An investigation of three-dimensional laminar boundary layer similarity solutions

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AN INVESTIGATION OF THREE-DIMENSIONAL LAMINAR
BOUNDARY LAYER SIMILARITY SOLUTIONS

by

Lisa A. Thomas

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Abstract

The behavior of self-similar flow in a three-dimensional laminar boundary layer is investigated. The governing equations are a set of coupled, nonlinear, ordinary differential equations that determine the behavior of the streamwise velocity and the cross-stream velocity. These equations depend on two parameters: $\beta_1$, the streamwise pressure gradient, and $\beta_2$, the cross-stream pressure gradient. Solutions are obtained for a broad range of these parameters. Results for adverse pressure gradient flow patterns are complex and exhibit traits such as backflow and separation; a separation criterion is proposed for self-similar flow that depends both on $\beta_1$ and $\beta_2$. In a favorable pressure gradient, the crossflow component approaches the freestream velocity as $\beta_2 \to \infty$. 
1. Introduction

Understanding of flow behavior in laminar boundary layers was significantly advanced when in 1931 V. M. Falkner and S. W. Skan discovered a subset of the boundary layer equations now known as the Falkner-Skan equation. This equation describes flow in two-dimensional boundary layers, and its solutions reflect a wide range of interesting phenomena, including velocity overshoot, backflow, separation, non-uniqueness, and algebraic decay. This thesis outlines the derivation of this equation in §2, as well as a description of its general properties, including a discussion of both the unique and non-unique velocity profiles associated with this equation.

The discussion of the Falkner-Skan equation is a precursor to the main topic of interest in this study, namely, three-dimensional laminar boundary layer similarity solutions. Such solutions depend not only on the streamwise pressure gradient as do Falkner-Skan profiles, but also on a cross-stream pressure gradient. The combination of these parameters leads to some interesting and complex velocity distributions, which exhibit similar phenomena to the two-dimensional case but with some added complications.

In §3 the derivation of the equations governing three-dimensional laminar similar boundary layer flow is described, as well as the numerical scheme required to obtain both regular and backflow solutions. Flows having favorable streamwise pressure gradients are characterized by unique solutions for streamwise and cross-stream velocity profiles. On the other hand, flows with adverse streamwise pressure gradients generally exhibit multiple or non-unique solutions, at least one of which has backflow in the
streamwise direction. A general asymptotic solution for small cross-stream pressure gradients is obtained. The numerical scheme required to obtain the profile solutions is described in §3. Some general remarks and conclusions are made, and the study ends with appendices containing relevant figures and tables. The source code for all work performed for this investigation is filed with the Department of Mechanical Engineering and Mechanics, Lehigh University.
2. Two-Dimensional Similarity Solutions: The Falkner-Skan Equation

2.1 Overview

In this chapter, a brief derivation of the Falkner-Skan equation is given, as well as a discussion of the physical significance associated with a range of geometries and pressure gradients. The Falkner-Skan equation contains a single pressure gradient parameter $\beta$. For certain values of $\beta$, the solution for the streamwise velocity profile exhibits some interesting traits including overshoot and backflow. Here the terminology "overshoot" implies that the velocity profile achieves a maximum value in excess of the mainstream value somewhere in the boundary layer. On the other hand, "backflow" implies that a region of negative streamwise velocity occurs near the wall. The possible physical significance for the various ranges of $\beta$ is discussed. The Falkner-Skan equation is well known, but it is worthwhile to discuss it here in some detail because there are many analogies between the two-dimensional and three-dimensional cases; the latter is the main topic of interest in this study.

All tables and figures pertaining to the Falkner-Skan equation are included in Appendix A.

2.2 Derivation of the Falkner-Skan Equation

Consider the two-dimensional planar bluff body pictured in Figure A.1. A boundary layer originates from the stagnation point and serves to reduce the mainstream velocity $U_e(x)$ to zero at the surface of the body. The velocity distribution in the boundary layer along the body depends on the geometry of the body and may exhibit complex behavior. However, in the
immediate vicinity of the point of stagnation, the surface can be viewed as essentially flat over a limited range. We can expect a self-similar velocity profile in this region, and it is here, for example, that the Falkner-Skan equation (derived below) applies.

The dimensionless boundary layer equations for two dimensional, incompressible, $\mu =$ constant, steady flow are:

\[
\begin{align*}
\text{x-momentum:} & \quad u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U_e \frac{dU_e}{dx} + \frac{\partial^2 u}{\partial y^2} ; \\
\text{continuity:} & \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 .
\end{align*}
\]

Here $y$ and $v$ are magnified by $Re^{1/2}$, viz., $y = y^* Re^{1/2}$, $v = v^* Re^{1/2}$, where $y^*$ and $v^*$ are the actual normal coordinate and velocity respectively. In the scaled coordinates $y$ and $v$ are $O(1)$ in the boundary layer.

The boundary conditions are:

\[
\begin{align*}
\text{at } y = 0 : & \quad u = 0, \quad v = 0 , \\
\text{as } y \to \infty : & \quad u \to U_e(x) ,
\end{align*}
\]

where $U_e(x)$ denotes the mainstream velocity. Define

\[
\eta = \frac{y}{\delta(x)} ,
\]

where $\delta(x)$ is a function proportional to the boundary-layer thickness which is to be found. Define

\[
\begin{align*}
u = U_e(x) \frac{df}{d\eta} = U_e(x)f'(\eta) , \quad \text{or} \quad f'(\eta) = \frac{u}{U_e(x)} .
\end{align*}
\]
Here \( \eta \) is a similarity variable and the interest is in self-similar solutions for which the dependence of the velocity profile on \( x \) and \( y \) can be collapsed into one variable, \( \eta \). The boundary conditions transform to:

\[
f' = 0 \quad \text{at} \quad \eta = 0 ,
\]

\[
f' \to 1 \quad \text{as} \quad \eta \to \infty .
\]

From the continuity equation, a stream function \( \psi \) may be defined by

\[
\begin{align*}
  u &= \frac{\partial \psi}{\partial y}, \\
  v &= -\frac{\partial \psi}{\partial x}.
\end{align*}
\]

From the definition of \( u \), we obtain

\[
\frac{\partial \psi}{\partial y} = U_e(x) f'(\eta). \tag{2.8}
\]

Integrating equation (2.8) yields \( \psi \):

\[
\psi = U_e(x) \delta(x) f(\eta) + \psi_o(x). \tag{2.9}
\]

Here \( \psi_o(x) \) at this stage is an arbitrary function of integration. Differentiating equation (2.9) with respect to \( x \) yields:

\[
v = -(U_e \delta)' f + U_e \delta' f' \eta - \psi'_o. \tag{2.10}
\]
Here the prime (') indicates differentiation with respect to $x$ or $\eta$ as appropriate. The chain rule is used to obtain $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial y}$, and $\frac{\partial^2 u}{\partial y^2}$ in terms of $\eta$. Since we are searching for similarity solutions, it is assumed that $f = f(\eta)$. The resulting transformations are:

\[
\frac{\partial u}{\partial x} = U_e f' - \eta \frac{\delta}{\delta} U_e f'' ,
\]

(2.11)

\[
\frac{\partial u}{\partial y} = \frac{1}{\delta} U_e f'' ,
\]

(2.12)

\[
\frac{\partial v}{\partial y} = -U_e f' + \eta \frac{\delta}{\delta} U_e f'' ,
\]

(2.13)

\[
\frac{\partial^2 u}{\partial y^2} = \frac{1}{\delta^2} U_e f''' .
\]

(2.14)

After substitution and simplification of the equations, the following result is obtained:

\[
f''' + \delta(U_e \delta)' f'' + \delta^2 U'_e (1-f'^2) + \psi_0 \delta f'' = 0 .
\]

(2.15)

The only way (2.15) can be an ordinary differential equation is if the coefficients are not functions of $x$. Therefore, defining

\[
\delta(U_e \delta)' = \alpha = \text{constant} ,
\]

(2.16)
\[
\delta^2 U_e' = \beta = \text{constant} , \quad \tag{2.17}
\]

\[
\psi_0' \delta = \gamma = \text{constant} , \quad \tag{2.18}
\]

whereupon equation (2.15) becomes

\[
f'''' + \alpha f'' + \beta (1 - f'') + \gamma f'' = 0. \quad \tag{2.19}
\]

At this stage there are three arbitrary functions of \(x\), namely, \(U_e\), \(\delta\), and \(\psi_0'\). The following discussion determines the form of these functions so that equations (2.16) – (2.18) are satisfied. Consider \(\gamma\) first, for which there are two situations to consider. For a solid wall, then \(v = 0\) at \(y = 0\) for all \(x\), and it follows from equation (2.10) that \(\psi_0 = \text{constant}\), which may be taken equal to zero without loss of generality. It follows from equation (2.18) that \(\gamma = 0\). On the other hand, if the wall is not solid, we may write

\[
v = -v_s(x) \quad \text{at} \quad y = 0 , \quad \tag{2.20}
\]

where \(v_s > 0\) corresponds to suction and \(v_s < 0\) to injection. Referring to equation (2.10), we see that for this case

\[
v_s(x) = \psi_0' . \quad \tag{2.21}
\]

Consequently, from equation (2.18) similarity is possible only if
\[ v_e(x) = \frac{\gamma}{\delta(x)} = \text{constant} \cdot \frac{\delta(x)}{\delta(x)}. \] (2.22)

We see that \( \delta \) and \( U_e \) are fixed by equations (2.16) and (2.17), so this condition is very restrictive. The injection must be done in a very specific way to obtain a similarity solution.

The form of \( U_e(x) \) required for similarity will now be addressed. Combining equations (2.16) and (2.17) yields

\[
2\alpha - \beta = \delta^2 U_e' + 2\delta\delta' U_e = (\delta^2 U_e)',
\] (2.23)

and two cases may be considered. If \( (2\alpha - \beta) \neq 0 \) integration of equation (2.23) yields

\[
\delta^2 U_e = (2\alpha - \beta)(x - x_o).
\] (2.24)

Here \( x = x_o \) is a stagnation point of the inviscid flow, so \( U_e = 0 \) at this point. Without loss of generality, choose \( x_o = 0 \).

Combining (2.17) and (2.24) yields

\[
\delta^2 U_e' = (2\alpha - \beta) \frac{U_e'}{U_e} x,
\] (2.25)

and integration gives

\[
U_e = u_o x^{2\alpha - \beta} = u_o x^m.
\] (2.26)
Here $u_o$ is a constant of integration and $m = \beta/(2\alpha - \beta)$. Thus for $2\alpha - \beta \neq 0$, a similarity solution is possible only when the mainstream velocity has the power law behavior in equation (2.26). To determine the form of $\delta(x)$ combine (2.17) and (2.26) to yield

$$
\delta(x) = \sqrt{\frac{\beta}{mu_o}} x^{(1-m)/2} = \left\{\frac{(2\alpha - \beta)x}{U_e(x)}\right\}^{1/2}.
$$

(2.27)

If in addition there is suction, we have the further requirement that the suction must be of the special form

$$
v_s(x) = (\text{constant}) x^{(m-1)/2}.
$$

(2.28)

For $(2\alpha - \beta) = 0$, equation (2.23) implies that

$$
\delta^2 U_e = \text{constant} = K.
$$

(2.29)

Combining equations (2.29) and (2.17) and integrating yields

$$
U_e = u_o e^{\beta x/K}.
$$

(2.30)

The scale factor $\delta(x)$ is found by combining (2.30) and (2.17), and noting that $\beta = 2\alpha$ to obtain:

$$
\delta(x) = \left(\frac{K}{u_o}\right)^{1/2} e^{-\alpha x/K}.
$$

(2.31)
If there is suction, it must be of the special form

$$v_\delta(x) = \text{(constant)} \ e^{\alpha x/K}.$$  \hfill (2.32)

The form of the Falkner-Skan equation most frequently seen in the literature is:

$$f'''' + ff'' + \beta(1-f'^2) = 0,$$ \hfill (2.33)

for which \( \alpha \) has been taken to equal 1. The boundary conditions are:

$$f = f' = 0 \quad \text{at} \ \eta = 0,$$ \hfill (2.34)

$$f' \rightarrow \infty \quad \text{as} \ \eta \rightarrow \infty.$$  

Generally, a solid wall is assumed so that \( \gamma \) vanishes. A constant value for \( \alpha = 1 \) may be chosen since the value of \( \alpha \) affects only the scale for \( \eta \). The solution of equation (2.33) depends only on the parameter \( \beta \) which is a measure of the streamwise pressure gradient. This derivation was adapted from Walker, 1992, pp. 40-49.

2.3 General Properties

Applications of the Falkner-Skan equation cover a wide range of interesting and physically important problems. Boundary layer phenomena
described by this equation include the influence of favorable and adverse pressure gradients, velocity overshoots, and backflow. While the conditions which make the equation valid may seem restrictive, the Falkner-Skan equation does provide a basis for understanding the physics of boundary layers, and solutions can be used as initiators or terminators of more complex flows.

