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The older cluster replacement rule and the parallel replacement problem

Pinar Keles
Lehigh University

Keles, Pinar

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The Older Cluster Replacement Rule and the Parallel Replacement Problem

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Pinar Keles

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Chairperson of the Industrial and

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Contents

List of Tables

List of Figures

Abstract

The Parallel Replacement Problem (PRP) requires replacement schedules for situations where assets are economically interdependent and operate in parallel. Previous research has developed dynamic programming algorithms which have been shown to be efficient under two rules, including the Older Cluster Replacement Rule (OCRR) where older assets are replaced before newer assets. However, this rule requires strict assumptions concerning the lifetime operating costs and salvage values of assets. In this thesis, we show that the OCRR is valid in many situations under the assumptions of homogeneous assets and stationary costs. The results are dependent on the economic life of a cluster, or optimal time to replace a cluster given that the decision is independent of other clusters. We also investigate various aspects of economic life and its implications in the PRP.

Keywords: Equipment replacement, parallel replacement analysis, economic life, older cluster replacement rule, dynamic programming

Chapter 1

Introduction

This thesis considers the parallel replacement problem (PRP) first presented in Jones et al.[6]. The PRP seeks the optimal replacement policy for a population of machines whose total number remains constant over some horizon. That is, a keep or replace decision is required for each asset in each period over a finite or infinite horizon such that total costs are minimized. The problem with multiple assets is interesting when the assets are economically interdependent. In this PRP, it is assumed that a fixed cost is incurred in each period in which a replacement occurs, regardless of the number of assets replaced. Thus, the replacement of one asset cannot be determined independently from others.

Equipment replacement problems are generally motivated by the fact that assets become more costly to operate as they get older and their salvage values decrease with time. Thus, there may be some point in time where the owner can save money by replacing the equipment with new equipment. It is also possible that the new equipment available is technologically superior and more efficient, providing further reason to replace the current equipment.

The importance of equipment replacement cannot be understated as the costs of owning and operating equipment can be very expensive. This importance is furthered in PRP problems as there are a multitude of assets. Managing the equipment over its lifetime is an important function for any business entity.

A variety of parallel replacement problems have been studied in the literature. Jones et al. [6] examine PRP under economies of scale (fixed charge) in the purchase price with dynamic programming. They prove two very important rules for this problem: the (1) No-Splitting Rule (NSR) and the (2) Older Cluster Replacement Rule (NSR). With these two rules, they reduce the amount of computation required to identify an optimal replacement policy.

Tang and Tang [8] provide another important rule for the PRP under economies of scale termed the All Or None Rule (AONR). They and Hopp et al. [5] also discuss the differences in assumptions between OCRR and AONR.

Chen [3] reformulates the problem as a 0-1 integer program for the general finite horizon problem without any assumptions on problem parameters and solves it using Benders' decomposition. He shows that the finite horizon case with general parameters and assumptions is equivalent to a shortest path problem.

A number of authors have expanded the PRP under economies of scale to include expansion decisions (non-decreasing demand). Rajagopalan [7] uses 0-1 integer programming to model both replacement and expansion decisions and solves it with Lagrangian relaxation. Chand et al. [1] examine this problem with dynamic programming and a heuristic algorithm. They show that the NSR and OCRR hold in this environment.

Hartman [4] furthers this work to consider fluctuating demand and capital budgeting constraints. He formulates the problem as a general integer program and proves both the NSR for non-decreasing demand and the One-Cluster-Splitting-Rule for general demand.

This paper is motivated by results and questions posed in Jones et al. [6] and later research. Specifically, we are interested in the OCRR rule. We show that the OCRR drastically reduces the computational complexity of the dynamic programming solution approach to PRP under economies of scale. Unfortunately, the assumptions required to prove that OCRR holds in the literature do not always hold in practice. Our goal is to relax these assumptions as it is our belief that OCRR holds in general conditions for homogeneous asset problems.

This thesis is laid out as follows. In Chapter 1, our problem statement, a brief review of OCRR, the studies about OCRR, and our research motivation is presented. In Chapter 2, the definition and importance of economic life of clusters and some implications are provided. In Chapter 3, results of Chapter 2 is applied to solve PRP, and we analyze the OCRR and illustrate when it is optimal without restrictive assumptions. Finally, conclusions and future studies are considered in Chapter 4.

1.1 Problem Description

Our problem, PRP, is defined as follows, We have a population of homogenous assets (n_i) of various ages (i) at time zero. Demand (d_j) for each period (j) is constant and each asset provides the same level of service (homogenous).

In this paper, both infinite- and finite-horizon problems are studied. All assets must be salvaged at the end of last period (T) in finite horizon problems. All assets must be replaced before or at their maximum age (N). Initial assets are either utilized or immediately salvaged. After each period, assets are either replaced or retained for another period. Moreover, an operating cost (c_{ij}) is paid for each period that assets are operated. These tend to increase with age i. Each asset's salvage value (r_{ij}) reduces or stays constant when it ages. In any period of a purchase, a unit purchase price p_j is paid along with a fixed cost k_j, regardless of the number of assets purchase.

The objective of the problem is to minimize the sum of discounted costs which are purchase, operation and maintenance, and less salvage values over the problem horizon. In this paper, the two important rules of Jones et. al.

1.2 Review of OCRR in Literature

The OCRR (Older Cluster Replacement Rule) is first defined by Jones et. al. [6]. The definition of OCRR is that it is never optimal, in the same time period, to sell a cluster of like-aged assets unless all older assets are also sold. According to Jones et. al. [6], the OCRR ensures that clusters of older machines are replaced at least as soon as clusters of young machines. To prove this, they made some mild assumptions:

a: For fixed $\in\{0,1,2,...\}$, $m_i(t)$ is non decreasing function of i;

b: For fixed $t \in\{0,1,2,...\}$, $s_i(t)$ is non increasing function of i, , where $s_0(t) = p(t)$;

c: For fixed $t \in\{0,1,2,...\}$, $s_i(t) + m_i(t)$ is non decreasing function of i;

where;

- $m_i(t)$=unit maintenance and operating cost of an i year old machine in year $t(i = 0, 1, ..n - 1)$
- $s_i(t)$ =unit salvage value of an i year old machine in year t (i=0,1,..n)
- $p(t)$ =unit purchase price of a new machine in year t.

With these assumptions they provide the following Theorem:

Older Cluster Replacement Rule: In any period of a finite or infinite horizon Parallel Machine Replacement Problem that satisfies assumptions (**a**), (**b**), and (**c**); it is optimal to replace a machine of age i only if all machines of age greater than age i are replaced $(i < n)$.

They illustrated some examples for the PRP with time invariant economics over an infinite horizon, however in these examples, OCRR did not hold because assumption (**c**) was violated.

