Construction of low-dimensional models of thermo-fluid systems: a fully spectral technique

Younjong Kim
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Construction of Low-Dimensional Models of Thermo-Fluid Systems: A Fully...

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CONSTRUCTION OF LOW-DIMENSIONAL MODELS OF
THERMO-FLUID SYSTEMS: A FULLY SPECTRAL TECHNIQUE

by

Younjong Kim

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in
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A/30/96

Date

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Professor Antonios Liakopoulos

Thesis Advisor

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Professor Robert P. Wei

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Subscript b denotes reference value.

Bars denote time-averaged quantities.

Primes denote fluctuation quantities.
Low-dimensional models for transitional buoyancy-induced flows are developed. Two cases are studied: convection in an enclosure subject to differential side heating and convection in a vertical channel with periodically spaced discrete heaters. Proper Orthogonal Decomposition (POD) is applied to extract the most energetic empirical eigenfunctions from time-dependent data obtained by direct numerical simulation based on the full model equations. The computed eigenfunctions are employed for the optimal approximation of the original flow and temperature fields. In this study, a fully spectral implementation of the procedure is presented. For the fully spectral element method, Legendre-Gauss-Lobatto collocation points and weighting factors have been employed. It is shown that for the thermo-fluid systems under study almost all the flow energy can be captured by the first four modes. Eight-equation models based on the fully spectral element implementation are capable of tracing the stable, self-sustained oscillatory behavior of the original flow with correct amplitude. For the same level of accuracy, POD and LOM results based on the fully spectral technique require, in general, fewer collocation points than those based on finite difference techniques.
CHAPTER 1
INTRODUCTION

Direct numerical simulations of transitional and turbulent convective flows require a formidable amount of computational time and computer memory. Consequently, low-dimensional models of convective flows are highly desirable. By obtaining accurate, low-order models, parametric studies of such flows can be easily performed. In addition, low-order models (LOMs) are crucial in stability and bifurcation analyses since it is practically impossible to perform these tasks directly on the full model for complex configurations.

Using the Galerkin method, an evolution partial differential equation can be reduced to a system of first order ordinary differential equations. Liakopoulos and Hsu (1984) carried out this approach successfully for compressible laminar boundary-layer flows subject to adverse pressure gradients using a finite-element-differential method. The solutions were approximated by classical cubic spline functions. Trigonometric functions and orthogonal polynomials are also frequently used (Gottlieb and Orszag, 1977). In general, these methodologies lead to large systems of ODEs. Low-order representations may be developed by expanding the unknown functions in terms of empirical eigenfunctions for the velocity and temperature modes.

The proper orthogonal decomposition (POD) can be applied to data acquired from experiment or numerical simulation. To extract the time-independent velocity and temperature empirical eigenfunctions, we apply the POD methodology. By POD, we can
obtain insight into the behavior of the flow and its instabilities as well as reduce the
governing partial differential equations (PDEs) to ordinary differential equations (ODEs).
Identification of the most energetic eigenmodes enables us to reduce the data significantly
by retaining a small number of modes which capture most of the flow energy. Applications of POD to the formation of periodic and chaotic temperature patterns in
chemical systems have been studied by Graham et al. (1993). They conducted experiments
to reveal various spatiotemporal features by POD analysis. Analysis using 40 snapshots
showed that most of the energy (mean square fluctuation) is captured by the first two
modes, with mode 1 containing 88 %, and mode 2 containing 10 % of the energy. The
remaining 38 modes capture only 2 % of the energy. The POD method was first discussed
in the fluid mechanics and heat transfer community by Lumley (1967) in the context of
turbulent isothermal flows. The method of snapshots, which is an efficient way of
extracting the empirical eigenfunctions from large data sets, was proposed by Sirovich
channel and the wake of an isolated circular cylinder. Application of POD to transitional
flow with heat transfer in a periodically grooved channel was studied by Sahan et al.
(1994). Berkooz et al. (1993) reviewed the important issues in applying POD to the
analysis of turbulent flows.

In the present study, we apply the POD methodology to transitional buoyancy-
induced flows. This decomposition leads to low-dimensional representations that reflect
the dynamical structure of the thermo-fluid system. In this thesis, the snapshot version of
POD (or method of empirical eigenfunctions) is employed to extract the coherent
structures and to obtain low-order models of transitional flows in cavities and channels. In the enclosure problem, we consider two-dimensional natural convection in a rectangular cavity due to the temperature difference between the left and right wall. The top and bottom walls are kept adiabatic. In the channel case, periodically-spaced heat sources are flush-mounted on the left vertical wall and the right vertical wall is kept isothermal. In performing the POD and in developing the LOMs, differentiations and integrations are carried out by the spectral element method. Using the computed empirical eigenfunctions we are capable of reconstructing the dynamics of the original flow and temperature fields in an optimal way.

This thesis is organized as follows: In chapter 2, the numerical methods developed in this study are presented. In chapter 3, we discuss the results of the two study cases described in the previous paragraph. POD and LOMs methodologies, using the spectral element method, are applied to both cases. The results based on the spectral element method and the finite difference method are compared. Finally, we draw conclusions and propose future work in chapter 4.
Numerical procedures for the calculation of derivatives or evaluation of integrals of a function differ from the analytical methods of calculus in that they are applicable to functions that are known only as table of values as well as to analytically defined functions. There are various well-known numerical schemes for differentiation such as methods based on divided-difference tables, central-difference formulas, extrapolation techniques, etc. Similarly, for integration, the trapezoidal rule, Simpson's rule, Newton-Cotes integration formulas are most frequently used (Gerald and Wheatley, 1994). In this study, the spectral element method is used for differentiation and integration and approximations are based on truncated series of orthogonal polynomials. In particular, the spectral element method adopted in this study is based on Legendre polynomials. Consequently, it is necessary to understand the properties of Legendre polynomials. In the next two sections, the general formulas for POD and LOMs procedures introduced in chapter 1 will be presented in detail. In section 2.3, some important formulas and basic properties of Legendre polynomials are summarized.
2.1 Proper Orthogonal Decomposition (POD)

In applying the POD methodology, the spatiotemporal data obtained through direct numerical simulation are decomposed into time-averaged (\( \overline{U}, \overline{V} \) and \( \overline{\Theta} \)) and time-varying (\( U', V' \) and \( \Theta' \)) parts, i.e.,

for velocity,

\[
U(x,y,t) = \overline{U}(x,y) + U'(x,y,t),
\]
\[
V(x,y,t) = \overline{V}(x,y) + V'(x,y,t),
\]

and for temperature,

\[
\Theta(x,y,t) = \overline{\Theta}(x,y) + \Theta'(x,y,t).
\]

The time-averaged values are obtained by calculating the simple arithmetic mean of \( M \) snapshots of the velocity and temperature fields, i.e.,

\[
\overline{U}(x,y) = \frac{1}{M} \sum_{i=1}^{M} U(x,y,t_i),
\]
\[
\overline{V}(x,y) = \frac{1}{M} \sum_{i=1}^{M} V(x,y,t_i),
\]
\[
\overline{\Theta}(x,y) = \frac{1}{M} \sum_{i=1}^{M} \Theta(x,y,t_i).
\]

In turn, the fluctuating parts of the velocity and temperature fields can be computed by

\[
U'(x,y,t) = U(x,y,t) - \overline{U}(x,y),
\]
\[
V'(x,y,t) = V(x,y,t) - \overline{V}(x,y),
\]
\[
\Theta'(x,y,t) = \Theta(x,y,t) - \overline{\Theta}(x,y).
\]

The empirical eigenfunctions are constructed by an appropriate superposition of the time-varying parts of the field variables (Sirovich, 1987), i.e.,
\[ \Phi_{uk}(x,y) = \sum_{i=1}^{M} A_{ki} U'_i(x,y,t_i), \quad (2.10) \]

\[ \Phi_{vk}(x,y) = \sum_{i=1}^{M} A_{ki} V'_i(x,y,t_i), \quad (2.11) \]

\[ \Psi_k(x,y) = \sum_{i=1}^{M} A_{ki} \Theta'_i(x,y,t_i), \quad (2.12) \]

where \( A_k \) denotes the \( k \)-th eigenvector of the matrix eigenvalue problem

\[ CA = \lambda A, \quad (2.13) \]

and the elements of matrix \( C \) are given by

\[ C_{mn} = \frac{1}{M} \int_{\Omega} F_{mn}(x,y,t) \, d\Omega. \quad (2.14) \]

In eq. (2.14),

\[ F_{mn} = \vec{V}'_m \cdot \vec{V}'_n \]

\[ = (U'_m(x,y,t) + V'_m(x,y,t)) \cdot (U'_n(x,y,t) + V'_n(x,y,t)) \]

\[ = U'_m(x,y,t)U'_n(x,y,t) + V'_m(x,y,t)V'_n(x,y,t) \quad (2.15) \]

for the velocity field, and

\[ F_{mn} = \Theta'_m(x,y,t)\Theta'_n(x,y,t) \quad (2.16) \]

for the temperature field. \( C_{mn} \) is \([M \times M]\) square symmetric matrix and has real elements. \( C_{mn} \) can be written as follows:

For velocity field,

\[ C_{mn} = \frac{1}{M} \iint_{\Omega} \{ U'_m(x,y,t_m)U'_n(x,y,t_n) + V'_m(x,y,t_m)V'_n(x,y,t_n) \} \, d\Omega \quad (2.17) \]
and for temperature field,

\[
C_{mn} = \frac{1}{M} \int \{ \Theta'_m(x, y, t_m) \Theta'_n(x, y, t_n) \} \, d\Omega.
\] (2.18)

Since \( C \) is known, the matrix eigenvalue problem (2.13) can be solved to obtain the eigenvalues and eigenvectors. Note the eq. (2.13) is equivalent to

\[(C - \lambda I) A = 0,
\] (2.19)

where \( I \) is the \([M \times M]\) identity matrix, or

\[
\begin{pmatrix}
C_{11} - \lambda & C_{12} & \cdots & C_{1M} \\
C_{21} & C_{22} - \lambda & \cdots & C_{2M} \\
\vdots & \vdots & \ddots & \vdots \\
C_{M1} & C_{M2} & \cdots & C_{MM} - \lambda
\end{pmatrix}
\begin{pmatrix}
A_1 \\
A_2 \\
\vdots \\
A_M
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
\vdots \\
0
\end{pmatrix}.
\] (2.20)

Note that matrix \( C \) is symmetric, i.e., \( C_{mn} = C_{nm} \) (see eqs. (2.17) and (2.18)), so efficient methods are available for finding its eigenvalues \((\lambda_1, \lambda_2, ..., \lambda_M)\) and corresponding eigenvectors \((A_1, A_2, ..., A_M)\).