The parameter $\beta$ is a measure of the pressure gradient $dp/dx$. For $\beta > 0$, the pressure gradient is favorable (negative). A value of $\beta < 0$ signifies an adverse (positive) pressure gradient. For $\beta > 0$, the solutions of equation (2.33) are unique and all increase monotonically towards the freestream velocity. These are the so-called "regular" solutions since the profiles generally behave as expected. In the range $-0.198838 \leq \beta \leq 0$ there are two solutions, one of which is regular and one which exhibits backflow. It is known (Stewartson, 1954) that only two solutions exist in this range. At $\beta = -0.198838$ the two solutions coalesce into one having zero shear at the wall; this case is often referred to as a separation profile.

For $\beta < -0.198838$ there exists an infinite family of solutions (Stewartson, 1954). One interesting type of solution is one which exhibits velocity overshoot, i.e., there are certain values of $\eta$ for which $f' > 1$. There is still much debate regarding the physical significance of such solutions. Some researchers argue that neither the main stream nor the pressure differential provides a mechanism for producing overshoot, thereby deeming the solutions as being physically unacceptable. Others claim that overshoot profiles can be explained by flows with streamwise blowing through an upstream slot, such as a wall jet (Libby and Liu, 1967). Libby and Liu
explain the existence of overshoot profiles by suggesting that this general behavior is generated over an initial length which governs the type of flow throughout the boundary layer.

Below is a compilation of various values of $\beta$ and their physical significance:

1. $\beta = 0$: Flat plate (Blasius solution).
2. $0 \leq \beta \leq 2$: Flow against a wedge of half angle $\frac{\beta \pi}{2}$.
3. $\beta = 1$: Plane stagnation point flow (Hiemenz flow).
4. $\beta = 2$: Flow diverted through an angle of $\pi$.
5. $\beta = 4$: Doublet flow near a plane wall.
6. $\beta = 5$: Doublet flow near a right angle corner.
7. $\beta \to \infty$: Flow toward a point sink.
8. $-2 \leq \beta \leq 0$: Flow around an expansion corner of turning angle $\frac{\beta \pi}{2}$.
9. $\beta = -2$: Flow around the edge of a thin plate.
10. $\beta = -1$: Flow around a right angle corner. (Puhak, 1992, pp. 5-6).

Figures A.2 – A.7 contain sketches of selected cases. They are discussed in greater detail in the next section. Finally, all solutions discussed so far exhibit exponential decay which means that $f'$ approaches 1 very rapidly for large values of $\eta$.

2.4 Favorable Pressure Gradients

As mentioned in the previous section, for $\beta > 0$ solutions are found to exist and to represent unique boundary layer profiles. Some values of $\beta$
reflect velocity profiles more physically significant than others, which will be discussed further.

The range $0 \leq \beta \leq 2$ represents flow against a wedge of half angle $\beta \pi / 2$, depicted in Figure A.3. These wedge flows are important in a physical sense because they correspond to geometries which occur frequently in industrial applications. This is especially true for $\beta \leq 1$. Consider the two dimensional representation of a missile, depicted in Figure A.8. The point of attachment occurs at $x = 0$, and the boundary layer develops symmetrically from this point. The Falkner-Skan solution is valid over the leading portion of the missile. Thus, a similarity solution is an initiator to start off a numerical simulation of the complex flow field over the balance of the missile.

A value of $\beta = 1$ corresponds to stagnation point flow (Hiemenz flow), portrayed in Figure A.4. For any two-dimensional planar bluff body, flow in the region of the point of attachment of the inviscid flow will exhibit Falkner-Skan velocity profiles. This assumption is justified since locally the surface appears flat, reflecting the geometry of Hiemenz flow. Thus, a Falkner-Skan solution initiates the numerical integration for any bluff body, such as flow occurring near the stagnation line of a cylinder.

Velocity profiles resulting from $\beta = 2, 4, 5,$ and $\infty$ represent situations which are more difficult to justify on physical grounds. Even though the similarity assumptions and restrictions indicate that these flows are possible and behave according to the Falkner-Skan predictions, in reality these flows are very complex and generally cannot be accurately described by a similarity solution.
2.5 Adverse Pressure Gradients

As briefly discussed previously, Falkner-Skan profiles for which $\beta \leq 0$ exhibit non-uniqueness, meaning that for each $\beta$ selected, there is more than one solution which satisfies the differential equation and boundary conditions. The most important (and most controversial) traits displayed by these solutions are backflow ($f' < 0$) and velocity overshoot ($f' > 1$). As in the case of favorable pressure gradients, some geometries which correspond to a particular $\beta$ are more questionable on physical grounds than others. The case $\beta = 0$ corresponds to the famous Blasius, or semi-infinite flat plate, solution. Of course, this case corresponds to no pressure gradient. The similar boundary layer equations for this case reduce to

$$f''' + ff'' = 0.$$  \hspace{1cm} (2.35)

This equation appears very simple, but no closed form analytical solution exists.

The regular solution to the Blasius equation shows the expected behavior with a monotonic increase of $f'$ from $0 \rightarrow 1$. However, in recent years it has been shown that the Blasius boundary layer also has non-unique solutions (Smith, 1984) and (Laine and Reinhart, 1984). This has been inferred from the observation that as $\beta \rightarrow 0^-$, it is exceedingly more difficult to obtain a solution for a very small negative $\beta$. Generally, the velocity here experiences slight backflow from $\eta = 0$ to large values of $\eta$, and then sharply increases to $f' = 1$. In the limit, the fluid in the region of reversed flow comes to rest and suddenly diverts to $f' = 1$ at $\eta = \infty$. Suggested profiles
which render the nature of this solution can be seen in Figure A.9.

The range $-2 \leq \beta \leq 0$ corresponds to flow around an expansion corner of turning angle $\beta \pi/2$, depicted in Figure A.5. Apparently the most physically plausible regime is $-1 \leq \beta \leq 0$, where $\beta = 0$ corresponds to the Blasius solution discussed earlier, and $\beta = -1$ corresponds to flow around a right angle corner. As $\beta$ extends beyond $-1$, the flow has to turn back around, introducing complications and other flow phenomena that cannot be described by a similarity solution, i.e., recirculation zones that develop as the flow turns. The limit in this range is $\beta = -2$, which corresponds to flow around the edge of a thin plate.

For $-0.198838 \leq \beta \leq 0$, two solutions exist (Stewartson, 1954): the regular solution which increases monotonically to $f' = 1$, and one which experiences backflow. At $\beta = 0$, the flows are completely different, but as $|\beta|$ increases the range of $\eta$ in which $f' < 1$ decreases until the two solutions merge at $\beta = -0.198838$. It is at this value of $\beta$ where a separation profile with $f''(0) = 0$ occurs. For $\beta$ greater than the critical value a region of reversed flow extends from $\eta = 0$ (where $f'' < 0$) to some finite value of $\eta$, and then $f'$ increases toward $f' = 1$ at $\infty$. Thus, backflow occurs in the vicinity of the wall. The velocity in the region of reversed flow progressively comes to rest as $\beta \to 0^-$. Table A.1 lists tabular data for selected $\beta$ values in this range.

As previously mentioned, solutions for which $\beta < -0.198838$ have an infinite family of solutions. The solutions exhibit overshoot, in which $f' > 1$ at locations within the boundary layer. As mentioned earlier, these
solutions may be questionable with regards to their physical significance.

Included in Figures A.10–A.12 are typical velocity profiles exhibiting monotonic behavior, backflow, and overshoot, along with the corresponding shear stress profiles for monotonic flow.
3. Three Dimensional Similarity Solutions

3.1 Overview

In this chapter three-dimensional laminar boundary layer similarity solutions are introduced. A derivation of the governing equations and the numerical scheme used to compute both regular and backflow solutions is given. Solutions to the streamwise and cross-stream velocity profiles are computed for the following ranges of the streamwise pressure gradient parameter $\beta_1$: $-0.198838 \leq \beta_1 \leq 0$, and $0 \leq \beta_1 \leq 1$. Solutions for $\beta_1 < -0.198838$ are also obtained for values of $|\beta_2| > 0$. It seems that a separation streamwise profile can be delayed with the addition of a cross-stream pressure gradient. These ranges were considered because they are the most physically relevant. The asymptotic behavior of the solutions for small values of the cross-stream pressure gradient $\beta_2$ is also investigated. Finally, the section concludes with some general remarks and conclusions.

3.2 Derivation of the Governing Equations

A streamline coordinate system is selected with $x_1$ and $x_2$ corresponding to the streamwise and cross-stream directions respectively; $x_3$ is normal to both $x_1$ and $x_2$. The boundary layer equations in a general streamline orthogonal coordinate system are:

$$\frac{\partial}{\partial x_1} (h_2 u_1) + \frac{\partial}{\partial x_2} (h_1 u_2) + h_1 h_2 \frac{\partial u_3}{\partial x_3} = 0, \quad (3.1)$$

$$\frac{u_1}{h_1} \frac{\partial u_1}{\partial x_1} + \frac{u_2}{h_2} \frac{\partial u_1}{\partial x_2} + u_3 \frac{\partial u_1}{\partial x_3} - K_2 u_1 u_2 + K_1 u_2^2 = \frac{U_e}{h_1} \frac{\partial U_e}{\partial x_1} + \frac{\partial^2 u_1}{\partial x_3^2}, \quad (3.2)$$
\[
\frac{u_1}{h_1} \frac{\partial u_2}{\partial x_1} + \frac{u_2}{h_2} \frac{\partial u_2}{\partial x_2} + u_3 \frac{\partial u_2}{\partial x_3} - K_1 u_1 u_2 + K_2 u_1^2 = K_e u_2^2 + \frac{\partial^2 u_2}{\partial x_3^2}. \tag{3.3}
\]

Here \( U_e \) is the free stream velocity. The pressure gradient term in the streamwise direction has been written in terms of \( U_e \) and is identical to that in the two-dimensional case; in addition, \( u_1, u_2, \) and \( u_3 \) are the velocities corresponding to the streamwise, cross-stream, and normal directions, respectively; \( h_1 \) and \( h_2 \) are the metrics which describe the geometry of the streamlines; \( K_1 \) and \( K_2 \) measure the curvature of constant \( x_1 \) and \( x_2 \) lines, respectively. It is assumed that \( h_3 = 1 \).

Define a similarity variable \( \eta \) such that

\[
\eta = \frac{x_3}{\delta(x_1, x_2)}. \tag{3.4}
\]

Using the chain rule, it is clear that

\[
\frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_1} - \frac{\eta}{\delta} \frac{\partial \delta}{\partial x_1} \frac{\partial}{\partial \eta}, \tag{3.5}
\]

\[
\frac{\partial}{\partial x_2} = \frac{\partial}{\partial x_2} - \frac{\eta}{\delta} \frac{\partial \delta}{\partial x_2} \frac{\partial}{\partial \eta}, \tag{3.6}
\]

\[
\frac{\partial}{\partial x_3} = \frac{1}{\delta} \frac{\partial}{\partial \eta}. \tag{3.7}
\]

Define the profile functions \( f'(\eta) \) and \( g'(\eta) \) by

\[
u_1 = U_e f'(\eta), \tag{3.8}
\]
\[ u_2 = U_e g'(\eta) , \]  
and a function \( q(x_1,x_2) \) by

\[ q = \frac{1}{U_e} \left| \frac{\partial u_3}{\partial x_3} \right|_{x_3 \to 0} . \]  

(3.10)

Since \( f' \to 1 \) and \( g' \to 0 \) as \( \eta \to \infty \) at the boundary layer edge, it follows from the continuity equation (at the edge of the boundary layer) that

\[ \frac{1}{h_1 h_2} \frac{\partial}{\partial x_1} (h_2 U_e) = -q U_e . \]

Therefore,

\[ K_1 = \frac{-1}{h_1 h_2} \frac{\partial h_2}{\partial x_1} = \frac{1}{U_e h_1} \frac{\partial U_e}{\partial x_1} + q . \]  

(3.11)

If it is assumed that the normal component of vorticity is zero in the boundary layer it can be shown that \( h_1 = \frac{1}{U_e} \) (Degani, Smith, and Walker, 1992) and

\[ K_2 = \frac{-1}{h_1 h_2} \frac{\partial h_1}{\partial x_2} = \frac{1}{U_e h_2} \frac{\partial U_e}{\partial x_2} . \]  

(3.12)

