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A Weitzenbock Formula for Compact Complex Manifolds and Applications to the Hopf Conjecture in Real Dimension 6

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Contents

Abstract

The Hopf Conjecture is a well-known problem in differential geometry which relates the geometry of a manifold to its topology [Hopf 1]. In this thesis, we investigate this problem on compact complex real 6-dimensional manifolds. First, we prove a Weitzenbock formula on a complex manifold involving the Hodge Laplacian Δ_H, the Bochner Laplacian of the Levi-Civita connection Δ_R, and another Laplacian we construct that is related to the Lefschetz operator and ∂ operator on a compact complex manifold Δ_K such that for any $(p,q)-$form in [2.11], $\Delta_K + \Delta_H - 2\Delta_R = F(R)+$"quadratic terms" where the curvature operator $F(R) : E^{p,q} \to E^{p,q}$ and the quadratic terms are given in [2.12]. This formula generalizes a Weitzenbock formula of Wu for Kähler manifolds in [Wu]. Then, under certain conditions, we show that the Weitzenbock formula provides vanishing theorems for the Dolbeault cohomology groups of complex differential $(p,q)-$forms and obtain information about the Hodge numbers of the manifold. We use these vanishing theorems to obtain information about the geometric and arithmetic genus and irregularity of a compact complex manifold under certain conditions. Earlier result of Alfred Gray shows that a hypothetical integrable almost complex structure on a 6-dimensional sphere, S^6, has to satisfy $h^{0,1} > 0$ [Gray]. We apply our vanishing theorem for $(0,1)-$forms to show that $h^{0,1} = 0$ and thus, under certain additional conditions a 6-dimensional sphere can not have integrable almost complex structure. We use the Frölicher spectral sequence to obtain the Hodge-deRham cohomology groups of any compact complex manifold of real dimension 6 and show that under certain conditions, the Euler characteristic of a compact complex manifold of real dimension 6 is positive to prove the Hopf conjecture.

Introduction

In 1926, H.Hopf proved that a compact, simply connected Riemannian manifold with constant sectional curvature 1 is necessarily isometric to the round sphere S^n, [Hopf 1], [Hopf 2]. The motivation for this thesis comes from a generalization of Hopf's results which claims a relationship between the curvature and the topology of even dimensional compact Riemannian manifolds:

Conjecture 0.0.1 (Hopf) *Let M^{2n} be a compact orientable $2n-$dimensional Riemannian manifold.*

1^0 *if all sectional curvatures of M are nonnegative (resp. strictly positive), then $\chi(M)$ is also nonnegative (resp. strictly positive).*

2^0 *if all sectional curvatures of M are nonpositive (resp. strictly negative), then $(-1)^n \chi(M)$ is nonnegative (resp. strictly positive).*

J. Milnor proved the conjecture for all 4-dimensional manifolds. His result was mentioned by S. S. Chern in [Chern]. D. L. Johnson, proved that the conjecture holds for all 6-dimensional Kähler manifolds [Johnson].

In this thesis, we show that $\chi(M)$ is nonnegative for a compact complex manifold with real dimension 6 under certain additional assumptions by using the results of Hodge theory and using our version of the Weitzenbock formula, which then provides vanishing theorems on the Dolbeault cohomology groups under those additional assumptions. We use the vanishing of Dolbeault cohomology groups in the Frölicher spectral sequence to obtain the deRham cohomology groups of the manifold.

Another open question is whether the 6-dimensional sphere, S^6, admits an integrable almost complex structure. The almost complex structure induced on it by the unit imaginary Cayley numbers is not integrable since the Nijenhuis tensor does not vanish [Nirenberg & Newlander]. However, A. Gray has a result that any hypothetical complex structure on S^6 has the property that $h^{0,1}\left(S^6\right) \geq 1$ [Gray]. The Weitzenbock formula we have found for $(0,1)-$forms on a 6-dimensional complex manifold provides a vanishing under certain conditions, thus giving a negative result under under these conditions for the question about the existence of a hypothetical integrable almost complex structure on S^6.

In Chapter 1, given a unitary local frame, we complexify it to a frame with $(1,0)$ and $(0,1)$ frame fields. For any $(p,q)-$form $\alpha \in E^{p,q}$, we define a new operator $L_{\overline{\partial}\Omega} : E^{p,q} \to E^{p+1,q+2}$ wedging any $(p,q)-$form by $\overline{\partial}\Omega$, the (1,2)-part of the derivative of the fundamental form Ω and similarly the conjugate of this operator $L_{\partial\Omega} : E^{p,q} \to E^{p+2,q+1}$ which is wedging any $(p,q)-$form by $\partial\Omega$, the (2,1)-part of the derivative of the fundamental form Ω, $\left(L_{\overline{\partial}\Omega}\right)^* : E^{p,q} \to E^{p-1,q-2}$ and $\left(L_{\partial\Omega}\right)^* : E^{p,q} \to E^{p-2,q-1}$ as their L^2- adjoints, respectively. Then we use the generalized Kähler identities proved by Demailly [Demailly]

$$\begin{aligned}\overline{\partial}^{*} &= -\sqrt{-1}\,[\Lambda,\partial]+\left[L,\left(L_{\overline{\partial}\Omega}\right)^{*}\right]\\ \partial^{*} &= \sqrt{-1}\,\left[\Lambda,\overline{\partial}\right]+\left[L,\left(L_{\partial\Omega}\right)^{*}\right]\end{aligned}$$

to obtain formulas for the adjoint derivative operators $\partial^{*}: E^{p,q}\to E^{p-1,q}$ and $\overline{\partial}^{*}: E^{p,q}\to E^{p,q-1}$. We also introduce a new derivative operator $\overline{\partial}_K := [\partial^{*}, L] : E^{p,q}\to E^{p,q+1}$ and its L^2-adjoint $\overline{\partial}_K^{*} := [\Lambda,\partial] : E^{p,q}\to E^{p,q-1}$, which play a crucial role in our Weitzenbock formula. We define tensors T_1, T_2 and compute the commutator of these two tensor with their L^2-adjoints, $\left[T_1,(T_1)^{*}\right]$ and $\left[T_2,(T_2)^{*}\right]$ respectively. We introduce a rough Laplacian $\Delta_R : E^{p,q}\to E^{p,q}$, show that its real part is positive semi-definite for any $(p,q)-$form $\alpha\in E^{p,q}$. We compute the Hodge Laplacian $\Delta_H = \overline{\partial}\circ\overline{\partial}^{*}+\overline{\partial}^{*}\circ\overline{\partial} : E^{p,q}\to E^{p,q}$ for any $(p,q)-$form. Finally, we introduce a new Laplacian $\Delta_K := \overline{\partial}_K\circ\overline{\partial}_K^{*}+\overline{\partial}_K^{*}\circ\overline{\partial}_K : E^{p,q}\to E^{p,q}$, show that it is positive semi-definite for all $(p,q)-$forms.

In Chapter 2, we prove a Weitzenbock formula for any $(p,q)-$form in [2.11], that involves the Laplacians $\Delta_K, \Delta_H, \Delta_R$ on M defined in [2.1] and the algebraic tensors $\left[L_{\overline{\partial}\Omega},\Lambda\right]$, T_1, T_2 defined in [1.7], [2.6], [2.8], respectively. Then we have the following Weitzenbock formula

$$\begin{aligned}&\Delta_K(\alpha)+\Delta_H(\alpha)-2\Delta_R(\alpha)\\ =\;& F(R)(\alpha)+\frac{1}{2}\left[\left[L,\left(L_{\overline{\partial}\Omega}\right)^{*}\right],\left[L_{\overline{\partial}\Omega},\Lambda\right]\right](\alpha)+\frac{1}{2}\left[T_1,(T_1)^{*}\right](\alpha)-\frac{1}{2}\left[T_2,(T_2)^{*}\right](\alpha)\end{aligned}$$

where $F(R) : E^{p,q}\to E^{p,q}$ is a curvature operator given in [2.12] generalizes a Weitzenbock formula of Wu for Kähler manifolds in [Wu]. Then under certain conditions on a compact complex manifold, we prove vanishing theorems by using these Weitzenbock formulas. We use these vanishing theorems to obtain information about the geometric and arithmetic genus and irregularity of a compact complex manifold under certain conditions. Earlier result of Alfred Gray shows that a hypothetical integrable almost complex structure on a 6-dimensional sphere, S^6, has to satisfy $h^{0,1}>0$ [Gray]. We apply our vanishing theorem for $(0,1)-$forms to show that $h^{0,1}=0$ and thus, under certain additional conditions we prove the nonexistence of a hypothetical integrable almost complex structure on the 6-dimensional sphere by using A. Gray's result [Gray].

In Chapter 3, we use the Frölicher spectral sequence to obtain the Hodge-deRham cohomology groups of any compact complex manifold of real dimension 6 and show that $\chi(M) = \sum_{0\le p,q\le 3}(-1)^{p+q}\,h^{p,q}$ where $h^{p,q}$ is the $(p,q)-$Hodge number. Finally we show that the Hopf conjecture is true for a compact complex manifold with real dimension 6 satisfying certain conditions.

Chapter 1

Preliminaries, Frame setup, Computations of the Operators

1.1 Hodge Theory

Definition 1.1.1 *The Euler characteristic $\chi(M)$ of a smooth manifold M of dimension n can be defined as the alternating sum of the ranks of its singular cohomology groups over the real numbers, $b^k = \dim H^k(M, \mathbb{R})$, also known as the Betti numbers, i.e.*

$$\chi(M) = \sum_{k=0}^{n} b^k = b^0 - b^1 + b^2 - b^3 + \cdots + (-1)^n b^n$$

Let $\mathcal{H}^k(M, \mathbb{C}) = \left\{\alpha \in E^k(M) \mid \Delta\alpha = 0\right\}$ be the space of complex-valued harmonic $k-$forms on M equipped with a Hermitian metric. Then we have the important result of W. V. Hodge [Hodge]:

Theorem 1.1.1 *If M is a compact complex manifold equipped with a hermitian metric, then*
(1) $\dim \mathcal{H}^k(M, \mathbb{C}) < \infty$,
(2) $E^k(M) = \mathcal{H}^k(M, \mathbb{C}) \oplus d^* E^{k+1}(M) \oplus dE^{k-1}(M)$,
(3) $\mathcal{H}^k(M, \mathbb{C}) \cong H^k_{dR}(M, \mathbb{C})$.

The space of complexified $k-$forms on M decomposes as the direct sum of space of the $(p,q)-$forms on M,

$$E^k \otimes \mathbb{C} = \oplus_{p+q=k} E^{p,q} = E^{k,0} \oplus E^{k-1,1} \oplus \cdots \oplus E^{1,k-1} \oplus E^{0,k}$$

Let $\mathcal{H}^{p,q}(M, \mathbb{C}) = \left\{\alpha \in E^{p,q}(M) \mid \Delta_{\overline{\partial}}\alpha = 0\right\}$ be the space of complex-valued harmonic $k-$forms on M. Set $H^{p,q}(M, \mathbb{C}) = \frac{\ker\left\{\overline{\partial} \,:\, E^{p,q}(M) \to E^{p,q+1}(M)\right\}}{\overline{\partial}(E^{p,q-1}(M))}$ to be the Dolbeault cohomology group of

$(p,q)-$forms, and let $h^{p,q} = \dim H^{p,q}(M,\mathbb{C})$ denote its rank, which is also known as the $(p,q)-$Hodge number. Hodge Theory also provides a decomposition for $E^{p,q}$.

Theorem 1.1.2 *[Hodge] If M is a compact complex manifold equipped with a hermitian metric, then*
(1) $\dim \mathcal{H}^{p,q}(M,\mathbb{C}) < \infty$,
(2) $E^{p,q}(M) = \mathcal{H}^{p,q}(M,\mathbb{C}) \oplus \overline{\partial}^* E^{p+1,q}(M) \oplus \overline{\partial} E^{p-1,q}(M)$,
(3) $\mathcal{H}^{p,q}(M,\mathbb{C}) \cong H^{p,q}(M,\mathbb{C})$.

For a compact Kähler manifold M, let $\mathcal{H}^k = \oplus_{p+q=k}\mathcal{H}^{p,q}$. Furthermore, we have the following relations between the Betti and Hodge numbers for a Kähler manifold:

$$\begin{aligned} b^k &= \sum_{p+q=k} h^{p,q} \\ h^{p,q} &= h^{q,p} \text{ (Serre Duality)} \\ h^{p,p} &\geq 1 \text{ for all } 0 \leq p \leq \dim_{\mathbb{C}} M \text{ (Kähler form)} \end{aligned}$$

We do not necessarily have these relations for compact complex manifolds since $d-$harmonicity is a stronger condition than $\overline{\partial}-$harmonicity.

Theorem 1.1.3 *[Frölicher] Given a complex manifold M of complex dimension n, there is a spectral sequence converging strongly to $H^k_{dR}(M,\mathbb{C})$ with $(E_0^{p,q}, d_0)$ term equal to $\left(E^{p,q}, \overline{\partial}\right)$, and $(E_1^{p,q}, d_1)$ term equal to $\left(H^{p,q}_{\overline{\partial}}, [\partial]\right)$. Furthermore, $H^k_{dR}(M,\mathbb{C}) = \oplus_{p+q=k} E^{p,q}_{n+2}$.*

The deRham cohomology can then be written as the direct sum of the kernels and the cokernels of the differentials of the Frölicher spectral sequence.

Let M be a compact complex manifold of real dimension 6. The Euler characteristic of M is given by

$$\chi(M) = \sum_{k=0}^{n} b^k = b^0 - b^1 + b^2 - b^3 + b^4 - b^5 + b^6$$

If we assume the manifold is connected, then $b^0 = 1$ and again by the Poincaré duality $b^6 = 1$. We would like to provide vanishing theorems for some of the Dolbeault cohomology groups and by using them and the Frölicher spectral sequence, obtain the Betti numbers to compute $\chi(M)$.

1.2 Compact Complex Manifolds

Let (M, g, J) be a compact Riemannian manifold with a Riemannian metric g and an integrable almost complex structure $J \in End(T_*M)$ with $J^2 = -Id_{T_*M}$, such that the Nijenhuis tensor vanishes

$$N_J(X,Y) = [JX, JY] - J[X, JY] - J[JX, Y] - [X,Y] \equiv 0$$

for any $X, Y \in \Gamma(M, T_*M)$. Assume in addition that the metric is compatible with the integrable almost complex structure J, i.e.

$$g(JX, JY) = g(X, Y),$$

for any $X, Y \in \Gamma(M, T_*M)$.

The complexified tangent bundle $T_*M \otimes_{\mathbb{R}} \mathbb{C}$ has a Hermitian metric induced by the Riemannian metric g and the compatible integrable almost complex structure J given by

$$h(X, Y) = g(X, Y) - \sqrt{-1}g(JX, Y)$$

for any $X, Y \in \Gamma(M, T_*M \otimes_{\mathbb{R}} \mathbb{C})$. The imaginary part of the Hermitian metric is called the Kähler form $\Omega(X, Y) = g(JX, Y)$.

The Kähler form $\Omega \in E^2$ is skew-symmetric since

$$\Omega(X, Y) = \langle JX, Y\rangle = \langle Y, JX\rangle = -\langle JJY, JX\rangle = -\langle JY, X\rangle = -\Omega(Y, X)$$

and it is $J-$invariant

$$\Omega(JX, JY) = \langle JJX, JY\rangle = \langle JX, Y\rangle = \Omega(X, Y)$$

Therefore, h is also $J-$invariant.

In particular, if $d\Omega \equiv 0$ then (M, g, J, Ω) is called a Kähler manifold.

1.2.1 Frame Setup, Operators, Laplacians

Let (M, J) be an almost complex manifold. (T_*M, J) can be complexified by defining scalar multiplication by complex numbers as follows:

$$\left(a + \sqrt{-1}b\right) X := aX + bJX, \text{ for } X \in \Gamma(M, TM), \ a, b \in \mathbb{R}$$

For $T_*M \otimes \mathbb{C}$, we can then find a local unitary frame fields $\{v_1, \cdots, v_n, Jv_1, \cdots, Jv_n\}$ for the real $2n-$dimensional vector space T_*M.

$$\begin{aligned} g(v_i, v_j) &= \delta^i_j, \text{ for all } 1 \le i, j \le n \\ g(Jv_i, Jv_j) &= g(v_i, v_j) = \delta^i_j, \text{ for all } 1 \le i, j \le n \end{aligned}$$

and also

$$g(v_i, Jv_j) = 0$$

1.2.2 $(1, 0)$ and $(0, 1)$ Frame Fields

If we consider the complexification of T_*M, i.e.

$$T_*M \otimes_{\mathbb{R}} \mathbb{C} = \left\{X + \sqrt{-1}Y \mid X, Y \in T_*M\right\},$$

realizing the fact that the eigenvalues of J are $\pm\sqrt{-1}$ by $J^2 = -I$, we get a direct sum decomposition for the complexification of T_*M

$$T_*M \otimes_{\mathbb{R}} \mathbb{C} = T_{1,0}M \oplus T_{0,1}M$$

by setting

$$\begin{aligned} T_{1,0}M &= span\left\{V_i \in T_*M^{\mathbb{C}} \mid JV_i = \sqrt{-1}\ V_i\right\}, \\ T_{0,1}M &= span\left\{\overline{V}_i \in T_*M^{\mathbb{C}} \mid J\overline{V}_i = -\sqrt{-1}\ \overline{V}_i\right\}. \end{aligned}$$

By making use of the above frame, we will get more precisely

$$\begin{aligned} T_{1,0}M &= span\left\{V_i = \frac{1}{\sqrt{2}}\left(v_i - \sqrt{-1}Jv_i\right) \mid i = 1, \cdots, n\right\}, \\ T_{0,1}M &= span\left\{\overline{V}_i = \frac{1}{\sqrt{2}}\left(v_i + \sqrt{-1}Jv_i\right) \mid i = 1, \cdots, n\right\}. \end{aligned}$$

1.2.3 Dual $(1,0)$ and $(0,1)$ Coframe Fields

Now let T^*M be the cotangent bundle of M, the dual bundle of T_*M, and let its complexification $T^*M \otimes_{\mathbb{R}} \mathbb{C}$ be the dual of the complexification of T_*M. Then with respect to the eigenvalues $\pm\sqrt{-1}$ of J, similarly we have a direct sum decomposition for $T^*M \otimes_{\mathbb{R}} \mathbb{C}$,

$$T^*M \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$$

by setting $i = 1, \cdots, n$

$$\begin{aligned} E^{1,0} &:= \Gamma\left(M, T^{1,0}M\right) \\ &= span\left\{\omega^j \in T^*M^{\mathbb{C}} \mid \omega^j\left(V_i\right) = \delta^i_j,\ \forall V_i \in T_{1,0}M \text{ and } \omega^j\left(\overline{V}_i\right) = 0,\ \forall \overline{V}_i \in T_{0,1}M\right\}, \\ E^{0,1} &:= \Gamma\left(M, T^{0,1}M\right) \\ &= span\left\{\overline{\omega}^j \in T^*M^{\mathbb{C}} \mid \overline{\omega}^j\left(\overline{V}_i\right) = \delta^i_j,\ \forall \overline{V}_i \in T_{0,1}M \text{ and } \overline{\omega}^j\left(V_i\right) = 0,\ \forall V_i \in T_{1,0}M\right\}. \end{aligned}$$

By using this, we can also define the space of complex (p,q) −forms $E^{p,q}$ on M, for any $1 \leq p, q \leq \dim_{\mathbb{C}} M$,

$$E^{p,q} := \underbrace{E^{1,0} \wedge \ldots \wedge E^{1,0}}_{p \text{ times}} \wedge \underbrace{E^{0,1} \wedge \ldots \wedge E^{0,1}}_{q \text{ times}}.$$

1.2.4 Complexified Metric and the $(1,0)$ and $(0,1)$ Frame Fields

From now on, we will denote the complexified $\mathbb{C}$−bilinear Riemannian metric on $T_*M \otimes_{\mathbb{R}} \mathbb{C}$ by $\langle \cdot, \cdot \rangle$. For the frame $\left\{V_i, \overline{V}_i\right\}$, we have

$$\begin{aligned} \langle V_i, V_j \rangle &= \langle JV_i, JV_j \rangle = \left\langle \sqrt{-1}V_i, \sqrt{-1}V_j \right\rangle \\ &= \sqrt{-1}\left(\sqrt{-1}\right)\langle V_i, V_j \rangle = -\langle V_i, V_j \rangle \\ &\Rightarrow \langle V_i, V_j \rangle = 0 \end{aligned}$$

and similarly $\langle \overline{V}_i, \overline{V}_j \rangle = 0$. Also

$$\begin{aligned}
\langle V_i, \overline{V}_j \rangle &= \left\langle \frac{1}{\sqrt{2}} \left(v_i - \sqrt{-1} J v_i \right), \frac{1}{\sqrt{2}} \left(v_j + \sqrt{-1} J v_j \right) \right\rangle = \delta^i_j \\
\langle \overline{V}_i, V_j \rangle &= \left\langle \frac{1}{\sqrt{2}} \left(v_i + \sqrt{-1} J v_i \right), \frac{1}{\sqrt{2}} \left(v_j - \sqrt{-1} J v_j \right) \right\rangle = \delta^i_j
\end{aligned}$$

Thus we have

$$\langle V_i, V_j \rangle = 0 = \langle \overline{V}_i, \overline{V}_j \rangle, \qquad \langle V_i, \overline{V}_j \rangle = \delta^i_j = \langle \overline{V}_i, V_j \rangle.$$

1.2.5 Induced Inner Product on $(p,q)-$forms

The induced inner product on $(p,q)-$forms has the following property:

If $I = (i_1, \cdots, i_p, j_1, \cdots, j_q)$ is a multi-index, then denote the $(p,q)-$form by ω^I and the $(q,p)-$form by $\overline{\omega}^I$ which are indexed by I as follows:

$$\omega^I = \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q},$$

$$\overline{\omega}^I = \overline{\omega}^{i_1} \wedge \cdots \wedge \overline{\omega}^{i_p} \wedge \omega^{j_1} \wedge \cdots \wedge \omega^{j_q}.$$

If I, J are two such multi-indices, we have

$$\begin{aligned}
\langle \omega^I, \omega^J \rangle &= \langle \overline{\omega}^I, \overline{\omega}^J \rangle = 0, \\
\langle \omega^I, \overline{\omega}^J \rangle &= \begin{cases} 0 & \text{if } I \neq J \\ 1 & \text{if } I = J \end{cases}.
\end{aligned} \tag{1.1}$$

1.2.6 L^2-product on $(p,q)-$forms

By using the induced inner product on $E^{p,q}(M)$, one can also define the L^2 metric on $E^{p,q}(M)$ given by

$$(\alpha, \beta)_{L^2} := \int_M \langle \alpha, \overline{\beta} \rangle \, dvol_M \tag{1.2}$$

for any $\alpha, \beta \in E^{p,q}(M)$ and the induced inner product $\langle \alpha, \overline{\beta} \rangle$ at a point $x \in M$ is induced by the evaluation of the forms at the point x.

1.2.7 The Kähler form and the $(1,0)$ and $(0,1)$ Frame Fields

For the frame $\{V_i, \overline{V}_i\}$, the Kähler form $\Omega \in E^2$ becomes

$$\begin{aligned}
\Omega(V_i, V_j) &= \langle J V_i, V_j \rangle = \sqrt{-1} \langle V_i, V_j \rangle = 0 \\
\Omega(\overline{V}_i, \overline{V}_j) &= \langle J \overline{V}_i, \overline{V}_j \rangle = -\sqrt{-1} \langle \overline{V}_i, \overline{V}_j \rangle = 0
\end{aligned}$$

This means that Ω is a $(1,1)-$form. We can express it by

$$\Omega = \sqrt{-1} \sum_{1 \le i \le n} \omega^i \wedge \overline{\omega}^i$$

and we also have

$$\overline{\Omega} = \overline{\sqrt{-1} \sum_{1 \le i \le n} \omega^i \wedge \overline{\omega}^i} = -\sqrt{-1} \sum_{1 \le i \le n} \overline{\omega}^i \wedge \omega^i = \sqrt{-1} \sum_{1 \le i \le n} \omega^i \wedge \overline{\omega}^i = \Omega.$$

1.2.8 Nijenhuis Tensor and the $(1,0)$ and $(0,1)$ Frame Fields

The Nijenhuis tensor vanishes for $(1,0)$ frame fields, so we have

$$\begin{aligned} 0 &= \langle N_J(V_i, V_j), V_k \rangle \\ &= \langle [JV_i, JV_j] - J[V_i, JV_j] - J[JV_i, V_j] - [V_i, V_j], V_k \rangle \\ &= -2\langle [V_i, V_j], V_k \rangle - 2\sqrt{-1}\langle J[V_i, V_j], V_k \rangle \\ &= -4\left\langle \nabla_{V_i} V_j - \nabla_{V_j} V_i, V_k \right\rangle \end{aligned}$$

and so

$$\Rightarrow \langle \nabla_{V_i} V_j, V_k \rangle = \left\langle \nabla_{V_j} V_i, V_k \right\rangle.$$

Then, using the compatibility of the metric with the connection, we get

$$\begin{aligned} \langle \nabla_{V_i} V_j, V_k \rangle &= \left\langle \nabla_{V_j} V_i, V_k \right\rangle = -\left\langle V_i, \nabla_{V_j} V_k \right\rangle \\ &= -\langle V_i, \nabla_{V_k} V_j \rangle = \langle \nabla_{V_k} V_i, V_j \rangle = \langle \nabla_{V_i} V_k, V_j \rangle \end{aligned}$$

so that

$$\Rightarrow \langle \nabla_{V_i} V_j, V_k \rangle = \langle \nabla_{V_i} V_k, V_j \rangle = -\langle V_k, \nabla_{V_i} V_j \rangle = 0.$$

Similarly, the Nijenhuis tensor vanishes for $(0,1)$ frame fields, so

$$\begin{aligned} 0 &= \left\langle N_J\left(\overline{V}_i, \overline{V}_j\right), \overline{V}_k \right\rangle \\ &= \left\langle \left[J\overline{V}_i, J\overline{V}_j\right] - J\left[\overline{V}_i, J\overline{V}_j\right] - J\left[J\overline{V}_i, \overline{V}_j\right] - \left[\overline{V}_i, \overline{V}_j\right], \overline{V}_k \right\rangle \\ &= -2\left\langle \left[\overline{V}_i, \overline{V}_j\right], \overline{V}_k \right\rangle + 2\sqrt{-1}\left\langle J\left[\overline{V}_i, \overline{V}_j\right], \overline{V}_k \right\rangle \\ &= -2\left\langle \nabla_{\overline{V}_i} \overline{V}_j - \nabla_{\overline{V}_j} \overline{V}_i, \overline{V}_k \right\rangle - 2\sqrt{-1}\left\langle \nabla_{\overline{V}_i} \overline{V}_j - \nabla_{\overline{V}_j} \overline{V}_i, J\overline{V}_k \right\rangle \\ &= -4\left\langle \nabla_{\overline{V}_i} \overline{V}_j - \nabla_{\overline{V}_j} \overline{V}_i, \overline{V}_k \right\rangle \end{aligned}$$

thus

$$\Rightarrow \left\langle \nabla_{\overline{V}_i} \overline{V}_j, \overline{V}_k \right\rangle = \left\langle \nabla_{\overline{V}_j} \overline{V}_i, \overline{V}_k \right\rangle,$$

then using the compatibility of the metric with the connection, we get

$$\begin{aligned} \left\langle \nabla_{\overline{V}_i} \overline{V}_j, \overline{V}_k \right\rangle &= \left\langle \nabla_{\overline{V}_j} \overline{V}_i, \overline{V}_k \right\rangle = -\left\langle \overline{V}_i, \nabla_{\overline{V}_j} \overline{V}_k \right\rangle \\ &= -\left\langle \overline{V}_i, \nabla_{\overline{V}_k} \overline{V}_j \right\rangle = \left\langle \nabla_{\overline{V}_k} \overline{V}_i, \overline{V}_j \right\rangle = \left\langle \nabla_{\overline{V}_i} \overline{V}_k, \overline{V}_j \right\rangle \end{aligned}$$

and hence

$$\Rightarrow \left\langle \nabla_{\overline{V}_i}\overline{V}_j, \overline{V}_k \right\rangle = \left\langle \nabla_{\overline{V}_i}\overline{V}_k, \overline{V}_j \right\rangle = -\left\langle \overline{V}_k, \nabla_{\overline{V}_i}\overline{V}_j \right\rangle = 0.$$

1.2.9 The Curvature Tensor on a Complex Manifold

For a complex manifold, we have at any point for any set of $(1,0)$, respectively $(0,1)$, frame fields,

$$\left\langle \nabla_{V_j} V_i, V_k \right\rangle \equiv 0 \equiv \left\langle \nabla_{\overline{V}_i}\overline{V}_j, \overline{V}_k \right\rangle$$

This implies that some of the complex curvature terms vanish while some others could be simplified and written in terms of real curvature terms.

$$\begin{aligned}
\left\langle R_{V_iV_j}V_i, V_j \right\rangle &= -\left\langle \nabla_{V_i}\nabla_{V_j}V_i, V_j \right\rangle + \left\langle \nabla_{V_j}\nabla_{V_i}V_i, V_j \right\rangle + \left\langle \nabla_{[V_i,V_j]}V_i, V_j \right\rangle \\
&= -V_i \left\langle \nabla_{V_j}V_i, V_j \right\rangle + V_j \left\langle \nabla_{V_i}V_i, V_j \right\rangle \\
&\quad + \left\langle \nabla_{V_j}V_i, V_k \right\rangle \left\langle \nabla_{V_i}V_j, \overline{V}_k \right\rangle + \left\langle \nabla_{V_j}V_i, \overline{V}_k \right\rangle \left\langle \nabla_{V_i}V_j, V_k \right\rangle \\
&\quad - \left\langle \nabla_{V_i}V_i, V_k \right\rangle \left\langle \nabla_{V_j}V_j, \overline{V}_k \right\rangle - \left\langle \nabla_{V_i}V_i, \overline{V}_k \right\rangle \left\langle \nabla_{V_j}V_j, V_k \right\rangle \\
&\quad - \left\langle \nabla_{\overline{V}_k}V_i, V_j \right\rangle \left\langle \nabla_{V_j}V_i, V_k \right\rangle - \left\langle \nabla_{V_k}V_i, V_j \right\rangle \left\langle \nabla_{V_j}V_i, \overline{V}_k \right\rangle \\
&\quad + \left\langle \nabla_{\overline{V}_k}V_i, V_j \right\rangle \left\langle \nabla_{V_i}V_j, V_k \right\rangle + \left\langle \nabla_{V_k}V_i, V_j \right\rangle \left\langle \nabla_{V_i}V_j, \overline{V}_k \right\rangle \\
&\equiv 0.
\end{aligned}$$

We denote Jv_i by v_{i^*} and for simplicity, we use $R_{ij^*k^*l}$ for the curvature term $\left\langle R_{v_iv_{j^*}}v_{k^*}, v_l \right\rangle$.

$$\begin{aligned}
&\left\langle R_{V_iV_j}V_i, V_j \right\rangle \\
= \;& +\frac{1}{4}\left[R_{ijij} - R_{iji^*j^*} - R_{ij^*i^*j} - R_{ij^*ij^*} - R_{i^*ji^*j} - R_{i^*jij^*} - R_{i^*j^*ij} + R_{i^*j^*i^*j^*}\right] \\
& + \frac{\sqrt{-1}}{4}\left[-R_{ijij^*} - R_{iji^*j} - R_{ij^*ij} + R_{ij^*i^*j^*} - R_{i^*jij} + R_{i^*ji^*j^*} + R_{i^*j^*ij^*} + R_{i^*j^*i^*j}\right]
\end{aligned}$$

using the symmetries of the curvature tensor would simplify this expression to

$$\begin{aligned}
&\left\langle R_{V_iV_j}V_i, V_j \right\rangle \\
= \;& +\frac{1}{4}\left[R_{ijij} - R_{ij^*ij^*} - R_{i^*ji^*j} + R_{i^*j^*i^*j^*} - 2R_{iji^*j^*} - 2R_{ij^*i^*j}\right] \\
& + \frac{\sqrt{-1}}{2}\left[-R_{ijij^*} - R_{iji^*j} + R_{ij^*i^*j^*} + R_{i^*ji^*j^*}\right].
\end{aligned}$$

This gives a relation on a complex manifold as below,

$$\begin{aligned}
R_{ijij} - R_{ij^*ij^*} - R_{i^*ji^*j} + R_{i^*j^*i^*j^*} - 2R_{iji^*j^*} - 2R_{ij^*i^*j} &= 0 \qquad \text{(R)} \\
-R_{ijij^*} - R_{iji^*j} + R_{ij^*i^*j^*} + R_{i^*ji^*j^*} &= 0.
\end{aligned}$$

In a similar fashion, we also obtain

$$\begin{aligned}
&\left\langle R_{\overline{V}_i\overline{V}_j}\overline{V}_i, \overline{V}_j\right\rangle \\
= \;& +\frac{1}{4}\left[R_{ijij} - R_{iji^*j^*} - R_{ij^*i^*j} - R_{ij^*ij^*} - R_{i^*ji^*j} - R_{i^*jij^*} - R_{i^*j^*ij} + R_{i^*j^*i^*j^*}\right] \\
& +\frac{\sqrt{-1}}{4}\left[+R_{ijij^*} + R_{iji^*j} + R_{ij^*ij} - R_{ij^*i^*j^*} + R_{i^*jij} - R_{i^*ji^*j^*} - R_{i^*j^*ij^*} - R_{i^*j^*i^*j}\right] \\
= \;& 0.
\end{aligned}$$

Definition 1.2.1 (isotropic curvature) *The curvature term,* $\left\langle R_{V_iV_j}\overline{V}_i, \overline{V}_j\right\rangle$ *or equivalently* $\left\langle R_{\overline{V}_i\overline{V}_j}V_i, V_j\right\rangle$ *is called isotropic curvature.*

In terms of real curvature terms, it could be written as

$$\begin{aligned}
&\left\langle R_{V_iV_j}\overline{V}_i, \overline{V}_j\right\rangle \\
&= +\frac{1}{4}\left[R_{ijij} + R_{ij^*ij^*} + R_{i^*ji^*j} - R_{iji^*j^*} - R_{i^*j^*ij} + R_{ij^*i^*j} + R_{i^*jij^*} + R_{i^*j^*i^*j^*}\right] \\
&+ \frac{\sqrt{-1}}{4}\left[+R_{ijij^*} - R_{ij^*ij} + R_{iji^*j} + R_{ij^*i^*j^*} - R_{i^*jij} + R_{i^*ji^*j^*} - R_{i^*j^*ij^*} - R_{i^*j^*i^*j}\right] \\
&= +\frac{1}{4}\left[R_{ijij} + (R_{ij^*ij^*} + R_{i^*ji^*j}) + R_{i^*j^*i^*j^*} - 2R_{iji^*j^*} + 2R_{ij^*i^*j}\right] \\
&\overset{(R)}{=} +\frac{1}{2}\left[R_{ijij} + R_{i^*j^*i^*j^*} - 2R_{iji^*j^*}\right].
\end{aligned}$$

Similarly its complex conjugate could be written as

$$\begin{aligned}
&\left\langle R_{\overline{V}_i\overline{V}_j}V_i, V_j\right\rangle \\
&= +\frac{1}{4}\left[R_{ijij} - R_{iji^*j^*} + R_{ij^*i^*j} + R_{ij^*ij^*} + R_{i^*ji^*j} + R_{i^*jij^*} - R_{i^*j^*ij} + R_{i^*j^*i^*j^*}\right] \\
&+ \frac{\sqrt{-1}}{4}\left[-R_{ijij^*} - R_{iji^*j} + R_{ij^*ij} - R_{ij^*i^*j^*} + R_{i^*jij} - R_{i^*ji^*j^*} + R_{i^*j^*ij^*} + R_{i^*j^*i^*j}\right] \\
&= +\frac{1}{4}\left[R_{ijij} + (R_{ij^*ij^*} + R_{i^*ji^*j}) + R_{i^*j^*i^*j^*} - 2R_{iji^*j^*} + 2R_{ij^*i^*j}\right] \\
&\overset{(R)}{=} +\frac{1}{2}\left[R_{ijij} + R_{i^*j^*i^*j^*} - 2R_{iji^*j^*}\right].
\end{aligned}$$

We also have

$$\begin{aligned}
&\left\langle R_{\overline{V}_iV_j}V_i, \overline{V}_j\right\rangle \\
&= +\frac{1}{4}\left[R_{ijij} + R_{iji^*j^*} - R_{ij^*i^*j} + R_{ij^*ij^*} + R_{i^*ji^*j} - R_{i^*jij^*} + R_{i^*j^*ij} + R_{i^*j^*i^*j^*}\right] \\
&+ \frac{\sqrt{-1}}{4}\left[+R_{ijij^*} - R_{iji^*j} - R_{ij^*ij} - R_{ij^*i^*j^*} + R_{i^*jij} + R_{i^*ji^*j^*} + R_{i^*j^*ij^*} - R_{i^*j^*i^*j}\right] \\
&= +\frac{1}{4}\left[R_{ijij} + R_{ij^*ij^*} + R_{i^*ji^*j} + R_{i^*j^*i^*j^*} + 2R_{iji^*j^*} - 2R_{i^*jij^*}\right] \\
&\overset{(R)}{=} +\frac{1}{2}\left[R_{ij^*ij^*} + R_{i^*ji^*j} + 2R_{iji^*j^*}\right].
\end{aligned}$$

We note that

$$
\begin{aligned}
& \left\langle R_{\overline{V}_i V_i} V_j, \overline{V}_j \right\rangle \\
= & +\frac{1}{4}\left[R_{iijj} + R_{iij^*j^*} - R_{ii^*j^*j} + R_{ii^*jj^*} + R_{i^*ij^*j} - R_{i^*ijj^*} + R_{i^*i^*jj} + R_{i^*i^*j^*j^*}\right] \\
& +\frac{\sqrt{-1}}{4}\left[+R_{iijj^*} - R_{iij^*j} - R_{ii^*jj} - R_{ii^*j^*j^*} + R_{i^*ijj} + R_{i^*ij^*j^*} + R_{i^*i^*jj^*} - R_{i^*i^*j^*j}\right] \\
= & +\frac{1}{4}\left[-R_{ii^*j^*j} + R_{ii^*jj^*} + R_{i^*ij^*j} - R_{i^*ijj^*}\right] = R_{ii^*jj^*}.
\end{aligned}
$$

By the Bianchi identity,

$$
0 = \left\langle R_{\overline{V}_i V_i} V_j, \overline{V}_j \right\rangle + \left\langle R_{\overline{V}_i V_j} \overline{V}_j, V_i \right\rangle + \left\langle R_{\overline{V}_i \overline{V}_j} V_i, V_j \right\rangle,
$$

which gives

$$
\begin{aligned}
0 &= R_{ii^*jj^*} - \frac{1}{2}R_{ij^*ij^*} - \frac{1}{2}R_{i^*ji^*j} - R_{iji^*j^*} + \frac{1}{2}R_{ijij} + \frac{1}{2}R_{i^*j^*i^*j^*} - R_{iji^*j^*} \\
&\Rightarrow\ 2R_{iji^*j^*} = R_{ii^*jj^*} + \frac{1}{2}R_{ijij} + \frac{1}{2}R_{i^*j^*i^*j^*} - \frac{1}{2}R_{ij^*ij^*} - \frac{1}{2}R_{i^*ji^*j}.
\end{aligned}
$$

So if we plug this into $\left\langle R_{\overline{V}_i V_j} V_i, \overline{V}_j \right\rangle$, we obtain

$$
\left\langle R_{\overline{V}_i V_j} V_i, \overline{V}_j \right\rangle = +\frac{1}{4}\left[2R_{ii^*jj^*} + R_{ijij} + R_{i^*j^*i^*j^*} + R_{ij^*ij^*} + R_{i^*ji^*j}\right]
$$

and if we plug this into $\left\langle R_{\overline{V}_i \overline{V}_j} V_i, V_j \right\rangle$, we obtain

$$
\left\langle R_{\overline{V}_i \overline{V}_j} V_i, V_j \right\rangle = +\frac{1}{4}\left[-2R_{ii^*jj^*} + R_{ijij} + R_{i^*j^*i^*j^*} + R_{ij^*ij^*} + R_{i^*ji^*j}\right].
$$

Definition 1.2.2 (sectional curvature) *For orthonormal vector fields X, Y we define the sectional curvature of the 2-plane spanned by X and Y as $\sigma(XY) = \langle R_{XY}X, Y\rangle$.*

Definition 1.2.3 (holomorphic sectional curvature) *For a unit vector field X, we define its holomorphic sectional curvature $H(X)$ to be the sectional curvature of the 2-plane spanned by X and JX, i.e. $H(X) = \langle R_{XJX}X, JX\rangle$.*

Thus in our setting $R_{ii^*ii^*}$ is a holomorphic sectional curvature term. Also we note that

$$
\left\langle R_{\overline{V}_i V_i} V_i, \overline{V}_i \right\rangle = +\frac{1}{2}\left[R_{ii^*ii^*} + R_{i^*ii^*i} + 2R_{iii^*i^*}\right] = R_{ii^*ii^*}.
$$

Definition 1.2.4 (holomorphic bisectional curvature) *For orthonormal vector fields X, Y we define their holomorphic bisectional curvature $H(X, Y) = \langle R_{XJX}Y, JY\rangle$.*

In our setting $R_{ii^*jj^*}$ is a holomorphic bisectional curvature term. and also that

$$
\left\langle R_{V_i V_j} \overline{V}_i, \overline{V}_j \right\rangle + \left\langle R_{\overline{V}_i V_j} V_i, \overline{V}_j \right\rangle = +\frac{1}{2}\left[R_{ijij} + R_{i^*j^*i^*j^*} + R_{ij^*ij^*} + R_{i^*ji^*j}\right].
$$

1.3 Differential operators on Complex Manifolds

Throughout this thesis, we use the standard Einstein summation convention that expressions with repeated indices will be summed over those repeated indices from 1 to n. Exceptions, such as the definition of sectional curvature above, should be clear from the context.

1.3.1 Derivative operators

Let $E^{p,q}(M)$ be the space of complex $(p,q)-$forms on M and ∇ be the Levi-Civita connection extended to $T_*M \otimes_{\mathbb{R}} \mathbb{C}$. Then we have the following differential operators defined by projecting the image of the exterior derivative operator d onto the $(1,0)$ and the $(0,1)$ parts respectively.

Denote the projection onto $E^{p,q}$ by $proj_{p,q}$. For any $(p,q)-$form $\alpha \in E^{p,q}$, we can define $\partial : E^{p,q} \to E^{p+1,q}$ by

$$\begin{aligned}\partial\left(\alpha^{p,q}\right) &= proj_{p+1,q}\left(d\left(\alpha\right)\right) \qquad (1.3)\\ &= proj_{p+1,q}\left(\overline{\omega}^j \wedge \nabla_{\overline{V}_j}\left(\alpha\right)\right) + proj_{p+1,q}\left(\omega^j \wedge \nabla_{V_j}\left(\alpha\right)\right)\\ &= \overline{\omega}^j \wedge proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}\left(\alpha\right)\right) + \omega^j \wedge proj_{p,q}\left(\nabla_{V_j}\left(\alpha\right)\right)\end{aligned}$$

and define $\overline{\partial} : E^{p,q} \to E^{p,q+1}$ by

$$\begin{aligned}\overline{\partial}\left(\alpha^{p,q}\right) &= proj_{p,q+1}\left(d\left(\alpha\right)\right) \qquad (1.4)\\ &= proj_{p,q+1}\left(\overline{\omega}^j \wedge \nabla_{\overline{V}_j}\left(\alpha\right)\right) + proj_{p,q+1}\left(\omega^j \wedge \nabla_{V_j}\left(\alpha\right)\right)\\ &= \overline{\omega}^j \wedge proj_{p,q}\left(\nabla_{\overline{V}_j}\left(\alpha\right)\right) + \omega^j \wedge proj_{p-1,q+1}\left(\nabla_{V_j}\left(\alpha\right)\right).\end{aligned}$$

1.3.2 The Lefschetz operator L and adjoint Lefschetz operator Λ

We define the Lefschetz operator $L : E^{p,q} \to E^{p+1,q+1}$ by

$$L\left(\alpha\right) = \Omega \wedge \alpha = \sqrt{-1}\omega^k \wedge \overline{\omega}^k \wedge \alpha \qquad (1.5)$$

and we define the adjoint Lefschetz operator $\Lambda : E^{p,q} \to E^{p-1,q-1}$ by

$$\Lambda\left(\alpha\right) = -\sqrt{-1}\ \iota_{\overline{V}_k}\left(\iota_{V_k}\left(\alpha\right)\right). \qquad (1.6)$$

1.3.3 The antisymplectic operators $L_{\overline{\partial}\Omega}$ and $L_{\partial\Omega}$

Let $\Omega = \sqrt{-1}\omega^i \wedge \overline{\omega}^i$ be the fundamental $(1,1)-$form. Then we have

$$\begin{aligned}
\overline{\partial}\Omega &= \overline{\partial}\left(\sqrt{-1}\omega^i \wedge \overline{\omega}^i\right) \\
&= proj_{1,2}\left(\overline{\omega}^j \wedge \nabla_{\overline{V}_j}\left(\sqrt{-1}\omega^i \wedge \overline{\omega}^i\right) + \omega^j \wedge \nabla_{V_j}\left(\sqrt{-1}\omega^i \wedge \overline{\omega}^i\right)\right) \\
&= \overline{\omega}^j \wedge proj_{1,1}\left(\nabla_{\overline{V}_j}\left(\sqrt{-1}\omega^i \wedge \overline{\omega}^i\right)\right) + \omega^j \wedge proj_{0,2}\left(\nabla_{V_j}\left(\sqrt{-1}\omega^i \wedge \overline{\omega}^i\right)\right) \\
&= +\sqrt{-1}\left\langle \nabla_{\overline{V}_j}\omega^i, \overline{\omega}^k\right\rangle \overline{\omega}^j \wedge \omega^k \wedge \overline{\omega}^i + \sqrt{-1}\left\langle \nabla_{\overline{V}_j}\overline{\omega}^i, \omega^k\right\rangle \overline{\omega}^j \wedge \omega^i \wedge \overline{\omega}^k \\
&\quad +\sqrt{-1}\left\langle \nabla_{V_j}\omega^i, \omega^k\right\rangle \omega^j \wedge \overline{\omega}^k \wedge \overline{\omega}^i \\
&= +\sqrt{-1}\left\langle \nabla_{\overline{V}_j}\omega^k, \overline{\omega}^i\right\rangle \overline{\omega}^j \wedge \omega^i \wedge \overline{\omega}^k + \sqrt{-1}\left\langle \nabla_{\overline{V}_j}\overline{\omega}^i, \omega^k\right\rangle \overline{\omega}^j \wedge \omega^i \wedge \overline{\omega}^k \\
&\quad +\sqrt{-1}\left\langle \nabla_{V_j}\omega^i, \omega^k\right\rangle \omega^j \wedge \overline{\omega}^k \wedge \overline{\omega}^i \\
&= +\sqrt{-1}\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \omega^j \wedge \overline{\omega}^k \wedge \overline{\omega}^i
\end{aligned}$$

Thus we define $L_{\overline{\partial}\Omega} : E^{p,q} \rightarrow E^{p+1,q+2}$ by

$$L_{\overline{\partial}\Omega}(\alpha) := \overline{\partial}\Omega \wedge \alpha = +\sqrt{-1}\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \omega^j \wedge \overline{\omega}^k \wedge \overline{\omega}^i \wedge \alpha$$

for $\alpha \in E^{p,q}$.

Similarly, we also have

$$\begin{aligned}
\partial\Omega &= \partial\left(\sqrt{-1}\omega^i \wedge \overline{\omega}^i\right) \\
&= proj_{2,1}\left(\overline{\omega}^j \wedge \nabla_{\overline{V}_j}\left(\sqrt{-1}\omega^i \wedge \overline{\omega}^i\right) + \omega^j \wedge \nabla_{V_j}\left(\sqrt{-1}\omega^i \wedge \overline{\omega}^i\right)\right) \\
&= \overline{\omega}^j \wedge proj_{2,0}\left(\nabla_{\overline{V}_j}\left(\sqrt{-1}\omega^i \wedge \overline{\omega}^i\right)\right) + \omega^j \wedge proj_{1,1}\left(\nabla_{V_j}\left(\sqrt{-1}\omega^i \wedge \overline{\omega}^i\right)\right) \\
&= +\sqrt{-1}\overline{\omega}^j \wedge \omega^i \wedge \left\langle \nabla_{\overline{V}_j}\overline{\omega}^i, \overline{\omega}^k\right\rangle \omega^k + \sqrt{-1}\omega^j \wedge \left\langle \nabla_{V_j}\omega^i, \overline{\omega}^k\right\rangle \omega^k \wedge \overline{\omega}^i \\
&\quad +\sqrt{-1}\omega^j \wedge \omega^i \wedge \left\langle \nabla_{V_j}\overline{\omega}^i, \omega^k\right\rangle \overline{\omega}^k \\
&= +\sqrt{-1}\overline{\omega}^j \wedge \omega^i \wedge \left\langle \nabla_{\overline{V}_j}\overline{\omega}^i, \overline{\omega}^k\right\rangle \omega^k + \sqrt{-1}\omega^j \wedge \left\langle \nabla_{V_j}\omega^i, \overline{\omega}^k\right\rangle \omega^k \wedge \overline{\omega}^i \\
&\quad +\sqrt{-1}\omega^j \wedge \omega^k \wedge \left\langle \nabla_{V_j}\overline{\omega}^k, \omega^i\right\rangle \overline{\omega}^i \\
&= +\sqrt{-1}\left\langle \nabla_{\overline{V}_j}V_i, V_k\right\rangle \omega^i \wedge \omega^k \wedge \overline{\omega}^j
\end{aligned}$$

and we also define $L_{\partial\Omega} : E^{p,q} \rightarrow E^{p+2,q+1}$ by

$$L_{\partial\Omega}(\alpha) := \partial\Omega \wedge \alpha = +\sqrt{-1}\left\langle \nabla_{\overline{V}_j}V_k, V_i\right\rangle \omega^k \wedge \omega^i \wedge \overline{\omega}^j \wedge \alpha$$

for $\alpha \in E^{p,q}$.

1.3.4 The Adjoint antisymplectic operators $(L_{\overline{\partial}\Omega})^*$ and $(L_{\partial\Omega})^*$

The adjoint antisymplectic operator $\left(L_{\overline{\partial}\Omega}\right)^* : E^{p,q} \to E^{p-1,q-2}$ is defined by

$$\begin{aligned}
\left(\left(L_{\overline{\partial}\Omega}\right)^* (\alpha), \beta\right)_{L^2} &= \left(\alpha, L_{\overline{\partial}\Omega}(\beta)\right)_{L^2} = \left(\alpha, \overline{\partial}\Omega \wedge \beta\right)_{L^2} \\
&= \left(\alpha, \sqrt{-1}\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \omega^j \wedge \overline{\omega}^k \wedge \overline{\omega}^i \wedge \beta\right)_{L^2} \\
&= \left(-\sqrt{-1}\left\langle \nabla_{\overline{V}_j} V_i, V_k\right\rangle \iota_{\overline{V}_i}\iota_{\overline{V}_k}\iota_{V_j}(\alpha), \beta\right)_{L^2}
\end{aligned}$$

for $\alpha \in E^{p,q}$, $\beta \in E^{p-1,q-2}$.

Similarly, the adjoint antisymplectic operator $(L_{\partial\Omega})^* : E^{p,q} \to E^{p-2,q-1}$ is defined by

$$\begin{aligned}
\left((L_{\partial\Omega})^* (\alpha), \beta\right)_{L^2} &= \left(\alpha, L_{\partial\Omega}(\beta)\right)_{L^2} = \left(\alpha, \partial\Omega \wedge \beta\right)_{L^2} \\
&= \left(\alpha, \sqrt{-1}\left\langle \nabla_{\overline{V}_j} V_k, V_i\right\rangle \omega^k \wedge \omega^i \wedge \overline{\omega}^j \wedge \beta\right)_{L^2} \\
&= \left(-\sqrt{-1}\left\langle \nabla_{V_j}\overline{V}_k, \overline{V}_i\right\rangle \iota_{\overline{V}_j}\iota_{V_i}\iota_{V_k}(\alpha), \beta\right)_{L^2}
\end{aligned}$$

for $\alpha \in E^{p,q}$, $\beta \in E^{p-2,q-1}$.

1.3.5 $[L_{\overline{\partial}\Omega}, \Lambda]$ operator

Define the commutator operator $\left[L_{\overline{\partial}\Omega}, \Lambda\right](\alpha) = L_{\overline{\partial}\Omega}(\Lambda(\alpha)) - \Lambda\left(L_{\overline{\partial}\Omega}(\alpha)\right)$.

$$\begin{aligned}
&\left[L_{\overline{\partial}\Omega}, \Lambda\right](\alpha) \\
=\ & L_{\overline{\partial}\Omega}(\Lambda(\alpha)) - \Lambda\left(L_{\overline{\partial}\Omega}(\alpha)\right) \\
=\ & -\sqrt{-1} L_{\overline{\partial}\Omega}\left(\iota_{\overline{V}_l}(\iota_{V_l}\alpha)\right) - \Lambda\left(\sqrt{-1}\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \omega^j \wedge \overline{\omega}^k \wedge \overline{\omega}^i \wedge \alpha\right) \\
=\ & +\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \omega^j \wedge \overline{\omega}^k \wedge \overline{\omega}^i \wedge \left(\iota_{\overline{V}_l}(\iota_{V_l}\alpha)\right) - \iota_{\overline{V}_j}\left(\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \overline{\omega}^i \wedge \alpha\right) \\
& +\iota_{\overline{V}_l}\left(\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \omega^j \wedge \overline{\omega}^k \wedge \overline{\omega}^i \wedge \iota_{V_l}(\alpha)\right) \\
=\ & -\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_j\right\rangle \overline{\omega}^i \wedge \alpha + \left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \alpha - \left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_j}(\alpha) \\
& -\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \omega^j \wedge \overline{\omega}^i \wedge \iota_{V_k}(\alpha) + \left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \omega^j \wedge \overline{\omega}^k \wedge \iota_{V_i}(\alpha)
\end{aligned}$$

Equivalently, we have

$$\begin{aligned}
\left[L_{\overline{\partial}\Omega}, \Lambda\right](\alpha) &= +2\left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \alpha \\
&\quad +2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \omega^j \wedge \overline{\omega}^k \wedge \iota_{V_i}(\alpha) \\
&\quad -\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_j}(\alpha)
\end{aligned} \tag{1.7}$$

1.3.6 The adjoint operator, $[L,(L_{\overline{\partial}\Omega})^*]$

Define the commutator operator $\left[L,(L_{\overline{\partial}\Omega})^*\right](\alpha)=L\left((L_{\overline{\partial}\Omega})^*(\alpha)\right)-(L_{\overline{\partial}\Omega})^*(L(\alpha))$. If we compute the commutator for a $(p,q)-$form $\alpha\in E^{p,q}$

$$\begin{aligned}
&\left[L,(L_{\overline{\partial}\Omega})^*\right](\alpha)\\
=\;& L\left((L_{\overline{\partial}\Omega})^*(\alpha)\right)-(L_{\overline{\partial}\Omega})^*(L(\alpha))\\
=\;& L\left(-\sqrt{-1}\left\langle\nabla_{\overline{V}_j}V_i,V_k\right\rangle\iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right)\right)\right)-(L_{\overline{\partial}\Omega})^*\left(\sqrt{-1}\omega^l\wedge\overline{\omega}^l\wedge\alpha\right)\\
=\;& \left\langle\nabla_{\overline{V}_j}V_i,V_k\right\rangle\omega^l\wedge\overline{\omega}^l\wedge\iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right)\right)\\
&-\left\langle\nabla_{\overline{V}_j}V_i,V_k\right\rangle\iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}\left(\iota_{V_j}\left(\omega^l\wedge\overline{\omega}^l\wedge\alpha\right)\right)\right)
\end{aligned}$$

$$\begin{aligned}
=\;& \left\langle\nabla_{\overline{V}_j}V_i,V_k\right\rangle\omega^l\wedge\overline{\omega}^l\wedge\iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right)\right)\\
&-\left\langle\nabla_{\overline{V}_j}V_i,V_j\right\rangle\iota_{\overline{V}_i}(\alpha)+\left\langle\nabla_{\overline{V}_j}V_j,V_k\right\rangle\iota_{\overline{V}_k}(\alpha)-\left\langle\nabla_{\overline{V}_j}V_i,V_k\right\rangle\overline{\omega}^j\wedge\iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}(\alpha)\right)\\
&-\left\langle\nabla_{\overline{V}_j}V_i,V_k\right\rangle\omega^k\wedge\iota_{\overline{V}_i}\left(\iota_{V_j}(\alpha)\right)+\left\langle\nabla_{\overline{V}_j}V_i,V_k\right\rangle\omega^i\wedge\iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right)\\
&-\left\langle\nabla_{\overline{V}_j}V_i,V_k\right\rangle\omega^l\wedge\overline{\omega}^l\wedge\iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right)\right)
\end{aligned}$$

Thus we have

$$\begin{aligned}
\left[L,(L_{\overline{\partial}\Omega})^*\right] =\;& +2\left\langle\nabla_{\overline{V}_j}V_j,V_i\right\rangle\iota_{\overline{V}_i}(\alpha)\\
&+2\left\langle\nabla_{\overline{V}_j}V_i,V_k\right\rangle\omega^i\wedge\iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right)\\
&-\left\langle\nabla_{\overline{V}_j}V_i,V_k\right\rangle\overline{\omega}^j\wedge\iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}(\alpha)\right).
\end{aligned}\tag{1.8}$$

1.3.7 $[L_{\partial\Omega},\Lambda]$ operator

Define the commutator operator $[L_{\partial\Omega},\Lambda](\alpha)=L_{\partial\Omega}(\Lambda(\alpha))-\Lambda(L_{\partial\Omega}(\alpha))$.

$$\begin{aligned}
&[L_{\partial\Omega},\Lambda](\alpha)\\
=\;& L_{\partial\Omega}(\Lambda(\alpha))-\Lambda(L_{\partial\Omega}(\alpha))\\
=\;& -\sqrt{-1}L_{\partial\Omega}\left(\iota_{\overline{V}_l}(\iota_{V_l}\alpha)\right)-\Lambda\left(\sqrt{-1}\left\langle\nabla_{\overline{V}_j}V_k,V_i\right\rangle\omega^k\wedge\omega^i\wedge\overline{\omega}^j\wedge\alpha\right)\\
=\;& +\left\langle\nabla_{\overline{V}_j}V_k,V_i\right\rangle\omega^k\wedge\omega^i\wedge\overline{\omega}^j\wedge\left(\iota_{\overline{V}_l}(\iota_{V_l}\alpha)\right)-\iota_{\overline{V}_k}\left(\left\langle\nabla_{\overline{V}_j}V_k,V_i\right\rangle\omega^i\wedge\overline{\omega}^j\wedge\alpha\right)\\
&+\iota_{\overline{V}_i}\left(\left\langle\nabla_{\overline{V}_j}V_k,V_i\right\rangle\omega^k\wedge\overline{\omega}^j\wedge\alpha\right)+\iota_{\overline{V}_l}\left(\left\langle\nabla_{\overline{V}_j}V_k,V_i\right\rangle\omega^k\wedge\omega^i\wedge\overline{\omega}^j\wedge\iota_{V_l}(\alpha)\right)
\end{aligned}$$

$$\begin{aligned}
[L_{\partial\Omega},\Lambda](\alpha) &= +\left\langle \nabla_{\overline{V}_j} V_k, V_i \right\rangle \omega^k \wedge \omega^i \wedge \overline{\omega}^j \wedge \left(\iota_{\overline{V}_l}\left(\iota_{V_l}\alpha\right)\right) \\
&\quad + \left\langle \nabla_{\overline{V}_j} V_j, V_i \right\rangle \omega^i \wedge \alpha - \left\langle \nabla_{\overline{V}_j} V_k, V_i \right\rangle \omega^i \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_k}(\alpha) \\
&\quad - \left\langle \nabla_{\overline{V}_j} V_k, V_j \right\rangle \omega^k \wedge \alpha + \left\langle \nabla_{\overline{V}_j} V_k, V_i \right\rangle \omega^k \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_i}(\alpha) \\
&\quad + \left\langle \nabla_{\overline{V}_j} V_k, V_i \right\rangle \omega^k \wedge \omega^i \wedge \overline{\omega}^j \wedge \iota_{V_j}(\alpha) \\
&\quad - \left\langle \nabla_{\overline{V}_j} V_k, V_i \right\rangle \omega^k \wedge \omega^i \wedge \overline{\omega}^j \wedge \left(\iota_{\overline{V}_l}\left(\iota_{V_l}(\alpha)\right)\right).
\end{aligned}$$

Equivalently, we have

$$\begin{aligned}
[L_{\partial\Omega},\Lambda](\alpha) &= +2\left\langle \nabla_{\overline{V}_j} V_j, V_i \right\rangle \omega^i \wedge \alpha \\
&\quad +2\left\langle \nabla_{\overline{V}_j} V_i, V_k \right\rangle \omega^i \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_k}(\alpha) \\
&\quad + \left\langle \nabla_{\overline{V}_j} V_k, V_i \right\rangle \omega^k \wedge \omega^i \wedge \iota_{V_j}(\alpha)
\end{aligned}$$