The empirical eigenfunctions (eqs. (2.10) - (2.12)) are orthogonal to each other (Gunes et al., 1995) and, after normalization, form an orthonormal set of basis eigenfunctions. If the flow is incompressible, the velocity empirical eigenfunctions are divergence-free. The eigenvalues \(\lambda_k \) \((k=1,2, ..., M)\) are non-negative and each eigenvalue represents the energy that the corresponding eigenfunction contributes to the flow or temperature field. The crucial outcome of applying the POD method is the extraction of these orthogonal eigenfunctions for the velocity and temperature fields. These
eigenfunctions play an important role in the construction of LOMs as described in the next section.

2.2 Low-Order Models (LOMs)

Having obtained the empirical eigenfunctions for velocity and temperature, eqs. (2.10)–(2.12), low-order models can be constructed by the following procedure. After normalizing the eigenfunctions, we expand the time-varying parts of the velocity and temperature fields in terms of the eigenfunctions, i.e.,

\begin{align*}
U'(x,y,t) &= \sum_{k=1}^{M_1} a_k(t) \Phi_{uk}(x,y), \\
V'(x,y,t) &= \sum_{k=1}^{M_1} a_k(t) \Phi_{vk}(x,y), \\
\Theta'(x,y,t) &= \sum_{k=1}^{M_2} b_k(t) \Psi_k(x,y),
\end{align*}

where \(\Phi_{uk}\) and \(\Phi_{vk}\) denote the x- and y-components of the \(k^{th}\) eigenfunction of the velocity field respectively and \(\Psi_k\) denotes the \(k^{th}\) eigenfunction of the temperature field.

The instantaneous velocity and temperature fields can now be expressed as

\begin{align*}
U(x,y,t) &= \bar{U}(x,y) + \sum_{k=1}^{M_1} a_k(t) \Phi_{uk}(x,y), \\
V(x,y,t) &= \bar{V}(x,y) + \sum_{k=1}^{M_1} a_k(t) \Phi_{vk}(x,y), \\
\Theta(x,y,t) &= \bar{\Theta}(x,y) + \sum_{k=1}^{M_2} b_k(t) \Psi_k(x,y).
\end{align*}
Note that $a_k$ and $b_k$ are unspecified at this stage. To determine $a_k$ and $b_k$, which are time-
dependent, we substitute eqs. (2.24) - (2.26) into the governing equations. Even if we restrict our attention to thermal convection problem, the governing equations are problem-
dependent due to different nondimensionalization required for specific problems.

In this study, the governing partial differential equations for transitional convective flow are written in the form:

Continuity equation:

$$\nabla \cdot \vec{V} = 0$$  \hspace{1cm} (2.27)

Momentum equation:

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} + \nabla P = \Theta j + \frac{1}{\sqrt{Gr}} \nabla^2 \vec{V}$$  \hspace{1cm} (2.28)

Energy equation:

$$\frac{\partial \Theta}{\partial t} + \vec{V} \cdot \nabla \Theta = \frac{1}{Pr \sqrt{Gr}} \nabla^2 \Theta.$$  \hspace{1cm} (2.29)

Dimensionless variables are defined here as follows:

$$(X,Y) = (x,y)/l,$$  \hspace{1cm} (2.30)

$$t = \frac{u_e t^*}{l},$$  \hspace{1cm} (2.31)

$$(U,V) = \frac{(u,v)}{u_e},$$  \hspace{1cm} (2.32)

$$P = \frac{p}{\rho u_e^2},$$  \hspace{1cm} (2.33)
\[ \Theta = \begin{cases} \frac{T - T_1}{T_2 - T_1} & : \text{Enclosure case,} \\
\frac{k}{q''} (T - T_b) & : \text{Channel case.} \end{cases} \quad (2.34) \]

Note that \( u_e = \sqrt{\beta g I (T_2 - T_1)} \) for the enclosure flow and \( \sqrt{\beta g I q''/k} \) for the channel flow. Furthermore, the Prandtl number and Grashof number are defined as

\[ \text{Pr} = \frac{\frac{\beta g I (T_2 - T_1) t^3}{v^2}}{\frac{\beta g q'' I^4}{v^2 k}} \quad \text{Enclosure,} \]
\[ \text{Gr} = \begin{cases} \frac{\beta g I (T_2 - T_1) t^3}{v^2} & : \text{Enclosure,} \\
\frac{\beta g q'' I^4}{v^2 k} & : \text{Channel,} \end{cases} \quad (2.36) \]

where \( \alpha \) is the thermal diffusivity, \( \beta \) is the thermal expansion coefficient, \( v \) is the kinematic viscosity and \( g \) is the acceleration of gravity. For the enclosure case, \( T_1, T_2 \) are the cold and hot wall temperature respectively. For the channel case, \( T_b \) is the reference temperature, \( k \) is the thermal conductivity of the fluid and \( q'' \) denotes the uniform input heat flux of the heat source. The width of enclosure or channel is denoted by \( l \). For further details see sections 3.1.1 and 3.2.1. Substituting eqs. (2.24) - (2.26) into the dimensionless \( x \)- and \( y \)-momentum equations and the dimensionless energy equation, applying Galerkin method and using the orthonormality property of the empirical eigenfunctions, we obtain the following \( (M_1 + M_2) \) non-linear ODEs:

From momentum equation:

\[ \frac{d\alpha_k}{dt} = A_k + \frac{1}{\sqrt{Gr}} B_k + \sum_{i=1}^{M_1} a_i C_{ki} + \frac{1}{\sqrt{Gr}} \sum_{i=1}^{M_1} a_i D_{ki} \]
\[ + \sum_{i=1}^{M_1} \sum_{j=1}^{M_1} a_{ij} E_{ki} + \sum_{i=1}^{M_1} b_i F_{ki}, \quad k=1,2,\ldots,M_1. \]  

(2.37)

From energy equation:

\[ \frac{db_k}{dt} = G_k + \frac{1}{Pr\sqrt{Gr}} H_k + \sum_{i=1}^{M_2} a_i I_{ki} + \frac{1}{Pr\sqrt{Gr}} \sum_{i=1}^{M_2} b_i J_{ki} \]

\[ + \sum_{i=1}^{M_1} \sum_{j=1}^{M_1} a_{ij} K_{ki} + \sum_{i=1}^{M_1} b_i L_{ki}, \quad k=1,2,\ldots,M_2. \]  

(2.38)

Generally, \( M_1 \) and \( M_2 \) are chosen to be much smaller than \( M \). The coefficients shown in the above equations come from the various inner products among eigenmodes and/or mean flow and temperature. Equations (2.37) and (2.38) are integrated numerically to obtain \( a_k(t) \) and \( b_k(t) \) using a fourth order Runge-Kutta solver.

### 2.3 Legendre polynomials

The Legendre polynomials \( \{ L_k(x), k=0,1,\ldots \} \) are defined as the eigenfunctions of the singular Sturm-Liouville problem (Canuto et al., 1988)

\[ \frac{d}{dx} \left[ (1-x^2) \frac{dL_k(x)}{dx} \right] + k(k+1) L_k(x) = 0, \quad -1 \leq x \leq 1. \]  

(2.39)

\( L_k(x) \) is even if \( k \) is even, and odd if \( k \) is odd.

The Legendre polynomials satisfy the following recurrence relation

\[ (k+1)L_{k+1}(x) = (2k+1) xL_k(x) - k L_{k-1}(x), \quad k=1,2,\ldots, \]  

(2.40)

where \( L_0(x) = 1 \) and \( L_1(x) = x \). By using the recurrence relation, eq. (2.40), we find that

\[ L_2(x) = \frac{1}{2} (3x^2 - 1), \]
Furthermore, \( L_k(\pm 1) = (\pm 1)^k \) and \( L'_k(\pm 1) = (\pm 1)^k \frac{1}{2} k(k + 1) \).

The Legendre polynomials satisfy the orthogonality relation over the interval \([-1, 1]\), i.e.,

\[
\int_{-1}^{1} L_n(x)L_m(x) \, dx = \frac{2}{2n + 1} \delta_{nm},
\]

where \( \delta_{nm} \) is Kronecker's delta. Note also that a well-behaved function \( f(x) \) defined in \([-1, 1]\) can be expanded in Legendre polynomials series

\[
f(x) = \sum_{k=0}^{\infty} a_k L_k(x),
\]

where

\[
a_k = \frac{2k + 1}{2} \int_{-1}^{1} f(x)L_k(x) \, dx.
\]

In this section, we have presented some important properties of Legendre polynomials. In the next section, the general concept of Gaussian quadrature will be studied.

### 2.4 Gaussian quadrature method

Trapezoidal rule, Simpson's rule and Newton-Cotes integration formulas are widely used for numerical integration (quadrature). Those schemes are usually
implemented on evenly spaced x-values and the integral of a function is approximated by the sum of its functional values multiplied by appropriate coefficients.

The Gaussian quadrature method chooses the location of points at which the function is to be computed as well as the weighting factors in a way that reduces significantly the error associated with the numerical evaluation of the integral. The collocation points in the Gaussian quadrature method are no longer evenly spaced. The quadrature nodes are computed from the zeros of Legendre polynomials or modified formulas. There are three well-known choices of collocation points and corresponding quadrature weights: Legendre-Gauss, Legendre-Gauss-Radau and Legendre-Gauss-Lobatto (Canuto et al., 1988). The collocation points \( x_j \) \( (j = 0, 1, \ldots, N) \) in the Legendre-Gauss and Legendre-Gauss-Radau method are the zeros of \( L_{N+1} \) and \( L_N L_{N+1} \) respectively. In this study, Legendre-Gauss-Lobatto collocation points and quadrature weights are used. The collocation points \( x_j \) \( (j = 1, \ldots, N-1) \) are the zeros of the first derivative of \( L_N \) augmented by points \( x_0 = -1, x_N = 1 \), and the weights are given by

\[
w_j = \frac{2}{N(N+1)} \frac{1}{[L_n(x_j)]^2}, \quad j = 0, \ldots, N. \tag{2.45}
\]

In this section, we have discussed the Gaussian quadrature method and the Legendre-Gauss-Lobatto formula for the quadrature weights. In the next two sections, spectral differentiation and integration using the Legendre-Gauss-Lobatto nodes will be presented.
2.5 Spectral differentiation

Let us consider first spectral differentiation in the physical space. In this study, the Legendre polynomial is employed for the spectral implementation. The finite series defined by discrete transform is the interpolant of $u$ at the nodes (Canuto et al., 1988).

If the function $u$ is known at one set of nodes, we can compute an approximation of the first derivative at the nodes through matrix multiplication, namely,

$$ (D_N u)(x_i) = \sum_{j=0}^{N} (D_N)_{ij} u(x_j), \quad i = 0, \ldots, N, \tag{2.46} $$

where $N$ is the order of interpolants. The $[N \times N]$ differentiation matrix is formed by (Canuto et al., 1988)

$$ (D_N)_{ij} = \begin{cases} 
\frac{1}{L_N'(x_j)} \cdot \frac{1}{L_N'(x_i)} \cdot \frac{x_i - x_j}{x_i - x_j} & \text{if } i \neq j \\
\frac{1}{4} (N+1)N & \text{if } i = j = 0 \\
-\frac{1}{4} (N+1)N & \text{if } i = j = N \\
0 & \text{if } i = j \text{ (i,j = 1,\ldots,N-1)}
\end{cases} \tag{2.47} $$

We should note that the collocation points in equations (2.46) and (2.47) are the zeros of the first derivative of $L_N$ and $x_0 = -1$ and $x_N = 1$ (see section 2.4).