In their example, the behavior of the system depends on the initial ages of clusters. For example, if there are three clusters of ages 5, 6, and 9 years, it is optimal to replace all assets at time zero. On the other hand, if there are three clusters of ages 1, 8, and 9 years at time zero, then it is optimal to keep the 1- year old and combine the 8- and 9- year old clusters into a single cluster of 20 machines. It is then optimal to keep the two clusters for a total of 7 years until the smaller is 9 and larger is

8, when both are replaced and combined to form a single cluster. Finally, if there are three clusters of age 5, 8, 9, it is optimal to combine the two older clusters. Subsequent replacements occur when each cluster reaches its natural replacement age. The "natural" replacement age of a cluster defined to be the age at which the cluster would be replaced if it were considered to be isolated from the economic influence of other clusters. In these example, there is no violation of OCRR in the solution. However, because of their assumption (**c**), they cannot use OCRR rule in their solution procedure.

Later, Tang, and Tang [8] prove that the three assumptions of Jones et. al. [6] imply that optimal replacement policies must satisfy the All-or-None Rule (AONR). AONR states that in any period, an optimal policy is to keep or to replace all the machines regardless of age in each period.

They rewrite the assumption (**c**) of [1] as:

c1: For fixed $t \in \{0, 1, 2, ...\}$, $m_i(t) + m_{i-1}(t) \geq s_{i-1}(t) + s_i(t)$ $\forall i \geq 0$

This implies that the increase in the maintenance cost is not smaller than the decrease in the salvage value of the machine. They prove AONR using an assumption milder the than (**c1**):

c2: For fixed $t \in \{0, 1, 2, ...\}$, $m_i(t) - s_i(t) \geq s_0(t)$ $\forall i \geq 0$

Since (**c1**) implies (**c2**), this also proves the AONR under the assumptions of [1].

They present a new theorem:

All-Or-None Rule (AONR): In any period of either a finite- or infinite-horizon PRP satisfying (**a**), (**b**), and (**c2**), an optimal policy is to keep or to replace all the machines regardless of age.

In their article, they claimed that they proved the AONR using a slightly weaker set of assumptions than that used in [6] to prove OCRR. Hopp et. al. [5] published another article to prove the OCRR by using the weaker version of assumption of (**c**).

c3: For fixed $t \in \{0, 1, 2, .\}$ $m_i(t) + s_i(t)$ is non decreasing function of $i > 0$.

They use an example to illustrate the difference between these two assumptions. It is a simple problem with $p(t)$, $m_i(t)$, and $s_i(t)$ which are all constant with respect to t. Table 1.1 gives the data for the example from Hoop et. al. used[5]

Table 1.1: *Example Data from JZH*

i	m_i	s_i	$m_i + s_i$
0	0	p	p
1	5	5	10
2	10	3	13
3	20	2	22
4	30	1	31
5	50	0	50

Clearly, no matter what value is chosen for p, (**c3**) holds. However, for $p > 10$, (**c**) violated. The key difference is that assumption (**c**) places a restriction on the purchase price of a new machine, while (**c3**) does not.

Another assumption for OCRR comes from McClurg and Chand [2]. Their paper assumes that the "operating cost" for using an older machine in a period is at least as large as the operating cost for using a newer machine, except perhaps in the first period of operation of a new machine. The operating cost of a i period old machine in a period t; O_i(t), is defined as the sum of maintenance cost, labor cost, and the decline in the salvage value over the period. The operating cost for a brand new machine in its first period of usage could be very high because most machines experience a sharp decline in salvage value as soon as they leave the dealer's storage area.

They show $O_i(t)$;

$O_i(t) = m_i(t) + s_i(t) - s_{i+1}(t+1); i \in 0; n-1; t \in 0; T$, ($n$ is the maximum allowable age).

Note that $\triangle s_i(t) = s_i(t) - s_{i+1}(t+1)$ denotes the reduction in salvage value of a i-period-old machine during period t for $i \in 0; T$. Thus, in the first period the reduction in salvage value is the initial purchase price minus the salvage value at the end of the first period. In the second period, the reduction in salvage value is the salvage value at the end of the first period minus the salvage value at the end of the second period. The result is that, over the periods in which the asset was used, the loss in the value of the asset is the purchase price at the start of the first period of use minus the salvage value at the end of the last period of use. This process ensures that the actual economic depreciation is recognized as a cost of operating the machine in each period. Thus, the cost $O_i(t)$ is representative of the total cost associated with operating a i period old machine in period t [2].

As a result their new assumption is:

c4: For any $t \in 1; T; O_{i+1}(t) - O_i(t)$ for $i > 0$.

It is reasonable to assume that it costs more to use an older machine in a period than it does to use a newer machine. It is not required that inequality hold for $j = 1$ because a new machine can experience a very steep decline in salvage value in its first period of operation.

The difference of these assumption from others is that the change in salvage value is considered rather than the absolute value (i.e $\Delta s_i(t)$ instead of $s_i(t)$). They claimed that this assumption is much stronger.

1.3 Research Motivation

In this section, we discuss why this paper is important and our motivations. Specifically the following three factors motivate this research.

1. PRP is an important problem. As we explained in the first section, PRP is required in many applications. When assets are economically are interdependent and operate in parallel, PRP must be considered. When there are demand requirements, budget constraints, or other service requirements, PRP determines the optimal solution because of the interdependence of assets.

   Today, there are many examples of this problem. JZH mentioned many examples in communication industry. Investment in communication equipment shows economies of scale. The cost function of these systems has a fixed component, including the right of way, building and power plant costs, plus a variable component that depends on the number of communication channels added per year. Another example can be given from transportation industry. They have to make decisions for keeping and replacing vehicles in order to reduce costs and increase profitability. Here, assets are also interdependent, through demand constraints and economies of scale. Additionally, these industries are characterized by the operations of many (often thousands) assets that can be worth considerable value (i.e., $50,000 for a delivery truck or $150 million for a shipping vessel).

2. JZH said they generated numerous examples which followed the OCRR rule. They claimed

   > *Note that in both Examples 1 and 2, the NSR holds but the OCRR does not necessarily [because assumption (c) is violated]. We have been unable to generate numerical examples with the types of optimal behavior discussed in these examples in which assumption (c) does hold. On the other hand, we have not been able to show the assumption (c) implies that these types of optimal behavior cannot occur.*([6], p.30)

   Thus, in this paper, we attempt to reduce limitations and expand the application of OCRR to more problems.

3. Jones et. al [6] solved the PRP using dynamic programming. Their formulation was defined with the following parameters:

   n = Maximum allowable age

   n_i= number of assets of age i

   $p(t)$= unit purchase price of a new machine in year t, fixed

   $K(t)$=cost for any asset purchase in year t,

   $m_i(t)$= O-M cost for an i-period old asset in use in year t.

$s_i(t)$= revenue for an i-period old asset salvaged at the end of year t.

δ_t = discount factor for year t. relative to some preceding year τ used in calculation of present worth.

In this formulation, they are assuming there is only one alternative cost structure available for all machines in each year.