1.3.8 The adjoint operator, $\left[L,(L_{\partial\Omega})^*\right]$

Define the commutator operator $\left[L,(L_{\partial\Omega})^*\right](\alpha) = L\left((L_{\partial\Omega})^*(\alpha)\right) - (L_{\partial\Omega})^*(L(\alpha))$. If we compute the commutator for a $(p,q)-$form $\alpha \in E^{p,q}$

$$\begin{aligned}
& \left[L,(L_{\partial\Omega})^*\right](\alpha) \\
=\ & L\left((L_{\partial\Omega})^*(\alpha)\right) - (L_{\partial\Omega})^*(L(\alpha)) \\
=\ & L\left(-\sqrt{-1}\left\langle \nabla_{V_j}\overline{V}_k, \overline{V}_i\right\rangle \iota_{\overline{V}_j}\left(\iota_{V_i}\left(\iota_{V_k}(\alpha)\right)\right)\right) - (L_{\partial\Omega})^*\left(\sqrt{-1}\omega^l \wedge \overline{\omega}^l \wedge \alpha\right) \\
=\ & \left\langle \nabla_{V_j}\overline{V}_k, \overline{V}_i\right\rangle \omega^l \wedge \overline{\omega}^l \wedge \iota_{\overline{V}_j}\left(\iota_{V_i}\left(\iota_{V_k}(\alpha)\right)\right) \\
& - \left\langle \nabla_{V_j}\overline{V}_k, \overline{V}_i\right\rangle \iota_{\overline{V}_j}\left(\iota_{V_i}\left(\iota_{V_k}\left(\omega^l \wedge \overline{\omega}^l \wedge \alpha\right)\right)\right) \\
=\ & + \left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_i\right\rangle \iota_{V_i}(\alpha) - \left\langle \nabla_{V_j}\overline{V}_k, \overline{V}_i\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_j}\left(\iota_{V_i}(\alpha)\right) \\
& - \left\langle \nabla_{V_j}\overline{V}_k, \overline{V}_j\right\rangle \iota_{V_k}(\alpha) + \left\langle \nabla_{V_j}\overline{V}_k, \overline{V}_i\right\rangle \overline{\omega}^i \wedge \iota_{\overline{V}_j}\left(\iota_{V_k}(\alpha)\right) \\
& + \left\langle \nabla_{V_j}\overline{V}_k, \overline{V}_i\right\rangle \omega^j \wedge \iota_{V_i}\left(\iota_{V_k}(\alpha)\right)
\end{aligned}$$

Thus,

$$\begin{aligned}
& \left[L,(L_{\partial\Omega})^*\right] \\
=\ & +2\left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_i\right\rangle \iota_{V_i}(\alpha) + 2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_j}\left(\iota_{V_i}(\alpha)\right) \\
& + \left\langle \nabla_{V_j}\overline{V}_k, \overline{V}_i\right\rangle \omega^j \wedge \iota_{V_i}\left(\iota_{V_k}(\alpha)\right)
\end{aligned}$$

1.3.9 $[\Lambda, \partial]$ Operator

Define the commutator operator $[\Lambda, \partial] : E^{p,q} \to E^{p,q-1}$ by $[\Lambda, \partial](\alpha) = \Lambda(\partial(\alpha)) - \partial(\Lambda(\alpha))$. If we compute the commutator, we would have

$$\begin{aligned}
& [\Lambda, \partial](\alpha) \\
= \; & \Lambda(\partial(\alpha)) - \partial(\Lambda(\alpha)) \\
= \; & \Lambda\left(\overline{\omega}^j \wedge proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right) + \omega^j \wedge proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right) - \partial\left(-\sqrt{-1}\iota_{\overline{V}_k}\left(\iota_{V_k}(\alpha)\right)\right) \\
= \; & -\sqrt{-1}\iota_{\overline{V}_k}\left(\iota_{V_k}\left(\overline{\omega}^j \wedge proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right) + \omega^j \wedge proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \\
& -\overline{\omega}^j \wedge proj_{p,q-2}\left(\nabla_{\overline{V}_j}\left(-\sqrt{-1}\iota_{\overline{V}_k}\left(\iota_{V_k}(\alpha)\right)\right)\right) \\
& -\omega^j \wedge proj_{p-1,q-1}\left(\nabla_{V_j}\left(-\sqrt{-1}\iota_{\overline{V}_k}\left(\iota_{V_k}(\alpha)\right)\right)\right)
\end{aligned}$$

Then

$$\begin{aligned}
[\Lambda, \partial](\alpha) = \; & -\sqrt{-1}\overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{V_k}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \\
& -\sqrt{-1}\iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right) - \sqrt{-1}\omega^j \wedge \iota_{\overline{V}_k}\left(\iota_{V_k}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \\
& +\sqrt{-1}\overline{\omega}^j \wedge proj_{p,q-2}\left(\iota_{\overline{V}_k}\left(\iota_{V_k}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \\
& +\sqrt{-1}\overline{\omega}^j \wedge proj_{p,q-2}\left(\iota_{\overline{V}_k}\left(\iota_{\nabla_{\overline{V}_j}V_k}(\alpha)\right)\right) \\
& +\sqrt{-1}\overline{\omega}^j \wedge proj_{p,q-2}\left(\iota_{\left\langle\nabla_{\overline{V}_j}\overline{V}_k, V_i\right\rangle\overline{V}_i + \left\langle\nabla_{\overline{V}_j}\overline{V}_k, \overline{V}_i\right\rangle V_i}\left(\iota_{V_k}(\alpha)\right)\right) \\
& +\sqrt{-1}\omega^j \wedge proj_{p-1,q-1}\left(\iota_{\overline{V}_k}\left(\iota_{V_k}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \\
& +\sqrt{-1}\omega^j \wedge proj_{p-1,q-1}\left(\iota_{\overline{V}_k}\left(\iota_{\nabla_{V_j}V_k}(\alpha)\right)\right) \\
& +\sqrt{-1}\omega^j \wedge proj_{p-1,q-1}\left(\iota_{\left\langle\nabla_{V_j}\overline{V}_k, V_i\right\rangle\overline{V}_i + \left\langle\nabla_{V_j}\overline{V}_k, \overline{V}_i\right\rangle V_i}\left(\iota_{V_k}(\alpha)\right)\right)
\end{aligned}$$

Simplifying,

$$\begin{aligned}
& [\Lambda, \partial](\alpha) \\
= & +\sqrt{-1}\iota_{V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) - \sqrt{-1}\iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right) \\
& +\sqrt{-1}\overline{\omega}^j \wedge proj_{p,q-2}\left(\iota_{\overline{V}_k}\left(\iota_{\left\langle\nabla_{\overline{V}_j}V_k, V_i\right\rangle\overline{V}_i + \left\langle\nabla_{\overline{V}_j}V_k, \overline{V}_i\right\rangle V_i}(\alpha)\right)\right) \\
& +\sqrt{-1}\overline{\omega}^j \wedge proj_{p,q-2}\left(\iota_{\left\langle\nabla_{\overline{V}_j}\overline{V}_k, V_i\right\rangle\overline{V}_i + \left\langle\nabla_{\overline{V}_j}\overline{V}_k, \overline{V}_i\right\rangle V_i}\left(\iota_{V_k}(\alpha)\right)\right) \\
& +\sqrt{-1}\omega^j \wedge proj_{p-1,q-1}\left(\iota_{\overline{V}_k}\left(\iota_{\left\langle\nabla_{V_j}V_k, V_i\right\rangle\overline{V}_i + \left\langle\nabla_{V_j}V_k, \overline{V}_i\right\rangle V_i}(\alpha)\right)\right) \\
& +\sqrt{-1}\omega^j \wedge proj_{p-1,q-1}\left(\iota_{\left\langle\nabla_{V_j}\overline{V}_k, V_i\right\rangle\overline{V}_i + \left\langle\nabla_{V_j}\overline{V}_k, \overline{V}_i\right\rangle V_i}\left(\iota_{V_k}(\alpha)\right)\right) \\
= & +\sqrt{-1}\iota_{V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) - \sqrt{-1}\iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right) \\
& +\sqrt{-1}\left\langle\nabla_{\overline{V}_j}V_k, V_i\right\rangle\overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right) + \sqrt{-1}\left\langle\nabla_{V_j}V_k, \overline{V}_i\right\rangle\omega^j \wedge \iota_{\overline{V}_k}\left(\iota_{V_i}(\alpha)\right) \\
& +\sqrt{-1}\left\langle\nabla_{V_j}\overline{V}_i, V_k\right\rangle\omega^j \wedge \iota_{\overline{V}_k}\left(\iota_{V_i}(\alpha)\right)
\end{aligned}$$

Finally we obtain

$$\begin{aligned}
[\Lambda, \partial](\alpha) &= +\sqrt{-1}\partial^*(\alpha) - \sqrt{-1}\left[L, \left(L_{\overline{\partial}\Omega}\right)^*\right](\alpha) \qquad (*) \\
&= +\sqrt{-1}\iota_{V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) - \sqrt{-1}\iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right) \\
&\quad +\sqrt{-1}\left\langle\nabla_{\overline{V}_j}V_k, V_i\right\rangle\overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right)
\end{aligned}$$

1.3.10 $\left[\Lambda, \overline{\partial}\right]$ Operator

Define the commutator operator $\left[\Lambda, \overline{\partial}\right] : E^{p,q} \to E^{p-1,q}$ by $\left[\Lambda, \overline{\partial}\right](\alpha) = \Lambda\left(\overline{\partial}(\alpha)\right) - \overline{\partial}(\Lambda(\alpha))$. If we compute the commutator, we have

$$\begin{aligned}
& \left[\Lambda, \overline{\partial}\right](\alpha) \\
= & \Lambda\left(\overline{\partial}(\alpha)\right) - \overline{\partial}(\Lambda(\alpha)) \\
= & \Lambda\left(\overline{\omega}^j \wedge proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right) + \omega^j \wedge proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right) - \overline{\partial}\left(-\sqrt{-1}\iota_{\overline{V}_k}\left(\iota_{V_k}(\alpha)\right)\right) \\
= & -\sqrt{-1}\iota_{\overline{V}_k}\left(\iota_{V_k}\left(\overline{\omega}^j \wedge proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) - \sqrt{-1}\iota_{\overline{V}_k}\left(\iota_{V_k}\left(\omega^j \wedge proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \\
& +\sqrt{-1}\overline{\omega}^j \wedge proj_{p-1,q-1}\left(\nabla_{\overline{V}_j}\left(\iota_{\overline{V}_k}\left(\iota_{V_k}(\alpha)\right)\right)\right) + \sqrt{-1}\omega^j \wedge proj_{p-2,q}\left(\nabla_{V_j}\left(\iota_{\overline{V}_k}\left(\iota_{V_k}(\alpha)\right)\right)\right)
\end{aligned}$$

If we separate the interior products into directions, this simplifies to

$$
\begin{aligned}
& \left[\Lambda, \overline{\partial}\right](\alpha) \\
= \;& +\sqrt{-1}\iota_{V_j}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) - \sqrt{-1}\overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{V_k}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \\
& -\sqrt{-1}\iota_{\overline{V}_j}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right) - \sqrt{-1}\omega^j \wedge \iota_{\overline{V}_k}\left(\iota_{V_k}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \\
& +\sqrt{-1}\overline{\omega}^j \wedge proj_{p-1,q-1}\left(\iota_{\overline{V}_k}\left(\iota_{V_k}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) + \sqrt{-1}\overline{\omega}^j \wedge proj_{p-1,q-1}\left(\iota_{\overline{V}_k}\left(\iota_{\nabla_{\overline{V}_j}V_k}(\alpha)\right)\right) \\
& +\sqrt{-1}\overline{\omega}^j \wedge proj_{p-1,q-1}\left(\iota_{\left\langle\nabla_{\overline{V}_j}\overline{V}_k, V_i\right\rangle\overline{V}_i + \left\langle\nabla_{\overline{V}_j}\overline{V}_k, \overline{V}_i\right\rangle V_i}\left(\iota_{V_k}(\alpha)\right)\right) \\
& +\sqrt{-1}\omega^j \wedge proj_{p-2,q}\left(\iota_{\overline{V}_k}\left(\iota_{V_k}\left(\nabla_{V_j}(\alpha)\right)\right)\right) + \sqrt{-1}\omega^j \wedge proj_{p-2,q}\left(\iota_{\overline{V}_k}\left(\iota_{\nabla_{V_j}V_k}(\alpha)\right)\right) \\
& +\sqrt{-1}\omega^j \wedge proj_{p-2,q}\left(\iota_{\left\langle\nabla_{V_j}\overline{V}_k, V_i\right\rangle\overline{V}_i + \left\langle\nabla_{V_j}\overline{V}_k, \overline{V}_i\right\rangle V_i}\left(\iota_{V_k}(\alpha)\right)\right)
\end{aligned}
$$

Next simplifying the like-terms and taking the projections

$$
\begin{aligned}
& \left[\Lambda, \overline{\partial}\right](\alpha) \\
= \;& +\sqrt{-1}\iota_{V_j}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) - \sqrt{-1}\iota_{\overline{V}_j}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right) \\
& +\sqrt{-1}\left\langle\nabla_{\overline{V}_j}V_k, \overline{V}_i\right\rangle\overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{V_i}(\alpha)\right) + \sqrt{-1}\left\langle\nabla_{\overline{V}_j}\overline{V}_k, V_i\right\rangle\overline{\omega}^j \wedge \iota_{\overline{V}_i}\left(\iota_{V_k}(\alpha)\right) \\
& +\sqrt{-1}\left\langle\nabla_{V_j}\overline{V}_k, \overline{V}_i\right\rangle\omega^j \wedge \iota_{V_i}\left(\iota_{V_k}(\alpha)\right)
\end{aligned}
$$

Finally we obtain

$$
\begin{aligned}
& \left[\Lambda, \overline{\partial}\right](\alpha) \\
= \;& +\sqrt{-1}\iota_{V_j}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) - \sqrt{-1}\iota_{\overline{V}_j}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right) \\
& +\sqrt{-1}\left\langle\nabla_{V_j}\overline{V}_k, \overline{V}_i\right\rangle\omega^j \wedge \iota_{V_i}\left(\iota_{V_k}(\alpha)\right)
\end{aligned}
$$

1.3.11 Generalized Kähler identities

Let $\Omega = \sqrt{-1}\omega^i \wedge \overline{\omega}^i$ be the fundamental $(1,1)-$form on M. By the results of Le Potier [Le Potier] in his thesis, the Kähler identities generalize to the following on a non-Kähler complex manifold: [Demailly]

$$
\begin{aligned}
\left[\partial^*, L\right](\alpha) &= -\sqrt{-1}\overline{\partial}(\alpha) + \sqrt{-1}\left[L_{\overline{\partial}\Omega}, \Lambda\right](\alpha) \\
\left[\overline{\partial}^*, L\right](\alpha) &= +\sqrt{-1}\partial(\alpha) - \sqrt{-1}\left[L_{\partial\Omega}, \Lambda\right](\alpha) \\
\left[\Lambda, \partial\right](\alpha) &= +\sqrt{-1}\overline{\partial}^*(\alpha) - \sqrt{-1}\left[L, \left(L_{\overline{\partial}\Omega}\right)^*\right](\alpha) \\
\left[\Lambda, \overline{\partial}\right](\alpha) &= -\sqrt{-1}\partial^*(\alpha) + \sqrt{-1}\left[L, \left(L_{\partial\Omega}\right)^*\right](\alpha)
\end{aligned}
$$

1.3.12 Adjoint Derivative operators

Using the Kähler identities, we define the adjoint derivative operators as

$$\begin{aligned}[\Lambda,\partial](\alpha) &= \sqrt{-1}\overline{\partial}^*(\alpha)-\sqrt{-1}\left[L,\left(L_{\overline{\partial}\Omega}\right)^*\right](\alpha)\\ &\Rightarrow\ \overline{\partial}^*(\alpha):=-\sqrt{-1}[\Lambda,\partial](\alpha)+\left[L,\left(L_{\overline{\partial}\Omega}\right)^*\right](\alpha)\\ \left[\Lambda,\overline{\partial}\right](\alpha) &= -\sqrt{-1}\partial^*(\alpha)+\sqrt{-1}\left[L,\left(L_{\partial\Omega}\right)^*\right](\alpha)\\ &\Rightarrow\ \partial^*(\alpha):=\sqrt{-1}\left[\Lambda,\overline{\partial}\right](\alpha)+\left[L,\left(L_{\partial\Omega}\right)^*\right](\alpha)\end{aligned}$$

So if we write the adjoint derivative operators explicitly, we get

$$\overline{\partial}^*(\alpha):=-\sqrt{-1}[\Lambda,\partial](\alpha)+\left[L,\left(L_{\overline{\partial}\Omega}\right)^*\right](\alpha).$$

Also

$$\begin{aligned}\overline{\partial}^*(\alpha) &= -\sqrt{-1}[\Lambda,\partial](\alpha)+\left[L,\left(L_{\overline{\partial}\Omega}\right)^*\right](\alpha)\\ &= +\iota_{V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)-\iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\\ &+2\left\langle\nabla_{\overline{V}_j}V_j,V_i\right\rangle\iota_{\overline{V}_i}(\alpha)+2\left\langle\nabla_{\overline{V}_j}V_i,V_k\right\rangle\omega^i\wedge\iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right)\end{aligned}\tag{1.9}$$

Similarly

$$\partial^*(\alpha):=\sqrt{-1}\left[\Lambda,\overline{\partial}\right](\alpha)+\left[L,\left(L_{\partial\Omega}\right)^*\right](\alpha)$$

implies

$$\begin{aligned}\partial^*(\alpha) &= \sqrt{-1}\left[\Lambda,\overline{\partial}\right](\alpha)+\left[L,\left(L_{\partial\Omega}\right)^*\right](\alpha)\\ &= -\iota_{V_j}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)+\iota_{\overline{V}_j}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\\ &+2\left\langle\nabla_{V_j}\overline{V}_j,\overline{V}_i\right\rangle\iota_{V_i}(\alpha)+2\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\overline{\omega}^k\wedge\iota_{\overline{V}_j}\left(\iota_{V_i}(\alpha)\right)\end{aligned}\tag{1.10}$$

1.3.13 The Kähler $\overline{\partial}$ operator

Then let's define the Kähler $\overline{\partial}$ operator $\overline{\partial}_K : E^{p,q}\to E^{p,q+1}$ as the commutator of these two operators as

$$\overline{\partial}_K(\alpha):=[\partial^*,L](\alpha)=-\sqrt{-1}\,\overline{\partial}(\alpha)+\sqrt{-1}\left[L_{\overline{\partial}\Omega},\Lambda\right](\alpha)$$

and since

$$\begin{aligned}\left[L_{\overline{\partial}\Omega},\Lambda\right](\alpha) &= +2\left\langle\nabla_{V_j}\overline{V}_j,\overline{V}_k\right\rangle\overline{\omega}^k\wedge\alpha\\ &\quad+2\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\omega^j\wedge\overline{\omega}^k\wedge\iota_{V_i}(\alpha)\\ &\quad-\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\overline{\omega}^k\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_j}(\alpha)\end{aligned}$$

Thus

$$\overline{\partial}_K(\alpha) = -\sqrt{-1}\overline{\omega}^j \wedge proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right) \tag{1.11}$$
$$-\sqrt{-1}\omega^j \wedge proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)$$
$$+2\sqrt{-1}\left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \alpha + 2\sqrt{-1}\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \omega^j \wedge \overline{\omega}^k \wedge \iota_{V_i}(\alpha)$$
$$-\sqrt{-1}\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_j}(\alpha)$$

Or alternatively,

$$\overline{\partial}_K(\alpha) := [\partial^*, L](\alpha) = -\sqrt{-1}\partial(\alpha) + \sqrt{-1}\left[L_{\overline{\partial}\Omega}, \Lambda\right](\alpha)$$

By the equations [1.5] and [1.10], the commutator of these operators would be

$$[\partial^*, L](\alpha) = \partial^* \circ L(\alpha) - L \circ \partial^*(\alpha) = \partial^*\left(\sqrt{-1}\omega^l \wedge \overline{\omega}^l \wedge \alpha\right)$$
$$-L\left(-\iota_{V_j}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) + \iota_{\overline{V}_j}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right)$$
$$-L\left(+2\left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_i\right\rangle \iota_{V_i}(\alpha) + 2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_j}\left(\iota_{V_i}(\alpha)\right)\right)$$

$$[\partial^*, L](\alpha) = -\iota_{V_j}\left(proj_{p+1,q+1}\left(\nabla_{\overline{V}_j}\left(\sqrt{-1}\omega^l \wedge \overline{\omega}^l \wedge \alpha\right)\right)\right)$$
$$+\iota_{\overline{V}_j}\left(proj_{p,q+2}\left(\nabla_{V_j}\left(\sqrt{-1}\omega^l \wedge \overline{\omega}^l \wedge \alpha\right)\right)\right)$$
$$+2\left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_i\right\rangle \iota_{V_i}\left(\sqrt{-1}\omega^l \wedge \overline{\omega}^l \wedge \alpha\right)$$
$$+2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_j}\left(\iota_{V_i}\left(\sqrt{-1}\omega^l \wedge \overline{\omega}^l \wedge \alpha\right)\right)$$
$$-\sqrt{-1}\omega^l \wedge \overline{\omega}^l \wedge \left(-\iota_{V_j}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) + \iota_{\overline{V}_j}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right)$$
$$-\sqrt{-1}\omega^l \wedge \overline{\omega}^l \wedge \left(+2\left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_i\right\rangle \iota_{V_i}(\alpha) + 2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_j}\left(\iota_{V_i}(\alpha)\right)\right)$$

We separate the covariant derivative terms into (1,0) and (0,1) directions and apply the interior derivatives and take projections to get

$$
\begin{aligned}
& [\partial^*, L](\alpha) \\
= \quad & -\sqrt{-1}\iota_{V_j}\left(\left\langle \nabla_{\overline{V}_j}\omega^l, \overline{\omega}^k\right\rangle \omega^k \wedge \overline{\omega}^l \wedge \alpha\right) \\
& -\sqrt{-1}\iota_{V_j}\left(\omega^l \wedge \left\langle \nabla_{\overline{V}_j}\overline{\omega}^l, \omega^k\right\rangle \overline{\omega}^k \wedge \alpha\right) \\
& -\sqrt{-1}\iota_{V_j}\left(\omega^l \wedge \overline{\omega}^l \wedge \left(proj_{p,q}\left(\nabla_{\overline{V}_j}\alpha\right)\right)\right) \\
& +\sqrt{-1}\iota_{\overline{V}_j}\left(\left\langle \nabla_{V_j}\omega^l, \omega^k\right\rangle \overline{\omega}^k \wedge \overline{\omega}^l \wedge \alpha\right) \\
& +\sqrt{-1}\iota_{\overline{V}_j}\left(\omega^l \wedge \overline{\omega}^l \wedge \left(proj_{p-1,q+1}\left(\nabla_{V_j}\alpha\right)\right)\right) \\
& +2\sqrt{-1}\left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_i\right\rangle \overline{\omega}^i \wedge \alpha + 2\sqrt{-1}\left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_i\right\rangle \omega^l \wedge \overline{\omega}^l \wedge \iota_{V_i}(\alpha) \\
& +2\sqrt{-1}\left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \alpha - 2\sqrt{-1}\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_j}(\alpha) \\
& -2\sqrt{-1}\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \omega^j \wedge \iota_{V_i}(\alpha) \\
& +2\sqrt{-1}\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \omega^l \wedge \overline{\omega}^l \wedge \iota_{\overline{V}_j}\left(\iota_{V_i}(\alpha)\right) \\
& +\sqrt{-1}\omega^l \wedge \overline{\omega}^l \wedge \iota_{V_j}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) \\
& -\sqrt{-1}\omega^l \wedge \overline{\omega}^l \wedge \iota_{\overline{V}_j}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right) \\
& -2\sqrt{-1}\left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_i\right\rangle \omega^l \wedge \overline{\omega}^l \wedge \iota_{V_i}(\alpha) \\
& -2\sqrt{-1}\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \omega^l \wedge \overline{\omega}^l \wedge \overline{\omega}^k \wedge \iota_{\overline{V}_j}\left(\iota_{V_i}(\alpha)\right)
\end{aligned}
$$

Finally simplifying this gives

$$
\begin{aligned}
\overline{\partial}_K(\alpha) &:= [\partial^*, L](\alpha) \\
&= -\sqrt{-1}\overline{\omega}^j \wedge proj_{p,q}\left(\nabla_{\overline{V}_j}\alpha\right) - \sqrt{-1}\omega^j \wedge proj_{p-1,q+1}\left(\nabla_{V_j}\alpha\right) \qquad (1.12) \\
&+ 2\sqrt{-1}\left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_i\right\rangle \overline{\omega}^i \wedge \alpha - \sqrt{-1}\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_j}(\alpha) \\
&+ 2\sqrt{-1}\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \omega^j \wedge \overline{\omega}^k \wedge \iota_{V_i}(\alpha) \\
&= -\sqrt{-1}\,\overline{\partial}(\alpha) + \sqrt{-1}\left[L_{\overline{\partial}\Omega}, \Lambda\right](\alpha).
\end{aligned}
$$

1.3.14 The adjoint Kähler $\overline{\partial}$ operator

Define the adjoint Kähler $\overline{\partial}$ operator $\overline{\partial}_K^* : E^{p,q} \to E^{p,q-1}$ as the commutator of the two operators Λ, ∂ as

$$\overline{\partial}_K^*(\alpha) := [\Lambda, \partial](\alpha) = \Lambda(\partial(\alpha)) - \partial(\Lambda(\alpha))$$

then by [*]

$$
\begin{aligned}
\overline{\partial}_K^*(\alpha) \quad &= \quad [\Lambda, \partial](\alpha) \qquad (1.13) \\
&= \quad +\sqrt{-1}\iota_{V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) - \sqrt{-1}\iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right) \\
&\quad\quad +\sqrt{-1}\left\langle \nabla_{\overline{V}_j}V_k, V_i\right\rangle \overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right).
\end{aligned}
$$

Chapter 2

A Weitzenbock Formula for Compact Complex Manifolds

2.1 Hodge, Kähler and rough Laplacians of a $(p,q)-$form

Definition 2.1.1 *Using these operators defined in Chapter 1, we can define Laplacians* $\Delta_H, \Delta_K, \Delta_R : E^{p,q} \to E^{p,q}$. *More precisely, for any* $\alpha \in E^{p,q}$,

$$\begin{aligned} \Delta_K(\alpha) &= ([\partial^*, L] \circ [\Lambda, \partial] + [\Lambda, \partial] \circ [\partial^*, L])(\alpha) \\ \Delta_H(\alpha) &= \left(\overline{\partial} \circ \overline{\partial}^* + \overline{\partial}^* \circ \overline{\partial}\right)(\alpha) \\ \Delta_R(\alpha) &= proj_{p,q}\left(\left(-\nabla_{V_i}\nabla_{\overline{V}_i} + \nabla_{\nabla_{V_i}\overline{V}_i}\right)(\alpha)\right) \end{aligned} \tag{2.1}$$

2.1.1 The Hodge Laplacian of a $(p,q)-$form

Let $\alpha \in E^{p,q}$ be any $(p,q)-$form, using the definitions for $\overline{\partial}$ and $\overline{\partial}^*$ in equation [1.4] and equation [1.9], the Hodge Laplacian of the $(p,q)-$form would be

$$\begin{aligned} &\Delta_H(\alpha) = \overline{\partial}^* \circ \overline{\partial}(\alpha) + \overline{\partial} \circ \overline{\partial}^*(\alpha) \\ &= -\overline{\omega}^a \wedge \iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (H1) \\ &+ \overline{\omega}^j \wedge \iota_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{V_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (H2) \\ &- \overline{\omega}^j \wedge \iota_{V_a}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (H3) \\ &+ \overline{\omega}^a \wedge \iota_{V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_a}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (H4) \end{aligned} \tag{2.2}$$

$$- \omega^a \wedge \iota_{\overline{V}_j} \left(proj_{p-1,q+1}\left(\nabla_{V_a}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (H5)$$

$$+ \omega^j \wedge \iota_{\overline{V}_a} \left(proj_{p-1,q+1}\left(\nabla_{V_a}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (H6)$$

$$- \omega^j \wedge \iota_{V_a} \left(proj_{p,q}\left(\nabla_{\overline{V}_a}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (H7)$$

$$+ \omega^a \wedge \iota_{V_j} \left(proj_{p,q}\left(\nabla_{V_a}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (H8)$$

$$- proj_{p,q}\left(\nabla_{V_j}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (H9)$$

$$+ proj_{p,q}\left(\nabla_{\overline{V}_j}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (H10)$$

$$+ 2\left\langle \nabla_{\overline{V}_a}\nabla_{\overline{V}_j}V_j, V_i\right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_i}(\alpha) \quad (H11)$$

$$+ 2\left\langle \nabla_{\overline{V}_a}\nabla_{\overline{V}_j}V_i, V_k\right\rangle \overline{\omega}^a \wedge \omega^i \wedge \iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right) \quad (H12)$$

$$+ \left\langle \nabla_{\overline{V}_a}V_a, V_j\right\rangle proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right) \quad (H13)$$

$$- \left\langle \nabla_{V_k}V_j, \overline{V}_k\right\rangle proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right) \quad (H14)$$

$$+ 2\left\langle \nabla_{\overline{V}_j}V_b, V_c\right\rangle \omega^b \wedge \iota_{\overline{V}_c}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right) \quad (H15)$$

$$+ \left\langle \nabla_{\overline{V}_a}V_j, V_k\right\rangle \omega^k \wedge \iota_{V_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) \quad (H16)$$

$$- \left\langle \nabla_{V_a}\overline{V}_j, \overline{V}_k\right\rangle \omega^a \wedge \iota_{V_k}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right) \quad (H17)$$

$$+ \left\langle \nabla_{V_a}\overline{V}_j, V_k\right\rangle \omega^k \wedge \iota_{\overline{V}_a}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right) \quad (H18)$$

$$+ \left\langle \nabla_{\overline{V}_a}V_j, \overline{V}_k\right\rangle \overline{\omega}^a \wedge \iota_{V_k}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) \quad (H19)$$

$$- \left\langle \nabla_{\overline{V}_a}\overline{V}_j, V_k\right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_k}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right) \quad (H20)$$

$$+ \left\langle \nabla_{V_a}V_j, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) \quad (H21)$$

$$+ 2\left\langle \nabla_{\overline{V}_j}V_j, \overline{V}_k\right\rangle \left\langle \nabla_{\overline{V}_a}V_i, V_k\right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_i}(\alpha) \quad (H22)$$

$$+ 2\left\langle \nabla_{\overline{V}_j}V_j, V_i\right\rangle \left\langle \nabla_{V_a}\overline{V}_i, \overline{V}_k\right\rangle \omega^a \wedge \iota_{V_k}(\alpha) \quad (H23)$$

$$+ 2\left\langle \nabla_{\overline{V}_j}V_i, \overline{V}_b\right\rangle \left\langle \nabla_{\overline{V}_a}V_k, V_b\right\rangle \overline{\omega}^a \wedge \omega^i \wedge \iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right) \quad (H24)$$

$$+ 2\left\langle \nabla_{\overline{V}_j}V_b, V_k\right\rangle \left\langle \nabla_{\overline{V}_a}\overline{V}_b, V_i\right\rangle \overline{\omega}^a \wedge \omega^i \wedge \iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right) \quad (H25)$$

$$+ 2\left\langle \nabla_{\overline{V}_j}V_i, V_k\right\rangle \left\langle \nabla_{\overline{V}_a}V_j, \overline{V}_b\right\rangle \overline{\omega}^a \wedge \omega^i \wedge \iota_{\overline{V}_k}\left(\iota_{V_b}(\alpha)\right) \quad (H26)$$

$$+ 2\left\langle \nabla_{\overline{V}_j}V_i, V_k\right\rangle \left\langle \nabla_{V_a}\overline{V}_i, \overline{V}_b\right\rangle \omega^a \wedge \overline{\omega}^b \wedge \iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right) \quad (H27)$$

$$+ 2\left\langle \nabla_{\overline{V}_j}V_i, V_k\right\rangle \left\langle \nabla_{V_a}\overline{V}_k, \overline{V}_b\right\rangle \omega^a \wedge \omega^i \wedge \left(\iota_{V_b}\left(\iota_{V_j}(\alpha)\right)\right) \quad (H28)$$

See the appendix A.1 for the details of the computation which gives the expression above.

Positive Semi-Definiteness of the Hodge Laplacian

Let $\alpha \in E^{p,q}$, then we have

$$\begin{aligned}
(\Delta_H \alpha, \alpha)_{L^2} &= \left(\overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^*\alpha, \alpha\right)_{L^2} \\
&= \left(\overline{\partial}^*\overline{\partial}\alpha, \alpha\right)_{L^2} + \left(\overline{\partial}\overline{\partial}^*\alpha, \alpha\right)_{L^2} \\
&= \left(\overline{\partial}\alpha, \overline{\partial}\alpha\right)_{L^2} + \left(\overline{\partial}^*\alpha, \overline{\partial}^*\alpha\right)_{L^2} \geq 0
\end{aligned}$$

2.1.2 The Kähler Laplacian of a $(p, q)-$form

Let $\alpha \in E^{p,q}$ be any $(p, q)-$form, using the definitions for $\overline{\partial}_K$ and $\overline{\partial}_K^*$ in equation [1.11] and equation [1.13], the Kähler Laplacian would be given by

$$\begin{aligned}
\Delta_K(\alpha) &= \overline{\partial}_K\left(\overline{\partial}_K^*(\alpha)\right) + \overline{\partial}_K^*\left(\overline{\partial}_K(\alpha)\right) \qquad (2.3) \\
&= +\overline{\omega}^a \wedge \iota_{V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_a}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (K1) \\
&\quad -\overline{\omega}^a \wedge \iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (K2) \\
&\quad -\overline{\omega}^j \wedge \iota_{V_a}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (K3) \\
&\quad +\overline{\omega}^j \wedge \iota_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{V_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (K4) \\
&\quad +\omega^a \wedge \iota_{V_j}\left(proj_{p,q}\left(\nabla_{V_a}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (K5) \\
&\quad -\omega^a \wedge \iota_{\overline{V}_j}\left(proj_{p-1,q+1}\left(\nabla_{V_a}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (K6) \\
&\quad -\omega^j \wedge \iota_{V_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_a}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (K7) \\
&\quad +\omega^j \wedge \iota_{\overline{V}_a}\left(proj_{p-1,q+1}\left(\nabla_{V_a}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (K8) \\
&\quad -proj_{p,q}\left(\nabla_{V_j}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (K9) \\
&\quad +proj_{p,q}\left(\nabla_{\overline{V}_j}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (K10)
\end{aligned}$$

$$+2\left\langle\nabla_{V_k}\nabla_{V_j}\overline{V}_j,\overline{V}_k\right\rangle\alpha\quad(K11)$$
$$-2\left\langle\nabla_{V_a}\nabla_{V_j}\overline{V}_j,\overline{V}_k\right\rangle\overline{\omega}^k\wedge\iota_{\overline{V}_a}(\alpha)\quad(K12)$$
$$-2\left\langle\nabla_{V_k}\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\omega^j\wedge\iota_{V_i}(\alpha)\quad(K13)$$
$$-\left\langle\nabla_{V_k}\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\overline{\omega}^i\wedge\iota_{\overline{V}_j}(\alpha)\quad(K14)$$
$$+\left\langle\nabla_{\overline{V}_a}\nabla_{\overline{V}_j}V_k,V_i\right\rangle\overline{\omega}^a\wedge\overline{\omega}^j\wedge\iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right)\quad(K15)$$
$$+\left\langle\nabla_{V_i}\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\overline{\omega}^k\wedge\iota_{\overline{V}_j}(\alpha)\quad(K16)$$
$$-\left\langle\nabla_{V_a}\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\overline{\omega}^k\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}(\alpha)\right)\quad(K17)$$
$$+2\left\langle\nabla_{V_a}\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\omega^j\wedge\overline{\omega}^k\wedge\iota_{\overline{V}_a}\left(\iota_{V_i}(\alpha)\right)\quad(K18)$$
$$+\left\langle\nabla_{\overline{V}_a}V_j,V_a\right\rangle proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\quad(K19)$$
$$-\left\langle\nabla_{V_a}V_j,\overline{V}_a\right\rangle proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\quad(K20)$$
$$+2\left\langle\nabla_{V_j}\overline{V}_j,\overline{V}_a\right\rangle proj_{p,q}\left(\nabla_{V_a}\alpha\right)\quad(K21)$$
$$-\left\langle\nabla_{\overline{V}_a}V_j,V_b\right\rangle\omega^b\wedge\iota_{V_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\quad(K22)$$
$$-\left\langle\nabla_{V_a}\overline{V}_j,\overline{V}_b\right\rangle\omega^a\wedge\iota_{V_b}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\quad(K23)$$
$$-2\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_a\right\rangle\omega^j\wedge\iota_{V_i}\left(proj_{p,q}\left(\nabla_{V_a}(\alpha)\right)\right)\quad(K24)$$
$$-\left\langle\nabla_{\overline{V}_a}\overline{V}_j,V_b\right\rangle\overline{\omega}^a\wedge\iota_{\overline{V}_b}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\quad(K25)$$
$$+\left\langle\nabla_{V_a}V_j,\overline{V}_b\right\rangle\overline{\omega}^b\wedge\iota_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\quad(K26)$$
$$+\left\langle\nabla_{\overline{V}_a}V_c,V_j\right\rangle\overline{\omega}^a\wedge\iota_{\overline{V}_c}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\quad(K27)$$
$$-\left\langle\nabla_{\overline{V}_a}V_j,V_b\right\rangle\overline{\omega}^a\wedge\iota_{\overline{V}_b}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\quad(K28)$$
$$-\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_a\right\rangle\overline{\omega}^i\wedge\iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{V_a}(\alpha)\right)\right)\quad(K29)$$
$$+\left\langle\nabla_{V_j}\overline{V}_a,\overline{V}_k\right\rangle\overline{\omega}^k\wedge\iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{V_a}(\alpha)\right)\right)\quad(K30)$$
$$-2\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\overline{\omega}^k\wedge\iota_{V_i}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\quad(K31)$$
$$+\left\langle\nabla_{\overline{V}_a}V_j,\overline{V}_b\right\rangle\overline{\omega}^a\wedge\iota_{V_b}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\quad(K32)$$
$$+\left\langle\nabla_{V_a}\overline{V}_j,V_b\right\rangle\omega^b\wedge\iota_{\overline{V}_a}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\quad(K33)$$
$$-2\left\langle\nabla_{V_j}\overline{V}_j,\overline{V}_k\right\rangle\left\langle\nabla_{\overline{V}_a}V_k,V_a\right\rangle\alpha\quad(K34)$$
$$+2\left\langle\nabla_{V_j}\overline{V}_j,V_b\right\rangle\left\langle\nabla_{V_k}\overline{V}_k,\overline{V}_b\right\rangle\alpha\quad(K35)$$
$$+2\left\langle\nabla_{V_j}\overline{V}_j,\overline{V}_k\right\rangle\left\langle\nabla_{\overline{V}_a}V_k,V_b\right\rangle\omega^b\wedge\iota_{V_a}(\alpha)\quad(K36)$$
$$-2\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\left\langle\nabla_{\overline{V}_j}V_k,V_b\right\rangle\omega^b\wedge\iota_{V_i}(\alpha)\quad(K37)$$
$$+2\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\left\langle\nabla_{\overline{V}_a}V_k,V_a\right\rangle\omega^j\wedge\iota_{V_i}(\alpha)\quad(K38)$$
$$-2\left\langle\nabla_{V_j}\overline{V}_i,V_b\right\rangle\left\langle\nabla_{V_k}\overline{V}_k,\overline{V}_b\right\rangle\omega^j\wedge\iota_{V_i}(\alpha)\quad(K39)$$
$$-2\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\left\langle\nabla_{V_k}\overline{V}_j,V_b\right\rangle\omega^b\wedge\iota_{V_i}(\alpha)\quad(K40)$$

$$-2\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_a\right\rangle\left\langle \nabla_{V_a}V_i,\overline{V}_b\right\rangle\omega^j\wedge\iota_{V_b}(\alpha)\quad (K41)$$
$$-2\left\langle \nabla_{V_j}\overline{V}_j,V_b\right\rangle\left\langle \nabla_{V_a}\overline{V}_k,\overline{V}_b\right\rangle\overline{\omega}^k\wedge\iota_{\overline{V}_a}(\alpha)\quad (K42)$$
$$-2\left\langle \nabla_{\overline{V}_a}V_c,V_k\right\rangle\left\langle \nabla_{V_j}\overline{V}_j,\overline{V}_k\right\rangle\overline{\omega}^a\wedge\iota_{\overline{V}_c}(\alpha)\quad (K43)$$
$$+2\left\langle \nabla_{\overline{V}_a}V_k,V_b\right\rangle\left\langle \nabla_{V_j}\overline{V}_j,\overline{V}_k\right\rangle\overline{\omega}^a\wedge\iota_{\overline{V}_b}(\alpha)\quad (K44)$$
$$-2\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\left\langle \nabla_{\overline{V}_j}V_i,V_b\right\rangle\overline{\omega}^k\wedge\iota_{\overline{V}_b}(\alpha)\quad (K45)$$
$$+\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\left\langle \nabla_{\overline{V}_a}V_k,V_a\right\rangle\overline{\omega}^i\wedge\iota_{\overline{V}_j}(\alpha)\quad (K46)$$
$$-\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\left\langle \nabla_{\overline{V}_a}V_i,V_a\right\rangle\overline{\omega}^k\wedge\iota_{\overline{V}_j}(\alpha)\quad (K47)$$
$$-\left\langle \nabla_{V_j}\overline{V}_i,V_b\right\rangle\left\langle \nabla_{V_k}\overline{V}_k,\overline{V}_b\right\rangle\overline{\omega}^i\wedge\iota_{\overline{V}_j}(\alpha)\quad (K48)$$
$$+\left\langle \nabla_{V_j}\overline{V}_i,V_b\right\rangle\left\langle \nabla_{V_i}\overline{V}_k,\overline{V}_b\right\rangle\overline{\omega}^k\wedge\iota_{\overline{V}_j}(\alpha)\quad (K49)$$
$$-\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\left\langle \nabla_{V_k}V_i,\overline{V}_b\right\rangle\overline{\omega}^b\wedge\iota_{\overline{V}_j}(\alpha)\quad (K50)$$
$$+\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\left\langle \nabla_{V_a}V_i,\overline{V}_a\right\rangle\overline{\omega}^k\wedge\iota_{\overline{V}_j}(\alpha)\quad (K51)$$
$$-\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_a\right\rangle\left\langle \nabla_{V_a}\overline{V}_j,V_b\right\rangle\overline{\omega}^i\wedge\iota_{\overline{V}_b}(\alpha)\quad (K52)$$
$$+\left\langle \nabla_{V_j}\overline{V}_a,\overline{V}_k\right\rangle\left\langle \nabla_{V_a}\overline{V}_j,V_b\right\rangle\overline{\omega}^k\wedge\iota_{\overline{V}_b}(\alpha)\quad (K53)$$
$$+2\left\langle \nabla_{\overline{V}_a}V_i,V_k\right\rangle\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\overline{\omega}^a\wedge\iota_{\overline{V}_j}(\alpha)\quad (K54)$$
$$+\left\langle \nabla_{\overline{V}_j}V_k,V_i\right\rangle\left\langle \nabla_{\overline{V}_a}V_j,\overline{V}_b\right\rangle\overline{\omega}^a\wedge\overline{\omega}^b\wedge\iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right)\quad (K55)$$
$$+\left\langle \nabla_{V_j}\overline{V}_b,\overline{V}_c\right\rangle\left\langle \nabla_{\overline{V}_j}V_k,V_i\right\rangle\overline{\omega}^c\wedge\overline{\omega}^b\wedge\iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right)\quad (K56)$$
$$-\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\left\langle \nabla_{V_a}\overline{V}_j,V_b\right\rangle\overline{\omega}^k\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_a}\left(\iota_{\overline{V}_b}(\alpha)\right)\quad (K57)$$
$$-2\left\langle \nabla_{\overline{V}_a}V_c,V_k\right\rangle\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\overline{\omega}^a\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_c}\left(\iota_{\overline{V}_j}(\alpha)\right)\quad (K58)$$
$$+2\left\langle \nabla_{\overline{V}_a}V_k,V_b\right\rangle\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\overline{\omega}^a\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_b}\left(\iota_{\overline{V}_j}(\alpha)\right)\quad (K59)$$
$$+\left\langle \nabla_{\overline{V}_j}V_k,V_i\right\rangle\left\langle \nabla_{V_a}\overline{V}_k,\overline{V}_b\right\rangle\omega^a\wedge\overline{\omega}^j\wedge\iota_{V_b}\left(\iota_{\overline{V}_i}(\alpha)\right)\quad (K60)$$
$$+\left\langle \nabla_{\overline{V}_j}V_k,V_i\right\rangle\left\langle \nabla_{V_a}\overline{V}_i,\overline{V}_b\right\rangle\omega^a\wedge\overline{\omega}^j\wedge\iota_{\overline{V}_k}\left(\iota_{V_b}(\alpha)\right)\quad (K61)$$
$$-2\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\left\langle \nabla_{\overline{V}_a}V_i,V_b\right\rangle\omega^j\wedge\overline{\omega}^k\wedge\iota_{V_a}\left(\iota_{\overline{V}_b}(\alpha)\right)\quad (K62)$$
$$+2\left\langle \nabla_{V_j}\overline{V}_i,V_b\right\rangle\left\langle \nabla_{V_a}\overline{V}_k,\overline{V}_b\right\rangle\omega^j\wedge\overline{\omega}^k\wedge\iota_{\overline{V}_a}\left(\iota_{V_i}(\alpha)\right)\quad (K63)$$
$$+2\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\left\langle \nabla_{V_a}\overline{V}_j,V_b\right\rangle\omega^b\wedge\overline{\omega}^k\wedge\iota_{\overline{V}_a}\left(\iota_{V_i}(\alpha)\right)\quad (K64)$$
$$+2\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\left\langle \nabla_{V_a}V_i,\overline{V}_b\right\rangle\omega^j\wedge\overline{\omega}^k\wedge\iota_{\overline{V}_a}\left(\iota_{V_b}(\alpha)\right)\quad (K65)$$
$$-2\left\langle \nabla_{\overline{V}_a}V_c,V_b\right\rangle\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_b\right\rangle\overline{\omega}^a\wedge\omega^j\wedge\iota_{\overline{V}_c}\left(\iota_{V_i}(\alpha)\right)\quad (K66)$$
$$+2\left\langle \nabla_{\overline{V}_a}V_k,V_b\right\rangle\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\overline{\omega}^a\wedge\omega^j\wedge\iota_{\overline{V}_b}\left(\iota_{V_i}(\alpha)\right)\quad (K67)$$
$$+\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\left\langle \nabla_{\overline{V}_a}V_k,V_b\right\rangle\omega^b\wedge\overline{\omega}^i\wedge\iota_{V_a}\left(\iota_{\overline{V}_j}(\alpha)\right)\quad (K68)$$
$$+\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\left\langle \nabla_{\overline{V}_a}V_i,V_b\right\rangle\overline{\omega}^k\wedge\omega^b\wedge\iota_{V_a}\left(\iota_{\overline{V}_j}(\alpha)\right)\quad (K69)$$
$$-2\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\left\langle \nabla_{\overline{V}_a}V_k,V_b\right\rangle\omega^j\wedge\omega^b\wedge\iota_{V_a}\left(\iota_{V_i}(\alpha)\right)\quad (K70)$$

See the appendix A.2 for the details of the computation which gives the expression above.

Positive Semi-Definiteness of the Kähler Laplacian

Let $\alpha \in E^{p,q}$, then we have

$$\begin{aligned}
(\Delta_K \alpha, \alpha)_{L^2} &= \left(\overline{\partial}_K \circ \overline{\partial}_K^* + \overline{\partial}_K^* \circ \overline{\partial}_K\right)(\alpha) \\
&= ([\partial^*, L] \circ [\Lambda, \partial]\, \alpha, \alpha)_{L^2} + ([\Lambda, \partial] \circ [\partial^*, L]\, \alpha, \alpha)_{L^2} \\
&= (L \circ [\Lambda, \partial]\, \alpha, \partial\alpha)_{L^2} - (\partial^* \circ [\Lambda, \partial]\, \alpha, \Lambda\alpha)_{L^2} \\
&\quad + (\partial \circ [\partial^*, L]\, \alpha, L\alpha)_{L^2} - (\Lambda \circ [\partial^*, L]\, \alpha, \partial^*\alpha)_{L^2} \\
&= ([\Lambda, \partial]\, \alpha, \Lambda \circ \partial\alpha)_{L^2} - ([\Lambda, \partial]\, \alpha, \partial \circ \Lambda\alpha)_{L^2} \\
&\quad + ([\partial^*, L]\, \alpha, \partial^* L\alpha)_{L^2} - ([\partial^*, L]\, \alpha, L \circ \partial^*\alpha)_{L^2} \\
&= ([\Lambda, \partial]\, \alpha, [\Lambda, \partial]\, \alpha)_{L^2} + ([\partial^*, L]\, \alpha, [\partial^*, L]\, \alpha)_{L^2} \\
&= \left(\overline{\partial}_K(\alpha), \overline{\partial}_K(\alpha)\right)_{L^2} + \left(\overline{\partial}_K^*(\alpha), \overline{\partial}_K^*(\alpha)\right)_{L^2} \geq 0.
\end{aligned}$$

2.1.3 The rough Laplacian of a $(p, q)-$form

Let $\alpha \in E^{p,q}$ be any $(p, q)-$form on M. We define the rough Laplacian of a $(p, q)-$form on M by

$$\begin{aligned}
\Delta_R(\alpha) &= +proj_{p,q}\left(-\nabla_{V_j}\left(\nabla_{\overline{V}_j}(\alpha)\right) + \nabla_{\nabla_{V_j}\overline{V}_j}(\alpha)\right) \qquad (2.4) \\
&= -proj_{p,q}\left(\nabla_{V_j}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \\
&\quad -proj_{p,q}\left(\nabla_{V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \\
&\quad +proj_{p,q}\left(\nabla_{\left\langle \nabla_{V_j}\overline{V}_j, V_k\right\rangle \overline{V}_k + \left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle V_k}(\alpha)\right) \\
&= -proj_{p,q}\left(\nabla_{V_j}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (R1) \\
&\quad -proj_{p,q}\left(\nabla_{V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (R2) \\
&\quad + \left\langle \nabla_{V_j}\overline{V}_j, V_k\right\rangle proj_{p,q}\left(\nabla_{\overline{V}_k}(\alpha)\right) \quad (R3) \\
&\quad + \left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle proj_{p,q}\left(\nabla_{V_k}(\alpha)\right) \quad (R4)
\end{aligned}$$

Positive Semi-Definiteness of the rough Laplacian

Let $\alpha \in E^{p,q}$ be any $(p, q)-$form on M. Consider the 1-form

$$\beta(X) = \langle \nabla_X \alpha, \overline{\alpha} \rangle$$

for any vector field $X \in \Gamma(M, T_* M)$. Then with respect to a $(1, 0)$ and antiholomorpic frame field $\left\{V_i, \overline{V}_i\right\}$, we can write β as

$$\beta = \langle \nabla_{V_k} \alpha, \overline{\alpha} \rangle \omega^k + \left\langle \nabla_{\overline{V}_k} \alpha, \overline{\alpha} \right\rangle \overline{\omega}^k$$

since

$$\begin{aligned}
\beta\left(V_j\right) &= \left\langle\nabla_{V_k}\alpha,\overline{\alpha}\right\rangle\omega^k\left(V_j\right)+\left\langle\nabla_{\overline{V}_k}\alpha,\overline{\alpha}\right\rangle\overline{\omega}^k\left(V_j\right)=\left\langle\nabla_{V_k}\alpha,\overline{\alpha}\right\rangle\\
\beta\left(\overline{V}_j\right) &= \left\langle\nabla_{V_k}\alpha,\overline{\alpha}\right\rangle\omega^k\left(\overline{V}_j\right)+\left\langle\nabla_{\overline{V}_k}\alpha,\overline{\alpha}\right\rangle\overline{\omega}^k\left(\overline{V}_j\right)=\left\langle\nabla_{\overline{V}_k}\alpha,\overline{\alpha}\right\rangle.
\end{aligned}$$

The adjoint $\overline{\partial}$ operator $\overline{\partial}^*: E^{0,1}\to E^{0,0}$ is defined by

$$\begin{aligned}
\overline{\partial}^*(\alpha) &= +\iota_{V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)-\iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\\
&\quad +2\left\langle\nabla_{\overline{V}_j}V_j,V_i\right\rangle\iota_{\overline{V}_i}(\alpha)
\end{aligned}$$

and which's clearly 0 for $(1,0)-$forms. So if we apply $\overline{\partial}^*$ on β, we get

$$\begin{aligned}
\overline{\partial}^*(\beta) &= +\iota_{V_j}\left(proj_{1,0}\left(\nabla_{\overline{V}_j}(\beta)\right)\right)-\iota_{\overline{V}_j}\left(proj_{0,1}\left(\nabla_{V_j}(\beta)\right)\right)\\
&\quad +2\left\langle\nabla_{\overline{V}_j}V_j,V_i\right\rangle\iota_{\overline{V}_i}(\beta)\\
&= +\iota_{V_j}\left(proj_{1,0}\left(\nabla_{\overline{V}_j}\left(\left\langle\nabla_{\overline{V}_k}\alpha,\overline{\alpha}\right\rangle\overline{\omega}^k\right)\right)\right)-\iota_{\overline{V}_j}\left(proj_{0,1}\left(\nabla_{V_j}\left(\left\langle\nabla_{\overline{V}_k}\alpha,\overline{\alpha}\right\rangle\overline{\omega}^k\right)\right)\right)\\
&\quad +2\left\langle\nabla_{\overline{V}_j}V_j,V_i\right\rangle\iota_{\overline{V}_i}\left(\left\langle\nabla_{\overline{V}_k}\alpha,\overline{\alpha}\right\rangle\overline{\omega}^k\right)\\
&= +\iota_{V_j}\left(proj_{1,0}\left(\left\langle\nabla_{\overline{V}_j}\nabla_{\overline{V}_k}\alpha,\overline{\alpha}\right\rangle\overline{\omega}^k\right)\right)+\iota_{V_j}\left(proj_{1,0}\left(\left\langle\nabla_{\overline{V}_k}\alpha,\nabla_{\overline{V}_j}\overline{\alpha}\right\rangle\overline{\omega}^k\right)\right)\\
&\quad +\iota_{V_j}\left(proj_{1,0}\left(\left\langle\nabla_{\overline{V}_k}\alpha,\overline{\alpha}\right\rangle\nabla_{\overline{V}_j}\overline{\omega}^k\right)\right)-\iota_{\overline{V}_j}\left(proj_{0,1}\left(\left\langle\nabla_{V_j}\nabla_{\overline{V}_k}\alpha,\overline{\alpha}\right\rangle\overline{\omega}^k\right)\right)\\
&\quad -\iota_{\overline{V}_j}\left(proj_{0,1}\left(\left\langle\nabla_{\overline{V}_k}\alpha,\nabla_{V_j}\overline{\alpha}\right\rangle\overline{\omega}^k\right)\right)-\iota_{\overline{V}_j}\left(proj_{0,1}\left(\left\langle\nabla_{\overline{V}_k}\alpha,\overline{\alpha}\right\rangle\nabla_{V_j}\overline{\omega}^k\right)\right)\\
&\quad +2\left\langle\nabla_{\overline{V}_j}V_j,V_i\right\rangle\left\langle\nabla_{\overline{V}_i}\alpha,\overline{\alpha}\right\rangle.
\end{aligned}$$

Then we have

$$\begin{aligned}
\overline{\partial}^*(\beta) &= +\delta_i^j\left\langle\nabla_{\overline{V}_k}\alpha,\overline{\alpha}\right\rangle\left\langle\nabla_{\overline{V}_j}\overline{\omega}^k,\overline{\omega}^i\right\rangle-\delta_k^j\left\langle\nabla_{V_j}\nabla_{\overline{V}_k}\alpha,\overline{\alpha}\right\rangle\\
&\quad -\delta_k^j\left\langle\nabla_{\overline{V}_k}\alpha,\nabla_{V_j}\overline{\alpha}\right\rangle-\delta_i^j\left\langle\nabla_{\overline{V}_k}\alpha,\overline{\alpha}\right\rangle\left\langle\nabla_{V_j}\overline{\omega}^k,\omega^i\right\rangle\\
&\quad +2\left\langle\nabla_{\overline{V}_j}V_j,V_i\right\rangle\left\langle\nabla_{\overline{V}_i}\alpha,\overline{\alpha}\right\rangle\\
&= +\left\langle\nabla_{\overline{V}_k}\alpha,\overline{\alpha}\right\rangle\left\langle\nabla_{\overline{V}_i}V_k,V_i\right\rangle-\left\langle\nabla_{V_j}\nabla_{\overline{V}_j}\alpha,\overline{\alpha}\right\rangle-\left\langle\nabla_{\overline{V}_j}\alpha,\nabla_{V_j}\overline{\alpha}\right\rangle\\
&\quad -\left\langle\nabla_{\overline{V}_k}\alpha,\overline{\alpha}\right\rangle\left\langle\nabla_{V_i}V_k,\overline{V}_i\right\rangle+2\left\langle\nabla_{\overline{V}_j}V_j,V_i\right\rangle\left\langle\nabla_{\overline{V}_i}\alpha,\overline{\alpha}\right\rangle\\
&= -\left\langle\nabla_{V_j}\nabla_{\overline{V}_j}\alpha,\overline{\alpha}\right\rangle-\left\langle\nabla_{\overline{V}_j}\alpha,\nabla_{V_j}\overline{\alpha}\right\rangle\\
&\quad +\left\langle\nabla_{\overline{V}_k}\alpha,\overline{\alpha}\right\rangle\left\langle\nabla_{\overline{V}_j}V_j,V_k\right\rangle-\left\langle\nabla_{\overline{V}_k}\alpha,\overline{\alpha}\right\rangle\left\langle\nabla_{V_j}V_k,\overline{V}_j\right\rangle\\
&= -\left\langle\nabla_{V_j}\nabla_{\overline{V}_j}\alpha,\overline{\alpha}\right\rangle-\left\langle\nabla_{\overline{V}_j}\alpha,\nabla_{V_j}\overline{\alpha}\right\rangle+\left\langle\nabla_{\overline{V}_k}\alpha,\overline{\alpha}\right\rangle\left\langle\nabla_{\overline{V}_j}V_j+\nabla_{V_j}\overline{V}_j,V_k\right\rangle\\
&= -\left\langle\nabla_{V_j}\nabla_{\overline{V}_j}\alpha,\overline{\alpha}\right\rangle-\left\langle\nabla_{\overline{V}_j}\alpha,\nabla_{V_j}\overline{\alpha}\right\rangle+\left\langle\nabla_{\overline{V}_k}\alpha,\overline{\alpha}\right\rangle\left\langle\nabla_{\overline{V}_j}V_j+\nabla_{V_j}\overline{V}_j,V_k\right\rangle.
\end{aligned}$$

Now, by the divergence theorem, since M is a compact manifold without boundary, we have

$$0=\int_M\left(\overline{\partial}^*\beta\right)dvol_M$$

Inserting $\overline{\partial}^*\beta$ into the formula, we get

$$\begin{aligned}0 &= \int_M \left(\overline{\partial}^*\beta\right) dvol_M \\ &= -\int_M \left\langle \nabla_{V_j}\nabla_{\overline{V}_j}\alpha, \overline{\alpha}\right\rangle dvol_M - \int_M \left\langle \nabla_{\overline{V}_j}\alpha, \nabla_{V_j}\overline{\alpha}\right\rangle dvol_M \\ &\quad + \int_M \left\langle \nabla_{\overline{V}_k}\alpha, \overline{\alpha}\right\rangle \left\langle \nabla_{\overline{V}_j}V_j + \nabla_{V_j}\overline{V}_j, V_k\right\rangle dvol_M.\end{aligned}$$

Taking the second integral to the left side above gives

$$\begin{aligned}0 &\leq \int_M \left\langle \nabla_{\overline{V}_j}\alpha, \nabla_{V_j}\overline{\alpha}\right\rangle dvol_M \\ &= \int_M \left[-\left\langle \nabla_{V_j}\nabla_{\overline{V}_j}\alpha, \overline{\alpha}\right\rangle + \left\langle \nabla_{\overline{V}_k}\alpha, \overline{\alpha}\right\rangle \left\langle \nabla_{\overline{V}_j}V_j + \nabla_{V_j}\overline{V}_j, V_k\right\rangle\right] dvol_M \\ &= \int_M \left[-\left\langle \nabla_{V_j}\nabla_{\overline{V}_j}\alpha, \overline{\alpha}\right\rangle + \left\langle \nabla_{\left\langle \nabla_{\overline{V}_j}V_j + \nabla_{V_j}\overline{V}_j, V_k\right\rangle \overline{V}_k}\alpha, \overline{\alpha}\right\rangle\right] dvol_M.\end{aligned}$$

Thus for any $\alpha \in E^{p,q}$, we have shown that the following quantity is positive semi-definite.

$$\left\langle \Delta_{\widetilde{R}}(\alpha), \overline{\alpha}\right\rangle := \left\langle proj_{p,q}\left(-\nabla_{V_j}\left(\nabla_{\overline{V}_j}\alpha\right) + \nabla_{\left\langle \nabla_{\overline{V}_j}V_j + \nabla_{V_j}\overline{V}_j, V_k\right\rangle \overline{V}_k}\alpha\right), \overline{\alpha}\right\rangle.$$

We could also write this as

$$\begin{aligned}\left\langle \Delta_{\widetilde{R}}(\alpha), \overline{\alpha}\right\rangle &= \left\langle proj_{p,q}\left(-\nabla_{V_j}\left(\nabla_{\overline{V}_j}\alpha\right) + \nabla_{\left\langle \nabla_{\overline{V}_j}V_j + \nabla_{V_j}\overline{V}_j, V_k\right\rangle \overline{V}_k}\alpha\right), \overline{\alpha}\right\rangle \\ &= \left\langle proj_{p,q}\left(-\nabla_{V_j}\left(\nabla_{\overline{V}_j}\alpha\right) + \nabla_{\left(\nabla_{\overline{V}_j}V_j + \nabla_{V_j}\overline{V}_j\right)^{(0,1)}}\alpha\right), \overline{\alpha}\right\rangle.\end{aligned}$$

However, we have defined the rough Laplacian by

$$\Delta_R(\alpha) := +proj_{p,q}\left(-\nabla_{V_j}\left(\nabla_{\overline{V}_j}(\alpha)\right) + \nabla_{\nabla_{V_j}\overline{V}_j}(\alpha)\right)$$

and we will show that the difference of $\left\langle \Delta_{\widetilde{R}}(\alpha), \overline{\alpha}\right\rangle$ and $\left\langle \Delta_R(\alpha), \overline{\alpha}\right\rangle$ is imaginary.