Each term in the continuity equation may be written as:

\[ \frac{\partial}{\partial x_1} (h_2 u_1) = \frac{\partial}{\partial x_1} (h_2 U_e f') = \frac{\partial}{\partial x_1} (h_2 U_e f') + h_2 U_e \frac{\partial f'}{\partial x_1} , \]
\[ \frac{\partial}{\partial x_2} (h_1 u_2) = \frac{\partial}{\partial x_2} (h_1 U_e g') = \frac{\partial}{\partial x_2} (h_1 U_e g') + h_1 U_e \frac{\partial g'}{\partial x_2}; \]

since

\[ \frac{\partial f'}{\partial x_1} = -\frac{\eta}{\delta} \frac{\partial \delta}{\partial x_1} f''; \]

\[ \frac{\partial g'}{\partial x_2} = -\frac{\eta}{\delta} \frac{\partial \delta}{\partial x_2} g''; \]

it follows that

\[ \frac{1}{h_1 h_2} \frac{\partial}{\partial x_1} (h_2 U_e) f' - \frac{U_e}{\delta h_1} \frac{\partial \delta}{\partial x_1} (\eta f'' - f) - \frac{U_e}{\delta h_2} \frac{\partial \delta}{\partial x_2} (\eta g'' - g) + \frac{1}{\delta} \frac{\partial u_3}{\partial \eta} = 0. \]

Upon integration the following expression is obtained for the normal velocity \( u_3 \):

\[ u_3 = \delta \left\{ q U_e f' + \frac{U_e}{\delta h_1} \frac{\partial \delta}{\partial x_1} (\eta f'' - f) + \frac{U_e}{\delta h_2} \frac{\partial \delta}{\partial x_2} (\eta g'' - g) \right\}. \quad (3.13) \]

In the streamwise momentum equation, each term becomes as follows:

\[ \frac{u_1}{h_1} \frac{\partial u_1}{\partial x_1} = \frac{U_e f'}{h_1} \frac{\partial}{\partial x_1} (U_e f') = \frac{U_e}{h_1} \frac{\partial U_e}{\partial x_1} (f')^2 - \frac{U_e^2}{\delta h_1} \frac{\partial \delta}{\partial x_1} \eta f''', \]

\[ \frac{u_2}{h_2} \frac{\partial u_1}{\partial x_2} = \frac{U_e g'}{h_2} \frac{\partial}{\partial x_2} (U_e g') = \frac{U_e}{h_2} \frac{\partial U_e}{\partial x_2} f' g' - \frac{U_e^2}{\delta h_2} \frac{\partial \delta}{\partial x_2} \eta g'''', \]
\[
\frac{u_3}{\partial x_3} = \left\{ qU_e f + \frac{U_e}{\delta h_1} \frac{\partial \delta}{\partial x_1} (\eta' - f) + \frac{U_e}{\delta h_2} \frac{\partial \delta}{\partial x_2} (\eta' - g) \right\} U_e'. \\
\]

Therefore,

\[
\frac{u_1}{h_1 \frac{\partial u_1}{\partial x_1}} + \frac{u_2}{h_2 \frac{\partial u_1}{\partial x_2}} + \frac{u_3}{\partial x_3} = \frac{U_e}{h_1} \frac{\partial U_e}{\partial x_1} (f')^2, \\
+ \frac{U_e}{h_2} \frac{\partial U_e}{\partial x_2} f'g' + qU_e^2 f'' - \frac{U_e^2}{\delta h_1} \frac{\partial \delta}{\partial x_1} f'' - \frac{U_e^2}{\delta h_2} \frac{\partial \delta}{\partial x_2} g f'', \\
(3.14)
\]

\[
-K_1u_1u_2 = -K_2U_e^2 f'g' = -\frac{U_e}{h_2} \frac{\partial U_e}{\partial x_2} f'g', \\
(3.15)
\]

\[
K_1u_2^2 = K_1U_e^2 (g')^2 = \left( qU_e^2 + \frac{U_e}{h_1} \frac{\partial U_e}{\partial x_1} \right) (g')^2, \\
(3.16)
\]

\[
\frac{\partial^2 u_1}{\partial x_3^2} = \frac{U_e}{\delta^2} \frac{\partial^2}{\partial \eta^2} f' = \frac{U_e}{\delta^2} f'''. \\
(3.17)
\]

Combining equations (3.14) – (3.17) and multiplying though by $\delta^2 U_e^{-1}$ yields

\[
\frac{\delta^2}{h_1} \frac{\partial U_e}{\partial x_1} (f')^2 + \left( \delta^2 qU_e - \frac{U_e \delta}{h_1} \frac{\partial \delta}{\partial x_1} \right) f'' - \frac{U_e \delta}{h_2} \frac{\partial \delta}{\partial x_2} g f'' \\
+ \left( qU_e \delta^2 + \frac{\delta^2}{h_1} \frac{\partial U_e}{\partial x_1} \right) (g')^2 = \frac{\delta^2}{h_1} \frac{\partial U_e}{\partial x_1} + f'''.
\]

Define the following quantities:

\[
\alpha_1 = \frac{U_e \delta}{h_1} \frac{\partial \delta}{\partial x_1} = \frac{\sigma}{\delta h_1} \frac{\partial \delta}{\partial x_1}, \quad (3.18)
\]

22
where

\[ \sigma = U_e \delta^2. \]  

Upon rearranging the momentum equation, it is easily shown that

\[ f'''' + (\alpha_1 - \nu) ff''' - \beta_1 (1 - (f')^2) + \alpha_2 g g'' + (\beta_1 - \nu) (g')^2 = 0. \]  

Now consider the cross-stream equation for which the individual terms transform as follows:

\[ \frac{u_1}{h_1} \frac{\partial u_2}{\partial x_1} = \frac{U_e}{h_1} \frac{\partial}{\partial x_1} \left( U_e g' \right) = \frac{U_e}{h_1} \frac{\partial U_e}{\partial x_1} f' g' - \frac{U_e^2}{\delta h_1} \frac{\partial \delta}{\partial x_1} \eta' g''', \]

\[ \frac{u_2}{h_2} \frac{\partial u_2}{\partial x_2} = \frac{U_e g'}{h_2} \frac{\partial}{\partial x_2} \left( U_e g' \right) = \frac{U_e}{h_2} \frac{\partial U_e}{\partial x_2} (g')^2 - \frac{U_e^2}{\delta h_2} \frac{\partial \delta}{\partial x_2} \eta g'''', \]

\[ u_3 \frac{\partial u_2}{\partial x_3} = \left\{ q U_e f + \frac{U_e}{\delta h_1} \frac{\partial \delta}{\partial x_1} (\eta'' - f) + \frac{U_e}{\delta h_2} \frac{\partial \delta}{\partial x_2} (\eta' - g) \right\} U_e g''. \]
Therefore, the convective terms become

\[
\frac{u_1}{h_1} \frac{\partial u_2}{\partial x_1} + \frac{u_2}{h_2} \frac{\partial u_2}{\partial x_2} + u_3 \frac{\partial u_2}{\partial x_3} = \frac{U_e}{h_1} \frac{\partial U_e}{\partial x_1} (g')^2 + \frac{U_e}{h_2} \frac{\partial U_e}{\partial x_2} (g')^2
\]

\[
+ q U_e^2 f g'' - \frac{U_e^2}{\delta h_1} \frac{\partial \delta}{\partial x_1} f g'' - \frac{U_e^2}{\delta h_2} \frac{\partial \delta}{\partial x_2} g g'' .
\]

Likewise, the terms on the right side become

\[
-K_1 u_1 u_2 = -K_1 U_e^2 (g') ,
\]

\[
K_2 U_1^2 = K_2 U_e^2 (f')^2 ,
\]

\[
\frac{\partial^2 u_2}{\partial x_3^2} = \frac{U_e}{\delta^2} g g'' .
\]

Combining equations (3.25 - 3.28) and multiplying through by \( \delta^2 U_e^{-1} \)
yields

\[
\left\{ \frac{\delta^2}{h_1} \frac{\partial U_e}{\partial x_1} - K_1 U_e \delta^2 \right\} f g' + \left\{ \frac{\delta^2}{h_2} \frac{\partial U_e}{\partial x_2} (g')^2 \right\} + \left\{ q U_e \delta^2 - \frac{\delta U_e}{h_1} \frac{\partial \delta}{\partial x_1} \right\} f g''
\]

\[
- \frac{\delta U_e}{h_2} \frac{\partial \delta}{\partial x_2} g g'' + K_2 U_e \delta^2 (f')^2 = K_2 U_e \delta^2 + g'' .
\]

Rearranging the above and substituting equations (3.18) - (3.22) yields
Note that the boundary conditions corresponding to the streamwise and cross-stream equations are:

\[ f = f' = g = g' = 0, \quad \text{at } \eta = 0, \quad (3.30) \]

\[ f' \rightarrow 1, \quad g' \rightarrow 0, \quad \text{as } \eta \rightarrow \infty. \]  

The wall is assumed to be solid with no suction.

### 3.3 External Flows Leading to Similarity and Selection of the Mathematical Model

Consider the quantity \( \sigma \), as defined in equation (3.23). It follows from differentiation with respect to \( x_1 \) and \( x_2 \) that

\[ \frac{1}{h_i} \frac{\partial \sigma}{\partial x_i} = \frac{\delta}{h_1} \frac{\partial u_e}{\partial x_i} + \frac{2u_e \delta}{h_i} \frac{\partial \delta}{\partial x_i} = 2\alpha_i - \beta_i, \quad i = 1,2. \]

Consider the case \( 2\alpha_1 - \beta_1 \neq 0 \). Integration yields \( \sigma = (2\alpha_1 - \beta_1)(s_1 - s_0) \), where \( s_1 \) is a coordinate in the streamwise direction such that \( ds_1 = h_1 dx_1 \), and \( s_0 \) is a constant. From the definition of \( \beta_1 \),

\[ \frac{1}{U_e h_1} \frac{\partial U_e}{\partial x_1} = \frac{-\beta_1}{\sigma} = \frac{-\beta_1}{(2\alpha_1 - \beta_1)(s_1 - s_0)}. \]
Therefore, for this situation the mainstream velocity obeys a power law according to

\[ U_e = u_o |s_1 - s_0|^{-\beta_1/2\alpha_1 - \beta_1} \]  

(3.31)

where \( u_o \) is a function of \( x_2 \) and \( s_0 \) is the point of attachment of the inviscid flow (i.e., \( U_e = 0 \) at \( s_1 = s_0 \)).

Since \( \sigma = U_e \delta^2 \), then

\[ \delta^2 = \frac{\sigma}{U_e} = \frac{(2\alpha_1 - \beta_1)(s_1 - s_0)}{u_o |s_1 - s_0|^{-\beta_1/2\alpha_1 - \beta_1}} \]

and the following expression is obtained for the thickness function:

\[ \delta = \sqrt{\frac{2\alpha_1 - \beta_1}{u_o}} \cdot |s_1 - s_0|^{-\beta_1/2\alpha_1 - \beta_1} \]  

(3.32)

Because the definition of \( \eta \) is arbitrary within a constant, the same procedure discussed for the Falkner-Skan equation may be followed and the second coefficient in both equations (3.24) and (3.29) may be selected as

\[ (\alpha_2 - \nu) = 1. \]  

(3.33)
Next, two other conditions are derived by requiring that $d\sigma$ and $dU_e$ be perfect differentials. Since
\[
d\sigma = \frac{\partial \sigma}{\partial x_1} dx_1 + \frac{\partial \sigma}{\partial x_2} dx_2 = h_1(2\alpha_1 - \beta_1)dx_1 - h_2(2\alpha_2 - \beta_2)dx_2,
\]
it follows that for $d\sigma$ to be a perfect differential,
\[
\frac{\partial}{\partial x_2} \{h_1(2\alpha_1 - \beta_1)\} = \frac{\partial}{\partial x_2} \{h_2(2\alpha_2 - \beta_2)\},
\]
or
\[
K_2(2\alpha_1 - \beta_1) = K_1(2\alpha_2 - \beta_2). \tag{3.34}
\]

But from equation (3.11) it follows that
\[
K_1 = \frac{\nu - \beta_1}{\sigma}, \tag{3.35}
\]
and from equation (3.12) we obtain
\[
K_2 = \frac{1}{U_e h_2} \frac{\partial U_e}{\partial x_2} = \frac{-\beta_2}{\sigma}. \tag{3.36}
\]

Therefore, equation (3.34) becomes
\[
(2\alpha_1 - \beta_1) \cdot \beta_2 = (2\alpha_2 - \beta_2)(\beta_1 - \nu). \tag{3.37}
\]
Now we consider $U_e$:

$$dU_e = \frac{\partial U_e}{\partial x_1} dx_1 + \frac{\partial U_e}{\partial x_2} dx_2 = -\frac{\beta_1 U_e h_1}{\sigma} dx_1 - \frac{\beta_2 U_e h_2}{\sigma} dx_2 ,$$

and this implies that for $dU_e$ to be a perfect differential,

$$\frac{\partial}{\partial x_2} \left( \frac{\beta_1 U_e h_1}{\sigma} \right) = \frac{\partial}{\partial x_1} \left( \frac{\beta_2 U_e h_2}{\sigma} \right).$$