The formulation:

> Let $f_\tau(i[1], n_1; ...i[k], n_k)$ denote the optimal cost in years τ through $t+T$ discounted back to do beginning of year τ with n_j machines of ages i[j](i=1,....,k), where each $i[j]$ is an integer between 1 and n, and two of the $i[j]$s are equal. Assuming that machines salvaged at the end of the study year $t+T$:
>
> $$f_{t+T}(i[1], n_1; ...; i[k], n_k) = -\sum_{j=1}^{k}(n_j s_i[j](t+T))$$
>
> These optimal cost functions may be expressed recursively. The NSR ensures that 2^k keep/replace options must be considered in calculating each cost function.[6]

OCRR is also very important, because it reduces the computational burden of solving the DP. If both OCRR and NSR are used, the keep/replace options reduces to k options for k number of challengers. Without these two rules, the options are $[\sum_{x=0}^{n} C(n_i; x)k^x]^T$. (C(a; b) means a chooses b.) This is really a large decrease in options, and that makes easier to solve PRP when we use OCRR.

Theorem 1 *The DP formulation from JZH can be solved on the order of $O(n^n * 2^n * T)$ time.*

Proof:

The time calculated from the worst case scenario:

Maximum possible states × Maximum possible decisions × Number of period(T)

N represents the maximum age, and total number of assets demanded is $n = \sum_{i=1}^{N}(n_i)$. The maximum possible number of states are how many ways one can spread n assets into N ages slots. This equals $n!$ states. $n! = n*(n-1)*(n-2)*...2*1$. Since the worst case is used in $O()$notation, it is assumed that $n! = n^n$ in $O()$ notation. The maximum number of decisions are $2^n - 1$ per state. This comes from $C(n;1) + C(n;2) + ... + C(n;n)(C(n;n)$ means n chooses n). In the worst case scenario, it becomes 2^n. Thus, the DP can be solved in $O(n^n * 2^n * T)$ time without using neither NSR, nor OCRR.$\star$

Theorem 2 *The DP formulation from JZH can be solved on the order of $O((N^N * 2^N * T))$ time using the NSR.*

Proof:

When we use NSR, we assume we have N clusters, as opposed to n machines. Because, combinations of all machines are not allowed. We only look all possible states and decisions in N clusters. Maximum possible states are how many ways to spread N clusters into N slots. It is $N!$ states. $N! = N*(N-1)*(N-2)*...2*1$, since the worst case is used in O()notation. It is assumed that $N! = N^N$ in O() notation. This equals to N^N. Maximum possible decisions are all possible combinations of N clusters, and this equals to $2^N - 1$. This comes from $C(N;1) + C(N;2) + ... + C(N;N)$. In the worst case scenario, it becomes 2^N. Thus, the DP can be solved in $O((N^N) * 2^N * T))$ time using only NSR. $\star$

Theorem 3 *The DP formulation from JZH can be solved on the order of $O(2^{N^2} * N * N * T)$ time using the NSR and OCRR.*

Proof:

In this case, we have N clusters, and we need to care about their ages. We can't split clusters. So, when we spread N clusters into N slots with respect to their ages, we need to consider some factors like grouping of clusters. In order to calculate number of states, we need to calculate the all possible combinations of clusters. When there are both NSR and OCRR, we know that only adjacent cluster can combine. If there are 4 clusters, $N1$ and $N2$ can combine, but $N1$ and $N3$ cannot combine. To calculate the states, we need to look for the slots with no clusters, because they determine the grouping of clusters. We are calling this as number of zeros. For an i number of zeros in N clusters, the number of combinations is $C(N;i)$. The all clusters in slots can change places with each other, so we need to multiply by $(N-i)$ which number of slots which are not zero. Finally the zeros can have $C(N;i)$ different position in the states. This holds for the number of zeros up to $N-2$. For N zeros there is no state, and for $N-1$ zeros, the number of states equals N, because there is only one slot which is not zero, and the N is the possible number of positions that zeros can have. As a result, the number of states are $\sum_{i=0}^{N-2}(C(N;i)^2*(N-i))+N$. In the worst case, it becomes $2^{N^2}*N$ states. For each state, we have N decisions, over T periods. Thus, the algorithm is exponential in N, on the order of $O(2^{N^2}*N*N*T)$. $\star$

These can be explained with some examples of different cluster sizes. For 4 cluster case, there can be no zero, 1 zero, 2 zeros, and 3 zeros in the states. For no zero, there is no aggregation in N clusters, so $C(4;0)=1$, but these 4 clusters can change places between each other so $(4-0)$ places are multiplied by $C(4;0)=1$. The second $C(4;0)$ represents the positions in the states that zeros can have. In this case, it is 1, because there is no zero. For 1 zero, there is 2-cluster aggregation in N clusters so the number of possibilities are $C(4;1)=4$, but these 3 clusters can change places between each other so $(4-1)$ places are multiplied by $C(4;1)=4$. The second $C(4;1)$ represents the places that zeros can be placed. In this case, it is 4, because there is 1 zero and it can only go in 4 places. This is same way in the 2 zeros case. But, for 3 zeros or $(N-1)$ zeros case, there can only be aggregation of N clusters, and the possibility of this is

1, not $C(4;3)$. The only concern in this case is the number of positions that zeros can have, and it equals to $C(4;3) = 4$. So, the $N-1$ zeros don't included in summation, it is just added at the end of summation as N. This expression holds for any number of clusters. For 3 clusters case, when there is no zero, the states are 3, when there is 1 zero, the number of states is 18, when there is 2 zero, the number of states are 3. The total number of states are 24.

Clearly, using NSR and OCRR drastically reduces the problem space. Thus, our motivation is threefold.

1. Practical application
2. Experimental results from JZH
3. Computational reduction of DP

Chapter 2

Economic Life of Clusters

In this chapter, we will discuss the economic life of clusters, and some implications of fixed economic life. In the first part, it is proven that economic life is not constant with cluster size. In the second part, some implications are given for fixed economic life.

2.1 Economic Life of Clusters

The economic life of an asset is the optimal age to salvage an asset in order to minimize the equivalent annual costs (EAC) over an infinite horizon. Replacing at any other age results in higher costs. To calculate the EAC, the fixed cost, investment cost, operation cost and salvage value must be considered. The EAC is calculated by annualizing the present value all possible asset lives. The economic life is the minimum EAC over all possible ages, N, where N is the maximum allowable age of an asset.

Economic life is the age where clusters have a tendency to be replaced. Jones et. al. [6] defines this as the natural replacement age. They claim that if a cluster is independent from the economic influence of other clusters, the natural replacement age of a cluster is defined to be the age at which the cluster is replaced. They consider it as an economic life of a cluster, as defined above. For a single asset, there is no economic influence of others, but for a cluster, it must be considered if it is isolated from the other clusters' economic influence.

Under the assumption of stationary costs, the economic life is constant. We analyze economic life of clusters in this chapter. First, we examine the effects of the cluster size on economic life, as the cost parameters, except the fixed cost, in the annual cost equation differ with the cluster size. Thus, for a cluster of assets, the fixed cost is not influenced by an increase in assets. On the other hand, other costs are affected with the cluster size, as they are defined as costs per assets. Hence, in the case of a cluster of assets, the economic life may change depending on the cluster size.

Theorem 4 *n^* is non-increasing with increasing k under time invariant costs.*

where,

n^* is the economic life of clusters

k is the cluster size

Proof:

In order to prove this, we need to examine the EAC equations, and determine if $k' \geq k$ results in $n' \leq n$.