First, consider the Lie bracket, where we recall $V_j = \frac{1}{\sqrt{2}}\left(v_j - \sqrt{-1}Jv_j\right)$

$$\begin{aligned}\nabla_{V_j}\overline{V}_j - \nabla_{\overline{V}_j}V_j &= \left[V_j, \overline{V}_j\right] \\ &= \frac{1}{2}\left[v_j - \sqrt{-1}Jv_j, v_j + \sqrt{-1}Jv_j\right] \\ &= \frac{1}{2}[v_j, v_j] + \frac{1}{2}\sqrt{-1}[v_j, Jv_j] - \frac{1}{2}\sqrt{-1}[Jv_j, v_j] + \frac{1}{2}[Jv_j, Jv_j] \\ &= +\sqrt{-1}[v_j, Jv_j]\end{aligned}$$

which gives

$$\nabla_{\overline{V}_j}V_j = \nabla_{V_j}\overline{V}_j - \sqrt{-1}[v_j, Jv_j]$$

If we plug this into the expression for $\left\langle \Delta_{\widetilde{R}}(\alpha), \overline{\alpha} \right\rangle$ we get

$$\begin{aligned}
\left\langle \Delta_{\widetilde{R}}(\alpha), \overline{\alpha} \right\rangle &= \left\langle proj_{p,q}\left(-\nabla_{V_j}\left(\nabla_{\overline{V}_j}\alpha \right) + \nabla_{\left(\nabla_{\overline{V}_j}V_j + \nabla_{V_j}\overline{V}_j\right)^{(0,1)}}\alpha \right), \overline{\alpha} \right\rangle \\
&= \left\langle proj_{p,q}\left(-\nabla_{V_j}\left(\nabla_{\overline{V}_j}\alpha \right) + \nabla_{\left(2\nabla_{V_j}\overline{V}_j - 2\sqrt{-1}[v_j, Jv_j]\right)^{(0,1)}}\alpha \right), \overline{\alpha} \right\rangle
\end{aligned}$$

If we subtract it from

$$\left\langle \Delta_R(\alpha), \overline{\alpha} \right\rangle = \left\langle proj_{p,q}\left(-\nabla_{V_j}\left(\nabla_{\overline{V}_j}(\alpha) \right) + \nabla_{\nabla_{V_j}\overline{V}_j}(\alpha) \right), \overline{\alpha} \right\rangle$$

this gives

$$\begin{aligned}
& \left\langle \Delta_R(\alpha), \overline{\alpha} \right\rangle - \left\langle \Delta_{\widetilde{R}}(\alpha), \overline{\alpha} \right\rangle \\
= & \left\langle proj_{p,q}\left(\nabla_{\nabla_{V_j}\overline{V}_j}(\alpha) \right), \overline{\alpha} \right\rangle - \left\langle proj_{p,q}\left(\nabla_{\left(\nabla_{\overline{V}_j}V_j + \nabla_{V_j}\overline{V}_j\right)^{(0,1)}}\alpha \right), \overline{\alpha} \right\rangle \\
= & \left\langle proj_{p,q}\left(\nabla_{\left(\nabla_{\overline{V}_j}V_j\right)^{(1,0)} - \left(\nabla_{V_j}\overline{V}_j\right)^{(0,1)}}\alpha \right), \overline{\alpha} \right\rangle
\end{aligned}$$

Let $\nabla_{\overline{V}_j}V_j = a + \sqrt{-1}b$ for some $a, b \in \mathbb{R}$. Then $\nabla_{V_j}\overline{V}_j = \overline{\left(\nabla_{\overline{V}_j}V_j\right)} = a - \sqrt{-1}b$ and

$$\begin{aligned}
\left(\nabla_{\overline{V}_j}V_j\right)^{(1,0)} &= \frac{1}{\sqrt{2}}\left(a - \sqrt{-1}Ja\right) + \frac{1}{\sqrt{2}}\sqrt{-1}\left(b - \sqrt{-1}Jb\right) \\
\left(\nabla_{V_j}\overline{V}_j\right)^{(0,1)} &= \frac{1}{\sqrt{2}}\left(a + \sqrt{-1}Ja\right) - \frac{1}{\sqrt{2}}\sqrt{-1}\left(b + \sqrt{-1}Jb\right)
\end{aligned}$$

Thus the difference is equal to

$$\begin{aligned}
& \left\langle \Delta_R(\alpha), \overline{\alpha} \right\rangle - \left\langle \Delta_{\widetilde{R}}(\alpha), \overline{\alpha} \right\rangle \\
= & \left\langle proj_{p,q}\left(\nabla_{\left(\nabla_{\overline{V}_j}V_j\right)^{(1,0)} - \left(\nabla_{V_j}\overline{V}_j\right)^{(0,1)}}\alpha \right), \overline{\alpha} \right\rangle \\
= & \left\langle \nabla_{\sqrt{-1}(-Ja+b)}\alpha, \overline{\alpha} \right\rangle
\end{aligned}$$

which is an imaginary quantity. Therefore

$$\operatorname{Re}\left(\left\langle \Delta_R(\alpha), \overline{\alpha} \right\rangle - \left\langle \Delta_{\widetilde{R}}(\alpha), \overline{\alpha} \right\rangle \right) = 0 \text{ i.e. } \operatorname{Re}\left(\left\langle \Delta_R(\alpha), \overline{\alpha} \right\rangle \right) = \operatorname{Re}\left(\left\langle \Delta_{\widetilde{R}}(\alpha), \overline{\alpha} \right\rangle \right).$$

Thus we have the real part of the rough Laplacian in our Weitzenbock formula is positive semi-definite.

2.2 Commutators of Algebraic Tensors

We have defined $\left[L_{\overline{\partial}\Omega}, \Lambda\right] : E^{p,q} \to E^{p,q+1}$ in equation [1.7] and $\left[L, \left(L_{\overline{\partial}\Omega}\right)^*\right] : E^{p,q} \to E^{p,q-1}$ in equation [1.8] before. Then we have

$$
\begin{aligned}
& \left[L, \left(L_{\overline{\partial}\Omega}\right)^*\right] \left[L_{\overline{\partial}\Omega}, \Lambda\right](\alpha) + \left[L_{\overline{\partial}\Omega}, \Lambda\right] \left[L, \left(L_{\overline{\partial}\Omega}\right)^*\right](\alpha) \\
= \quad & +4 \left\langle \nabla_{\overline{V}_j} V_j, V_c \right\rangle \left\langle \nabla_{V_a} \overline{V}_a, \overline{V}_c \right\rangle \alpha \qquad (A) \\
& +4 \left\langle \nabla_{\overline{V}_j} V_j, V_b \right\rangle \left\langle \nabla_{V_a} \overline{V}_b, \overline{V}_c \right\rangle \overline{\omega}^c \wedge \iota_{\overline{V}_a}(\alpha) \qquad (B) \\
& +4 \left\langle \nabla_{\overline{V}_j} V_c, V_i \right\rangle \left\langle \nabla_{V_a} \overline{V}_a, \overline{V}_c \right\rangle \overline{\omega}^j \wedge \iota_{\overline{V}_i}(\alpha) \qquad (C) \\
& +2 \left\langle \nabla_{\overline{V}_j} V_b, V_c \right\rangle \left\langle \nabla_{V_a} \overline{V}_b, \overline{V}_c \right\rangle \overline{\omega}^j \wedge \iota_{\overline{V}_a}(\alpha) \qquad (D) \\
& -4 \left\langle \nabla_{\overline{V}_j} V_j, V_c \right\rangle \left\langle \nabla_{V_a} \overline{V}_b, \overline{V}_c \right\rangle \omega^a \wedge \iota_{V_b}(\alpha) \qquad (E) \\
& -4 \left\langle \nabla_{\overline{V}_j} V_i, V_c \right\rangle \left\langle \nabla_{V_a} \overline{V}_a, \overline{V}_c \right\rangle \omega^i \wedge \iota_{V_j}(\alpha) \qquad (F) \\
& +4 \left\langle \nabla_{\overline{V}_a} V_i, V_c \right\rangle \left\langle \nabla_{V_a} \overline{V}_b, \overline{V}_c \right\rangle \omega^i \wedge \iota_{V_b}(\alpha) \qquad (G) \\
& -4 \left\langle \nabla_{\overline{V}_j} V_i, V_c \right\rangle \left\langle \nabla_{V_a} \overline{V}_b, \overline{V}_c \right\rangle \omega^i \wedge \omega^a \wedge \iota_{V_j}\left(\iota_{V_b}(\alpha)\right) \qquad (H) \\
& -4 \left\langle \nabla_{\overline{V}_a} V_i, V_k \right\rangle \left\langle \nabla_{V_a} \overline{V}_b, \overline{V}_c \right\rangle \omega^i \wedge \overline{\omega}^c \wedge \iota_{\overline{V}_k}\left(\iota_{V_b}(\alpha)\right) \qquad (I) \\
& +4 \left\langle \nabla_{\overline{V}_j} V_i, V_c \right\rangle \left\langle \nabla_{V_a} \overline{V}_c, \overline{V}_b \right\rangle \omega^i \wedge \overline{\omega}^b \wedge \iota_{V_j}\left(\iota_{\overline{V}_a}(\alpha)\right) \qquad (J) \\
& +4 \left\langle \nabla_{\overline{V}_j} V_c, V_i \right\rangle \left\langle \nabla_{V_a} \overline{V}_b, \overline{V}_c \right\rangle \overline{\omega}^j \wedge \omega^a \wedge \iota_{\overline{V}_i}\left(\iota_{V_b}(\alpha)\right) \qquad (K) \\
& -4 \left\langle \nabla_{\overline{V}_j} V_i, V_k \right\rangle \left\langle \nabla_{V_a} \overline{V}_i, \overline{V}_c \right\rangle \overline{\omega}^c \wedge \omega^a \wedge \iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right) \qquad (L) \\
& +4 \left\langle \nabla_{\overline{V}_j} V_i, V_c \right\rangle \left\langle \nabla_{V_a} \overline{V}_c, \overline{V}_b \right\rangle \overline{\omega}^j \wedge \overline{\omega}^b \wedge \iota_{\overline{V}_i}\left(\iota_{\overline{V}_a}(\alpha)\right) \qquad (M) \\
& + \left\langle \nabla_{\overline{V}_j} V_i, V_k \right\rangle \left\langle \nabla_{V_j} \overline{V}_b, \overline{V}_c \right\rangle \overline{\omega}^c \wedge \overline{\omega}^b \wedge \iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}(\alpha)\right) \qquad (N)
\end{aligned} \tag{2.5}
$$

See the appendix A.3 for the details of the computation which gives the expression above.

If we define $T_1 : E^{p,q} \to E^{p,q-1}$ by

$$
T_1(\alpha) = \left\langle \nabla_{\overline{V}_j} V_k, V_i \right\rangle \overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right), \tag{2.6}
$$

then L^2-adjoint of T_1, $(T_1)^* : E^{p,q} \to E^{p,q+1}$ is equal to

$$
(T_1)^*(\alpha) = \left\langle \nabla_{V_j} \overline{V}_k, \overline{V}_i \right\rangle \overline{\omega}^i \wedge \overline{\omega}^k \wedge \iota_{\overline{V}_j}(\alpha).
$$

Thus we see

$$\begin{aligned} & \left[T_1, (T_1)^*\right](\alpha) && (2.7) \\ = \; & T_1 (T_1)^* (\alpha) + (T_1)^* (T_1 (\alpha)) \\ = \; & +2 \left\langle \nabla_{\overline{V}_j} V_c, V_b \right\rangle \left\langle \nabla_{V_a} \overline{V}_c, \overline{V}_b \right\rangle \overline{\omega}^j \wedge \iota_{\overline{V}_a} (\alpha) \qquad (O) \\ & +4 \left\langle \nabla_{\overline{V}_j} V_i, V_c \right\rangle \left\langle \nabla_{V_a} \overline{V}_c, \overline{V}_b \right\rangle \overline{\omega}^j \wedge \overline{\omega}^b \wedge \iota_{\overline{V}_i} \left(\iota_{\overline{V}_a} (\alpha) \right) \qquad (P) \\ & + \left\langle \nabla_{V_j} \overline{V}_c, \overline{V}_b \right\rangle \left\langle \nabla_{\overline{V}_j} V_k, V_i \right\rangle \overline{\omega}^b \wedge \overline{\omega}^c \wedge \iota_{\overline{V}_k} \left(\iota_{\overline{V}_i} \alpha \right) \qquad (Q) \end{aligned}$$

See the appendix A.3 for the details of the computation which gives the expression above.

If we define $T_2 : E^{p,q} \to E^{p+1,q}$ by

$$T_2 (\alpha) = +2 \left\langle \nabla_{\overline{V}_j} V_i, V_k \right\rangle \overline{\omega}^j \wedge \omega^k \wedge \iota_{\overline{V}_i} (\alpha) \tag{2.8}$$

then its L^2-adjoint $(T_2)^* : E^{p,q} \to E^{p-1,q}$ would be given by

$$(T_2)^* (\alpha) = +2 \left\langle \nabla_{V_a} \overline{V}_b, \overline{V}_c \right\rangle \overline{\omega}^b \wedge \iota_{V_c} \left(\iota_{\overline{V}_a} (\alpha) \right).$$

Then we have

$$\begin{aligned} & \left[T_2, (T_2)^*\right](\alpha) && (2.9) \\ = \; & T_2 \left((T_2)^* (\alpha) \right) + (T_2)^* (T_2 (\alpha)) \\ = \; & +4 \left\langle \nabla_{\overline{V}_j} V_i, V_k \right\rangle \left\langle \nabla_{V_j} \overline{V}_b, \overline{V}_k \right\rangle \overline{\omega}^b \wedge \iota_{\overline{V}_i} (\alpha) \qquad (R) \\ & +4 \left\langle \nabla_{\overline{V}_j} V_i, V_k \right\rangle \left\langle \nabla_{V_a} \overline{V}_i, \overline{V}_c \right\rangle \overline{\omega}^j \wedge \omega^k \wedge \iota_{V_c} \left(\iota_{\overline{V}_a} (\alpha) \right) \qquad (S) \\ & -4 \left\langle \nabla_{\overline{V}_j} V_i, V_k \right\rangle \left\langle \nabla_{V_a} \overline{V}_b, \overline{V}_k \right\rangle \overline{\omega}^b \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_a} \left(\iota_{\overline{V}_i} (\alpha) \right) \qquad (T) \\ & -4 \left\langle \nabla_{\overline{V}_j} V_i, V_k \right\rangle \left\langle \nabla_{V_j} \overline{V}_b, \overline{V}_c \right\rangle \overline{\omega}^b \wedge \omega^k \wedge \iota_{V_c} \left(\iota_{\overline{V}_i} (\alpha) \right). \qquad (U) \end{aligned}$$

See the appendix A.3 for the details of the computation which gives the expression above.

This gives

$$
\begin{aligned}
&\frac{1}{2}\left[\left[L,\left(L_{\overline{\partial}\Omega}\right)^{*}\right],\left[L_{\overline{\partial}\Omega},\Lambda\right]\right](\alpha)+\frac{1}{2}\left[T_1,(T_1)^{*}\right](\alpha)-\frac{1}{2}\left[T_2,(T_2)^{*}\right](\alpha) \qquad (2.10)\\
= \;& +2\left\langle\nabla_{\overline{V}_j}V_j,V_c\right\rangle\left\langle\nabla_{V_a}\overline{V}_a,\overline{V}_c\right\rangle\alpha \qquad (A)\\
& -2\left\langle\nabla_{\overline{V}_j}V_j,V_c\right\rangle\left\langle\nabla_{V_a}\overline{V}_b,\overline{V}_c\right\rangle\omega^a\wedge\iota_{V_b}(\alpha) \qquad (E)\\
& -2\left\langle\nabla_{\overline{V}_j}V_i,V_c\right\rangle\left\langle\nabla_{V_a}\overline{V}_a,\overline{V}_c\right\rangle\omega^i\wedge\iota_{V_j}(\alpha) \qquad (F)\\
& +2\left\langle\nabla_{\overline{V}_a}V_i,V_c\right\rangle\left\langle\nabla_{V_a}\overline{V}_b,\overline{V}_c\right\rangle\omega^i\wedge\iota_{V_b}(\alpha) \qquad (G)\\
& +2\left\langle\nabla_{\overline{V}_j}V_j,V_b\right\rangle\left\langle\nabla_{V_a}\overline{V}_b,\overline{V}_c\right\rangle\overline{\omega}^c\wedge\iota_{\overline{V}_a}(\alpha) \qquad (B)\\
& +2\left\langle\nabla_{\overline{V}_j}V_c,V_i\right\rangle\left\langle\nabla_{V_a}\overline{V}_a,\overline{V}_c\right\rangle\overline{\omega}^j\wedge\iota_{\overline{V}_i}(\alpha) \qquad (C)\\
& +2\left\langle\nabla_{\overline{V}_j}V_b,V_c\right\rangle\left\langle\nabla_{V_a}\overline{V}_b,\overline{V}_c\right\rangle\overline{\omega}^j\wedge\iota_{\overline{V}_a}(\alpha) \qquad (D)+(O)\\
& +2\left\langle\nabla_{\overline{V}_j}V_i,V_c\right\rangle\left\langle\nabla_{V_a}\overline{V}_c,\overline{V}_b\right\rangle\omega^i\wedge\overline{\omega}^b\wedge\iota_{V_j}\left(\iota_{\overline{V}_a}(\alpha)\right) \qquad (J)\\
& +2\left\langle\nabla_{\overline{V}_j}V_i,V_k\right\rangle\left\langle\nabla_{V_a}\overline{V}_i,\overline{V}_c\right\rangle\omega^a\wedge\overline{\omega}^c\wedge\iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right) \qquad (K)\\
& -2\left\langle\nabla_{\overline{V}_j}V_c,V_i\right\rangle\left\langle\nabla_{V_a}\overline{V}_b,\overline{V}_c\right\rangle\omega^a\wedge\overline{\omega}^j\wedge\iota_{\overline{V}_i}\left(\iota_{V_b}(\alpha)\right) \qquad (L)\\
& -2\left\langle\nabla_{\overline{V}_j}V_i,V_c\right\rangle\left\langle\nabla_{V_a}\overline{V}_b,\overline{V}_c\right\rangle\omega^i\wedge\omega^a\wedge\iota_{V_j}\left(\iota_{V_b}(\alpha)\right) \qquad (H)\\
& +2\left\langle\nabla_{\overline{V}_j}V_i,V_c\right\rangle\left\langle\nabla_{V_a}\overline{V}_c,\overline{V}_b\right\rangle\overline{\omega}^j\wedge\overline{\omega}^b\wedge\iota_{\overline{V}_i}\left(\iota_{\overline{V}_a}(\alpha)\right) \qquad (M)+(P)-(T)\\
& +\left\langle\nabla_{\overline{V}_j}V_i,V_k\right\rangle\left\langle\nabla_{V_j}\overline{V}_b,\overline{V}_c\right\rangle\overline{\omega}^b\wedge\overline{\omega}^c\wedge\iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}\alpha\right) \qquad (N)+(Q)\\
& -2\left\langle\nabla_{\overline{V}_j}V_b,V_i\right\rangle\left\langle\nabla_{V_j}\overline{V}_k,\overline{V}_i\right\rangle\overline{\omega}^k\wedge\iota_{\overline{V}_b}(\alpha) \qquad (R)\\
& +2\left\langle\nabla_{\overline{V}_a}V_i,V_k\right\rangle\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_c\right\rangle\overline{\omega}^a\wedge\omega^k\wedge\iota_{\overline{V}_j}\left(\iota_{V_c}(\alpha)\right) \qquad (S)
\end{aligned}
$$

also

$$(I)-(U)=0$$

2.3 Weitzenbock Formula for a $(p,q)-$form

Theorem 2.3.1 *Let $\alpha\in E^{p,q}$ be any $(p,q)-$form on a compact complex manifold and let $\Delta_H,\Delta_K,\Delta_R : E^{p,q}\to E^{p,q}$ be the Hodge, Kähler and rough Laplacians on M defined in equations [2.1] and let $\left[L_{\overline{\partial}\Omega},\Lambda\right] : E^{p,q}\to E^{p,q+1}$ be the operator defined in equation [1.7], $T_1 : E^{p,q}\to E^{p,q-1}$ be the operator defined in equation [2.6] and $T_2 : E^{p,q}\to E^{p+1,q}$ be the operator defined in equation [2.8]. Then we have the following Weitzenbock formula*

$$
\begin{aligned}
&\Delta_K(\alpha)+\Delta_H(\alpha)-2\Delta_R(\alpha) \qquad (2.11)\\
= \;& F(R)(\alpha)+\frac{1}{2}\left[\left[L,\left(L_{\overline{\partial}\Omega}\right)^{*}\right],\left[L_{\overline{\partial}\Omega},\Lambda\right]\right](\alpha)+\frac{1}{2}\left[T_1,(T_1)^{*}\right](\alpha)-\frac{1}{2}\left[T_2,(T_2)^{*}\right](\alpha)
\end{aligned}
$$

where the curvature operator $F(R) : E^{p,q} \to E^{p,q}$ is given by

$$\begin{aligned} F(R)\,(\alpha) &= +\left\langle R_{V_j V_k}\overline{V}_j, \overline{V}_k\right\rangle \alpha + 2\left\langle R_{\overline{V}_j V_j} V_a, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}(\alpha) \\ &\quad +2\left\langle R_{V_a \overline{V}_k}\overline{V}_d, V_c\right\rangle \overline{\omega}^k \wedge \omega^c \wedge \iota_{\overline{V}_a}\left(\iota_{V_d}(\alpha)\right) \\ &\quad +2\left\langle R_{V_a \overline{V}_k} V_b, \overline{V}_j\right\rangle \overline{\omega}^k \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_b}(\alpha)\right) \\ &\quad +\left\langle R_{V_a V_b}\overline{V}_j, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_b}(\alpha)\right) \end{aligned} \tag{2.12}$$

Proof 2.3.1 *Let $\alpha \in E^{p,q}$ be any $(p,q)-$form on a compact complex manifold and let $\Delta_H(\alpha)$ be the Hodge Laplacian on M given in equation [2.2], let Δ_K be the Kähler Laplacian on M given in equation [2.3] and let Δ_R be the rough Laplacian on M given in equation [2.4]. If we combine them as $\Delta_K(\alpha) + \Delta_H(\alpha) - 2\Delta_R(\alpha)$, then we have the terms below remaining after a long, yet straightforward computation. The details of the computation are given in the appendix A.4.*

$$\begin{aligned} &\Delta_K(\alpha) + \Delta_H(\alpha) - 2\Delta_R(\alpha) \\ = \;& -\left\langle R_{V_j V_k}\overline{V}_j, \overline{V}_k\right\rangle \alpha \quad (W23) \\ & +2\left\langle R_{V_i V_j}\overline{V}_k, \overline{V}_j\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_i}(\alpha) \quad (W24) \\ & +2\left\langle R_{\overline{V}_a V_j} V_i, \overline{V}_j\right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_i}(\alpha) \quad (W7c2b + 8c2b - a) \\ & +\frac{1}{2}\left\langle R_{\overline{V}_a \overline{V}_j} V_k, V_i\right\rangle \overline{\omega}^a \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}(\alpha)\right) \quad (W25) \\ & +\frac{1}{2}\left\langle R_{V_a V_j}\overline{V}_i, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}(\alpha)\right) \quad (W26) \\ & -2\left\langle R_{\overline{V}_a V_j}\overline{V}_i, V_b\right\rangle \omega^b \wedge \overline{\omega}^a \wedge \iota_{V_i}\left(\iota_{\overline{V}_j}(\alpha)\right) \quad (W7b2 + 8b2 - a) \\ & -2\left\langle R_{\overline{V}_a V_i}\overline{V}_j, V_b\right\rangle \overline{\omega}^a \wedge \overline{\omega}^b \wedge \iota_{\overline{V}_i}\left(\iota_{\overline{V}_j}(\alpha)\right) \quad (W7c2a + 8c2a - a) \end{aligned}$$

$$-2\left\langle\nabla_{\overline{V}_a}V_k,V_a\right\rangle\left\langle\nabla_{V_j}\overline{V}_j,\overline{V}_k\right\rangle\alpha\quad(W27)$$

$$+2\left\langle\nabla_{\overline{V}_i}V_k,V_b\right\rangle\left\langle\nabla_{V_j}\overline{V}_j,\overline{V}_k\right\rangle\omega^b\wedge\iota_{V_i}(\alpha)\quad(W28)$$

$$-2\left\langle\nabla_{\overline{V}_j}V_k,V_b\right\rangle\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\omega^b\wedge\iota_{V_i}(\alpha)\quad(W29)$$

$$+2\left\langle\nabla_{\overline{V}_j}V_j,V_i\right\rangle\left\langle\nabla_{V_a}\overline{V}_i,\overline{V}_k\right\rangle\omega^a\wedge\iota_{V_k}(\alpha)\quad(W3a1)+(W30)$$

$$+2\left\langle\nabla_{\overline{V}_a}V_i,V_k\right\rangle\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\overline{\omega}^a\wedge\iota_{\overline{V}_j}(\alpha)\quad(W31)$$

$$+2\left\langle\nabla_{\overline{V}_a}V_k,V_c\right\rangle\left\langle\nabla_{V_j}\overline{V}_j,\overline{V}_k\right\rangle\overline{\omega}^a\wedge\iota_{\overline{V}_c}(\alpha)\quad(W8c3b2)+(W32)$$

$$+2\left\langle\nabla_{\overline{V}_a}V_k,V_a\right\rangle\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\overline{\omega}^i\wedge\iota_{\overline{V}_j}(\alpha)\quad(W33)$$

$$-2\left\langle\nabla_{\overline{V}_j}V_i,V_a\right\rangle\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\overline{\omega}^k\wedge\iota_{\overline{V}_a}(\alpha)\quad(W34)$$

$$-\left\langle\nabla_{\overline{V}_j}V_k,V_a\right\rangle\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_c\right\rangle\overline{\omega}^c\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_a}\left(\iota_{\overline{V}_k}(\alpha)\right)\quad(W39)$$

$$+2\left\langle\nabla_{\overline{V}_j}V_k,V_c\right\rangle\left\langle\nabla_{V_a}\overline{V}_i,\overline{V}_k\right\rangle\overline{\omega}^j\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_c}\left(\iota_{\overline{V}_a}(\alpha)\right)\quad(W8c3a2)+(W40)$$

$$-2\left\langle\nabla_{\overline{V}_a}V_k,V_b\right\rangle\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\omega^j\wedge\omega^b\wedge\iota_{V_a}\left(\iota_{V_i}(\alpha)\right)\quad(W41)+(W3a2)$$

$$-2\left\langle\nabla_{\overline{V}_j}V_k,V_i\right\rangle\left\langle\nabla_{V_a}\overline{V}_k,\overline{V}_b\right\rangle\omega^a\wedge\overline{\omega}^j\wedge\iota_{V_b}\left(\iota_{\overline{V}_i}(\alpha)\right)\quad(W42)$$

$$+2\left\langle\nabla_{\overline{V}_a}V_d,V_b\right\rangle\left\langle\nabla_{V_j}\overline{V}_c,\overline{V}_b\right\rangle\overline{\omega}^a\wedge\omega^d\wedge\iota_{\overline{V}_j}\left(\iota_{V_c}(\alpha)\right)\quad(W7b3a)$$

$$-2\left\langle\nabla_{\overline{V}_j}V_k,V_i\right\rangle\left\langle\nabla_{V_a}\overline{V}_k,\overline{V}_b\right\rangle\omega^a\wedge\overline{\omega}^b\wedge\iota_{V_j}\left(\iota_{\overline{V}_i}(\alpha)\right)\quad(W4b)+(W43)$$

$$+2\left\langle\nabla_{\overline{V}_a}V_k,V_b\right\rangle\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\omega^b\wedge\overline{\omega}^i\wedge\iota_{V_a}\left(\iota_{\overline{V}_j}(\alpha)\right)\quad(W44)$$

Also let $\left[L_{\overline{\partial}\Omega},\Lambda\right]:E^{p,q}\to E^{p,q+1}$ *be the operator defined in equation [1.7], then its commutator is given in equation [2.5]. Let* $T_1:E^{p,q}\to E^{p,q-1}$ *be the operator defined in equation [2.6], then its commutator is given in equation [2.7] and let* $T_2:E^{p,q}\to E^{p+1,q}$ *be the operator defined in* *equation [2.8], then its commutator is given in equation [2.9]. If we combine these commutators as*

$$+\frac{1}{2}\left[\left[L,\left(L_{\overline{\partial}\Omega}\right)^{*}\right],\left[L_{\overline{\partial}\Omega},\Lambda\right]\right](\alpha)+\frac{1}{2}\left[T_{1},(T_{1})^{*}\right](\alpha)-\frac{1}{2}\left[T_{2},(T_{2})^{*}\right](\alpha)$$

given in equation [2.10], we obtain

$$\begin{aligned}
(W20)+(W3a1) &= \frac{1}{2}(E)\\
(W27) &= \frac{1}{2}(A)\\
(W28) &= \frac{1}{2}(F)\\
(W29) &= \frac{1}{2}(G)\\
(W31) &= \frac{1}{2}(D)+\frac{1}{2}(O)\\
(W8c3b2)+(W32) &= \frac{1}{2}(C)\\
(W33) &= \frac{1}{2}(B)\\
(W39) &= \frac{1}{2}(N)+\frac{1}{2}(Q)\\
(W3a2)+(W41) &= \frac{1}{2}(H)\\
(W42) &= \frac{1}{2}(L)\\
(W4b)+(W43) &= \frac{1}{2}(K)\\
(W44) &= \frac{1}{2}(J)\\
(W34) &= -\frac{1}{2}(R)\\
(W7b3a) &= -\frac{1}{2}(S)\\
(W8c3a2)+(W40) &= +\frac{1}{2}(M)+\frac{1}{2}(P)-\frac{1}{2}(T)\\
0 &= \frac{1}{2}(I)-\frac{1}{2}(U)
\end{aligned}$$

This implies

$$\begin{aligned}
&\Delta_{K}(\alpha)+\Delta_{H}(\alpha)-2\Delta_{R}(\alpha)\\
=\;& F(R)(\alpha)+\frac{1}{2}\left[\left[L,\left(L_{\overline{\partial}\Omega}\right)^{*}\right],\left[L_{\overline{\partial}\Omega},\Lambda\right]\right](\alpha)\\
&+\frac{1}{2}\left[T_{1},(T_{1})^{*}\right](\alpha)-\frac{1}{2}\left[T_{2},(T_{2})^{*}\right](\alpha)
\end{aligned}$$

where the curvature operator $F(R) : E^{p,q} \to E^{p,q}$ is given by

$$
\begin{aligned}
F(R)(\alpha) &= +\left\langle R_{V_j V_k}\overline{V}_j, \overline{V}_k\right\rangle \alpha + 2\left\langle R_{\overline{V}_j V_j} V_a, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}(\alpha) \\
&\quad +2\left\langle R_{V_a \overline{V}_k}\overline{V}_d, V_c\right\rangle \overline{\omega}^k \wedge \omega^c \wedge \iota_{\overline{V}_a}\left(\iota_{V_d}(\alpha)\right) \\
&\quad +2\left\langle R_{V_a \overline{V}_k} V_b, \overline{V}_j\right\rangle \overline{\omega}^k \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_b}(\alpha)\right) \\
&\quad +\left\langle R_{V_a V_b}\overline{V}_j, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_b}(\alpha)\right)
\end{aligned}
$$

Here $F(R) : E^{p,q} \to E^{p,q}$ is simplified after switching the roles of the indices, using the symmetries of the curvature tensor and the Bianchi identity, the details are given on the last page of the appendices. This completes the proof of the theorem. □

Remark 2.3.1 (Weitzenbock formula for a (0,0)-form) *Let $\alpha \in E^{0,0}$ be a $(0,0)-$form, i.e. a complex valued function on M. Then the rough Laplacian $\Delta_R\left(\alpha^{0,0}\right) : E^{0,0} \to E^{0,0}$ vanishes since*

$$
\begin{aligned}
\Delta_R(\alpha) &= -\nabla_{V_j}\left(\overline{V}_j(\alpha)\right) \quad (R1) \\
&\quad +\overline{V}_k(\alpha)\left\langle \nabla_{V_j}\overline{V}_j, V_k\right\rangle \quad (R3) \\
&\quad +V_k(\alpha)\left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle \quad (R4) \\
&= 0
\end{aligned}
$$

Suppose that $\alpha \in E^{0,0}$ is a $\overline{\partial}-$harmonic $(0,0)-$form, i.e. $\overline{\partial}(\alpha) = \overline{V}_i(\alpha)\overline{\omega}^i = 0$. Then the Hodge Laplacian vanishes, $\Delta_H(\alpha) = 0$, and the Kähler Laplacian $\Delta_K(\alpha)$ simplifies to
$\left[\left[L, \left(L_{\overline{\partial}\Omega}\right)^*\right], \left[L_{\overline{\partial}\Omega}, \Lambda\right]\right](\alpha)$

$$\Delta_K(\alpha) = \left[\left[L, \left(L_{\overline{\partial}\Omega}\right)^*\right], \left[L_{\overline{\partial}\Omega}, \Lambda\right]\right](\alpha) = +4\left\langle \nabla_{\overline{V}_j} V_j, V_c\right\rangle \left\langle \nabla_{V_a}\overline{V}_a, \overline{V}_c\right\rangle \alpha.$$

$F(R)(\alpha)$ would simplify to

$$F(R)(\alpha) = +\left\langle R_{V_j V_k}\overline{V}_j, \overline{V}_k\right\rangle \alpha$$

and

$$\left[T_1, (T_1)^*\right](\alpha) = 0 = \left[T_2, (T_2)^*\right](\alpha).$$

Thus the Weitzenbock formula for a $(0,0)-$form would become

$$0 = +\left\langle R_{V_j V_k}\overline{V}_j, \overline{V}_k\right\rangle \alpha - 2\left\langle \nabla_{\overline{V}_j} V_j, V_c\right\rangle \left\langle \nabla_{V_k}\overline{V}_k, \overline{V}_c\right\rangle \alpha.$$

Since α does not need to vanish, we must have

$$0 = +\left\langle R_{V_j V_k}\overline{V}_j, \overline{V}_k\right\rangle - 2\left\langle \nabla_{\overline{V}_j} V_j, V_c\right\rangle \left\langle \nabla_{V_k}\overline{V}_k, \overline{V}_c\right\rangle.$$

Remark 2.3.2 (Weitzenbock formula for a (n,n)-form) *Let* $\alpha = f\omega^1 \wedge \cdots \wedge \omega^n$ $\wedge \overline{\omega}^1 \wedge \cdots \wedge \overline{\omega}^n \in E^{n,n}$ be any $(n,n)-$form where f is a complex-valued function on M, the rough Laplacian $\Delta_R : E^{n,n} \to E^{n,n}$ would be given by

$$
\begin{aligned}
& \Delta_R(\alpha) \\
= \quad & -proj_{E^{n,n}}\left(\nabla_{V_j}\left(proj_{E^{n,n}}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (R1) \\
& + \left\langle \nabla_{V_j}\overline{V}_j, V_k\right\rangle proj_{E^{n,n}}\left(\nabla_{\overline{V}_k}(\alpha)\right) \quad (R3) \\
& + \left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle proj_{E^{n,n}}\left(\nabla_{V_k}(\alpha)\right) \quad (R4)
\end{aligned}
$$

Denote $vol := \omega^1 \wedge \cdots \wedge \omega^n \wedge \overline{\omega}^1 \wedge \cdots \wedge \overline{\omega}^n$. Then

$$
\begin{aligned}
& proj_{E^{n,n}}\left(\nabla_{\overline{V}_k}(\alpha)\right) \\
= \quad & +\overline{V}_k(f)\, vol \\
& + f\left\langle \nabla_{\overline{V}_k}\omega^r, \overline{\omega}^r\right\rangle \omega^1 \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^r} \wedge \cdots \wedge \omega^n \wedge \overline{\omega}^1 \wedge \cdots \wedge \overline{\omega}^n \\
& + f\left\langle \nabla_{\overline{V}_k}\overline{\omega}^r, \omega^r\right\rangle \overline{\omega}^1 \wedge \cdots \wedge \omega^n \wedge \overline{\omega}^1 \wedge \cdots \wedge \underset{r-th \text{ place}}{\overline{\omega}^r} \wedge \cdots \wedge \overline{\omega}^n \\
= \quad & +\overline{V}_k(f)\, vol
\end{aligned}
$$

and

$$
\begin{aligned}
& proj_{E^{n,n}}\left(\nabla_{V_k}(\alpha)\right) \\
= \quad & +V_k(f)\, vol \\
& + f\langle \nabla_{V_k}\omega^r, \overline{\omega}^r\rangle \omega^1 \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^r} \wedge \cdots \wedge \omega^n \wedge \overline{\omega}^1 \wedge \cdots \wedge \overline{\omega}^n \\
& + f\langle \nabla_{V_k}\overline{\omega}^r, \omega^r\rangle \omega^1 \wedge \cdots \wedge \omega^n \wedge \overline{\omega}^1 \wedge \cdots \wedge \underset{r-th \text{ place}}{\overline{\omega}^r} \wedge \cdots \wedge \overline{\omega}^n \\
= \quad & +V_k(f)\, vol
\end{aligned}
$$

Therefore the rough Laplacian would be

$$
\begin{aligned}
& \Delta_R(\alpha) \\
= \quad & -proj_{E^{n,n}}\left(\nabla_{V_j}\left(\overline{V}_j(f)\, vol\right)\right) - proj_{E^{n,n}}\left(\nabla_{V_j}\left(f\left\langle \nabla_{\overline{V}_j}\overline{V}_r, V_r\right\rangle vol\right)\right) \\
& -proj_{E^{n,n}}\left(\nabla_{V_j}\left(f\left\langle \nabla_{\overline{V}_j}V_r, \overline{V}_r\right\rangle vol\right)\right) \\
& +\overline{V}_k(f)\left\langle \nabla_{V_j}\overline{V}_j, V_k\right\rangle vol + f\left\langle \nabla_{\overline{V}_k}\overline{V}_r, V_r\right\rangle \left\langle \nabla_{V_j}\overline{V}_j, V_k\right\rangle vol \\
& + f\left\langle \nabla_{\overline{V}_k}V_r, \overline{V}_r\right\rangle \left\langle \nabla_{V_j}\overline{V}_j, V_k\right\rangle vol \\
& + V_k(f)\left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle vol + f\left\langle \nabla_{V_k}\overline{V}_r, V_r\right\rangle \left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle vol \\
& + f\left\langle \nabla_{V_k}V_r, \overline{V}_r\right\rangle \left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle vol
\end{aligned}
$$

$$
\begin{aligned}
= \quad & +f\left\langle\nabla_{\overline{V}_j}V_r,V_a\right\rangle\left\langle\nabla_{V_j}\overline{V}_r,\overline{V}_a\right\rangle vol+f\left\langle\nabla_{V_j}V_r,\overline{V}_a\right\rangle\left\langle\nabla_{\overline{V}_j}\overline{V}_r,V_a\right\rangle vol\\
& -f\left\langle\nabla_{\overline{V}_j}\overline{V}_r,\overline{V}_a\right\rangle\left\langle\nabla_{V_j}V_r,V_a\right\rangle vol-f\left\langle\nabla_{\overline{V}_j}\overline{V}_r,V_a\right\rangle\left\langle\nabla_{V_j}V_r,\overline{V}_a\right\rangle vol\\
& -f\left\langle\nabla_{\overline{V}_j}\overline{V}_r,V_r\right\rangle\left\langle\nabla_{V_j}\overline{V}_s,V_s\right\rangle vol-f\left\langle\nabla_{\overline{V}_j}\overline{V}_r,V_r\right\rangle\left\langle\nabla_{V_j}V_s,\overline{V}_s\right\rangle vol\\
& -f\left\langle\nabla_{\overline{V}_j}V_r,\overline{V}_a\right\rangle\left\langle\nabla_{V_j}\overline{V}_r,V_a\right\rangle vol-f\left\langle\nabla_{\overline{V}_j}V_r,V_a\right\rangle\left\langle\nabla_{V_j}\overline{V}_r,\overline{V}_a\right\rangle vol\\
& -f\left\langle\nabla_{\overline{V}_j}V_r,\overline{V}_r\right\rangle\left\langle\nabla_{V_j}\overline{V}_s,V_s\right\rangle vol-f\left\langle\nabla_{\overline{V}_j}V_r,\overline{V}_r\right\rangle\left\langle\nabla_{V_j}V_s,\overline{V}_s\right\rangle vol\\
& +f\left\langle\nabla_{\overline{V}_k}\overline{V}_r,V_r\right\rangle\left\langle\nabla_{V_j}\overline{V}_j,V_k\right\rangle vol+f\left\langle\nabla_{\overline{V}_k}V_r,\overline{V}_r\right\rangle\left\langle\nabla_{V_j}\overline{V}_j,V_k\right\rangle vol\\
& +f\left\langle\nabla_{V_k}\overline{V}_r,V_r\right\rangle\left\langle\nabla_{V_j}\overline{V}_j,\overline{V}_k\right\rangle vol+f\left\langle\nabla_{V_k}V_r,\overline{V}_r\right\rangle\left\langle\nabla_{V_j}\overline{V}_j,\overline{V}_k\right\rangle vol
\end{aligned}
$$

This simplifies to

$$
\begin{aligned}
= \quad & -f\left\langle\nabla_{V_j}\nabla_{\overline{V}_j}\overline{V}_r,V_r\right\rangle vol+f\left\langle\nabla_{V_j}\nabla_{\overline{V}_j}\overline{V}_r,V_r\right\rangle vol\\
& +f\left\langle\nabla_{V_j}\overline{V}_r,\nabla_{\overline{V}_j}V_r\right\rangle vol+f\left\langle\nabla_{\overline{V}_j}\overline{V}_r,\nabla_{V_j}V_r\right\rangle vol\\
& -2f\left\langle\nabla_{\overline{V}_j}\overline{V}_a,V_r\right\rangle\left\langle\nabla_{V_j}V_a,\overline{V}_r\right\rangle vol-f\left\langle\nabla_{\overline{V}_j}V_r,V_a\right\rangle\left\langle\nabla_{V_j}\overline{V}_r,\overline{V}_a\right\rangle vol
\end{aligned}
$$

Then we have

$$
\begin{aligned}
= \quad & +f\left\langle\nabla_{V_j}\overline{V}_r,V_a\right\rangle\left\langle\nabla_{\overline{V}_j}V_r,\overline{V}_a\right\rangle vol+f\left\langle\nabla_{V_j}\overline{V}_r,\overline{V}_a\right\rangle\left\langle\nabla_{\overline{V}_j}V_r,V_a\right\rangle vol\\
& -f\left\langle\nabla_{V_j}\overline{V}_r,\overline{V}_a\right\rangle\left\langle\nabla_{\overline{V}_j}V_r,V_a\right\rangle vol+f\left\langle\nabla_{\overline{V}_j}\overline{V}_r,V_a\right\rangle\left\langle\nabla_{V_j}V_r,\overline{V}_a\right\rangle vol\\
& -2f\left\langle\nabla_{\overline{V}_j}\overline{V}_a,V_r\right\rangle\left\langle\nabla_{V_j}V_a,\overline{V}_r\right\rangle vol
\end{aligned}
$$

If we simplify again, we finally obtain

$$\Delta_R(\alpha)=0.$$

For a $\overline{\partial}-$harmonic $(n,n)-$form,

$$
\begin{aligned}
0 \quad &= \quad \overline{\partial}^*(\alpha)\\
&= \quad -\iota_{\overline{V}_a}\left(proj_{E^{n,n}}\left(\nabla_{V_a}(\alpha)\right)\right)+2\left\langle\nabla_{\overline{V}_a}V_a,V_b\right\rangle\iota_{\overline{V}_b}(\alpha)\\
&\quad\quad +2\left\langle\nabla_{\overline{V}_a}V_b,V_c\right\rangle\omega^b\wedge\iota_{\overline{V}_c}\left(\iota_{V_a}(\alpha)\right)\\
&= \quad -\iota_{\overline{V}_a}\left(proj_{E^{n,n}}\left(\nabla_{V_a}(\alpha)\right)\right)+2\left\langle\nabla_{\overline{V}_a}V_a,V_b\right\rangle\iota_{\overline{V}_b}(\alpha)\\
&\quad\quad -2\left\langle\nabla_{\overline{V}_a}V_a,V_c\right\rangle\iota_{\overline{V}_c}(\alpha)\\
&= \quad -\iota_{\overline{V}_a}\left(V_a(f)\,vol\right)\\
&= \quad (-1)^{n+a}V_a(f)\,\omega^1\wedge\cdots\wedge\omega^n\wedge\overline{\omega}^1\wedge\cdots\wedge\underset{a-th\text{ place}}{\widehat{\overline{\omega}^a}}\wedge\cdots\wedge\overline{\omega}^n\\
&\Rightarrow \quad V_a(f)=0
\end{aligned}
$$

For any $\overline{\partial}-$harmonic $(n,n)-$form $\alpha \in E^{n,n}$, the Hodge Laplacian vanishes $\Delta_H(\alpha_{harmonic}) = 0$ and the Kähler Laplacian $\Delta_K : E^{n,n} \to E^{n,n}$ would simplify to $\Delta_K(\alpha) = \left[\left[L, \left(L_{\overline{\partial}\Omega}\right)^*\right], \left[L_{\overline{\partial}\Omega}, \Lambda\right]\right](\alpha)$ and so the Weitzenbock formula simplifies to

$$0 = F(R)(\alpha) - \frac{1}{2}\left[\left[L, \left(L_{\overline{\partial}\Omega}\right)^*\right], \left[L_{\overline{\partial}\Omega}, \Lambda\right]\right](\alpha) + \frac{1}{2}\left[T_1, (T_1)^*\right](\alpha) - \frac{1}{2}\left[T_2, (T_2)^*\right](\alpha)$$

Then using the identities for any $(n,n)-$form

$$\begin{aligned}
\omega^j \wedge \iota_{V_i}(\alpha) &= \delta_i^j \alpha \\
\overline{\omega}^j \wedge \iota_{\overline{V}_i}(\alpha) &= \delta_i^j \alpha \\
\omega^j \wedge \overline{\omega}^a \wedge \iota_{\overline{V}_c}\left(\iota_{V_i}(\alpha)\right) &= -\delta_i^j \delta_c^a \alpha \\
\omega^j \wedge \omega^b \wedge \iota_{V_i}\left(\iota_{V_a}(\alpha)\right) &= \delta_a^j \delta_i^b \alpha - \delta_i^j \delta_a^b \alpha \\
\overline{\omega}^j \wedge \overline{\omega}^b \wedge \iota_{\overline{V}_i}\left(\iota_{\overline{V}_a}(\alpha)\right) &= \delta_a^j \delta_i^b \alpha - \delta_i^j \delta_a^b \alpha
\end{aligned}$$

The curvature operator $F(R) : E^{n,n} \to E^{n,n}$ simplifies to

$$F(R)(\alpha) = +\left\langle R_{V_j V_k} \overline{V}_j, \overline{V}_k \right\rangle \alpha.$$

and the quadratic terms simplify to

$$\begin{aligned}
& -\frac{1}{2}\left[\left[L, \left(L_{\overline{\partial}\Omega}\right)^*\right], \left[L_{\overline{\partial}\Omega}, \Lambda\right]\right](\alpha) + \frac{1}{2}\left[T_1, (T_1)^*\right](\alpha) - \frac{1}{2}\left[T_2, (T_2)^*\right](\alpha) \\
= \quad & +2\left\langle \nabla_{\overline{V}_j} V_j, V_k \right\rangle \left\langle \nabla_{V_a} \overline{V}_k, \overline{V}_a \right\rangle \alpha
\end{aligned}$$

Thus the Weitzenbock formula for a $(n,n)-$form becomes

$$0 = +\left\langle R_{V_j V_k} \overline{V}_j, \overline{V}_k \right\rangle \alpha + 2\left\langle \nabla_{\overline{V}_j} V_j, V_k \right\rangle \left\langle \nabla_{V_a} \overline{V}_k, \overline{V}_a \right\rangle \alpha.$$

Since α does not need to vanish, we must have

$$0 = +\left\langle R_{V_j V_k} \overline{V}_j, \overline{V}_k \right\rangle + 2\left\langle \nabla_{\overline{V}_j} V_j, V_k \right\rangle \left\langle \nabla_{V_a} \overline{V}_k, \overline{V}_a \right\rangle.$$

The previous remarks imply the following proposition:

Proposition 2.3.1 *Let M be a compact, complex manifold. Then we must have*

$$0 = +\left\langle R_{V_j V_k} \overline{V}_j, \overline{V}_k \right\rangle + 2\left\langle \nabla_{\overline{V}_j} V_j, V_k \right\rangle \left\langle \nabla_{V_a} \overline{V}_k, \overline{V}_a \right\rangle.$$

2.4 Vanishing Theorems and Plurigenera of a compact complex manifold

We first give the definition of a diagonally dominant matrix.

Definition 2.4.1 *[Diagonally Dominant Matrix]A matrix $A = (a_{ij})$ is said to be diagonally dominant if for every row of the matrix, the magnitude of each diagonal entry in a row is larger than or equal to the sum of the magnitudes of all the other (non-diagonal) entries in that row. More precisely, the matrix A is strictly diagonally dominant if*

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|$$

for all i where a_{ij} denotes the entry in the $i-th$ row and $j-th$ column.

We will use the following Horn's lemma in our vanishing theorems.

Lemma 2.4.1 *[Horn] Every Hermitian strictly diagonally dominant matrix $A = (a_{ij})$ with real non-negative diagonal entries is positive definite.*

Theorem 2.4.1 (Vanishing Theorem) *Let M be a compact complex mfd. Let $\alpha \in \mathcal{H}^{p,q}$ be any harmonic $(p, q)-$form. If*

$$\begin{aligned} &+2\operatorname{Re}\left(\Delta_R\left(\alpha\right),\alpha\right)_{L^2} - \frac{1}{2}\left(\left[\left[L_{\overline{\partial}\Omega},\Lambda\right],\left[L,\left(L_{\overline{\partial}\Omega}\right)^*\right]\right]\left(\alpha\right),\alpha\right)_{L^2} \\ &+\frac{1}{2}\left(\left[T_1,(T_1)^*\right]\left(\alpha\right),\alpha\right)_{L^2} - \frac{1}{2}\left(\left[T_2,(T_2)^*\right]\left(\alpha\right),\alpha\right)_{L^2} \geq 0 \end{aligned}$$

and if $F(R)(\alpha)$ is strictly diagonally dominant then $\alpha^{p,q}$ has to vanish.

Proof 2.4.1 *For any harmonic $(p, q)-$form, the Weitzenbock formula becomes*

$$\begin{aligned} 0 &= F(R)\left(\alpha\right) + 2\Delta_R\left(\alpha\right) - \frac{1}{2}\left[\left[L_{\overline{\partial}\Omega},\Lambda\right],\left[L,\left(L_{\overline{\partial}\Omega}\right)^*\right]\right]\left(\alpha\right) \\ &\quad +\frac{1}{2}\left[T_1,(T_1)^*\right]\left(\alpha\right) - \frac{1}{2}\left[T_2,(T_2)^*\right]\left(\alpha\right) \end{aligned}$$

$F(R)(\alpha)$ is a hermitian matrix and if $F(R)(\alpha)$ is strictly diagonally dominant, then by Horn's lemma $F(R)(\alpha)$ is positive definite and so $(F(R)(\alpha), \alpha)_{L^2} > 0$ and if

$$\begin{aligned} &+2\operatorname{Re}\left(\Delta_R\left(\alpha\right),\alpha\right)_{L^2} - \frac{1}{2}\left(\left[\left[L_{\overline{\partial}\Omega},\Lambda\right],\left[L,\left(L_{\overline{\partial}\Omega}\right)^*\right]\right]\left(\alpha\right),\alpha\right)_{L^2} \\ &+\frac{1}{2}\left(\left[T_1,(T_1)^*\right]\left(\alpha\right),\alpha\right)_{L^2} - \frac{1}{2}\left(\left[T_2,(T_2)^*\right]\left(\alpha\right),\alpha\right)_{L^2} \geq 0 \end{aligned}$$

then this implies $\alpha \in \mathcal{H}^{p,q}$ has to vanish. $\square$

Remark 2.4.1 *$(F(R)(\alpha), \alpha)_{L^2}$ is a hermitian matrix and the diagonal of the hermitian matrix consists of biholomorphic sectional curvature, isotropic curvature and sectional curvature terms. On the diagonal, we have the terms*

$$\begin{aligned}
& +\left(\left\langle R_{V_j V_i}\overline{V}_j, \overline{V}_i\right\rangle \alpha, \alpha\right)_{L^2} +2\left(\left\langle R_{\overline{V}_i V_j} V_i, \overline{V}_j\right\rangle \alpha, \alpha\right)_{L^2} +2\left(\left\langle R_{\overline{V}_j V_j} V_i, \overline{V}_i\right\rangle \alpha,\alpha\right)_{L^2} \\
= & +\frac{1}{4}\left[-2R_{ii^*jj^*} + R_{ijij} + R_{i^*j^*i^*j^*} + R_{ij^*ij^*} + R_{i^*ji^*j}\right](\alpha,\alpha)_{L^2} \\
& +\frac{1}{2}\left[2R_{ii^*jj^*} + R_{ijij} + R_{i^*j^*i^*j^*} + R_{ij^*ij^*} + R_{i^*ji^*j}\right](\alpha,\alpha)_{L^2} \\
& +2R_{ii^*jj^*}(\alpha,\alpha)_{L^2} \\
= & \frac{5}{2}R_{ii^*jj^*}(\alpha,\alpha)_{L^2} + \frac{3}{4}\left(R_{ijij} + R_{i^*j^*i^*j^*} + R_{ij^*ij^*} + R_{i^*ji^*j}\right)(\alpha,\alpha)_{L^2}.
\end{aligned}$$

Theorem 2.4.2 *Let M be a compact complex manifold. Let $\alpha \in \mathcal{H}^{p,0}$ be any harmonic $(p,0)-$form. If M admits positive isotropic curvature and if*

$$+2\operatorname{Re}\left(\Delta_R(\alpha), \alpha\right)_{L^2} - \frac{1}{2}\left(\left[L_{\overline{\partial}\Omega}, \Lambda\right](\alpha), \left[L_{\overline{\partial}\Omega}, \Lambda\right](\alpha)\right)_{L^2} \geq 0$$

then α has to vanish.

Proof 2.4.2 *For any harmonic $(p,0)-$form $\alpha \in \mathcal{H}^{p,0}$, $F(R)(\alpha)$ simplifies to*

$$F(R)\left(\alpha^{p,0}\right) = +\left\langle R_{V_j V_k}\overline{V}_j, \overline{V}_k\right\rangle \alpha^{p,0}$$

which the curvature terms are independent from the coefficients of the $(p,0)-$form and so it becomes a diagonal matrix where the diagonal entries are sums of isotropic curvature terms times $\alpha^{p,0}$ and since M admits positive isotropic curvature, i.e. $\left\langle R_{V_i V_j}\overline{V}_i, \overline{V}_j\right\rangle > 0$, then $\left(F(R)\left(\alpha^{p,0}\right), \alpha^{p,0}\right)_{L^2} > 0$. Also

$$\left[T_1, (T_1)^*\right]\left(\alpha^{p,0}\right) = 0 = \left[T_2, (T_2)^*\right]\left(\alpha^{p,0}\right).$$

Then the Weitzenbock formula simplifies to

$$0 = 2\Delta_R(\alpha) + F(R)(\alpha) - \frac{1}{2}\left[L, \left(L_{\overline{\partial}\Omega}\right)^*\right] \circ \left[L_{\overline{\partial}\Omega}, \Lambda\right](\alpha)$$

and if

$$+2\operatorname{Re}\left(\Delta_R(\alpha), \alpha\right)_{L^2} - \frac{1}{2}\left(\left[L_{\overline{\partial}\Omega}, \Lambda\right](\alpha), \left[L_{\overline{\partial}\Omega}, \Lambda\right](\alpha)\right)_{L^2} \geq 0$$

this implies $\alpha^{p,0}$ has to vanish. $\square$

Definition 2.4.2 *The geometric genus of a complex manifold of complex dimension* n *is defined by the quantity*

$$\mathfrak{p}_g = h^{n,0}$$

Theorem 2.4.3 *Let* M *be a compact complex manifold of complex dimension* n*. If* M *admits positive isotropic curvature, i.e.* $\left\langle R_{V_iV_j}\overline{V}_i, \overline{V}_j \right\rangle > 0$*, then* $\mathfrak{p}_g = 0$*.*

Proof 2.4.3 *Let* M *be a compact complex manifold of complex dimension* n *which admits positive isotropic curvature, i.e.* $\left\langle R_{V_iV_j}\overline{V}_i, \overline{V}_j \right\rangle > 0$*.*

Let $\alpha \in \mathcal{H}^{n,0}$ *be an* $(n,0)-$*harmonic form, then the Weitzenbock formula simplifies to*

$$0 = 2\Delta_R(\alpha) + F(R)(\alpha).$$

The Hermitian matrix $F(R)\left(\alpha^{n,0}\right) = \left\langle R_{V_iV_j}\overline{V}_i, \overline{V}_j \right\rangle \alpha^{n,0}$ *simplifies to a* 1×1 *matrix with the entry* $\left\langle R_{V_iV_j}\overline{V}_i, \overline{V}_j \right\rangle > 0$ *since* M *admits isotropic curvature, this implies that* $F(R)\left(\alpha^{n,0}\right)$ *is positive definite and* $\left(F(R)\left(\alpha^{n,0}\right), \alpha^{n,0}\right)_{L^2} > 0$ *and since* Δ_R *is positive semi-definite, thus we have* $\mathfrak{p}_g = h^{n,0} = 0$*.* □

Definition 2.4.3 *The arithmetic genus of a complex manifold of complex dimension* n *is defined by the quantity*

$$\mathfrak{p}_a = h^{n,0} - h^{n-1,0} + \cdots + (-1)^{n-1} h^{1,0}.$$

Theorem 2.4.4 *Let* M *be a compact complex manifold of complex dimension* n*. If* M *admits positive isotropic curvature, i.e.* $\left\langle R_{V_iV_j}\overline{V}_i, \overline{V}_j \right\rangle > 0$ *and*

$$+2\,\mathrm{Re}\left(\Delta_R\left(\alpha^{p,0}\right), \alpha^{p,0}\right)_{L^2} - \frac{1}{2}\left(\left[L_{\overline{\partial}\Omega}, \Lambda\right]\left(\alpha^{p,0}\right), \left[L_{\overline{\partial}\Omega}, \Lambda\right]\left(\alpha^{p,0}\right)\right)_{L^2} \geq 0$$

for each $p = 1, ..., n-1$ *then* $\mathfrak{p}_a = 0$*.*

Proof 2.4.4 *If* M *is a compact complex manifold of complex dimension* n *which admits positive isotropic curvature, i.e.* $\left\langle R_{V_iV_j}\overline{V}_i, \overline{V}_j \right\rangle > 0$ *and also if*

$$+2\,\mathrm{Re}\left(\Delta_R\left(\alpha^{p,0}\right), \alpha^{p,0}\right)_{L^2} - \frac{1}{2}\left(\left[\left[L, \left(L_{\overline{\partial}\Omega}\right)^*\right], \left[L_{\overline{\partial}\Omega}, \Lambda\right]\right]\left(\alpha^{p,0}\right), \left(\alpha^{p,0}\right)\right)_{L^2} \geq 0$$

for each $p = 1, ..., n-1$ *and since* $F(R)\left(\alpha^{p,0}\right) = \left\langle R_{V_iV_j}\overline{V}_i, \overline{V}_j \right\rangle \alpha^{p,0}$ *simplifies to a diagonal matrix where the entries are isotropic curvature terms which are independent from* $\alpha^{p,0}$ *and since* M *admits positive isotropic curvature,*

$$\left(F(R)\left(\alpha^{p,0}\right), \alpha^{p,0}\right)_{L^2} = \left(\left\langle R_{V_iV_j}\overline{V}_i, \overline{V}_j \right\rangle \alpha^{p,0}, \alpha^{p,0}\right)_{L^2} > 0,$$

this implies $h^{p,0} = 0$ *for each* $p = 1, ..., n$*. Thus we have*

$$\mathfrak{p}_a = h^{n,0} - h^{n-1,0} + \cdots + (-1)^{n-1} h^{1,0} = 0.$$

□

2.5 Existence of an hypothetical integrable almost complex structure on $\mathbb{S}^6$

Definition 2.5.1 *The irregularity of a complex manifold of complex dimension n is defined by the quantity*

$$\mathfrak{q} = h^{0,1}$$

Theorem 2.5.1 (Gray) *If the six-dimensional sphere $\mathbb{S}^6$ has a hypothetical integrable complex structure then the irregularity of $\mathbb{S}^6$ would be nonzero.*

In [Gray], A. Gray proves that for any hypothetical complex structure on $\mathbb{S}^6$, the Hodge number $h^{0,3}\left(\mathbb{S}^6\right)$ and the arithmetic genus $1 - h^{0,1}\left(\mathbb{S}^6\right) + h^{0,2}\left(\mathbb{S}^6\right) - h^{0,3}\left(\mathbb{S}^6\right)$ of $\mathbb{S}^6$ would be zero, and therefore $h^{0,1}\left(\mathbb{S}^6\right) \geq 1$.

Corollary 2.5.1 *Let $M = \mathbb{S}^6$ be the six-dimensional sphere and let $\alpha \in E^{0,1}\left(\mathbb{S}^6\right)$ be any harmonic $(0,1)-$form. If $F(R)(\alpha)$ is strictly diagonally dominant and if the following inequality holds*

$$+2\operatorname{Re}\left(\Delta_R\left(\alpha^{0,1}\right), \alpha^{0,1}\right)_{L^2} - \frac{1}{2}\left(\left[\left[L, \left(L_{\overline{\partial}\Omega}\right)^*\right], \left[L_{\overline{\partial}\Omega}, \Lambda\right]\right]\left(\alpha^{0,1}\right), \alpha^{0,1}\right)_{L^2}$$

$$+\frac{1}{2}\left(\left[T_1, (T_1)^*\right]\left(\alpha^{0,1}\right), \alpha^{0,1}\right)_{L^2} - \frac{1}{2}\left(\left[T_2, (T_2)^*\right]\left(\alpha^{0,1}\right), \alpha^{0,1}\right)_{L^2} \geq 0$$

then this implies the nonexistence of a hypothetical integrable almost complex structure on $\mathbb{S}^6$ for the given conditions.