Using eq. (2.46), we can also compute the second derivative at the nodes. If we calculate the first derivative of $u(x_j)$, then the multiplication of differentiation matrix $D_N$ with the first derivative of $u(x_j)$ yields the second derivative of $u$ at points $x_i$, i.e.,

$$ \frac{d^2}{dx^2} [u(x_i)] = \sum_{j=0}^{N} (D_N)_{ij} \frac{du(x_j)}{dx}, \quad i = 0, \ldots, N. \tag{2.48} $$
In general, when we deal with an arbitrary interval \( a \leq x \leq b \), the Legendre-Gauss-Lobatto collocation points, \(-1 \leq t \leq 1\), need to be transformed to \( a \leq x \leq b \). This is easily accomplished by the linear transformation

\[
x = \frac{1}{2} [(b - a)t + (b + a)].
\]

(2.49)

Then,

\[
dx = \frac{b - a}{2} \, dt.
\]

(2.50)

By chain rule, \( f'( = df/dx ) \) can be calculated by

\[
f' = \frac{df}{dx} = \frac{df}{dt} \frac{dt}{dx},
\]

(2.51)

where \( df/dt \) is computed based on eq. (2.46). By eq. (2.50), \( a \leq x \leq b \), we obtain

\[
f' = \frac{df}{dx} = \frac{2}{b - a} \frac{df}{dt}.
\]

(2.52)

### 2.6 Spectral integration

#### 2.6.1 One-dimensional problems

If the values of the function \( f(t) \) at the Legendre-Gauss-Lobatto points are known, we can compute its integral by the following equation

\[
\int_{-1}^{1} f(t) \, dt \equiv \sum_{i=0}^{N} w_i f(t_i),
\]

(2.53)

where the weights \( w_i \) are given by eq. (2.45) and \( f(t_i) \) is the function value at the Legendre-Gauss-Lobatto points. The points \( t_i \) \( (i = 1,...,N-1) \) are the zeros of the first derivative of \( L_N \), as already explained in the previous sections.
2.6.2 Two-dimensional problems

To perform two-dimensional integration we can extend the Gaussian quadrature method described for one-dimensional problems, eq. (2.53). Let's consider that we desire to compute the following double integral of \( f(x,y) \)

\[
\int_{-1}^{1} \int_{-1}^{1} f(x,y) \, dy \, dx, \quad (2.54)
\]

and \( f(x,y) \) is known at the Gauss points \( x_i \) (i = 0, ..., m), \( y_j \) (j = 0, ..., n). Note that in this thesis \((x_i,y_j)\) are the Legendre-Gauss-Lobatto collocation points.

In order to compute the double integral of \( f(x,y) \) over the square \(-1 \leq x \leq 1\), \(-1 \leq y \leq 1\), we perform the integration along the y-direction first and then along the x-direction. Let \( S_i \) be the corresponding value obtained using Gaussian quadrature for a fixed \( x_i \). Then \( S_i \) (i fixed) can be computed

\[
S_i = \int_{-1}^{1} f(x_i, y) \, dy
\]

\[
= \sum_{j=0}^{n} w_j f(x_i, y_j). \quad (2.55)
\]

Now we compute the value of the double integral after doing integration along the x-direction, namely,

\[
I_{xy} = \int_{-1}^{1} \int_{-1}^{1} f(x,y) \, dy \, dx
\]

\[
= \sum_{i=0}^{m} w_i S_i, \quad (2.56)
\]

where \( I_{xy} \) denotes the double integral of \( f(x,y) \) considered in eq. (2.54).

If we have arbitrary upper and lower limits in the double integral such as,
we can relate those arbitrary upper and lower limits with variable \( t \) in a manner similar to that shown in eq. (2.49).

If \(-1 \leq t \leq 1\), the appropriate transformations are:

\[
x = \frac{1}{2} [(b - a)t + (b + a)],
\]

\[
y = \frac{1}{2} [(d - c)t + (d + c)].
\]

(2.58)

Then,

\[
\frac{dx}{dt} = \frac{b-a}{2},
\]

and

\[
\frac{dy}{dt} = \frac{d-c}{2}.
\]

(2.59)

Also, by chain rule we have that,

\[
\frac{df}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt} = \frac{b-a}{2} \cdot \frac{df}{dx},
\]

\[
\frac{df}{dt} = \frac{df}{dy} \cdot \frac{dy}{dt} = \frac{d-c}{2} \cdot \frac{df}{dy}.
\]

(2.60)

Combining eq. (2.60) with eqs. (2.55) and (2.56), we obtain the following equations in the case of arbitrary limits

\[
I_{xy} = \int_{a}^{b} \int_{c}^{d} f(x, y) dy dx
\]

\[
= \frac{b-a}{2} \cdot \frac{d-c}{2} \sum_{i=0}^{m} w_i S_i,
\]

(2.61)
where \( S_i \) can be calculated as in eq. (2.55).

2.7 Weak forms of differential equations

In the process of performing differentiation and integration numerically, it is important to recognize the effect of propagation of truncation and round-off errors on the accuracy of the results.

It is known that the procedure of differentiation is basically an "unstable" one in the sense that even small errors in the initial data can cause significant errors in the final results. Differentiation of noisy data from laboratory measurements can pose a similar problem. On the other hand, the process of integration sums up function values together. The errors can be cancelled out because of different sign. In general, the integration is a smoothing process which means that we can regard the integration process as inherently "stable".

In constructing LOMs, we need to compute the double integrals that correspond to various inner products among eigenmodes and/or mean flow and temperature to obtain the coefficients shown in eqs. (2.37) and (2.38) (See also Appendices I and II). Some double integrals contain the second derivative in the integrand. Using the well-known relations for integration by parts, we can reduce the order of the highest derivative appearing in the integrand, thus obtaining formulas that lead to smaller numerical errors.

Let's consider the integration by parts of the following two-dimensional expressions:
The general expressions for the integration by parts (see Zienkiewicz, 1977) are:

\[
\iint_{\Omega} \frac{\partial^2 \Phi_u}{\partial x^2} \cdot \Phi_u \, dx \, dy ,
\]

\[
\iint_{\Omega} \frac{\partial^2 \Phi_u}{\partial y^2} \cdot \Phi_u \, dx \, dy. \tag{2.62}
\]

where the last term in the right hand side is the contour integral taken around an anticlockwise direction and \( n_x, n_y \) are the direction cosines between the outward normal and the \( x, y \) axes respectively. By applying eq. (2.63), the weak form of eq. (2.62) is:

\[
\iint_{\Omega} \frac{\partial^2 \Phi_u}{\partial x^2} \cdot \Phi_u \, dx \, dy = -\iint_{\Omega} \frac{\partial \Phi_u}{\partial x} \cdot \frac{\partial \Phi_u}{\partial x} \, dx \, dy \\
+ \int_{\Gamma} \Phi_u \cdot \frac{\partial \Phi_u}{\partial x} \, n_x \, d\Gamma,
\]

\[
\iint_{\Omega} \frac{\partial^2 \Phi_u}{\partial y^2} \cdot \Phi_u \, dx \, dy = -\iint_{\Omega} \frac{\partial \Phi_u}{\partial y} \cdot \frac{\partial \Phi_u}{\partial y} \, dx \, dy \\
+ \int_{\Gamma} \Phi_u \cdot \frac{\partial \Phi_u}{\partial y} \, n_y \, d\Gamma. \tag{2.64}
\]

Note that the closed contour integral is problem-dependent. Consider, for example, the cavity which is one of the physical domains discussed in this study (Fig. 2.1). In this case, we should take into account the specific boundary conditions along the boundary \( \Gamma \) which is composed of \( \Gamma_1, \Gamma_2, \Gamma_3 \) and \( \Gamma_4 \).
Figure 2.1 Domain for integration by parts.
CHAPTER 3 -
RESULTS

In chapter 2, the methodology of proper orthogonal decomposition (POD) and the construction of low-order models (LOMs) were described. This chapter presents the results of two different buoyancy-driven, transitional flows: convection in an enclosure (section 3.1) and convection in a vertical channel (section 3.2). The results of the fully spectral element method will be compared with those of the finite difference method.

3.1 Case study 1: Thermal convection in an enclosure

Natural convection air cooling is regarded as an effective mode of cooling of electronic devices in many low heat flux applications ($q'' \leq 0.1 \text{ W/cm}^2$). Peterson and Ortega (1990) presented thermal control schemes for electronic equipment and a comprehensive review of convection cooling options has been published by Incropera (1988). Liakopoulos et al. (1990) and Huang and Liakopoulos (1993) performed numerical investigations of two-dimensional thermally-driven flows in cavities and vertical channels for $Pr=0.71$ (air-filled) and a wide range of Gr. Lee et al. (1987) and Kelleher et al. (1987) studied the effect of the placement of a heat source on an adiabatic wall of a two-dimensional rectangular enclosure.
In this study, a low-order model of buoyancy-induced flow in an enclosure subject to differential side heating is presented. Furthermore, suggestions will be given on how low-dimensional predictions can be improved by applying the spectral element method in the POD and Galerkin procedure.

3.1.1 Mathematical model

Fig. 3.1 shows the computational domain of the problem. The spectral element mesh (Fig. 3.2) is composed of 80 elements and each element has 11 mesh points in x and y direction.

The dimensionless governing partial differential equations for time-dependent buoyancy-driven flow can be written as follows:

Continuity equation:
\[ \nabla \cdot \vec{V} = 0 \]  
(3.1)

Momentum equation:
\[ \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} + \nabla P = \Theta \cdot \vec{V} + \frac{1}{\sqrt{\text{Gr}}} \nabla^2 \vec{V} \]  
(3.2)

Energy equation:
\[ \frac{\partial \Theta}{\partial t} + \vec{V} \cdot \nabla \Theta = \frac{1}{\text{Pr} \sqrt{\text{Gr}}} \nabla^2 \Theta. \]  
(3.3)

Here, dimensionless variables are defined as

\[ (X, Y) = (x, y)/1, \]
\[ t = \frac{u_c}{1} t^*, \]
(U, V) = \frac{(u, v)}{u_c},

P = \frac{p}{\rho u_c^2},

\Theta = \frac{T - T_1}{T_2 - T_1}, \quad (3.4)

where, \( u_c = \sqrt{\beta g (T_2 - T_1)} \), and \( T_1, T_2 \) are the cold and hot wall temperatures respectively. The governing dimensionless parameters, Prandtl and Grashof numbers, are defined as,

\[ Pr = \frac{v}{\alpha}, \]

\[ Gr = \frac{\beta g (T_2 - T_1) l^3}{v^2}, \quad (3.5) \]

where, \( v \) is the kinematic viscosity, \( \alpha \) is the thermal diffusivity, \( \beta \) is the thermal expansion coefficient, and \( g \) is the acceleration of gravity.