The total costs can be written as follows:

$$AEC_n = min[P + \frac{K}{k} + \sum_{i=1}^{n} (\frac{C(i-1)}{(1+r)^{i-1}}) - \frac{S(i)}{(1+r)^n}](A/P, r, n)$$

$$AECn' = min[P + \frac{K}{k'} + \sum_{i=1}^{n'} (\frac{C(i-1)}{(1+r)^{i-1}}) - \frac{S(i)}{(1+r)^{n'}}](A/P, r, n')$$

Where,

P= investment cost

K= fixed cost

r= interest rate

i= age

$C(i)$= Operation and maintenance cost for each age

$S(i)$= Salvage cost for each i

When we compare these two total cost equations, It is easy to see that ,

$$AEC_n \geq AEC'_n, \bigvee n = 1, 2, .., N \text{ as}$$
$$\tfrac{K}{k} \geq \tfrac{K}{k'}.$$

But, the difference between AEC_n and AEC'_n is not constant. If we look at a typical example, we can see that it decreases.

$$AEC_n - AEC'_n = (A/P, r, n)K(\frac{1}{k} - \frac{1}{k'})$$

The above equation can only change if n changes, because the other parameters are constant($K(\frac{1}{k} - \frac{1}{k'})$ is constant and positive when $k \leq k'$). Note the $(A/P, r, n)$ is defined as $[\frac{r(1+r)^n}{(1+r)^n-1}]$. As $(A/P, r, n)$ decreases in r, it is the only way to change the difference in $(AEC_n - AEC'_n)$. As a result, it is obvious that the difference reduces in n. Since the difference decreases, AEC_n cannot start increasing before AEC'_n starts increasing. Hence, AEC_n cannot reach its minimum before AEC'_n, and it follows that n for AEC' cannot be larger then n for AEC, when $k \leq k'$, or:

$$AEC_n - AEC_{n-1} \geq 0 \Rightarrow AEC'_n - AEC'_{n-1} \geq 0$$

But:

$$AEC'_n - AEC'_{n-1} \geq 0$$

does not mean:

$$AEC_n - AEC_{n-1} \geq 0$$

Hence:

$$n \text{ for } AEC' \leq n \text{ for } AEC.$$

So, the economic life can differ with cluster size, and the economic life decreases with increasing k. ⋆

2.2 Implications for Fixed Economic Life

Economic life can be affected from many factors. Especially, the fixed cost and cluster size are the important factors. As proved, with increasing cluster size, EL stays constant or decreases. For increasing fixed cost, EL stays constant or increases. Fixed cost also affects the optimal solution for the infinite horizon problem. With high fixed cost, clusters tend to combine in early periods. Thus, with some fixed cost EL differs, and optimal infinite horizon solution is one cluster which all clusters are combined into. On the other hand, when EL is fixed, optimal infinite horizon solution can be separate clusters, group of some combine clusters, or 1 cluster.

Moreover, with increasing fixed cost or decreasing cluster size, economic life approaches to maximum allowable age, and at some points economic life becomes maximum allowable age, after that point further increase in fixed cost or decrease in cluster size no more change the EL. The following corollary is a result of this.

COROLLARY 1:

If EL $= N$ (maximum allowable age) for some K (Fixed Cost) and/or k (cluster size), EL=$N$for all $K' > K$ and/or $k' < k$.

Proof:

When K increases, or k decreases, EL increases. If EL is the maximum allowable age, then increasing K, or decreasing k can make EL any bigger. Thus, Economic Life equals to maximum allowable age for all $K' > K$ and/or $k' < k$. $\star$

Chapter 3

Economic Life and OCRR

In this chapter, we apply results of chapter 2 to solve PRP. We illustrate it with examples for finite and infinite horizons. Moreover, some theorems are provided for OCRR assuming clusters have the same or different economic lives.

In this chapter, we also examine the effects of the economic life on the Older Cluster Replacement Rule. It is obvious that if all clusters replace at their economic life, then they replace with the minimum cost and the optimal replacement policy can be found without any violation of OCRR. However, initial clusters can have different economic lives, because they do not have the same cost behavior. Hence, OCRR may be violated until all initial clusters are replaced. But after period N, if all clusters have the same EL, then OCRR cannot be violated.

3.1 Infinite Horizon Analysis

3.1.1 Fixed Economic Life:

In this section, an example is illustrated for the infinite horizon. An asset is defined by, its age $i = 0, 1, \ldots, N$, and the period $j = 0, 1, \ldots, T$

The costs are:

p_j = per-unit cost for a new asset purchased at the end of period j,

k_j = fixed cost for any asset purchase at the end of period j,

c_{ij}= O-M cost for an i-period old asset in use from the end of period j to $j+1$,

r_{ij} = revenue for an i-period old asset salvaged at the end of period j.

Other relevant parameters include:

n_i = the number of i-period old assets available (in inventory) at time zero,

d_j = number of assets demanded from the end of period j to $j+1$,

δ_j = discount factor for period j.

The solution for this example was found with AMPL on a data follows N=6 and T=100, to simulate an infinite horizon problem. Table 3.1 provides the salvage values and operating costs with other data as follows;

Table 3.1: *Operation Costs and Salvage Value*

i	0	1	2	3	4	5	6
r_i	25	20	19	19	14	14	10
c_i	0	5	5	6	6	7	7

Example with Fixed Cost=0:

$$p = 25$$

$$n_j = 10$$

$$n_0 = 0$$

$$K_j = 0$$

$$\delta = 0.91$$

$$d_j = 50$$

According to the solution, the initial 2, 4, and 5-year old clusters are replaced and combined into a cluster of 30 assets at time zero and the initial 1 and 3-year old clusters are kept. In the second period, 1- year old cluster is kept and the 2 and 4-year old clusters are replaced and combined into a cluster of 20 assets. In the third period, both clusters (now initial and 1-year old) are kept. After this point, these clusters are replaced at their economic lives, which is age three for both clusters in this case. The solution network for this example can be seen at Figure 3.1

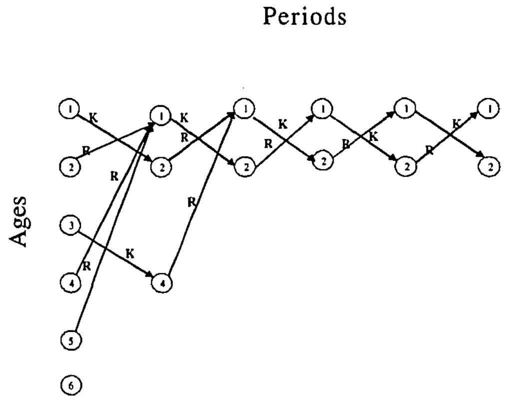

Figure 3.1: Solution Network for Example with Fixed Cost=0

From this answer, we can see that OCRR does not hold in the first period, because the initial 3-year old assets replace after the initial 2-year old machines. However, the OCRR holds after the second period. From here we can understand that the OCRR depends strongly on initial values of assets. The reason of this behavior can be described in Figure 3.1 which depicts the EAC of retaining each cluster owned at time zero for 1,2,..,5 years until it reaches its maximum age N. As seen in Figure 3.2, the 2-year old machine has its lowest EAC in the first period while the 3-year old machine has its lowest cost in the second period. As a result, the 2-year old machine replaces before the 3-year old machine, violating the OCRR.