Expand this expression to obtain

$$\frac{\beta_1}{\sigma} \frac{\partial}{\partial x_2} (U_e h_1) - \frac{\beta_1 h_1}{\sigma^2} \frac{\partial \sigma}{\partial x_2} = \frac{\beta_2}{\sigma} \frac{\partial}{\partial x_1} (U_e h_2) - \frac{\beta_2 h_2}{\sigma^2} \frac{\partial \sigma}{\partial x_1} .$$

Multiplying through by $\frac{\sigma^2}{U_e h_1 h_2}$ yields:

$$-\frac{\beta_1}{h_2} \frac{\partial \sigma}{\partial x_2} = \frac{\beta_2}{U_e h_1 h_2} \frac{\partial}{\partial x_2} (U_e h_2) - \frac{\beta_2}{h_1} \frac{\partial \sigma}{\partial x_1} .$$

But

$$\frac{\sigma}{U_e h_1 h_2} \frac{\partial}{\partial x_1} (U_e h_2) = -\sigma q = -\nu ,$$

and therefore

$$-\beta_1 (2 \alpha_2 - \beta_2) = -\beta_2 \nu - \beta_2 (2 \alpha_1 - \beta_1) ,$$

or
\[(2\alpha_2 - \beta_2)\beta_1 = (2\alpha_1 - \beta_1 + \nu)\beta_2.\] 

(3.38)

Multiplying equation (3.37) by \(\beta_1\) and equation (3.38) by \((\beta_1 - \nu)\) and eliminating \(\alpha_2\) yields

\[\beta_2 \nu(2\alpha_1 + \nu - 2\beta_1) = 0.\] 

(3.39)

Equations (3.33), (3.38) and (3.39) describe a number of cases which are now discussed individually.

**Case 1: \(\beta_2 = 0\)**

Since equation (3.38) is valid for all \(\beta_1\), it follows that \(\alpha_2 = 0\). Consequently, this situation describes a two-dimensional boundary layer since the cross-stream pressure gradient vanishes. Hence, \(\nu = \beta_1\) and from equation (3.33) it follows that \(\alpha_1 = 1 + \beta_1\). With these values of the constants, the cross-stream equation is identically satisfied by \(g = 0\), and the streamwise equation reduces to the Falkner-Skan equation if \(\beta_1\) is replaced by \(\beta_1 = -\beta_1\). Performing this step, equations (3.24) and (3.29) become:

\[f''' + ff'' + \overline{\beta}_1(1 - f^2) = 0,\] 

(3.40)

\[g = 0.\] 

(3.41)

**Case 2: \(2\alpha_1 + \nu - 2\beta_1 = 0\).**

Substitution of equation (3.33) yields
\[
\alpha_1 = \frac{1 + 2\beta_1}{3} = \frac{1 - 2\bar{\beta}_1}{3},
\]

\[
\nu = \frac{2(\beta_1 - 1)}{3} = \frac{2(-1 - \bar{\beta}_1)}{3},
\]

where again \(\bar{\beta}_1 = -\beta_1\). Substitution into equation (3.37) shows that \(\alpha_2 = \beta_2\). If these values are substituted into (3.24) and (3.29), the momentum equations in the streamwise and cross-stream directions are:

\[
f'''' + ff''' + \bar{\beta}_1(1-f'^2) + \beta_2 g f'' + \left(\frac{2-\bar{\beta}_1}{3}\right)(g')^2 = 0, \tag{3.42}
\]

\[
g'''' + fg''' + \beta_2 gg'' - \frac{2}{3}(\bar{\beta}_1 + 1)fg' + \beta_2 g'^2 - \beta_2(1-f'^2) = 0. \tag{3.43}
\]

It may be noted that if \(\beta_2\) is set equal to zero, equation (3.43) is identically satisfied by \(g' = 0\) and equation (3.42) again reduces to the well-known Falkner-Skan equation.

**Case 3: \(\nu = 0\).**

From equation (3.33) it follows that \(\alpha_1 = 1\), and from equation (3.37) we obtain \(\alpha_2 = \frac{\beta_2}{\bar{\beta}_1}\), with \(\beta_1 \neq 0\). For these values of the constants, the governing equations reduce to:

\[
f'''' + ff''' + \bar{\beta}_1(1-f'^2) - \frac{\beta_2}{\bar{\beta}_1} g f'' - \bar{\beta}_1 g'^2 = 0, \tag{3.44}
\]

\[
g'''' + fg''' - \frac{\beta_2}{\bar{\beta}_1} gg'' + \beta_2 g'^2 - \beta_2(1-f'^2) = 0. \tag{3.45}
\]
It may be noted that these equations are apparently irregular as $\tilde{\beta}_1 \to 0$, and this behavior is not expected. As a consequence, the equations corresponding to Case 3 will not be considered further and instead Case 2 will be addressed. These equations appear to be well-behaved and are consistent with the two-dimensional case. Thus, the mathematical model adopted for self-similarity in three-dimensional laminar boundary layers is (with $\tilde{\beta}_1 \to \beta_1$):

\begin{align}
  f''' + ff'' + \beta_1(1-f^2) + \beta_2gf'' + \frac{(2-\beta_1)}{3} g''^2 &= 0, \quad (3.46) \\
  g''' + fg'' + \beta_2gg'' - \frac{2}{3} (\beta_1 + 1) f'g' + \beta_2g'^2 - \beta_2(1-f^2) &= 0. \quad (3.47)
\end{align}

The boundary conditions are:

\begin{align}
  f = f' = g = g' = 0, \quad \text{at } \eta = 0, \\
  f' \to 1, \quad g' \to 0, \quad \text{as } \eta \to \infty. \quad (3.48)
\end{align}

Information relative to all cases is summarized in tabular form in Appendix E, Figure E.1.
4. Numerical Methods

The equations to be solved for various ranges of the streamwise pressure gradient $\beta_1$ and the cross-stream pressure gradient $\beta_2$ are:

\begin{align*}
&f''' + ff'' + \beta_1 (1-f'^2) + \beta_2 g f'' + \left(\frac{2-\beta_1}{3}\right) g'^2 = 0, \quad (4.1) \\
g''' + fg'' + \beta_2 gg'' - \frac{g'}{3} (1+\beta_1) f g' + \beta_2 g'^2 - \beta_2 (1-f'^2) = 0. \quad (4.2)
\end{align*}

The boundary conditions are:

\begin{align*}
f = f' = g = g' = 0, \quad &\text{at } \eta = 0, \quad (4.3) \\
f' \rightarrow 1, \quad g' \rightarrow 0, \quad &\text{as } \eta \rightarrow \infty.
\end{align*}

Two approaches had to be taken in order to obtain the regular solutions and the backflow solutions. They are described in synopsis form below, and then the details of the numerical algorithms are presented.

The regular solutions were obtained by applying iterative boundary value techniques to solve for both the cross-stream velocity $g'$ and the streamwise velocity $f'$. Since the equation set is strongly coupled and nonlinear, initial guesses are made first for $f'$ and $g'$. The equations also depend on $f$ and $g$, and the trapezoidal rule was used to obtain estimates of these functions over the mesh from the current estimates of $f'$ and $g'$ respectively. The Thomas algorithm was used to solve the difference equations for all $f'$ and $g'$. In practice, a "large" value of $\eta$ was picked as
the place to apply the end boundary condition. A relative test for convergence was applied on \( f' \) only, and the results were considered to have converged when two successive iterates agreed to within six significant figures. This assumes that the convergence of \( g' \) will "follow" that of \( f' \), and that in general \( g' \) converges faster than \( f' \). Relative convergence testing on \( g' \) is more difficult since both boundary conditions are zero.

The solutions involving backflow streamwise profiles were obtained by iteratively applying a shooting method on \( f' \) and a boundary value technique on \( g' \). Other methods were attempted with discouraging results. The regular solution technique was tried by making educated guesses for \( f' \) and \( g' \), but the method latched on to the regular solution and would not produce profiles characteristic of backflow. Two shooting methods were attempted by guessing both \( f''(0) \) and \( g''(0) \), but the method was unstable and no solutions were obtained.

The scheme that was successful was initiated by making an initial guess for \( g' \) at each mesh point, as well as a guess for \( f''(0) \); \( g \) itself was then evaluated using the trapezoidal rule. A four-step Runge Kutta technique was used to calculate the solution of \( f, f', \) and \( f'' \) in a step-by-step manner starting from \( \eta = 0 \). The current values of \( g \) and \( g' \) were utilized in this procedure. Using the new values of \( f \) and \( f' \) and the current values of \( g \) and \( g' \), the Thomas algorithm was used to compute a new estimate for \( g' \). A relative convergence test was then performed on \( f' \). If convergence had not been achieved, a linear interpolation is used to re-estimate \( f''(0) \) from the previous two values and the corresponding boundary values of \( f' \) at \( \eta = \infty \); \( g \) and \( g' \) were then updated, and the iteration was continued until convergence.
was achieved. The numerical details for the regular solution technique are outlined below.

Equations (4.1) and (4.2) are rewritten as a four equation set of two first-order differential equations and two second-order differential equations using the substitutions:

\[ f' = y, \quad (4.4) \]
\[ g' = z, \quad (4.5) \]
\[ y'' + f y' + \beta_1 (1 - y^2) + \beta_2 g y' + \frac{2}{3} (2 - \beta_1) z^2 = 0, \quad (4.6) \]
\[ z'' + f z' + \beta_2 g z' - \frac{2}{3} (1 + \beta_1) y z + \beta_2 z^2 - \beta_2 (1 - y^2) = 0. \quad (4.7) \]

Using central differences to approximate equation (4.6) at a typical mesh point, it is easily shown that:

\[ b_{1j} y_{j+1} + a_{1j} y_j + c_{1j} y_{j-1} = d_{1j}, \quad (4.8) \]

where

\[ b_{1j} = 1 + \frac{h}{2} f_j + \frac{h}{2} \beta_2 g_j, \quad (4.9) \]
\[ c_{1j} = 1 - \frac{h}{2} f_j - \frac{h}{2} \beta_2 g_j, \quad (4.10) \]
\[ a_{1j} = -2 - h^2 \beta_1 y_j, \quad (4.11) \]
\[ d_{1j} = -h^2 \beta_1 - h^2 \left( \frac{2 - \beta_1}{3} \right) z_j^2, \quad (4.12) \]

where \( h \) is the mesh size. A similar procedure applied to equation (4.7) yields:

\[ b_{2j} z_{j+1} + a_{2j} z_j + c_{2j} z_{j-1} = d_{2j}, \quad (4.13) \]

where
Equations (4.8) and (4.13) constitute a tri-diagonal matrix problem which was solved using the Thomas algorithm. Let $n+1$ denote the last point in the $\eta$ mesh, while $\eta=0$ is at $n=1$. To utilize the Thomas algorithm two arrays $(F_1$, $\delta_1)$ and $(F_2$, $\delta_2)$ must be defined for each equation. The boundary values of these arrays at $\eta=0$ are:

\begin{align*}
F_{1,1} &= 0, \\
F_{2,1} &= 0, \\
\delta_{1,1} &= y_1, \\
\delta_{2,1} &= z_1.
\end{align*}

A forward elimination is then carried out by taking $j = 1, 2, 3, ...$ successively in:
\[ F_{1,j} = -b_{1j}\{a_{1j} + c_{1j} + F_{1,j-1}\}^{-1}, \]  
\[ (4.22) \]

\[ F_{2,j} = -b_{2j}\{a_{2j} + c_{2j} + F_{2,j-1}\}^{-1}, \]  
\[ (4.23) \]

\[ \delta_{1,j} = \{d_{1j} - c_{1j} \delta_{1,j-1}\}\{a_{1j} + c_{1j} + F_{1,j-1}\}^{-1}, \]  
\[ (4.24) \]

\[ \delta_{2,j} = \{d_{2j} - c_{2j} \delta_{2,j-1}\}\{a_{2j} + c_{2j} + F_{2,j-1}\}^{-1}. \]  
\[ (4.25) \]

At the outer boundary \( j = n + 1 \), the boundary conditions at infinity were applied as an approximation according to:

\[ y_{n+1} = 1, \]  
\[ (4.26) \]

\[ z_{n+1} = 0. \]  
\[ (4.27) \]

The back substitution step to obtain new estimates for \( y \) and \( z \) is accomplished by taking \( j = 1, 2, 3, \ldots \) successively in:

\[ y_{n+2-j} = F_{1,n+2-j} y_{n+3-j} + \delta_{1,n+2-j}, \]  
\[ (4.28) \]

\[ z_{n+2-j} = F_{2,n+2-j} z_{n+3-j} + \delta_{2,n+2-j}. \]  
\[ (4.29) \]

Convergence was then checked at each point in the mesh by insisting that \( y_j \) agree with the previous iterate to a specified number of significant figures. If the test was not met at every mesh point, the iteration was continued.