Proof 2.5.1 *Assume $\mathbb{S}^6$ has integrable almost complex structure and let $\alpha \in E^{0,1}\left(\mathbb{S}^6\right)$ be any harmonic $(0,1)-$form on $\mathbb{S}^6$. Then by our main vanishing theorem for the case of $p = 0,\ q = 1$ if $F(R)(\alpha)$ is strictly diagonally dominant and if the following inequality holds*

$$+2\operatorname{Re}\left(\Delta_R\left(\alpha^{0,1}\right), \alpha^{0,1}\right)_{L^2} - \frac{1}{2}\left(\left[\left[L, \left(L_{\overline{\partial}\Omega}\right)^*\right], \left[L_{\overline{\partial}\Omega}, \Lambda\right]\right]\left(\alpha^{0,1}\right), \alpha^{0,1}\right)_{L^2}$$

$$+\frac{1}{2}\left(\left[T_1, (T_1)^*\right]\left(\alpha^{0,1}\right), \alpha^{0,1}\right)_{L^2} - \frac{1}{2}\left(\left[T_2, (T_2)^*\right]\left(\alpha^{0,1}\right), \alpha^{0,1}\right)_{L^2} \geq 0$$

this implies that α has to vanish and therefore $\mathfrak{q} = h^{0,1} = 0$ and but this contradicts with Gray's result for the existence of hypothetical integrable almost complex structure. Thus for the given conditions, there exists no hypothetical integrable almost complex structure on $\mathbb{S}^6$. □

Chapter 3

Frölicher Spectral Sequence Analysis on a Compact Complex 6-dimensional Manifold

Let $E^{p,q}$ be the space of $(p,q)-$forms on a compact complex 2n-dimensional manifold. Then there exists a spectral sequence (FSS) that relates the Dolbeault cohomology groups of the complex manifold to its Hodge-deRham cohomology groups:

Theorem 3.0.2 *[Frölicher] Given a complex manifold M of complex dimension n, there is a spectral sequence converging strongly to $H^*_{DR}(M,\mathbb{C})$ with $(E_0^{p,q}, d_0)$ term is equal to $\left(E^{p,q}, \overline{\partial}\right)$ and $(E_1^{p,q}, d_1)$ term is equal to $\left(H^{p,q}_{\overline{\partial}}, [\partial]\right)$. Furthermore, $H^*_{DR}(M,\mathbb{C}) = \oplus_{p+q=k} E^{p,q}_{n+2}$.*

The deRham cohomology can be written as the direct sum of the kernels and the cokernels of the differentials of the Frölicher spectral sequence [Frölicher]. In particular, if M is a compact complex manifold of real dimension 6, then the Euler Characteristic is given by

$$\chi(M^6) = \sum_{k=0}^{6} b^k = b^0 - b^1 + b^2 - b^3 + b^4 - b^5 + b^6$$

If we assume the manifold is connected, then $b^0 = 1$ and again by the Poincaré duality $b^6 = 1$. So we would like to provide vanishing theorems for some of the Dolbeault cohomology groups and by using them and the Frölicher spectral sequence, we'd like to obtain $\chi(M^6)$.

The following proposition was stated without a proof in [Wells] for any compact complex manifold of any complex dimension. We prove it for a compact complex manifold of complex dimension 3.

Proposition 3.0.1 *Let M be a compact connected complex non-Kähler 6-dimensional manifold. Then $\chi(M) = \sum_{0 \leq p,q \leq 3} (-1)^{p+q} h^{p,q}$ where $h^{p,q}$ is the Hodge number for each $0 \leq p, q \leq 3$.*

Proof 3.0.2 *Then the first page of the (FSS) would be as below:*

$$\begin{array}{ccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
0 & \stackrel{0}{\to} & E^{3,0} & \stackrel{d_0}{\to} & E^{3,1} & \stackrel{d_0}{\to} & E^{3,2} & \stackrel{d_0}{\to} & E^{3,3} & \stackrel{0}{\to} & 0 \\
0 & \stackrel{0}{\to} & E^{2,0} & \stackrel{d_0}{\to} & E^{2,1} & \stackrel{d_0}{\to} & E^{2,2} & \stackrel{d_0}{\to} & E^{2,3} & \stackrel{0}{\to} & 0 \\
0 & \stackrel{0}{\to} & E^{1,0} & \stackrel{d_0}{\to} & E^{1,1} & \stackrel{d_0}{\to} & E^{1,2} & \stackrel{d_0}{\to} & E^{1,3} & \stackrel{0}{\to} & 0 \\
0 & \stackrel{0}{\to} & E^{0,0} & \stackrel{d_0}{\to} & E^{0,1} & \stackrel{d_0}{\to} & E^{0,2} & \stackrel{d_0}{\to} & E^{0,3} & \stackrel{0}{\to} & 0 \\
0 & & 0 & & 0 & & 0 & & 0 & & 0
\end{array}$$

The groups on the first page are the Dolbeault cohomology groups:

$$\begin{array}{ccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
0 & \stackrel{0}{\to} & H^{0,3} & \stackrel{d_1^{0,3}}{\to} & H^{1,3} & \stackrel{d_1^{1,3}}{\to} & H^{2,3} & \stackrel{d_1^{2,3}}{\to} & H^{3,3} & \stackrel{d_1^{3,3}=0}{\to} & 0 \\
0 & \stackrel{0}{\to} & H^{0,2} & \stackrel{d_1^{0,2}}{\to} & H^{1,2} & \stackrel{d_1^{1,2}}{\to} & H^{2,2} & \stackrel{d_1^{2,2}}{\to} & H^{3,2} & \stackrel{d_1^{3,2}=0}{\to} & 0 \\
0 & \stackrel{0}{\to} & H^{0,1} & \stackrel{d_1^{0,1}}{\to} & H^{1,1} & \stackrel{d_1^{1,1}}{\to} & H^{2,1} & \stackrel{d_1^{2,1}}{\to} & H^{3,1} & \stackrel{d_1^{3,1}=0}{\to} & 0 \\
0 & \stackrel{0}{\to} & H^{0,0} & \stackrel{d_1^{0,0}}{\to} & H^{1,0} & \stackrel{d_1^{1,0}}{\to} & H^{2,0} & \stackrel{d_1^{2,0}}{\to} & H^{3,0} & \stackrel{d_1^{3,0}=0}{\to} & 0 \\
0 & & 0 & & 0 & & 0 & & 0 & & 0
\end{array}$$

or in other words

$$\begin{array}{ccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
0 & \xrightarrow{0} & \mathbb{C}^{h^{0,3}} & \xrightarrow{d_1^{0,3}} & \mathbb{C}^{h^{1,3}} & \xrightarrow{d_1^{1,3}} & \mathbb{C}^{h^{2,3}} & \xrightarrow{d_1^{2,3}} & \mathbb{C}^{h^{3,3}} = \mathbb{C} & \xrightarrow{d_1^{3,3}=0} & 0 \\
0 & \xrightarrow{0} & \mathbb{C}^{h^{0,2}} & \xrightarrow{d_1^{0,2}} & \mathbb{C}^{h^{1,2}} & \xrightarrow{d_1^{1,2}} & \mathbb{C}^{h^{2,2}} & \xrightarrow{d_1^{2,2}} & \mathbb{C}^{h^{3,2}} & \xrightarrow{d_1^{3,2}=0} & 0 \\
0 & \xrightarrow{0} & \mathbb{C}^{h^{0,1}} & \xrightarrow{d_1^{0,1}} & \mathbb{C}^{h^{1,1}} & \xrightarrow{d_1^{1,1}} & \mathbb{C}^{h^{2,1}} & \xrightarrow{d_1^{2,1}} & \mathbb{C}^{h^{3,1}} & \xrightarrow{d_1^{3,1}=0} & 0 \\
0 & \xrightarrow{0} & \mathbb{C}^{h^{0,0}} = \mathbb{C} & \xrightarrow{d_1^{0,0}} & \mathbb{C}^{h^{1,0}} & \xrightarrow{d_1^{1,0}} & \mathbb{C}^{h^{2,0}} & \xrightarrow{d_1^{2,0}} & \mathbb{C}^{h^{3,0}} & \xrightarrow{d_1^{3,0}=0} & 0 \\
0 & & 0 & & 0 & & 0 & & 0 & & 0
\end{array}$$

Now set the rank of $H^{p,q} := h^{p,q}$, *rank*$(\ker d_1^{p,q}) = k_1^{p,q}$, *rank*$(\operatorname{Im} d_1^{p,q}) = i_1^{p,q}$ *such that* $0 \leq k_1^{p,q} \leq h^{p,q}$ *and* $0 \leq i_1^{p,q} \leq h^{p+1,q}$. *If the complex manifold is connected, then we have* $h^{0,0} = 1$ *and* $h^{3,3} = 1$.

We have $i_1^{0,0} = 0$ *since otherwise we had to have* $h^{0,0} = 0$. *Similarly, we have* $i_1^{2,3} = 0$ *since otherwise we had to have* $h^{3,3} = 0$.

$$\begin{array}{lllll}
i_1^{0,0} & = & 0, & i_1^{1,0} = h^{1,0} - k_1^{1,0}, & i_1^{2,0} = h^{2,0} - k_1^{2,0}, \quad i_1^{3,0} = 0, \\
i_1^{0,1} & = & h^{0,1} - k_1^{0,1}, & i_1^{1,1} = h^{1,1} - k_1^{1,1}, & i_1^{2,1} = h^{2,1} - k_1^{2,1}, \quad i_1^{3,1} = 0, \\
i_1^{0,2} & = & h^{0,2} - k_1^{0,2}, & i_1^{1,2} = h^{1,2} - k_1^{1,2}, & i_1^{2,2} = h^{2,2} - k_1^{2,2}, \quad i_1^{3,2} = 0, \\
i_1^{0,3} & = & h^{0,3} - k_1^{0,3}, & i_1^{1,3} = h^{1,3} - k_1^{1,3}, & i_1^{2,3} = 0, \quad i_1^{3,3} = 0.
\end{array}$$

Then since we have

$$E_2^{p,q} = \frac{\ker d_1^{p,q}}{\operatorname{Im} d_1^{p-1,q}}$$

the second page would be equal to

$$\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & \\
0 & \mathbb{C}^{k_1^{0,3}} & \mathbb{C}^{k_1^{1,3}-h^{0,3}+k_1^{0,3}} & \mathbb{C}^{h^{2,3}-h^{1,3}+k_1^{1,3}} & \mathbb{C} & 0 \\
 & & \searrow & \searrow & & \\
0 & \mathbb{C}^{k_1^{0,2}} & \mathbb{C}^{k_1^{1,2}-h^{0,2}+k_1^{0,2}} & \mathbb{C}^{k_1^{2,2}-h^{1,2}+k_1^{1,2}} & \mathbb{C}^{h^{3,2}-h^{2,2}+k_1^{2,2}} & 0 \\
 & & \searrow & \searrow & & \\
0 & \mathbb{C}^{k_1^{0,1}} & \mathbb{C}^{k_1^{1,1}-h^{0,1}+k_1^{0,1}} & \mathbb{C}^{k_1^{2,1}-h^{1,1}+k_1^{1,1}} & \mathbb{C}^{h^{3,1}-h^{2,1}+k_1^{2,1}} & 0 \\
 & & \searrow & \searrow & & \\
0 & \mathbb{C} & \mathbb{C}^{k_1^{1,0}} & \mathbb{C}^{k_1^{2,0}-h^{1,0}+k_1^{1,0}} & \mathbb{C}^{h^{3,0}-h^{2,0}+k_1^{2,0}} & 0 \\
 & & & & & \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

Then we have the following maps on the second page:

$$\begin{array}{lll}
0 & \to & \mathbb{C}^{k_1^{0,3}} \xrightarrow{d_2^{0,3}} \mathbb{C}^{k_1^{2,2}-h^{1,2}+k_1^{1,2}} \to 0 \\
0 & \to & \mathbb{C}^{k_1^{1,3}-h^{0,3}+k_1^{0,3}} \xrightarrow{d_2^{1,3}} \mathbb{C}^{h^{3,2}-h^{2,2}+k_1^{2,2}} \to 0 \\
0 & \to & \mathbb{C}^{k_1^{0,2}} \xrightarrow{d_2^{0,2}} \mathbb{C}^{k_1^{2,1}-h^{1,1}+k_1^{1,1}} \to 0 \\
0 & \to & \mathbb{C}^{k_1^{1,2}-h^{0,2}+k_1^{0,2}} \xrightarrow{d_2^{1,2}} \mathbb{C}^{h^{3,1}-h^{2,1}+k_1^{2,1}} \to 0 \\
0 & \to & \mathbb{C}^{k_1^{0,1}} \xrightarrow{d_2^{0,1}} \mathbb{C}^{k_1^{2,0}-h^{1,0}+k_1^{1,0}} \to 0 \\
0 & \to & \mathbb{C}^{k_1^{1,1}-h^{0,1}+k_1^{0,1}} \xrightarrow{d_2^{1,1}} \mathbb{C}^{h^{3,0}-h^{2,0}+k_1^{2,0}} \to 0
\end{array}$$

Now let's denote rank$(\ker d_2^{p,q}) = k_2^{p,q}$, *rank*$(\operatorname{Im} d_2^{p,q}) = i_2^{p,q}$ *such that* $0 \le k_2^{p,q} \le k_1^{p,q}$ *and* $0 \le i_2^{p,q} \le k_1^{p+2,q-1} - h^{p+1,q-1} + k_1^{p+1,q-1}$. *Then we have*

$$\begin{array}{rcll}
i_2^{0,1} & = & k_1^{0,1} - k_2^{0,1}, & \qquad i_2^{1,1} = k_1^{1,1} - h^{0,1} + k_1^{0,1} - k_2^{1,1}, \\
i_2^{0,2} & = & k_1^{0,2} - k_2^{0,2}, & \qquad i_2^{1,2} = k_1^{1,2} - h^{0,2} + k_1^{0,2} - k_2^{1,2}, \\
i_2^{0,3} & = & k_1^{0,3} - k_2^{0,3}, & \qquad i_2^{1,3} = k_1^{1,3} - h^{0,3} + k_1^{0,3} - k_2^{1,3}
\end{array}$$

Then since we have

$$E_3^{p,q} = \frac{\ker d_2^{p,q}}{\operatorname{Im} d_2^{p+2,q-1}}$$

the third page would be equal to

$$\begin{array}{cccc}
\mathbb{C}^{k_2^{0,3}} & \mathbb{C}^{k_2^{1,3}} & \mathbb{C}^{h^{2,3}-h^{1,3}+k_1^{1,3}} & \mathbb{C} \\
\mathbb{C}^{k_2^{0,2}} & \mathbb{C}^{k_2^{1,2}} & \mathbb{C}^{k_1^{2,2}-h^{1,2}+k_1^{1,2}-k_1^{0,3}+k_2^{0,3}} & \mathbb{C}^{h^{3,2}-h^{2,2}+k_1^{2,2}-k_1^{1,3}+h^{0,3}-k_1^{0,3}+k_2^{1,3}} \\
\mathbb{C}^{k_2^{0,1}} & \mathbb{C}^{k_2^{1,1}} & \mathbb{C}^{k_1^{2,1}-h^{1,1}+k_1^{1,1}-k_1^{0,2}+k_2^{0,2}} & \mathbb{C}^{h^{3,1}-h^{2,1}+k_1^{2,1}-k_1^{1,2}+h^{0,2}-k_1^{0,2}+k_2^{1,2}} \\
\mathbb{C} & \mathbb{C}^{k_1^{1,0}} & \mathbb{C}^{k_1^{2,0}-h^{1,0}+k_1^{1,0}-k_1^{0,1}+k_2^{0,1}} & \mathbb{C}^{h^{3,0}-h^{2,0}+k_1^{2,0}-k_1^{1,1}+h^{0,1}-k_1^{0,1}+k_2^{1,1}}
\end{array}$$

Then we have the following maps on the third page:

$$\begin{array}{ccl}
0 & \to & \mathbb{C}^{k_2^{0,3}} \xrightarrow{d_3^{0,3}} \mathbb{C}^{h^{3,1}-h^{2,1}+k_1^{2,1}-k_1^{1,2}+h^{0,2}-k_1^{0,2}+k_2^{1,2}} \to 0 \\
0 & \to & \mathbb{C}^{k_2^{0,2}} \xrightarrow{d_3^{0,2}} \mathbb{C}^{h^{3,0}-h^{2,0}+k_1^{2,0}-k_1^{1,1}+h^{0,1}-k_1^{0,1}+k_2^{1,1}} \to 0
\end{array}$$

Now we denote rank$(\ker d_3^{p,q}) = k_3^{p,q}$, *rank*$(\operatorname{Im} d_3^{p,q}) = i_3^{p,q}$ *such that* $0 \le k_3^{p,q} \le k_2^{p,q}$ *and* $0 \le i_3^{p,q} \le h^{p+3,q-2} - h^{p+2,q-2} + k_1^{p+2,q-2} - k_1^{p+1,q-1} + h^{p,q-1} - k_1^{p,q-1} + k_2^{p+1,q-1}$. *Then we have*

$$\begin{aligned}
i_2^{0,3} &= k_2^{0,3} - k_3^{0,3}, \\
i_2^{0,2} &= k_2^{0,2} - k_3^{0,2},
\end{aligned}$$

Then since we have

$$E_4^{p,q} = \frac{\ker d_3^{p,q}}{\operatorname{Im} d_3^{p+3,q-2}}$$

the fourth page would be equal to

$$\begin{array}{cccc}
\mathbb{C}^{k_3^{0,3}} & \mathbb{C}^{k_2^{1,3}} & \mathbb{C}^{h^{2,3}-h^{1,3}+k_1^{1,3}} & \mathbb{C} \\
\mathbb{C}^{k_3^{0,2}} & \mathbb{C}^{k_2^{1,2}} & \mathbb{C}^{k_1^{2,2}-h^{1,2}+k_1^{1,2}-k_1^{0,3}+k_2^{0,3}} & \mathbb{C}^{h^{3,2}-h^{2,2}+k_1^{2,2}-k_1^{1,3}+h^{0,3}-k_1^{0,3}+k_2^{1,3}} \\
\mathbb{C}^{k_2^{0,1}} & \mathbb{C}^{k_2^{1,1}} & \mathbb{C}^{k_1^{2,1}-h^{1,1}+k_1^{1,1}-k_1^{0,2}+k_2^{0,2}} & \mathbb{C}^{h^{3,1}-h^{2,1}+k_1^{2,1}-k_1^{1,2}+h^{0,2}-k_1^{0,2}+k_2^{1,2}-k_2^{0,3}+k_3^{0,3}} \\
\mathbb{C} & \mathbb{C}^{k_1^{1,0}} & \mathbb{C}^{k_1^{2,0}-h^{1,0}+k_1^{1,0}-k_1^{0,1}+k_2^{0,1}} & \mathbb{C}^{h^{3,0}-h^{2,0}+k_1^{2,0}-k_1^{1,1}+h^{0,1}-k_1^{0,1}+k_2^{1,1}-k_2^{0,2}+k_3^{0,2}}
\end{array}$$

And it stabilizes at the fourth page since there exist no more nontrivial maps. So we can use the Frölicher's theorem to obtain the betti numbers now:

$$\begin{aligned}
b^1 &= k_1^{1,0} + k_2^{0,1} \\
b^2 &= k_1^{2,0} - h^{1,0} + k_1^{1,0} - k_1^{0,1} + k_2^{0,1} + k_2^{1,1} + k_3^{0,2} \\
b^3 &= h^{3,0} - h^{2,0} + k_1^{2,0} - k_1^{1,1} + h^{0,1} - k_1^{0,1} + k_2^{1,1} - k_2^{0,2} + k_3^{0,2} \\
&\quad + k_1^{2,1} - h^{1,1} + k_1^{1,1} - k_1^{0,2} + k_2^{0,2} + k_2^{1,2} + k_3^{0,3} \\
b^4 &= h^{3,1} - h^{2,1} + k_1^{2,1} - k_1^{1,2} + h^{0,2} - k_1^{0,2} + k_2^{1,2} - k_2^{0,3} + k_3^{0,3} \\
&\quad + k_1^{2,2} - h^{1,2} + k_1^{1,2} - k_1^{0,3} + k_2^{0,3} + k_2^{1,3} \\
b^5 &= h^{3,2} - h^{2,2} + k_1^{2,2} - k_1^{1,3} + h^{0,3} - k_1^{0,3} + k_2^{1,3} + h^{2,3} - h^{1,3} + k_1^{1,3}
\end{aligned}$$

Thus we get the Euler number of the complex manifold as

$$\begin{aligned}
\chi(M) &= b^0 - b^1 + b^2 - b^3 + b^4 - b^5 + b^6 \\
&= 1 - \left(k_1^{1,0} + k_2^{0,1}\right) + \left(k_1^{2,0} - h^{1,0} + k_1^{1,0} - k_1^{0,1} + k_2^{0,1} + k_2^{1,1} + k_3^{0,2}\right) \\
&\quad - \left(h^{3,0} - h^{2,0} + k_1^{2,0} - k_1^{1,1} + h^{0,1} - k_1^{0,1} + k_2^{1,1} - k_2^{0,2} + k_3^{0,2}\right) \\
&\quad - \left(k_1^{2,1} - h^{1,1} + k_1^{1,1} - k_1^{0,2} + k_2^{0,2} + k_2^{1,2} + k_3^{0,3}\right) \\
&\quad + \left(h^{3,1} - h^{2,1} + k_1^{2,1} - k_1^{1,2} + h^{0,2} - k_1^{0,2} + k_2^{1,2} - k_2^{0,3} + k_3^{0,3}\right) \\
&\quad + \left(k_1^{2,2} - h^{1,2} + k_1^{1,2} - k_1^{0,3} + k_2^{0,3} + k_2^{1,3}\right) \\
&\quad - \left(h^{3,2} - h^{2,2} + k_1^{2,2} - k_1^{1,3} + h^{0,3} - k_1^{0,3} + k_2^{1,3} + h^{2,3} - h^{1,3} + k_1^{1,3}\right) + 1 \\
&= 2 + h^{2,0} + h^{1,1} + h^{0,2} + h^{3,1} + h^{2,2} + h^{1,3} \\
&\quad - h^{1,0} - h^{0,1} - h^{3,0} - h^{2,1} - h^{1,2} - h^{3,2} - h^{0,3} - h^{2,3}.
\end{aligned}$$

□

Theorem 3.0.3 *Let M be a compact complex 6-dimensional manifold. Assume for each harmonic $(p,q) = (1,0),\ (0,1),\ (2,1),\ (1,2),\ (3,2),\ (2,3)-$forms, $F(R)\,\alpha^{p,q}$ is strictly diagonally dominant and the condition hold*

$$\begin{aligned}
0 \leq\ & +2\,\mathrm{Re}\left(\Delta_R\left(\alpha^{p,q}\right), \alpha^{p,q}\right)_{L^2} - \frac{1}{2}\left(\left[\left[L_{\overline{\partial}\Omega}, \Lambda\right], \left[L, \left(L_{\overline{\partial}\Omega}\right)^*\right]\right]\left(\alpha^{p,q}\right), \alpha^{p,q}\right)_{L^2} \\
& + \frac{1}{2}\left(\left[T_1, (T_1)^*\right]\left(\alpha^{p,q}\right), \alpha^{p,q}\right)_{L^2} - \frac{1}{2}\left(\left[T_2, (T_2)^*\right]\left(\alpha^{p,q}\right), \alpha^{p,q}\right)_{L^2}
\end{aligned}$$

then the FSS stabilizes at the second page and also $\chi(M) > 0$.

Proof 3.0.3 *Assume for each harmonic $(p,q) = (1,0),\ (0,1),\ (2,1),\ (1,2),\ (3,2),\ (2,3)-$forms,*

$F(R)\,\alpha^{p,q}$ is strictly diagonally dominant and the condition hold

$$\begin{aligned} 0 \;\leq\; & +2\operatorname{Re}\left(\Delta_R\left(\alpha^{p,q}\right),\alpha^{p,q}\right)_{L^2} - \frac{1}{2}\left(\left[\left[L_{\overline{\partial}\Omega},\Lambda\right],\left[L,\left(L_{\overline{\partial}\Omega}\right)^*\right]\right]\left(\alpha^{p,q}\right),\alpha^{p,q}\right)_{L^2} \\ & +\frac{1}{2}\left(\left[T_1,\left(T_1\right)^*\right]\left(\alpha^{p,q}\right),\alpha^{p,q}\right)_{L^2} - \frac{1}{2}\left(\left[T_2,\left(T_2\right)^*\right]\left(\alpha^{p,q}\right),\alpha^{p,q}\right)_{L^2} \end{aligned}$$

then by the Vanishing theorems for the corresponding Dolbeault cohomology groups are equal to 0. Then the first page of the (FSS) would be as below:

$$\begin{array}{ccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
0 & \overset{0}{\to} & E^{3,0} & \overset{d_0}{\to} & E^{3,1} & \overset{d_0}{\to} & E^{3,2} & \overset{d_0}{\to} & E^{3,3} & \overset{0}{\to} & 0 \\
0 & \overset{0}{\to} & E^{2,0} & \overset{d_0}{\to} & E^{2,1} & \overset{d_0}{\to} & E^{2,2} & \overset{d_0}{\to} & E^{2,3} & \overset{0}{\to} & 0 \\
0 & \overset{0}{\to} & E^{1,0} & \overset{d_0}{\to} & E^{1,1} & \overset{d_0}{\to} & E^{1,2} & \overset{d_0}{\to} & E^{1,3} & \overset{0}{\to} & 0 \\
0 & \overset{0}{\to} & E^{0,0} & \overset{d_0}{\to} & E^{0,1} & \overset{d_0}{\to} & E^{0,2} & \overset{d_0}{\to} & E^{0,3} & \overset{0}{\to} & 0 \\
0 & & 0 & & 0 & & 0 & & 0 & & 0
\end{array}$$

The groups on the first page are the Dolbeault cohomology groups:

$$\begin{array}{ccccccccccc}
0 & \overset{0}{\to} & H^{0,3}=0 & \overset{d_1^{0,3}}{\to} & H^{1,3} & \overset{d_1^{1,3}}{\to} & H^{2,3}=0 & \overset{d_1^{2,3}}{\to} & H^{3,3} & \overset{d_1^{3,3}=0}{\to} & 0 \\
0 & \overset{0}{\to} & H^{0,2} & \overset{d_1^{0,2}}{\to} & H^{1,2}=0 & \overset{d_1^{1,2}}{\to} & H^{2,2} & \overset{d_1^{2,2}}{\to} & H^{3,2}=0 & \overset{d_1^{3,2}=0}{\to} & 0 \\
0 & \overset{0}{\to} & H^{0,1}=0 & \overset{d_1^{0,1}}{\to} & H^{1,1} & \overset{d_1^{1,1}}{\to} & H^{2,1}=0 & \overset{d_1^{2,1}}{\to} & H^{3,1} & \overset{d_1^{3,1}=0}{\to} & 0 \\
0 & \overset{0}{\to} & H^{0,0} & \overset{d_1^{0,0}}{\to} & H^{1,0}=0 & \overset{d_1^{1,0}}{\to} & H^{2,0} & \overset{d_1^{2,0}}{\to} & H^{3,0}=0 & \overset{d_1^{3,0}=0}{\to} & 0
\end{array}$$

or in other words

$$\begin{array}{ccccccccccc}
0 & \overset{0}{\to} & 0 & \overset{0}{\to} & \mathbb{C}^{h^{1,3}} & \overset{0}{\to} & 0 & \overset{0}{\to} & \mathbb{C}^{h^{3,3}}=\mathbb{C} & \overset{0}{\to} & 0 \\
0 & \overset{0}{\to} & \mathbb{C}^{h^{0,2}} & \overset{0}{\to} & 0 & \overset{0}{\to} & \mathbb{C}^{h^{2,2}} & \overset{0}{\to} & 0 & \overset{0}{\to} & 0 \\
0 & \overset{0}{\to} & 0 & \overset{0}{\to} & \mathbb{C}^{h^{1,1}} & \overset{0}{\to} & 0 & \overset{0}{\to} & \mathbb{C}^{h^{3,1}} & \overset{0}{\to} & 0 \\
0 & \overset{0}{\to} & \mathbb{C}^{h^{0,0}}=\mathbb{C} & \overset{0}{\to} & 0 & \overset{0}{\to} & \mathbb{C}^{h^{2,0}} & \overset{0}{\to} & 0 & \overset{0}{\to} & 0
\end{array}$$

Now let's denote the rank of $H^{p,q} = h^{p,q}$, *rank*$(\ker d_1^{p,q}) = k_1^{p,q}$, *rank*$(\operatorname{Im} d_1^{p,q}) = i_1^{p,q}$ *such that* $0 \leq k_1^{p,q} \leq h^{p,q}$ *and* $0 \leq i_1^{p,q} \leq h^{p+1,q}$. *In particular, if we assume that the complex manifold is connected. Then we have* $h^{0,0} = 1$ *and* $h^{3,3} = 1$. *Due to the assumption that all* $h^{p,q}$ *with* $p + q = 2m + 1$ *are all zero, all the boundary maps have to be zero maps, thus the FSS stabilizes and we obtain the Betti numbers as equal to*

$$\begin{aligned} b^1 &= 0, \qquad b^2 = h^{2,0} + h^{1,1} + h^{0,2}, \qquad b^3 = 0, \\ b^4 &= h^{3,1} + h^{2,2} + h^{1,3}, \qquad b^5 = 0 \end{aligned}$$

Thus we get the Euler number of the complex manifold as

$$\begin{aligned} \chi(M) &= b^0 - b^1 + b^2 - b^3 + b^4 - b^5 + b^6 \\ &= 2 + \left(h^{2,0} + h^{1,1} + h^{0,2}\right) + \left(h^{3,1} + h^{2,2} + h^{1,3}\right) \end{aligned}$$

Thus we also have $\chi(M) > 0$. $\square$

Appendix A

Appendices

Throughout our computations, we labeled our terms in the following way:

$$\begin{aligned} &+A && (1)\\ &+B && (2)\\ = \;&+C && (1a)\\ &+D && (1b)\\ &+E && (2a)\\ &+F && (2b) \end{aligned}$$

if $A = C + D$ and $B = E + F$

$$\begin{aligned} = \;&+G && (1a1)\\ &+H && (1a2)\\ &+I && (1b)\\ &+J && (2a)\\ &+K && (2b1)\\ &+L && (2b2) \end{aligned}$$

if $C = G + H$ and $F = K + L$ and so on.

We use the following lemma in our computations for the Laplacians.

Lemma A.0.1 *Let α be a $(p,q)-$form. Then for any vector fields X, Y, we have*

$$\nabla_X(\iota_Y \alpha) = \iota_Y(\nabla_X \alpha) + \iota_{\nabla_X Y}\alpha$$

Proof A.0.4 *Let α be a $(p,q)-$form and X, Y be any two vector fields. For $(1,0)-$type vector fields $V_{i_1}, \ldots, V_{i_p}$ and $(0,1)-$type vector fields $\overline{V}_{j_1}, \ldots, \overline{V}_{j_q}$, we have*

$$\begin{aligned}
& \nabla_X(\iota_Y \alpha)\left(V_{i_1}, \ldots, V_{i_p}, \overline{V}_{j_1}, \ldots, \overline{V}_{j_q}\right) \\
= \; & X\left((\iota_Y \alpha)\left(V_{i_1}, \ldots, V_{i_p}, \overline{V}_{j_1}, \ldots, \overline{V}_{j_q}\right)\right) \\
& -(-1)^r (\iota_Y \alpha)\left(V_{i_1}, \ldots, \nabla_X V_{i_r}, \ldots, V_{i_p}, \overline{V}_{j_1}, \ldots, \overline{V}_{j_q}\right) \\
& -(-1)^{p+s} (\iota_Y \alpha)\left(V_{i_1}, \ldots, V_{i_p}, \overline{V}_{j_1}, \ldots, \nabla_X \overline{V}_{j_s}, \ldots, \overline{V}_{j_q}\right) \\
= \; & X\left(\alpha\left(Y, V_{i_1}, \ldots, V_{i_p}, \overline{V}_{j_1}, \ldots, \overline{V}_{j_q}\right)\right) \\
& +\alpha\left(\nabla_X Y, V_{i_1}, \ldots, V_{i_r}, \ldots, V_{i_p}\right) \\
& -(-1)^r \alpha\left(Y, V_{i_1}, \ldots, \nabla_X V_{i_r}, \ldots, V_{i_p}, \overline{V}_{j_1}, \ldots, \overline{V}_{j_q}\right) \\
& -(-1)^{p+s} \alpha\left(Y, V_{i_1}, \ldots, V_{i_p}, \overline{V}_{j_1}, \ldots, \nabla_X \overline{V}_{j_s}, \ldots, \overline{V}_{j_q}\right) \\
= \; & \iota_Y(\nabla_X \alpha)\left(V_{i_1}, \ldots, V_{i_p}, \overline{V}_{j_1}, \ldots, \overline{V}_{j_q}\right) \\
& +\iota_{\nabla_X Y}\alpha\left(V_{i_1}, \ldots, V_{i_p}, \overline{V}_{j_1}, \ldots, \overline{V}_{j_q}\right)
\end{aligned}$$

A.1 Hodge Laplacian of a $(p,q)-$form

Let $\alpha \in E^{p,q}$ be any $(p,q)-$form, using the definitions for $\overline{\partial}$ and $\overline{\partial}^*$ in 1.4 and 1.9, the Hodge Laplacian of the $(p,q)-$form would be equal to

$$\Delta_H(\alpha) = \overline{\partial}^* \circ \overline{\partial}(\alpha) + \overline{\partial} \circ \overline{\partial}^*(\alpha)$$

$$
\begin{aligned}
\Delta_H(\alpha) \quad = \quad & +\iota_{V_a}\left(proj_{p+1,q}\left(\nabla_{\overline{V}_a}\left(\overline{\omega}^j \wedge proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (h1)\\
& -\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\nabla_{V_a}\left(\overline{\omega}^j \wedge proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (h2)\\
& +\iota_{V_a}\left(proj_{p+1,q}\left(\nabla_{\overline{V}_a}\left(\omega^j \wedge proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (h3)\\
& -\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\nabla_{V_a}\left(\omega^j \wedge proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (h4)\\
& +2\left\langle\nabla_{\overline{V}_a}V_a, V_b\right\rangle \iota_{\overline{V}_b}\left(\overline{\omega}^j \wedge proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) \quad (h5)\\
& +2\left\langle\nabla_{\overline{V}_a}V_b, V_c\right\rangle \omega^b \wedge \iota_{\overline{V}_c}\left(\iota_{V_a}\left(\overline{\omega}^j \wedge proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (h6)\\
& +2\left\langle\nabla_{\overline{V}_a}V_a, V_b\right\rangle \iota_{\overline{V}_b}\left(\omega^j \wedge proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right) \quad (h7)\\
& +2\left\langle\nabla_{\overline{V}_a}V_b, V_c\right\rangle \omega^b \wedge \iota_{\overline{V}_c}\left(\iota_{V_a}\left(\omega^j \wedge proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (h8)\\
& +\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\nabla_{\overline{V}_a}\left(\iota_{V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (h9)\\
& +\omega^a \wedge proj_{E^{p-1,q}}\left(\nabla_{V_a}\left(\iota_{V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (h10)\\
& -\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\nabla_{\overline{V}_a}\left(\iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (h11)\\
& -\omega^a \wedge proj_{E^{p-1,q}}\left(\nabla_{V_a}\left(\iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (h12)\\
& +\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\nabla_{\overline{V}_a}\left(2\left\langle\nabla_{\overline{V}_j}V_j, V_i\right\rangle \iota_{\overline{V}_i}(\alpha)\right)\right) \quad (h13)\\
& +\omega^a \wedge proj_{E^{p-1,q}}\left(\nabla_{V_a}\left(2\left\langle\nabla_{\overline{V}_j}V_j, V_i\right\rangle \iota_{\overline{V}_i}(\alpha)\right)\right) \quad (h14)\\
& +\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\nabla_{\overline{V}_a}\left(2\left\langle\nabla_{\overline{V}_j}V_i, V_k\right\rangle \omega^i \wedge \iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right)\right)\right) \quad (h15)\\
& +\omega^a \wedge proj_{E^{p-1,q}}\left(\nabla_{V_a}\left(2\left\langle\nabla_{\overline{V}_j}V_i, V_k\right\rangle \omega^i \wedge \iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right)\right)\right) \quad (h16)
\end{aligned}
$$

If we apply the covariant derivative and the interior products, we have

$$
\begin{aligned}
\Delta_H(\alpha) \quad = \quad & +\iota_{V_a}\left(proj_{p+1,q}\left(\nabla_{\overline{V}_a}\overline{\omega}^j \wedge proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (h1a)\\
& +\iota_{V_a}\left(proj_{p+1,q}\left(\overline{\omega}^j \wedge \nabla_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (h1b)\\
& -\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\nabla_{V_a}\overline{\omega}^j \wedge proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (h2a)\\
& -\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\overline{\omega}^j \wedge \nabla_{V_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (h2b)\\
& +\iota_{V_a}\left(proj_{p+1,q}\left(\nabla_{\overline{V}_a}\omega^j \wedge proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (h3a)\\
& +\iota_{V_a}\left(proj_{p+1,q}\left(\omega^j \wedge \nabla_{\overline{V}_a}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (h3b)\\
& -\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\nabla_{V_a}\omega^j \wedge proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (h4a)\\
& -\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\omega^j \wedge \nabla_{V_a}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (h4b)\\
& +2\left\langle\nabla_{\overline{V}_a}V_a, V_b\right\rangle \iota_{\overline{V}_b}\left(\overline{\omega}^j\right) \wedge proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right) \quad (h5a)\\
& -2\left\langle\nabla_{\overline{V}_a}V_a, V_b\right\rangle \overline{\omega}^j \wedge \iota_{\overline{V}_b}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) \quad (h5b)
\end{aligned}
$$

$$-2\left\langle \nabla_{\overline{V}_a} V_b, V_c\right\rangle \omega^b \wedge \iota_{\overline{V}_c}\left(\overline{\omega}^j \wedge \iota_{V_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (h6)$$
$$-2\left\langle \nabla_{\overline{V}_a} V_a, V_b\right\rangle \omega^j \wedge \iota_{\overline{V}_b}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right) \quad (h7)$$
$$+2\left\langle \nabla_{\overline{V}_a} V_b, V_c\right\rangle \omega^b \wedge \iota_{\overline{V}_c}\left(\iota_{V_a}\left(\omega^j\right) \wedge proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right) \quad (h8a)$$
$$-2\left\langle \nabla_{\overline{V}_a} V_b, V_c\right\rangle \omega^b \wedge \iota_{\overline{V}_c}\left(\omega^j \wedge \iota_{V_a}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (h8b)$$
$$+\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\iota_{V_j}\left(\nabla_{\overline{V}_a}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (h9a)$$
$$+\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\iota_{\nabla_{\overline{V}_a} V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (h9b)$$
$$+\omega^a \wedge proj_{E^{p-1,q}}\left(\iota_{V_j}\left(\nabla_{V_a}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (h10a)$$
$$+\omega^a \wedge proj_{E^{p-1,q}}\left(\iota_{\nabla_{V_a} V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (h10b)$$
$$-\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\iota_{\overline{V}_j}\left(\nabla_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (h11a)$$
$$-\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\iota_{\nabla_{\overline{V}_a}\overline{V}_j}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (h11b)$$
$$-\omega^a \wedge proj_{E^{p-1,q}}\left(\iota_{\overline{V}_j}\left(\nabla_{V_a}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (h12a)$$
$$-\omega^a \wedge proj_{E^{p-1,q}}\left(\iota_{\nabla_{V_a}\overline{V}_j}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (h12b)$$
$$+2\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\left\langle \nabla_{\overline{V}_a}\nabla_{\overline{V}_j} V_j, V_i\right\rangle \iota_{\overline{V}_i}(\alpha)\right) \quad (h13a)$$
$$+2\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\left\langle \nabla_{\overline{V}_j} V_j, \nabla_{\overline{V}_a} V_i\right\rangle \iota_{\overline{V}_i}(\alpha)\right) \quad (h13b)$$
$$+2\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\left\langle \nabla_{\overline{V}_j} V_j, V_i\right\rangle \nabla_{\overline{V}_a}\iota_{\overline{V}_i}(\alpha)\right) \quad (h13c)$$
$$+2\omega^a \wedge proj_{E^{p-1,q}}\left(\left\langle \nabla_{V_a}\nabla_{\overline{V}_j} V_j, V_i\right\rangle \iota_{\overline{V}_i}(\alpha)\right) \quad (h14a)$$
$$+2\omega^a \wedge proj_{E^{p-1,q}}\left(\left\langle \nabla_{\overline{V}_j} V_j, \nabla_{V_a} V_i\right\rangle \iota_{\overline{V}_i}(\alpha)\right) \quad (h14b)$$
$$+2\omega^a \wedge proj_{E^{p-1,q}}\left(\left\langle \nabla_{\overline{V}_j} V_j, V_i\right\rangle \nabla_{V_a}\left(\iota_{\overline{V}_i}(\alpha)\right)\right) \quad (h14c)$$
$$+2\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\left(\left\langle \nabla_{\overline{V}_a}\nabla_{\overline{V}_j} V_i, V_k\right\rangle \omega^i \wedge \iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right)\right)\right) \quad (h15a)$$
$$+2\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\left(\left\langle \nabla_{\overline{V}_j} V_i, \nabla_{\overline{V}_a} V_k\right\rangle \omega^i \wedge \iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right)\right)\right) \quad (h15b)$$
$$+2\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\left(\left\langle \nabla_{\overline{V}_j} V_i, V_k\right\rangle \nabla_{\overline{V}_a}\omega^i \wedge \iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right)\right)\right) \quad (h15c)$$
$$+2\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\left\langle \nabla_{\overline{V}_j} V_i, V_k\right\rangle \omega^i \wedge \nabla_{\overline{V}_a}\left(\iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right)\right)\right) \quad (h15d)$$
$$+2\omega^a \wedge proj_{E^{p-1,q}}\left(\left(\left\langle \nabla_{V_a}\nabla_{\overline{V}_j} V_i, V_k\right\rangle \omega^i \wedge \iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right)\right)\right) \quad (h16a)$$
$$+2\omega^a \wedge proj_{E^{p-1,q}}\left(\left(\left\langle \nabla_{\overline{V}_j} V_i, \nabla_{V_a} V_k\right\rangle \omega^i \wedge \iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right)\right)\right) \quad (h16b)$$
$$+2\omega^a \wedge proj_{E^{p-1,q}}\left(\left(\left\langle \nabla_{\overline{V}_j} V_i, V_k\right\rangle \nabla_{V_a}\omega^i \wedge \iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right)\right)\right) \quad (h16c)$$
$$+2\omega^a \wedge proj_{E^{p-1,q}}\left(\left\langle \nabla_{\overline{V}_j} V_i, V_k\right\rangle \omega^i \wedge \nabla_{V_a}\left(\iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right)\right)\right) \quad (h16d)$$

If we separate the necessary covariant derivative terms into (1,0) and (0,1) directions and apply the interior products, we will get

$$
\begin{aligned}
&\Delta_H(\alpha) \\
= \ & +\iota_{V_a}\left(proj_{p+1,q}\left(\left[\left\langle \nabla_{\overline{V}_a}\overline{\omega}^j, \omega^k\right\rangle \overline{\omega}^k + \left\langle \nabla_{\overline{V}_a}\overline{\omega}^j, \overline{\omega}^k\right\rangle \omega^k\right] \wedge proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (h1a) \\
& +\iota_{V_a}\left(proj_{p+1,q}\left(\overline{\omega}^j \wedge \nabla_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (h1b) \\
& -\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\left[\left\langle \nabla_{V_a}\overline{\omega}^j, \omega^k\right\rangle \overline{\omega}^k + \left\langle \nabla_{V_a}\overline{\omega}^j, \overline{\omega}^k\right\rangle \omega^k\right] \wedge proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (h2a) \\
& -\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\overline{\omega}^j \wedge \nabla_{V_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (h2b) \\
& +\iota_{V_a}\left(proj_{p+1,q}\left(\left[\left\langle \nabla_{\overline{V}_a}\omega^j, \omega^k\right\rangle \overline{\omega}^k + \left\langle \nabla_{\overline{V}_a}\omega^j, \overline{\omega}^k\right\rangle \omega^k\right] \wedge proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (h3a) \\
& +\iota_{V_a}\left(proj_{p+1,q}\left(\omega^j \wedge \nabla_{\overline{V}_a}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (h3b) \\
& -\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\left[\left\langle \nabla_{V_a}\omega^j, \omega^k\right\rangle \overline{\omega}^k + \left\langle \nabla_{V_a}\omega^j, \overline{\omega}^k\right\rangle \omega^k\right] \wedge proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (h4a) \\
& -\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\omega^j \wedge \nabla_{V_a}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (h4b) \\
& +2\left\langle \nabla_{\overline{V}_a}V_a, V_b\right\rangle \iota_{\overline{V}_b}\left(\overline{\omega}^j\right) \wedge proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right) \quad (h5a) \\
& -2\left\langle \nabla_{\overline{V}_a}V_a, V_b\right\rangle \overline{\omega}^j \wedge \iota_{\overline{V}_b}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) \quad (h5b) \\
& -2\left\langle \nabla_{\overline{V}_a}V_b, V_c\right\rangle \omega^b \wedge \iota_{\overline{V}_c}\left(\overline{\omega}^j\right) \wedge \iota_{V_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) \quad (h6a) \\
& +2\left\langle \nabla_{\overline{V}_a}V_b, V_c\right\rangle \omega^b \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_c}\left(\iota_{V_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (h6b) \\
& -2\left\langle \nabla_{\overline{V}_a}V_a, V_b\right\rangle \omega^j \wedge \iota_{\overline{V}_b}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right) \quad (h7) \\
& +2\left\langle \nabla_{\overline{V}_j}V_b, V_c\right\rangle \omega^b \wedge \iota_{\overline{V}_c}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right) \quad (h8a) \\
& +2\left\langle \nabla_{\overline{V}_a}V_b, V_c\right\rangle \omega^b \wedge \omega^j \wedge \iota_{\overline{V}_c}\left(\iota_{V_a}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (h8b) \\
& +\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\iota_{V_j}\left(\nabla_{\overline{V}_a}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (h9a) \\
& +\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\iota_{\left\langle \nabla_{\overline{V}_a}V_j, V_k\right\rangle \overline{V}_k + \left\langle \nabla_{\overline{V}_a}V_j, \overline{V}_k\right\rangle V_k}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (h9b) \\
& +\omega^a \wedge proj_{E^{p-1,q}}\left(\iota_{V_j}\left(\nabla_{V_a}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (h10a) \\
& +\omega^a \wedge proj_{E^{p-1,q}}\left(\iota_{\left\langle \nabla_{V_a}V_j, V_k\right\rangle \overline{V}_k + \left\langle \nabla_{V_a}V_j, \overline{V}_k\right\rangle V_k}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (h10b) \\
& -\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\iota_{\overline{V}_j}\left(\nabla_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (h11a) \\
& -\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\iota_{\left\langle \nabla_{\overline{V}_a}\overline{V}_j, V_k\right\rangle \overline{V}_k + \left\langle \nabla_{\overline{V}_a}\overline{V}_j, \overline{V}_k\right\rangle V_k}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (h11b) \\
& -\omega^a \wedge proj_{E^{p-1,q}}\left(\iota_{\overline{V}_j}\left(\nabla_{V_a}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (h12a) \\
& -\omega^a \wedge proj_{E^{p-1,q}}\left(\iota_{\left\langle \nabla_{V_a}\overline{V}_j, V_k\right\rangle \overline{V}_k + \left\langle \nabla_{V_a}\overline{V}_j, \overline{V}_k\right\rangle V_k}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (h12b)
\end{aligned}
$$

$$+2\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\left\langle \nabla_{\overline{V}_a}\nabla_{\overline{V}_j}V_j, V_i\right\rangle \iota_{\overline{V}_i}(\alpha)\right) \quad (h13a)$$
$$+2\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\left\langle \nabla_{\overline{V}_j}V_j, \left\langle \nabla_{\overline{V}_a}V_i, V_k\right\rangle \overline{V}_k + \left\langle \nabla_{\overline{V}_a}V_i, \overline{V}_k\right\rangle V_k\right\rangle \iota_{\overline{V}_i}(\alpha)\right) \quad (h13b)$$
$$+2\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\left\langle \nabla_{\overline{V}_j}V_j, V_i\right\rangle \iota_{\overline{V}_i}\left(\nabla_{\overline{V}_a}(\alpha)\right)\right) \quad (h13c1)$$
$$+2\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\left\langle \nabla_{\overline{V}_j}V_j, V_i\right\rangle \left(\iota_{\nabla_{\overline{V}_a}\overline{V}_i}(\alpha)\right)\right) \quad (h13c2)$$
$$+2\omega^a \wedge proj_{E^{p-1,q}}\left(\left\langle \nabla_{V_a}\nabla_{\overline{V}_j}V_j, V_i\right\rangle \iota_{\overline{V}_i}(\alpha)\right) \quad (h14a)$$
$$+2\omega^a \wedge proj_{E^{p-1,q}}\left(\left\langle \nabla_{\overline{V}_j}V_j, \left\langle \nabla_{V_a}V_i, V_k\right\rangle \overline{V}_k + \left\langle \nabla_{V_a}V_i, \overline{V}_k\right\rangle V_k\right\rangle \iota_{\overline{V}_i}(\alpha)\right) \quad (h14b)$$
$$+2\omega^a \wedge proj_{E^{p-1,q}}\left(\left\langle \nabla_{\overline{V}_j}V_j, V_i\right\rangle \iota_{\overline{V}_i}\left(\nabla_{V_a}(\alpha)\right)\right) \quad (h14c1)$$
$$+2\omega^a \wedge proj_{E^{p-1,q}}\left(\left\langle \nabla_{\overline{V}_j}V_j, V_i\right\rangle \left(\iota_{\nabla_{V_a}\overline{V}_i}(\alpha)\right)\right) \quad (h14c2)$$
$$+2\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\left\langle \nabla_{\overline{V}_a}\nabla_{\overline{V}_j}V_i, V_k\right\rangle \omega^i \wedge \iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right)\right) \quad (h15a)$$
$$+2\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\left\langle \nabla_{\overline{V}_j}V_i, \left\langle \nabla_{\overline{V}_a}V_k, V_b\right\rangle \overline{V}_b + \left\langle \nabla_{\overline{V}_a}V_k, \overline{V}_b\right\rangle V_b\right\rangle \omega^i \wedge \iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right)\right) \quad (h15b)$$
$$+2\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\left\langle \nabla_{\overline{V}_j}V_i, V_k\right\rangle \left[\left\langle \nabla_{\overline{V}_a}\omega^i, \omega^b\right\rangle \overline{\omega}^b + \left\langle \nabla_{\overline{V}_a}\omega^i, \overline{\omega}^b\right\rangle \omega^b\right] \wedge \iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right)\right) \quad (h15c)$$
$$+2\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\left\langle \nabla_{\overline{V}_j}V_i, V_k\right\rangle \omega^i \wedge \iota_{\overline{V}_k}\left(\iota_{V_j}\left(\nabla_{\overline{V}_a}(\alpha)\right)\right)\right) \quad (h15d1)$$
$$+2\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\left\langle \nabla_{\overline{V}_j}V_i, V_k\right\rangle \omega^i \wedge \iota_{\overline{V}_k}\left(\iota_{\nabla_{\overline{V}_a}V_j}(\alpha)\right)\right) \quad (h15d2)$$
$$+2\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\left\langle \nabla_{\overline{V}_j}V_i, V_k\right\rangle \omega^i \wedge \iota_{\nabla_{\overline{V}_a}\overline{V}_k}\left(\iota_{V_j}(\alpha)\right)\right) \quad (h15d3)$$
$$+2\omega^a \wedge proj_{E^{p-1,q}}\left(\left\langle \nabla_{V_a}\nabla_{\overline{V}_j}V_i, V_k\right\rangle \omega^i \wedge \iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right)\right) \quad (h16a)$$
$$+2\omega^a \wedge proj_{E^{p-1,q}}\left(\left\langle \nabla_{\overline{V}_j}V_i, \left\langle \nabla_{V_a}V_k, V_b\right\rangle \overline{V}_b + \left\langle \nabla_{V_a}V_k, \overline{V}_b\right\rangle V_b\right\rangle \omega^i \wedge \iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right)\right) \quad (h16b)$$
$$+2\omega^a \wedge proj_{E^{p-1,q}}\left(\left\langle \nabla_{\overline{V}_j}V_i, V_k\right\rangle \left[\left\langle \nabla_{V_a}\omega^i, \omega^b\right\rangle \overline{\omega}^b + \left\langle \nabla_{V_a}\omega^i, \overline{\omega}^b\right\rangle \omega^b\right] \wedge \iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right)\right) \quad (h16c)$$
$$+2\omega^a \wedge proj_{E^{p-1,q}}\left(\left\langle \nabla_{\overline{V}_j}V_i, V_k\right\rangle \omega^i \wedge \iota_{\overline{V}_k}\left(\iota_{V_j}\left(\nabla_{V_a}(\alpha)\right)\right)\right) \quad (h16d1)$$
$$+2\omega^a \wedge proj_{E^{p-1,q}}\left(\left\langle \nabla_{\overline{V}_j}V_i, V_k\right\rangle \omega^i \wedge \iota_{\overline{V}_k}\left(\iota_{\nabla_{V_a}V_j}(\alpha)\right)\right) \quad (h16d2)$$
$$+2\omega^a \wedge proj_{E^{p-1,q}}\left(\left\langle \nabla_{\overline{V}_j}V_i, V_k\right\rangle \omega^i \wedge \iota_{\nabla_{V_a}\overline{V}_k}\left(\iota_{V_j}(\alpha)\right)\right) \quad (h16d3)$$

Separate the necessary covariant derivative terms into (1,0) and (0,1) directions and apply the interior products and take the projections to obtain

$$
\begin{aligned}
\Delta_H(\alpha) = \; & +\left\langle \nabla_{\overline{V}_a} V_j, V_a \right\rangle proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right) \quad (h1a1) \\
& -\left\langle \nabla_{\overline{V}_a} V_j, V_k \right\rangle \omega^k \wedge \iota_{V_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) \quad (h1a2) \\
& -\overline{\omega}^j \wedge \iota_{V_a}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (h1b) \\
& -\left\langle \nabla_{V_k} V_j, \overline{V}_k \right\rangle proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right) \quad (h2a1) \\
& +\left\langle \nabla_{V_a} V_j, \overline{V}_k \right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) \quad (h2a2) \\
& -proj_{p,q}\left(\nabla_{V_j}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (h2b1) \\
& +\overline{\omega}^j \wedge \iota_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{V_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (h2b2) \\
& +proj_{p,q}\left(\nabla_{\overline{V}_j}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (h3b1) \\
& -\omega^j \wedge \iota_{V_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_a}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (h3b2) \\
& +\left\langle \nabla_{V_a} \overline{V}_j, V_k \right\rangle \omega^k \wedge \iota_{\overline{V}_a}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right) \quad (h4a) \\
& +\omega^j \wedge \iota_{\overline{V}_a}\left(proj_{p-1,q+1}\left(\nabla_{V_a}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (h4b) \\
& +2\left\langle \nabla_{\overline{V}_a} V_a, V_j \right\rangle proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right) \quad (h5a) \\
& -2\left\langle \nabla_{\overline{V}_a} V_a, V_b \right\rangle \overline{\omega}^j \wedge \iota_{\overline{V}_b}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) \quad (h5b) \\
& -2\left\langle \nabla_{\overline{V}_a} V_b, V_j \right\rangle \omega^b \wedge \iota_{V_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) \quad (h6a) \\
& +2\left\langle \nabla_{\overline{V}_a} V_b, V_c \right\rangle \omega^b \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_c}\left(\iota_{V_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (h6b) \\
& -2\left\langle \nabla_{\overline{V}_a} V_a, V_b \right\rangle \omega^j \wedge \iota_{\overline{V}_b}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right) \quad (h7) \\
& +2\left\langle \nabla_{\overline{V}_j} V_b, V_c \right\rangle \omega^b \wedge \iota_{\overline{V}_c}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right) \quad (h8a) \\
& +2\left\langle \nabla_{\overline{V}_a} V_b, V_c \right\rangle \omega^b \wedge \omega^j \wedge \iota_{\overline{V}_c}\left(\iota_{V_a}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (h8b) \\
& +\overline{\omega}^a \wedge \iota_{V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_a}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (h9a) \\
& +\left\langle \nabla_{\overline{V}_a} V_j, \overline{V}_k \right\rangle \overline{\omega}^a \wedge \iota_{V_k}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) \quad (h9b) \\
& +\omega^a \wedge \iota_{V_j}\left(proj_{p,q}\left(\nabla_{V_a}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (h10a) \\
& -\overline{\omega}^a \wedge \iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (h11a) \\
& -\left\langle \nabla_{\overline{V}_a} \overline{V}_j, V_k \right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_k}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right) \quad (h11b) \\
& -\omega^a \wedge \iota_{\overline{V}_j}\left(proj_{p-1,q+1}\left(\nabla_{V_a}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (h12a) \\
& -\left\langle \nabla_{V_a} \overline{V}_j, \overline{V}_k \right\rangle \omega^a \wedge \iota_{V_k}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right) \quad (h12b)
\end{aligned}
$$

$$
\begin{aligned}
&+2\left\langle \nabla_{\overline{V}_a}\nabla_{\overline{V}_j}V_j, V_i\right\rangle \overline{\omega}^a\wedge\iota_{\overline{V}_i}(\alpha) \quad (h13a)\\
&+2\left\langle \nabla_{\overline{V}_j}V_j, V_k\right\rangle \left\langle \nabla_{\overline{V}_a}V_i, \overline{V}_k\right\rangle \overline{\omega}^a\wedge\iota_{\overline{V}_i}(\alpha) \quad (h13b1)\\
&+2\left\langle \nabla_{\overline{V}_j}V_j, \overline{V}_k\right\rangle \left\langle \nabla_{\overline{V}_a}V_i, V_k\right\rangle \overline{\omega}^a\wedge\iota_{\overline{V}_i}(\alpha) \quad (h13b2)\\
&+2\left\langle \nabla_{\overline{V}_j}V_j, V_i\right\rangle \overline{\omega}^a\wedge\iota_{\overline{V}_i}\left(proj_{p,q}\left(\nabla_{\overline{V}_a}(\alpha)\right)\right) \quad (h13c1)\\
&+2\left\langle \nabla_{\overline{V}_j}V_j, V_i\right\rangle \left\langle \nabla_{\overline{V}_a}\overline{V}_i, V_k\right\rangle \overline{\omega}^a\wedge\iota_{\overline{V}_k}(\alpha) \quad (h13c2)\\
&+2\left\langle \nabla_{\overline{V}_j}V_j, V_i\right\rangle \omega^a\wedge\iota_{\overline{V}_i}\left(proj_{p-1,q+1}\left(\nabla_{V_a}(\alpha)\right)\right) \quad (h14c1)\\
&+2\left\langle \nabla_{\overline{V}_j}V_j, V_i\right\rangle \left\langle \nabla_{V_a}\overline{V}_i, \overline{V}_k\right\rangle \omega^a\wedge\iota_{V_k}(\alpha) \quad (h14c2)\\
&+2\left\langle \nabla_{\overline{V}_a}\nabla_{\overline{V}_j}V_i, V_k\right\rangle \overline{\omega}^a\wedge\omega^i\wedge\iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right) \quad (h15a)\\
&+2\left\langle \nabla_{\overline{V}_j}V_i, \overline{V}_b\right\rangle \left\langle \nabla_{\overline{V}_a}V_k, V_b\right\rangle \overline{\omega}^a\wedge\omega^i\wedge\iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right) \quad (h15b1)\\
&+2\left\langle \nabla_{\overline{V}_j}V_i, V_b\right\rangle \left\langle \nabla_{\overline{V}_a}V_k, \overline{V}_b\right\rangle \overline{\omega}^a\wedge\omega^i\wedge\iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right) \quad (h15b2)\\
&+2\left\langle \nabla_{\overline{V}_j}V_i, V_k\right\rangle \left\langle \nabla_{\overline{V}_a}\overline{V}_i, V_b\right\rangle \overline{\omega}^a\wedge\omega^b\wedge\iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right) \quad (h15c)\\
&+2\left\langle \nabla_{\overline{V}_j}V_i, V_k\right\rangle \overline{\omega}^a\wedge\omega^i\wedge\iota_{\overline{V}_k}\left(\iota_{V_j}\left(proj_{p,q}\left(\nabla_{\overline{V}_a}(\alpha)\right)\right)\right) \quad (h15d1)\\
&+2\left\langle \nabla_{\overline{V}_j}V_i, V_k\right\rangle \left\langle \nabla_{\overline{V}_a}V_j, \overline{V}_b\right\rangle \overline{\omega}^a\wedge\omega^i\wedge\iota_{\overline{V}_k}\left(\iota_{V_b}(\alpha)\right) \quad (h15d2)\\
&+2\left\langle \nabla_{\overline{V}_j}V_i, V_k\right\rangle \left\langle \nabla_{\overline{V}_a}\overline{V}_k, V_b\right\rangle \overline{\omega}^a\wedge\omega^i\wedge\left(\iota_{\overline{V}_b}\left(\iota_{V_j}(\alpha)\right)\right) \quad (h15d3)\\
&+2\left\langle \nabla_{\overline{V}_j}V_i, V_k\right\rangle \left\langle \nabla_{V_a}\overline{V}_i, \overline{V}_b\right\rangle \omega^a\wedge\overline{\omega}^b\wedge\iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right) \quad (h16c)\\
&+2\left\langle \nabla_{\overline{V}_j}V_i, V_k\right\rangle \omega^a\wedge\omega^i\wedge\iota_{\overline{V}_k}\left(\iota_{V_j}\left(proj_{p-1,q+1}\left(\nabla_{V_a}(\alpha)\right)\right)\right) \quad (h16d1)\\
&+2\left\langle \nabla_{\overline{V}_j}V_i, V_k\right\rangle \left\langle \nabla_{V_a}\overline{V}_k, \overline{V}_b\right\rangle \omega^a\wedge\omega^i\wedge\left(\iota_{V_b}\left(\iota_{V_j}(\alpha)\right)\right) \quad (h16d3)
\end{aligned}
$$