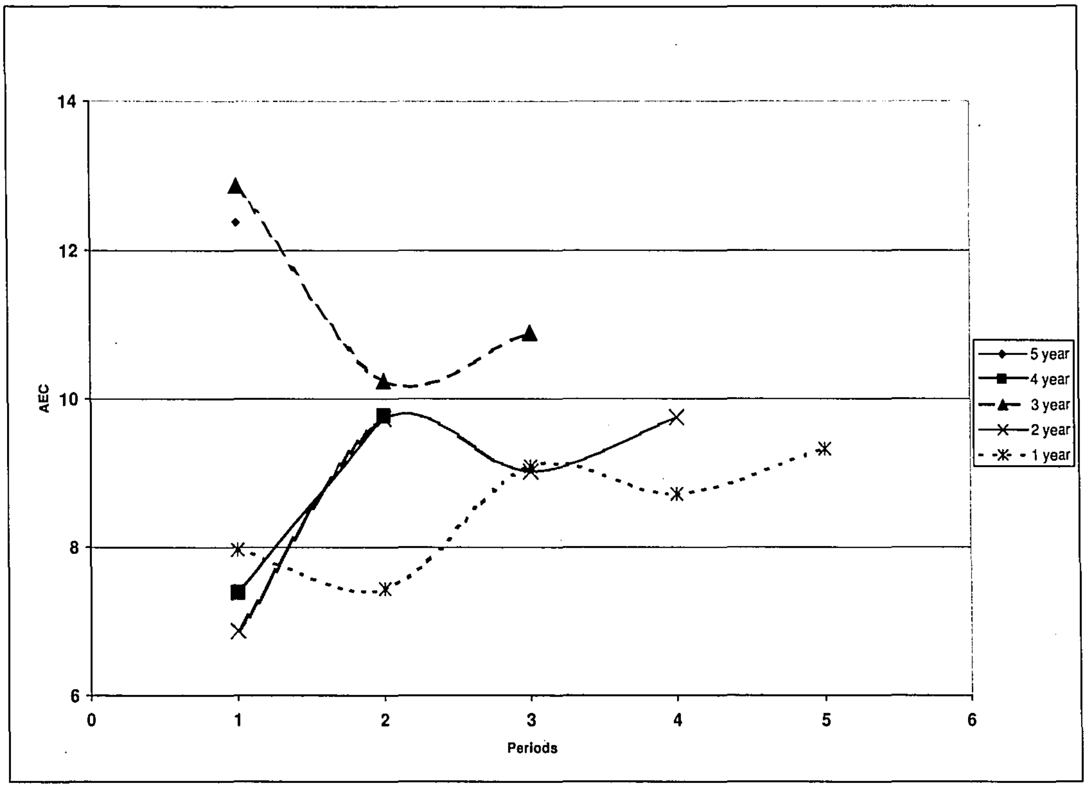

Figure 3.2: Graph of EAC for Owning and Renting an Initial Cluster for 1,2,3,4, and 5 years

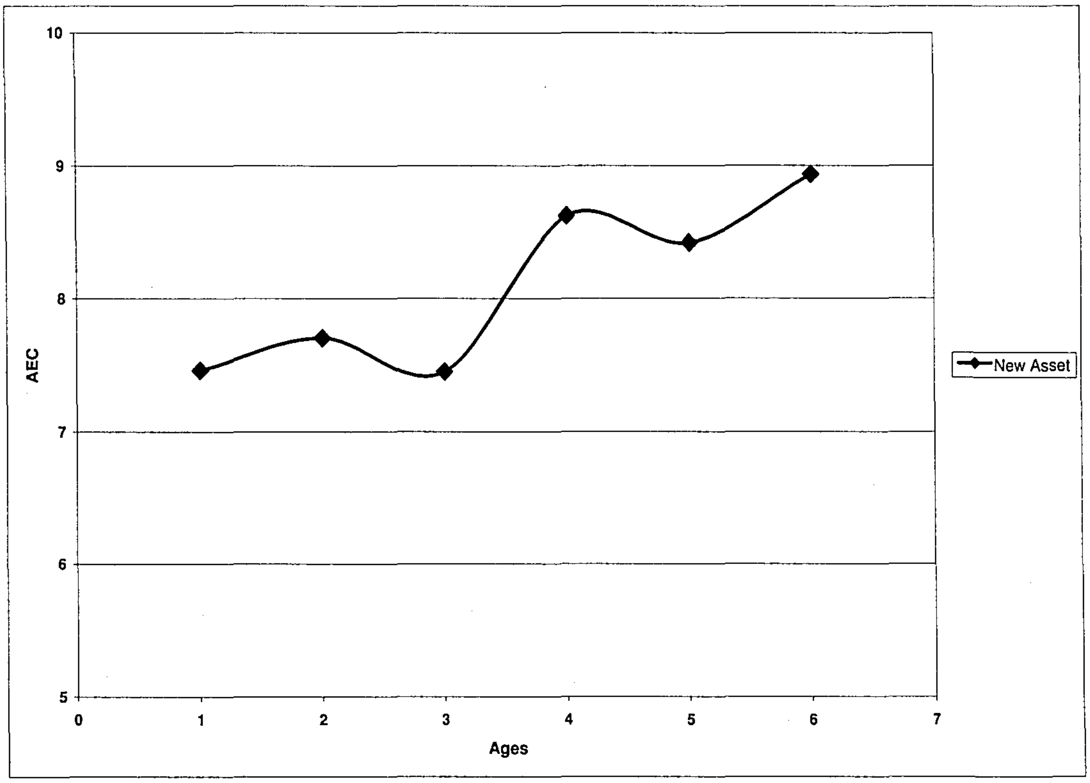

Figure 3.3: Graph of EAC for Owning a New Asset

As shown in Figure 3.3, after period N (6 in this case) the economic life for a new asset is three. In this example, all assets after period 2 are replaced at age 3. As a result, we can find new applications for OCRR which are not cost dependent.

Example with Fixed Cost=5:

In this example, we increase the fixed cost. When there is a fixed cost, the solution changes and all clusters replaces at the second period. The Figure 3.4 depicts the solution network for this example with fixed cost=5.

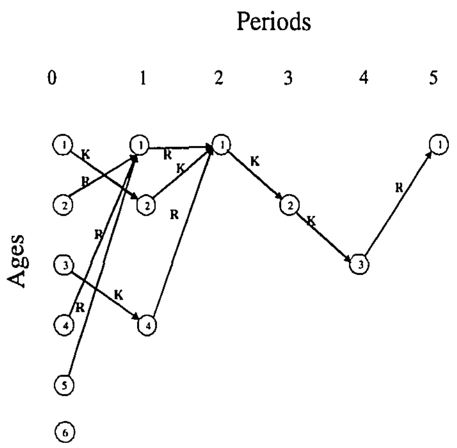

Figure 3.4: Solution Network with Fixed Cost=5

After grouping of clusters, they still are replaced at their economic lives. The OCRR is violated at the first period. According to the solution, the initial 2, 4, and 5-year old clusters are replaced and combined into a cluster of 30 assets at time zero and the initial 1 and 3-year old clusters are kept. In the second period, 1- year old cluster, the 2 and 4- year old clusters are replaced and combined into a cluster of 50 assets. In the third period, the only cluster is kept. After this point, a cluster is replaced at its economic life, which is age three.