The numerical scheme for the backflow solution is somewhat different and is now described. Recall that the backflow solution employs a shooting method for \( f' \) in conjunction with a boundary value technique for \( g' \).
Rewrite equation (4.2) as

\[ f' = y_1 , \]  
\[ g' = y_2 , \]  
\[ y_2'' + f y_2' + \beta_2 g y_2' - \frac{2}{3} (1 + \beta_1) y_1 y_2 + \beta_2 y_2^2 - \beta_2 (1 - y_1^2) = 0 \]  

(4.30)  
(4.31)  
(4.32)

Using central difference formulae, the finite difference approximation to equation (4.32) is of the form:

\[ b_j y_{2,j+1} + a_j y_{2,j} + c_j y_{1,j-1} = d_j , \]  

(4.33)

where

\[ b_j = 1 + \frac{h}{2} f_j + \frac{h}{2} \beta_2 g_j , \]  

(4.34)

\[ c_j = 1 - \frac{h}{2} f_j - \frac{h}{2} \beta_2 g_j , \]  

(4.35)

\[ a_j = -2 - h^2 \frac{2}{3} (1 + \beta_1) y_{1j} + h^2 \beta_2 y_{2j} , \]  

(4.36)

\[ d_j = h^2 \beta_2 (1 - y_{1j}^2) . \]  

(4.37)

Equation (4.1) can be rewritten in the form:

\[ f' = y_1 , \]  
\[ f'' = y_1' = z_1 , \]  
\[ f''' = z_1' , \]  
\[ g' = y_2 , \]  
\[ g'' = y_2' = z_2 , \]  

(4.38)  
(4.39)  
(4.40)  
(4.41)  
(4.42)
\[ g''' = z_1', \]  
\[ z_1' = -fz_1 - \beta_1(1 - y_1^2) - \beta_2 g z_1 - \left( \frac{2 - \beta_1}{3} \right) y_2. \]  
(4.43)  
(4.44)

To define a Runge-Kutta procedure for this system, let

\[ k_f_1 = h y_{1j}, \]  
(4.45)

\[ \ell_f_1 = h z_{1j}, \]  
(4.46)

\[ m_f_1 = h \left\{ -f z_{1j} - \beta_1[1 - y_{1j}^2] - \beta_2 g z_{1j} - \left( \frac{2 - \beta_1}{3} \right) (y_{2j})^2 \right\}, \]  
(4.47)

\[ k_f_2 = h(y_{1j} + \frac{\ell_f_1}{2}), \]  
(4.48)

\[ \ell_f_2 = h(z_{1j} + \frac{m_f_1}{2}), \]  
(4.49)

\[ m_f_2 = h \left\{ -\left( f_j + \frac{k_f_1}{2} \right) \left( z_{1j} + \frac{m_f_1}{2} \right) - \beta_1 \left[ 1 - \left( y_{1j} + \frac{\ell_f_1}{2} \right)^2 \right] \right. \]  
\[ - \beta_2 \frac{1}{2} \left( g_j + g_{j+1} \right) \left( z_{1j} + \frac{m_f_1}{2} \right) - \left( \frac{2 - \beta_1}{3} \right) \left[ \frac{1}{2} \left( y_{2j+1} + y_{2j+1} \right)^2 \right] \right\}, \]  
(4.50)

\[ k_f_3 = h(y_{1j} + \frac{\ell_f_2}{2}), \]  
(4.51)

\[ \ell_f_3 = h(z_{1j} + \frac{m_f_2}{2}), \]  
(4.52)

\[ m_f_3 = h \left\{ -\left( f_j + \frac{k_f_2}{2} \right) \left( z_{1j} + \frac{m_f_2}{2} \right) - \beta_1 \left[ 1 - \left( y_{1j} + \frac{\ell_f_2}{2} \right)^2 \right] \right. \]  
\[ - \beta_2 \frac{1}{2} \left( g_j + g_{j+1} \right) \left( z_{1j} + \frac{m_f_2}{2} \right) - \left( \frac{2 - \beta_1}{3} \right) \left[ \frac{1}{2} \left( y_{2j+1} + y_{2j+1} \right)^2 \right] \right\}, \]  
(4.53)

\[ k_f_4 = h(y_{1j} + \ell_f_3), \]  
(4.54)
\[ \ell f_4 = h(z_{1j} + mf_3), \quad (4.55) \]

\[ mf_4 = h \left\{ -\left( f_j + kf_3 \right) \left( z_{1j} + mf_3 \right) - \beta_1 \left[ 1 - \left( y_{1j} + \ell f_3 \right)^2 \right] \right. \]
\[ \left. - \beta_2 g_{j+1} \left( z_{1j} + mf_3 \right) - \left( \frac{2 - \beta_1}{3} \right) \left( y_{2j+1} \right)^2 \right\}. \quad (4.56) \]

In a Runge-Kutta method the solution is computed in a step-by-step manner using the formulae:

\[ f_{j+1} = f_j + \frac{1}{6} \left( kf_1 + 2kf_2 + 2kf_3 + kf_4 \right), \quad (4.57) \]

\[ y_{1,j+1} = y_{1,j} + \frac{1}{6} \left( \ell f_1 + 2\ell f_2 + 2\ell f_3 + \ell f_4 \right), \quad (4.58) \]

\[ z_{1,j+1} = z_{1,j} + \frac{1}{6} \left( mf_1 + 2mf_2 + 2mf_3 + mf_4 \right). \quad (4.59) \]

In the present scheme the solution of the streamwise momentum equation is computed by guessing a value for \( z_1(0) = z_{1,1} \), and then using equations \((4.57)-(4.59)\) to perform a step-by-step Runge-Kutta integration for increasing \( \eta \). The value of \( z_1(0) \) is systematically adjusted until the condition \( f' \to 1 \) for \( \eta \) large is realized. At any stage in this iteration, estimates of the cross-stream profile \( g' \) are used to evaluate the terms in equations \((4.45)-(4.56)\). An estimate of the cross-stream profile is evaluated at any stage by solving equation \((4.44)\) using the Thomas algorithm.

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The skew angle is the angle between the streamwise and cross-stream velocity vectors at any point which varies across the boundary layer. The skew angle is formally defined by

\[ \theta = \arctan \left( \frac{g'}{f'} \right). \]  

(4.60)

For the regular (monotonic) solutions, the angle starts out at some finite value at \( \eta = 0 \) and decreases to zero as \( \eta \to \infty \). On the other hand, \( \theta \) may exceed \( \pi/2 \) for backflow streamwise solutions. The skew angle is calculated for each flow type in the same manner. Once the solutions have converged, equation (4.60) was used to calculate \( \theta \) throughout the mesh. Near \( \eta = 0 \) (the wall), \( g'/f' \) will be indeterminate and L'Hopital's rule must be used, according to

\[ \lim_{\eta \to 0} \theta = \lim_{\eta \to 0} \left\{ \arctan \left( \frac{g''}{f''} \right) \right\}. \]  

(4.61)
5. Results

5.1 Favorable Streamwise Pressure Gradients

The equations governing similar velocity behavior in the boundary layer were solved numerically for a wide range of favorable streamwise pressure gradients and cross-stream pressure gradients; in this range the solutions are unique. The streamwise pressure gradient parameter \( \beta_1 \) was varied through 0, 0.2, 0.6, and 1.0; at each \( \beta_1 \), \( \beta_2 \) was increased from zero to some finite value so trends could be inferred. Appendix B contains some representative solutions. It is worthwhile to re-emphasize that velocity profiles in which \( f' \) increases monotonically from 0 to 1 through the boundary layer will be referred to as “regular” solutions. All unique solutions in the range \( \beta_1 > 0 \) are unique regular solutions. The mesh size used for the calculations reported in Appendix B was \( \Delta \eta = 0.05 \), and convergence is tested using a factor of \( \epsilon = 10^{-6} \). Note that various grid refinement studies were carried out and the results are believed to be grid independent.

Typical regular velocity profiles (\( f' \) and \( g' \)) for a favorable streamwise pressure gradient and a moderate cross-stream pressure gradient are pictured in Figure B.1.a, and the corresponding skew angle profile in Figure B.1.b. The representative value chosen was \( \beta_1 = 0.2 \), \( \beta_2 = -0.5 \). The distributions generally behave as expected with \( f' \) increasing monotonically from 0 to 1 through the boundary layer; \( g' \) increases from 0 at the wall to some maximum value midway through the layer, and then decays to 0 again as \( \eta \rightarrow \infty \). The vector sum of these two profiles represents the total velocity, and the influence of the cross-stream pressure gradient is to change the flow direction. The skew angle (\( \theta \)) profile measures the change in direction of the
resultant velocity from the external streamline. The skew angle is sketched in Figure B.2. In this parameter range, the skew angle takes on a maximum finite value at the wall and decays to zero at the freestream.

Typical regular velocity profiles for a favorable streamwise pressure gradient and a very large cross-stream pressure gradient are depicted in Figure B.3.a, with the corresponding skew angle profile shown in Figure B.3.b for $\beta_1 = 0.2, \beta_2 = -4.4$. There are some interesting characteristics that should be noted concerning these distributions. First, note that the maximum value of $g'$ is much larger than in the first case discussed: it is approximately 0.9 compared to 0.18. Also, note that this peak occurs extremely close to the wall. The value of $\eta_\infty$ where the uniform stream condition was applied had to be increased from approximately 7.0 in the first case compared to 500 for the second case, in order to ensure adequate decay of the streamwise profile to the mainstream distribution. Thus, the boundary layer grows much thicker as the magnitude of the cross-stream pressure gradient is increased. Upon close examination of $g'$ near $\eta = \eta_\infty$, it may seem that $g'$ appears to decrease rather suddenly to zero, and it appears that the validity of the solution may be in question. However, the data show that while the decay of $g'$ to zero is very rapid, it is also smooth. Also, a larger $\eta_\infty$ was selected to check the accuracy of the solution ($\eta_\infty = 500$ vs. $\eta_\infty = 1000$) and the exact same results were obtained.

The skew angle profile in Figure B.3.b reflects the high degree of change in the velocity direction. Note that near the wall, $\theta$ is close to 90°. Thus, the change in velocity direction is rather severe for a large cross-
stream pressure gradient, and decays to zero in the same general manner as $g'$. The important conclusion to be made here is that it appears that another boundary layer is developing near the wall. Because of the scales on the graphs, $g'$ appears to “spike” from zero to some finite value close to 1, and then smoothly decay to zero in the freestream. In the limit (as $\beta_2 \to \infty$), at the wall $f' \to 0$ and $g'$ appears to change from 0 to 1 over a relatively short distance, and the $\eta_\infty$ required for the application of the end boundary condition would be very large, closely approaching infinity. It is evident from these results that an increasing cross-stream pressure gradient leads to a high degree of skew near the wall, and as $\beta_2 \to \infty$, $\theta \to \pi/2$ so that the flow direction changes by the maximum of 90° across the boundary layer.

Figures B.4.a through B.9.b depict velocity distributions and skew angle profiles for streamwise pressure gradients with $\beta_1 = 0$ for cases with low, moderate, and large cross-stream pressure gradients. As already discussed, at a fixed value of $\beta_1$, a number of trends can be noted as $\beta_2$ increases. For $\beta_2 = 0$, $g'$ is identically zero, and the flow is entirely two-dimensional. For low values of $\beta_2$, good solutions can be obtained with values of $\eta_\infty$ which are $O(1)$ and the computed skew angles at the wall are small and generally less than 10°. As $\beta_2$ increases, the maximum value of $g'$ in the boundary layer increases along with the wall skew angle, and $\eta_\infty$ must be increased. In fact, at a large enough $\beta_2$, values of $\eta_\infty$ such as $10^3 - 10^4$ are required.

Some interesting trends can also be inferred by comparing the velocity and skew angle profiles as $\beta_1$ increases. Generally, for a fixed $\beta_2$, the
maximum value of $g'$ is lower, as well as the wall skew angle and the required value of $\eta_\infty$. Thus, a higher value of the streamwise pressure gradient $\beta_1$ yields a thinner, more stable boundary layer for fixed $\beta_2$.

So far, cases where the cross-stream pressure gradient $\beta_2$ is negative have been discussed. In Figures B.10.a and B.10.b, velocity and skew angle profiles for flows with equal streamwise pressure gradients and equal magnitude but opposite sign cross-stream pressure gradients ($\beta_1 = 0.2$, $\beta_2 = \pm 1.0$) are compared. As shown, the distributions of $f'$ are equal, while the $g'$ and $\theta$ profiles for both cases are simply mirror images. Thus, the flow is simply deflected the opposite way for $\beta_2 > 0$.