No slip boundary conditions are imposed along all walls. The left vertical wall is subject to constant temperature (\( \Theta_2 = 1 \)) and the right vertical wall is kept at \( \Theta_1 = 0 \). Also, the top and bottom walls are thermally insulated.

Direct numerical simulation is performed for \( A_y = 5 \), \( Gr = 2 \times 10^6 \), \( Pr = 0.71 \). Instantaneous streamlines and isotherms are shown in Figures 3.3 and 3.4. Typical history plots of the non-dimensional y-velocity and temperature at a fixed point are presented in Figure 3.5. The signals (functions) are periodic in time.
Figure 3.1 Computational domain and boundary conditions.
Figure 3.2 Spectral element mesh.
Figure 3.3 Instantaneous streamlines, $A_y = 5$, Pr = 0.71, Gr = $2 \times 10^6$. 
Figure 3.4 Instantaneous isotherms, \( A_y = 5, \text{Pr} = 0.71, \text{Gr} = 2 \times 10^6 \).
Figure 3.5 Non-dimensional y-velocity and temperature history plots;
(a) x = 0.258, y = 1.909,
(b) x = 0.662, y = -0.38,
(c) x = 0.162, y = -2.327.
3.1.2 Proper Orthogonal Decomposition (POD)

Table 3.1 shows the eigenvalues and the cumulative energy contribution of the four most energetic velocity and temperature modes as calculated by the spectral element method and the finite difference method. Recall that a normalized eigenvalue represents the contribution of the corresponding mode to the fluctuation energy of the system.

In Table 3.1, the first modes of velocity and temperature capture almost 50% of the fluctuation energy. The difference between the spectral element method and the finite difference method is that the first velocity mode calculated by the spectral element method indicates slightly more energy than the finite difference method. However, the finite difference method predicts slightly more energy associated with the first temperature mode.

The first two velocity modes show strong dominance over the other modes and capture approximately 98% of the whole energy. Similar trends can be observed in the temperature modes. Note that the eigenvalues $\lambda_1$ and $\lambda_2$ are almost equal and similar relation exists between $\lambda_3$ and $\lambda_4$ in both velocity and temperature modes. In both methods, the cumulative contribution of the first four modes reaches the energy level of 99.95% for the velocity field, and 99.89% for the temperature field. Thus, the first four eigenmodes enable us to predict the structure of the field and its dynamics with excellent accuracy.
Table 3.1. Comparison between spectral element method and finite difference method. Eigenvalues of the four most energetic modes and their cumulative energy contribution.

1. Spectral element method (N=11)
   a. Velocity modes

<table>
<thead>
<tr>
<th>Modes</th>
<th>Eigenvalue</th>
<th>Normalized Eigenvalue</th>
<th>Energy (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0007080</td>
<td>0.4973665</td>
<td>49.737</td>
</tr>
<tr>
<td>2</td>
<td>0.0006936</td>
<td>0.4872884</td>
<td>98.466</td>
</tr>
<tr>
<td>3</td>
<td>0.0000107</td>
<td>0.7483236×10⁻²</td>
<td>99.214</td>
</tr>
<tr>
<td>4</td>
<td>0.0000105</td>
<td>0.7399076×10⁻²</td>
<td>99.954</td>
</tr>
</tbody>
</table>

   b. Temperature modes

<table>
<thead>
<tr>
<th>Modes</th>
<th>Eigenvalue</th>
<th>Normalized Eigenvalue</th>
<th>Energy (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.12277×10⁻³</td>
<td>0.4892561</td>
<td>48.926</td>
</tr>
<tr>
<td>2</td>
<td>0.12065×10⁻³</td>
<td>0.4808142</td>
<td>97.007</td>
</tr>
<tr>
<td>3</td>
<td>0.36317×10⁻⁵</td>
<td>0.1447354×10⁻¹</td>
<td>98.455</td>
</tr>
<tr>
<td>4</td>
<td>0.35899×10⁻⁵</td>
<td>0.1430709×10⁻¹</td>
<td>99.885</td>
</tr>
</tbody>
</table>
2. Finite difference method (N=11)

a. Velocity modes

<table>
<thead>
<tr>
<th>Modes</th>
<th>Eigenvalue</th>
<th>Normalized Energy Value</th>
<th>Energy (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0007034</td>
<td>0.4969366</td>
<td>49.694</td>
</tr>
<tr>
<td>2</td>
<td>0.0006904</td>
<td>0.4877924</td>
<td>98.473</td>
</tr>
<tr>
<td>3</td>
<td>0.0000106</td>
<td>0.7453982×10⁻²</td>
<td>99.219</td>
</tr>
<tr>
<td>4</td>
<td>0.0000104</td>
<td>0.7356023×10⁻²</td>
<td>99.954</td>
</tr>
</tbody>
</table>

b. Temperature modes

<table>
<thead>
<tr>
<th>Modes</th>
<th>Eigenvalue</th>
<th>Normalized Energy Value</th>
<th>Energy (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.12194×10⁻³</td>
<td>0.4917965</td>
<td>49.179</td>
</tr>
<tr>
<td>2</td>
<td>0.11858×10⁻³</td>
<td>0.4782492</td>
<td>97.004</td>
</tr>
<tr>
<td>3</td>
<td>0.35986×10⁻⁵</td>
<td>0.1451317×10⁻¹</td>
<td>98.455</td>
</tr>
<tr>
<td>4</td>
<td>0.35425×10⁻⁵</td>
<td>0.1428692×10⁻¹</td>
<td>99.884</td>
</tr>
</tbody>
</table>

The first six empirical eigenfunctions for velocity and temperature are shown in Figs. 3.6 - 3.11. The two most energetic modes capture the larger scale features of the flow. Modes with smaller eigenvalues contain smaller scale features. Similar trend can be observed in the temperature eigenfunctions. Note that the velocity and temperature empirical eigenfunctions are centro-symmetric. Empirical eigenfunctions for velocity and temperature computed by the spectral element implementation are almost identical to those computed by the finite difference implementation.

Fig. 3.12 shows the computed temporal eigenmodes (temporal expansion coefficients) obtained by direct projection of the input velocity and temperature data by the spectral element method and the finite difference method. It can be noticed that they are almost identical.
Figure 3.6 Velocity empirical eigenfunctions.
(a) $\Phi_1$ (b) $\Phi_2$
Figure 3.7 Velocity empirical eigenfunctions.

(a) $\Phi_3$  (b) $\Phi_4$
Figure 3.8 Velocity empirical eigenfunctions.
(a) $\Phi_5$ (b) $\Phi_6$
Figure 3.9 Temperature empirical eigenfunctions.

(a) $\Psi_1$ (b) $\Psi_2$
Figure 3.10 Temperature empirical eigenfunctions.
(a) $\Psi_3$ (b) $\Psi_4$
Figure 3.11 Temperature empirical eigenfunctions.
(a) $\Psi_5$ (b) $\Psi_6$
Figure 3.12 Temporal expansion coefficients based on direct projection.
(a),(b): spectral element implementation,
(c),(d): finite difference implementation.
Line style: solid–a, or b, dotted–a, or b,
dashed–a, or b, long dashed–a, or b.
3.1.3 Low-Order Model predictions (LOM)

After applying the POD and LOM procedures, the following non-linear ODEs are obtained for the temporal expansion coefficients:

\[
\frac{da_k}{dt} = A_k + \frac{1}{\sqrt{Gr}}B_k + \sum_{i=1}^{M_1} a_i C_{ki} + \frac{1}{\sqrt{Gr}} \sum_{i=1}^{M_1} a_i D_{ki} + \sum_{i=1}^{M_1} \sum_{j=1}^{M_1} a_j E_{kij} + \sum_{i=1}^{M_1} b_j F_{ki}, \quad k=1,2,\ldots,M_1, \tag{3.6}
\]

\[
\frac{db_k}{dt} = G_k + \frac{1}{Pr\sqrt{Gr}}H_k + \sum_{i=1}^{M_1} a_i I_{ki} + \frac{1}{Pr\sqrt{Gr}} \sum_{i=1}^{M_2} b_j J_{ki} + \sum_{i=1}^{M_1} \sum_{j=1}^{M_1} a_j b_j K_{kij} + \sum_{i=1}^{M_1} b_j L_{ki}, \quad k=1,2,\ldots,M_2. \tag{3.7}
\]

The explicit form of eqs. (3.6) and (3.7) is provided in Appendix I for \( M_1 = M_2 = 4 \). The fully coupled eqs. (3.6) and (3.7) are integrated numerically using a fourth order Runge-Kutta solver.

The temporal expansion coefficients of \( a_i, b_i \) (\( i=1,\ldots,4 \)) of the low-order model predictions computed by the spectral element method are shown in Fig. 3.13. As time increases, \( a_i \) and \( b_i \) become stable, exhibiting self-sustained oscillations in time. Maximum and minimum values of \( a_1 \) and \( a_2 \) are initially 0.0365 and -0.0382. After reaching a stable, self-sustained, constant-amplitude oscillation state, the maximum value is decreased by approximately 10\% and the minimum value is decreased by 11\%. In the case of \( b_1 \), the amplitude is also reduced by about 9\% and 13\% for maximum and
minimum values respectively. The low-order model predictions of \(a_i\) and \(b_i\) by the spectral element method are slightly smaller than those predicted by the full model simulation, but in general the behavior of low-order model predictions for \(a_i\) and \(b_i\) by the spectral element method (Fig. 3.13) are in excellent agreement with those obtained by the direct simulation (see Fig. 3.12).

The low-order model predictions of \(a_i\) and \(b_i\) by the finite difference method are shown in Fig. 3.14. The amplitudes of \(a_i\) are initially rapidly decreasing (Fig. 3.14; a) and finally converge to zero as shown in Fig. 3.14 (b). Identical behavior for \(b_i\) can be observed in Fig. 3.14 (c) and (d). Again the amplitudes of \(b_i\) are reduced in a very short time interval and eventually converge to zero. The low-order model developed using the finite difference method is unable to reproduce the full model dynamical behavior which exhibits stable, self-sustained oscillations as shown in Fig. 3.12. Thus the spectral element method is superior to the finite difference method in identifying the correct dynamical behavior of the system.

Table 3.2 shows the ratios of the amplitudes of \(a_i\) and \(b_i\) as predicted by the low-order model and those predicted by the full model. The spectral element method and the finite difference method are compared. Note that if the low-order model reproduced exactly the dynamics of the full model, the ratio would be unity.
Table 3.2 Ratios of temporal expansion coefficients computed by direct projection (DP) and low-order model (LOM). Comparison of spectral element method (SEM) and finite difference method (FDM).