From this answer, we can see that OCRR does not hold in the first period, because the initial 3-year old asset replaces after the initial 2-year old machines. However, the OCRR holds after the first period. The reason for this behavior can be described with the figure below. Figure 3.5 depicts the EAC of retaining each cluster owned at time zero for 1,2,.. years until it reaches its maximum age N. As seen in Figure 3.5, the 2-year old machine has its lowest EAC in the first period while the 3-year old machine has its lowest cost in the second period. As a result, the 2-year old machine replaces before the 3-year old machine, violating the OCRR.

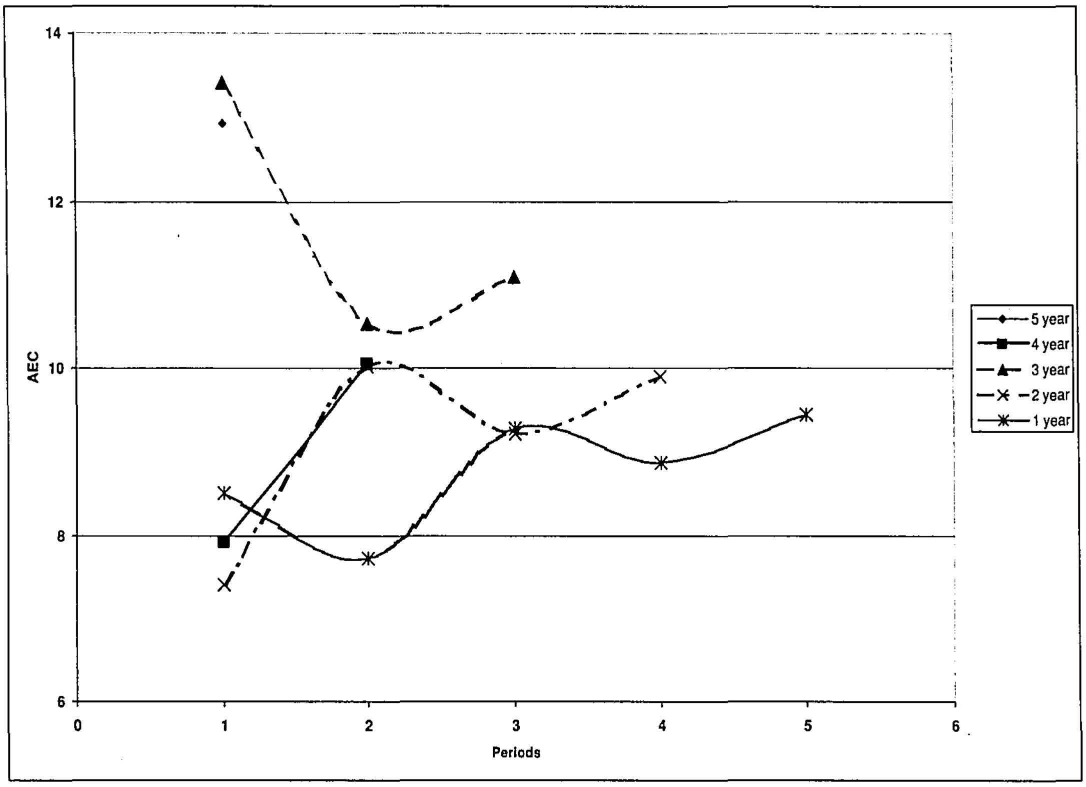

Figure 3.5: Graph of EAC for Owning and Renting an Initial Cluster for 1,2,3,4, and 5 years with Cluster Size of 10 and Fixed Cost of 5

Figure 3.6 depicts the economic lives of initial clusters with different cluster size. As you see, the economic life stays constant with cluster size, when there is a small fixed cost. Also, economic life is age 3 for a new cluster, and this supports the solution where clusters are replaced at age 3. After period N (Because before period N, there are still initial clusters and this affects the replacement time of clusters).

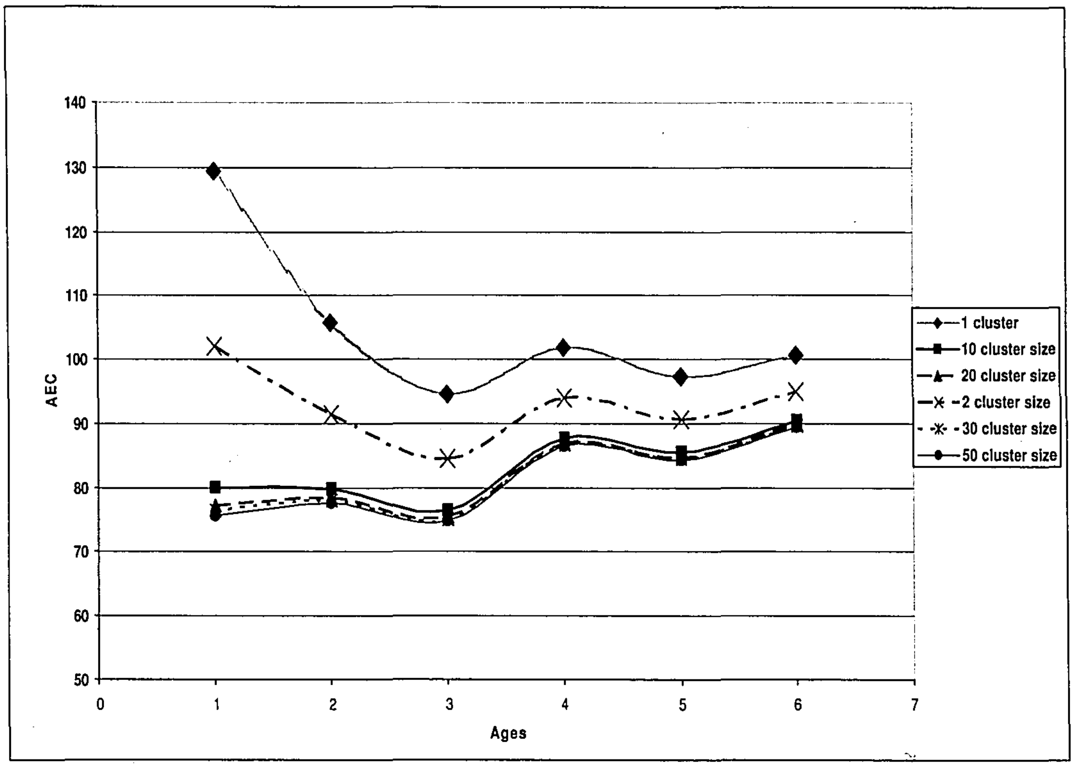

Figure 3.6: Graph of EAC for Owning a New Cluster with Fixed Cost=5

The fixed cost strongly affects the grouping in these two examples. Moreover, when we study examples with higher cluster sizes, it is seen that for a fixed cost greater than 11, all clusters combine at the first period to a cluster of 50 assets. Thus, there becomes only one cluster and it is replaced at its economic life over the problem horizon. Thus, there is no OCRR violation.