Thus if we relabel the terms, we have

$$
\begin{aligned}
\Delta_H(\alpha) &= -\overline{\omega}^a\wedge\iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (h11a\leftrightarrow H1)\\
&+\overline{\omega}^j\wedge\iota_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{V_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (h2b2\leftrightarrow H2)\\
&-\overline{\omega}^j\wedge\iota_{V_a}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (h1b\leftrightarrow H3)\\
&+\overline{\omega}^a\wedge\iota_{V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_a}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (h9a\leftrightarrow H4)
\end{aligned}
$$

$$
\begin{aligned}
&- \omega^a \wedge \iota_{\overline{V}_j}\left(proj_{p-1,q+1}\left(\nabla_{V_a}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (h12a \leftrightarrow H5)\\
&+ \omega^j \wedge \iota_{\overline{V}_a}\left(proj_{p-1,q+1}\left(\nabla_{V_a}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (h4b \leftrightarrow H6)\\
&- \omega^j \wedge \iota_{V_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_a}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (h3b2 \leftrightarrow H7)\\
&+ \omega^a \wedge \iota_{V_j}\left(proj_{p,q}\left(\nabla_{V_a}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (h10a \leftrightarrow H8)\\
&- proj_{p,q}\left(\nabla_{V_j}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (h2b1 \leftrightarrow H9)\\
&+ proj_{p,q}\left(\nabla_{\overline{V}_j}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (h3b1 \leftrightarrow H10)\\
&+ 2\left\langle \nabla_{\overline{V}_a}\nabla_{\overline{V}_j}V_j, V_i\right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_i}(\alpha) \quad (h13a \leftrightarrow H11)\\
&+ 2\left\langle \nabla_{\overline{V}_a}\nabla_{\overline{V}_j}V_i, V_k\right\rangle \overline{\omega}^a \wedge \omega^i \wedge \iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right) \quad (h15a \leftrightarrow H12)\\
&+ \left\langle \nabla_{\overline{V}_a}V_a, V_j\right\rangle proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right) \quad (h1a1 + h5a \leftrightarrow H13)\\
&- \left\langle \nabla_{V_k}V_j, \overline{V}_k\right\rangle proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right) \quad (h2a1 \leftrightarrow H14)\\
&+ 2\left\langle \nabla_{\overline{V}_j}V_b, V_c\right\rangle \omega^b \wedge \iota_{\overline{V}_c}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right) \quad (h8a \leftrightarrow H15)\\
&+ \left\langle \nabla_{\overline{V}_a}V_j, V_k\right\rangle \omega^k \wedge \iota_{V_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) \quad (h1a2 + h6a \leftrightarrow H16)\\
&- \left\langle \nabla_{V_a}\overline{V}_j, \overline{V}_k\right\rangle \omega^a \wedge \iota_{V_k}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right) \quad (h12b \leftrightarrow H17)\\
&+ \left\langle \nabla_{V_a}\overline{V}_j, V_k\right\rangle \omega^k \wedge \iota_{\overline{V}_a}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right) \quad (h4a \leftrightarrow H18)\\
&+ \left\langle \nabla_{\overline{V}_a}V_j, \overline{V}_k\right\rangle \overline{\omega}^a \wedge \iota_{V_k}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) \quad (h9b \leftrightarrow H19)\\
&- \left\langle \nabla_{\overline{V}_a}\overline{V}_j, V_k\right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_k}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right) \quad (h11b \leftrightarrow H20)\\
&+ \left\langle \nabla_{V_a}V_j, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) \quad (h2a2 \leftrightarrow H21)\\
&+ 2\left\langle \nabla_{\overline{V}_j}V_j, \overline{V}_k\right\rangle \left\langle \nabla_{\overline{V}_a}V_i, V_k\right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_i}(\alpha) \quad (h13b2 \leftrightarrow H22)\\
&+ 2\left\langle \nabla_{\overline{V}_j}V_j, V_i\right\rangle \left\langle \nabla_{V_a}\overline{V}_i, \overline{V}_k\right\rangle \omega^a \wedge \iota_{V_k}(\alpha) \quad (h14c2 \leftrightarrow H23)\\
&+ 2\left\langle \nabla_{\overline{V}_j}V_i, \overline{V}_b\right\rangle \left\langle \nabla_{\overline{V}_a}V_k, V_b\right\rangle \overline{\omega}^a \wedge \omega^i \wedge \iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right) \quad (h15b1 \leftrightarrow H24)\\
&+ 2\left\langle \nabla_{\overline{V}_j}V_b, V_k\right\rangle \left\langle \nabla_{\overline{V}_a}\overline{V}_b, V_i\right\rangle \overline{\omega}^a \wedge \omega^i \wedge \iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right) \quad (h15c \leftrightarrow H25)\\
&+ 2\left\langle \nabla_{\overline{V}_j}V_i, V_k\right\rangle \left\langle \nabla_{\overline{V}_a}V_j, \overline{V}_b\right\rangle \overline{\omega}^a \wedge \omega^i \wedge \iota_{\overline{V}_k}\left(\iota_{V_b}(\alpha)\right) \quad (h15d2 \leftrightarrow H26)\\
&+ 2\left\langle \nabla_{\overline{V}_j}V_i, V_k\right\rangle \left\langle \nabla_{V_a}\overline{V}_i, \overline{V}_b\right\rangle \omega^a \wedge \overline{\omega}^b \wedge \iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right) \quad (h16c \leftrightarrow H27)\\
&+ 2\left\langle \nabla_{\overline{V}_j}V_i, V_k\right\rangle \left\langle \nabla_{V_a}\overline{V}_k, \overline{V}_b\right\rangle \omega^a \wedge \omega^i \wedge \left(\iota_{V_b}\left(\iota_{V_j}(\alpha)\right)\right) \quad (h16d3 \leftrightarrow H28)
\end{aligned}
$$

and the rest of the terms vanish as

$$
\begin{aligned}
(h5b)+(h13c1) &= 0, \qquad (h6b)+(h15d1)=0\\
(h7)+(h14c1) &= 0, \qquad (h8b)+(h16d1)=0\\
(h13b1)+(h13c2) &= 0, \qquad (h15b2)+(h15d3)=0
\end{aligned}
$$

A.2 Kähler Laplacian of a $(p,q)-$form

Let $\alpha \in E^{p,q}$ be any $(p,q)-$form, using the definitions for $\overline{\partial}_K$ and $\overline{\partial}_K^*$ in 1.11 and 1.13, the Kähler Laplacian is given by

$$
\begin{aligned}
&\Delta_K(\alpha) = \overline{\partial}_K\left(\overline{\partial}_K^*(\alpha)\right) + \overline{\partial}_K^*\left(\overline{\partial}_K(\alpha)\right)\\
= \;&+\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\nabla_{\overline{V}_a}\left(\iota_{V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (k1)\\
&-\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\nabla_{\overline{V}_a}\left(\iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (k2)\\
&+\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\nabla_{\overline{V}_a}\left(\left\langle\nabla_{\overline{V}_j}V_k, V_i\right\rangle \overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right)\right)\right) \quad (k3)\\
&+\omega^a \wedge proj_{E^{p-1,q}}\left(\nabla_{V_a}\left(\iota_{V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (k4)\\
&-\omega^a \wedge proj_{E^{p-1,q}}\left(\nabla_{V_a}\left(\iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (k5)\\
&+\omega^a \wedge proj_{E^{p-1,q}}\left(\nabla_{V_a}\left(\left\langle\nabla_{\overline{V}_j}V_k, V_i\right\rangle \overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right)\right)\right) \quad (k6)
\end{aligned}
$$

$$
\begin{aligned}
&-2\left\langle\nabla_{V_a}\overline{V}_a, \overline{V}_c\right\rangle \overline{\omega}^c \wedge \iota_{V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) \quad (k7)\\
&+2\left\langle\nabla_{V_a}\overline{V}_a, \overline{V}_c\right\rangle \overline{\omega}^c \wedge \iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right) \quad (k8)\\
&-2\left\langle\nabla_{V_a}\overline{V}_a, \overline{V}_c\right\rangle \overline{\omega}^c \wedge \left(\left\langle\nabla_{\overline{V}_j}V_k, V_i\right\rangle \overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right)\right) \quad (k9)\\
&-2\left\langle\nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle \omega^a \wedge \overline{\omega}^c \wedge \iota_{V_b}\left(\iota_{V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (k10)\\
&+2\left\langle\nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle \omega^a \wedge \overline{\omega}^c \wedge \iota_{V_b}\left(\iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (k11)\\
&-2\left\langle\nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle \omega^a \wedge \overline{\omega}^c \wedge \iota_{V_b}\left(\left\langle\nabla_{\overline{V}_j}V_k, V_i\right\rangle \overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right)\right) \quad (k12)\\
&+\left\langle\nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle \overline{\omega}^c \wedge \overline{\omega}^b \wedge \iota_{\overline{V}_a}\left(\iota_{V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (k13)\\
&-\left\langle\nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle \overline{\omega}^c \wedge \overline{\omega}^b \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (k14)\\
&+\left\langle\nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle \overline{\omega}^c \wedge \overline{\omega}^b \wedge \iota_{\overline{V}_a}\left(\left\langle\nabla_{\overline{V}_j}V_k, V_i\right\rangle \overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right)\right) \quad (k15)
\end{aligned}
$$

$$+\iota_{V_a}\left(proj_{p+1,q}\left(\nabla_{\overline{V}_a}\left(\overline{\omega}^j \wedge proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (k16)$$
$$-\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\nabla_{V_a}\left(\overline{\omega}^j \wedge proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (k17)$$
$$+\left\langle \nabla_{\overline{V}_a} V_c, V_b \right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_c}\left(\iota_{\overline{V}_b}\left(\overline{\omega}^j \wedge proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (k18)$$
$$+\iota_{V_a}\left(proj_{p+1,q}\left(\nabla_{\overline{V}_a}\left(\omega^j \wedge proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (k19)$$
$$-\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\nabla_{V_a}\left(\omega^j \wedge proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (k20)$$
$$+\left\langle \nabla_{\overline{V}_a} V_c, V_b \right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_c}\left(\iota_{\overline{V}_b}\left(\omega^j \wedge proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (k21)$$
$$-2\iota_{V_a}\left(proj_{p+1,q}\left(\nabla_{\overline{V}_a}\left(\left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \alpha\right)\right)\right) \quad (k22)$$
$$+2\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\nabla_{V_a}\left(\left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \alpha\right)\right)\right) \quad (k23)$$
$$-2\left\langle \nabla_{\overline{V}_a} V_c, V_b \right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_c}\left(\iota_{\overline{V}_b}\left(\left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \alpha\right)\right) \quad (k24)$$
$$-2\iota_{V_a}\left(proj_{p+1,q}\left(\nabla_{\overline{V}_a}\left(\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \omega^j \wedge \overline{\omega}^k \wedge \iota_{V_i}(\alpha)\right)\right)\right) \quad (k25)$$
$$+2\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\nabla_{V_a}\left(\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \omega^j \wedge \overline{\omega}^k \wedge \iota_{V_i}(\alpha)\right)\right)\right) \quad (k26)$$
$$-2\left\langle \nabla_{\overline{V}_a} V_c, V_b \right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_c}\left(\iota_{\overline{V}_b}\left(\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \omega^j \wedge \overline{\omega}^k \wedge \iota_{V_i}(\alpha)\right)\right) \quad (k27)$$
$$+\iota_{V_a}\left(proj_{p+1,q}\left(\nabla_{\overline{V}_a}\left(\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (k28)$$
$$-\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\nabla_{V_a}\left(\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (k29)$$
$$+\left\langle \nabla_{\overline{V}_a} V_c, V_b \right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_c}\left(\iota_{\overline{V}_b}\left(\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_j}(\alpha)\right)\right) \quad (k30)$$

If we apply the covariant derivatives and the interior products and use the commutativity property between these two operators, we have

$$
\begin{aligned}
\Delta_K(\alpha) \quad = \quad & +\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\iota_{V_j}\left(\nabla_{\overline{V}_a}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (k1a)\\
& +\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\iota_{\nabla_{\overline{V}_a}V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (k1b)\\
& -\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\iota_{\overline{V}_j}\left(\nabla_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (k2a)\\
& -\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\iota_{\nabla_{\overline{V}_a}\overline{V}_j}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (k2b)\\
& +\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\left\langle\nabla_{\overline{V}_a}\nabla_{\overline{V}_j}V_k, V_i\right\rangle\overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right)\right) \quad (k3a)\\
& +\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\left\langle\nabla_{\overline{V}_j}V_k, \nabla_{\overline{V}_a}V_i\right\rangle\overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right)\right) \quad (k3b)\\
& +\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\left\langle\nabla_{\overline{V}_j}V_k, V_i\right\rangle\nabla_{\overline{V}_a}\overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right)\right) \quad (k3c)\\
& +\overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\left\langle\nabla_{\overline{V}_j}V_k, V_i\right\rangle\overline{\omega}^j \wedge \nabla_{\overline{V}_a}\left(\iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right)\right)\right) \quad (k3d)\\
& +\omega^a \wedge proj_{E^{p-1,q}}\left(\iota_{V_j}\left(\nabla_{V_a}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (k4a)\\
& +\omega^a \wedge proj_{E^{p-1,q}}\left(\iota_{\nabla_{V_a}V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (k4b)\\
& -\omega^a \wedge proj_{E^{p-1,q}}\left(\iota_{\overline{V}_j}\left(\nabla_{V_a}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (k5a)\\
& -\omega^a \wedge proj_{E^{p-1,q}}\left(\iota_{\nabla_{V_a}\overline{V}_j}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (k5b)\\
& +\omega^a \wedge proj_{E^{p-1,q}}\left(\left\langle\nabla_{V_a}\nabla_{\overline{V}_j}V_k, V_i\right\rangle\overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right)\right) \quad (k6a)\\
& +\omega^a \wedge proj_{E^{p-1,q}}\left(\left\langle\nabla_{\overline{V}_j}V_k, \nabla_{V_a}V_i\right\rangle\overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right)\right) \quad (k6b)\\
& +\omega^a \wedge proj_{E^{p-1,q}}\left(\left\langle\nabla_{\overline{V}_j}V_k, V_i\right\rangle\nabla_{V_a}\overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right)\right) \quad (k6c)\\
& +\omega^a \wedge proj_{E^{p-1,q}}\left(\left\langle\nabla_{\overline{V}_j}V_k, V_i\right\rangle\overline{\omega}^j \wedge \nabla_{V_a}\left(\iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right)\right)\right) \quad (k6d)\\
& -2\left\langle\nabla_{V_a}\overline{V}_a, \overline{V}_c\right\rangle\overline{\omega}^c \wedge \left(\iota_{V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (k7)\\
& +2\left\langle\nabla_{V_a}\overline{V}_a, \overline{V}_c\right\rangle\overline{\omega}^c \wedge \left(\iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (k8)\\
& -2\left\langle\nabla_{V_a}\overline{V}_a, \overline{V}_c\right\rangle\left\langle\nabla_{\overline{V}_j}V_k, V_i\right\rangle\overline{\omega}^c \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right) \quad (k9)\\
& -2\left\langle\nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle\omega^a \wedge \overline{\omega}^c \wedge \iota_{V_b}\left(\iota_{V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (k10)\\
& +2\left\langle\nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle\omega^a \wedge \overline{\omega}^c \wedge \iota_{V_b}\left(\iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (k11)\\
& +2\left\langle\nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle\left\langle\nabla_{\overline{V}_j}V_k, V_i\right\rangle\omega^a \wedge \overline{\omega}^c \wedge \overline{\omega}^j \wedge \iota_{V_b}\left(\iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right)\right) \quad (k12)\\
& +\left\langle\nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle\overline{\omega}^c \wedge \overline{\omega}^b \wedge \iota_{\overline{V}_a}\left(\iota_{V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (k13)\\
& -\left\langle\nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle\overline{\omega}^c \wedge \overline{\omega}^b \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (k14)
\end{aligned}
$$

$$+\left\langle \nabla_{V_j}\overline{V}_b, \overline{V}_c\right\rangle \left\langle \nabla_{\overline{V}_j}V_k, V_i\right\rangle \overline{\omega}^c \wedge \overline{\omega}^b \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right) \quad (k15a)$$

$$-\left\langle \nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle \left\langle \nabla_{\overline{V}_j}V_k, V_i\right\rangle \overline{\omega}^c \wedge \overline{\omega}^b \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right)\right) \quad (k15b)$$

$$+\iota_{V_a}\left(proj_{p+1,q}\left(\nabla_{\overline{V}_a}\overline{\omega}^j \wedge proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (k16a)$$

$$+\iota_{V_a}\left(proj_{p+1,q}\left(\overline{\omega}^j \wedge \nabla_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (k16b)$$

$$-\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\nabla_{V_a}\overline{\omega}^j \wedge proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (k17a)$$

$$-\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\overline{\omega}^j \wedge \nabla_{V_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (k17b)$$

$$+\left\langle \nabla_{\overline{V}_a}V_c, V_b\right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_c}\left(\iota_{\overline{V}_b}\left(\overline{\omega}^j\right) \wedge proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) \quad (k18a)$$

$$-\left\langle \nabla_{\overline{V}_a}V_c, V_b\right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_c}\left(\overline{\omega}^j \wedge \iota_{\overline{V}_b}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (k18b)$$

$$+\iota_{V_a}\left(proj_{p+1,q}\left(\nabla_{\overline{V}_a}\omega^j \wedge proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (k19a)$$

$$+\iota_{V_a}\left(proj_{p+1,q}\left(\omega^j \wedge \nabla_{\overline{V}_a}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (k19b)$$

$$-\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\nabla_{V_a}\omega^j \wedge proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (k20a)$$

$$-\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\omega^j \wedge \nabla_{V_a}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (k20b)$$

$$+\left\langle \nabla_{\overline{V}_a}V_c, V_b\right\rangle \overline{\omega}^a \wedge \omega^j \wedge \iota_{\overline{V}_c}\left(\iota_{\overline{V}_b}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (k21)$$

$$-2\iota_{V_a}\left(proj_{p+1,q}\left(\left\langle \nabla_{\overline{V}_a}\nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \alpha\right)\right) \quad (k22a)$$

$$-2\iota_{V_a}\left(proj_{p+1,q}\left(\left\langle \nabla_{V_j}\overline{V}_j, \nabla_{\overline{V}_a}\overline{V}_k\right\rangle \overline{\omega}^k \wedge \alpha\right)\right) \quad (k22b)$$

$$-2\iota_{V_a}\left(proj_{p+1,q}\left(\left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle \nabla_{\overline{V}_a}\overline{\omega}^k \wedge \alpha\right)\right) \quad (k22c)$$

$$-2\iota_{V_a}\left(proj_{p+1,q}\left(\left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \nabla_{\overline{V}_a}\alpha\right)\right) \quad (k22d)$$

$$+2\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\left\langle \nabla_{V_a}\nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \alpha\right)\right) \quad (k23a)$$

$$+2\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\left\langle \nabla_{V_j}\overline{V}_j, \nabla_{V_a}\overline{V}_k\right\rangle \overline{\omega}^k \wedge \alpha\right)\right) \quad (k23b)$$

$$+2\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle \nabla_{V_a}\overline{\omega}^k \wedge \alpha\right)\right) \quad (k23c)$$

$$+2\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \nabla_{V_a}\alpha\right)\right) \quad (k23d)$$

$$-2\left\langle \nabla_{\overline{V}_a}V_c, V_b\right\rangle \left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_c}\left(\iota_{\overline{V}_b}\left(\overline{\omega}^k\right) \wedge \alpha\right) \quad (k24a)$$

$$+2\left\langle \nabla_{\overline{V}_a}V_c, V_b\right\rangle \left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_c}\left(\overline{\omega}^k \wedge \iota_{\overline{V}_b}(\alpha)\right) \quad (k24b)$$

$$-2\iota_{V_a}\left(proj_{p+1,q}\left(\left\langle \nabla_{\overline{V}_a}\nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \omega^j \wedge \overline{\omega}^k \wedge \iota_{V_i}(\alpha)\right)\right) \quad (k25a)$$

$$-2\iota_{V_a}\left(proj_{p+1,q}\left(\left\langle \nabla_{V_j}\overline{V}_i, \nabla_{\overline{V}_a}\overline{V}_k\right\rangle \omega^j \wedge \overline{\omega}^k \wedge \iota_{V_i}(\alpha)\right)\right) \quad (k25b)$$

$$-2\iota_{V_a}\left(proj_{p+1,q}\left(\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \nabla_{\overline{V}_a}\omega^j \wedge \overline{\omega}^k \wedge \iota_{V_i}(\alpha)\right)\right) \quad (k25c)$$

$$-2\iota_{V_a}\left(proj_{p+1,q}\left(\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \omega^j \wedge \nabla_{\overline{V}_a}\overline{\omega}^k \wedge \iota_{V_i}(\alpha)\right)\right) \quad (k25d)$$

$$-2\iota_{V_a}\left(proj_{p+1,q}\left(\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \omega^j \wedge \overline{\omega}^k \wedge \nabla_{\overline{V}_a}\left(\iota_{V_i}(\alpha)\right)\right)\right) \quad (k25e)$$

$$+2\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\left\langle \nabla_{V_a}\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle \omega^j\wedge\overline{\omega}^k\wedge\iota_{V_i}(\alpha)\right)\right)\quad (k26a)$$
$$+2\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\left\langle \nabla_{V_j}\overline{V}_i,\nabla_{V_a}\overline{V}_k\right\rangle \omega^j\wedge\overline{\omega}^k\wedge\iota_{V_i}(\alpha)\right)\right)\quad (k26b)$$
$$+2\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle \nabla_{V_a}\omega^j\wedge\overline{\omega}^k\wedge\iota_{V_i}(\alpha)\right)\right)\quad (k26c)$$
$$+2\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle \omega^j\wedge\nabla_{V_a}\overline{\omega}^k\wedge\iota_{V_i}(\alpha)\right)\right)\quad (k26d)$$
$$+2\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle \omega^j\wedge\overline{\omega}^k\wedge\nabla_{V_a}\left(\iota_{V_i}(\alpha)\right)\right)\right)\quad (k26e)$$
$$+2\left\langle \nabla_{\overline{V}_a}V_c,V_b\right\rangle\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\overline{\omega}^a\wedge\iota_{\overline{V}_c}\left(\omega^j\wedge\iota_{\overline{V}_b}\left(\overline{\omega}^k\right)\wedge\iota_{V_i}(\alpha)\right)\quad (k27a)$$
$$-2\left\langle \nabla_{\overline{V}_a}V_c,V_b\right\rangle\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\overline{\omega}^a\wedge\iota_{\overline{V}_c}\left(\omega^j\wedge\overline{\omega}^k\wedge\iota_{\overline{V}_b}\left(\iota_{V_i}(\alpha)\right)\right)\quad (k27b)$$
$$+\iota_{V_a}\left(proj_{p+1,q}\left(\left\langle \nabla_{\overline{V}_a}\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle \overline{\omega}^k\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_j}(\alpha)\right)\right)\quad (k28a)$$
$$+\iota_{V_a}\left(proj_{p+1,q}\left(\left\langle \nabla_{V_j}\overline{V}_i,\nabla_{\overline{V}_a}\overline{V}_k\right\rangle \overline{\omega}^k\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_j}(\alpha)\right)\right)\quad (k28b)$$
$$+\iota_{V_a}\left(proj_{p+1,q}\left(\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle \nabla_{\overline{V}_a}\overline{\omega}^k\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_j}(\alpha)\right)\right)\quad (k28c)$$
$$+\iota_{V_a}\left(proj_{p+1,q}\left(\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle \overline{\omega}^k\wedge\nabla_{\overline{V}_a}\overline{\omega}^i\wedge\iota_{\overline{V}_j}(\alpha)\right)\right)\quad (k28d)$$
$$+\iota_{V_a}\left(proj_{p+1,q}\left(\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle \overline{\omega}^k\wedge\overline{\omega}^i\wedge\nabla_{\overline{V}_a}\left(\iota_{\overline{V}_j}(\alpha)\right)\right)\right)\quad (k28e)$$
$$-\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\left\langle \nabla_{V_a}\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle \overline{\omega}^k\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_j}(\alpha)\right)\right)\quad (k29a)$$
$$-\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\left\langle \nabla_{V_j}\overline{V}_i,\nabla_{V_a}\overline{V}_k\right\rangle \overline{\omega}^k\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_j}(\alpha)\right)\right)\quad (k29b)$$
$$-\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle \nabla_{V_a}\overline{\omega}^k\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_j}(\alpha)\right)\right)\quad (k29c)$$
$$-\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle \overline{\omega}^k\wedge\nabla_{V_a}\overline{\omega}^i\wedge\iota_{\overline{V}_j}(\alpha)\right)\right)\quad (k29d)$$
$$-\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle \overline{\omega}^k\wedge\overline{\omega}^i\wedge\nabla_{V_a}\left(\iota_{\overline{V}_j}(\alpha)\right)\right)\right)\quad (k29e)$$
$$+\left\langle \nabla_{\overline{V}_a}V_c,V_b\right\rangle\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\overline{\omega}^a\wedge\iota_{\overline{V}_c}\left(\iota_{\overline{V}_b}\left(\overline{\omega}^k\right)\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_j}(\alpha)\right)\quad (k30a)$$
$$-\left\langle \nabla_{\overline{V}_a}V_c,V_b\right\rangle\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\overline{\omega}^a\wedge\iota_{\overline{V}_c}\left(\overline{\omega}^k\wedge\iota_{\overline{V}_b}\left(\overline{\omega}^i\right)\wedge\iota_{\overline{V}_j}(\alpha)\right)\quad (k30b)$$
$$+\left\langle \nabla_{\overline{V}_a}V_c,V_b\right\rangle\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\overline{\omega}^a\wedge\iota_{\overline{V}_c}\left(\overline{\omega}^k\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_b}\left(\iota_{\overline{V}_j}(\alpha)\right)\right)\quad (k30c)$$

If we apply the covariant derivatives and the interior products and use the commutativity property between these two operators and separate the necessary terms into (1,0) and (0,1) components, we have

$$\begin{aligned}\Delta_K(\alpha) \quad = \quad & +\overline{\omega}^a\wedge\iota_{V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_a}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right)\quad (k1a)\\ & +\overline{\omega}^a\wedge proj_{E^{p,q-1}}\left(\iota_{\left\langle\nabla_{\overline{V}_a}V_j,\overline{V}_b\right\rangle V_b+\left\langle\nabla_{\overline{V}_a}V_j,V_b\right\rangle\overline{V}_b}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\quad (k1b)\\ & -\overline{\omega}^a\wedge\iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right)\quad (k2a)\\ & -\overline{\omega}^a\wedge proj_{E^{p,q-1}}\left(\iota_{\left\langle\nabla_{\overline{V}_a}\overline{V}_j,\overline{V}_b\right\rangle V_b+\left\langle\nabla_{\overline{V}_a}\overline{V}_j,V_b\right\rangle\overline{V}_b}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\quad (k2b)\end{aligned}$$

$$
\begin{aligned}
&+\left\langle \nabla_{\overline{V}_a}\nabla_{\overline{V}_j}V_k, V_i\right\rangle \overline{\omega}^a \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right) \quad (k3a)\\
&+\left\langle \nabla_{\overline{V}_j}V_k, V_b\right\rangle \left\langle \nabla_{\overline{V}_a}V_i, \overline{V}_b\right\rangle \overline{\omega}^a \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right) \quad (k3b1)\\
&+\left\langle \nabla_{\overline{V}_j}V_k, \overline{V}_b\right\rangle \left\langle \nabla_{\overline{V}_a}V_i, V_b\right\rangle \overline{\omega}^a \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right) \quad (k3b2)\\
&+\left\langle \nabla_{\overline{V}_j}V_k, V_i\right\rangle \overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\left\langle \nabla_{\overline{V}_a}\overline{\omega}^j, \omega^b\right\rangle \overline{\omega}^b \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right)\right) \quad (k3c1)\\
&+\left\langle \nabla_{\overline{V}_j}V_k, V_i\right\rangle \overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\left\langle \nabla_{\overline{V}_a}\overline{\omega}^j, \overline{\omega}^b\right\rangle \omega^b \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right)\right) \quad (k3c2)\\
&+\left\langle \nabla_{\overline{V}_j}V_k, V_i\right\rangle \overline{\omega}^a \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}\left(proj_{p,q}\left(\nabla_{\overline{V}_a}(\alpha)\right)\right)\right) \quad (k3d1)\\
&+\left\langle \nabla_{\overline{V}_j}V_k, V_i\right\rangle \overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\overline{\omega}^j \wedge \iota_{\nabla_{\overline{V}_a}\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right)\right) \quad (k3d2)\\
&+\left\langle \nabla_{\overline{V}_j}V_k, V_i\right\rangle \overline{\omega}^a \wedge proj_{E^{p,q-1}}\left(\overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{\nabla_{\overline{V}_a}\overline{V}_i}(\alpha)\right)\right) \quad (k3d3)\\
&+\omega^a \wedge proj_{E^{p-1,q}}\left(\iota_{V_j}\left(\nabla_{V_a}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (k4a)\\
&+\omega^a \wedge proj_{E^{p-1,q}}\left(\iota_{\left\langle \nabla_{V_a}V_j, \overline{V}_b\right\rangle V_b + \left\langle \nabla_{V_a}V_j, V_b\right\rangle \overline{V}_b}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (k4b)\\
&-\omega^a \wedge proj_{E^{p-1,q}}\left(\iota_{\overline{V}_j}\left(\nabla_{V_a}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (k5a)\\
&-\omega^a \wedge proj_{E^{p-1,q}}\left(\iota_{\left\langle \nabla_{V_a}\overline{V}_j, \overline{V}_b\right\rangle V_b + \left\langle \nabla_{V_a}\overline{V}_j, V_b\right\rangle \overline{V}_b}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (k5b)\\
&+\left\langle \nabla_{\overline{V}_j}V_k, V_i\right\rangle \omega^a \wedge proj_{E^{p-1,q}}\left(\left\langle \nabla_{V_a}\overline{\omega}^j, \overline{\omega}^b\right\rangle \omega^b \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right)\right) \quad (k6c1)\\
&+\left\langle \nabla_{\overline{V}_j}V_k, V_i\right\rangle \omega^a \wedge proj_{E^{p-1,q}}\left(\left\langle \nabla_{V_a}\overline{\omega}^j, \omega^b\right\rangle \overline{\omega}^b \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right)\right) \quad (k6c2)\\
&+\left\langle \nabla_{\overline{V}_j}V_k, V_i\right\rangle \omega^a \wedge proj_{E^{p-1,q}}\left(\overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}\left(\nabla_{V_a}(\alpha)\right)\right)\right) \quad (k6d1)\\
&+\left\langle \nabla_{\overline{V}_j}V_k, V_i\right\rangle \omega^a \wedge proj_{E^{p-1,q}}\left(\overline{\omega}^j \wedge \iota_{\nabla_{V_a}\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right)\right) \quad (k6d2)\\
&+\left\langle \nabla_{\overline{V}_j}V_k, V_i\right\rangle \omega^a \wedge proj_{E^{p-1,q}}\left(\overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{\nabla_{V_a}\overline{V}_i}(\alpha)\right)\right) \quad (k6d3)\\
&-2\left\langle \nabla_{V_a}\overline{V}_a, \overline{V}_c\right\rangle \overline{\omega}^c \wedge \iota_{V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) \quad (k7)\\
&+2\left\langle \nabla_{V_a}\overline{V}_a, \overline{V}_c\right\rangle \overline{\omega}^c \wedge \iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right) \quad (k8)\\
&-2\left\langle \nabla_{V_a}\overline{V}_a, \overline{V}_c\right\rangle \left\langle \nabla_{\overline{V}_j}V_k, V_i\right\rangle \overline{\omega}^c \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right) \quad (k9)\\
&-2\left\langle \nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle \omega^a \wedge \overline{\omega}^c \wedge \iota_{V_b}\left(\iota_{V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (k10)\\
&+2\left\langle \nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle \omega^a \wedge \overline{\omega}^c \wedge \iota_{V_b}\left(\iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (k11)\\
&+2\left\langle \nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle \left\langle \nabla_{\overline{V}_j}V_k, V_i\right\rangle \omega^a \wedge \overline{\omega}^c \wedge \overline{\omega}^j \wedge \iota_{V_b}\left(\iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right)\right) \quad (k12)\\
&+\left\langle \nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle \overline{\omega}^c \wedge \overline{\omega}^b \wedge \iota_{\overline{V}_a}\left(\iota_{V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (k13)\\
&-\left\langle \nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle \overline{\omega}^c \wedge \overline{\omega}^b \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (k14)
\end{aligned}
$$

$$+\left\langle \nabla_{V_j}\overline{V}_b,\overline{V}_c\right\rangle \left\langle \nabla_{\overline{V}_j}V_k,V_i\right\rangle \overline{\omega}^c\wedge\overline{\omega}^b\wedge\iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}\left(\alpha\right)\right)\quad (k15a)$$

$$-\left\langle \nabla_{V_a}\overline{V}_b,\overline{V}_c\right\rangle \left\langle \nabla_{\overline{V}_j}V_k,V_i\right\rangle \overline{\omega}^c\wedge\overline{\omega}^b\wedge\overline{\omega}^j\wedge\iota_{\overline{V}_a}\left(\iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}\left(\alpha\right)\right)\right)\quad (k15b)$$

$$+\iota_{V_a}\left(proj_{p+1,q}\left(\left\langle \nabla_{\overline{V}_a}\overline{\omega}^j,\overline{\omega}^b\right\rangle \omega^b\wedge proj_{p,q}\left(\nabla_{\overline{V}_j}\left(\alpha\right)\right)\right)\right)\quad (k16a1)$$

$$+\iota_{V_a}\left(proj_{p+1,q}\left(\left\langle \nabla_{\overline{V}_a}\overline{\omega}^j,\omega^b\right\rangle \overline{\omega}^b\wedge proj_{p,q}\left(\nabla_{\overline{V}_j}\left(\alpha\right)\right)\right)\right)\quad (k16a2)$$

$$-\overline{\omega}^j\wedge\iota_{V_a}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}\left(\alpha\right)\right)\right)\right)\right)\quad (k16b)$$

$$-\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\left\langle \nabla_{V_a}\overline{\omega}^j,\overline{\omega}^b\right\rangle \omega^b\wedge proj_{p,q}\left(\nabla_{\overline{V}_j}\left(\alpha\right)\right)\right)\right)\quad (k17a1)$$

$$-\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\left\langle \nabla_{V_a}\overline{\omega}^j,\omega^b\right\rangle \overline{\omega}^b\wedge proj_{p,q}\left(\nabla_{\overline{V}_j}\left(\alpha\right)\right)\right)\right)\quad (k17a2)$$

$$-\iota_{\overline{V}_a}\left(\overline{\omega}^j\right)\wedge proj_{p,q}\left(\nabla_{V_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}\left(\alpha\right)\right)\right)\right)\quad (k17b1)$$

$$+\overline{\omega}^j\wedge\iota_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{V_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}\left(\alpha\right)\right)\right)\right)\right)\quad (k17b2)$$

$$+\left\langle \nabla_{\overline{V}_a}V_c,V_j\right\rangle \overline{\omega}^a\wedge\iota_{\overline{V}_c}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}\left(\alpha\right)\right)\right)\quad (k18a)$$

$$-\left\langle \nabla_{\overline{V}_a}V_j,V_b\right\rangle \overline{\omega}^a\wedge\iota_{\overline{V}_b}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}\left(\alpha\right)\right)\right)\quad (k18b1)$$

$$+\left\langle \nabla_{\overline{V}_a}V_c,V_b\right\rangle \overline{\omega}^a\wedge\overline{\omega}^j\wedge\iota_{\overline{V}_c}\left(\iota_{\overline{V}_b}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}\left(\alpha\right)\right)\right)\right)\quad (k18b2)$$

$$+\iota_{V_a}\left(proj_{p+1,q}\left(\left\langle \nabla_{\overline{V}_a}\omega^j,\overline{\omega}^b\right\rangle \omega^b\wedge proj_{p-1,q+1}\left(\nabla_{V_j}\left(\alpha\right)\right)\right)\right)\quad (k19a1)$$

$$+\iota_{V_a}\left(proj_{p+1,q}\left(\left\langle \nabla_{\overline{V}_a}\omega^j,\omega^b\right\rangle \overline{\omega}^b\wedge proj_{p-1,q+1}\left(\nabla_{V_j}\left(\alpha\right)\right)\right)\right)\quad (k19a2)$$

$$+\iota_{V_a}\left(\omega^j\right)\wedge proj_{p,q}\left(\nabla_{\overline{V}_a}\left(proj_{p-1,q+1}\left(\nabla_{V_j}\left(\alpha\right)\right)\right)\right)\quad (k19b1)$$

$$-\omega^j\wedge\iota_{V_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_a}\left(proj_{p-1,q+1}\left(\nabla_{V_j}\left(\alpha\right)\right)\right)\right)\right)\quad (k19b2)$$

$$-\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\left\langle \nabla_{V_a}\omega^j,\overline{\omega}^b\right\rangle \omega^b\wedge proj_{p-1,q+1}\left(\nabla_{V_j}\left(\alpha\right)\right)\right)\right)\quad (k20a1)$$

$$-\iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\left\langle \nabla_{V_a}\omega^j,\omega^b\right\rangle \overline{\omega}^b\wedge proj_{p-1,q+1}\left(\nabla_{V_j}\left(\alpha\right)\right)\right)\right)\quad (k20a2)$$

$$+\omega^j\wedge\iota_{\overline{V}_a}\left(proj_{p-1,q+1}\left(\nabla_{V_a}\left(proj_{p-1,q+1}\left(\nabla_{V_j}\left(\alpha\right)\right)\right)\right)\right)\quad (k20b)$$

$$+\left\langle \nabla_{\overline{V}_a}V_c,V_b\right\rangle \overline{\omega}^a\wedge\omega^j\wedge\iota_{\overline{V}_c}\left(\iota_{\overline{V}_b}\left(proj_{p-1,q+1}\left(\nabla_{V_j}\left(\alpha\right)\right)\right)\right)\quad (k21)$$

$$-2\left\langle \nabla_{V_j}\overline{V}_j,\overline{V}_k\right\rangle \iota_{V_a}\left(proj_{p+1,q}\left(\left\langle \nabla_{\overline{V}_a}\overline{\omega}^k,\overline{\omega}^b\right\rangle \omega^b\wedge\alpha\right)\right)\quad (k22c1)$$

$$-2\left\langle \nabla_{V_j}\overline{V}_j,\overline{V}_k\right\rangle \iota_{V_a}\left(proj_{p+1,q}\left(\left\langle \nabla_{\overline{V}_a}\overline{\omega}^k,\omega^b\right\rangle \overline{\omega}^b\wedge\alpha\right)\right)\quad (k22c2)$$

$$-2\left\langle \nabla_{V_j}\overline{V}_j,\overline{V}_k\right\rangle \iota_{V_a}\left(proj_{p+1,q}\left(\overline{\omega}^k\wedge\nabla_{\overline{V}_a}\alpha\right)\right)\quad (k22d)$$

$$+2\left\langle \nabla_{V_a}\nabla_{V_j}\overline{V}_j,\overline{V}_k\right\rangle \iota_{\overline{V}_a}\left(\overline{\omega}^k\wedge\alpha\right)\quad (k23a)$$

$$+2\left\langle \nabla_{V_j}\overline{V}_j,\left[\left\langle \nabla_{V_a}\overline{V}_k,\overline{V}_b\right\rangle V_b+\left\langle \nabla_{V_a}\overline{V}_k,V_b\right\rangle \overline{V}_b\right]\right\rangle \iota_{\overline{V}_a}\left(\overline{\omega}^k\wedge\alpha\right)\quad (k23b)$$

$$+2\left\langle \nabla_{V_j}\overline{V}_j,\overline{V}_k\right\rangle \iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\left\langle \nabla_{V_a}\overline{\omega}^k,\overline{\omega}^b\right\rangle \omega^b\wedge\alpha\right)\right)\quad (k23c1)$$

$$+2\left\langle \nabla_{V_j}\overline{V}_j,\overline{V}_k\right\rangle \iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\left\langle \nabla_{V_a}\overline{\omega}^k,\omega^b\right\rangle \overline{\omega}^b\wedge\alpha\right)\right)\quad (k23c2)$$

$$+2\left\langle \nabla_{V_j}\overline{V}_j,\overline{V}_k\right\rangle \iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\overline{\omega}^k\wedge\nabla_{V_a}\alpha\right)\right)\quad (k23d)$$

$$-2\left\langle \nabla_{\overline{V}_a} V_c, V_k\right\rangle \left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_c}\left(\alpha\right) \quad (k24a)$$
$$+2\left\langle \nabla_{\overline{V}_a} V_k, V_b\right\rangle \left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_b}\left(\alpha\right) \quad (k24b1)$$
$$-2\left\langle \nabla_{\overline{V}_a} V_c, V_b\right\rangle \left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle \overline{\omega}^a \wedge \overline{\omega}^k \wedge \iota_{\overline{V}_c}\left(\iota_{\overline{V}_b}\left(\alpha\right)\right) \quad (k24b2)$$
$$-2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \iota_{V_a}\left(proj_{p+1,q}\left(\left\langle \nabla_{\overline{V}_a}\omega^j, \overline{\omega}^b\right\rangle \omega^b \wedge \overline{\omega}^k \wedge \iota_{V_i}\left(\alpha\right)\right)\right) \quad (k25c1)$$
$$-2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \iota_{V_a}\left(proj_{p+1,q}\left(\left\langle \nabla_{\overline{V}_a}\omega^j, \omega^b\right\rangle \overline{\omega}^b \wedge \overline{\omega}^k \wedge \iota_{V_i}\left(\alpha\right)\right)\right) \quad (k25c2)$$
$$-2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \iota_{V_a}\left(proj_{p+1,q}\left(\omega^j \wedge \left\langle \nabla_{\overline{V}_a}\overline{\omega}^k, \overline{\omega}^b\right\rangle \omega^b \wedge \iota_{V_i}\left(\alpha\right)\right)\right) \quad (k25d1)$$
$$-2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \iota_{V_a}\left(proj_{p+1,q}\left(\omega^j \wedge \left\langle \nabla_{\overline{V}_a}\overline{\omega}^k, \omega^b\right\rangle \overline{\omega}^b \wedge \iota_{V_i}\left(\alpha\right)\right)\right) \quad (k25d2)$$
$$-2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \iota_{V_a}\left(\omega^j \wedge \overline{\omega}^k \wedge proj_{E^{p,q-1}}\left(\iota_{V_i}\left(\nabla_{\overline{V}_a}\left(\alpha\right)\right)\right)\right) \quad (k25e1)$$
$$-2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \iota_{V_a}\left(\omega^j \wedge \overline{\omega}^k \wedge proj_{E^{p,q-1}}\left(\iota_{\nabla_{\overline{V}_a} V_i}\left(\alpha\right)\right)\right) \quad (k25e2)$$
$$+2\left\langle \nabla_{V_a}\nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \iota_{\overline{V}_a}\left(\omega^j \wedge \overline{\omega}^k \wedge \iota_{V_i}\left(\alpha\right)\right) \quad (k26a)$$
$$+2\left\langle \nabla_{V_j}\overline{V}_i, \left[\left\langle \nabla_{V_a}\overline{V}_k, \overline{V}_b\right\rangle V_b + \left\langle \nabla_{V_a}\overline{V}_k, V_b\right\rangle \overline{V}_b\right]\right\rangle \iota_{\overline{V}_a}\left(\omega^j \wedge \overline{\omega}^k \wedge \iota_{V_i}\left(\alpha\right)\right) \quad (k26b)$$
$$+2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\left\langle \nabla_{V_a}\omega^j, \overline{\omega}^b\right\rangle \omega^b \wedge \overline{\omega}^k \wedge \iota_{V_i}\left(\alpha\right)\right)\right) \quad (k26c1)$$
$$+2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\left\langle \nabla_{V_a}\omega^j, \omega^b\right\rangle \overline{\omega}^b \wedge \overline{\omega}^k \wedge \iota_{V_i}\left(\alpha\right)\right)\right) \quad (k26c2)$$
$$+2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\omega^j \wedge \left\langle \nabla_{V_a}\overline{\omega}^k, \overline{\omega}^b\right\rangle \omega^b \wedge \iota_{V_i}\left(\alpha\right)\right)\right) \quad (k26d1)$$
$$+2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\omega^j \wedge \left\langle \nabla_{V_a}\overline{\omega}^k, \omega^b\right\rangle \overline{\omega}^b \wedge \iota_{V_i}\left(\alpha\right)\right)\right) \quad (k26d2)$$
$$+2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \iota_{\overline{V}_a}\left(\omega^j \wedge \overline{\omega}^k \wedge proj_{E^{p-1,q}}\left(\iota_{V_i}\left(\nabla_{V_a}\left(\alpha\right)\right)\right)\right) \quad (k26e1)$$
$$+2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \iota_{\overline{V}_a}\left(\omega^j \wedge \overline{\omega}^k \wedge proj_{E^{p-1,q}}\left(\iota_{\nabla_{V_a} V_i}\left(\alpha\right)\right)\right) \quad (k26e2)$$
$$+2\left\langle \nabla_{\overline{V}_a} V_c, V_b\right\rangle \left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_c}\left(\omega^j \wedge \iota_{\overline{V}_b}\left(\overline{\omega}^k\right) \wedge \iota_{V_i}\left(\alpha\right)\right) \quad (k27a)$$
$$-2\left\langle \nabla_{\overline{V}_a} V_c, V_b\right\rangle \left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_c}\left(\omega^j \wedge \overline{\omega}^k \wedge \iota_{\overline{V}_b}\left(\iota_{V_i}\left(\alpha\right)\right)\right) \quad (k27b)$$
$$+\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \iota_{V_a}\left(proj_{p+1,q}\left(\left\langle \nabla_{\overline{V}_a}\overline{\omega}^k, \overline{\omega}^b\right\rangle \omega^b \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_j}\left(\alpha\right)\right)\right) \quad (k28c1)$$
$$+\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \iota_{V_a}\left(proj_{p+1,q}\left(\left\langle \nabla_{\overline{V}_a}\overline{\omega}^k, \omega^b\right\rangle \overline{\omega}^b \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_j}\left(\alpha\right)\right)\right) \quad (k28c2)$$
$$+\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \iota_{V_a}\left(proj_{p+1,q}\left(\overline{\omega}^k \wedge \left\langle \nabla_{\overline{V}_a}\overline{\omega}^i, \overline{\omega}^b\right\rangle \omega^b \wedge \iota_{\overline{V}_j}\left(\alpha\right)\right)\right) \quad (k28d1)$$
$$+\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \iota_{V_a}\left(proj_{p+1,q}\left(\overline{\omega}^k \wedge \left\langle \nabla_{\overline{V}_a}\overline{\omega}^i, \omega^b\right\rangle \overline{\omega}^b \wedge \iota_{\overline{V}_j}\left(\alpha\right)\right)\right) \quad (k28d2)$$
$$+\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \iota_{V_a}\left(proj_{p+1,q}\left(\overline{\omega}^k \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_j}\left(\nabla_{\overline{V}_a}\left(\alpha\right)\right)\right)\right) \quad (k28e1)$$
$$+\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \iota_{V_a}\left(proj_{p+1,q}\left(\overline{\omega}^k \wedge \overline{\omega}^i \wedge \iota_{\nabla_{\overline{V}_a}\overline{V}_j}\left(\alpha\right)\right)\right) \quad (k28e2)$$
$$-\left\langle \nabla_{V_a}\nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \iota_{\overline{V}_a}\left(\overline{\omega}^k \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_j}\left(\alpha\right)\right) \quad (k29a)$$
$$-\left\langle \nabla_{V_j}\overline{V}_i, \left[\left\langle \nabla_{V_a}\overline{V}_k, \overline{V}_b\right\rangle V_b + \left\langle \nabla_{V_a}\overline{V}_k, V_b\right\rangle \overline{V}_b\right]\right\rangle \iota_{\overline{V}_a}\left(\overline{\omega}^k \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_j}\left(\alpha\right)\right) \quad (k29b)$$
$$-\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\left\langle \nabla_{V_a}\overline{\omega}^k, \overline{\omega}^b\right\rangle \omega^b \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_j}\left(\alpha\right)\right)\right) \quad (k29c1)$$
$$-\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \iota_{\overline{V}_a}\left(proj_{p,q+1}\left(\left\langle \nabla_{V_a}\overline{\omega}^k, \omega^b\right\rangle \overline{\omega}^b \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_j}\left(\alpha\right)\right)\right) \quad (k29c2)$$

$$
\begin{aligned}
&- \left\langle \nabla_{V_j} \overline{V}_i, \overline{V}_k \right\rangle \iota_{\overline{V}_a} \left(proj_{p,q+1} \left(\overline{\omega}^k \wedge \left\langle \nabla_{V_a} \overline{\omega}^i, \overline{\omega}^b \right\rangle \omega^b \wedge \iota_{\overline{V}_j} (\alpha) \right) \right) \quad (k29d1) \\
&- \left\langle \nabla_{V_j} \overline{V}_i, \overline{V}_k \right\rangle \iota_{\overline{V}_a} \left(proj_{p,q+1} \left(\overline{\omega}^k \wedge \left\langle \nabla_{V_a} \overline{\omega}^i, \omega^b \right\rangle \overline{\omega}^b \wedge \iota_{\overline{V}_j} (\alpha) \right) \right) \quad (k29d2) \\
&- \left\langle \nabla_{V_j} \overline{V}_i, \overline{V}_k \right\rangle \iota_{\overline{V}_a} \left(proj_{p,q+1} \left(\overline{\omega}^k \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_j} \left(\nabla_{V_a} (\alpha) \right) \right) \right) \quad (k29e1) \\
&- \left\langle \nabla_{V_j} \overline{V}_i, \overline{V}_k \right\rangle \iota_{\overline{V}_a} \left(proj_{p,q+1} \left(\overline{\omega}^k \wedge \overline{\omega}^i \wedge \iota_{\nabla_{V_a} \overline{V}_j} (\alpha) \right) \right) \quad (k29e2) \\
&+ \left\langle \nabla_{\overline{V}_a} V_c, V_k \right\rangle \left\langle \nabla_{V_j} \overline{V}_i, \overline{V}_k \right\rangle \overline{\omega}^a \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_c} \left(\iota_{\overline{V}_j} (\alpha) \right) \quad (k30a1) \\
&- \left\langle \nabla_{\overline{V}_a} V_c, V_k \right\rangle \left\langle \nabla_{V_j} \overline{V}_i, \overline{V}_k \right\rangle \overline{\omega}^a \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_c} \left(\iota_{\overline{V}_j} (\alpha) \right) \quad (k30a2) \\
&- \left\langle \nabla_{\overline{V}_a} V_c, V_i \right\rangle \left\langle \nabla_{V_j} \overline{V}_i, \overline{V}_k \right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_c} \left(\overline{\omega}^k \right) \wedge \iota_{\overline{V}_j} (\alpha) \quad (k30b1) \\
&+ \left\langle \nabla_{\overline{V}_a} V_c, V_i \right\rangle \left\langle \nabla_{V_j} \overline{V}_i, \overline{V}_k \right\rangle \overline{\omega}^a \wedge \overline{\omega}^k \wedge \iota_{\overline{V}_c} \left(\iota_{\overline{V}_j} (\alpha) \right) \quad (k30b2) \\
&+ \left\langle \nabla_{\overline{V}_a} V_c, V_b \right\rangle \left\langle \nabla_{V_j} \overline{V}_i, \overline{V}_k \right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_c} \left(\overline{\omega}^k \right) \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_b} \left(\iota_{\overline{V}_j} (\alpha) \right) \quad (k30c1) \\
&- \left\langle \nabla_{\overline{V}_a} V_c, V_b \right\rangle \left\langle \nabla_{V_j} \overline{V}_i, \overline{V}_k \right\rangle \overline{\omega}^a \wedge \overline{\omega}^k \wedge \iota_{\overline{V}_c} \left(\overline{\omega}^i \right) \wedge \iota_{\overline{V}_b} \left(\iota_{\overline{V}_j} (\alpha) \right) \quad (k30c2) \\
&+ \left\langle \nabla_{\overline{V}_a} V_c, V_b \right\rangle \left\langle \nabla_{V_j} \overline{V}_i, \overline{V}_k \right\rangle \overline{\omega}^a \wedge \overline{\omega}^k \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_c} \left(\iota_{\overline{V}_b} \left(\iota_{\overline{V}_j} (\alpha) \right) \right) \quad (k30c3)
\end{aligned}
$$

Apply the covariant derivatives and the interior products and use the commutativity property between these two operators and separate the necessary terms into (1,0) and (0,1) components yielding

$$
\begin{aligned}
\Delta_K (\alpha) \quad = \quad &+\overline{\omega}^a \wedge \iota_{V_j} \left(proj_{p+1,q-1} \left(\nabla_{\overline{V}_a} \left(proj_{p+1,q-1} \left(\nabla_{\overline{V}_j} (\alpha) \right) \right) \right) \right) \quad (k1a) \\
&+ \left\langle \nabla_{\overline{V}_a} V_j, \overline{V}_b \right\rangle \overline{\omega}^a \wedge \iota_{V_b} \left(proj_{p+1,q-1} \left(\nabla_{\overline{V}_j} (\alpha) \right) \right) \quad (k1b) \\
&-\overline{\omega}^a \wedge \iota_{\overline{V}_j} \left(proj_{p,q} \left(\nabla_{\overline{V}_a} \left(proj_{p,q} \left(\nabla_{V_j} (\alpha) \right) \right) \right) \right) \quad (k2a) \\
&- \left\langle \nabla_{\overline{V}_a} \overline{V}_j, V_b \right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_b} \left(proj_{p,q} \left(\nabla_{V_j} (\alpha) \right) \right) \quad (k2b) \\
&+ \left\langle \nabla_{\overline{V}_a} \nabla_{\overline{V}_j} V_k, V_i \right\rangle \overline{\omega}^a \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_k} \left(\iota_{\overline{V}_i} (\alpha) \right) \quad (k3a) \\
&+ \left\langle \nabla_{\overline{V}_j} V_k, V_b \right\rangle \left\langle \nabla_{\overline{V}_a} V_i, \overline{V}_b \right\rangle \overline{\omega}^a \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_k} \left(\iota_{\overline{V}_i} (\alpha) \right) \quad (k3b1) \\
&+ \left\langle \nabla_{\overline{V}_j} V_k, \overline{V}_b \right\rangle \left\langle \nabla_{\overline{V}_a} V_i, V_b \right\rangle \overline{\omega}^a \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_k} \left(\iota_{\overline{V}_i} (\alpha) \right) \quad (k3b2) \\
&+ \left\langle \nabla_{\overline{V}_j} V_k, V_i \right\rangle \left\langle \nabla_{\overline{V}_a} V_j, \overline{V}_b \right\rangle \overline{\omega}^a \wedge \overline{\omega}^b \wedge \iota_{\overline{V}_k} \left(\iota_{\overline{V}_i} (\alpha) \right) \quad (k3c1) \\
&+ \left\langle \nabla_{\overline{V}_j} V_k, V_i \right\rangle \overline{\omega}^a \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_k} \left(\iota_{\overline{V}_i} \left(proj_{p,q} \left(\nabla_{\overline{V}_a} (\alpha) \right) \right) \right) \quad (k3d1) \\
&+ \left\langle \nabla_{\overline{V}_j} V_k, V_i \right\rangle \left\langle \nabla_{\overline{V}_a} \overline{V}_k, V_b \right\rangle \overline{\omega}^a \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_b} \left(\iota_{\overline{V}_i} (\alpha) \right) \quad (k3d2) \\
&+ \left\langle \nabla_{\overline{V}_j} V_k, V_i \right\rangle \left\langle \nabla_{\overline{V}_a} \overline{V}_i, V_b \right\rangle \overline{\omega}^a \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_k} \left(\iota_{\overline{V}_b} (\alpha) \right) \quad (k3d3) \\
&+\omega^a \wedge \iota_{V_j} \left(proj_{p,q} \left(\nabla_{V_a} \left(proj_{p+1,q-1} \left(\nabla_{\overline{V}_j} (\alpha) \right) \right) \right) \right) \quad (k4a)
\end{aligned}
$$

$$-\omega^a \wedge \iota_{\overline{V}_j}\left(proj_{p-1,q+1}\left(\nabla_{V_a}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (k5a)$$
$$-\left\langle \nabla_{V_a}\overline{V}_j, \overline{V}_b\right\rangle \omega^a \wedge \iota_{V_b}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right) \quad (k5b)$$
$$+\left\langle \nabla_{\overline{V}_j}V_k, V_i\right\rangle \omega^a \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}\left(proj_{p-1,q+1}\left(\nabla_{V_a}(\alpha)\right)\right)\right) \quad (k6d1)$$
$$+\left\langle \nabla_{\overline{V}_j}V_k, V_i\right\rangle \left\langle \nabla_{V_a}\overline{V}_k, \overline{V}_b\right\rangle \omega^a \wedge \overline{\omega}^j \wedge \iota_{V_b}\left(\iota_{\overline{V}_i}(\alpha)\right) \quad (k6d2)$$
$$+\left\langle \nabla_{\overline{V}_j}V_k, V_i\right\rangle \left\langle \nabla_{V_a}\overline{V}_i, \overline{V}_b\right\rangle \omega^a \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{V_b}(\alpha)\right) \quad (k6d3)$$
$$-2\left\langle \nabla_{V_a}\overline{V}_a, \overline{V}_c\right\rangle \overline{\omega}^c \wedge \iota_{V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) \quad (k7)$$
$$+2\left\langle \nabla_{V_a}\overline{V}_a, \overline{V}_c\right\rangle \overline{\omega}^c \wedge \iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right) \quad (k8)$$
$$-2\left\langle \nabla_{V_a}\overline{V}_a, \overline{V}_c\right\rangle \left\langle \nabla_{\overline{V}_j}V_k, V_i\right\rangle \overline{\omega}^c \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right) \quad (k9)$$
$$-2\left\langle \nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle \omega^a \wedge \overline{\omega}^c \wedge \iota_{V_b}\left(\iota_{V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (k10)$$
$$+2\left\langle \nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle \omega^a \wedge \overline{\omega}^c \wedge \iota_{V_b}\left(\iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (k11)$$
$$+2\left\langle \nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle \left\langle \nabla_{\overline{V}_j}V_k, V_i\right\rangle \omega^a \wedge \overline{\omega}^c \wedge \overline{\omega}^j \wedge \iota_{V_b}\left(\iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right)\right) \quad (k12)$$
$$+\left\langle \nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle \overline{\omega}^c \wedge \overline{\omega}^b \wedge \iota_{\overline{V}_a}\left(\iota_{V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (k13)$$
$$-\left\langle \nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle \overline{\omega}^c \wedge \overline{\omega}^b \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (k14)$$
$$+\left\langle \nabla_{V_j}\overline{V}_b, \overline{V}_c\right\rangle \left\langle \nabla_{\overline{V}_j}V_k, V_i\right\rangle \overline{\omega}^c \wedge \overline{\omega}^b \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right) \quad (k15a)$$
$$-\left\langle \nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle \left\langle \nabla_{\overline{V}_j}V_k, V_i\right\rangle \overline{\omega}^c \wedge \overline{\omega}^b \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right)\right) \quad (k15b)$$
$$+\left\langle \nabla_{\overline{V}_a}V_j, V_b\right\rangle \iota_{V_a}\left(\omega^b\right) \wedge proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right) \quad (k16a1a)$$
$$-\left\langle \nabla_{\overline{V}_a}V_j, V_b\right\rangle \omega^b \wedge \iota_{V_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) \quad (k16a1b)$$
$$-\overline{\omega}^j \wedge \iota_{V_a}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (k16b)$$
$$-\left\langle \nabla_{V_a}V_j, \overline{V}_b\right\rangle \iota_{\overline{V}_a}\left(\overline{\omega}^b\right) \wedge proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right) \quad (k17a2a)$$
$$+\left\langle \nabla_{V_a}V_j, \overline{V}_b\right\rangle \overline{\omega}^b \wedge \iota_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) \quad (k17a2b)$$
$$-\iota_{\overline{V}_a}\left(\overline{\omega}^j\right) \wedge proj_{p,q}\left(\nabla_{V_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (k17b1)$$
$$+\overline{\omega}^j \wedge \iota_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{V_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (k17b2)$$
$$+\left\langle \nabla_{\overline{V}_a}V_c, V_j\right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_c}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) \quad (k18a)$$
$$-\left\langle \nabla_{\overline{V}_a}V_j, V_b\right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_b}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) \quad (k18b1)$$
$$+\left\langle \nabla_{\overline{V}_a}V_c, V_b\right\rangle \overline{\omega}^a \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_c}\left(\iota_{\overline{V}_b}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (k18b2)$$
$$+\iota_{V_a}\left(\omega^j\right) \wedge proj_{p,q}\left(\nabla_{\overline{V}_a}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (k19b1)$$
$$-\omega^j \wedge \iota_{V_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_a}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (k19b2)$$

$$+\left\langle \nabla_{V_a}\overline{V}_j, V_b\right\rangle \omega^b \wedge \iota_{\overline{V}_a}\left(proj_{p,q+1}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (k20a1)$$

$$+\omega^j \wedge \iota_{\overline{V}_a}\left(proj_{p-1,q+1}\left(\nabla_{V_a}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (k20b)$$

$$+\left\langle \nabla_{\overline{V}_a}V_c, V_b\right\rangle \overline{\omega}^a \wedge \omega^j \wedge \iota_{\overline{V}_c}\left(\iota_{\overline{V}_b}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (k21)$$

$$-2\left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle \left\langle \nabla_{\overline{V}_a}V_k, V_b\right\rangle \iota_{V_a}\left(\omega^b\right) \wedge \alpha \quad (k22c1a)$$

$$+2\left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle \left\langle \nabla_{\overline{V}_a}V_k, V_b\right\rangle \omega^b \wedge \iota_{V_a}(\alpha) \quad (k22c1b)$$

$$+2\left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \iota_{V_a}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_a}\alpha\right)\right) \quad (k22d)$$

$$+2\left\langle \nabla_{V_a}\nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle \iota_{\overline{V}_a}\left(\overline{\omega}^k\right) \wedge \alpha \quad (k23a1)$$

$$-2\left\langle \nabla_{V_a}\nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}(\alpha) \quad (k23a2)$$

$$+2\left\langle \nabla_{V_j}\overline{V}_j, V_b\right\rangle \left\langle \nabla_{V_a}\overline{V}_k, \overline{V}_b\right\rangle \iota_{\overline{V}_a}\left(\overline{\omega}^k\right) \wedge \alpha \quad (k23b1)$$

$$+2\left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_b\right\rangle \left\langle \nabla_{V_a}\overline{V}_k, V_b\right\rangle \iota_{\overline{V}_a}\left(\overline{\omega}^k\right) \wedge \alpha \quad (k23b2)$$

$$-2\left\langle \nabla_{V_j}\overline{V}_j, V_b\right\rangle \left\langle \nabla_{V_a}\overline{V}_k, \overline{V}_b\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}(\alpha) \quad (k23b3)$$

$$-2\left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_b\right\rangle \left\langle \nabla_{V_a}\overline{V}_k, V_b\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}(\alpha) \quad (k23b4)$$

$$+2\left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle \left\langle \nabla_{V_a}V_k, \overline{V}_b\right\rangle \iota_{\overline{V}_a}\left(\overline{\omega}^b\right) \wedge \alpha \quad (k23c2a)$$