<table>
<thead>
<tr>
<th>Ratios</th>
<th>SEM</th>
<th>FDM</th>
</tr>
</thead>
<tbody>
<tr>
<td>MAX : $a_i$ (lom) / $a_i$ (dp)</td>
<td>0.0328/0.0369=0.889</td>
<td>0</td>
</tr>
<tr>
<td>MAX : $b_j$ (lom) / $b_j$ (dp)</td>
<td>0.0319/0.0156=0.891</td>
<td>0</td>
</tr>
<tr>
<td>MIN : $a_i$ (lom) / $a_i$ (dp)</td>
<td>-0.034/-0.0378=0.899</td>
<td>0</td>
</tr>
<tr>
<td>MIN : $b_i$ (lom) / $b_i$ (dp)</td>
<td>-0.0142/-0.016=0.888</td>
<td>0</td>
</tr>
</tbody>
</table>

For the spectral element method, the ratios of low-order model predictions over the full model predictions are approximately 0.9 for both $a_i$ and $b_i$. The low-order model predictions using spectral element method are 10% off the full model simulations. For the finite difference method, the ratios approach zero which means that the low-order model fails to predict the oscillations. Consequently, the spectral element method is highly recommended for the construction of accurate low-order models in this enclosure problem.
Figure 3.13 Low-order model prediction based on spectral element implementation.
(a),(b) : a(t)–velocity modes,
(c),(d) : b(t)–temperature modes.
Line style: solid–a₁ or b₁, dotted–a₂ or b₂,
dashed–a₃ or b₃, long dashed–a₄ or b₄.
Figure 3.14 Low-order model prediction based on finite difference implementation.
(a),(b) : a(t)–velocity modes,
(c),(d) : b(t)–temperature modes.
Line style : solid–a, or b, dotted–a, or b,
dashed–a, or b, long dashed–a, or b.
3.2 Case study 2: Thermal convection in a vertical channel

In this section, a low-dimensional representation of transitional buoyancy-induced flow in a vertical channel with spatially-periodic flush-mounted discrete heaters is developed. Comparisons between the fully spectral element implementation and the finite difference implementation procedures will be presented.

3.2.1 Mathematical model

The computational domain is shown in Fig. 3.15. The spectral element mesh is composed of 32 elements (see Fig. 3.16).

The non-dimensional time-dependent governing partial differential equations are as follows:

Continuity equation:

\[ \nabla \cdot \vec{\bar{V}} = 0 \]  
(3.8)

Momentum equation:

\[ \frac{\partial \vec{\bar{V}}}{\partial t} + (\vec{\bar{V}} \cdot \nabla) \vec{\bar{V}} + \nabla P = \Theta \bar{j} + \frac{1}{\sqrt{Gr}} \nabla^2 \vec{\bar{V}} \]  
(3.9)

Energy equation:

\[ \frac{\partial \Theta}{\partial t} + \vec{\bar{V}} \cdot \nabla \Theta = \frac{1}{\text{Pr} \sqrt{Gr}} \nabla^2 \Theta. \]  
(3.10)

Dimensionless variables are as follows:

\[ (X, Y) = (x, y)/1, \]

\[ t = \frac{u_e}{t^*}, \]

45
(U, V) = \frac{(u, v)}{u_c},

p = \frac{\rho}{\rho u_c^2},

\Theta = \frac{k}{q''l}(T - T_b),

(3.11)

where, \( u_c = \sqrt{\frac{\beta g l^2 q''}{k}} \). Here, the Prandtl and Grashof numbers are defined by

Pr = \frac{\nu}{\alpha},

Gr = \frac{\beta g q'' l^d}{\nu^2 k}. \quad (3.12)

In the above equations, (x,y) and (u,v) denote the dimensional Cartesian coordinates and velocity components respectively, \( p \) is the pressure relative to the background hydrostatic pressure distribution, \( l \) is the width of the channel, \( k \) is the thermal conductivity of the fluid, \( \alpha \) is the thermal diffusivity, \( \nu \) is the kinematic viscosity, \( \beta \) is the thermal expansion coefficient, \( g \) is the gravitational acceleration, \( \rho \) is the reference density, \( T_b \) is the reference temperature and \( q'' \) denotes the uniform heat flux of the heat sources. No slip conditions are imposed at the right and left vertical walls. Periodic boundary conditions are assumed at the top and bottom sides of the computational domain. The right vertical wall is kept at constant temperature, while the left vertical wall is divided into a constant heat flux heat source and an adiabatic segment.
The governing eqs. (3.8) - (3.10) are solved by a spectral element method. Direct numerical simulation is performed for \( A_c = \frac{l}{l} = 2, A_y = \frac{h}{l} = 4, \text{Gr}=22500, \text{Pr}=0.71. \)

Instantaneous streamlines and isotherms are shown in Figures 3.17 and 3.18. History plots for the non-dimensional \( y \)-velocity and temperature are presented in Figure 3.19. One period of the signals is shown in each case.
Figure 3.15 Computational domain and boundary conditions.
Figure 3.16 Spectral element mesh.
Figure 3.17 Instantaneous streamlines, $A_y = 4$, $A_z = 2$, $Gr = 22500$, $Pr = 0.71$. 
Figure 3.18 Instantaneous isotherms, $A_y = 4$, $A_x = 2$, $Gr = 22500$, $Pr = 0.71$. 

Figure 3.19 Non-dimensional \( y \)-velocity and temperature history plots;
(a) \( x = 0.054, y = 1.054 \),
(b) \( x = 0.008, y = -1.984 \),
(c) \( x = 0.777, y = 0.554 \).
3.2.2 Proper Orthogonal Decomposition (POD)

The velocity and temperature eigenvalues and eigenfunctions are obtained by applying the proper orthogonal decomposition. The eigenvalues are listed in Table 3.3 where the spectral element and finite difference based results are compared. The eigenvalues of the four most energetic modes and their respective cumulative energy contributions to the total fluctuation energy are shown. The eigenvalues are ordered based on their magnitudes and are listed for order of interpolants $N = 9$ and $11$. The eigenvalues occur in pairs of approximately equal magnitude. Slightly more energy is captured by each mode in the spectral element implementation. The cumulative energy contribution of the first four modes reaches the energy level of 99.98% of the velocity and temperature fields in both implementations ($N = 9, 11$). These four modes enable us to reconstruct the flow and temperature fields with very high degree of accuracy.

Figs. (3.20) and (3.21) show the first four empirical eigenfunctions for velocity and Figs. (3.22) and (3.23) depict the empirical eigenfunctions for the temperature field. All eigenfunctions have been computed by the spectral element method with $N = 9$. The large scale feature of the flow are captured by the first two most energetic modes while the smaller features are captured by the less energetic modes. Furthermore, the spatial structures are characterized by a phase shift of approximately $\frac{\pi}{2}$ in the streamwise direction. Similar observations can be made for the temperature eigenfunctions shown in Figs. (3.22) and (3.23). This type of behavior has also been observed by Deane et al. (1991) and Rajaee and Karlsson (1990).
Table 3.3  Case study 2.  Eigenvalues and cumulative energy contribution.

Comparison of spectral element implementation and finite difference implementation.

1. Spectral element implementation  (N=9)

   a. Velocity modes

<table>
<thead>
<tr>
<th>Modes</th>
<th>Eigenvalue</th>
<th>Normalized Eigenvalue</th>
<th>Energy (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.004044</td>
<td>0.5100265</td>
<td>51.003</td>
</tr>
<tr>
<td>2</td>
<td>0.0038592</td>
<td>0.4867129</td>
<td>99.674</td>
</tr>
<tr>
<td>3</td>
<td>0.000013</td>
<td>0.1636157×10^{-2}</td>
<td>99.838</td>
</tr>
<tr>
<td>4</td>
<td>0.0000117</td>
<td>0.1480585×10^{-2}</td>
<td>99.986</td>
</tr>
</tbody>
</table>

   b. Temperature modes

<table>
<thead>
<tr>
<th>Modes</th>
<th>Eigenvalue</th>
<th>Normalized Eigenvalue</th>
<th>Energy (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.50048×10^{-3}</td>
<td>0.50734</td>
<td>50.734</td>
</tr>
<tr>
<td>2</td>
<td>0.47899×10^{-3}</td>
<td>0.48555</td>
<td>99.289</td>
</tr>
<tr>
<td>3</td>
<td>0.35626×10^{-2}</td>
<td>0.36114×10^{-2}</td>
<td>99.650</td>
</tr>
<tr>
<td>4</td>
<td>0.33149×10^{-5}</td>
<td>0.33603×10^{-2}</td>
<td>99.986</td>
</tr>
</tbody>
</table>
2. Finite difference implementation (N=9)

a. Velocity modes

<table>
<thead>
<tr>
<th>Modes</th>
<th>Eigenvalue</th>
<th>Normalized Eigenvalue</th>
<th>Energy (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0040272</td>
<td>0.510018</td>
<td>51.002</td>
</tr>
<tr>
<td>2</td>
<td>0.0038431</td>
<td>0.4867058</td>
<td>99.672</td>
</tr>
<tr>
<td>3</td>
<td>0.0000130</td>
<td>0.1640699×10⁻²</td>
<td>99.836</td>
</tr>
<tr>
<td>4</td>
<td>0.0000118</td>
<td>0.1491043×10⁻²</td>
<td>99.986</td>
</tr>
</tbody>
</table>

b. Temperature modes

<table>
<thead>
<tr>
<th>Modes</th>
<th>Eigenvalue</th>
<th>Normalized Eigenvalue</th>
<th>Energy (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.50034×10⁻²</td>
<td>0.50733</td>
<td>50.733</td>
</tr>
<tr>
<td>2</td>
<td>0.47886×10⁻³</td>
<td>0.48555</td>
<td>99.288</td>
</tr>
<tr>
<td>3</td>
<td>0.35555×10⁻³</td>
<td>0.36052×10⁻²</td>
<td>99.649</td>
</tr>
<tr>
<td>4</td>
<td>0.33304×10⁻⁵</td>
<td>0.33769×10⁻²</td>
<td>99.986</td>
</tr>
</tbody>
</table>
3. Spectral element implementation (N=11)

a. Velocity modes

<table>
<thead>
<tr>
<th>Modes</th>
<th>Eigenvalue</th>
<th>Normalized Eigenvalue</th>
<th>Energy (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0039931</td>
<td>0.52369</td>
<td>52.369</td>
</tr>
<tr>
<td>2</td>
<td>0.0036048</td>
<td>0.47276</td>
<td>99.645</td>
</tr>
<tr>
<td>3</td>
<td>0.0000144</td>
<td>0.18821×10⁻²</td>
<td>99.833</td>
</tr>
<tr>
<td>4</td>
<td>0.0000117</td>
<td>0.15383×10⁻²</td>
<td>99.987</td>
</tr>
</tbody>
</table>