Tables B.1.a, b, c and Figure B.11 exhibit the peak values of $g'$ and associated values of $\eta$ and $f'$ as either $\beta_1$ or $\beta_2$ increase. Note as $\beta_1$ increases for a fixed $\beta_2$, the maximum value of $g'$ decreases, and corresponding values of $\eta$ decrease and $f'$ increase. This is indicative of a thinner boundary layer for larger $\beta_1$; the streamwise velocity $f'$ approaches the freestream more rapidly, and the peak on $g'$ occurs much closer to the wall. Tables B.1.a and B.1.b depict peak values for $\beta_2 = -0.5$ and $-4.0$.

The influence of $\beta_2$ for a fixed $\beta_1$ is displayed in tabular form in Table B.1.c and in graphical form in Figure B.11. Note that as $\beta_2$ increases, the maximum value of $g'$ increases from 0 at $\beta_2 = 0$ and takes on a value of 0.9 at $\beta_2 = -4.0$. Note also that the growth rate is greater for small $\beta_2$. As $\beta_2$ increases, $\eta_\infty$ increases which indicates a thicker boundary layer; in addition, the peak on $g'$ occurs further away from the wall.

Tables B.2 and B.3 depict velocity and skew angle profiles in tabular form for the following cases: $\beta_1 = 0.2$, $\beta_2 = -0.5$ and $\beta_1 = 0.2$, $\beta_2 = -4.4$. 
They are included so results could be verified by an interested reader.

5.2 Adverse Streamwise Pressure Gradients

The governing equations were solved numerically for a wide range of adverse streamwise and cross-stream pressure gradients. In particular, the values of $\beta_1$ considered were: $-0.001, -0.01, -0.025, -0.05, -0.10, -0.15, -0.18, -0.198838$; for each $\beta_1$, the value of $\beta_2$ was increased from zero to some finite value at which it was found that the solution would not converge. In this range of $\beta_1$, there are generally two solutions, one of which is regular and one which exhibits backflow. Some typical results are shown in Appendix C. The mesh size used was $\Delta \eta = 0.05$.

Typical regular velocity profiles for an adverse streamwise pressure gradient and a moderate cross-stream pressure gradient are pictured in Figure C.1.a, and the corresponding skew angle profile is shown in Figure C.1.b. The representative value chosen was $\beta_1 = -0.10, \beta_2 = -0.60$. The distributions behave similarly to that discussed for the favorable streamwise pressure gradient case: $f'$ increases monotonically from zero to 1; $g'$ rises from zero, through a peak, and decays to zero; $\theta$ begins at some finite value less than 90° at the wall and then decays to zero as $\eta$ approaches the freestream.

Typical backflow velocity profiles for an adverse streamwise pressure gradient and a moderate cross-stream pressure gradient are portrayed in Figure C.2.a, and the corresponding skew angle profile is shown in Figure C.2.b. The representative value chosen was $\beta_1 = -0.10, \beta_2 = -0.05$. Note that these profiles are markedly different than those discussed so far. The
crossflow $g'$ behaves as previously discussed. However, $f'$ begins at zero at the wall as expected, becomes negative in a region of reversed flow midway through the layer, and finally becomes positive again approaching 1. The flow reversal is due to the presence of an adverse streamwise pressure gradient. It should be noted that $f''(0)$ is less than zero at the wall.

The skew angle profile also reflects the modified flow behavior; it takes on a value greater than 90° at the wall, decreases through 90° midway through the boundary layer, and approaches zero at the freestream. The point in the boundary layer where the skew angle is 90° occurs where $f'$ passes through zero.

Figure C.3.a depicts the backflow velocity profile for $\beta_1 = -0.001$ and $\beta_2 = 0$. Note the large region of relatively weak reversed flow (up to approximately $\eta = 20$) and the sharp change in $f'$ that occurs midway through the boundary layer in order for the profile to meet the condition $f' \rightarrow 1$ as $\eta \rightarrow \infty$. An interesting observation is that while the region of reversed flow may extend to large values of $\eta$, the magnitude of $f'$ in this region is small and of $O(10^{-1})$. In the limit as $\beta_1 \rightarrow 0^-$, the fluid in the region of backflow comes to rest and instantaneously jumps to $f' = 1$ at $\eta = \infty$. This is the alternative solution to the Blasius equation (Smith, 1984). Figure C.3.b portrays the evolution of the velocity profile as $\beta_1 \rightarrow 0^-$, obtained from a paper by Laine and Reinhart (1984), which is consistent with the present results.

Attempts were made to calculate three-dimensional solutions with a cross-stream pressure gradient for $\beta_1 = -0.001$ with no success. Apparently, a flow with such a small adverse streamwise pressure gradient can support
little or no cross-stream pressure gradient. This concept of a value of $\beta_2$ beyond which solutions do not converge will be discussed in detail later in this section.

Figures C.4.a and C.4.b compare regular and backflow velocity and skew angle profiles for representative values of the adverse streamwise and cross-stream pressure gradients. The values chosen were $\beta_1 = -0.15$ and $\beta_2 = -0.05$. The regular solution has $f''(0) > 0$ and the crossflow $g'$ has a very small peak and decays back to zero for small $\eta$ values. The backflow solution has $f''(0) < 0$ and the crossflow displays a larger peak than in the former case. The streamwise profile $f'$ approaches 1 much faster for the regular solution than for the reversed flow solution. Generally, this signifies that for flows with the same adverse streamwise pressure gradient and the same cross-stream pressure gradient, the backflow solution yields a thicker boundary layer than the regular solution; in addition, the peak on $g'$ is larger. The upper possible limit on $\beta_2$ is also expected to be much lower for the reversed flow case than for the regular case, and this will be confirmed later in this section. This means that a regular flow appears to be able to support a larger cross-stream pressure gradient than can a reversed flow for the same adverse streamwise pressure gradient.

The corresponding skew angle profiles pictured in Figure C.4.b support these same conclusions. Note also that these same observations can be made for the full range of adverse streamwise pressure gradients and cross-stream pressure gradients considered.

Figures C.5.a through C.11.b depict regular and backflow velocity distributions and skew angle profiles for $\beta_1 = -0.05$ and $\beta_1 = -0.18$; for
each value of $\beta_1$, $\beta_2$ is increased from zero to the point at which the numerical scheme does not converge. A number of trends can be noted for each type of solution as $\beta_2$ increases for a fixed $\beta_1$. The crossflow $g'$ is identically zero at $\beta_2 = 0$ for both the regular and backflow cases. For the regular monotonic solution, an increasing value of $\beta_2$ yields an increasing value of the crossflow peak, and the required value of $\eta_{\infty}$ also increases. The skew angle is also identically zero at $\beta_2 = 0$, and as $\beta_2$ increases, the value of $\theta$ at the wall increases towards $90^\circ$. These behaviors are consistent with those discussed for favorable streamwise pressure gradients.

The backflow solution exhibits somewhat different trends, the most notable being the existence of a region of reversed streamwise flow near the wall. The crossflow $g'$ and the behavior of $\eta_{\infty}$ as $\beta_2$ increases behave similarly to that discussed for the regular flow case. The behavior of the skew angle, however, is markedly different than that discussed so far. For a value of $\beta_2$ equal to zero, the skew angle at the wall takes on a value of $180^\circ$ and sharply diverts towards zero as $f'$ passes through zero. The profiles for $\theta$ retain this general shape as the cross-stream pressure gradient increases for a fixed $\beta_1$, and $\theta_{wall}$ approaches $90^\circ$ from above instead of from below as was the case for regular solutions.

Some interesting trends can also be inferred by comparing the velocity and skew angle profiles for a fixed $\beta_2$ as $\beta_1$ increases for both the regular and backflow cases. The regular velocity profiles exhibit the same traits for increasing values of $\beta_1$ that were observed for the favorable pressure gradient cases. Reversed flow profiles follow somewhat different trends. For a fixed value of $\beta_2$, as $\beta_1$ increases the maximum value of $g'$ increases, as
well as the skew angle at the wall and the required value of \( \eta_\infty \) for a smooth streamwise decay to the freestream. This is exactly opposite to the trend noted for the regular flow case. It can be concluded that reversed flows with higher adverse streamwise pressure gradients appear to have thinner boundary layers with smaller peak crossflows.

The skew angle profiles for the backflow case possess some interesting characteristics. There appears to be a point \((\eta, \theta)\) at a fixed value of \( \beta_1 \) that remains constant for any changes in \( \beta_2 \). Alternatively, one could say that at some point defined by the position \( \eta_r \), the ratio \( g'/f' \) does not change. This point can be found from the backflow Falkner-Skan solution at \( \beta_2 = 0 \); the value of \( \eta \) inside the boundary layer where \( f' = 0 \) determines the value of \( \eta_r \). As the value of \( \beta_1 \) decreases for a fixed \( \beta_2 \), the position \( \eta_r \) in the boundary layer decreases, finally taking on a value of \( \eta_r = 0 \) at the Falkner-Skan separation point, \( \beta_1 = -0.198838 \). The constant value of \( \theta \) corresponding to \( \eta_r \) decreases with decreasing values of \( \beta_2 \); it terminates at the value of 90° at the Falkner-Skan separation point where the backflow and regular solutions merge.

Figures C.12.a through C.13.b represent regular and backflow velocity profiles at the Falkner-Skan separation point, \( \beta_1 = -0.198838 \). The behavior of the crossflow and the required values of \( \eta_\infty \) as \( \beta_2 \) increases are consistent with the cases discussed so far. However, the wall skew angle for a regular monotonic profile decreases with increasing \( \beta_2 \). This trend in \( \theta_{wall} \) is opposite to the observations made for regular solutions influenced by magnitudes of the adverse streamwise pressure gradient below that of Falkner-Skan separation. Note also that the flow will not separate if \( \beta_2 \) is
large enough. Likewise, the trends in the wall skew angle for reversed flow
profiles as $\beta_2$ increases are opposite to those previously observed. Another
interesting observation is that for fixed values of the streamwise and cross-
stream pressure gradients, the maximum value of $g'$ for the backflow
solution is less than the $g'$ peak for the regular solution. This is again
opposite to that seen for values of $\beta_1 > -0.198838$. Thus, the Falkner-
Skan separation point seems to represent a turning point for three-
dimensional solutions as well as two-dimensional solutions.

Figures C.14.a through C.15.b depict velocity and skew angle profiles
for values of $\beta_1 < -0.198838$. A representative set of solutions in this range
is pictured in C.14.a and C.14.b for $\beta_1 = -0.21$ with $\beta_2$ decreasing
from $-0.27$ through $-0.50$. Profiles for flows influenced by adverse
streamwise pressure gradients beyond the Falkner-Skan separation point
cannot be obtained from $\beta_2 = 0$ to some finite value defined by $\beta_2,\text{min}$. The
numerical scheme diverges below this point. This value of $\beta_2$ represents the
minimum cross-stream pressure gradient required for a solution. As $|\beta_2|$
increases beyond this minimum value, the skew angle decreases until
separation is reached (i.e., $f''(0) = 0$); further increases in $|\beta_2|$ cause $\theta_{wall}$ to
again increase and streamwise flow reversal to occur. Moreover, in this
range the crossflow peaks increase as expected but drop below those attained
in the regular solution. The combination of these behaviors lends a non-
symmetric appearance to the graphs of the velocity and skew angle profiles
in this parameter range.

It is this author's contention that beyond the Falkner-Skan separation
point and up to a new separation point (discussed next), the solutions are
unique; either regular or reversed flow is exhibited. It is important to recall that in the discussion of the double boundary value method, regular solutions could only be obtained because this scheme would "latch on" to the monotonic behavior of $f'$, and backflow profiles could not be computed even though they existed. Reversal of flow had to be determined using a shooting method for $f'$ in conjunction with a boundary value technique for $g'$. However, beyond the Falkner-Skan separation point, this latter method is unstable and yields no solutions. The double boundary value technique calculates regular solutions starting from a minimum value of $|\beta_2|$, and proceeds through separation and flow reversal as $|\beta_2|$ increases. Because this scheme tends to latch onto the most stable solutions, and since previously backflow solutions were not able to be computed using this method, it is proposed that the solutions in the present parameter range are unique. This conclusion is derived from observations of numerical experiments and would have to be verified by a mathematical proof.

Figures C.15.a and C.15.b represent the velocity and skew angle profiles at the proposed new approximate point of separation: $\beta_1 = -0.219$, $\beta_2 \approx -0.39$. It should be noted that $\theta_{\text{wall}} \approx 90^\circ$ here. Table C.1 depicts this data in a tabular format.