$$-2\left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle \left\langle \nabla_{V_a}V_k, \overline{V}_b\right\rangle \overline{\omega}^b \wedge \iota_{\overline{V}_a}(\alpha) \quad (k23c2b)$$

$$+2\left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle \iota_{\overline{V}_a}\left(\overline{\omega}^k\right) \wedge proj_{p,q}\left(\nabla_{V_a}\alpha\right) \quad (k23d1)$$

$$-2\left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{V_a}\alpha\right)\right) \quad (k23d2)$$

$$-2\left\langle \nabla_{\overline{V}_a}V_c, V_k\right\rangle \left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_c}(\alpha) \quad (k24a)$$

$$+2\left\langle \nabla_{\overline{V}_a}V_k, V_b\right\rangle \left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_b}(\alpha) \quad (k24b1)$$

$$-2\left\langle \nabla_{\overline{V}_a}V_c, V_b\right\rangle \left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle \overline{\omega}^a \wedge \overline{\omega}^k \wedge \iota_{\overline{V}_c}\left(\iota_{\overline{V}_b}(\alpha)\right) \quad (k24b2)$$

$$-2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \left\langle \nabla_{\overline{V}_a}V_k, V_b\right\rangle \iota_{V_a}\left(\omega^j\right) \wedge \omega^b \wedge \iota_{V_i}(\alpha) \quad (k25d1a)$$

$$+2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \left\langle \nabla_{\overline{V}_a}V_k, V_b\right\rangle \omega^j \wedge \iota_{V_a}\left(\omega^b\right) \wedge \iota_{V_i}(\alpha) \quad (k25d1b)$$

$$-2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \left\langle \nabla_{\overline{V}_a}V_k, V_b\right\rangle \omega^j \wedge \omega^b \wedge \iota_{V_a}\left(\iota_{V_i}(\alpha)\right) \quad (k25d1c)$$

$$-2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \iota_{V_a}\left(\omega^j\right) \wedge \overline{\omega}^k \wedge \iota_{V_i}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_a}(\alpha)\right)\right) \quad (k25e1a)$$

$$-2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \omega^j \wedge \overline{\omega}^k \wedge \iota_{V_a}\left(\iota_{V_i}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_a}(\alpha)\right)\right)\right) \quad (k25e1b)$$

$$-2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \left\langle \nabla_{\overline{V}_a}V_i, V_b\right\rangle \iota_{V_a}\left(\omega^j\right) \wedge \overline{\omega}^k \wedge \iota_{\overline{V}_b}(\alpha) \quad (k25e2a)$$

$$-2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \left\langle \nabla_{\overline{V}_a}V_i, V_b\right\rangle \omega^j \wedge \overline{\omega}^k \wedge \iota_{V_a}\left(\iota_{\overline{V}_b}(\alpha)\right) \quad (k25e2b)$$

$$-2\left\langle \nabla_{V_a}\nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \omega^j \wedge \iota_{\overline{V}_a}\left(\overline{\omega}^k\right) \wedge \iota_{V_i}(\alpha) \quad (k26a1)$$

$$+2\left\langle \nabla_{V_a}\nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \omega^j \wedge \overline{\omega}^k \wedge \iota_{\overline{V}_a}\left(\iota_{V_i}(\alpha)\right) \quad (k26a2)$$

$$-2\left\langle \nabla_{V_j}\overline{V}_i, V_b\right\rangle \left\langle \nabla_{V_a}\overline{V}_k, \overline{V}_b\right\rangle \omega^j \wedge \iota_{\overline{V}_a}\left(\overline{\omega}^k\right) \wedge \iota_{V_i}(\alpha) \quad (k26b1)$$

$$-2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_b\right\rangle \left\langle \nabla_{V_a}\overline{V}_k, V_b\right\rangle \omega^j \wedge \iota_{\overline{V}_a}\left(\overline{\omega}^k\right) \wedge \iota_{V_i}(\alpha) \quad (k26b2)$$

$$+2\left\langle \nabla_{V_j}\overline{V}_i, V_b\right\rangle \left\langle \nabla_{V_a}\overline{V}_k, \overline{V}_b\right\rangle \omega^j \wedge \overline{\omega}^k \wedge \iota_{\overline{V}_a}\left(\iota_{V_i}(\alpha)\right) \quad (k26b3)$$
$$+2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_b\right\rangle \left\langle \nabla_{V_a}\overline{V}_k, V_b\right\rangle \omega^j \wedge \overline{\omega}^k \wedge \iota_{\overline{V}_a}\left(\iota_{V_i}(\alpha)\right) \quad (k26b4)$$
$$-2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \left\langle \nabla_{V_a}\overline{V}_j, V_b\right\rangle \omega^b \wedge \iota_{\overline{V}_a}\left(\overline{\omega}^k\right) \wedge \iota_{V_i}(\alpha) \quad (k26c1a)$$
$$+2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \left\langle \nabla_{V_a}\overline{V}_j, V_b\right\rangle \omega^b \wedge \overline{\omega}^k \wedge \iota_{\overline{V}_a}\left(\iota_{V_i}(\alpha)\right) \quad (k26c1b)$$
$$-2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \left\langle \nabla_{V_a}V_k, \overline{V}_b\right\rangle \omega^j \wedge \iota_{\overline{V}_a}\left(\overline{\omega}^b\right) \wedge \iota_{V_i}(\alpha) \quad (k26d2a)$$
$$+2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \left\langle \nabla_{V_a}V_k, \overline{V}_b\right\rangle \omega^j \wedge \overline{\omega}^b \wedge \iota_{\overline{V}_a}\left(\iota_{V_i}(\alpha)\right) \quad (k26d2b)$$
$$-2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \omega^j \wedge \iota_{\overline{V}_a}\left(\overline{\omega}^k\right) \wedge \iota_{V_i}\left(proj_{p,q}\left(\nabla_{V_a}(\alpha)\right)\right) \quad (k26e1a)$$
$$+2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \omega^j \wedge \overline{\omega}^k \wedge \iota_{\overline{V}_a}\left(\iota_{V_i}\left(proj_{p,q}\left(\nabla_{V_a}(\alpha)\right)\right)\right) \quad (k26e1b)$$
$$-2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \left\langle \nabla_{V_a}V_i, \overline{V}_b\right\rangle \omega^j \wedge \iota_{\overline{V}_a}\left(\overline{\omega}^k\right) \wedge \iota_{V_b}(\alpha) \quad (k26e2a)$$
$$+2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \left\langle \nabla_{V_a}V_i, \overline{V}_b\right\rangle \omega^j \wedge \overline{\omega}^k \wedge \iota_{\overline{V}_a}\left(\iota_{V_b}(\alpha)\right) \quad (k26e2b)$$
$$-2\left\langle \nabla_{\overline{V}_a}V_c, V_k\right\rangle \left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \overline{\omega}^a \wedge \omega^j \wedge \iota_{\overline{V}_c}\left(\iota_{V_i}(\alpha)\right) \quad (k27a)$$
$$+2\left\langle \nabla_{\overline{V}_a}V_c, V_b\right\rangle \left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \overline{\omega}^a \wedge \omega^j \wedge \iota_{\overline{V}_c}\left(\overline{\omega}^k\right) \wedge \iota_{\overline{V}_b}\left(\iota_{V_i}(\alpha)\right) \quad (k27b1)$$
$$-2\left\langle \nabla_{\overline{V}_a}V_c, V_b\right\rangle \left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \overline{\omega}^a \wedge \omega^j \wedge \overline{\omega}^k \wedge \iota_{\overline{V}_c}\left(\iota_{\overline{V}_b}\left(\iota_{V_i}(\alpha)\right)\right) \quad (k27b2)$$
$$+\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \left\langle \nabla_{\overline{V}_a}V_k, V_b\right\rangle \iota_{V_a}\left(\omega^b\right) \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_j}(\alpha) \quad (k28c1a)$$
$$+\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \left\langle \nabla_{\overline{V}_a}V_k, V_b\right\rangle \omega^b \wedge \overline{\omega}^i \wedge \iota_{V_a}\left(\iota_{\overline{V}_j}(\alpha)\right) \quad (k28c1b)$$
$$-\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \left\langle \nabla_{\overline{V}_a}V_i, V_b\right\rangle \overline{\omega}^k \wedge \iota_{V_a}\left(\omega^b\right) \wedge \iota_{\overline{V}_j}(\alpha) \quad (k28d1a)$$
$$+\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \left\langle \nabla_{\overline{V}_a}V_i, V_b\right\rangle \overline{\omega}^k \wedge \omega^b \wedge \iota_{V_a}\left(\iota_{\overline{V}_j}(\alpha)\right) \quad (k28d1b)$$
$$+\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \overline{\omega}^i \wedge \iota_{V_a}\left(\iota_{\overline{V}_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_a}(\alpha)\right)\right)\right) \quad (k28e1)$$
$$-\left\langle \nabla_{V_a}\nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \iota_{\overline{V}_a}\left(\overline{\omega}^k\right) \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_j}(\alpha) \quad (k29a1)$$
$$+\left\langle \nabla_{V_a}\nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}\left(\overline{\omega}^i\right) \wedge \iota_{\overline{V}_j}(\alpha) \quad (k29a2)$$
$$-\left\langle \nabla_{V_a}\nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}(\alpha)\right) \quad (k29a3)$$
$$-\left\langle \nabla_{V_j}\overline{V}_i, V_b\right\rangle \left\langle \nabla_{V_a}\overline{V}_k, \overline{V}_b\right\rangle \iota_{\overline{V}_a}\left(\overline{\omega}^k\right) \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_j}(\alpha) \quad (k29b1)$$
$$-\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_b\right\rangle \left\langle \nabla_{V_a}\overline{V}_k, V_b\right\rangle \iota_{\overline{V}_a}\left(\overline{\omega}^k\right) \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_j}(\alpha) \quad (k29b2)$$
$$+\left\langle \nabla_{V_j}\overline{V}_i, V_b\right\rangle \left\langle \nabla_{V_a}\overline{V}_k, \overline{V}_b\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}\left(\overline{\omega}^i\right) \wedge \iota_{\overline{V}_j}(\alpha) \quad (k29b3)$$
$$+\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_b\right\rangle \left\langle \nabla_{V_a}\overline{V}_k, V_b\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}\left(\overline{\omega}^i\right) \wedge \iota_{\overline{V}_j}(\alpha) \quad (k29b4)$$
$$-\left\langle \nabla_{V_j}\overline{V}_i, V_b\right\rangle \left\langle \nabla_{V_a}\overline{V}_k, \overline{V}_b\right\rangle \overline{\omega}^k \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}(\alpha)\right) \quad (k29b5)$$
$$-\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_b\right\rangle \left\langle \nabla_{V_a}\overline{V}_k, V_b\right\rangle \overline{\omega}^k \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}(\alpha)\right) \quad (k29b6)$$
$$-\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \left\langle \nabla_{V_a}V_k, \overline{V}_b\right\rangle \iota_{\overline{V}_a}\left(\overline{\omega}^b\right) \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_j}(\alpha) \quad (k29c2a)$$
$$+\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \left\langle \nabla_{V_a}V_k, \overline{V}_b\right\rangle \overline{\omega}^b \wedge \iota_{\overline{V}_a}\left(\overline{\omega}^i\right) \wedge \iota_{\overline{V}_j}(\alpha) \quad (k29c2b)$$
$$-\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \left\langle \nabla_{V_a}V_k, \overline{V}_b\right\rangle \overline{\omega}^b \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}(\alpha)\right) \quad (k29c2c)$$

$$
\begin{aligned}
&-\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \left\langle \nabla_{V_a}V_i, \overline{V}_b\right\rangle \iota_{\overline{V}_a}\left(\overline{\omega}^k\right)\wedge\overline{\omega}^b\wedge\iota_{\overline{V}_j}(\alpha) \quad (k29d2a)\\
&+\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \left\langle \nabla_{V_a}V_i, \overline{V}_b\right\rangle \overline{\omega}^k\wedge\iota_{\overline{V}_a}\left(\overline{\omega}^b\right)\wedge\iota_{\overline{V}_j}(\alpha) \quad (k29d2b)\\
&-\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \left\langle \nabla_{V_a}V_i, \overline{V}_b\right\rangle \overline{\omega}^k\wedge\overline{\omega}^b\wedge\iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}(\alpha)\right) \quad (k29d2c)\\
&-\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \iota_{\overline{V}_a}\left(\overline{\omega}^k\right)\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{V_a}(\alpha)\right)\right) \quad (k29e1a)\\
&+\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \overline{\omega}^k\wedge\iota_{\overline{V}_a}\left(\overline{\omega}^i\right)\wedge\iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{V_a}(\alpha)\right)\right) \quad (k29e1b)\\
&-\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \overline{\omega}^k\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{V_a}(\alpha)\right)\right)\right) \quad (k29e1c)\\
&-\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \left\langle \nabla_{V_a}\overline{V}_j, V_b\right\rangle \iota_{\overline{V}_a}\left(\overline{\omega}^k\right)\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_b}(\alpha) \quad (k29e2a)\\
&+\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \left\langle \nabla_{V_a}\overline{V}_j, V_b\right\rangle \overline{\omega}^k\wedge\iota_{\overline{V}_a}\left(\overline{\omega}^i\right)\wedge\iota_{\overline{V}_b}(\alpha) \quad (k29e2b)\\
&-\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \left\langle \nabla_{V_a}\overline{V}_j, V_b\right\rangle \overline{\omega}^k\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_a}\left(\iota_{\overline{V}_b}(\alpha)\right) \quad (k29e2c)\\
&+\left\langle \nabla_{\overline{V}_a}V_c, V_k\right\rangle \left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \overline{\omega}^a\wedge\iota_{\overline{V}_c}\left(\overline{\omega}^i\right)\wedge\iota_{\overline{V}_j}(\alpha) \quad (k30a1)\\
&-\left\langle \nabla_{\overline{V}_a}V_c, V_k\right\rangle \left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \overline{\omega}^a\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_c}\left(\iota_{\overline{V}_j}(\alpha)\right) \quad (k30a2)\\
&-\left\langle \nabla_{\overline{V}_a}V_c, V_i\right\rangle \left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \overline{\omega}^a\wedge\iota_{\overline{V}_c}\left(\overline{\omega}^k\right)\wedge\iota_{\overline{V}_j}(\alpha) \quad (k30b1)\\
&+\left\langle \nabla_{\overline{V}_a}V_c, V_i\right\rangle \left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \overline{\omega}^a\wedge\overline{\omega}^k\wedge\iota_{\overline{V}_c}\left(\iota_{\overline{V}_j}(\alpha)\right) \quad (k30b2)\\
&+\left\langle \nabla_{\overline{V}_a}V_c, V_b\right\rangle \left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \overline{\omega}^a\wedge\iota_{\overline{V}_c}\left(\overline{\omega}^k\right)\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_b}\left(\iota_{\overline{V}_j}(\alpha)\right) \quad (k30c1)\\
&-\left\langle \nabla_{\overline{V}_a}V_c, V_b\right\rangle \left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \overline{\omega}^a\wedge\overline{\omega}^k\wedge\iota_{\overline{V}_c}\left(\overline{\omega}^i\right)\wedge\iota_{\overline{V}_b}\left(\iota_{\overline{V}_j}(\alpha)\right) \quad (k30c2)\\
&+\left\langle \nabla_{\overline{V}_a}V_c, V_b\right\rangle \left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \overline{\omega}^a\wedge\overline{\omega}^k\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_c}\left(\iota_{\overline{V}_b}\left(\iota_{\overline{V}_j}(\alpha)\right)\right) \quad (k30c3)
\end{aligned}
$$

Thus if we relabel the terms, the Kähler Laplacian becomes

$$
\begin{aligned}
\Delta_K(\alpha) \quad = \quad & +\overline{\omega}^a\wedge\iota_{V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_a}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (k1a\leftrightarrow K1)\\
& -\overline{\omega}^a\wedge\iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (k2a\leftrightarrow K2)\\
& -\overline{\omega}^j\wedge\iota_{V_a}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (k16b\leftrightarrow K3)\\
& +\overline{\omega}^j\wedge\iota_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{V_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (k17b2\leftrightarrow K4)\\
& +\omega^a\wedge\iota_{V_j}\left(proj_{p,q}\left(\nabla_{V_a}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (k4a\leftrightarrow K5)\\
& -\omega^a\wedge\iota_{\overline{V}_j}\left(proj_{p-1,q+1}\left(\nabla_{V_a}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (k5a\leftrightarrow K6)\\
& -\omega^j\wedge\iota_{V_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_a}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (k19b2\leftrightarrow K7)\\
& +\omega^j\wedge\iota_{\overline{V}_a}\left(proj_{p-1,q+1}\left(\nabla_{V_a}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (k20b\leftrightarrow K8)\\
& -proj_{p,q}\left(\nabla_{V_j}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (k17b1\leftrightarrow K9)\\
& +proj_{p,q}\left(\nabla_{\overline{V}_j}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (k19b1\leftrightarrow K10)
\end{aligned}
$$

$+2\left\langle \nabla_{V_k}\nabla_{V_j}\overline{V}_j,\overline{V}_k\right\rangle \alpha \quad (k23a1 \leftrightarrow K11)$

$-2\left\langle \nabla_{V_a}\nabla_{V_j}\overline{V}_j,\overline{V}_k\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}(\alpha) \quad (k23a2 \leftrightarrow K12)$

$-2\left\langle \nabla_{V_k}\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle \omega^j \wedge \iota_{V_i}(\alpha) \quad (k26a1 \leftrightarrow K13)$

$-\left\langle \nabla_{V_k}\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle \overline{\omega}^i \wedge \iota_{\overline{V}_j}(\alpha) \quad (k29a1 \leftrightarrow K14)$

$+\left\langle \nabla_{\overline{V}_a}\nabla_{\overline{V}_j}V_k,V_i\right\rangle \overline{\omega}^a \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right) \quad (k3a \leftrightarrow K15)$

$+\left\langle \nabla_{V_i}\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_j}(\alpha) \quad (k29a2 \leftrightarrow K16)$

$-\left\langle \nabla_{V_a}\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle \overline{\omega}^k \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}(\alpha)\right) \quad (k29a3 \leftrightarrow K17)$

$+2\left\langle \nabla_{V_a}\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle \omega^j \wedge \overline{\omega}^k \wedge \iota_{\overline{V}_a}\left(\iota_{V_i}(\alpha)\right) \quad (k26a2 \leftrightarrow K18)$

$+\left\langle \nabla_{\overline{V}_a}V_j,V_a\right\rangle proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right) \quad (k16a1a \leftrightarrow K19)$

$-\left\langle \nabla_{V_a}V_j,\overline{V}_a\right\rangle proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right) \quad (k17a2a \leftrightarrow K20)$

$+2\left\langle \nabla_{V_j}\overline{V}_j,\overline{V}_a\right\rangle proj_{p,q}\left(\nabla_{V_a}\alpha\right) \quad (k23d1 \leftrightarrow K21)$

$-\left\langle \nabla_{\overline{V}_a}V_j,V_b\right\rangle \omega^b \wedge \iota_{V_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) \quad (k16a1b \leftrightarrow K22)$

$-\left\langle \nabla_{V_a}\overline{V}_j,\overline{V}_b\right\rangle \omega^a \wedge \iota_{V_b}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right) \quad (k5b \leftrightarrow K23)$

$-2\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_a\right\rangle \omega^j \wedge \iota_{V_i}\left(proj_{p,q}\left(\nabla_{V_a}(\alpha)\right)\right) \quad (k26e1a \leftrightarrow K24)$

$-\left\langle \nabla_{\overline{V}_a}\overline{V}_j,V_b\right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_b}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right) \quad (k2b \leftrightarrow K25)$

$+\left\langle \nabla_{V_a}V_j,\overline{V}_b\right\rangle \overline{\omega}^b \wedge \iota_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) \quad (k17a2b \leftrightarrow K26)$

$+\left\langle \nabla_{\overline{V}_a}V_c,V_j\right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_c}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) \quad (k18a \leftrightarrow K27)$

$-\left\langle \nabla_{\overline{V}_a}V_j,V_b\right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_b}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) \quad (k18b1 \leftrightarrow K28)$

$-\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_a\right\rangle \overline{\omega}^i \wedge \iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{V_a}(\alpha)\right)\right) \quad (k29e1a \leftrightarrow K29)$

$+\left\langle \nabla_{V_j}\overline{V}_a,\overline{V}_k\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{V_a}(\alpha)\right)\right) \quad (k29e1b \leftrightarrow K30)$

$-2\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle \overline{\omega}^k \wedge \iota_{V_i}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) \quad (k25e1a \leftrightarrow K31)$

$+\left\langle \nabla_{\overline{V}_a}V_j,\overline{V}_b\right\rangle \overline{\omega}^a \wedge \iota_{V_b}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) \quad (k1b \leftrightarrow K32)$

$+\left\langle \nabla_{V_a}\overline{V}_j,V_b\right\rangle \omega^b \wedge \iota_{\overline{V}_a}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right) \quad (k20a1 \leftrightarrow K33)$

$-2\left\langle \nabla_{V_j}\overline{V}_j,\overline{V}_k\right\rangle \left\langle \nabla_{\overline{V}_a}V_k,V_a\right\rangle \alpha \quad (k22c1a \leftrightarrow K34)$

$+2\left\langle \nabla_{V_j}\overline{V}_j,V_b\right\rangle \left\langle \nabla_{V_k}\overline{V}_k,\overline{V}_b\right\rangle \alpha \quad (k23b1 \leftrightarrow K35)$

$+2\left\langle \nabla_{V_j}\overline{V}_j,\overline{V}_k\right\rangle \left\langle \nabla_{\overline{V}_a}V_k,V_b\right\rangle \omega^b \wedge \iota_{V_a}(\alpha) \quad (k22c1b \leftrightarrow K36)$

$-2\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle \left\langle \nabla_{\overline{V}_j}V_k,V_b\right\rangle \omega^b \wedge \iota_{V_i}(\alpha) \quad (k25d1a \leftrightarrow K37)$

$+2\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle \left\langle \nabla_{\overline{V}_a}V_k,V_a\right\rangle \omega^j \wedge \iota_{V_i}(\alpha) \quad (k25d1b \leftrightarrow K38)$

$-2\left\langle \nabla_{V_j}\overline{V}_i,V_b\right\rangle \left\langle \nabla_{V_k}\overline{V}_k,\overline{V}_b\right\rangle \omega^j \wedge \iota_{V_i}(\alpha) \quad (k26b1 \leftrightarrow K39)$

$-2\left\langle \nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle \left\langle \nabla_{V_k}\overline{V}_j,V_b\right\rangle \omega^b \wedge \iota_{V_i}(\alpha) \quad (k26c1a \leftrightarrow K40)$

$$-2\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_a\right\rangle\left\langle\nabla_{V_a}V_i,\overline{V}_b\right\rangle\omega^j\wedge\iota_{V_b}(\alpha)\quad(k26e2a\leftrightarrow K41)$$
$$-2\left\langle\nabla_{V_j}\overline{V}_j,V_b\right\rangle\left\langle\nabla_{V_a}\overline{V}_k,\overline{V}_b\right\rangle\overline{\omega}^k\wedge\iota_{\overline{V}_a}(\alpha)\quad(k23b3\leftrightarrow K42)$$
$$-2\left\langle\nabla_{\overline{V}_a}V_c,V_k\right\rangle\left\langle\nabla_{V_j}\overline{V}_j,\overline{V}_k\right\rangle\overline{\omega}^a\wedge\iota_{\overline{V}_c}(\alpha)\quad(k24a\leftrightarrow K43)$$
$$+2\left\langle\nabla_{\overline{V}_a}V_k,V_b\right\rangle\left\langle\nabla_{V_j}\overline{V}_j,\overline{V}_k\right\rangle\overline{\omega}^a\wedge\iota_{\overline{V}_b}(\alpha)\quad(k24b1\leftrightarrow K44)$$
$$-2\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\left\langle\nabla_{\overline{V}_j}V_i,V_b\right\rangle\overline{\omega}^k\wedge\iota_{\overline{V}_b}(\alpha)\quad(k25e2a\leftrightarrow K45)$$
$$+\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\left\langle\nabla_{\overline{V}_a}V_k,V_a\right\rangle\overline{\omega}^i\wedge\iota_{\overline{V}_j}(\alpha)\quad(k28c1a\leftrightarrow K46)$$
$$-\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\left\langle\nabla_{\overline{V}_a}V_i,V_a\right\rangle\overline{\omega}^k\wedge\iota_{\overline{V}_j}(\alpha)\quad(k28d1a\leftrightarrow K47)$$
$$-\left\langle\nabla_{V_j}\overline{V}_i,V_b\right\rangle\left\langle\nabla_{V_k}\overline{V}_k,\overline{V}_b\right\rangle\overline{\omega}^i\wedge\iota_{\overline{V}_j}(\alpha)\quad(k29b1\leftrightarrow K48)$$
$$+\left\langle\nabla_{V_j}\overline{V}_i,V_b\right\rangle\left\langle\nabla_{V_i}\overline{V}_k,\overline{V}_b\right\rangle\overline{\omega}^k\wedge\iota_{\overline{V}_j}(\alpha)\quad(k29b3\leftrightarrow K49)$$
$$-\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\left\langle\nabla_{V_k}V_i,\overline{V}_b\right\rangle\overline{\omega}^b\wedge\iota_{\overline{V}_j}(\alpha)\quad(k29d2a\leftrightarrow K50)$$
$$+\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\left\langle\nabla_{V_a}V_i,\overline{V}_a\right\rangle\overline{\omega}^k\wedge\iota_{\overline{V}_j}(\alpha)\quad(k29d2b\leftrightarrow K51)$$
$$-\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_a\right\rangle\left\langle\nabla_{V_a}\overline{V}_j,V_b\right\rangle\overline{\omega}^i\wedge\iota_{\overline{V}_b}(\alpha)\quad(k29e2a\leftrightarrow K52)$$
$$+\left\langle\nabla_{V_j}\overline{V}_a,\overline{V}_k\right\rangle\left\langle\nabla_{V_a}\overline{V}_j,V_b\right\rangle\overline{\omega}^k\wedge\iota_{\overline{V}_b}(\alpha)\quad(k29e2b\leftrightarrow K53)$$
$$+2\left\langle\nabla_{\overline{V}_a}V_i,V_k\right\rangle\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\overline{\omega}^a\wedge\iota_{\overline{V}_j}(\alpha)\quad(k30a1+k30b1\leftrightarrow K54)$$
$$+\left\langle\nabla_{\overline{V}_j}V_k,V_i\right\rangle\left\langle\nabla_{\overline{V}_a}V_j,\overline{V}_b\right\rangle\overline{\omega}^a\wedge\overline{\omega}^b\wedge\iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right)\quad(k3c1\leftrightarrow K55)$$
$$+\left\langle\nabla_{V_j}\overline{V}_b,\overline{V}_c\right\rangle\left\langle\nabla_{\overline{V}_j}V_k,V_i\right\rangle\overline{\omega}^c\wedge\overline{\omega}^b\wedge\iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right)\quad(k15a\leftrightarrow K56)$$
$$-\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\left\langle\nabla_{V_a}\overline{V}_j,V_b\right\rangle\overline{\omega}^k\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_a}\left(\iota_{\overline{V}_b}(\alpha)\right)\quad(k29e2c\leftrightarrow K57)$$
$$-2\left\langle\nabla_{\overline{V}_a}V_c,V_k\right\rangle\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\overline{\omega}^a\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_c}\left(\iota_{\overline{V}_j}(\alpha)\right)\quad(k30a2+k30b2\leftrightarrow K58)$$
$$+2\left\langle\nabla_{\overline{V}_a}V_k,V_b\right\rangle\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\overline{\omega}^a\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_b}\left(\iota_{\overline{V}_j}(\alpha)\right)\quad(k30c1+k30c2\leftrightarrow K59)$$
$$+\left\langle\nabla_{\overline{V}_j}V_k,V_i\right\rangle\left\langle\nabla_{V_a}\overline{V}_k,\overline{V}_b\right\rangle\omega^a\wedge\overline{\omega}^j\wedge\iota_{V_b}\left(\iota_{\overline{V}_i}(\alpha)\right)\quad(k6d2\leftrightarrow K60)$$
$$+\left\langle\nabla_{\overline{V}_j}V_k,V_i\right\rangle\left\langle\nabla_{V_a}\overline{V}_i,\overline{V}_b\right\rangle\omega^a\wedge\overline{\omega}^j\wedge\iota_{\overline{V}_k}\left(\iota_{V_b}(\alpha)\right)\quad(k6d3\leftrightarrow K61)$$
$$-2\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\left\langle\nabla_{\overline{V}_a}V_i,V_b\right\rangle\omega^j\wedge\overline{\omega}^k\wedge\iota_{V_a}\left(\iota_{\overline{V}_b}(\alpha)\right)\quad(k25e2b\leftrightarrow K62)$$
$$+2\left\langle\nabla_{V_j}\overline{V}_i,V_b\right\rangle\left\langle\nabla_{V_a}\overline{V}_k,\overline{V}_b\right\rangle\omega^j\wedge\overline{\omega}^k\wedge\iota_{\overline{V}_a}\left(\iota_{V_i}(\alpha)\right)\quad(k26b3\leftrightarrow K63)$$
$$+2\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\left\langle\nabla_{V_a}\overline{V}_j,V_b\right\rangle\omega^b\wedge\overline{\omega}^k\wedge\iota_{\overline{V}_a}\left(\iota_{V_i}(\alpha)\right)\quad(k26c1b\leftrightarrow K64)$$
$$+2\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\left\langle\nabla_{V_a}V_i,\overline{V}_b\right\rangle\omega^j\wedge\overline{\omega}^k\wedge\iota_{\overline{V}_a}\left(\iota_{V_b}(\alpha)\right)\quad(k26e2b\leftrightarrow K65)$$
$$-2\left\langle\nabla_{\overline{V}_a}V_c,V_b\right\rangle\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_b\right\rangle\overline{\omega}^a\wedge\omega^j\wedge\iota_{\overline{V}_c}\left(\iota_{V_i}(\alpha)\right)\quad(k27a\leftrightarrow K66)$$
$$+2\left\langle\nabla_{\overline{V}_a}V_k,V_b\right\rangle\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\overline{\omega}^a\wedge\omega^j\wedge\iota_{\overline{V}_b}\left(\iota_{V_i}(\alpha)\right)\quad(k27b1\leftrightarrow K67)$$
$$+\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\left\langle\nabla_{\overline{V}_a}V_k,V_b\right\rangle\omega^b\wedge\overline{\omega}^i\wedge\iota_{V_a}\left(\iota_{\overline{V}_j}(\alpha)\right)\quad(k28c1b\leftrightarrow K68)$$
$$+\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\left\langle\nabla_{\overline{V}_a}V_i,V_b\right\rangle\overline{\omega}^k\wedge\omega^b\wedge\iota_{V_a}\left(\iota_{\overline{V}_j}(\alpha)\right)\quad(k28d1b\leftrightarrow K69)$$
$$-2\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\left\langle\nabla_{\overline{V}_a}V_k,V_b\right\rangle\omega^j\wedge\omega^b\wedge\iota_{V_a}\left(\iota_{V_i}(\alpha)\right)\quad(k25d1c\leftrightarrow K70)$$

Here we have

$$\begin{aligned}
(k3b1)+(k3b2)+(k3d2)+(k3d3) &= 0, \qquad (k3d1)+(k18b2)=0\\
(k7)+(k22d) &= 0, \qquad (k6d1)+(k21)=0\\
(k8)+(k23d2) &= 0, \qquad (k9)+(k24b2)=0\\
(k10)+(k25e1b) &= 0, \qquad (k11)+(k26e1b)=0\\
(k12)+(k27b2) &= 0, \qquad (k13)+(k28e1)=0\\
(k14)+(k29e1c) &= 0, \qquad (k15b)+(k30c3)=0\\
(k23b2)+(k23c2a) &= 0, \qquad (k23b4)+(k23c2b)=0\\
(k26b2)+(k26d2a) &= 0, \qquad (k26b4)+(k26d2b)=0\\
(k29b2)+(k29c2a) &= 0, \qquad (k29b4)+(k29c2b)=0\\
(k29b5)+(k29d2c) &= 0, \qquad (k29b6)+(k29c2c)=0
\end{aligned}$$

A.3 Commutators of Algebraic Tensors for a $(p,q)-$form

We have defined $\left[L_{\overline{\partial}\Omega},\Lambda\right]: E^{p,q}\to E^{p,q+1}$ 1.7 and $\left[L,\left(L_{\overline{\partial}\Omega}\right)^{*}\right]: E^{p,q}\to E^{p,q-1}$ 1.8 before. Then we have

$$\begin{aligned}
&\left[L,\left(L_{\overline{\partial}\Omega}\right)^{*}\right]\left[L_{\overline{\partial}\Omega},\Lambda\right](\alpha)\\
= \quad &+4\left\langle\nabla_{\overline{V}_j}V_j,V_i\right\rangle\iota_{\overline{V}_i}\left(\left\langle\nabla_{V_a}\overline{V}_a,\overline{V}_c\right\rangle\overline{\omega}^c\wedge\alpha\right)\\
&+4\left\langle\nabla_{\overline{V}_j}V_j,V_i\right\rangle\iota_{\overline{V}_i}\left(\left\langle\nabla_{V_a}\overline{V}_b,\overline{V}_c\right\rangle\omega^a\wedge\overline{\omega}^c\wedge\iota_{V_b}(\alpha)\right)\\
&-2\left\langle\nabla_{\overline{V}_j}V_j,V_i\right\rangle\iota_{\overline{V}_i}\left(\left\langle\nabla_{V_a}\overline{V}_b,\overline{V}_c\right\rangle\overline{\omega}^c\wedge\overline{\omega}^b\wedge\iota_{\overline{V}_a}(\alpha)\right)\\
&+4\left\langle\nabla_{\overline{V}_j}V_i,V_k\right\rangle\omega^i\wedge\iota_{\overline{V}_k}\left(\iota_{V_j}\left(\left\langle\nabla_{V_a}\overline{V}_a,\overline{V}_c\right\rangle\overline{\omega}^c\wedge\alpha\right)\right)\\
&+4\left\langle\nabla_{\overline{V}_j}V_i,V_k\right\rangle\omega^i\wedge\iota_{\overline{V}_k}\left(\iota_{V_j}\left(\left\langle\nabla_{V_a}\overline{V}_b,\overline{V}_c\right\rangle\omega^a\wedge\overline{\omega}^c\wedge\iota_{V_b}(\alpha)\right)\right)\\
&-2\left\langle\nabla_{\overline{V}_j}V_i,V_k\right\rangle\omega^i\wedge\iota_{\overline{V}_k}\left(\iota_{V_j}\left(\left\langle\nabla_{V_a}\overline{V}_b,\overline{V}_c\right\rangle\overline{\omega}^c\wedge\overline{\omega}^b\wedge\iota_{\overline{V}_a}(\alpha)\right)\right)\\
&-2\left\langle\nabla_{\overline{V}_j}V_i,V_k\right\rangle\overline{\omega}^j\wedge\iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}\left(\left\langle\nabla_{V_a}\overline{V}_a,\overline{V}_c\right\rangle\overline{\omega}^c\wedge\alpha\right)\right)\\
&-2\left\langle\nabla_{\overline{V}_j}V_i,V_k\right\rangle\overline{\omega}^j\wedge\iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}\left(\left\langle\nabla_{V_a}\overline{V}_b,\overline{V}_c\right\rangle\omega^a\wedge\overline{\omega}^c\wedge\iota_{V_b}(\alpha)\right)\right)\\
&+\left\langle\nabla_{\overline{V}_j}V_i,V_k\right\rangle\overline{\omega}^j\wedge\iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}\left(\left\langle\nabla_{V_a}\overline{V}_b,\overline{V}_c\right\rangle\overline{\omega}^c\wedge\overline{\omega}^b\wedge\iota_{\overline{V}_a}(\alpha)\right)\right)
\end{aligned}$$

$$
\begin{aligned}
& \left[L,\left(L_{\overline{\partial}\Omega}\right)^{*}\right]\left[L_{\overline{\partial}\Omega},\Lambda\right](\alpha) \\
= \quad & +4\left\langle\nabla_{\overline{V}_j}V_j,V_c\right\rangle\left\langle\nabla_{V_a}\overline{V}_a,\overline{V}_c\right\rangle\alpha \\
& -4\left\langle\nabla_{\overline{V}_j}V_j,V_i\right\rangle\left\langle\nabla_{V_a}\overline{V}_a,\overline{V}_c\right\rangle\overline{\omega}^c\wedge\iota_{\overline{V}_i}(\alpha) \\
& +4\left\langle\nabla_{\overline{V}_j}V_j,V_b\right\rangle\left\langle\nabla_{V_a}\overline{V}_b,\overline{V}_c\right\rangle\overline{\omega}^c\wedge\iota_{\overline{V}_a}(\alpha) \\
& +4\left\langle\nabla_{\overline{V}_j}V_c,V_i\right\rangle\left\langle\nabla_{V_a}\overline{V}_a,\overline{V}_c\right\rangle\overline{\omega}^j\wedge\iota_{\overline{V}_i}(\alpha) \\
& +2\left\langle\nabla_{\overline{V}_j}V_b,V_c\right\rangle\left\langle\nabla_{V_a}\overline{V}_b,\overline{V}_c\right\rangle\overline{\omega}^j\wedge\iota_{\overline{V}_a}(\alpha) \\
& -4\left\langle\nabla_{\overline{V}_j}V_j,V_c\right\rangle\left\langle\nabla_{V_a}\overline{V}_b,\overline{V}_c\right\rangle\omega^a\wedge\iota_{V_b}(\alpha) \\
& -4\left\langle\nabla_{\overline{V}_j}V_i,V_c\right\rangle\left\langle\nabla_{V_a}\overline{V}_a,\overline{V}_c\right\rangle\omega^i\wedge\iota_{V_j}(\alpha) \\
& +4\left\langle\nabla_{\overline{V}_a}V_i,V_c\right\rangle\left\langle\nabla_{V_a}\overline{V}_b,\overline{V}_c\right\rangle\omega^i\wedge\iota_{V_b}(\alpha) \\
& +4\left\langle\nabla_{\overline{V}_j}V_j,V_i\right\rangle\left\langle\nabla_{V_a}\overline{V}_b,\overline{V}_c\right\rangle\omega^a\wedge\overline{\omega}^c\wedge\iota_{\overline{V}_i}\left(\iota_{V_b}(\alpha)\right) \\
& +4\left\langle\nabla_{\overline{V}_j}V_i,V_k\right\rangle\left\langle\nabla_{V_a}\overline{V}_a,\overline{V}_c\right\rangle\omega^i\wedge\overline{\omega}^c\wedge\iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right) \\
& -4\left\langle\nabla_{\overline{V}_a}V_i,V_k\right\rangle\left\langle\nabla_{V_a}\overline{V}_b,\overline{V}_c\right\rangle\omega^i\wedge\overline{\omega}^c\wedge\iota_{\overline{V}_k}\left(\iota_{V_b}(\alpha)\right) \\
& +4\left\langle\nabla_{\overline{V}_j}V_i,V_c\right\rangle\left\langle\nabla_{V_a}\overline{V}_c,\overline{V}_b\right\rangle\omega^i\wedge\overline{\omega}^b\wedge\iota_{V_j}\left(\iota_{\overline{V}_a}(\alpha)\right) \\
& +4\left\langle\nabla_{\overline{V}_j}V_c,V_i\right\rangle\left\langle\nabla_{V_a}\overline{V}_b,\overline{V}_c\right\rangle\overline{\omega}^j\wedge\omega^a\wedge\iota_{\overline{V}_i}\left(\iota_{V_b}(\alpha)\right) \\
& -4\left\langle\nabla_{\overline{V}_j}V_i,V_c\right\rangle\left\langle\nabla_{V_a}\overline{V}_b,\overline{V}_c\right\rangle\omega^i\wedge\omega^a\wedge\iota_{V_j}\left(\iota_{V_b}(\alpha)\right) \\
& -2\left\langle\nabla_{\overline{V}_j}V_j,V_i\right\rangle\left\langle\nabla_{V_a}\overline{V}_b,\overline{V}_c\right\rangle\overline{\omega}^c\wedge\overline{\omega}^b\wedge\iota_{\overline{V}_i}\left(\iota_{\overline{V}_a}(\alpha)\right) \\
& +4\left\langle\nabla_{\overline{V}_j}V_i,V_c\right\rangle\left\langle\nabla_{V_a}\overline{V}_c,\overline{V}_b\right\rangle\overline{\omega}^j\wedge\overline{\omega}^b\wedge\iota_{\overline{V}_i}\left(\iota_{\overline{V}_a}(\alpha)\right) \\
& -2\left\langle\nabla_{\overline{V}_j}V_i,V_k\right\rangle\left\langle\nabla_{V_a}\overline{V}_a,\overline{V}_c\right\rangle\overline{\omega}^j\wedge\overline{\omega}^c\wedge\iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}(\alpha)\right) \\
& +4\left\langle\nabla_{\overline{V}_j}V_i,V_k\right\rangle\left\langle\nabla_{V_a}\overline{V}_b,\overline{V}_c\right\rangle\omega^i\wedge\omega^a\wedge\overline{\omega}^c\wedge\iota_{\overline{V}_k}\left(\iota_{V_j}\left(\iota_{V_b}(\alpha)\right)\right) \\
& -2\left\langle\nabla_{\overline{V}_j}V_i,V_k\right\rangle\left\langle\nabla_{V_a}\overline{V}_b,\overline{V}_c\right\rangle\omega^i\wedge\overline{\omega}^c\wedge\overline{\omega}^b\wedge\iota_{\overline{V}_k}\left(\iota_{V_j}\left(\iota_{\overline{V}_a}(\alpha)\right)\right) \\
& -2\left\langle\nabla_{\overline{V}_j}V_i,V_k\right\rangle\left\langle\nabla_{V_a}\overline{V}_b,\overline{V}_c\right\rangle\omega^a\wedge\overline{\omega}^c\wedge\overline{\omega}^j\wedge\iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}\left(\iota_{V_b}(\alpha)\right)\right) \\
& +\left\langle\nabla_{\overline{V}_j}V_i,V_k\right\rangle\left\langle\nabla_{V_a}\overline{V}_b,\overline{V}_c\right\rangle\overline{\omega}^j\wedge\overline{\omega}^c\wedge\overline{\omega}^b\wedge\iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}\left(\iota_{\overline{V}_a}(\alpha)\right)\right)
\end{aligned}
$$

And also

$$
\begin{aligned}
& \left[L_{\overline{\partial}\Omega}, \Lambda\right]\left[L, \left(L_{\overline{\partial}\Omega}\right)^*\right](\alpha) \\
= \; & +4\left\langle \nabla_{V_a}\overline{V}_a, \overline{V}_c\right\rangle \overline{\omega}^c \wedge \left(\left\langle \nabla_{\overline{V}_j} V_j, V_i\right\rangle \iota_{\overline{V}_i}(\alpha)\right) \\
& +4\left\langle \nabla_{V_a}\overline{V}_a, \overline{V}_c\right\rangle \overline{\omega}^c \wedge \left(\left\langle \nabla_{\overline{V}_j} V_i, V_k\right\rangle \omega^i \wedge \iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right)\right) \\
& -2\left\langle \nabla_{V_a}\overline{V}_a, \overline{V}_c\right\rangle \overline{\omega}^c \wedge \left(\left\langle \nabla_{\overline{V}_j} V_i, V_k\right\rangle \overline{\omega}^j \wedge \iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}(\alpha)\right)\right) \\
& +4\left\langle \nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle \omega^a \wedge \overline{\omega}^c \wedge \iota_{V_b}\left(\left\langle \nabla_{\overline{V}_j} V_j, V_i\right\rangle \iota_{\overline{V}_i}(\alpha)\right) \\
& +4\left\langle \nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle \omega^a \wedge \overline{\omega}^c \wedge \iota_{V_b}\left(\left\langle \nabla_{\overline{V}_j} V_i, V_k\right\rangle \omega^i \wedge \iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right)\right) \\
& -2\left\langle \nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle \omega^a \wedge \overline{\omega}^c \wedge \iota_{V_b}\left(\left\langle \nabla_{\overline{V}_j} V_i, V_k\right\rangle \overline{\omega}^j \wedge \iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}(\alpha)\right)\right) \\
& -2\left\langle \nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle \overline{\omega}^c \wedge \overline{\omega}^b \wedge \iota_{\overline{V}_a}\left(\left\langle \nabla_{\overline{V}_j} V_j, V_i\right\rangle \iota_{\overline{V}_i}(\alpha)\right) \\
& -2\left\langle \nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle \overline{\omega}^c \wedge \overline{\omega}^b \wedge \iota_{\overline{V}_a}\left(\left\langle \nabla_{\overline{V}_j} V_i, V_k\right\rangle \omega^i \wedge \iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right)\right) \\
& +\left\langle \nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle \overline{\omega}^c \wedge \overline{\omega}^b \wedge \iota_{\overline{V}_a}\left(\left\langle \nabla_{\overline{V}_j} V_i, V_k\right\rangle \overline{\omega}^j \wedge \iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}(\alpha)\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
= \; & +4\left\langle \nabla_{V_a}\overline{V}_a, \overline{V}_c\right\rangle \left\langle \nabla_{\overline{V}_j} V_j, V_i\right\rangle \overline{\omega}^c \wedge \iota_{\overline{V}_i}(\alpha) \\
& +4\left\langle \nabla_{V_a}\overline{V}_a, \overline{V}_c\right\rangle \left\langle \nabla_{\overline{V}_j} V_i, V_k\right\rangle \overline{\omega}^c \wedge \omega^i \wedge \iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right) \\
& +4\left\langle \nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle \left\langle \nabla_{\overline{V}_j} V_j, V_i\right\rangle \omega^a \wedge \overline{\omega}^c \wedge \iota_{V_b}\left(\iota_{\overline{V}_i}(\alpha)\right) \\
& +4\left\langle \nabla_{V_a}\overline{V}_i, \overline{V}_c\right\rangle \left\langle \nabla_{\overline{V}_j} V_i, V_k\right\rangle \omega^a \wedge \overline{\omega}^c \wedge \iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right) \\
& -2\left\langle \nabla_{V_a}\overline{V}_a, \overline{V}_c\right\rangle \left\langle \nabla_{\overline{V}_j} V_i, V_k\right\rangle \overline{\omega}^c \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}(\alpha)\right) \\
& -2\left\langle \nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle \left\langle \nabla_{\overline{V}_j} V_j, V_i\right\rangle \overline{\omega}^c \wedge \overline{\omega}^b \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_i}(\alpha)\right) \\
& +\left\langle \nabla_{V_j}\overline{V}_b, \overline{V}_c\right\rangle \left\langle \nabla_{\overline{V}_j} V_i, V_k\right\rangle \overline{\omega}^c \wedge \overline{\omega}^b \wedge \iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}(\alpha)\right) \\
& -4\left\langle \nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle \left\langle \nabla_{\overline{V}_j} V_i, V_k\right\rangle \omega^a \wedge \overline{\omega}^c \wedge \omega^i \wedge \iota_{V_b}\left(\iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right)\right) \\
& +2\left\langle \nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle \left\langle \nabla_{\overline{V}_j} V_i, V_k\right\rangle \omega^a \wedge \overline{\omega}^c \wedge \overline{\omega}^j \wedge \iota_{V_b}\left(\iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}(\alpha)\right)\right) \\
& +2\left\langle \nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle \left\langle \nabla_{\overline{V}_j} V_i, V_k\right\rangle \overline{\omega}^c \wedge \overline{\omega}^b \wedge \omega^i \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right)\right) \\
& -\left\langle \nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle \left\langle \nabla_{\overline{V}_j} V_i, V_k\right\rangle \overline{\omega}^c \wedge \overline{\omega}^b \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}(\alpha)\right)\right)
\end{aligned}
$$

So we have

$$
\begin{aligned}
& \left[L, \left(L_{\overline{\partial}\Omega}\right)^*\right]\left[L_{\overline{\partial}\Omega}, \Lambda\right](\alpha) + \left[L_{\overline{\partial}\Omega}, \Lambda\right]\left[L, \left(L_{\overline{\partial}\Omega}\right)^*\right](\alpha) \\
= \quad & +4\left\langle \nabla_{\overline{V}_j} V_j, V_c \right\rangle \left\langle \nabla_{V_a} \overline{V}_a, \overline{V}_c \right\rangle \alpha \qquad (A) \\
& +4\left\langle \nabla_{\overline{V}_j} V_j, V_b \right\rangle \left\langle \nabla_{V_a} \overline{V}_b, \overline{V}_c \right\rangle \overline{\omega}^c \wedge \iota_{\overline{V}_a}(\alpha) \qquad (B) \\
& +4\left\langle \nabla_{\overline{V}_j} V_c, V_i \right\rangle \left\langle \nabla_{V_a} \overline{V}_a, \overline{V}_c \right\rangle \overline{\omega}^j \wedge \iota_{\overline{V}_i}(\alpha) \qquad (C) \\
& +2\left\langle \nabla_{\overline{V}_j} V_b, V_c \right\rangle \left\langle \nabla_{V_a} \overline{V}_b, \overline{V}_c \right\rangle \overline{\omega}^j \wedge \iota_{\overline{V}_a}(\alpha) \qquad (D) \\
& -4\left\langle \nabla_{\overline{V}_j} V_j, V_c \right\rangle \left\langle \nabla_{V_a} \overline{V}_b, \overline{V}_c \right\rangle \omega^a \wedge \iota_{V_b}(\alpha) \qquad (E) \\
& -4\left\langle \nabla_{\overline{V}_j} V_i, V_c \right\rangle \left\langle \nabla_{V_a} \overline{V}_a, \overline{V}_c \right\rangle \omega^i \wedge \iota_{V_j}(\alpha) \qquad (F) \\
& +4\left\langle \nabla_{\overline{V}_a} V_i, V_c \right\rangle \left\langle \nabla_{V_a} \overline{V}_b, \overline{V}_c \right\rangle \omega^i \wedge \iota_{V_b}(\alpha) \qquad (G)
\end{aligned}
$$

$$
\begin{aligned}
& -4\left\langle \nabla_{\overline{V}_j} V_i, V_c \right\rangle \left\langle \nabla_{V_a} \overline{V}_b, \overline{V}_c \right\rangle \omega^i \wedge \omega^a \wedge \iota_{V_j}\left(\iota_{V_b}(\alpha)\right) \qquad (H) \\
& -4\left\langle \nabla_{\overline{V}_a} V_i, V_k \right\rangle \left\langle \nabla_{V_a} \overline{V}_b, \overline{V}_c \right\rangle \omega^i \wedge \overline{\omega}^c \wedge \iota_{\overline{V}_k}\left(\iota_{V_b}(\alpha)\right) \qquad (I) \\
& +4\left\langle \nabla_{\overline{V}_j} V_i, V_c \right\rangle \left\langle \nabla_{V_a} \overline{V}_c, \overline{V}_b \right\rangle \omega^i \wedge \overline{\omega}^b \wedge \iota_{V_j}\left(\iota_{\overline{V}_a}(\alpha)\right) \qquad (J) \\
& +4\left\langle \nabla_{\overline{V}_j} V_c, V_i \right\rangle \left\langle \nabla_{V_a} \overline{V}_b, \overline{V}_c \right\rangle \overline{\omega}^j \wedge \omega^a \wedge \iota_{\overline{V}_i}\left(\iota_{V_b}(\alpha)\right) \qquad (K) \\
& -4\left\langle \nabla_{\overline{V}_j} V_i, V_k \right\rangle \left\langle \nabla_{V_a} \overline{V}_i, \overline{V}_c \right\rangle \overline{\omega}^c \wedge \omega^a \wedge \iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right) \qquad (L) \\
& +4\left\langle \nabla_{\overline{V}_j} V_i, V_c \right\rangle \left\langle \nabla_{V_a} \overline{V}_c, \overline{V}_b \right\rangle \overline{\omega}^j \wedge \overline{\omega}^b \wedge \iota_{\overline{V}_i}\left(\iota_{\overline{V}_a}(\alpha)\right) \qquad (M) \\
& +\left\langle \nabla_{\overline{V}_j} V_i, V_k \right\rangle \left\langle \nabla_{V_j} \overline{V}_b, \overline{V}_c \right\rangle \overline{\omega}^c \wedge \overline{\omega}^b \wedge \iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}(\alpha)\right) \qquad (N)
\end{aligned}
$$

If we define $T_1 : E^{p,q} \to E^{p,q-1}$ by

$$T_1(\alpha) = \left\langle \nabla_{\overline{V}_j} V_k, V_i \right\rangle \overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right)$$

The L^2-adjoint of T_1, $(T_1)^* : E^{p,q} \to E^{p,q+1}$ is equal to

$$(T_1)^*(\alpha) = \left\langle \nabla_{V_j} \overline{V}_k, \overline{V}_i \right\rangle \overline{\omega}^i \wedge \overline{\omega}^k \wedge \iota_{\overline{V}_j}(\alpha)$$

then

$$
\begin{aligned}
& T_1\left((T_1)^*(\alpha)\right) \\
= \; & \left\langle \nabla_{\overline{V}_j} V_k, V_i \right\rangle \overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}\left(\left\langle \nabla_{V_a}\overline{V}_c, \overline{V}_b\right\rangle \overline{\omega}^b \wedge \overline{\omega}^c \wedge \iota_{\overline{V}_a}(\alpha)\right)\right) \\
= \; & +\left\langle \nabla_{\overline{V}_j} V_k, V_i \right\rangle \left\langle \nabla_{V_a}\overline{V}_c, \overline{V}_b\right\rangle \overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}\left(\overline{\omega}^b\right) \wedge \overline{\omega}^c \wedge \iota_{\overline{V}_a}(\alpha)\right) \\
& -\left\langle \nabla_{\overline{V}_j} V_k, V_i \right\rangle \left\langle \nabla_{V_a}\overline{V}_c, \overline{V}_b\right\rangle \overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\overline{\omega}^b \wedge \iota_{\overline{V}_i}\left(\overline{\omega}^c\right) \wedge \iota_{\overline{V}_a}(\alpha)\right) \\
& +\left\langle \nabla_{\overline{V}_j} V_k, V_i \right\rangle \left\langle \nabla_{V_a}\overline{V}_c, \overline{V}_b\right\rangle \overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\overline{\omega}^b \wedge \overline{\omega}^c \wedge \iota_{\overline{V}_i}\left(\iota_{\overline{V}_a}(\alpha)\right)\right) \\
= \; & +2\left\langle \nabla_{\overline{V}_j} V_c, V_b \right\rangle \left\langle \nabla_{V_a}\overline{V}_c, \overline{V}_b\right\rangle \overline{\omega}^j \wedge \iota_{\overline{V}_a}(\alpha) \\
& +4\left\langle \nabla_{\overline{V}_j} V_i, V_c \right\rangle \left\langle \nabla_{V_a}\overline{V}_c, \overline{V}_b\right\rangle \overline{\omega}^j \wedge \overline{\omega}^b \wedge \iota_{\overline{V}_i}\left(\iota_{\overline{V}_a}(\alpha)\right) \\
& +\left\langle \nabla_{\overline{V}_j} V_k, V_i \right\rangle \left\langle \nabla_{V_a}\overline{V}_c, \overline{V}_b\right\rangle \overline{\omega}^j \wedge \overline{\omega}^b \wedge \overline{\omega}^c \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}\left(\iota_{\overline{V}_a}(\alpha)\right)\right)
\end{aligned}
$$

Also

$$
\begin{aligned}
& (T_1)^*\left(T_1(\alpha)\right) \\
= \; & \left\langle \nabla_{V_a}\overline{V}_c, \overline{V}_b\right\rangle \overline{\omega}^b \wedge \overline{\omega}^c \wedge \iota_{\overline{V}_a}\left(\left\langle \nabla_{\overline{V}_j} V_k, V_i \right\rangle \overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}\alpha\right)\right) \\
= \; & \left\langle \nabla_{V_a}\overline{V}_c, \overline{V}_b\right\rangle \left\langle \nabla_{\overline{V}_j} V_k, V_i \right\rangle \overline{\omega}^b \wedge \overline{\omega}^c \wedge \iota_{\overline{V}_a}\left(\overline{\omega}^j\right) \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}\alpha\right) \\
& -\left\langle \nabla_{V_a}\overline{V}_c, \overline{V}_b\right\rangle \left\langle \nabla_{\overline{V}_j} V_k, V_i \right\rangle \overline{\omega}^b \wedge \overline{\omega}^c \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}\alpha\right)\right) \\
= \; & \left\langle \nabla_{V_j}\overline{V}_c, \overline{V}_b\right\rangle \left\langle \nabla_{\overline{V}_j} V_k, V_i \right\rangle \overline{\omega}^b \wedge \overline{\omega}^c \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}\alpha\right) \\
& -\left\langle \nabla_{V_a}\overline{V}_c, \overline{V}_b\right\rangle \left\langle \nabla_{\overline{V}_j} V_k, V_i \right\rangle \overline{\omega}^b \wedge \overline{\omega}^c \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}\alpha\right)\right)
\end{aligned}
$$

thus

$$
\begin{aligned}
& \left[T_1, (T_1)^*\right](\alpha) \\
= \; & T_1 (T_1)^*(\alpha) + (T_1)^*\left(T_1(\alpha)\right) \\
= \; & +2\left\langle \nabla_{\overline{V}_j} V_c, V_b \right\rangle \left\langle \nabla_{V_a}\overline{V}_c, \overline{V}_b\right\rangle \overline{\omega}^j \wedge \iota_{\overline{V}_a}(\alpha) \qquad (O) \\
& +4\left\langle \nabla_{\overline{V}_j} V_i, V_c \right\rangle \left\langle \nabla_{V_a}\overline{V}_c, \overline{V}_b\right\rangle \overline{\omega}^j \wedge \overline{\omega}^b \wedge \iota_{\overline{V}_i}\left(\iota_{\overline{V}_a}(\alpha)\right) \qquad (P) \\
& +\left\langle \nabla_{V_j}\overline{V}_c, \overline{V}_b\right\rangle \left\langle \nabla_{\overline{V}_j} V_k, V_i \right\rangle \overline{\omega}^b \wedge \overline{\omega}^c \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}\alpha\right) \qquad (Q)
\end{aligned}
$$

If we define $T_2 : E^{p,q} \to E^{p+1,q}$ by

$$T_2(\alpha) = +2\left\langle \nabla_{\overline{V}_j} V_i, V_k \right\rangle \overline{\omega}^j \wedge \omega^k \wedge \iota_{\overline{V}_i}(\alpha)$$

then its L^2-adjoint $(T_2)^* : E^{p,q} \to E^{p-1,q}$ would be given by

$$(T_2)^*(\alpha) = +2\left\langle \nabla_{V_a}\overline{V}_b, \overline{V}_c\right\rangle \overline{\omega}^b \wedge \iota_{V_c}\left(\iota_{\overline{V}_a}(\alpha)\right)$$

Then we have

$$\begin{aligned}
T_2\left((T_2)^*(\alpha)\right) &= +2\left\langle \nabla_{\overline{V}_j} V_i, V_k \right\rangle \overline{\omega}^j \wedge \omega^k \wedge \iota_{\overline{V}_i}\left(2\left\langle \nabla_{V_a} \overline{V}_b, \overline{V}_c\right\rangle \overline{\omega}^b \wedge \iota_{V_c}\left(\iota_{\overline{V}_a}(\alpha)\right)\right) \\
&= +4\left\langle \nabla_{\overline{V}_j} V_i, V_k \right\rangle \left\langle \nabla_{V_a} \overline{V}_b, \overline{V}_c\right\rangle \overline{\omega}^j \wedge \omega^k \wedge \iota_{\overline{V}_i}\left(\overline{\omega}^b\right) \wedge \iota_{V_c}\left(\iota_{\overline{V}_a}(\alpha)\right) \\
&\quad -4\left\langle \nabla_{\overline{V}_j} V_i, V_k \right\rangle \left\langle \nabla_{V_a} \overline{V}_b, \overline{V}_c\right\rangle \overline{\omega}^j \wedge \omega^k \wedge \overline{\omega}^b \wedge \iota_{\overline{V}_i}\left(\iota_{V_c}\left(\iota_{\overline{V}_a}(\alpha)\right)\right) \\
&= +4\left\langle \nabla_{\overline{V}_j} V_i, V_k \right\rangle \left\langle \nabla_{V_a} \overline{V}_i, \overline{V}_c\right\rangle \overline{\omega}^j \wedge \omega^k \wedge \iota_{V_c}\left(\iota_{\overline{V}_a}(\alpha)\right) \\
&\quad -4\left\langle \nabla_{\overline{V}_j} V_i, V_k \right\rangle \left\langle \nabla_{V_a} \overline{V}_b, \overline{V}_c\right\rangle \overline{\omega}^j \wedge \omega^k \wedge \overline{\omega}^b \wedge \iota_{\overline{V}_i}\left(\iota_{V_c}\left(\iota_{\overline{V}_a}(\alpha)\right)\right)
\end{aligned}$$

Also we have

$$\begin{aligned}
&(T_2)^*\left(T_2(\alpha)\right) \\
=\ & +2\left\langle \nabla_{V_a} \overline{V}_b, \overline{V}_c\right\rangle \overline{\omega}^b \wedge \iota_{V_c}\left(\iota_{\overline{V}_a}\left(2\left\langle \nabla_{\overline{V}_j} V_i, V_k \right\rangle \overline{\omega}^j \wedge \omega^k \wedge \iota_{\overline{V}_i}(\alpha)\right)\right) \\
=\ & +4\left\langle \nabla_{V_a} \overline{V}_b, \overline{V}_c\right\rangle \left\langle \nabla_{\overline{V}_j} V_i, V_k \right\rangle \overline{\omega}^b \wedge \iota_{V_c}\left(\iota_{\overline{V}_a}\left(\overline{\omega}^j\right) \wedge \omega^k \wedge \iota_{\overline{V}_i}(\alpha)\right) \\
& +4\left\langle \nabla_{V_a} \overline{V}_b, \overline{V}_c\right\rangle \left\langle \nabla_{\overline{V}_j} V_i, V_k \right\rangle \overline{\omega}^b \wedge \iota_{V_c}\left(\overline{\omega}^j \wedge \omega^k \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_i}(\alpha)\right)\right) \\
=\ & +4\left\langle \nabla_{V_j} \overline{V}_b, \overline{V}_c\right\rangle \left\langle \nabla_{\overline{V}_j} V_i, V_k \right\rangle \overline{\omega}^b \wedge \iota_{V_c}\left(\omega^k \wedge \iota_{\overline{V}_i}(\alpha)\right) \\
& +4\left\langle \nabla_{V_a} \overline{V}_b, \overline{V}_c\right\rangle \left\langle \nabla_{\overline{V}_j} V_i, V_k \right\rangle \overline{\omega}^b \wedge \iota_{V_c}\left(\overline{\omega}^j \wedge \omega^k \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_i}(\alpha)\right)\right) \\
=\ & +4\left\langle \nabla_{V_j} \overline{V}_b, \overline{V}_k\right\rangle \left\langle \nabla_{\overline{V}_j} V_i, V_k \right\rangle \overline{\omega}^b \wedge \iota_{\overline{V}_i}(\alpha) \\
& -4\left\langle \nabla_{V_j} \overline{V}_b, \overline{V}_c\right\rangle \left\langle \nabla_{\overline{V}_j} V_i, V_k \right\rangle \overline{\omega}^b \wedge \omega^k \wedge \iota_{V_c}\left(\iota_{\overline{V}_i}(\alpha)\right) \\
& -4\left\langle \nabla_{V_a} \overline{V}_b, \overline{V}_k\right\rangle \left\langle \nabla_{\overline{V}_j} V_i, V_k \right\rangle \overline{\omega}^b \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_i}(\alpha)\right) \\
& +4\left\langle \nabla_{V_a} \overline{V}_b, \overline{V}_c\right\rangle \left\langle \nabla_{\overline{V}_j} V_i, V_k \right\rangle \overline{\omega}^b \wedge \overline{\omega}^j \wedge \omega^k \wedge \iota_{V_c}\left(\iota_{\overline{V}_a}\left(\iota_{\overline{V}_i}(\alpha)\right)\right)
\end{aligned}$$