b. Temperature modes

<table>
<thead>
<tr>
<th>Modes</th>
<th>Eigenvalue</th>
<th>Normalized Eigenvalue</th>
<th>Energy (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.500611×10⁻³</td>
<td>0.52233</td>
<td>52.233</td>
</tr>
<tr>
<td>2</td>
<td>0.45138×10⁻³</td>
<td>0.47096</td>
<td>99.329</td>
</tr>
<tr>
<td>3</td>
<td>0.33709×10⁻⁵</td>
<td>0.35171×10⁻²</td>
<td>99.681</td>
</tr>
<tr>
<td>4</td>
<td>0.29804×10⁻⁵</td>
<td>0.31097×10⁻²</td>
<td>99.992</td>
</tr>
</tbody>
</table>
4. Finite difference implementation (N=11)

a. Velocity modes

<table>
<thead>
<tr>
<th>Modes</th>
<th>Eigenvalue</th>
<th>Normalized Eigenvalue</th>
<th>Energy (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0039823</td>
<td>0.52368</td>
<td>52.368</td>
</tr>
<tr>
<td>2</td>
<td>0.003595</td>
<td>0.47275</td>
<td>99.643</td>
</tr>
<tr>
<td>3</td>
<td>0.0000144</td>
<td>0.18885×10^{-2}</td>
<td>99.832</td>
</tr>
<tr>
<td>4</td>
<td>0.0000117</td>
<td>0.15424×10^{-2}</td>
<td>99.986</td>
</tr>
</tbody>
</table>

b. Temperature modes

<table>
<thead>
<tr>
<th>Modes</th>
<th>Eigenvalue</th>
<th>Normalized Eigenvalue</th>
<th>Energy (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.500516×10^{-3}</td>
<td>0.52232</td>
<td>52.232</td>
</tr>
<tr>
<td>2</td>
<td>0.451301×10^{-3}</td>
<td>0.47096</td>
<td>99.328</td>
</tr>
<tr>
<td>3</td>
<td>0.33722×10^{-5}</td>
<td>0.3519×10^{-2}</td>
<td>99.680</td>
</tr>
<tr>
<td>4</td>
<td>0.29845×10^{-5}</td>
<td>0.3115×10^{-2}</td>
<td>99.991</td>
</tr>
</tbody>
</table>

Figure 3.24 shows the computed temporal eigenmodes (temporal expansion coefficients) obtained by direct projection of the input velocity and temperature data by the spectral element method (N=9 and N=11).
Figure 3.20 Velocity empirical eigenfunctions.

(a) $\Phi_1$  (b) $\Phi_2$
Figure 3.21 Velocity empirical eigenfunctions.
(a) $\Phi_3$  (b) $\Phi_4$
Figure 3.22 Temperature empirical eigenfunctions.
(a) $\Psi_1$  (b) $\Psi_2$
Figure 3.23 Temperature empirical eigenfunctions.
(a) $\Psi_3$  (b) $\Psi_4$
Figure 3.24 Direct projection results based on spectral element implementation. (a),(b), N=9 ; (c),(d), N=11. Line style: solid–a, or b, dotted–a, or b, dashed–a, or b, long dashed–a, or b.
3.2.3 Low-Order Model predictions (LOM)

After applying the POD and LOM methodologies, a system of non-linear ODEs for the expansion coefficients can be obtained:

\[
\frac{\text{d}a_k}{\text{d}t} = A_k + \frac{1}{\sqrt{\text{Gr}}} B_k + \sum_{i=1}^{M_1} a_i C_{ki} + \frac{1}{\sqrt{\text{Gr}}} \sum_{i=1}^{M_1} a_i D_{ki} \\
+ \sum_{i=1}^{M_1} \sum_{j=1}^{M_1} a_i a_j E_{kij} + \sum_{i=1}^{M_1} b_i F_{ki}, \quad k=1,2,\ldots,M_1, 
\]

(3.13)

\[
\frac{\text{d}b_k}{\text{d}t} = G_k + \frac{1}{\text{Pr}\sqrt{\text{Gr}}} H_k + \sum_{i=1}^{M_2} a_i I_{ki} + \frac{1}{\text{Pr}\sqrt{\text{Gr}}} \sum_{i=1}^{M_2} b_i J_{ki} \\
+ \sum_{i=1}^{M_2} \sum_{j=1}^{M_2} a_i b_j K_{kij} + \sum_{i=1}^{M_2} b_i L_{ki}, \quad k=1,2,\ldots,M_2. 
\]

(3.14)

The explicit form of eqs. (3.13) and (3.14) is listed in Appendix II. The non-linear ODEs shown in eqs. (3.13) and (3.14) are integrated using a fourth order Runge-Kutta solver. The constants are obtained from the various inner products among the eigenmodes and/or mean flow, and the second derivatives are used for computing the double integrals. Reduction of the order of the derivatives using the weak form discussed in section 2.7 has also been employed. The resulting coefficients are different between the spectral element based implementation and the finite difference based implementation because of the difference in differentiation and integration methodology. In order to apply the weak form of the differential equations, consider the boundary contour integration discussed in section 2.7. The application of the weak form leads to differences in the values of the expansion coefficients in eqs. (3.13) and (3.14). It can be noticed that the \(\frac{1}{\sqrt{\text{Gr}}}\) terms in...
the equations for $\dot{a}_i(t)$ and the $\frac{1}{Pr\sqrt{Gr}}$ terms in the equations for $\dot{b}_i(t)$ are different between the spectral element implementation using second derivatives and the spectral element method applied to the weak form (see Appendix II). These differences obviously affect the low-order model predictions.

The low-order model predictions for the flow and temperature fields by spectral element implementation with $N=9$ and $11$ are shown in Figs. (3.25) and (3.26). Full model results by the spectral element implementation are illustrated in Fig. (3.24). Note that the LOM predictions when $N=9$ (Fig. 3.25; a,b) predict very well the behavior for the thermo-fluid system (compare with the direct projection results shown in Fig. 3.24). However, surprisingly, the LOM predictions based on the spectral element implementation turn out to be less accurate when the order of the interpolant is increased (see Fig. 3.25; c,d). This discrepancy is discussed in chapter 4.

When applying the weak formulation scheme, while there is a little improvement in the magnitude of amplitude of LOM predictions in the case of $N=11$ (Fig.3.26; c,d), the detailed features of the predicted behaviors of flow system are significantly different from the full model simulation shown in Fig. 3.24.

In general, the LOM predictions by finite difference implementation can be improved by increasing the interpolant as shown in Fig. 3.27. In addition, LOM predictions by the spectral element implementation with $N=9$ (Fig. 3.25; a,b) are much more accurate than those based on finite difference implementation with $N=11$ (Fig. 3.27; c,d).
Figure 3.25 Low-order model prediction based on spectral element implementation.
(a),(b), N=9 ; (c),(d), N=11.
Line style: solid—\( \alpha \) or \( \beta \), dotted—\( \alpha \) or \( \beta \),
dashed—\( \alpha \) or \( \beta \), long dashed—\( \alpha \) or \( \beta \).
Figure 3.26 Low-order model prediction based on spectral element implementation and integration by parts.
(a), (b), N=9; (c), (d), N=11.
Line style: solid—a, or b, dotted—a, or b, 
dashed—a, or b, long dashed—a, or b.
Figure 3.27 Low-order model prediction based on finite difference implementation. (a), (b), N=9; (c), (d), N=11. Line style: solid— a, or b, dotted— a, or b, dashed— a, or b, long dashed— a, or b.
CHAPTER 4

CONCLUSIONS

Two buoyancy-driven two-dimensional flows have been numerically investigated: flow in an enclosure (section 3.1) and flow in a channel (section 3.2). The general concepts and procedure of Proper Orthogonal Decomposition (POD) and Low-Order Models (LOMs) methodologies were discussed in section 2.1 and section 2.2 respectively. Time-dependent solutions in the transitional regime have been analyzed by the snapshot version of POD (or method of empirical eigenfunctions). The extracted empirical eigenfunctions associated with the largest eigenvalues are the modes that determine the coherent dynamical structures of the original flow. By the LOMs methodology, the full governing partial differential equations were reduced to a system of first order non-linear ordinary differential equations for the time-dependent amplitudes of the eigenmodes. In performing the POD and in developing the LOMs, spectral differentiation and spectral integration based on the Legendre-Gauss-Lobatto collocation points and quadrature weights were employed.

In chapter 3, the POD and LOM results for the two transitional flows were presented. In the case of convection in an enclosure subject to differential side heating, the first two modes contain approximately 97% of the energy of the flow and temperature fluctuations (see Table 3.1). The velocity and temperature eigenfunctions are centro-
symmetric (see Figures 3.6 - 3.11). It was found that small scale features of the
eigenfunctions correspond to low levels of fluctuation energy and vice versa. The validity
of the 8-equation models based on the spectral element implementation was ascertained by
comparing the full model results (Fig. 3.12) and the reduced model predictions
(Fig. 3.13). As shown in Fig. 3.14, the LOM based on the finite difference method was
incapable of reproducing the stable, self-sustained oscillations of the original flow.

In the case of convection in a vertical channel with spatially-periodic flush-
mounted discrete heaters, the phase-shifted empirical eigenfunctions occur in pairs at
Pr= 0.71, Gr= 22500. The LOMs constructed by the fully spectral technique and N=9
reproduced almost identically to the dynamical behavior of the full system. However, the
LOM results based on the fully spectral technique deteriorated when higher order
interpolants were used (Fig. 3.25; c,d). These inconsistent LOM predictions are possibly
due to the discontinuity in the thermal boundary conditions and the noise involved when N
is increased. It was also found that the weak formulation scheme did not improve the
LOM predictions (Fig. 3.26). LOMs based on the finite difference method achieved more
accurate results when N was increased (Fig. 3.27). Note that the low-order model based
on the fully spectral technique with N=9 was more accurate than that based on the finite
difference implementation with N=11.

Since the fully spectral technique can produce, in general, more accurate LOMs for
transitional flows than finite difference techniques, numerical investigations using the
spectral element implementation are very promising. To the author's knowledge, the present study is the first effort to understand the effect of spectral implementation on the quality of low-order models of transitional thermal convection systems. In the future, it will be very useful to compare our numerical results to experimental results. Furthermore, the numerical methods employed in this study can be extended to different values of the geometric parameters ($A_e$ and $A_y$) and the flow controlling parameters (Grashof number and Prandtl number). Finally, the stability analysis of the developed LOMs can be regarded as highly desirable future research task.
REFERENCES


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APPENDIX I

List of non-linear ODEs. Enclosure, $A_y = 5$, $Gr = 2 \times 10^6$, $Pr = 0.71$, $M_1 = M_2 = 4$.

