encoded in these amazing polynomials.

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