Then we have

$$\begin{aligned}
&\left[T_2, (T_2)^*\right](\alpha) \\
=\ & T_2\left((T_2)^*(\alpha)\right) + (T_2)^*\left(T_2(\alpha)\right) \\
=\ & +4\left\langle \nabla_{\overline{V}_j} V_i, V_k \right\rangle \left\langle \nabla_{V_j} \overline{V}_b, \overline{V}_k\right\rangle \overline{\omega}^b \wedge \iota_{\overline{V}_i}(\alpha) \qquad (R) \\
& +4\left\langle \nabla_{\overline{V}_j} V_i, V_k \right\rangle \left\langle \nabla_{V_a} \overline{V}_i, \overline{V}_c\right\rangle \overline{\omega}^j \wedge \omega^k \wedge \iota_{V_c}\left(\iota_{\overline{V}_a}(\alpha)\right) \qquad (S) \\
& -4\left\langle \nabla_{\overline{V}_j} V_i, V_k \right\rangle \left\langle \nabla_{V_a} \overline{V}_b, \overline{V}_k\right\rangle \overline{\omega}^b \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_i}(\alpha)\right) \qquad (T) \\
& -4\left\langle \nabla_{\overline{V}_j} V_i, V_k \right\rangle \left\langle \nabla_{V_j} \overline{V}_b, \overline{V}_c\right\rangle \overline{\omega}^b \wedge \omega^k \wedge \iota_{V_c}\left(\iota_{\overline{V}_i}(\alpha)\right) \qquad (U)
\end{aligned}$$

A.4 Proof of the Weitzenbock Formula for a $(p,q)-$form

Theorem A.4.1 *Let $\alpha \in E^{p,q}$ be any $(p,q)-$form on a compact complex manifold and let $\Delta_K, \Delta_H, \Delta_R : E^{p,q} \to E^{p,q}$ be the Laplacians on M defined in equations [2.1] and let $\left[L_{\overline{\partial}\Omega}, \Lambda\right]$, T_1, T_2 be the* *algebraic tensors defined in equation [1.7], equation [2.6], equation [2.8], respectively. Then we have the following Weitzenbock formula*

$$\begin{aligned} & \Delta_K(\alpha) + \Delta_H(\alpha) - 2\Delta_R(\alpha) \\ = \;& F(R)(\alpha) + \frac{1}{2}\left[\left[L, (L_{\overline{\partial}\Omega})^*\right], [L_{\overline{\partial}\Omega}, \Lambda]\right](\alpha) + \frac{1}{2}\left[T_1, (T_1)^*\right](\alpha) - \frac{1}{2}\left[T_2, (T_2)^*\right](\alpha) \end{aligned}$$

where $F(R)(\alpha) : E^{p,q} \to E^{p,q}$ *is the curvature operator given by*

$$\begin{aligned} F(R)(\alpha) = \;& +2\left\langle R_{V_a\overline{V}_j}\overline{V}_d, V_c\right\rangle \overline{\omega}^j \wedge \omega^c \wedge \iota_{\overline{V}_a}(\iota_{V_d}(\alpha)) \\ & +2\left\langle R_{\overline{V}_a V_j}V_c, \overline{V}_j\right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_c}(\alpha) + 2\left\langle R_{V_aV_j}\overline{V}_j, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}(\alpha) \\ & +2\left\langle R_{V_a\overline{V}_j}V_c, \overline{V}_d\right\rangle \overline{\omega}^j \wedge \overline{\omega}^d \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_c}(\alpha)\right) + \left\langle R_{V_jV_k}\overline{V}_j, \overline{V}_k\right\rangle \alpha \\ & +\frac{1}{2}\left\langle R_{\overline{V}_a\overline{V}_j}V_k, V_i\right\rangle \overline{\omega}^a \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}(\alpha)\right) \\ & +\frac{1}{2}\langle R_{V_aV_j}\overline{V}_i, \overline{V}_k\rangle \overline{\omega}^k \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}(\alpha)\right) \end{aligned}$$

Proof A.4.1 *Claim 1: We have the following terms canceling each other:*

$$\begin{aligned} (K23) + (K24) + (H17) &= 0, \qquad (K19) + (H13) = 0 \\ (K9) + (H9) - 2(R1) &= 0, \qquad (K22) + (H16) = 0 \\ (K20) + (H14) - 2(R3) &= 0, \qquad (H16) + (H17) = 0 \\ (K21) - 2(R4) &= 0 \end{aligned}$$

Claim 2: The terms $(K15), (K17), (K55), (K57)$ *all combine to give the curvature terms*

$$+\frac{1}{2}\left\langle R_{\overline{V}_a\overline{V}_j}V_k, V_i\right\rangle \overline{\omega}^a \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}(\alpha)\right) + \frac{1}{2}\langle R_{V_aV_j}\overline{V}_i, \overline{V}_k\rangle \overline{\omega}^k \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}(\alpha)\right)$$

Since

$$\begin{aligned} & +\left\langle \nabla_{\overline{V}_a}\nabla_{\overline{V}_j}V_k, V_i\right\rangle \overline{\omega}^a \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_i}(\alpha)\right) \quad (K15) \\ = \;& -\frac{1}{2}\left\langle \nabla_{\overline{V}_a}\nabla_{\overline{V}_j}V_k, V_i\right\rangle \overline{\omega}^a \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}(\alpha)\right) - \frac{1}{2}\left\langle \nabla_{\overline{V}_a}\nabla_{\overline{V}_j}V_k, V_i\right\rangle \overline{\omega}^a \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}(\alpha)\right) \\ = \;& -\frac{1}{2}\left\langle \nabla_{\overline{V}_a}\nabla_{\overline{V}_j}V_k, V_i\right\rangle \overline{\omega}^a \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}(\alpha)\right) - \frac{1}{2}\left\langle \nabla_{\overline{V}_j}\nabla_{\overline{V}_a}V_k, V_i\right\rangle \overline{\omega}^j \wedge \overline{\omega}^a \wedge \iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}(\alpha)\right) \\ = \;& -\frac{1}{2}\left\langle \nabla_{\overline{V}_a}\nabla_{\overline{V}_j}V_k, V_i\right\rangle \overline{\omega}^a \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}(\alpha)\right) + \frac{1}{2}\left\langle \nabla_{\overline{V}_j}\nabla_{\overline{V}_a}V_k, V_i\right\rangle \overline{\omega}^a \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}(\alpha)\right) \\ & +\frac{1}{2}\left\langle \nabla_{[\overline{V}_a,\overline{V}_j]}V_k, V_i\right\rangle \overline{\omega}^a \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}(\alpha)\right) - \frac{1}{2}\left\langle \nabla_{[\overline{V}_a,\overline{V}_j]}V_k, V_i\right\rangle \overline{\omega}^a \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}(\alpha)\right) \\ = \;& +\frac{1}{2}\left\langle R_{\overline{V}_a\overline{V}_j}V_k, V_i\right\rangle \overline{\omega}^a \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}(\alpha)\right) \\ & -\frac{1}{2}\left\langle \nabla_{\overline{V}_a}\overline{V}_j, V_b\right\rangle \left\langle \nabla_{\overline{V}_b}V_k, V_i\right\rangle \overline{\omega}^a \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}(\alpha)\right) \quad (K15b) \\ & +\frac{1}{2}\left\langle \nabla_{\overline{V}_j}\overline{V}_a, V_b\right\rangle \left\langle \nabla_{\overline{V}_b}V_k, V_i\right\rangle \overline{\omega}^a \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}(\alpha)\right) \quad (K15c) \end{aligned}$$

Similarly,

$$
\begin{aligned}
& -\left\langle \nabla_{V_a}\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle \overline{\omega}^k\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}(\alpha)\right) \quad (K17)\\
= & -\frac{1}{2}\left\langle \nabla_{V_a}\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle \overline{\omega}^k\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}(\alpha)\right)-\frac{1}{2}\left\langle \nabla_{V_a}\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle \overline{\omega}^k\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}(\alpha)\right)\\
= & -\frac{1}{2}\left\langle \nabla_{V_a}\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle \overline{\omega}^k\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}(\alpha)\right)-\frac{1}{2}\left\langle \nabla_{V_j}\nabla_{V_a}\overline{V}_i,\overline{V}_k\right\rangle \overline{\omega}^k\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_j}\left(\iota_{\overline{V}_a}(\alpha)\right)\\
= & -\frac{1}{2}\left\langle \nabla_{V_a}\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle \overline{\omega}^k\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}(\alpha)\right)+\frac{1}{2}\left\langle \nabla_{V_j}\nabla_{V_a}\overline{V}_i,\overline{V}_k\right\rangle \overline{\omega}^k\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}(\alpha)\right)\\
& +\frac{1}{2}\left\langle \nabla_{[V_a,V_j]}\overline{V}_i,\overline{V}_k\right\rangle \overline{\omega}^k\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}(\alpha)\right)-\frac{1}{2}\left\langle \nabla_{[V_a,V_j]}\overline{V}_i,\overline{V}_k\right\rangle \overline{\omega}^k\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}(\alpha)\right)\\
= & +\frac{1}{2}\left\langle R_{V_aV_j}\overline{V}_i,\overline{V}_k\right\rangle \overline{\omega}^k\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}(\alpha)\right) \quad (K17a)\\
& -\frac{1}{2}\left\langle \nabla_{V_a}V_j,\overline{V}_b\right\rangle\left\langle \nabla_{V_b}\overline{V}_i,\overline{V}_k\right\rangle \overline{\omega}^k\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}(\alpha)\right) \quad (K17b)\\
& +\frac{1}{2}\left\langle \nabla_{V_j}V_a,\overline{V}_b\right\rangle\left\langle \nabla_{V_b}\overline{V}_i,\overline{V}_k\right\rangle \overline{\omega}^k\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}(\alpha)\right) \quad (K17c)
\end{aligned}
$$

These last five terms are equal to

$$
\begin{aligned}
& +\frac{1}{2}\left\langle R_{\overline{V}_a\overline{V}_j}V_k,V_i\right\rangle \overline{\omega}^a\wedge\overline{\omega}^j\wedge\iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}(\alpha)\right) \quad (K15)\\
& +\frac{1}{2}\left\langle \nabla_{\overline{V}_b}V_k,V_i\right\rangle\left\langle \nabla_{\overline{V}_a}\overline{V}_j,V_b\right\rangle \overline{\omega}^a\wedge\overline{\omega}^j\wedge\iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}(\alpha)\right) \quad (K15a)+(K55)\\
& +\frac{1}{2}\left\langle \nabla_{\overline{V}_b}V_k,V_i\right\rangle\left\langle \nabla_{\overline{V}_j}\overline{V}_a,V_b\right\rangle \overline{\omega}^a\wedge\overline{\omega}^j\wedge\iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}(\alpha)\right) \quad (K15b)\\
& +\frac{1}{2}\left\langle R_{V_aV_j}\overline{V}_i,\overline{V}_k\right\rangle \overline{\omega}^k\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}(\alpha)\right) \quad (K17)\\
& +\frac{1}{2}\left\langle \nabla_{V_b}\overline{V}_i,\overline{V}_k\right\rangle\left\langle \nabla_{V_a}V_j,\overline{V}_b\right\rangle \overline{\omega}^k\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}(\alpha)\right) \quad (K17a)+(K57)\\
& +\frac{1}{2}\left\langle \nabla_{V_b}\overline{V}_i,\overline{V}_k\right\rangle\left\langle \nabla_{V_j}V_a,\overline{V}_b\right\rangle \overline{\omega}^k\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}(\alpha)\right) \quad (K17b)
\end{aligned}
$$

Finally if we switch the roles of the indices a and j in $(K15b)$ and $(K17b)$ then we get

$$
\begin{aligned}
(K15a)+(K55)+(K15b) &= 0\\
(K17a)+(K57)+(K17b) &= 0
\end{aligned}
$$

$(K15), (K17), (K55), (K57)$ *all combine to give the curvature terms*

$$
+\frac{1}{2}\left\langle R_{\overline{V}_a\overline{V}_j}V_k,V_i\right\rangle \overline{\omega}^a\wedge\overline{\omega}^j\wedge\iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}(\alpha)\right)+\frac{1}{2}\left\langle R_{V_aV_j}\overline{V}_i,\overline{V}_k\right\rangle \overline{\omega}^k\wedge\overline{\omega}^i\wedge\iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}(\alpha)\right)
$$

Claim 3: $(K11)$ *could be written as*

$$\begin{aligned}
& +2\left\langle \nabla_{V_k}\nabla_{V_j}\overline{V}_j,\overline{V}_k\right\rangle \alpha \quad (K11) \\
= \; & +\left\langle \nabla_{V_k}\nabla_{V_j}\overline{V}_j,\overline{V}_k\right\rangle \alpha \quad (K11a) \\
& +\left\langle \nabla_{V_j}\nabla_{V_k}\overline{V}_k,\overline{V}_j\right\rangle \alpha \quad (K11b) \\
= \; & +\left\langle \nabla_{V_k}\nabla_{V_j}\overline{V}_j,\overline{V}_k\right\rangle \alpha \quad (K11a) \\
& -\left\langle \nabla_{V_j}\nabla_{V_k}\overline{V}_j,\overline{V}_k\right\rangle \alpha \quad (K11b1) \\
& -\left\langle \nabla_{V_j}\overline{V}_j,\nabla_{V_k}\overline{V}_k\right\rangle \alpha \quad (K11b2) \\
& -\left\langle \nabla_{V_k}\overline{V}_j\nabla_{V_j},\overline{V}_k\right\rangle \alpha \quad (K11b3) \\
= \; & +\left\langle \nabla_{V_k}\nabla_{V_j}\overline{V}_j,\overline{V}_k\right\rangle \alpha \quad (K11a) \\
& -\left\langle \nabla_{V_j}\nabla_{V_k}\overline{V}_j,\overline{V}_k\right\rangle \alpha \quad (K11b1) \\
& -\left\langle \nabla_{V_j}\overline{V}_j,V_b\right\rangle \left\langle \nabla_{V_k}\overline{V}_k,\overline{V}_b\right\rangle \alpha \quad (K11b2a) \\
& -\left\langle \nabla_{V_j}\overline{V}_j,\overline{V}_b\right\rangle \left\langle \nabla_{V_k}\overline{V}_k,V_b\right\rangle \alpha \quad (K11b2b) \\
& -\left\langle \nabla_{V_k}\overline{V}_j,V_b\right\rangle \left\langle \nabla_{V_j}\overline{V}_k,\overline{V}_b\right\rangle \alpha \quad (K11b3a) \\
& -\left\langle \nabla_{V_k}\overline{V}_j,\overline{V}_b\right\rangle \left\langle \nabla_{V_j}\overline{V}_k,V_b\right\rangle \alpha \quad (K11b3b) \\
& -\left\langle R_{V_jV_k}\overline{V}_j,\overline{V}_k\right\rangle \alpha \quad (K11a+K11b1-a)
\end{aligned}$$

and we have the following cancelations

$$\begin{aligned}
0 &= (K11a+K11b1-b1)+(K11a+K11b1-b2)+(K11b3a)+(K11b3b) \\
0 &= (K11b2a)+(K11b2b)+(K35)
\end{aligned}$$

So the term $(K11)$ *is equal to*

$$\begin{aligned}
(K11) = \; & -\left\langle R_{V_jV_k}\overline{V}_j,\overline{V}_k\right\rangle \alpha \quad (K11a+K11b1-a) \\
& -\left\langle \nabla_{V_b}\overline{V}_j,\overline{V}_k\right\rangle \left\langle \nabla_{V_j}V_k,\overline{V}_b\right\rangle \alpha \quad (K11a+K11b1-b1) \\
& +\left\langle \nabla_{V_b}\overline{V}_j,\overline{V}_k\right\rangle \left\langle \nabla_{V_k}V_j,\overline{V}_b\right\rangle \alpha \quad (K11a+K11b1-b2) \\
& -\left\langle \nabla_{V_k}\overline{V}_k,\overline{V}_b\right\rangle \left\langle \nabla_{V_j}\overline{V}_j,V_b\right\rangle \alpha \quad (K11b2a) \\
& -\left\langle \nabla_{V_j}\overline{V}_j,\overline{V}_b\right\rangle \left\langle \nabla_{V_k}\overline{V}_k,V_b\right\rangle \alpha \quad (K11b2b) \\
& -\left\langle \nabla_{V_j}\overline{V}_k,\overline{V}_b\right\rangle \left\langle \nabla_{V_k}\overline{V}_j,V_b\right\rangle \alpha \quad (K11b3a) \\
& -\left\langle \nabla_{V_k}\overline{V}_j,\overline{V}_b\right\rangle \left\langle \nabla_{V_j}\overline{V}_k,V_b\right\rangle \alpha \quad (K11b3b)
\end{aligned}$$

Claim 4:

$$\begin{aligned}
& -\left\langle \nabla_{V_k}\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle \overline{\omega}^i\wedge\iota_{\overline{V}_j}(\alpha) \quad (K14) \\
& +\left\langle \nabla_{V_i}\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle \overline{\omega}^k\wedge\iota_{\overline{V}_j}(\alpha) \quad (K16)
\end{aligned}$$

If we switch the roles of i and k in $(K14)$; j and a in both $(K14)$ and $(K16)$ first we have

$$\begin{aligned}
&-\left\langle \nabla_{V_i}\nabla_{V_a}\overline{V}_k, \overline{V}_i\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}(\alpha) \quad (K14) \\
&+\left\langle \nabla_{V_i}\nabla_{V_a}\overline{V}_i, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}(\alpha) \quad (K16)
\end{aligned}$$

then switching i to j in both $(K14)$ and $(K16)$ gives

$$\begin{aligned}
&-\left\langle \nabla_{V_j}\nabla_{V_a}\overline{V}_k, \overline{V}_j\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}(\alpha) \quad (K14) \\
&+\left\langle \nabla_{V_j}\nabla_{V_a}\overline{V}_j, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}(\alpha) \quad (K16) \\
=\ &+2\left\langle \nabla_{V_j}\nabla_{V_a}\overline{V}_j, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}(\alpha) \quad (K14+K16-c1) \\
&+\left\langle \nabla_{V_a}\overline{V}_j, \nabla_{V_j}\overline{V}_k\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}(\alpha) \quad (K14+K16-c2) \\
&+\left\langle \nabla_{V_j}\overline{V}_j, \nabla_{V_a}\overline{V}_k\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}(\alpha) \quad (K14+K16-c3) \\
=\ &+2\left\langle \nabla_{V_j}\nabla_{V_a}\overline{V}_j, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}(\alpha) \quad (K14+K16-c1) \\
&+\left\langle \nabla_{V_j}\overline{V}_k, V_b\right\rangle \left\langle \nabla_{V_a}\overline{V}_j, \overline{V}_b\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}(\alpha) \quad (K14+K16-c2a) \\
&+\left\langle \nabla_{V_j}\overline{V}_k, \overline{V}_b\right\rangle \left\langle \nabla_{V_a}\overline{V}_j, V_b\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}(\alpha) \quad (K14+K16-c2b) \\
&+\left\langle \nabla_{V_a}\overline{V}_k, V_b\right\rangle \left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_b\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}(\alpha) \quad (K14+K16-c3a) \\
&+\left\langle \nabla_{V_a}\overline{V}_k, \overline{V}_b\right\rangle \left\langle \nabla_{V_j}\overline{V}_j, V_b\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}(\alpha) \quad (K14+K16-c3b)
\end{aligned}$$

this with the term $(K12)$ gives

$$\begin{aligned}
=\ &-2\left\langle R_{V_aV_j}\overline{V}_j, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}(\alpha) \quad (K12+(K14+K16-c1)-a) \\
&-2\left\langle \nabla_{V_b}\overline{V}_j, \overline{V}_k\right\rangle \left\langle \nabla_{V_a}V_j, \overline{V}_b\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}(\alpha) \quad (K12+(K14+K16-c1)-b1) \\
&+2\left\langle \nabla_{V_b}\overline{V}_j, \overline{V}_k\right\rangle \left\langle \nabla_{V_j}V_a, \overline{V}_b\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}(\alpha) \quad (K12+(K14+K16-c1)-b2) \\
&+2\left\langle \nabla_{V_a}\overline{V}_j, \overline{V}_b\right\rangle \left\langle \nabla_{V_j}\overline{V}_k, V_b\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}(\alpha) \quad (K14+K16-c2a) \\
&+2\left\langle \nabla_{V_j}\overline{V}_k, \overline{V}_b\right\rangle \left\langle \nabla_{V_a}\overline{V}_j, V_b\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}(\alpha) \quad (K14+K16-c2b) \\
&+2\left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_b\right\rangle \left\langle \nabla_{V_a}\overline{V}_k, V_b\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}(\alpha) \quad (K14+K16-c3a) \\
&+2\left\langle \nabla_{V_a}\overline{V}_k, \overline{V}_b\right\rangle \left\langle \nabla_{V_j}\overline{V}_j, V_b\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}(\alpha) \quad (K14+K16-c3b)
\end{aligned}$$

We also have some of the terms combined with other terms

$$\begin{aligned}
&-2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_b\right\rangle \left\langle \nabla_{V_b}V_i, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_j}(\alpha) \quad (K50)+(K16a2) \\
&-2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_b\right\rangle \left\langle \nabla_{V_k}\overline{V}_k, V_b\right\rangle \overline{\omega}^i \wedge \iota_{\overline{V}_j}(\alpha) \quad (K16b1)+(K92) \\
&-2\left\langle \nabla_{V_k}\overline{V}_k, \overline{V}_b\right\rangle \left\langle \nabla_{V_j}\overline{V}_i, V_b\right\rangle \overline{\omega}^i \wedge \iota_{\overline{V}_j}(\alpha) \quad (K16b2)+(K48)
\end{aligned}$$

which give the following cancellations

$$\begin{aligned}
0 &= (K12 + (K14 + K16 - c1) - b2) + (K52) + (K53) \\
0 &= (K12 + (K14 + K16 - c1) - b1) + (K14 + K16 - c2b) \\
0 &= (K14 + K16 - c2a) + (K50) + (K16a2) \\
0 &= (K14 + K16 - c3a) + (K16b2) + (K48) \\
0 &= (K14 + K16 - c3b) + (K42) \\
0 &= (K16b1) + (K51)
\end{aligned}$$

So if we combine the Laplacians in the following way, use the remarks 1-4 then label the terms, we get

$$\begin{aligned}
& \Delta_H(\alpha) + \Delta_K(\alpha) - 2\Delta_R(\alpha) \\
= & -2\overline{\omega}^j \wedge \iota_{V_a}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (K3 + H3 \longleftrightarrow W1) \\
& +2\overline{\omega}^a \wedge \iota_{V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_a}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (K1 + H4 \longleftrightarrow W2) \\
& -2\omega^j \wedge \iota_{V_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_a}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (K7 + H7 \longleftrightarrow W3) \\
& +2\omega^a \wedge \iota_{V_j}\left(proj_{p,q}\left(\nabla_{V_a}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (K5 + H8 \longleftrightarrow W4) \\
& -2\omega^a \wedge \iota_{\overline{V}_j}\left(proj_{p-1,q+1}\left(\nabla_{V_a}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (K6 + H5 \longleftrightarrow W5) \\
& +2\omega^j \wedge \iota_{\overline{V}_a}\left(proj_{p-1,q+1}\left(\nabla_{V_a}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (K8 + H6 \longleftrightarrow W6) \\
& -2\overline{\omega}^a \wedge \iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right)\right)\right) \quad (K2 + H1 \longleftrightarrow W7) \\
& +2\overline{\omega}^j \wedge \iota_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{V_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right)\right) \quad (K4 + H2 \longleftrightarrow W8) \\
& +2proj_{E^{p,q}}\left(\nabla_{V_j}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right)\right) \quad (R2 \longleftrightarrow W9) \\
& +2proj_{E^{p,q}}\left(\nabla_{\overline{V}_j}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right)\right) \quad (K10 + H10 \longleftrightarrow W10)
\end{aligned}$$

$$+2\left\langle \nabla_{\overline{V}_j} V_b, V_c \right\rangle \omega^b \wedge \iota_{\overline{V}_c}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right) \quad (H15 \longleftrightarrow W11)$$
$$+2\left\langle \nabla_{V_a} \overline{V}_j, V_b \right\rangle \omega^b \wedge \iota_{\overline{V}_a}\left(proj_{p-1,q+1}\left(\nabla_{V_j}(\alpha)\right)\right) \quad (K33 + H18 \longleftrightarrow W12)$$
$$+2\left\langle \nabla_{\overline{V}_a} V_j, \overline{V}_b \right\rangle \overline{\omega}^a \wedge \iota_{V_b}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) \quad (K32 + H19 \longleftrightarrow W13)$$
$$-2\left\langle \nabla_{V_j} \overline{V}_i, \overline{V}_k \right\rangle \overline{\omega}^k \wedge \iota_{V_i}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) \quad (K31 \longleftrightarrow W14)$$
$$-2\left\langle \nabla_{\overline{V}_a} \overline{V}_j, V_b \right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_b}\left(proj_{p,q}\left(\nabla_{V_j}(\alpha)\right)\right) \quad (K25 + H20 \longleftrightarrow W15)$$
$$+2\left\langle \nabla_{V_j} \overline{V}_a, \overline{V}_k \right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{V_a}(\alpha)\right)\right) \quad (K29 + K30 \longleftrightarrow W16)$$
$$+2\left\langle \nabla_{V_a} V_j, \overline{V}_k \right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) \quad (K26 + H21 \longleftrightarrow W17)$$
$$+2\left\langle \nabla_{\overline{V}_a} V_b, V_j \right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_b}\left(proj_{p,q}\left(\nabla_{\overline{V}_j}(\alpha)\right)\right) \quad (K27 + K28 \longleftrightarrow W18)$$
$$+2\left\langle \nabla_{\overline{V}_a} \nabla_{\overline{V}_j} V_j, V_i \right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_i}(\alpha) \quad (H11 \longleftrightarrow W19)$$
$$+2\left\langle \nabla_{\overline{V}_a} \nabla_{\overline{V}_j} V_i, V_k \right\rangle \overline{\omega}^a \wedge \omega^i \wedge \iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right) \quad (H12 \longleftrightarrow W20)$$
$$-2\left\langle \nabla_{V_k} \nabla_{V_j} \overline{V}_i, \overline{V}_k \right\rangle \omega^j \wedge \iota_{V_i}(\alpha) \quad (K13 \longleftrightarrow W21)$$
$$+2\left\langle \nabla_{V_a} \nabla_{V_j} \overline{V}_i, \overline{V}_k \right\rangle \omega^j \wedge \overline{\omega}^k \wedge \iota_{\overline{V}_a}\left(\iota_{V_i}(\alpha)\right) \quad (K18 \longleftrightarrow W22)$$

$$-\left\langle R_{V_j V_k} \overline{V}_j, \overline{V}_k \right\rangle \alpha \quad ((K11a + K11b1) - a \longleftrightarrow W23)$$
$$-2\left\langle R_{V_a V_j} \overline{V}_j, \overline{V}_k \right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}(\alpha) \quad (K12 + (K14 + K16 - c1) - a \longleftrightarrow W24)$$
$$+\frac{1}{2}\left\langle R_{\overline{V}_a \overline{V}_j} V_k, V_i \right\rangle \overline{\omega}^a \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}(\alpha)\right) \quad (K15 \longleftrightarrow W25)$$
$$+\frac{1}{2}\left\langle R_{V_a V_j} \overline{V}_i, \overline{V}_k \right\rangle \overline{\omega}^k \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}(\alpha)\right) \quad (K17 \longleftrightarrow W26)$$
$$-2\left\langle \nabla_{\overline{V}_a} V_k, V_a \right\rangle \left\langle \nabla_{V_j} \overline{V}_j, \overline{V}_k \right\rangle \alpha \quad (K34 \longleftrightarrow W27)$$
$$+2\left\langle \nabla_{\overline{V}_a} V_k, V_b \right\rangle \left\langle \nabla_{V_j} \overline{V}_j, \overline{V}_k \right\rangle \omega^b \wedge \iota_{V_a}(\alpha) \quad (K36 \longleftrightarrow W28)$$
$$-2\left\langle \nabla_{\overline{V}_j} V_k, V_b \right\rangle \left\langle \nabla_{V_j} \overline{V}_i, \overline{V}_k \right\rangle \omega^b \wedge \iota_{V_i}(\alpha) \quad (K37 \longleftrightarrow W29)$$
$$+4\left\langle \nabla_{\overline{V}_j} V_i, V_j \right\rangle \left\langle \nabla_{V_a} \overline{V}_k, \overline{V}_i \right\rangle \omega^a \wedge \iota_{V_k}(\alpha) \quad (K38 + H23 \longleftrightarrow W30)$$
$$+2\left\langle \nabla_{\overline{V}_a} V_i, V_k \right\rangle \left\langle \nabla_{V_j} \overline{V}_i, \overline{V}_k \right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_j}(\alpha) \quad (K54 \longleftrightarrow W31)$$
$$+4\left\langle \nabla_{\overline{V}_a} V_k, V_c \right\rangle \left\langle \nabla_{V_j} \overline{V}_j, \overline{V}_k \right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_c}(\alpha) \quad (K43 + K44 \longleftrightarrow W32)$$
$$+2\left\langle \nabla_{\overline{V}_a} V_k, V_a \right\rangle \left\langle \nabla_{V_j} \overline{V}_i, \overline{V}_k \right\rangle \overline{\omega}^i \wedge \iota_{\overline{V}_j}(\alpha) \quad (K46 + K47 \longleftrightarrow W33)$$
$$-2\left\langle \nabla_{\overline{V}_j} V_i, V_b \right\rangle \left\langle \nabla_{V_j} \overline{V}_i, \overline{V}_k \right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_b}(\alpha) \quad (K45 \longleftrightarrow W34)$$

$$-2\left\langle \nabla_{V_k}\overline{V}_k, \overline{V}_b\right\rangle \left\langle \nabla_{V_j}\overline{V}_i, V_b\right\rangle \omega^j \wedge \iota_{V_i}(\alpha) \quad (K39 \longleftrightarrow W35)$$
$$-2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \left\langle \nabla_{V_k}\overline{V}_j, V_b\right\rangle \omega^b \wedge \iota_{V_i}(\alpha) \quad (K40 \longleftrightarrow W36)$$
$$-2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_a\right\rangle \left\langle \nabla_{V_a}V_i, \overline{V}_b\right\rangle \omega^j \wedge \iota_{V_b}(\alpha) \quad (K41 \longleftrightarrow W37)$$
$$+2\left\langle \nabla_{\overline{V}_a}V_i, V_k\right\rangle \left\langle \nabla_{\overline{V}_j}V_j, \overline{V}_k\right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_i}(\alpha) \quad (H22 \longleftrightarrow W38)$$
$$+\left\langle \nabla_{\overline{V}_j}V_k, V_b\right\rangle \left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_c\right\rangle \overline{\omega}^c \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_k}\left(\iota_{\overline{V}_b}(\alpha)\right) \quad (K56 \longleftrightarrow W39)$$
$$+4\left\langle \nabla_{\overline{V}_a}V_k, V_c\right\rangle \left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \overline{\omega}^a \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_c}\left(\iota_{\overline{V}_j}(\alpha)\right) \quad (K58 + K59 \longleftrightarrow W40)$$
$$-4\left\langle \nabla_{\overline{V}_a}V_k, V_b\right\rangle \left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \omega^j \wedge \omega^b \wedge \iota_{V_a}\left(\iota_{V_i}(\alpha)\right) \quad (K70 + H28 \longleftrightarrow W41)$$
$$-2\left\langle \nabla_{\overline{V}_j}V_k, V_i\right\rangle \left\langle \nabla_{V_a}\overline{V}_k, \overline{V}_b\right\rangle \omega^a \wedge \overline{\omega}^j \wedge \iota_{V_b}\left(\iota_{\overline{V}_i}(\alpha)\right)$$
$$(K60 + K61 + K66 + K67 \longleftrightarrow W42)$$
$$-4\left\langle \nabla_{\overline{V}_j}V_k, V_i\right\rangle \left\langle \nabla_{V_a}\overline{V}_k, \overline{V}_b\right\rangle \omega^a \wedge \overline{\omega}^b \wedge \iota_{V_j}\left(\iota_{\overline{V}_i}(\alpha)\right) \quad (H27 + K62 \longleftrightarrow W43)$$
$$+2\left\langle \nabla_{\overline{V}_a}V_k, V_b\right\rangle \left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \omega^b \wedge \overline{\omega}^i \wedge \iota_{V_a}\left(\iota_{\overline{V}_j}(\alpha)\right) \quad (K68 + K69 \longleftrightarrow W44)$$
$$+2\left\langle \nabla_{V_a}\overline{V}_k, \overline{V}_b\right\rangle \left\langle \nabla_{V_j}\overline{V}_i, V_b\right\rangle \omega^j \wedge \overline{\omega}^k \wedge \iota_{\overline{V}_a}\left(\iota_{V_i}(\alpha)\right) \quad (K63 \longleftrightarrow W45)$$
$$+2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \left\langle \nabla_{V_a}\overline{V}_j, V_b\right\rangle \omega^b \wedge \overline{\omega}^k \wedge \iota_{\overline{V}_a}\left(\iota_{V_i}(\alpha)\right) \quad (K64 \longleftrightarrow W46)$$
$$+2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \left\langle \nabla_{V_a}V_i, \overline{V}_b\right\rangle \omega^j \wedge \overline{\omega}^k \wedge \iota_{\overline{V}_a}\left(\iota_{V_b}(\alpha)\right) \quad (K65 \longleftrightarrow W47)$$
$$+2\left\langle \nabla_{\overline{V}_a}V_k, V_b\right\rangle \left\langle \nabla_{\overline{V}_j}V_i, \overline{V}_b\right\rangle \overline{\omega}^a \wedge \omega^i \wedge \iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right) \quad (H24 \longleftrightarrow W48)$$
$$+2\left\langle \nabla_{\overline{V}_j}V_b, V_k\right\rangle \left\langle \nabla_{\overline{V}_a}\overline{V}_b, V_i\right\rangle \overline{\omega}^a \wedge \omega^i \wedge \iota_{\overline{V}_k}\left(\iota_{V_j}(\alpha)\right) \quad (H25 \longleftrightarrow W49)$$
$$+2\left\langle \nabla_{\overline{V}_j}V_i, V_k\right\rangle \left\langle \nabla_{\overline{V}_a}V_j, \overline{V}_b\right\rangle \overline{\omega}^a \wedge \omega^i \wedge \iota_{\overline{V}_k}\left(\iota_{V_b}(\alpha)\right) \quad (H26 \longleftrightarrow W50)$$

First we simplify the terms from $(W1)$ *to* $(W10)$*. If we take the first covariant derivatives, we would get*

$$
\begin{aligned}
& (W1) \text{ through } (W10) \\
= \quad & -2\overline{\omega}^j \wedge \iota_{V_a}\left(proj_{p+1,q-1}\left(\nabla_{\overline{V}_a}\left(\overline{V}_j\left(\alpha^{IJ}\right)\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\right)\right)\right) \quad (W1a) \\
& -2\overline{\omega}^j \wedge \iota_{V_a}\Bigg(proj_{p+1,q-1}\Bigg(\nabla_{\overline{V}_a}\Bigg(\alpha^{IJ}\left\langle\nabla_{\overline{V}_j}\omega^{i_r},\overline{\omega}^c\right\rangle\omega^{i_1}\wedge\cdots \\
& \wedge \underset{r-th\text{ place}}{\omega^c}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\Bigg)\Bigg)\Bigg) \quad (W1b) \\
& -2\overline{\omega}^j \wedge \iota_{V_a}\Bigg(proj_{p+1,q-1}\Bigg(\nabla_{\overline{V}_a}\Bigg(\alpha^{IJ}\left\langle\nabla_{\overline{V}_j}\overline{\omega}^{j_s},\omega^d\right\rangle\omega^{i_1}\wedge\cdots \\
& \wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^d}\wedge\cdots\wedge\overline{\omega}^{j_q}\Bigg)\Bigg)\Bigg) \quad (W1c) \\
& +2\overline{\omega}^a \wedge \iota_{V_j}\Bigg(proj_{p+1,q-1}\Bigg(\nabla_{\overline{V}_a}\Bigg(\alpha^{IJ}\left\langle\nabla_{\overline{V}_j}\overline{\omega}^{j_s},\overline{\omega}^d\right\rangle\omega^{i_1}\wedge\cdots \\
& \wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\omega^d}\wedge\cdots\wedge\overline{\omega}^{j_q}\Bigg)\Bigg)\Bigg) \quad (W2) \\
& -2\omega^j \wedge \iota_{V_a}\left(proj_{p,q}\left(\nabla_{\overline{V}_a}\left(\alpha^{IJ}\left\langle\nabla_{V_j}\omega^{i_r},\omega^d\right\rangle\omega^{i_1}\wedge\cdots\right.\right.\right. \\
& \wedge\underset{r-th\text{ place}}{\overline{\omega}^d}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\Bigg)\Bigg)\Bigg) \quad (W3) \\
& +2\omega^a \wedge \iota_{V_j}\Bigg(proj_{p,q}\Bigg(\nabla_{V_a}\Bigg(\alpha^{IJ}\left\langle\nabla_{\overline{V}_j}\overline{\omega}^{j_s},\overline{\omega}^d\right\rangle\omega^{i_1}\wedge\cdots \\
& \wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\omega^d}\wedge\cdots\wedge\overline{\omega}^{j_q}\Bigg)\Bigg)\Bigg) \quad (W4) \\
& -2\omega^a \wedge \iota_{\overline{V}_j}\left(proj_{p-1,q+1}\left(\nabla_{V_a}\left(V_j\left(\alpha^{IJ}\right)\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\right)\right)\right) \quad (W5a) \\
& -2\omega^a \wedge \iota_{\overline{V}_j}\left(proj_{p-1,q+1}\left(\nabla_{V_a}\left(\alpha^{IJ}\left\langle\nabla_{V_j}\omega^{i_r},\overline{\omega}^c\right\rangle\omega^{i_1}\wedge\cdots\right.\right.\right. \\
& \wedge\underset{r-th\text{ place}}{\omega^c}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\Bigg)\Bigg)\Bigg) \quad (W5b) \\
& -2\omega^a \wedge \iota_{\overline{V}_j}\left(proj_{p-1,q+1}\left(\nabla_{V_a}\left(\alpha^{IJ}\left\langle\nabla_{V_j}\overline{\omega}^{j_s},\omega^c\right\rangle\omega^{i_1}\wedge\cdots\right.\right.\right. \\
& \wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^c}\wedge\cdots\wedge\overline{\omega}^{j_q}\Bigg)\Bigg)\Bigg) \quad (W5c)
\end{aligned}
$$

$$+2\omega^j \wedge \iota_{\overline{V}_a}\left(proj_{p-1,q+1}\left(\nabla_{V_a}\left(\alpha^{IJ}\left\langle\nabla_{V_j}\omega^{i_r},\omega^b\right\rangle\omega^{i_1}\wedge\cdots\right.\right.\right.$$

$$\left.\left.\left.\wedge \underset{r-th\text{ place}}{\overline{\omega}^b} \wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\right)\right)\right) \quad (W6)$$

$$-2\overline{\omega}^a \wedge \iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{\overline{V}_a}\left(V_j\left(\alpha^{IJ}\right)\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\right)\right)\right) \quad (W7a)$$

$$-2\overline{\omega}^a \wedge \iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{\overline{V}_a}\left(\alpha^{IJ}\left\langle\nabla_{V_j}\omega^{i_r},\overline{\omega}^d\right\rangle\omega^{i_1}\wedge\cdots\right.\right.\right.$$

$$\left.\left.\left.\wedge \underset{r-th\text{ place}}{\omega^d} \wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\right)\right)\right) \quad (W7b)$$

$$-2\overline{\omega}^a \wedge \iota_{\overline{V}_j}\left(proj_{p,q}\left(\nabla_{\overline{V}_a}\left(\alpha^{IJ}\left\langle\nabla_{V_j}\overline{\omega}^{j_s},\omega^b\right\rangle\omega^{i_1}\wedge\cdots\right.\right.\right.$$

$$\left.\left.\left.\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge \underset{s-th\text{ place}}{\overline{\omega}^b} \wedge\cdots\wedge\overline{\omega}^{j_q}\right)\right)\right) \quad (W7c)$$

$$+2\overline{\omega}^j \wedge \iota_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{V_a}\left(\overline{V}_j\left(\alpha^{IJ}\right)\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\right)\right)\right) \quad (W8a)$$

$$+2\overline{\omega}^j \wedge \iota_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{V_a}\left(\alpha^{IJ}\left\langle\nabla_{\overline{V}_j}\omega^{i_r},\overline{\omega}^c\right\rangle\omega^{i_1}\wedge\cdots\right.\right.\right.$$

$$\left.\left.\left.\wedge \underset{r-th\text{ place}}{\omega^c} \wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\right)\right)\right) \quad (W8b)$$

$$+2\overline{\omega}^j \wedge \iota_{\overline{V}_a}\left(proj_{p,q}\left(\nabla_{V_a}\left(\alpha^{IJ}\left\langle\nabla_{\overline{V}_j}\overline{\omega}^{j_s},\omega^d\right\rangle\omega^{i_1}\wedge\cdots\right.\right.\right.$$

$$\left.\left.\left.\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge \underset{s-th\text{ place}}{\overline{\omega}^d} \wedge\cdots\wedge\overline{\omega}^{j_q}\right)\right)\right) \quad (W8c)$$

$$+2proj_{E^{p,q}}\left(\nabla_{V_j}\left(\alpha^{IJ}\left\langle\nabla_{\overline{V}_j}\overline{\omega}^{j_s},\overline{\omega}^d\right\rangle\omega^{i_1}\wedge\cdots\right.\right.$$

$$\left.\left.\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge \underset{s-th\text{ place}}{\omega^d} \wedge\cdots\wedge\overline{\omega}^{j_q}\right)\right)\Big) \quad (W9)$$

$$+2proj_{E^{p,q}}\left(\nabla_{\overline{V}_j}\left(\alpha^{IJ}\left\langle\nabla_{V_j}\omega^{i_r},\omega^d\right\rangle\omega^{i_1}\wedge\cdots\right.\right.$$

$$\left.\left.\wedge \underset{r-th\text{ place}}{\overline{\omega}^d} \wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\right)\right)\Big) \quad (W10)$$

Then if we apply the second covariant derivatives and the interior products, this gives

$$
\begin{aligned}
& (W1) \text{ through } (W10) \\
= \; & -2\overline{\omega}^j \wedge \iota_{V_a}\left(\overline{V}_j\left(\alpha^{IJ}\right)\left\langle\nabla_{\overline{V}_a}\overline{\omega}^{j_s},\overline{\omega}^c\right\rangle\omega^{i_1}\wedge\cdots\right.\\
& \left.\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\omega^c}\wedge\cdots\wedge\overline{\omega}^{j_q}\right) \quad (W1a)\\
& -2\overline{\omega}^j \wedge \iota_{V_a}\left(\alpha^{IJ}\left\langle\nabla_{\overline{V}_j}\omega^{i_r},\overline{\omega}^c\right\rangle\left\langle\nabla_{\overline{V}_a}\overline{\omega}^{j_s},\overline{\omega}^d\right\rangle\omega^{i_1}\wedge\cdots\right.\\
& \left.\wedge\underset{r-th\text{ place}}{\omega^c}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\omega^d}\wedge\cdots\wedge\overline{\omega}^{j_q}\right) \quad (W1b)\\
& -2\overline{\omega}^j \wedge \iota_{V_a}\left(\alpha^{IJ}\left\langle\nabla_{\overline{V}_j}\overline{\omega}^{j_s},\omega^d\right\rangle\left\langle\nabla_{\overline{V}_a}\overline{\omega}^{j_u},\overline{\omega}^c\right\rangle\omega^{i_1}\wedge\cdots\right.\\
& \left.\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\omega^c}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^d}\wedge\cdots\wedge\overline{\omega}^{j_q}\right) \quad (W1c1)\\
& -2\overline{\omega}^j \wedge \iota_{V_a}\left(\alpha^{IJ}\left\langle\nabla_{\overline{V}_j}\overline{\omega}^{j_s},\omega^d\right\rangle\left\langle\nabla_{\overline{V}_a}\overline{\omega}^{d},\overline{\omega}^c\right\rangle\omega^{i_1}\wedge\cdots\right.\\
& \left.\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\omega^c}\wedge\cdots\wedge\overline{\omega}^{j_q}\right) \quad (W1c2)\\
& +2\overline{\omega}^a \wedge \iota_{V_j}\left(\overline{V}_a\left(\alpha^{IJ}\right)\left\langle\nabla_{\overline{V}_j}V_{j_s},V_d\right\rangle\omega^{i_1}\wedge\cdots\right.\\
& \left.\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\omega^d}\wedge\cdots\wedge\overline{\omega}^{j_q}\right) \quad (W2a)\\
& +2\overline{\omega}^a \wedge \iota_{V_j}\left(\alpha^{IJ}\left\langle\nabla_{\overline{V}_a}\nabla_{\overline{V}_j}V_{j_s},V_d\right\rangle\omega^{i_1}\wedge\cdots\right.\\
& \left.\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\omega^d}\wedge\cdots\wedge\overline{\omega}^{j_q}\right) \quad (W2b)\\
& +2\overline{\omega}^a \wedge \iota_{V_j}\left(\alpha^{IJ}\left\langle\nabla_{\overline{V}_j}V_{j_s},\nabla_{\overline{V}_a}V_d\right\rangle\omega^{i_1}\wedge\cdots\right.\\
& \left.\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\omega^d}\wedge\cdots\wedge\overline{\omega}^{j_q}\right) \quad (W2c)\\
& +2\overline{\omega}^a \wedge \iota_{V_j}\left(\alpha^{IJ}\left\langle\nabla_{\overline{V}_j}V_{j_s},V_d\right\rangle\left\langle\nabla_{\overline{V}_a}\omega^{i_r},\overline{\omega}^c\right\rangle\omega^{i_1}\wedge\cdots\right.\\
& \left.\wedge\underset{r-th\text{ place}}{\omega^c}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\omega^d}\wedge\cdots\wedge\overline{\omega}^{j_q}\right) \quad (W2d)\\
& +2\overline{\omega}^a \wedge \iota_{V_j}\left(\alpha^{IJ}\left\langle\nabla_{\overline{V}_j}V_{j_s},V_d\right\rangle\left\langle\nabla_{\overline{V}_a}\omega^{d},\overline{\omega}^c\right\rangle\omega^{i_1}\wedge\cdots\right.\\
& \left.\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\omega^c}\wedge\cdots\wedge\overline{\omega}^{j_q}\right) \quad (W2e)
\end{aligned}
$$

$$+2\overline{\omega}^a \wedge \iota_{V_j}\left(\alpha^{IJ}\left\langle\nabla_{\overline{V}_j}V_{j_s},V_d\right\rangle\left\langle\nabla_{\overline{V}_a}\overline{\omega}^{j_u},\omega^c\right\rangle\omega^{i_1}\wedge\cdots\right.$$
$$\left.\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\omega^d}\wedge\cdots\wedge\underset{u-th\text{ place}}{\overline{\omega}^c}\wedge\cdots\wedge\overline{\omega}^{j_q}\right)\quad(W2f)$$

$$-2\omega^j \wedge \iota_{V_a}\left(\alpha^{IJ}\left\langle\nabla_{V_j}\omega^{i_r},\omega^d\right\rangle\left\langle\nabla_{\overline{V}_a}\overline{\omega}^d,\overline{\omega}^c\right\rangle\omega^{i_1}\wedge\cdots\right.$$
$$\left.\wedge\underset{r-th\text{ place}}{\omega^c}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\right)\quad(W3a)$$

$$-2\omega^j \wedge \iota_{V_a}\left(\alpha^{IJ}\left\langle\nabla_{V_j}\omega^{i_r},\omega^d\right\rangle\left\langle\nabla_{\overline{V}_a}\overline{\omega}^{j_s},\overline{\omega}^c\right\rangle\omega^{i_1}\wedge\cdots\right.$$
$$\left.\wedge\underset{r-th\text{ place}}{\overline{\omega}^d}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\omega^c}\wedge\cdots\wedge\overline{\omega}^{j_q}\right)\quad(W3b)$$

$$+2\omega^a \wedge \iota_{V_j}\left(\alpha^{IJ}\left\langle\nabla_{\overline{V}_j}\overline{\omega}^{j_s},\overline{\omega}^d\right\rangle\left\langle\nabla_{V_a}\omega^{i_r},\omega^c\right\rangle\omega^{i_1}\wedge\cdots\right.$$
$$\left.\wedge\underset{r-th\text{ place}}{\overline{\omega}^c}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\omega^d}\wedge\cdots\wedge\overline{\omega}^{j_q}\right)\quad(W4a)$$

$$+2\omega^a \wedge \iota_{V_j}\left(\alpha^{IJ}\left\langle\nabla_{\overline{V}_j}\overline{\omega}^{j_s},\overline{\omega}^d\right\rangle\left\langle\nabla_{V_a}\omega^d,\omega^c\right\rangle\omega^{i_1}\wedge\cdots\right.$$
$$\left.\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^c}\wedge\cdots\wedge\overline{\omega}^{j_q}\right)\quad(W4b)$$

$$-2\omega^a \wedge \iota_{\overline{V}_j}\left(V_j\left(\alpha^{IJ}\right)\left\langle\nabla_{V_a}\omega^{i_r},\omega^c\right\rangle\omega^{i_1}\wedge\cdots\right.$$
$$\left.\wedge\underset{r-th\text{ place}}{\overline{\omega}^c}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\right)\quad(W5a)$$

$$-2\omega^a \wedge \iota_{\overline{V}_j}\left(\alpha^{IJ}\left\langle\nabla_{V_j}\omega^{i_r},\overline{\omega}^c\right\rangle\left\langle\nabla_{V_a}\omega^{i_t},\omega^b\right\rangle\omega^{i_1}\wedge\cdots\right.$$
$$\left.\wedge\underset{t-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^c}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\right)\quad(W5b1)$$

$$-2\omega^a \wedge \iota_{\overline{V}_j}\left(\alpha^{IJ}\left\langle\nabla_{V_j}\omega^{i_r},\overline{\omega}^c\right\rangle\left\langle\nabla_{V_a}\omega^c,\omega^b\right\rangle\omega^{i_1}\wedge\cdots\right.$$
$$\left.\wedge\underset{r-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\right)\quad(W5b2)$$

$$-2\omega^a \wedge \iota_{\overline{V}_j}\left(\alpha^{IJ}\left\langle\nabla_{V_j}\overline{\omega}^{j_s},\omega^b\right\rangle\left\langle\nabla_{V_a}\omega^{i_r},\omega^c\right\rangle\omega^{i_1}\wedge\cdots\right.$$
$$\left.\wedge\underset{r-th\text{ place}}{\overline{\omega}^c}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q}\right)\quad(W5c)$$

$$+2\omega^j \wedge \iota_{\overline{V}_a} \left(V_a \left(\alpha^{IJ} \right) \left\langle \nabla_{V_j} \omega^{i_r}, \omega^b \right\rangle \omega^{i_1} \wedge \cdots \right.$$
$$\left. \wedge \underset{r-th \text{ place}}{\overline{\omega}^b} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q} \right) \quad (W6a)$$

$$+2\omega^j \wedge \iota_{\overline{V}_a} \left(\alpha^{IJ} \left\langle \nabla_{V_a} \nabla_{V_j} \omega^{i_r}, \omega^b \right\rangle \omega^{i_1} \wedge \cdots \right.$$
$$\left. \wedge \underset{r-th \text{ place}}{\overline{\omega}^b} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q} \right) \quad (W6b)$$

$$+2\omega^j \wedge \iota_{\overline{V}_a} \left(\alpha^{IJ} \left\langle \nabla_{V_j} \omega^{i_r}, \nabla_{V_a} \omega^b \right\rangle \omega^{i_1} \wedge \cdots \right.$$
$$\left. \wedge \underset{r-th \text{ place}}{\overline{\omega}^b} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q} \right) \quad (W6c)$$

$$+2\omega^j \wedge \iota_{\overline{V}_a} \left(\alpha^{IJ} \left\langle \nabla_{V_j} \omega^{i_r}, \omega^b \right\rangle \left\langle \nabla_{V_a} \omega^{i_t}, \overline{\omega}^c \right\rangle \omega^{i_1} \wedge \cdots \right.$$
$$\left. \wedge \underset{t-th \text{ place}}{\omega^c} \wedge \cdots \wedge \underset{r-th \text{ place}}{\overline{\omega}^b} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q} \right) \quad (W6d)$$

$$+2\omega^j \wedge \iota_{\overline{V}_a} \left(\alpha^{IJ} \left\langle \nabla_{V_j} \omega^{i_r}, \omega^b \right\rangle \left\langle \nabla_{V_a} \overline{\omega}^b, \omega^c \right\rangle \omega^{i_1} \wedge \cdots \right.$$
$$\left. \wedge \underset{r-th \text{ place}}{\overline{\omega}^c} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q} \right) \quad (W6e)$$

$$+2\omega^j \wedge \iota_{\overline{V}_a} \left(\alpha^{IJ} \left\langle \nabla_{V_j} \omega^{i_r}, \omega^b \right\rangle \left\langle \nabla_{V_a} \overline{\omega}^{j_s}, \omega^c \right\rangle \omega^{i_1} \wedge \cdots \right.$$
$$\left. \wedge \underset{r-th \text{ place}}{\overline{\omega}^b} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^c} \wedge \cdots \wedge \overline{\omega}^{j_q} \right) \quad (W6f)$$

$$-2\overline{\omega}^a \wedge \iota_{\overline{V}_j} \left(\overline{V}_a \left[V_j \left(\alpha^{IJ} \right) \right] \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q} \right) \quad (W7a1)$$

$$-2\overline{\omega}^a \wedge \iota_{\overline{V}_j} \left(V_j \left(\alpha^{IJ} \right) \left\langle \nabla_{\overline{V}_a} \omega^{i_r}, \overline{\omega}^c \right\rangle \omega^{i_1} \wedge \cdots \right.$$
$$\left. \wedge \underset{r-th \text{ place}}{\omega^c} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q} \right) \quad (W7a2)$$

$$-2\overline{\omega}^a \wedge \iota_{\overline{V}_j} \left(V_j \left(\alpha^{IJ} \right) \left\langle \nabla_{\overline{V}_a} \overline{\omega}^{j_s}, \omega^c \right\rangle \omega^{i_1} \wedge \cdots \right.$$
$$\left. \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^c} \wedge \cdots \wedge \overline{\omega}^{j_q} \right) \quad (W7a3)$$

$$-2\overline{\omega}^a \wedge \iota_{\overline{V}_j} \left(\overline{V}_a \left(\alpha^{IJ} \right) \left\langle \nabla_{V_j} \omega^{i_r}, \overline{\omega}^b \right\rangle \omega^{i_1} \wedge \cdots \right.$$
$$\left. \wedge \underset{r-th \text{ place}}{\omega^b} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q} \right) \quad (W7b1)$$

$$-2\overline{\omega}^a \wedge \iota_{\overline{V}_j}\left(\alpha^{IJ}\left\langle \nabla_{\overline{V}_a}\nabla_{V_j}\omega^{i_r},\overline{\omega}^b\right\rangle \omega^{i_1}\wedge\cdots\right.$$

$$\left.\wedge \underset{r-th\text{ place}}{\omega^b}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\right) \quad (W7b2)$$

$$-2\overline{\omega}^a \wedge \iota_{\overline{V}_j}\left(\alpha^{IJ}\left\langle \nabla_{V_j}\omega^{i_r},\nabla_{\overline{V}_a}\overline{\omega}^b\right\rangle \omega^{i_1}\wedge\cdots\right.$$

$$\left.\wedge \underset{r-th\text{ place}}{\omega^b}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\right) \quad (W7b3)$$

$$-2\overline{\omega}^a \wedge \iota_{\overline{V}_j}\left(\alpha^{IJ}\left\langle \nabla_{V_j}\omega^{i_r},\overline{\omega}^b\right\rangle\left\langle \nabla_{\overline{V}_a}\omega^b,\overline{\omega}^c\right\rangle \omega^{i_1}\wedge\cdots\right.$$

$$\left.\wedge \underset{r-th\text{ place}}{\omega^c}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\right) \quad (W7b4)$$

$$-2\overline{\omega}^a \wedge \iota_{\overline{V}_j}\left(\alpha^{IJ}\left\langle \nabla_{V_j}\omega^{i_r},\overline{\omega}^b\right\rangle\left\langle \nabla_{\overline{V}_a}\omega^{i_t},\overline{\omega}^c\right\rangle \omega^{i_1}\wedge\cdots\right.$$

$$\left.\wedge \underset{r-th\text{ place}}{\omega^b}\wedge\cdots\wedge\underset{t-th\text{ place}}{\omega^c}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\right) \quad (W7b5)$$

$$-2\overline{\omega}^a \wedge \iota_{\overline{V}_j}\left(\alpha^{IJ}\left\langle \nabla_{V_j}\omega^{i_r},\overline{\omega}^b\right\rangle\left\langle \nabla_{\overline{V}_a}\overline{\omega}^{j_s},\omega^c\right\rangle \omega^{i_1}\wedge\cdots\right.$$

$$\left.\wedge \underset{r-th\text{ place}}{\omega^b}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^c}\wedge\cdots\wedge\overline{\omega}^{j_q}\right) \quad (W7b6)$$

$$-2\overline{\omega}^a \wedge \iota_{\overline{V}_j}\left(\overline{V}_a\left(\alpha^{IJ}\right)\left\langle \nabla_{V_j}\overline{\omega}^{j_s},\omega^b\right\rangle \omega^{i_1}\wedge\cdots\right.$$

$$\left.\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q}\right) \quad (W7c1)$$

$$-2\overline{\omega}^a \wedge \iota_{\overline{V}_j}\left(\alpha^{IJ}\left\langle \nabla_{\overline{V}_a}\nabla_{V_j}\overline{\omega}^{j_s},\omega^b\right\rangle \omega^{i_1}\wedge\cdots\right.$$

$$\left.\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q}\right) \quad (W7c2)$$

$$-2\overline{\omega}^a \wedge \iota_{\overline{V}_j}\left(\alpha^{IJ}\left\langle \nabla_{V_j}\overline{\omega}^{j_s},\nabla_{\overline{V}_a}\omega^b\right\rangle \omega^{i_1}\wedge\cdots\right.$$

$$\left.\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q}\right) \quad (W7c3)$$

$$-2\overline{\omega}^a \wedge \iota_{\overline{V}_j}\left(\alpha^{IJ}\left\langle \nabla_{V_j}\overline{\omega}^{j_s},\omega^b\right\rangle\left\langle \nabla_{\overline{V}_a}\omega^{i_r},\overline{\omega}^c\right\rangle \omega^{i_1}\wedge\cdots\right.$$

$$\left.\wedge \underset{r-th\text{ place}}{\omega^c}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q}\right) \quad (W7c4)$$

$$-2\overline{\omega}^a \wedge \iota_{\overline{V}_j}\left(\alpha^{IJ}\left\langle \nabla_{V_j}\overline{\omega}^{j_s},\omega^b\right\rangle\left\langle \nabla_{\overline{V}_a}\overline{\omega}^{j_u},\omega^c\right\rangle \omega^{i_1}\wedge\cdots\right.$$
$$\left.\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\overline{\omega}^c}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q}\right)\quad (W7c5)$$

$$-2\overline{\omega}^a \wedge \iota_{\overline{V}_j}\left(\alpha^{IJ}\left\langle \nabla_{V_j}\overline{\omega}^{j_s},\omega^b\right\rangle\left\langle \nabla_{\overline{V}_a}\overline{\omega}^{b},\omega^c\right\rangle \omega^{i_1}\wedge\cdots\right.$$
$$\left.\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^c}\wedge\cdots\wedge\overline{\omega}^{j_q}\right)\quad (W7c6)$$

$$+2\overline{\omega}^j \wedge \iota_{\overline{V}_a}\left(V_a\left[\overline{V}_j\left(\alpha^{IJ}\right)\right]\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\right)\quad (W8a1)$$

$$+2\overline{\omega}^j \wedge \iota_{\overline{V}_a}\left(\overline{V}_j\left(\alpha^{IJ}\right)\left\langle \nabla_{V_a}\omega^{i_r},\overline{\omega}^b\right\rangle \omega^{i_1}\wedge\cdots\right.$$
$$\left.\wedge\underset{r-th\text{ place}}{\omega^b}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\right)\quad (W8a2)$$

$$+2\overline{\omega}^j \wedge \iota_{\overline{V}_a}\left(\overline{V}_j\left(\alpha^{IJ}\right)\left\langle \nabla_{V_a}\overline{\omega}^{j_s},\omega^b\right\rangle \omega^{i_1}\wedge\cdots\right.$$
$$\left.\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q}\right)\quad (W8a3)$$

$$+2\overline{\omega}^j \wedge \iota_{\overline{V}_a}\left(V_a\left(\alpha^{IJ}\right)\left\langle \nabla_{\overline{V}_j}\omega^{i_r},\overline{\omega}^c\right\rangle \omega^{i_1}\wedge\cdots\right.$$
$$\left.\wedge\underset{r-th\text{ place}}{\omega^c}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\right)\quad (W8b1)$$

$$+2\overline{\omega}^j \wedge \iota_{\overline{V}_a}\left(\alpha^{IJ}\left\langle \nabla_{V_a}\nabla_{\overline{V}_j}\omega^{i_r},\overline{\omega}^c\right\rangle \omega^{i_1}\wedge\cdots\right.$$
$$\left.\wedge\underset{r-th\text{ place}}{\omega^c}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\right)\quad (W8b2)$$

$$+2\overline{\omega}^j \wedge \iota_{\overline{V}_a}\left(\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}\omega^{i_r},\nabla_{V_a}\overline{\omega}^c\right\rangle \omega^{i_1}\wedge\cdots\right.$$
$$\left.\wedge\underset{r-th\text{ place}}{\omega^c}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\right)\quad (W8b3)$$

$$+2\overline{\omega}^j \wedge \iota_{\overline{V}_a}\left(\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}\omega^{i_r},\overline{\omega}^c\right\rangle\left\langle \nabla_{V_a}\omega^{i_t},\overline{\omega}^b\right\rangle \omega^{i_1}\wedge\cdots\right.$$
$$\left.\wedge\underset{t-th\text{ place}}{\omega^b}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^c}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\right)\quad (W8b4)$$

$$+2\overline{\omega}^j \wedge \iota_{\overline{V}_a}\left(\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}\omega^{i_r},\overline{\omega}^c\right\rangle\left\langle \nabla_{V_a}\omega^{c},\overline{\omega}^b\right\rangle \omega^{i_1}\wedge\cdots\right.$$
$$\left.\wedge\underset{r-th\text{ place}}{\omega^b}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\right)\quad (W8b5)$$

$$+2\overline{\omega}^j \wedge \iota_{\overline{V}_a} \left(\alpha^{IJ} \left\langle \nabla_{\overline{V}_j} \omega^{i_r}, \overline{\omega}^c \right\rangle \left\langle \nabla_{V_a} \overline{\omega}^{j_s}, \omega^b \right\rangle \omega^{i_1} \wedge \cdots \right.$$

$$\left. \wedge \underset{r-th \text{ place}}{\omega^c} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^b} \wedge \cdots \wedge \overline{\omega}^{j_q} \right) \quad (W8b6)$$

$$+2\alpha^{IJ} \left\langle \nabla_{V_a} \nabla_{\overline{V}_j} \overline{\omega}^{j_s}, \omega^d \right\rangle \delta^a_{j_u} (-1)^{p+u-1} \overline{\omega}^j \wedge \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$$

$$\wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{u-th \text{ place}}{\widehat{\overline{\omega}^{j_u}}} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^d} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W8c2a)$$

$$+2\alpha^{IJ} \left\langle \nabla_{V_a} \nabla_{\overline{V}_j} \overline{\omega}^{j_s}, \omega^d \right\rangle \delta^a_d (-1)^{p+s-1} \overline{\omega}^j \wedge \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$$

$$\wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\widehat{\overline{\omega}^d}} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W8c2b)$$

$$+2\alpha^{IJ} \left\langle \nabla_{\overline{V}_j} \overline{\omega}^{j_s}, \nabla_{V_a} \omega^d \right\rangle \delta^a_{j_u} (-1)^{p+u-1} \overline{\omega}^j \wedge \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$$

$$\wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{u-th \text{ place}}{\widehat{\overline{\omega}^{j_u}}} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^d} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W8c3a)$$

$$+2\alpha^{IJ} \left\langle \nabla_{\overline{V}_j} \overline{\omega}^{j_s}, \nabla_{V_a} \omega^d \right\rangle \delta^a_d (-1)^{p+s-1} \overline{\omega}^j \wedge \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$$

$$\wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\widehat{\overline{\omega}^d}} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W8c3b)$$

$$+2\alpha^{IJ} \left\langle \nabla_{\overline{V}_j} \overline{\omega}^{j_s}, \omega^d \right\rangle \left\langle \nabla_{V_a} \omega^{i_r}, \overline{\omega}^b \right\rangle \delta^a_{j_u} (-1)^{p+u-1} \overline{\omega}^j \wedge \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^b}$$