I.1 $\dot{a}(t)$ from momentum equation.

\[
\dot{a}_1 = -0.0002 + 0.0833 (Gr)^{0.5} + 0.4081a_1 + 1.8245a_2 - 0.0047a_3 + 0.1295a_4 \\
+ (Gr)^{0.5} (-542.9456a_1 + 2.6948a_2 - 6.5496a_3 - 0.6769a_4) \\
- 0.0001a_1^2 + 0.1375a_2^2 + 0.0735a_3^2 + 0.079a_4^2 \\
+ 0.0491a_1a_2 - 1.0187a_1a_3 + 2.3283a_1a_4 + 2.3299a_2a_3 + 0.8997a_2a_4 - 0.018a_3a_4 \\
+ 0.0008b_1 + 0.4386b_2 + 0.0204b_3 + 0.0015b_4,
\]

\[
\dot{a}_2 = 0.0007 - 1.5432 (Gr)^{0.5} - 1.8114a_1 + 0.4156a_2 + 0.1364a_3 - 0.006a_4 \\
+ (Gr)^{0.5} (-1.2070a_1 - 546.5506a_2 + 1.8103a_3 + 4.8937a_4) \\
- 0.0492a_1^2 - 0.0002a_2^2 - 0.1083a_3^2 - 0.1049a_4^2 \\
- 0.1376a_1a_2 + 2.4326a_1a_3 + 1.0829a_1a_4 + 0.9628a_2a_3 - 2.4318a_2a_4 + 0.1986a_3a_4 \\
- 0.4343b_1 + 0.0018b_2 + 0.0026b_3 - 0.0236b_4,
\]
\[ a_3 = -0.003 + 0.39092 (Gr)^{-0.5} + 0.0071a_1 - 0.003a_2 + 0.2054a_3 - 4.5438a_4 \]
\[ + (Gr)^{0.5} (-5.425a_1 + 1.8825a_2 - 873.6982a_3 + 3.2534a_4) \]
\[ + 0.9976a_1^2 - 0.9415a_2^2 - 0.003a_3^2 + 0.1945a_4^2 \]
\[ - 4.706a_1a_2 - 0.0832a_1a_3 + 0.0877a_1a_4 + 0.107a_2a_3 - 0.1913a_2a_4 + 0.1386a_3a_4 \]
\[ - 0.0109b_1 - 0.0097b_2 + 0.0712b_3 - 0.1388b_4, \]

\[ a_4 = -0.0005 + 0.8033 (Gr)^{-0.5} + 0.0087a_1 - 0.0078a_2 + 4.5473a_3 + 0.2051a_4 \]
\[ + (Gr)^{0.5} (-0.564a_1 + 3.815a_2 - 2.6776a_3 - 871.4557a_4) \]
\[ - 2.2996a_1^2 + 2.4038a_2^2 - 0.1395a_3^2 - 0.0012a_4^2 \]
\[ - 1.9404a_1a_2 - 0.069a_1a_3 - 0.0869a_1a_4 + 0.1725a_2a_3 + 0.1031a_2a_4 - 0.1968a_3a_4 \]
\[ - 0.0098b_1 + 0.0119b_2 + 0.1385b_3 + 0.0718b_4. \]

1.2 \( \dot{b}(t) \) from energy equation.

\[ \dot{b}_1 = -0.0007 + 0.6059 (Pr\sqrt{Gr})^{-1} + 0.2154a_1 - 0.3641a_2 + 0.0036a_3 - 0.0121a_4 \]
\[ + (Pr\sqrt{Gr})^{-1} (-386.1237b_1 + 0.654b_2 - 13.9873b_3 + 13.0393b_4) \]
\[ + 0.0001a_1b_1 + 0.0367a_1b_2 - 4.6609a_1b_3 - 0.0801a_1b_4 - 0.0001a_2b_1 + 0.3042a_2b_2 \]
\[ - 0.1332a_2b_3 + 4.5346a_2b_4 + 0.0021a_3b_1 - 0.1546a_3b_2 + 0.3236a_3b_3 - 0.0426a_3b_4 \]
\[ - 0.0102a_4b_1 - 0.1302a_4b_2 + 0.081a_4b_3 + 0.3144a_4b_4 \]
\[ - 0.000001b_1 + 2.9147b_2 + 0.0054b_3 + 0.0723b_4, \]
\[ \dot{b}_2 = -0.0012 + 1.0446 \left( \frac{Pr \sqrt{Gr}}{Re} \right)^1 + 0.3672a_1 + 0.2215a_2 - 0.0143a_3 - 0.006a_4 \\
+ \left( \frac{Pr \sqrt{Gr}}{Re} \right)^1 \left( -0.8531b_1 - 388.4186b_2 + 11.9043b_3 + 13.0962b_4 \right) \\
- 0.0368a_1b_1 - 0.00001a_1b_2 - 0.0336a_1b_3 + 4.6472a_1b_4 - 0.3041a_2b_1 - 0.0001a_2b_2 \\
+ 4.5155a_2b_3 + 0.091a_2b_4 + 0.1339a_3b_1 - 0.0021a_3b_2 - 0.0489a_3b_3 - 0.273a_3b_4 \\
+ 0.126a_4b_1 + 0.0104a_4b_2 + 0.2597a_4b_3 - 0.0788a_4b_4 \\
- 2.9147b_1 - 0.00001b_2 + 0.0847b_3 - 0.0076b_4 ,
\]

\[ \dot{b}_3 = -0.0003 + 0.2389 \left( \frac{Pr \sqrt{Gr}}{Re} \right)^1 + 0.0055a_1 + 0.0299a_2 - 0.0136a_3 + 0.5917a_4 \\
+ \left( \frac{Pr \sqrt{Gr}}{Re} \right)^1 \left( -13.9049b_1 + 11.4148b_2 - 833.1021b_3 + 1.1899b_4 \right) \\
+ 4.6676a_1b_1 + 0.0373a_1b_2 - 0.0035a_1b_3 + 0.5798a_1b_4 + 0.1372a_2b_1 - 4.5214a_2b_2 \\
+ 0.0004a_3b_3 - 0.1104a_3b_4 - 0.3384a_3b_1 + 0.0491a_3b_2 - 0.0035a_3b_3 + 0.2937a_3b_4 \\
- 0.0861a_4b_1 - 0.2478a_4b_2 + 0.0041a_4b_3 + 0.4808a_4b_4 \\
- 0.0054b_1 - 0.0847b_2 + 0.0002b_3 - 4.6849b_4 ,
\]

\[ \dot{b}_4 = 0.0001 - 0.0131 \left( \frac{Pr \sqrt{Gr}}{Re} \right)^1 + 0.0296a_1 - 0.0077a_2 - 0.5911a_3 - 0.013a_4 \\
+ \left( \frac{Pr \sqrt{Gr}}{Re} \right)^1 \left( 12.53b_1 + 12.658b_2 + 0.9734b_3 - 833.203b_4 \right) \\
+ 0.084a_1b_1 - 4.6536a_1b_2 - 0.5797a_1b_3 - 0.0019a_1b_4 - 4.5408a_2b_1 - 0.0948a_2b_2 \\
+ 0.1115a_2b_3 - 0.0001a_2b_4 + 0.0476a_3b_1 + 0.2609a_3b_2 - 0.3022a_3b_3 + 0.0031a_3b_4 \\
- 0.329a_4b_1 + 0.0791a_4b_2 - 0.4872a_4b_3 - 0.004a_4b_4 \\
- 0.0722b_1 + 0.0076b_2 + 4.6849b_3 + 0.0002b_4 .
\]
APPENDIX II

List of non-linear ODEs. Vertical channel, \( A_y = 4, A_z = 2, Gr = 22500, \)
\( Pr = 0.71, M_1 = M_2 = 4. \)

II.1 \( \dot{a}(t) \) based on the strong form of the momentum equation.

\[
\dot{a}_1 = -0.0029 + 0.5527 (Gr)^{0.5} + 0.429 a_1 + 0.3971 a_2 - 0.0579 a_3 - 0.0144 a_4 \\
+ (Gr)^{0.5} (-64.015 a_1 - 0.0324 a_2 - 0.5417 a_3 + 0.566 a_4 ) \\
- 0.0035 a_2^2 - 0.0205 a_3^2 - 0.0102 a_4^2 \\
- 0.0461 a_1 a_2 + 0.0816 a_1 a_3 - 0.4463 a_1 a_4 - 0.1745 a_2 a_3 + 0.002 a_2 a_4 - 0.0465 a_3 a_4 \\
+ 0.0008 b_1 - 0.0125 b_2 - 0.0173 b_3 + 0.0133 b_4 ,
\]

\[
\dot{a}_2 = -0.0053 + 0.7647 (Gr)^{0.5} - 0.3971 a_1 + 0.4288 a_2 - 0.0174 a_3 + 0.635 a_4 \\
+ (Gr)^{0.5} (-0.0304 a_1 - 64.0518 a_2 + 0.6365 a_3 + 0.2208 a_4 ) \\
+ 0.0461 a_1^2 - 0.0585 a_2^2 - 0.0165 a_4^2 \\
+ 0.0035 a_1 a_2 - 0.7005 a_1 a_3 - 0.0143 a_1 a_4 + 0.0701 a_2 a_3 + 0.4542 a_2 a_4 + 0.0075 a_3 a_4 \\
+ 0.014 b_1 + 0.0012 b_2 + 0.0074 b_3 + 0.1495 b_4 ,
\]
\[ a_3 = 0.3043 - 4.5032 (Gr)^{-0.5} - 0.002a_1 + 0.0061a_2 - 0.2013a_3 - 0.8177a_4 \]

\[ + (Gr)^{-0.5} (-0.5371a_1 + 0.6409a_2 - 102.8225a_3 - 0.0236a_4) \]

\[ - 0.0816a_1^2 - 0.0701a_2^2 + 0.0531a_4^2 \]

\[ + 0.8751a_1a_2 - 0.0205a_1a_3 - 0.0193a_1a_4 + 0.0585a_2a_3 + 0.0149a_2a_4 - 0.5419a_3a_4 \]

\[ + 0.0207b_1 - 0.0159b_2 + 0.4226b_3 - 0.0254b_4 , \]

\[ a_4 = -0.0002 + 0.0272 (Gr)^{-0.5} - 0.0001a_1 + 0.0058a_2 + 0.8175a_3 - 0.218a_4 \]

\[ + (Gr)^{-0.5} (0.5667a_1 + 0.2239a_2 - 0.0242a_3 - 107.4783a_4) \]

\[ + 0.4463a_1^2 - 0.4542a_2^2 + 0.5419a_3^2 \]

\[ + 0.0123a_1a_2 + 0.0658a_1a_3 + 0.1019a_1a_4 - 0.0224a_2a_3 + 0.0165a_2a_4 - 0.0531a_3a_4 \]

\[ - 0.0119b_1 - 0.0216b_2 + 0.0322b_3 + 0.4219b_4 . \]

II.2 \[ \dot{b}(t) \] based on the strong form of the energy equation.