Throughout this section, the reader may have noticed that no references were made to nor conclusions drawn based on flow behavior as $\beta_2 \to \infty$. For every value of $\beta_1$, there exists an upper limit on $\beta_2$ beyond which no solutions are obtainable. This is evidenced by a diverging numerical scheme near this critical value. There also exists a minimum $\beta_2$ below which no solutions can be obtained, as discussed earlier. Figure C.16.a portrays these
(\beta_1, \beta_2) \text{ pairs for solutions obtained by a double boundary value method and Figure C.16.b depicts these pairs for solutions obtained by the shooting/boundary value method.}

In Figure C.16.a, the squares represent minimum \beta_2 limits for each value of \beta_1; \beta_2 cannot be less than the displayed values for profiles to exist. The asterisks represent maximum \beta_2 values which must be achieved for a solution to exist at a certain value of \beta_1. Note that this curve begins at the Falkner-Skan separation point; for \beta_1 > -0.198838, the maximum value of \beta_2 is identically zero. As \beta_1 values decrease, the \beta_{2,\text{min}} values increase and the \beta_{2,\text{max}} values decrease until they meet at the new proposed separation point: \beta_1 = -0.219, \beta_2 = -0.39.

Figure C.16.b depicts \beta_1 vs. \beta_{2,\text{min}} values for backflow solution cases. Note that \beta_{2,\text{min}} decreases as \beta_1 decreases, and the curve ends at the Falkner-Skan separation point. Below this value of \beta_1, solution via the shooting/boundary value technique is impossible. From these two graphs it is easily seen that for the same value of \beta_1, the regular solution is able to withstand a much larger cross-stream pressure gradient than the backflow solution, as was previously discussed.

Table C.2 compares the two numerical schemes used to obtain regular and backflow solutions for \beta_1 = 0.2, \beta_2 = -0.1. The results are in excellent agreement. Tables C.3 and C.4 depict regular and backflow profiles in tabular format for verification by the interested reader. The regular solution is computed at \beta_1 = -0.10, \beta_2 = -0.60; the backflow solution is tabulated at \beta_1 = -0.10, \beta_2 = -0.05.

Appendix E, Table E.2 depicts a chart which summarizes the
conclusions made in this and in the preceding section.

5.3 Asymptotic Behavior for $|\beta_2| < 1$

The analytical structure of the profile functions $f'$ and $g'$ is desired. Candidate solutions for $f$ and $g$ are proposed for small $\beta_2$ for equations (4.1) and (4.2). The proposed solution is a regular perturbation expansion around the small parameter $\beta_2$, which takes the following form:

\begin{align}
  f &= f_o + \beta_2 f_1 + \beta_2^2 f_2 + \ldots , \\
  g &= \beta_2 \{ g_o + \beta_2 g_1 + \beta_2^2 g_2 + \ldots \} .
\end{align}

The method of solution for the terms in the expansion is now discussed. Equations (4.1) and (4.2) are substituted into equations (5.1) and (5.2), which results in a set of equations in terms of $f_o, f_1, f_2, \ldots , g_o, g_1, g_2 \ldots$. Like coefficients of powers of $\beta_2$ are grouped together, and each coefficient must be identically zero to satisfy the differential equations. This results in a set of differential equations which can be sequentially solved to yield $f_o, g_o, f_1, g_1, f_2, g_2 \ldots$. The boundary conditions for the leading order terms in these equations are:

\begin{align}
  f_o &= f_o' = g_o = g_o' = 0 , \quad \text{at } \eta = 0 , \\
  f_o' &\rightarrow 1 , \quad g_o' \rightarrow 0 , \quad \text{as } \eta \rightarrow \infty .
\end{align}

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The boundary conditions for \( f_1, g_1, f_2, g_2, \ldots \) and their derivatives are:

\[
\frac{d^n f_i}{d\eta^n} = \frac{d^n g_i}{d\eta^n} = 0 \quad \text{at} \quad \eta = 0; \quad n = 0, 1; \quad i = 1, 2, 3, \ldots
\]  
(5.4)

\[
\frac{df_i}{d\eta} = \frac{dg_i}{d\eta} \to 0 \quad \text{as} \quad \eta \to \infty, \quad \quad i = 1, 2, 3, \ldots
\]

For simplicity, only the terms \( f_o, g_o, f_1, g_1, f_2, g_2 \) are kept in the expansion. Upon substitution and simplification of the equations, it can be easily shown that the following differential equations govern the solution of \( f_o, g_o, f_1, g_1, f_2, g_2 \):

\[
f_o''' + f_o f_o'' + \beta_1 (1 - f_o^2) = 0, \quad (5.5)
\]

\[
g_o''' + f_o g_o'' - \frac{2}{3} (\beta_1 + 1) f_o g_o' - (1 - f_o^2) = 0, \quad (5.6)
\]

\[
f_1''' + f_o f_1'' + f_1 f_o'' - 2\beta_1 f_o f_1 = 0, \quad (5.7)
\]

\[
g_1''' + f_o g_1'' + g_1 f_o'' - \frac{2}{3} (\beta_1 + 1) (f_o g_1' + f_1 g_o') + 2f_o f_1' = 0, \quad (5.8)
\]

\[
f_2''' + f_1 f_2'' + f_2 f_1'' + g_o f_o'' + g_2 f_2'' + \left(\frac{2 - \beta_1}{3}\right) g_o^2
\]

\[- \beta_1 (f_o^2 + 2f_o f_2') = 0, \quad (5.9)
\]

\[
g_2''' + f_o g_2'' + f_1 g_1'' + f_2 g_o' + g_o g_2' + g_o^2
\]
There are a few interesting points to note regarding this set of equations. First, the solution is independent of $\beta_2$ and is dependent on $\beta_1$. Note also that equation (5.7) is identically satisfied by $f_1 \equiv 0$ when coupled with the relevant boundary conditions. Likewise, equation (5.8) is identically satisfied by $g_1 \equiv 0$. Thus, the solutions for $f$ and $g$ reduce to:

$$f = f_o + \beta_2^2 f_2,$$

$$g = \beta_2 (g_o + \beta_2^2 g_2).$$

Equations (5.11) and (5.12) will only converge for $\beta_2 < |1|$, which is the limitation for small $\beta_2$ solutions. Equation (5.58) is of the form of the Falkner-Skan equation, which makes sense; to a first approximation, the three-dimensional boundary layer will exhibit a velocity profile similar to a two-dimensional boundary layer. Another important note is that the equations are still nonlinear and must be solved numerically.

One may question the usefulness of the exercise just performed if solutions still must be evaluated numerically. It is always important to obtain the analytical structure of flow field solutions so that general flow behavior and properties can be understood. Also, it is expected that this kind of exercise will yield similar results to those obtained by numerically solving the governing equations directly, which would result in a validation of the chosen numerical method. Finally, this kind of approach decreases
the number of numerical simulations that must be performed for each value of \( \beta_1 \). If the expansion terms \( f_0, f_2, g_0, \) and \( g_2 \) are known at each value of \( \beta_1 \), then the solution of \( f \) and \( g \) and their derivatives over the range \( 0 \leq \beta_2 < 1 \) are also known. This knowledge can significantly decrease the computational time necessary to obtain velocity profiles over any region of interest.

The equations are solved numerically in sequence (i.e., \( f_0 \) is found first, then \( g_0, f_2, \) and so on) at each value of \( \beta_1 \) by employing the Thomas algorithm to solve for each expansion term; the previous stage's results are successively used for the solution of the next term. This type of method has been discussed in detail in the preceding sections. For the interested reader, the Thomas algorithm coefficients necessary for the solution of \( f_0, g_0, f_2, \) and \( g_2 \) are included below. Generally, an approximation to the differential equations via central differences was used. For the special cases at the wall (mesh point 1) and at \( \eta = \infty \) (mesh point \( n+1 \)), sloping differences were used to approximate \( f_0'', g_0'' \) where necessary. The following formulae contain only the central difference approximations.

For the solution of \( f_0 \), define \( y_o = f_0' \) and

\[
\begin{align*}
\quad b_j &= 1 + \frac{h}{2} f_{o,j} , \\
\quad c_j &= 1 - \frac{h}{2} f_{o,j} , \\
\quad a_j &= -2 - h^2 \beta_1 y_{o,j} , \\
\quad d_j &= - h^2 \beta_1 .
\end{align*}
\]

(5.13)
For the solution of $g_o$, define $z_o = g'_o$ and

\begin{align}
 b_j &= 1 + \frac{h}{2} f_{o,j}, \\
 c_j &= 1 - \frac{h}{2} f_{o,j}, \\
 a_j &= -2 - \frac{2}{3} h^2 (\beta_1 + 1) y_{o,j}, \\
 d_j &= h^2 (1 - y_{o,j})^2.
\end{align}

(5.14)

For the solution of $f_1$, it has been noted that $f_1 \equiv 0$; likewise, $g_1 \equiv 0$ also.

For the solution of $f_2$, define $y_2 = f_2$ and

\begin{align}
 b_j &= 1 + \frac{h}{2} f_{o,j}, \\
 c_j &= 1 - \frac{h}{2} f_{o,j}, \\
 a_j &= -2 - 2 h^2 \beta_1 y_{o,j}, \\
 d_j &= -\frac{h}{2} (f_{2,j} + g_{o,j})(y_{o,j+1} - y_{o,j-1}) - \frac{h^2}{3} (2 - \beta_1) z_{o,j}^2.
\end{align}

(5.15)

For the solution of $g_2$, define $z_2 = g'_2$ and

\begin{align}
 b_j &= 1 + \frac{h}{2} f_{o,j}, \\
 c_j &= 1 - \frac{h}{2} f_{o,j},
\end{align}

(5.16)
\[ a_j = -2 - \frac{2}{3} h^2 (\beta_1 + 1) y_o, j, \]

\[ d_j = -\frac{h}{2} (f_{2, j} + g_{o, j}) (z_{o, j+1} - z_{o, j-1}) - h^2 z_o, j^2 \]

\[ + \frac{2}{3} h^2 (\beta_1 + 1) y_{2, j} z_o, j - 2h^2 y_o, j y_{2, j}. \]

Tabular data is included in Appendix D, Table D.1, which compares the full numerical solution to the asymptotic solution for $\beta_1 = 0.2$, $\beta_2 = -0.05$. Note that the agreement is excellent. Also included in Appendix D is Table D.2 depicting the regular perturbation expansion terms $f'_o, g'_o, f'_2, g'_2$ that define the velocity profiles for $\beta_1 = 0.2, \beta_2$ small.
6. Conclusion

As evidenced in this study, the behavior of fluid flow in three-dimensional laminar similar boundary layers is complex and depends upon the streamwise pressure gradient $\beta_2$ and the cross-stream pressure gradient $\beta_1$. Two-dimensional flow is exhibited for a vanishing cross-stream pressure gradient, and the three-dimensional effect of a cross-stream velocity is introduced for a non-zero value of $\beta_2$. For favorable streamwise pressure gradients, the solutions to the governing equations represent unique boundary layer profiles, and as the value of $\beta_1 \to \infty$, the crossflow approaches the mainstream velocity near the wall, resulting in a large deflection of the resultant velocity from the external streamline. Flows influenced by adverse streamwise pressure gradients exhibit non-uniqueness; both a regular monotonic profile and one which experiences flow reversal near the wall appear to exist. For negative values of $\beta_1$, there seems to be a limit imposed upon the magnitude of the cross-stream pressure parameter $\beta_2$, and beyond this point no solutions were found to exist. Results also indicate that traditional Falkner-Skan separation can be delayed by the introduction of a cross-stream pressure gradient, and a new separation point has been proposed that depends on both $\beta_1$ and $\beta_2$.

There are some limitations associated with the use of three-dimensional similarity solutions in the physical world. Defining the external flow and curvature required for similarity is not as easily derived as in the two-dimensional case. Thus, determining the flow geometry from given values of $\beta_1$ and $\beta_2$ would be very complicated. Even though these limitations do exist, obtaining a general understanding of boundary layer flow behavior
under the influence of favorable and adverse streamwise pressure gradients and cross-stream pressure gradients is important and was achieved in this study. Also, a major motivation driving the investigation of three-dimensional boundary layer similarity is that these solutions are expected to be able to be used as initiators or terminators of more complicated three-dimensional flows that may not be able to be solved without an adequate profile to initiate a numerical integration.

Much work still needs to be done to more fully understand three-dimensional boundary layer similar flows, and investigation of fluid flow behavior under the influence of the various parameter ranges not considered in this study is encouraged.
5. References


10. Stewartson, K. "Further Solutions of the Falkner-Skan Equation."


Appendix A: The Falkner-Skan Equation

Tables
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<td>$0.999$</td>
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</table>

Table A.1: Falkner-Skan solutions with reversed flow (Stewartson, 1954).
Appendix A: The Falkner-Skan Equation

Figures
Figure A.1: Two-dimensional planar bluff body. (Puhak, 1992).