As a result, there is no violation of OCRR regardless of fixed cost. Thus, the following theorem generalizes this result.

Theorem 5 *If the EL is the same for all possible cluster sizes in an infinite horizon PRP problem, in any period $t \geq N$, an optimal solution exists such that a machine of age i is replaced only if all machines of age greater than i are replaced.*

Proof:

This is a proof by contradiction and construction. It should be obvious that only two clusters are required for OCRR to be violated. Here, we assume that we have just two clusters and will later generalize the result.

Note that with two clusters, the only possible steady state solutions in the infinite horizon problem include (1) staying separate clusters or (2) combining into one cluster.

Assume an optimal solution exists according to (1) in which the clusters stay separate through the horizon. Further assume an optimal solution exists such that the OCRR is violated in some period $t \geq N$. This means that there is some period where the younger cluster is sold in a period that the older cluster is retained. As both clusters have the same economic life, one cluster has to have been replaced at a time other than their economic life.

Consider the cluster that has been sold at an age not equal to its economic life. Trace it back to the period in which it was purchased and replace the solution with one in which the asset is repeatedly sold at its economic life. If the clusters remain separate in an optimal solution, this newly constructed solution must have a lower cost, by the definition of economic life, than the original solution, which by contradiction, cannot be optimal.

Now assume an optimal solution exists according to (2) in which the clusters group in the steady state solution and further assume an optimal solution exists such that the OCRR is violated in some period $t \geq N$. Note that the violation of the OCRR must occur in a period before the clusters group. Also note that the steady state solution is defined by a single cluster repeatedly replaced at its economic life. As this continues for an infinite number of periods, the net present value (starting at any time period) can be defined at a fixed value F (as it is merely the net present value of an infinite stream of equal cash flows). Note that if this sequence is started in any period, t or t', the cost of that period on is F.

Now, consider the period in which the OCRR is violated. We can separate this solution into three parts. They are before the violation, after the violation before the grouping, and after the grouping (where steady state solution starts). Now, assume that the costs up to the period of the violation sum to the value of A. Assume that the costs inbetween the period of the violation through the time period when the steady state solution starts sum to the value of B. Replace this solution with one in which the optimal steady state solution begins at the time period of the violation. We claim that this new solution, which does not violate OCRR, can be no worse that the original. The new solution has a cost of F beyond the period. The cost of the network up to this point is A - salvage value of the second asset. Thus, this new solution has a cost of $A + F - SV$, which is less than $A + B + F$, so our solution is lower and the solution with a violation cannot be optimal. $\star$

If there are more than two clusters, each with the same Economic Life, we claim the results from previous theorem hold. For example, if we have three clusters and a violation occurs it means that an asset is replaced at a period not equal to its economic life. For three clusters, there are three possible steady state solutions: (1) single cluster (2) three individual clusters and (3) two clusters of size one and two assets. There are two possible solutions for (3) depending on which clusters group. Given this situation and the assumption that all cluster size have the same EL, it should be clear that OCRR will not be violated as selling at an age other than EL should lead to grouping. If grouping is optimal in steady state, then argument is similar to that in the previous theorem can be constructed.

3.1.2 Non-constant Economic Life:

$$
\begin{aligned}
p &= 30 \\
s_0 &= 30 \\
n_0 &= 0 \\
n_1 &= 1 \\
n_2 &= 0 \\
n_3 &= 15 \\
n_4 &= 15 \\
n_5 &= 0 \\
K_j &= 20 \\
\delta &= 0.91 \\
d_j &= 31
\end{aligned}
$$

The other cost values are same as in previous examples. According to solution, all clusters are combined to one cluster of 31 assets at second period after they are kept for one period. This one cluster is replaced at its economic life, thus there is no violation of OCRR, even cluster of 1 asset has an economic life of 6 for both at time zero and later, while other clusters (15 assets) have an economic life of 5 at time zero and economic life of 3 as a new asset.

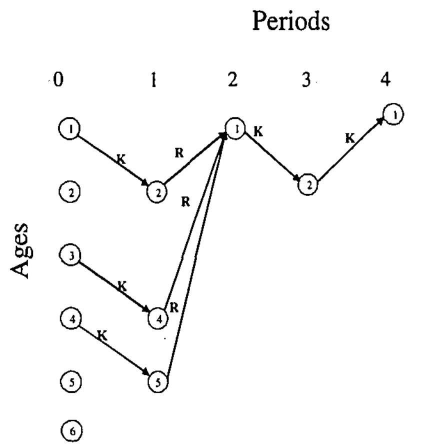

Figure 3.7: Solution Network for Example 1 for non-constant economic life

When we examine another example with different cluster sizes:

$$
\begin{aligned}
n_0 &= 0 \\
n_1 &= 4 \\
n_2 &= 0 \\
n_3 &= 0 \\
n_4 &= 15 \\
n_5 &= 12
\end{aligned}
$$

In this example, clusters of 4 assets have an economic life of 4 at time zero, and economic life of 5 as a new asset, clusters of 15 assets and 12 assets (greater than 12) have an economic life of 5 at time zero and economic life of 3 as a new asset. According to the solution, 5 and 4- year old clusters are combined to one cluster of 27 assets at the first period. This new cluster is kept until the fourth period, and the initial 1- year old cluster is kept until fourth period. Then, these two clusters are combined into one cluster of 31 assets at fourth period. After this time, the new cluster is replaced at its economic life, thus there is no violation of OCRR.

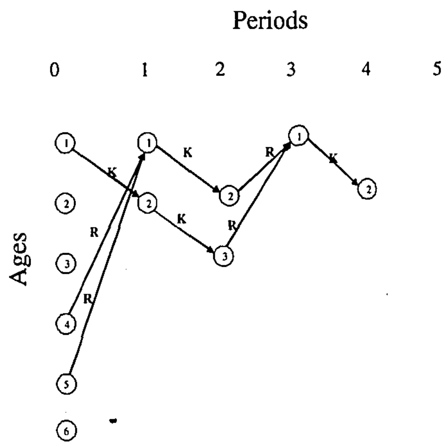

Figure 3.8: Solution Network for Example 2 for non-constant economic life

From, these two examples, and several others we have generated, it can be concluded that when the economic life is non-constant, clusters are combined at very early periods. The reason for this depends on the fixed cost. Clusters are combined due to high fixed cost. Moreover, the economic life usually differs when there is a high fixed costs. Thus, clusters are combined at early periods when their economic lives are different.

Our other studies also have showed us that when fixed cost is higher than some specific value, all clusters are combined at the first period to one cluster, and the only way to prevent this to decrease the investment value. Other cost considerations have also affect on grouping, but not as much as fixed cost and investment cost.