\[ \dot{b}_1 = 0.0017 - 0.153 (Pr\sqrt{Gr})^{-1} + 0.1326a_1 - 0.0319a_2 + 0.0011a_3 - 0.0112a_4 \]

\[ + (Pr\sqrt{Gr})^{-1} (-38.8994b_1 + 0.0298b_2 + 0.2776b_3 - 0.244b_4) \]

\[ - 0.0542a_1b_2 + 0.0521a_1b_3 - 1.5947a_1b_4 - 0.0198a_3b_2 \]

\[ - 1.5708a_3b_3 + 0.0047a_3b_4 + 0.3738a_3b_5 + 0.0974a_3b_6 + 0.1863a_3b_4 \]

\[ + 0.0218a_4b_2 - 0.1374a_4b_3 + 0.1236a_4b_4 \]

\[ + 0.4854b_2 - 0.0447b_3 + 0.0154b_4 , \]
\[
\dot{b}_2 = -0.0009 + 0.0877 (\Pr \sqrt{Gr})^{-1} + 0.0316a_1 + 0.1332a_2 - 0.0078a_3 + 0.0008a_4
\]
\[
+ (\Pr \sqrt{Gr})^{-1} \left( 0.3325b_1 - 38.8436b_2 + 0.0104b_3 + 0.5601b_4 \right)
\]
\[
+ 0.0542a_1b_1 - 1.5527a_1b_3 - 0.0003a_1b_4 + 0.0198a_2b_1
\]
\[
+ 0.0635a_2b_3 + 1.5672a_2b_4 - 0.3738a_3b_1 + 0.0827a_3b_3 + 0.0101a_3b_4
\]
\[- 0.0217a_4b_1 + 0.0121a_4b_3 + 0.0354a_4b_4
\]
\[- 0.4854b_1 + 0.0104b_3 + 0.0435b_4 ,
\]
\[
\dot{b}_3 = 0.001 - 0.095 (\Pr \sqrt{Gr})^{-1} - 0.0147a_1 + 0.0057a_2 - 0.1018a_3 - 0.0357a_4
\]
\[
+ (\Pr \sqrt{Gr})^{-1} \left( 0.2913b_1 + 0.0056b_2 - 150.2811b_3 - 0.3111b_4 \right)
\]
\[- 0.0521a_1b_1 + 1.5527a_1b_3 + 0.194a_1b_4 + 1.5708a_2b_1 - 0.0635a_2b_2
\]
\[- 0.0434a_2b_4 - 0.0975a_3b_1 - 0.0828a_3b_2 - 0.5713a_3b_4
\]
\[+ 0.1374a_4b_1 - 0.0121a_4b_3 + 0.1367a_4b_4
\]
\[+ 0.0447b_1 - 0.0104b_3 - 0.899b_4 ,
\]
\[
\dot{b}_4 = 0.0007 - 0.0887 (\Pr \sqrt{Gr})^{-1} + 0.0063a_1 + 0.0139a_2 + 0.0335a_3 - 0.1059a_4
\]
\[
+ (\Pr \sqrt{Gr})^{-1} \left( -0.2626b_1 + 0.555b_2 - 0.29b_3 - 153.1422b_4 \right)
\]
\[+ 1.5947a_1b_1 + 0.0003a_1b_2 - 0.194a_1b_3 - 0.0047a_2b_1 - 1.5672a_2b_2
\]
\[+ 0.0434a_2b_3 - 0.1863a_3b_1 - 0.0101a_3b_2 + 0.5713a_3b_3
\]
\[- 0.1236a_4b_1 - 0.0354a_4b_2 - 0.1367a_4b_3
\]
\[- 0.0154b_1 - 0.0435b_2 + 0.899b_3 .
\]
II.3 \( \dot{a}(t) \) based on the weak form of the momentum equation.

\[
\dot{a}_1 = -0.0029 + 0.5527 \ (Gr)^{0.5} + 0.429a_1 + 0.3971a_2 - 0.0579a_3 - 0.0144a_4 \\
+ (Gr)^{0.5} (-64.0179a_1 - 0.0312a_2 - 0.5378a_3 + 0.5673a_4) \\
- 0.0035a_2^2 - 0.0205a_3^2 - 0.0102a_4^2 \\
- 0.0461a_1a_2 + 0.0816a_1a_3 - 0.4463a_1a_4 - 0.1745a_2a_3 + 0.002a_2a_4 - 0.0465a_3a_4 \\
+ 0.0008b_1 - 0.0125b_2 - 0.0173b_3 + 0.0133b_4 ,
\]

\[
\dot{a}_2 = -0.0053 + 0.7647 \ (Gr)^{0.5} - 0.3971a_1 + 0.4288a_2 - 0.0174a_3 + 0.635a_4 \\
+ (Gr)^{0.5} (-0.0312a_1 - 64.0546a_2 + 0.6418a_3 + 0.2245a_4) \\
+ 0.0461a_1^2 - 0.0585a_3^2 - 0.0165a_4^2 \\
+ 0.0035a_1a_2 - 0.7005a_1a_3 - 0.0143a_1a_4 + 0.0701a_2a_3 + 0.4542a_2a_4 + 0.0075a_3a_4 \\
+ 0.014b_1 + 0.0012b_2 + 0.0074b_3 + 0.1495b_4 ,
\]

\[
\dot{a}_3 = 0.3043 - 4.5031 \ (Gr)^{0.5} - 0.002a_1 + 0.0061a_2 - 0.2013a_3 - 0.8177a_4 \\
+ (Gr)^{0.5} (-0.5378a_1 + 0.6418a_2 - 102.833a_3 - 0.0286a_4) \\
- 0.0816a_1^2 - 0.0701a_2^2 + 0.0531a_4^2 \\
+ 0.8751a_1a_2 + 0.0205a_1a_3 - 0.0193a_1a_4 + 0.0585a_2a_3 + 0.0149a_2a_4 - 0.5419a_3a_4 \\
+ 0.0207b_1 - 0.0159b_2 + 0.4226b_3 - 0.0254b_4 ,
\]
\[ \dot{a}_4 = -0.0002 + 0.0271 \text{ (Gr)}^{0.5} - 0.0001a_1 + 0.0058a_2 + 0.8175a_3 - 0.218a_4 \\
+ \text{ (Gr)}^{0.5} (0.5673a_1 + 0.2245a_2 - 0.0286a_3 - 107.4913a_4) \\
+ 0.4463a_1^2 - 0.4542a_2^2 + 0.5419a_3^2 \\
+ 0.0123a_1a_2 + 0.0658a_1a_3 + 0.1019a_1a_4 - 0.0224a_2a_3 + 0.0165a_2a_4 - 0.0531a_3a_4 \\
- 0.0119b_1 - 0.0216b_2 + 0.0322b_3 + 0.4219b_4. \]

II.4 \( \dot{b}(t) \) based on the weak form of the energy equation.

\[ \dot{b}_1 = 0.0017 + 0.349 \frac{\text{Pr} \sqrt{\text{Gr}}} - 0.0319a_1 + 0.0011a_3 - 0.0112a_4 \\
+ \frac{\text{Pr} \sqrt{\text{Gr}}}{ - 38.8987b_1 + 0.0319b_2 + 0.2901b_3 - 0.2632b_4} \\
- 0.0542a_1b_2 + 0.0521a_1b_3 - 1.5947a_1b_4 - 0.0198a_2b_2 \\
- 1.5708a_2b_3 + 0.0047a_2b_4 + 0.3738a_3b_2 + 0.0974a_3b_3 + 0.1863a_3b_4 \\
+ 0.0218a_4b_2 - 0.1374a_4b_3 + 0.1236a_4b_4 \\
+ 0.4854b_2 - 0.0447b_3 + 0.0154b_4, \]

\[ \dot{b}_2 = -0.0009 - 0.3039 \frac{\text{Pr} \sqrt{\text{Gr}}} - 0.0316a_1 + 0.1332a_2 - 0.0078a_3 + 0.0008a_4 \\
+ \frac{\text{Pr} \sqrt{\text{Gr}}}{ - 0.0319b_1 - 38.8422b_2 + 0.0045b_3 + 0.5548b_4} \\
+ 0.0542a_1b_1 - 1.5527a_1b_3 - 0.0003a_1b_4 + 0.0198a_2b_1 \\
+ 0.0635a_2b_3 + 1.5672a_2b_4 - 0.3738a_3b_1 + 0.0827a_3b_3 + 0.0101a_3b_4 \\
- 0.0217a_4b_1 + 0.0121a_4b_3 + 0.0354a_4b_4 \\
- 0.4854b_1 + 0.0104b_3 + 0.0435b_4, \]
\[ b_3 = 0.001 - 0.1232 (Pr \sqrt{Gr})^{-1} - 0.0147a_1 + 0.0057a_2 - 0.1018a_3 - 0.0357a_4 \]

\[ + (Pr \sqrt{Gr})^{-1} \left( 0.2901b_1 + 0.0045b_2 - 150.2884b_3 - 0.2985b_4 \right) \]

\[ - 0.0521a_1b_1 + 1.5527a_1b_2 + 0.194a_1b_4 + 1.5708a_2b_1 - 0.0635a_2b_2 \]

\[ - 0.0434a_2b_4 - 0.0975a_3b_1 - 0.0828a_3b_2 - 0.5713a_3b_4 \]

\[ + 0.1374a_4b_1 - 0.0121a_4b_2 + 0.1367a_4b_4 \]

\[ + 0.0447b_1 - 0.0104b_2 - 0.899b_4 , \]

\[ b_4 = 0.0007 + 0.1345 (Pr \sqrt{Gr})^{-1} + 0.0063a_1 + 0.0139a_2 + 0.0335a_3 - 0.1059a_4 \]

\[ + (Pr \sqrt{Gr})^{-1} \left( - 0.2632b_1 + 0.5548b_2 - 0.2985b_3 - 153.133b_4 \right) \]

\[ + 1.5947a_1b_1 + 0.0003a_1b_2 - 0.194a_1b_3 - 0.0047a_2b_1 - 1.5672a_2b_2 \]

\[ + 0.0434a_2b_3 - 0.1863a_3b_1 - 0.0101a_3b_2 + 0.5713a_3b_3 \]

\[ - 0.1236a_4b_1 - 0.0354a_4b_2 - 0.1367a_4b_3 \]

\[ - 0.0154b_1 - 0.0435b_2 + 0.899b_3 . \]
VITA

Younjong Kim was born February 20, 1969 in Seoul, Korea to parents of Mr. Kim, Hyunbae who is an architect and Mrs. Ahn, Boksik who is a doctor. He has one younger sister Kim, Jinsu.

The author completed his undergraduate studies at Hanyang University in Korea with a Bachelor of Science degree in Mechanical Engineering in 1994. He is pursuing his Master of Science degree under the direction of Dr. Antonios Liakopoulos in Mechanical Engineering at Lehigh University with major in Thermo-Fluids.

Presently, the author plans to gain practical engineering experience in industry.
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