Figure A.2: Flow over a flat plate. (Puhak, 1992).

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Figure A.3: Flow against a wedge of half-angle $\beta \pi/2$. (Evans, 1968).

Figure A.4: Plane stagnation point flow (Hiemenz flow). (Evans, 1968).
Figure A.5: Flow around an expansion corner of turning angle $\beta \pi/2$. (White, 1974).

Figure A.6: Flow around the edge of a thin plate. (Evans, 1968).
Figure A.7: Flow around a right angle corner. (Puhak, 1992).

Figure A.8: Approximation to the flow at the tip of a missile. (Schlichting, 1960).
Figure A.9: Alternative backflow solution to the Blasius equation. (Smith, 1984).

Figure A.10: Typical regular velocity profiles for the Falkner-Skan equation. (White, 1974).

\[ u = 1 \]

\[ \eta = \gamma \sqrt{\frac{U(1+m)}{2\nu x}} \]
Figure A.11: Typical Falkner-Skan overshoot profiles. (Libby and Liu, 1967).

Figure A.12: Typical Falkner-Skan shear stress profiles. (White, 1974).
Appendix B: Favorable Streamwise Pressure Gradients

Tables
### Table B.1.a

Table B.1: Crossflow peaks and corresponding position and streamwise flow for $\beta_2 = -4.0$ and -0.05, $\beta_1 = 0.0$ to 1.0.

<table>
<thead>
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<th>$\beta_1$</th>
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<th>$g'$</th>
<th>$\beta_2$</th>
<th>n</th>
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<th>$g'$</th>
</tr>
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<tbody>
<tr>
<td>0</td>
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<td>0.901</td>
<td>0</td>
<td>1.1</td>
<td>0.482</td>
<td>0.228</td>
</tr>
<tr>
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<td>0.42</td>
<td>0.855</td>
<td>0.2</td>
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<td>0.8</td>
<td>0.604</td>
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<td>1</td>
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<td>0.591</td>
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<td>0.627</td>
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### Table B.1.b

Table B.1: Crossflow peaks and corresponding position and streamwise flow for $\beta_2 = -0.5$.

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<th>n</th>
<th>$f'$</th>
<th>$g'$</th>
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<td>-0.2</td>
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<td>0.482</td>
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### Table B.1.c

Table B.1.c: Crossflow peaks and corresponding position and streamwise flow for $\beta_1 = 0.0$, $\beta_2 = -0.05$ to -4.0.

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Table B.2: Typical velocity profile for a favorable streamwise pressure gradient and a moderate cross-stream pressure gradient: $\beta_1 = 0.2, \beta_2 = -0.5$.
Table B.3: Typical velocity profile for a favorable streamwise pressure gradient and a large cross-stream pressure gradient: $\beta_1 = 0.2$, $\beta_2 = -4.4$. 

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<th>$\eta^*$</th>
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</tr>
</thead>
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<td>0.0000D+00</td>
<td>0.0000D+00</td>
</tr>
<tr>
<td>0.1000D+02</td>
<td>0.6938D+02</td>
<td>0.7187D+02</td>
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<td>0.2000D+02</td>
<td>0.7853D+02</td>
<td>0.6178D+02</td>
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<td>0.3000D+02</td>
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<td>0.5584D+02</td>
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<tr>
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<td>0.8616D+02</td>
<td>0.5073D+02</td>
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<td>0.4099D+02</td>
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<td>0.9225D+02</td>
<td>0.3857D+02</td>
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<td>0.9313D+02</td>
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<td>0.0283D+02</td>
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Appendix B: Favorable Streamwise Pressure Gradients

Figures
Figure B.1: Typical velocity and skew angle profiles for a favorable streamwise pressure gradient and moderate cross-stream pressure gradient: $\beta_1 = 0.2$, $\beta_2 = -0.5$. 
Figure B.2: Definition of the skew angle.
Figure B.3: Typical velocity and skew angle profiles for a favorable streamwise pressure gradient and a large cross-stream pressure gradient: $\beta_1 = 0.2$, $\beta_2 = -4.4$. 
Figure B.4: Velocity and skew angle profiles for $\beta_1 = 0$, $-1.0 < \beta_2 < 0$. 
Figure B.5: Velocity and skew angle profiles for $\beta_1 = 0$, $-2.8 < \beta_2 < -1.2$. 
Figure B.6: Velocity and skew angle profiles for $\beta_1 = 0$, $-3.5 < \beta_2 < -3.0$
Figure B.7: Velocity and skew angle profiles for $\beta_1 = 1.0, -2.0 < \beta_2 < 0$. 

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Figure B.8: Velocity and skew angle profiles for $\beta_1 = 1.0$, $-4.5 < \beta_2 < -3.0$. 

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Figure B.9: Velocity and skew angle profiles for $\beta_1 = 1.0$, $-6.0 < \beta_2 < -5.0$. 

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Figure B.10: Velocity and skew angle profiles comparing flows with equal streamwise pressure gradients and with equal cross-stream pressure gradients having the same magnitude but opposite sign: $\beta_1 = 0.2$, $\beta_2 = \pm 1.0$. 
Figure B.11: Peak values of $g'$ and corresponding values of $\eta$ and $f'$ as $\beta_2$ increases for $\beta_1 = 0$. 
Appendix C: Adverse Streamwise Pressure Gradients

Tables
\[
\beta_1 = -0.219, \quad \beta_2 = -0.39
\]

<table>
<thead>
<tr>
<th>( \eta )</th>
<th>( f' )</th>
<th>( g' )</th>
<th>( f )</th>
<th>( g )</th>
<th>( \theta )</th>
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Table C.1: Velocity profile for the proposed new separation point: \( \beta_1 = -0.219, \beta_2 = -0.39 \).
\( \beta_1 = 0.2, \beta_2 = -0.1 \)

<table>
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<th>( g' ) (shoot/bv)</th>
<th>( f' ) (bv/bv)</th>
<th>( g' ) (bv/bv)</th>
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Table C.2: Comparison of the double boundary value method and the shooting/boundary value method: \( \beta_1 = 0.2, \beta_2 = -0.10 \).
Table C.3: Typical regular velocity profile for an adverse streamwise pressure gradient: \( \beta_1 = -0.10, \beta_2 = -0.60 \).
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Table C.4: Typical backflow velocity profile for an adverse streamwise pressure gradient: $\beta_1 = -0.10$, $\beta_2 = -0.05$. 

Appendix C: Adverse Streamwise Pressure Gradients

Figures
Figure C.1: Typical regular velocity and skew angle profiles for an adverse streamwise pressure gradient and a moderate cross-stream pressure gradient: $\beta_1 = -0.10$, $\beta_2 = -0.60$. 
Figure C.2: Typical backflow velocity and skew angle profiles for an adverse streamwise pressure gradient and a moderate cross-stream pressure gradient: $\beta_1 = -0.10$, $\beta_2 = -0.05$. 
Figure C.3.a: Backflow velocity profile for $\beta_1 = -0.001$, $\beta_2 = 0.0$.

Figure C.3.b: Velocity profile as $\beta_1 \to 0^\circ$. (Laine and Reinhart, 1984).
Figure C.4: Comparison of the regular and backflow velocity and skew angle profiles for representative values of the adverse streamwise and cross-stream pressure gradients:

\[ \beta_1 = -0.15, \beta_2 = -0.05. \]
Figure C.5: Regular velocity and skew angle profiles for $\beta_1 = -0.05$, $-0.5 < \beta_2 < 0$. 
Figure C.6: Regular velocity and skew angle profiles for $\beta_1 = -0.05$, $-1.06 < \beta_2 < -0.70$. 

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Figure C.7: Backflow velocity and skew angle profiles for $\beta_1 = -0.05$, $-0.017 < \beta_2 < 0$. 
Figure C.8: Regular velocity and skew angle profiles for $\beta_1 = -0.18$, $-0.4 < \beta_2 < 0$. 
Figure C.9: Regular velocity and skew angle profiles for $\beta_1 = -0.18$, $-0.659 < \beta_2 < -0.50$. 
Figure C.10: Backflow velocity and skew angle profiles for $\beta_1 = -0.18$, $-0.05 < \beta_2 < 0$. 

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Figure C.11: Backflow velocity and skew angle profiles for $\beta_1 = -0.18$, $-0.16 < \beta_2 < -0.075$. 
Figure C.12: Regular velocity and skew angle profiles for $\beta_1 = -0.198838$, $-0.3 < \beta_2 < -0.15$. 

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Figure C.13: Backflow velocity and skew angle profiles for $\beta_1 = -0.198838$, $-0.22 < \beta_2 < 0$. 

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Figure C.14: Velocity and skew angle profiles for $\beta_1 = 0.21, -0.5 < \beta_2 < -0.27$. 

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Figure C.15: Velocity and skew angle profiles for $\beta_1 = -0.219$, $-0.39 < \beta_2 < -0.37$. 
Figure C.16.a: $\beta_2$ limits for regular solutions.

Figure C.16.b: $\beta_2$ limits for backflow solutions.
Appendix D: Asymptotic Solution for $|\beta_2| < 1$

Tables
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Table D.1: Comparison of the full numerical solution and the asymptotic solution for small \( \beta_2 \):
\[ \beta_1 = 0.2, \beta_2 = -0.05. \]
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Table D.2: Numerical solution of the velocity terms in the regular perturbation expansion for small β_2: β_1 = 0.2.
Appendix E: Summary Charts

Tables
Table E.1: Summary of the governing equations for cases 1, 2, and 3 ($2\alpha_1 - \beta_1 = 0$). Note that cases 2 and 3 are identical if $\beta_1 = 1.0$. (Degani, 1991).
<table>
<thead>
<tr>
<th>FAVORABLE</th>
<th>0 &lt; ( B_1 \leq 1.0 )</th>
<th>FLOW TYPE</th>
<th>ADVERSE TO SEPARATION -0.198838 &lt; ( B_1 \leq 0 )</th>
<th>BEYOND SEPARATION -0.219 &lt; ( B_1 \leq -0.198838 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B_1 ) fixed (</td>
<td>B_2</td>
<td>) increasing ( 0 \rightarrow \infty ) | ( B_2 ) fixed ( B_1 ) increasing ( 0 \rightarrow 1.0 )</td>
<td>Reg. increasing ( \eta_\infty )</td>
<td>increasing ( g' ) peak ( \eta = \infty )</td>
</tr>
<tr>
<td>( \eta_\infty ) increasing limit: ( B_2 = \infty )</td>
<td>( \eta = \infty ) decreasing</td>
<td>( \eta_\infty ) increasing ( \eta = \infty ) decreasing</td>
<td>( f' ) takes longer to reach ( f' = 1 ) less to reach ( f' = 1 ) longer to reach ( f' = 1 )</td>
<td>( \theta ) at wall ( \theta = 90^\circ )</td>
</tr>
<tr>
<td>( g' ) peak increasing limit: ( B_2 = \infty ) ( g'(0) ) instantaneously jumps from ( 0 ) to ( 1 )</td>
<td>( \eta_\infty ) increasing ( \eta = \infty ) decreasing</td>
<td>( f' ) takes longer to reach ( f' = 1 ) less to reach ( f' = 1 ) longer to reach ( f' = 1 )</td>
<td>( \theta ) at wall ( \theta = 90^\circ )</td>
<td></td>
</tr>
<tr>
<td>( f' ) ( \theta ) at wall increasing limit: ( B_2 = \infty ) ( \theta ) = ( 90^\circ ) | ( f' ) ( \theta ) at wall increasing limit: ( B_2 = \infty ) ( \theta ) = ( 90^\circ )</td>
<td>( \eta_\infty ) increasing ( \eta = \infty ) decreasing</td>
<td>( f' ) ( \theta ) at wall increasing limit: ( B_2 = \infty ) ( \theta ) = ( 90^\circ ) decreasing</td>
<td>( f' ) ( \theta ) at wall increasing limit: ( B_2 = \infty ) ( \theta ) = ( 90^\circ ) decreasing</td>
<td></td>
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</tbody>
</table>

**Remarks:**
- **General:**
  - Thicker boundary layer
  - Thinner boundary layer
- **Remarks:**
  - Regular and Reversed flow solutions merge at this point.
7. Vita

Lisa Anne Iannone Thomas was born on September 25, 1968, in Schenectady, New York. Her parents are Louis Iannone and the late Diane Iannone. She grew up in a small town north of Albany, New York, where she attended Shenendehowa High School and graduated in June, 1986, as Salutatorian. After high school, she attended Hudson Valley Community College, Troy, New York, and received her A.S. in Mathematics and Science. She continued her undergraduate studies at Rensselaer Polytechnic Institute, Troy, New York, where she received her B.S. in Mechanical Engineering, graduating Summa Cum Laude. She has been pursuing her M.S. in Mechanical Engineering at Lehigh University since September, 1991.