Theorem 6 *If the EL is non-fixed for all possible cluster sizes in infinite horizon PRP problem in any period* $t \geq LCM$ *of EL +N, an optimal solution exists such that* *a machine of age i is replaced only if all machines of age greater than i are replaced.*

Proof: With two clusters, there are naturally two possible optimal solutions in the infinite horizon case: (1) the clusters remain separate and replace at their respective economic lives and (2) the clusters combine and replace at the economic life of the new cluster. Note that in (1), the clusters must replace at their economic lives for the optimal solution of keeping separate clusters.

Consider (1) and note that a replacement of each cluster must occur within the first N periods. After this time, each cluster is replaced at their economic life in an optimal solution for keeping the clusters separate. However, there exists a period $t < EL + N$ where each cluster will replace in the same period, as their economic lives occur in the same period. Thus, an optimal infinite horizon solution where the clusters have different ELs cannot result in separate clusters and they must group.

It is possible that replacements before this period can violate the OCRR rule – as a younger cluster may have a shorter economic life. However, if the LCM period is reached, each cluster must replace at its economic life in the optimal solution. Therefore, OCRR cannot be violated past period t.

For (2), it should be clear that the clusters can combine at any time. However, at the latest, they will combine at some period $t < LCM + N$, as here their economic life replacement times will coincide. Any grouping after this period is dominated by this solution and cannot be optimal. As there is only one cluster after this time, OCRR cannot be violated after this period. $\star$

Multiple cluster case with non-fixed ELs is more complex, because there can be two possibilities:

- Three clusters, different EL for all combinations, infinite horizon: The problem here is in sequencing. If two clusters combine, then it is the LCM of the remaining to clusters After they combine, then it is the greatest LCM after all possible combinations. For three clusters it would be summing two LCMS depending on the order. Might be possible.

- Mix of ELs for three clusters: There exists some period where OCRR is followed. if there is a mix, then the clusters will different ELs will eventually group and if it reaches a point where all clusters have the same EL.

From this point, we can reach a general proof:

Theorem 7 *: In any infinite horizon PRP problem with stationary costs, there exists a period t' where the OCRR is followed for all $t > t'$.*

Proof:

As noted earlier, if clusters have different ELs, then they will group at some point, the latest being their LCM of ELs. This continues until their remains one cluster or multiple clusters with the same ELs, at which time they will either cluster or remain separate, but not in violation of OCRR, as noted in Theorem 5. $\star$

Given Theorem 7, we are motivated to alter our DP algorithm for solving PRP in order to take advantage of the computational benefits of using OCRR without making any cost assumptions. This online algorithm can be used if we know that clusters with different Els will eventually group and they reach a point where all clusters have the same EL. Thus, for multiple cluster case with different cluster sizes, if we know that optimal steady state solution is one cluster solution, an online algorithm can be used to take advantage of OCRR rule.

3.2 Finite Horizon Analysis

For the finite horizon case, in this example we have same costs as example 1, but this time p=30, and initial clusters sizes are different:

$$
\begin{aligned}
n_0 &= 0 \\
n_1 &= 5 \\
n_2 &= 10 \\
n_3 &= 5 \\
n_4 &= 10 \\
n_5 &= 25
\end{aligned}
$$

The solution network and the AEC figure is provided for the last N period. Figure 3.9 displays the solution network.

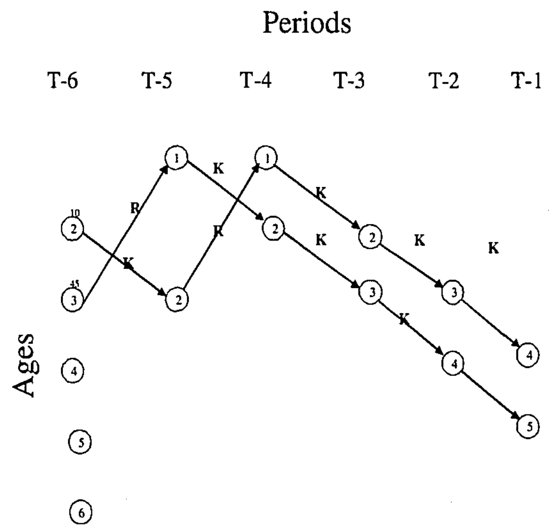

Figure 3.9: The Solution Network for Last N period

As seen in Figure 3.9, there is no violation of OCRR after period $N - T$. But cluster replaces after their economic lives which is 3 ages for this example. The cluster which is at age of 3 at period $T - N$ is kept until the age of 5, and the cluster which is at age of 2 at period $T - N$ is kept until the age of 4. However, they must be replaced at age 3, so there is a violation of constant EL, but there is no violation of OCRR.

If EL is non-fixed in the finite horizon problem, the OCRR can only be violated before the the period $t < LCM + N$. After this period OCRR always holds as proved in theorem 6. In the last N periods, there is no problem with OCRR since the optimal steady state solution is one cluster solution in non-fixed EL problem.

PROPOSITION 1:

If the EL is fixed for all possible cluster sizes in a finite horizon PRP problem, in any period $t \geq N$ and $t < T - N$, an optimal solution exists such that a machine of age i is replaced only if all machines of age greater than i are replaced.

In periods $t \geq N$, the proof follows that of Theorem 5. After period $T - N$, clusters can be forced to be replaced before their economic life, since period can last before their economic life. Hence, OCRR can be violated after period $T - N$.

Chapter 4

Conclusion and Directions for Future Study

The importance of Older Cluster Replacement Rule for reducing solution time for DP is analyzed. It is concluded that OCRR is important because we can reduce the number of decisions in a period from the $2^N - 1$ choices assuming only NSR to N choices assuming both NSR and OCRR.

The economic life (EL) of clusters is investigated with different cluster sizes, and it is concluded that EL is a non-increasing function of cluster size. Moreover, it is observed that the fixed cost has a significant effect on the EL of clusters. When there is a high fixed cost, clusters tend to replace earlier.

The restrictions on costs assumed for OCRR in former studies are relaxed using results of our EL study. Theorems are proved for clusters with both fixed and non-fixed ELs over finite and infinite horizons. According to these theorems, OCRR always holds after some period which depends on the economic lives and maximum allowable ages of clusters over an infinite horizon. Before this period, OCRR can be violated because of the initial conditions and economic lives of clusters. We believe this to hold in the finite horizon case too, with the exception of the final N periods. This case is more difficult to prove.

From this study, we have still some limitations in the OCRR, OCRR still can be violated before some period. We are wondering that if it is possible to reduce the bound on the last period after which OCRR always holds when the costs are stationary.

The other problem, which should be investigated further, is the non-stationary cost case. It could be very useful if it is proved that the results found from this study can hold with the non-stationary cost case. The results may hold in this case maybe if EL is held constant for any cluster size over time. If it is proved that this result cannot hold with non-stationary costs, the reasons for the violation can be investigated. For other concerns, like technology change or deterioration in our measure of capacity when costs are stationary, it can be useful to try to implement the results of this study into these problems.

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VITA

Pinar Keles was born on Konya, Turkey in 1979. She got her B.S. degree in Chemical Engineering from Bogazici University, Istanbul, Turkey. During her stay in Lehigh University, she worked as a Teaching Assistant for the Industrial and System Engineering Department.

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