$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{u-th \text{ place}}{\widehat{\overline{\omega}^{j_u}}} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^d} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W8c4a)$$

$$+2\alpha^{IJ} \left\langle \nabla_{\overline{V}_j} \overline{\omega}^{j_s}, \omega^d \right\rangle \left\langle \nabla_{V_a} \omega^{i_r}, \overline{\omega}^b \right\rangle \delta^a_d (-1)^{p+s-1} \overline{\omega}^j \wedge \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^b}$$

$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\widehat{\overline{\omega}^d}} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W8c4b)$$

$$+2\alpha^{IJ} \left\langle \nabla_{\overline{V}_j} \overline{\omega}^{j_s}, \omega^d \right\rangle \left\langle \nabla_{V_a} \overline{\omega}^{j_u}, \omega^b \right\rangle \delta^a_{j_x} (-1)^{p+x-1} \overline{\omega}^j \wedge \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$$

$$\wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{x-th \text{ place}}{\widehat{\overline{\omega}^{j_x}}} \wedge \cdots \wedge \underset{u-th \text{ place}}{\overline{\omega}^b} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^d} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W8c5a)$$

$$+2\alpha^{IJ} \left\langle \nabla_{\overline{V}_j} \overline{\omega}^{j_s}, \omega^d \right\rangle \left\langle \nabla_{V_a} \overline{\omega}^{j_u}, \omega^b \right\rangle \delta^a_b (-1)^{p+u-1} \overline{\omega}^j \wedge \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$$

$$\wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{u-th \text{ place}}{\widehat{\overline{\omega}^b}} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^d} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W8c5b)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}\overline{\omega}^{j_s},\omega^d\right\rangle \left\langle \nabla_{V_a}\overline{\omega}^{j_u},\omega^b\right\rangle \delta_d^a\left(-1\right)^{p+s-1}\overline{\omega}^j\wedge\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\underset{s-th\text{ place}}{\widehat{\overline{\omega}^d}}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W8c5c)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}\overline{\omega}^{j_s},\omega^d\right\rangle \left\langle \nabla_{V_a}\overline{\omega}^{d},\omega^b\right\rangle \delta_{j_u}^a\left(-1\right)^{p+u-1}\overline{\omega}^j\wedge\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\widehat{\overline{\omega}^{j_u}}}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W8c6a)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}\overline{\omega}^{j_s},\omega^d\right\rangle \left\langle \nabla_{V_a}\overline{\omega}^{d},\omega^b\right\rangle \delta_b^a\left(-1\right)^{p+s-1}\overline{\omega}^j\wedge\omega^{i_1}\wedge\cdots$$
$$\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\widehat{\overline{\omega}^b}}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W8c6b)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}\overline{\omega}^{j_s},\overline{\omega}^d\right\rangle \left\langle \nabla_{V_j}\omega^{i_r},\omega^b\right\rangle \omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\overline{\omega}^b}\wedge\cdots$$
$$\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\omega^d}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W9a)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}\overline{\omega}^{j_s},\overline{\omega}^d\right\rangle \left\langle \nabla_{V_j}\omega^{d},\omega^b\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W9b)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_j}\overline{V}_{i_r},\overline{V}_d\right\rangle \left\langle \nabla_{\overline{V}_j}V_d,V_c\right\rangle \omega^{i_1}\wedge\cdots$$
$$\wedge\underset{r-th\text{ place}}{\omega^c}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W10a)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_j}\overline{V}_{i_r},\overline{V}_d\right\rangle \left\langle \nabla_{\overline{V}_j}V_{j_s},V_c\right\rangle \left(-1\right)^{2(p+s+r)-1}\omega\wedge\cdots$$
$$\wedge\underset{r-th\text{ place}}{\omega^c}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^d}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W10b)$$

If we carry the wedge products and apply the Kronecker deltas, this yields

$$\begin{aligned}&(W1)\ \text{ through }\ (W10)\\ =\ &-2\overline{V}_j\left(\alpha^{IJ}\right)\left\langle \nabla_{\overline{V}_{i_r}}V_{j_s},V_c\right\rangle \left(-1\right)^{2(p+s+r)-1}\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^c}\\ &\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^j}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W1a1)\end{aligned}$$

$$\begin{aligned}&-2\overline{V}_j\left(\alpha^{IJ}\right)\left\langle \nabla_{\overline{V}_a}V_{j_s},V_a\right\rangle \left(-1\right)^{2(p+s-1)}\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\\ &\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^j}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W1a2)\end{aligned}$$

$$-2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}\overline{V}_{i_r}, V_c\right\rangle \left\langle \nabla_{\overline{V}_{i_t}} V_{j_s}, V_d\right\rangle (-1)^{2(p+s+t)-1}\,\omega^{i_1}\wedge\cdots\wedge \underset{t-th\text{ place}}{\omega^d}$$
$$\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^c}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge \underset{s-th\text{ place}}{\overline{\omega}^j}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad (W1b1)$$

$$-2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}\overline{V}_{i_r}, V_a\right\rangle \left\langle \nabla_{\overline{V}_a} V_{j_s}, V_d\right\rangle (-1)^{2(p+s+r)-1}\omega^{i_1}\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^d}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge \underset{s-th\text{ place}}{\overline{\omega}^j}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad (W1b2)$$

$$-2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}\overline{V}_{i_r}, V_c\right\rangle \left\langle \nabla_{\overline{V}_a} V_{j_s}, V_a\right\rangle (-1)^{2(p+s-1)}\omega^{i_1}\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^c}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge \underset{s-th\text{ place}}{\overline{\omega}^j}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad (W1b3)$$

$$-2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j} V_{j_s}, \overline{V}_d\right\rangle \left\langle \nabla_{\overline{V}_{i_r}} V_{j_u}, V_c\right\rangle (-1)^{2(p+u+r)-1}\omega^{i_1}\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^c}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge \underset{u-th\text{ place}}{\overline{\omega}^j}\wedge\cdots\wedge \underset{s-th\text{ place}}{\overline{\omega}^d}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad (W1c1a)$$

$$-2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j} V_{j_s}, \overline{V}_d\right\rangle \left\langle \nabla_{\overline{V}_a} V_{j_u}, V_a\right\rangle (-1)^{2(p+u-1)}\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge \underset{u-th\text{ place}}{\overline{\omega}^j}\wedge\cdots\wedge \underset{s-th\text{ place}}{\overline{\omega}^d}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad (W1c1b)$$

$$-2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j} V_{j_s}, \overline{V}_d\right\rangle \left\langle \nabla_{\overline{V}_{i_r}} V_d, V_c\right\rangle (-1)^{2(p+s+r)-1}\omega^{i_1}\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^c}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge \underset{s-th\text{ place}}{\overline{\omega}^j}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad (W1c2a)$$

$$-2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j} V_{j_s}, \overline{V}_d\right\rangle \left\langle \nabla_{\overline{V}_a} V_d, V_a\right\rangle (-1)^{2(p+s-1)}\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge \underset{s-th\text{ place}}{\overline{\omega}^j}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad (W1c2b)$$

$$+2\overline{V}_a\left(\alpha^{IJ}\right)\left\langle \nabla_{\overline{V}_{i_r}} V_{j_s}, V_d\right\rangle (-1)^{2(p+s+r)-1}\,\omega^{i_1}\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^d}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge \underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad (W2a1)$$

$$+2\overline{V}_a\left(\alpha^{IJ}\right)\left\langle \nabla_{\overline{V}_j} V_{j_s}, V_j\right\rangle (-1)^{2(p+s-1)}\,\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge \underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad (W2a2)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_a}\nabla_{\overline{V}_{i_r}}V_{j_s},V_d\right\rangle(-1)^{2(p+s+r)-1}\,\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^d}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W2b1)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_a}\nabla_{\overline{V}_j}V_{j_s},V_j\right\rangle(-1)^{2(p+s-1)}\,\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W2b2)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_{i_r}}V_{j_s},\overline{V}_b\right\rangle\left\langle \nabla_{\overline{V}_a}V_d,V_b\right\rangle(-1)^{2(p+s+r)-1}\,\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^d}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W2c1a)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_{i_r}}V_{j_s},V_b\right\rangle\left\langle \nabla_{\overline{V}_a}V_d,\overline{V}_b\right\rangle(-1)^{2(p+s+r)-1}\,\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^d}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W2c1b)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}V_{j_s},\overline{V}_b\right\rangle\left\langle \nabla_{\overline{V}_a}V_j,V_b\right\rangle(-1)^{2(p+s-1)}\,\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W2c2a)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}V_{j_s},V_b\right\rangle\left\langle \nabla_{\overline{V}_a}V_j,\overline{V}_b\right\rangle(-1)^{2(p+s-1)}\,\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W2c2b)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_{i_t}}V_{j_s},V_d\right\rangle\left\langle \nabla_{\overline{V}_a}\overline{V}_{i_r},V_c\right\rangle(-1)^{2(p+s+t)-1}\omega^{i_1}\wedge\cdots\wedge\underset{t-th\text{ place}}{\omega^d}$$
$$\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^c}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W2d1)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}V_{j_s},V_d\right\rangle\left\langle \nabla_{\overline{V}_a}\overline{V}_{i_r},V_j\right\rangle(-1)^{2(p+s+r)-1}\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^d}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W2d2)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}V_{j_s},V_j\right\rangle\left\langle \nabla_{\overline{V}_a}\overline{V}_{i_r},V_c\right\rangle(-1)^{2(p+s-1)}\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^c}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W2d3)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_{i_r}}V_{j_s},V_d\right\rangle\left\langle \nabla_{\overline{V}_a}\overline{V}_d,V_c\right\rangle(-1)^{2(p+s+r)-1}\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^c}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W2e1)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j} V_{j_s}, V_d \right\rangle \left\langle \nabla_{\overline{V}_a} \overline{V}_d, V_j \right\rangle (-1)^{2(p+s-1)} \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$$
$$\wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th\text{ place}}{\overline{\omega}^{a}} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W2e2)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_{i_r}} V_{j_s}, V_d \right\rangle \left\langle \nabla_{\overline{V}_a} V_{j_u}, \overline{V}_c \right\rangle (-1)^{2(p+s+r)-1} \omega^{i_1} \wedge \cdots \wedge \underset{r-th\text{ place}}{\omega^{d}}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th\text{ place}}{\overline{\omega}^{a}} \wedge \cdots \wedge \underset{u-th\text{ place}}{\overline{\omega}^{c}} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W2f1)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j} V_{j_s}, V_j \right\rangle \left\langle \nabla_{\overline{V}_a} V_{j_u}, \overline{V}_c \right\rangle (-1)^{2(p+s-1)} \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$$
$$\wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th\text{ place}}{\overline{\omega}^{a}} \wedge \cdots \wedge \underset{u-th\text{ place}}{\overline{\omega}^{c}} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W2f2)$$

$$-2\alpha^{IJ}(-1)^{2(r-1)} \left\langle \nabla_{V_j} \overline{V}_{i_r}, \overline{V}_d \right\rangle \left\langle \nabla_{\overline{V}_a} V_d, V_a \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th\text{ place}}{\omega^{j}}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W3a1)$$

$$-2\alpha^{IJ}(-1)^{2(t-1)} \left\langle \nabla_{V_j} \overline{V}_{i_r}, \overline{V}_d \right\rangle \left\langle \nabla_{\overline{V}_{i_t}} V_d, V_c \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{t-th\text{ place}}{\omega^{j}}$$
$$\wedge \cdots \wedge \underset{r-th\text{ place}}{\omega^{c}} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W3a2)$$

$$-2\alpha^{IJ}(-1)^{2(p+s+r)-1} \left\langle \nabla_{V_j} \overline{V}_{i_r}, \overline{V}_d \right\rangle \left\langle \nabla_{\overline{V}_a} V_{j_s}, V_a \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th\text{ place}}{\omega^{j}}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th\text{ place}}{\overline{\omega}^{d}} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W3b1)$$

$$-2\alpha^{IJ}(-1)^{2(p+s+r+t)-1} \left\langle \nabla_{V_j} \overline{V}_{i_r}, \overline{V}_d \right\rangle \left\langle \nabla_{\overline{V}_{i_t}} V_{j_s}, V_c \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{t-th\text{ place}}{\omega^{j}}$$
$$\wedge \cdots \wedge \underset{r-th\text{ place}}{\omega^{c}} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th\text{ place}}{\overline{\omega}^{d}} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W3b2)$$

$$+2\alpha^{IJ}(-1)^{2(p+r+t+s)-1} \left\langle \nabla_{\overline{V}_{i_t}} V_{j_s}, V_d \right\rangle \left\langle \nabla_{V_a} \overline{V}_{i_r}, \overline{V}_c \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{t-th\text{ place}}{\omega^{a}}$$
$$\wedge \cdots \wedge \underset{r-th\text{ place}}{\omega^{d}} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th\text{ place}}{\overline{\omega}^{c}} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W4a1)$$

$$+2\alpha^{IJ}(-1)^{2(p+r+s)-1} \left\langle \nabla_{\overline{V}_j} V_{j_s}, V_j \right\rangle \left\langle \nabla_{V_a} \overline{V}_{i_r}, \overline{V}_c \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th\text{ place}}{\omega^{a}}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th\text{ place}}{\overline{\omega}^{c}} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W4a2)$$

$$+2\alpha^{IJ}\,(-1)^{2(r-1)}\left\langle\nabla_{\overline{V}_{i_r}}V_{j_s},V_d\right\rangle\left\langle\nabla_{V_a}\overline{V}_d,\overline{V}_c\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^a}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^c}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W4b)$$

$$-2V_j\left(\alpha^{IJ}\right)\left\langle\nabla_{V_a}\overline{V}_{i_r},\overline{V}_j\right\rangle(-1)^{2(r-1)}\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^a}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W5a1)$$

$$-2V_{j_s}\left(\alpha^{IJ}\right)\left\langle\nabla_{V_a}\overline{V}_{i_r},\overline{V}_c\right\rangle(-1)^{2(p+s+r)-1}\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^a}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^c}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W5a2)$$

$$-2\alpha^{IJ}(-1)^{2(t-1)}\left\langle\nabla_{V_j}\overline{V}_{i_r},V_c\right\rangle\left\langle\nabla_{V_a}\overline{V}_{i_t},\overline{V}_j\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{t-th\text{ place}}{\omega^a}$$
$$\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^c}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W5b1a)$$

$$-2\alpha^{IJ}(-1)^{2(p+s+t)-1}\left\langle\nabla_{V_{j_s}}\overline{V}_{i_r},V_c\right\rangle\left\langle\nabla_{V_a}\overline{V}_{i_t},\overline{V}_b\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{t-th\text{ place}}{\omega^a}$$
$$\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^c}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W5b1b)$$

$$-2\alpha^{IJ}(-1)^{2(r-1)}\left\langle\nabla_{V_j}\overline{V}_{i_r},V_c\right\rangle\left\langle\nabla_{V_a}\overline{V}_c,\overline{V}_j\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^a}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W5b2a)$$

$$-2\alpha^{IJ}(-1)^{2(p+s+r)-1}\left\langle\nabla_{V_{j_s}}\overline{V}_{i_r},V_c\right\rangle\left\langle\nabla_{V_a}\overline{V}_c,\overline{V}_b\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^a}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W5b2b)$$

$$-2\alpha^{IJ}(-1)^{2(r-1)}\left\langle\nabla_{V_j}V_{j_s},\overline{V}_b\right\rangle\left\langle\nabla_{V_a}\overline{V}_{i_r},\overline{V}_j\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^a}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W5c1)$$

$$-2\alpha^{IJ}(-1)^{2(p+u+r)-1}\left\langle\nabla_{V_{j_u}}V_{j_s},\overline{V}_b\right\rangle\left\langle\nabla_{V_a}\overline{V}_{i_r},\overline{V}_c\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^a}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\overline{\omega}^c}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W5c2)$$

$$-2\alpha^{IJ}(-1)^{2(p+s+r)-1}\left\langle\nabla_{V_j}V_{j_s},\overline{V}_j\right\rangle\left\langle\nabla_{V_a}\overline{V}_{i_r},\overline{V}_c\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^a}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^c}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W5c3)$$

$$+2V_a\left(\alpha^{IJ}\right)\left\langle\nabla_{V_j}\overline{V}_{i_r},\overline{V}_a\right\rangle(-1)^{2(r-1)}\,\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^j}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W6a1)$$

$$+2V_{j_s}\left(\alpha^{IJ}\right)\left\langle\nabla_{V_j}\overline{V}_{i_r},\overline{V}_b\right\rangle(-1)^{2(p+s+r)-1}\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^j}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W6a2)$$

$$+2\alpha^{IJ}\left\langle\nabla_{V_a}\nabla_{V_j}\overline{V}_{i_r},\overline{V}_a\right\rangle(-1)^{2(r-1)}\,\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^j}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W6b1)$$

$$+2\alpha^{IJ}\left\langle\nabla_{V_{j_s}}\nabla_{V_j}\overline{V}_{i_r},\overline{V}_b\right\rangle(-1)^{2(p+s+r)-1}\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^j}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W6b2)$$

$$+2\alpha^{IJ}\left\langle\nabla_{V_j}\overline{V}_{i_r},\overline{V}_c\right\rangle\left\langle\nabla_{V_a}\overline{V}_a,V_c\right\rangle(-1)^{2(r-1)}\,\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^j}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W6c1a)$$

$$+2\alpha^{IJ}\left\langle\nabla_{V_j}\overline{V}_{i_r},V_c\right\rangle\left\langle\nabla_{V_a}\overline{V}_a,\overline{V}_c\right\rangle(-1)^{2(r-1)}\,\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^j}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W6c1b)$$

$$+2\alpha^{IJ}\left\langle\nabla_{V_j}\overline{V}_{i_r},\overline{V}_c\right\rangle\left\langle\nabla_{V_{j_s}}\overline{V}_b,V_c\right\rangle(-1)^{2(p+s+r)-1}\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^j}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W6c2a)$$

$$+2\alpha^{IJ}\left\langle\nabla_{V_j}\overline{V}_{i_r},V_c\right\rangle\left\langle\nabla_{V_{j_s}}\overline{V}_b,\overline{V}_c\right\rangle(-1)^{2(p+s+r)-1}\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^j}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W6c2b)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_j}\overline{V}_{i_r},\overline{V}_a\right\rangle\left\langle \nabla_{V_a}\overline{V}_{i_t},V_c\right\rangle(-1)^{2(r-1)}\,\omega^{i_1}\wedge\cdots\wedge\underset{t-th\text{ place}}{\omega^c}$$
$$\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^j}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W6d1)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_j}\overline{V}_{i_r},\overline{V}_b\right\rangle\left\langle \nabla_{V_{j_s}}\overline{V}_{i_t},V_c\right\rangle(-1)^{2(p+s+r)-1}\omega^{i_1}\wedge\cdots\wedge\underset{t-th\text{ place}}{\omega^c}$$
$$\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^j}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W6d2)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_j}\overline{V}_{i_r},\overline{V}_b\right\rangle\left\langle \nabla_{V_a}V_b,\overline{V}_a\right\rangle(-1)^{2(r-1)}\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^j}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W6e1)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_j}\overline{V}_{i_r},\overline{V}_b\right\rangle\left\langle \nabla_{V_{j_s}}V_b,\overline{V}_c\right\rangle(-1)^{2(p+s+r)-1}\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^j}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^c}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W6e2)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_j}\overline{V}_{i_r},\overline{V}_a\right\rangle\left\langle \nabla_{V_a}V_{j_s},\overline{V}_c\right\rangle(-1)^{2(r-1)}\,\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^j}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^c}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W6f1)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_j}\overline{V}_{i_r},\overline{V}_b\right\rangle\left\langle \nabla_{V_a}V_{j_s},\overline{V}_a\right\rangle(-1)^{2(p+s+r)-1}\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^j}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W6f2)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_j}\overline{V}_{i_r},\overline{V}_b\right\rangle\left\langle \nabla_{V_{j_u}}V_{j_s},\overline{V}_c\right\rangle(-1)^{2(p+u+r)-1}\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^j}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^c}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W6f3)$$

$$-2\overline{V}_a\left[V_{j_s}\left(\alpha^{IJ}\right)\right](-1)^{2(p+s-1)}\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W7a1)$$

$$-2V_{j_s}\left(\alpha^{IJ}\right)\left\langle \nabla_{\overline{V}_a}\overline{V}_{i_r},V_c\right\rangle(-1)^{2(p+s-1)}\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^c}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W7a2)$$

$$-2V_{j_u}\left(\alpha^{IJ}\right)\left\langle \nabla_{\overline{V}_a} V_{j_s}, \overline{V}_c\right\rangle (-1)^{2(p+u-1)} \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$$
$$\wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{u-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^c} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W7a3a)$$

$$-2V_{j}\left(\alpha^{IJ}\right)\left\langle \nabla_{\overline{V}_a} V_{j_s}, \overline{V}_j\right\rangle (-1)^{2(p+s-1)} \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$$
$$\wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W7a3b)$$

$$-2\overline{V}_a\left(\alpha^{IJ}\right)\left\langle \nabla_{V_{j_s}} \overline{V}_{i_r}, V_b\right\rangle (-1)^{2(p+s-1)} \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^b}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W7b1)$$

$$-2\alpha^{IJ}\left\langle \nabla_{\overline{V}_a} \nabla_{V_{j_s}} \overline{V}_{i_r}, V_b\right\rangle (-1)^{2(p+s-1)} \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^b}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W7b2)$$

$$-2\alpha^{IJ}\left\langle \nabla_{V_{j_s}} \overline{V}_{i_r}, \overline{V}_c\right\rangle \left\langle \nabla_{\overline{V}_a} V_b, V_c\right\rangle (-1)^{2(p+s-1)} \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^b}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W7b3a)$$

$$-2\alpha^{IJ}\left\langle \nabla_{V_{j_s}} \overline{V}_{i_r}, V_c\right\rangle \left\langle \nabla_{\overline{V}_a} V_b, \overline{V}_c\right\rangle (-1)^{2(p+s-1)} \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^b}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W7b3b)$$

$$-2\alpha^{IJ}\left\langle \nabla_{V_{j_s}} \overline{V}_{i_r}, V_b\right\rangle \left\langle \nabla_{\overline{V}_a} \overline{V}_b, V_c\right\rangle (-1)^{2(p+s-1)} \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^c}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W7b4)$$

$$-2\alpha^{IJ}\left\langle \nabla_{V_{j_s}} \overline{V}_{i_r}, V_b\right\rangle \left\langle \nabla_{\overline{V}_a} \overline{V}_{i_t}, V_c\right\rangle (-1)^{2(p+s-1)} \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^b}$$
$$\wedge \cdots \wedge \underset{t-th \text{ place}}{\omega^c} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W7b5)$$

$$-2\alpha^{IJ}\left\langle \nabla_{V_{j_u}} \overline{V}_{i_r}, V_b\right\rangle \left\langle \nabla_{\overline{V}_a} V_{j_s}, \overline{V}_c\right\rangle (-1)^{2(p+u-1)} \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^b}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{u-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^c} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W7b6a)$$

$$-2\alpha^{IJ}\left\langle \nabla_{V_j}\overline{V}_{i_r}, V_b\right\rangle \left\langle \nabla_{\overline{V}_a} V_{j_s}, \overline{V}_j\right\rangle (-1)^{2(p+s-1)}\omega^{i_1}\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^b}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge \underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q} \qquad (W7b6b)$$

$$-2\overline{V}_a\left(\alpha^{IJ}\right)\left\langle \nabla_{V_{j_u}} V_{j_s}, \overline{V}_b\right\rangle (-1)^{(2p+u-1)}\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots$$
$$\wedge \underset{u-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge \underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q} \qquad (WL7c1a)$$

$$-2\overline{V}_a\left(\alpha^{IJ}\right)\left\langle \nabla_{V_j} V_{j_s}, \overline{V}_j\right\rangle (-1)^{2(p+s-1)}\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge \underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q} \qquad (WL7c1b)$$

$$-2\alpha^{IJ}\left\langle \nabla_{\overline{V}_a}\nabla_{V_{j_u}} V_{j_s}, \overline{V}_b\right\rangle (-1)^{2(p+u-1)}\,\overline{\omega}^a\wedge\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}$$
$$\wedge\cdots\wedge \underset{u-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge \underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q} \qquad (WL7c2a)$$

$$-2\alpha^{IJ}\left\langle \nabla_{\overline{V}_a}\nabla_{V_j} V_{j_s}, \overline{V}_j\right\rangle (-1)^{2(p+s-1)}\,\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots$$
$$\wedge \underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q} \qquad (WL7c2b)$$

$$-2\alpha^{IJ}\left\langle \nabla_{V_{j_u}} V_{j_s}, \overline{V}_c\right\rangle \left\langle \nabla_{\overline{V}_a}\overline{V}_b, V_c\right\rangle (-1)^{2(p+u-1)}\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}$$
$$\wedge\cdots\wedge \underset{u-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge \underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q} \qquad (WL7c3a)$$

$$-2\alpha^{IJ}\left\langle \nabla_{V_j} V_{j_s}, \overline{V}_c\right\rangle \left\langle \nabla_{\overline{V}_a}\overline{V}_j, V_c\right\rangle (-1)^{2(p+s-1)}\,\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}$$
$$\wedge\cdots\wedge \underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q} \qquad (WL7c3b)$$

$$-2\alpha^{IJ}\left\langle \nabla_{V_{j_u}} V_{j_s}, \overline{V}_b\right\rangle \left\langle \nabla_{\overline{V}_a}\overline{V}_{i_r}, V_c\right\rangle (-1)^{2(p+u-1)}\,\omega^{i_1}\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^c}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge \underset{u-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge \underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q} \qquad (WL7c4a)$$

$$-2\alpha^{IJ}\left\langle \nabla_{V_j} V_{j_s}, \overline{V}_j\right\rangle \left\langle \nabla_{\overline{V}_a}\overline{V}_{i_r}, V_c\right\rangle (-1)^{2(p+s-1)}\,\omega^{i_1}\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^c}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge \underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q} \qquad (WL7c4b)$$

$$-2\alpha^{IJ}\left\langle\nabla_{V_{j_x}}V_{j_s},\overline{V}_b\right\rangle\left\langle\nabla_{\overline{V}_a}V_{j_u},\overline{V}_c\right\rangle(-1)^{2(p+x-1)}\,\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}$$
$$\wedge\cdots\wedge\underset{x-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\underset{u-th\text{ place}}{\overline{\omega}^c}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(WL7c5a)$$

$$-2\alpha^{IJ}\left\langle\nabla_{V_j}V_{j_s},\overline{V}_b\right\rangle\left\langle\nabla_{\overline{V}_a}V_{j_u},\overline{V}_j\right\rangle(-1)^{2(p+u-1)}\,\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(WL7c5b)$$

$$-2\alpha^{IJ}\left\langle\nabla_{V_j}V_{j_s},\overline{V}_j\right\rangle\left\langle\nabla_{\overline{V}_a}V_{j_u},\overline{V}_c\right\rangle(-1)^{2(p+s-1)}\,\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\overline{\omega}^c}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(WL7c5c)$$

$$-2\alpha^{IJ}\left\langle\nabla_{V_{j_u}}V_{j_s},\overline{V}_b\right\rangle\left\langle\nabla_{\overline{V}_a}V_b,\overline{V}_c\right\rangle(-1)^{2(p+u-1)}\,\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^c}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(WL7c6a)$$

$$-2\alpha^{IJ}\left\langle\nabla_{V_j}V_{j_s},\overline{V}_b\right\rangle\left\langle\nabla_{\overline{V}_a}V_b,\overline{V}_j\right\rangle(-1)^{2(p+s-1)}\,\omega^{i_1}\wedge\cdots$$
$$\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(WL7c6b)$$

$$+2V_{j_s}\left[\overline{V}_j\left(\alpha^{IJ}\right)\right](-1)^{2(p+s-1)}\,\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^j}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W8a1)$$

$$+2\overline{V}_j\left(\alpha^{IJ}\right)\left\langle\nabla_{V_{j_s}}\overline{V}_{i_r},V_b\right\rangle(-1)^{2(p+s-1)}\,\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^b}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^j}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W8a2)$$

$$+2\overline{V}_j\left(\alpha^{IJ}\right)\left\langle\nabla_{V_{j_u}}V_{j_s},\overline{V}_b\right\rangle(-1)^{2(p+u-1)}\,\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots$$
$$\wedge\underset{u-th\text{ place}}{\overline{\omega}^j}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W8a3a)$$

$$+2\overline{V}_j\left(\alpha^{IJ}\right)\left\langle\nabla_{V_a}V_{j_s},\overline{V}_a\right\rangle(-1)^{2(p+s-1)}\,\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^j}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W8a3b)$$

$$+2V_{j_s}\left(\alpha^{IJ}\right)\left\langle\nabla_{\overline{V}_j}\overline{V}_{i_r},V_c\right\rangle(-1)^{2(p+s-1)}\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^c}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^j}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W8b1)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_{j_s}}\nabla_{\overline{V}_j}\overline{V}_{i_r}, V_c\right\rangle (-1)^{2(p+s-1)}\omega^{i_1}\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^c}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^j}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W8b2)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}\overline{V}_{i_r}, V_a\right\rangle \left\langle \nabla_{V_{j_s}}V_c, \overline{V}_a\right\rangle (-1)^{2(p+s-1)}\omega^{i_1}\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^c}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^j}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W8b3)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}\overline{V}_{i_r}, V_c\right\rangle \left\langle \nabla_{V_{j_s}}\overline{V}_{i_t}, V_b\right\rangle (-1)^{2(p+s-1)}\omega^{i_1}\wedge\cdots\wedge \underset{t-th\text{ place}}{\omega^b}$$
$$\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^c}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^j}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W8b4)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}\overline{V}_{i_r}, V_c\right\rangle \left\langle \nabla_{V_{j_s}}\overline{V}_c, V_b\right\rangle (-1)^{2(p+s-1)}\omega^{i_1}\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^b}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^j}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W8b5)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}\overline{V}_{i_r}, V_c\right\rangle \left\langle \nabla_{V_{j_u}}V_{j_s}, \overline{V}_b\right\rangle (-1)^{2(p+u-1)}\omega^{i_1}\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^c}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\overline{\omega}^j}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W8b6a)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}\overline{V}_{i_r}, V_c\right\rangle \left\langle \nabla_{V_a}V_{j_s}, \overline{V}_a\right\rangle (-1)^{2(p+s-1)}\omega^{i_1}\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^c}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^j}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W8b6b)$$

$$+2V_{j_u}\left(\alpha^{IJ}\right)\left\langle \nabla_{\overline{V}_j}V_{j_s}, \overline{V}_d\right\rangle (-1)^{2(p+u-1)}\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\overline{\omega}^j}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^d}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W8c1a)$$

$$+2V_a\left(\alpha^{IJ}\right)\left\langle \nabla_{\overline{V}_j}V_{j_s}, \overline{V}_a\right\rangle (-1)^{2(p+s-1)}\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^j}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W8c1b)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_{j_u}}\nabla_{\overline{V}_j}V_{j_s},\overline{V}_d\right\rangle(-1)^{2(p+u-1)}\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\overline{\omega}^{j}}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{d}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W8c2a)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_a}\nabla_{\overline{V}_j}V_{j_s},\overline{V}_a\right\rangle(-1)^{2(p+s-1)}\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{j}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W8c2b)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}V_{j_s},\overline{V}_b\right\rangle\left\langle \nabla_{V_{j_u}}\overline{V}_d,V_b\right\rangle(-1)^{2(p+u-1)}\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\overline{\omega}^{j}}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{d}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W8c3a1)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}V_{j_s},V_b\right\rangle\left\langle \nabla_{V_{j_u}}\overline{V}_d,\overline{V}_b\right\rangle(-1)^{2(p+u-1)}\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\overline{\omega}^{j}}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{d}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W8c3a2)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}V_{j_s},\overline{V}_b\right\rangle\left\langle \nabla_{V_a}\overline{V}_a,V_b\right\rangle(-1)^{2(p+s-1)}\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{j}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W8c3b1)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}V_{j_s},V_b\right\rangle\left\langle \nabla_{V_a}\overline{V}_a,\overline{V}_b\right\rangle(-1)^{2(p+s-1)}\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{j}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W8c3b2)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}V_{j_s},\overline{V}_d\right\rangle\left\langle \nabla_{V_{j_u}}\overline{V}_{i_r},V_b\right\rangle(-1)^{2(p+u-1)}\omega^{i_1}\wedge..\wedge\underset{r-th\text{ place}}{\omega^{b}}\wedge$$
$$..\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge..\wedge\underset{u-th\text{ place}}{\overline{\omega}^{j}}\wedge..\wedge\underset{s-th\text{ place}}{\overline{\omega}^{d}}\wedge..\wedge\overline{\omega}^{j_q}\quad (W8c4a)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}V_{j_s},\overline{V}_a\right\rangle\left\langle \nabla_{V_a}\overline{V}_{i_r},V_b\right\rangle(-1)^{2(p+s-1)}\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^{b}}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{j}}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad (W8c4b)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j} V_{j_s}, \overline{V}_d \right\rangle \left\langle \nabla_{V_{j_x}} V_{j_u}, \overline{V}_b \right\rangle (-1)^{2(p+x-1)} \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$$
$$\wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{x-th \text{ place}}{\overline{\omega}^j} \wedge \cdots \wedge \underset{u-th \text{ place}}{\overline{\omega}^b} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^d} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W8c5a)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j} V_{j_s}, \overline{V}_d \right\rangle \left\langle \nabla_{V_a} V_{j_u}, \overline{V}_a \right\rangle (-1)^{2(p+u-1)} \omega^{i_1} \wedge .. \wedge \omega^{i_p}$$
$$\wedge \overline{\omega}^{j_1} \wedge .. \wedge \underset{u-th \text{ place}}{\overline{\omega}^j} \wedge .. \wedge \underset{s-th \text{ place}}{\overline{\omega}^d} \wedge .. \wedge \overline{\omega}^{j_q} \qquad (W8c5b)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j} V_{j_s}, \overline{V}_a \right\rangle \left\langle \nabla_{V_a} V_{j_u}, \overline{V}_b \right\rangle (-1)^{2(p+s-1)} \omega^{i_1} \wedge .. \wedge \omega^{i_p}$$
$$\wedge \overline{\omega}^{j_1} \wedge .. \wedge \underset{u-th \text{ place}}{\overline{\omega}^b} \wedge .. \wedge \underset{s-th \text{ place}}{\overline{\omega}^j} \wedge .. \wedge \overline{\omega}^{j_q} \qquad (W8c5c)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j} V_{j_s}, \overline{V}_d \right\rangle \left\langle \nabla_{V_{j_u}} V_d, \overline{V}_b \right\rangle (-1)^{2(p+u-1)} \omega^{i_1} \wedge .. \wedge \omega^{i_p}$$
$$\wedge \overline{\omega}^{j_1} \wedge .. \wedge \underset{u-th \text{ place}}{\overline{\omega}^j} \wedge .. \wedge \underset{s-th \text{ place}}{\overline{\omega}^b} \wedge .. \wedge \overline{\omega}^{j_q} \qquad (W8c6a)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j} V_{j_s}, \overline{V}_d \right\rangle \left\langle \nabla_{V_a} V_d, \overline{V}_a \right\rangle (-1)^{2(p+s-1)} \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$$
$$\wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^j} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W8c6b)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j} V_{j_s}, V_d \right\rangle \left\langle \nabla_{V_j} \overline{V}_{i_r}, \overline{V}_b \right\rangle (-1)^{2(p+s+r)-1} \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^d}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^b} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W9a)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j} V_{j_s}, V_d \right\rangle \left\langle \nabla_{V_j} \overline{V}_d, \overline{V}_b \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$$
$$\wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^b} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W9b)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_j} \overline{V}_{i_r}, \overline{V}_d \right\rangle \left\langle \nabla_{\overline{V}_j} V_d, V_c \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^c}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W10a)$$

$$-2\alpha^{IJ}\left\langle \nabla_{V_j} \overline{V}_{i_r}, \overline{V}_d \right\rangle \left\langle \nabla_{\overline{V}_j} V_{j_s}, V_c \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^c}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^d} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W10b)$$

Then simplifying the signs gives

$$
\begin{aligned}
& (W1) \text{ through } (W10) \\
= \; & +2\overline{V}_j \left(\alpha^{IJ}\right) \left\langle \nabla_{\overline{V}_{i_r}} V_{j_s}, V_c \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^c} \wedge \cdots \wedge \omega^{i_p} \\
& \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^j} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W1a1) \\
& -2\overline{V}_j \left(\alpha^{IJ}\right) \left\langle \nabla_{\overline{V}_a} V_{j_s}, V_a \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \\
& \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^j} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W1a2) \\
& +2\alpha^{IJ} \left\langle \nabla_{\overline{V}_j} \overline{V}_{i_r}, V_c \right\rangle \left\langle \nabla_{\overline{V}_{i_t}} V_{j_s}, V_d \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{t-th \text{ place}}{\omega^d} \\
& \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^c} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^j} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W1b1) \\
& +2\alpha^{IJ} \left\langle \nabla_{\overline{V}_j} \overline{V}_{i_r}, V_a \right\rangle \left\langle \nabla_{\overline{V}_a} V_{j_s}, V_d \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^d} \\
& \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^j} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W1b2) \\
& -2\alpha^{IJ} \left\langle \nabla_{\overline{V}_j} \overline{V}_{i_r}, V_c \right\rangle \left\langle \nabla_{\overline{V}_a} V_{j_s}, V_a \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^c} \\
& \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^j} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W1b3) \\
& +2\alpha^{IJ} \left\langle \nabla_{\overline{V}_j} V_{j_s}, \overline{V}_d \right\rangle \left\langle \nabla_{\overline{V}_{i_r}} V_{j_u}, V_c \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^c} \\
& \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{u-th \text{ place}}{\overline{\omega}^j} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^d} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W1c1a) \\
& -2\alpha^{IJ} \left\langle \nabla_{\overline{V}_j} V_{j_s}, \overline{V}_d \right\rangle \left\langle \nabla_{\overline{V}_a} V_{j_u}, V_a \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \\
& \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{u-th \text{ place}}{\overline{\omega}^j} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^d} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W1c1b) \\
& +2\alpha^{IJ} \left\langle \nabla_{\overline{V}_j} V_{j_s}, \overline{V}_d \right\rangle \left\langle \nabla_{\overline{V}_{i_r}} V_d, V_c \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^c} \\
& \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^j} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W1c2a) \\
& -2\alpha^{IJ} \left\langle \nabla_{\overline{V}_j} V_{j_s}, \overline{V}_d \right\rangle \left\langle \nabla_{\overline{V}_a} V_d, V_a \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \\
& \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^j} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W1c2b)
\end{aligned}
$$

$$-2\overline{V}_a\left(\alpha^{IJ}\right)\left\langle\nabla_{\overline{V}_{i_r}}V_{j_s},V_d\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^d}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W2a1)$$

$$+2\overline{V}_a\left(\alpha^{IJ}\right)\left\langle\nabla_{\overline{V}_j}V_{j_s},V_j\right\rangle\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W2a2)$$

$$-2\alpha^{IJ}\left\langle\nabla_{\overline{V}_a}\nabla_{\overline{V}_{i_r}}V_{j_s},V_d\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^d}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W2b1)$$

$$+2\alpha^{IJ}\left\langle\nabla_{\overline{V}_a}\nabla_{\overline{V}_j}V_{j_s},V_j\right\rangle\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W2b2)$$

$$-2\alpha^{IJ}\left\langle\nabla_{\overline{V}_{i_r}}V_{j_s},\overline{V}_b\right\rangle\left\langle\nabla_{\overline{V}_a}V_d,V_b\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^d}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W2c1a)$$

$$-2\alpha^{IJ}\left\langle\nabla_{\overline{V}_{i_r}}V_{j_s},V_b\right\rangle\left\langle\nabla_{\overline{V}_a}V_d,\overline{V}_b\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^d}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W2c1b)$$

$$+2\alpha^{IJ}\left\langle\nabla_{\overline{V}_j}V_{j_s},\overline{V}_b\right\rangle\left\langle\nabla_{\overline{V}_a}V_j,V_b\right\rangle\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W2c2a)$$

$$+2\alpha^{IJ}\left\langle\nabla_{\overline{V}_j}V_{j_s},V_b\right\rangle\left\langle\nabla_{\overline{V}_a}V_j,\overline{V}_b\right\rangle\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W2c2b)$$

$$-2\alpha^{IJ}\left\langle \nabla_{\overline{V}_{i_t}} V_{j_s}, V_d \right\rangle \left\langle \nabla_{\overline{V}_a} \overline{V}_{i_r}, V_c \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{t-th \text{ place}}{\omega^d}$$
$$\wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^c} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W2d1)$$

$$-2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j} V_{j_s}, V_d \right\rangle \left\langle \nabla_{\overline{V}_a} \overline{V}_{i_r}, V_j \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^d}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W2d2)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j} V_{j_s}, V_j \right\rangle \left\langle \nabla_{\overline{V}_a} \overline{V}_{i_r}, V_c \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^c}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W2d3)$$

$$-2\alpha^{IJ}\left\langle \nabla_{\overline{V}_{i_r}} V_{j_s}, V_d \right\rangle \left\langle \nabla_{\overline{V}_a} \overline{V}_d, V_c \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^c}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W2e1)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j} V_{j_s}, V_d \right\rangle \left\langle \nabla_{\overline{V}_a} \overline{V}_d, V_j \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$$
$$\wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W2e2)$$

$$-2\alpha^{IJ}\left\langle \nabla_{\overline{V}_{i_r}} V_{j_s}, V_d \right\rangle \left\langle \nabla_{\overline{V}_a} V_{j_u}, \overline{V}_c \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^d}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \underset{u-th \text{ place}}{\overline{\omega}^c} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W2f1)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j} V_{j_s}, V_j \right\rangle \left\langle \nabla_{\overline{V}_a} V_{j_u}, \overline{V}_c \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$$
$$\wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \underset{u-th \text{ place}}{\overline{\omega}^c} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W2f2)$$

$$-2\alpha^{IJ}\left\langle \nabla_{V_j}\overline{V}_{i_r},\overline{V}_d\right\rangle \left\langle \nabla_{\overline{V}_a}V_d,V_a\right\rangle \omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^j}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad (W3a1)$$

$$-2\alpha^{IJ}\left\langle \nabla_{V_j}\overline{V}_{i_r},\overline{V}_d\right\rangle \left\langle \nabla_{\overline{V}_{i_t}}V_d,V_c\right\rangle \omega^{i_1}\wedge\cdots\wedge\underset{t-th\text{ place}}{\omega^j}$$
$$\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^c}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad (W3a2)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_j}\overline{V}_{i_r},\overline{V}_d\right\rangle \left\langle \nabla_{\overline{V}_a}V_{j_s},V_a\right\rangle \omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^j}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^d}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad (W3b1)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_j}\overline{V}_{i_r},\overline{V}_d\right\rangle \left\langle \nabla_{\overline{V}_{i_t}}V_{j_s},V_c\right\rangle \omega^{i_1}\wedge\cdots\wedge\underset{t-th\text{ place}}{\omega^j}$$
$$\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^c}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^d}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad (W3b2)$$

$$-2\alpha^{IJ}\left\langle \nabla_{\overline{V}_{i_t}}V_{j_s},V_d\right\rangle \left\langle \nabla_{V_a}\overline{V}_{i_r},\overline{V}_c\right\rangle \omega^{i_1}\wedge\cdots\wedge\underset{t-th\text{ place}}{\omega^a}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^d}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^c}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad (W4a1)$$

$$-2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}V_{j_s},V_j\right\rangle \left\langle \nabla_{V_a}\overline{V}_{i_r},\overline{V}_c\right\rangle \omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^a}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^c}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad (W4a2)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_{i_r}}V_{j_s},V_d\right\rangle \left\langle \nabla_{V_a}\overline{V}_d,\overline{V}_c\right\rangle \omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^a}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^c}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad (W4b)$$

$$-2V_j\left(\alpha^{IJ}\right)\left\langle \nabla_{V_a}\overline{V}_{i_r},\overline{V}_j\right\rangle \omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^a}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad (W5a1)$$

$$+2V_{j_s}\left(\alpha^{IJ}\right)\left\langle \nabla_{V_a}\overline{V}_{i_r},\overline{V}_c\right\rangle \omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^a}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^c}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad (W5a2)$$

$$-2\alpha^{IJ}\left\langle \nabla_{V_j}\overline{V}_{i_r},V_c\right\rangle \left\langle \nabla_{V_a}\overline{V}_{i_t},\overline{V}_j\right\rangle \omega^{i_1}\wedge\cdots\wedge \underset{t-th\text{ place}}{\omega^a}$$

$$\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^c}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W5b1a)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_{j_s}}\overline{V}_{i_r},V_c\right\rangle \left\langle \nabla_{V_a}\overline{V}_{i_t},\overline{V}_b\right\rangle \omega^{i_1}\wedge\cdots\wedge \underset{t-th\text{ place}}{\omega^a}\wedge\cdots$$

$$\wedge \underset{r-th\text{ place}}{\omega^c}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W5b1b)$$

$$-2\alpha^{IJ}\left\langle \nabla_{V_j}\overline{V}_{i_r},V_c\right\rangle \left\langle \nabla_{V_a}\overline{V}_c,\overline{V}_j\right\rangle \omega^{i_1}\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^a}$$

$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W5b2a)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_{j_s}}\overline{V}_{i_r},V_c\right\rangle \left\langle \nabla_{V_a}\overline{V}_c,\overline{V}_b\right\rangle \omega^{i_1}\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^a}$$

$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W5b2b)$$

$$-2\alpha^{IJ}\left\langle \nabla_{V_j}V_{j_s},\overline{V}_b\right\rangle \left\langle \nabla_{V_a}\overline{V}_{i_r},\overline{V}_j\right\rangle \omega^{i_1}\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^a}$$

$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W5c1)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_{j_u}}V_{j_s},\overline{V}_b\right\rangle \left\langle \nabla_{V_a}\overline{V}_{i_r},\overline{V}_c\right\rangle \omega^{i_1}\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^a}\wedge\cdots$$

$$\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\overline{\omega}^c}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W5c2)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_j}V_{j_s},\overline{V}_j\right\rangle \left\langle \nabla_{V_a}\overline{V}_{i_r},\overline{V}_c\right\rangle \omega^{i_1}\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^a}$$

$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^c}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W5c3)$$

$$+2V_a\left(\alpha^{IJ}\right)\left\langle \nabla_{V_j}\overline{V}_{i_r},\overline{V}_a\right\rangle \omega^{i_1}\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^j}$$

$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W6a1)$$

$$-2V_{j_s}\left(\alpha^{IJ}\right)\left\langle \nabla_{V_j}\overline{V}_{i_r},\overline{V}_b\right\rangle \omega^{i_1}\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^j}$$

$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W6a2)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_a}\nabla_{V_j}\overline{V}_{i_r},\overline{V}_a\right\rangle \omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^j}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W6b1)$$

$$-2\alpha^{IJ}\left\langle \nabla_{V_{j_s}}\nabla_{V_j}\overline{V}_{i_r},\overline{V}_b\right\rangle \omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^j}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W6b2)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_j}\overline{V}_{i_r},\overline{V}_c\right\rangle\left\langle \nabla_{V_a}\overline{V}_a,V_c\right\rangle \omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^j}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W6c1a)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_j}\overline{V}_{i_r},V_c\right\rangle\left\langle \nabla_{V_a}\overline{V}_a,\overline{V}_c\right\rangle \omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^j}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W6c1b)$$

$$-2\alpha^{IJ}\left\langle \nabla_{V_j}\overline{V}_{i_r},\overline{V}_c\right\rangle\left\langle \nabla_{V_{j_s}}\overline{V}_b,V_c\right\rangle \omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^j}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W6c2a)$$

$$-2\alpha^{IJ}\left\langle \nabla_{V_j}\overline{V}_{i_r},V_c\right\rangle\left\langle \nabla_{V_{j_s}}\overline{V}_b,\overline{V}_c\right\rangle \omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^j}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W6c2b)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_j}\overline{V}_{i_r},\overline{V}_a\right\rangle\left\langle \nabla_{V_a}\overline{V}_{i_t},V_c\right\rangle \omega^{i_1}\wedge\cdots\wedge\underset{t-th\text{ place}}{\omega^c}$$
$$\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^j}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W6d1)$$

$$-2\alpha^{IJ}\left\langle \nabla_{V_j}\overline{V}_{i_r},\overline{V}_b\right\rangle\left\langle \nabla_{V_{j_s}}\overline{V}_{i_t},V_c\right\rangle \omega^{i_1}\wedge\cdots\wedge\underset{t-th\text{ place}}{\omega^c}\wedge\cdots$$
$$\wedge\underset{r-th\text{ place}}{\omega^j}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W6d2)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_j}\overline{V}_{i_r},\overline{V}_b\right\rangle\left\langle \nabla_{V_a}V_b,\overline{V}_a\right\rangle \omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^j}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W6e1)$$

$$-2\alpha^{IJ}\left\langle \nabla_{V_j}\overline{V}_{i_r},\overline{V}_b\right\rangle \left\langle \nabla_{V_{j_s}}V_b,\overline{V}_c\right\rangle \omega^{i_1}\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^j}$$

$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^c}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad (W6e2)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_j}\overline{V}_{i_r},\overline{V}_a\right\rangle \left\langle \nabla_{V_a}V_{j_s},\overline{V}_c\right\rangle \omega^{i_1}\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^j}$$

$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^c}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad (W6f1)$$

$$-2\alpha^{IJ}\left\langle \nabla_{V_j}\overline{V}_{i_r},\overline{V}_b\right\rangle \left\langle \nabla_{V_a}V_{j_s},\overline{V}_a\right\rangle \omega^{i_1}\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^j}$$

$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad (W6f2)$$

$$-2\alpha^{IJ}\left\langle \nabla_{V_j}\overline{V}_{i_r},\overline{V}_b\right\rangle \left\langle \nabla_{V_{j_u}}V_{j_s},\overline{V}_c\right\rangle \omega^{i_1}\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^j}$$

$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^c}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W6f3)$$

$$-2\overline{V}_a\left[V_{j_s}\left(\alpha^{IJ}\right)\right]\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$

$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad (W7a1)$$

$$-2V_{j_s}\left(\alpha^{IJ}\right)\left\langle \nabla_{\overline{V}_a}\overline{V}_{i_r},V_c\right\rangle \omega^{i_1}\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^c}$$

$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad (W7a2)$$

$$-2V_{j_u}\left(\alpha^{IJ}\right)\left\langle \nabla_{\overline{V}_a}V_{j_s},\overline{V}_c\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\overline{\omega}^a}$$

$$\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^c}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad (W7a3a)$$

$$-2V_j\left(\alpha^{IJ}\right)\left\langle \nabla_{\overline{V}_a}V_{j_s},\overline{V}_j\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$

$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad (W7a3b)$$

$$-2\overline{V}_a\left(\alpha^{IJ}\right)\left\langle\nabla_{V_{j_s}}\overline{V}_{i_r},V_b\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^b}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W7b1)$$

$$-2\alpha^{IJ}\left\langle\nabla_{\overline{V}_a}\nabla_{V_{j_s}}\overline{V}_{i_r},V_b\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^b}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W7b2)$$

$$-2\alpha^{IJ}\left\langle\nabla_{V_{j_s}}\overline{V}_{i_r},\overline{V}_c\right\rangle\left\langle\nabla_{\overline{V}_a}V_b,V_c\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^b}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W7b3a)$$

$$-2\alpha^{IJ}\left\langle\nabla_{V_{j_s}}\overline{V}_{i_r},V_c\right\rangle\left\langle\nabla_{\overline{V}_a}V_b,\overline{V}_c\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^b}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W7b3b)$$

$$-2\alpha^{IJ}\left\langle\nabla_{V_{j_s}}\overline{V}_{i_r},V_b\right\rangle\left\langle\nabla_{\overline{V}_a}\overline{V}_b,V_c\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^c}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W7b4)$$

$$-2\alpha^{IJ}\left\langle\nabla_{V_{j_s}}\overline{V}_{i_r},V_b\right\rangle\left\langle\nabla_{\overline{V}_a}\overline{V}_{i_t},V_c\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^b}\wedge\cdots$$
$$\wedge\underset{t-th\text{ place}}{\omega^c}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W7b5)$$

$$-2\alpha^{IJ}\left\langle\nabla_{V_{j_u}}\overline{V}_{i_r},V_b\right\rangle\left\langle\nabla_{\overline{V}_a}V_{j_s},\overline{V}_c\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^b}\wedge\cdots$$
$$\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^c}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W7b6a)$$

$$-2\alpha^{IJ}\left\langle\nabla_{V_j}\overline{V}_{i_r},V_b\right\rangle\left\langle\nabla_{\overline{V}_a}V_{j_s},\overline{V}_j\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^b}\wedge\cdots$$
$$\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W7b6b)$$

$$-2\overline{V}_a\left(\alpha^{IJ}\right)\left\langle\nabla_{V_{j_u}}V_{j_s},\overline{V}_b\right\rangle\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots$$
$$\wedge\underbrace{\overline{\omega}^a}_{u-th\text{ place}}\wedge\cdots\wedge\underbrace{\overline{\omega}^b}_{s-th\text{ place}}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W7c1a)$$

$$-2\overline{V}_a\left(\alpha^{IJ}\right)\left\langle\nabla_{V_j}V_{j_s},\overline{V}_j\right\rangle\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underbrace{\overline{\omega}^a}_{s-th\text{ place}}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W7c1b)$$

$$-2\alpha^{IJ}\left\langle\nabla_{\overline{V}_a}\nabla_{V_{j_u}}V_{j_s},\overline{V}_b\right\rangle\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underbrace{\overline{\omega}^a}_{u-th\text{ place}}$$
$$\wedge\cdots\wedge\underbrace{\overline{\omega}^b}_{s-th\text{ place}}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W7c2a)$$

$$-2\alpha^{IJ}\left\langle\nabla_{\overline{V}_a}\nabla_{V_j}V_{j_s},\overline{V}_j\right\rangle\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}$$
$$\wedge\cdots\wedge\underbrace{\overline{\omega}^a}_{s-th\text{ place}}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W7c2b)$$

$$-2\alpha^{IJ}\left\langle\nabla_{V_{j_u}}V_{j_s},\overline{V}_c\right\rangle\left\langle\nabla_{\overline{V}_a}\overline{V}_b,V_c\right\rangle\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underbrace{\overline{\omega}^a}_{u-th\text{ place}}$$
$$\wedge\cdots\wedge\underbrace{\overline{\omega}^b}_{s-th\text{ place}}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W7c3a)$$

$$-2\alpha^{IJ}\left\langle\nabla_{V_j}V_{j_s},\overline{V}_c\right\rangle\left\langle\nabla_{\overline{V}_a}\overline{V}_j,V_c\right\rangle\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}$$
$$\wedge\cdots\wedge\underbrace{\overline{\omega}^a}_{s-th\text{ place}}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W7c3b)$$

$$-2\alpha^{IJ}\left\langle\nabla_{V_{j_u}}V_{j_s},\overline{V}_b\right\rangle\left\langle\nabla_{\overline{V}_a}\overline{V}_{i_r},V_c\right\rangle\omega^{i_1}\wedge\cdots\wedge\underbrace{\omega^c}_{r-th\text{ place}}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underbrace{\overline{\omega}^a}_{u-th\text{ place}}\wedge..\wedge\underbrace{\overline{\omega}^b}_{s-th\text{ place}}\wedge..\wedge\overline{\omega}^{j_q}\quad(W7c4a)$$

$$-2\alpha^{IJ}\left\langle\nabla_{V_j}V_{j_s},\overline{V}_j\right\rangle\left\langle\nabla_{\overline{V}_a}\overline{V}_{i_r},V_c\right\rangle\omega^{i_1}\wedge\cdots\wedge\underbrace{\omega^c}_{r-th\text{ place}}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underbrace{\overline{\omega}^a}_{s-th\text{ place}}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W7c4b)$$

$$-2\alpha^{IJ}\left\langle \nabla_{V_{j_x}} V_{j_s}, \overline{V}_b \right\rangle \left\langle \nabla_{\overline{V}_a} V_{j_u}, \overline{V}_c \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{x-th \text{ place}}{\overline{\omega}^a}$$

$$\wedge .. \wedge \underset{u-th \text{ place}}{\overline{\omega}^c} \wedge .. \wedge \underset{s-th \text{ place}}{\overline{\omega}^b} \wedge .. \wedge \overline{\omega}^{j_q} \quad (W7c5a)$$

$$-2\alpha^{IJ}\left\langle \nabla_{V_j} V_{j_s}, \overline{V}_b \right\rangle \left\langle \nabla_{\overline{V}_a} V_{j_u}, \overline{V}_j \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1}$$

$$\wedge \cdots \wedge \underset{u-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^b} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W7c5b)$$

$$-2\alpha^{IJ}\left\langle \nabla_{V_j} V_{j_s}, \overline{V}_j \right\rangle \left\langle \nabla_{\overline{V}_a} V_{j_u}, \overline{V}_c \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1}$$

$$\wedge \cdots \wedge \underset{u-th \text{ place}}{\overline{\omega}^c} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W7c5c)$$

$$-2\alpha^{IJ}\left\langle \nabla_{V_{j_u}} V_{j_s}, \overline{V}_b \right\rangle \left\langle \nabla_{\overline{V}_a} V_b, \overline{V}_c \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1}$$

$$\wedge \cdots \wedge \underset{u-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^c} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W7c6a)$$

$$-2\alpha^{IJ}\left\langle \nabla_{V_j} V_{j_s}, \overline{V}_b \right\rangle \left\langle \nabla_{\overline{V}_a} V_b, \overline{V}_j \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1}$$

$$\wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W7c6b)$$

$$+2V_{j_s}\left[\overline{V}_j\left(\alpha^{IJ}\right)\right] \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1}$$

$$\wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^j} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W8a1)$$

$$+2\overline{V}_j\left(\alpha^{IJ}\right)\left\langle \nabla_{V_{j_s}} \overline{V}_{i_r}, V_b \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^b} \wedge \cdots \wedge \omega^{i_p}$$

$$\wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^j} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W8a2)$$

$$+2\overline{V}_j\left(\alpha^{IJ}\right)\left\langle \nabla_{V_{j_u}} V_{j_s}, \overline{V}_b \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{u-th \text{ place}}{\overline{\omega}^j}$$

$$\wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^b} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W8a3a)$$

$$+2\overline{V}_j\left(\alpha^{IJ}\right)\left\langle \nabla_{V_a} V_{j_s}, \overline{V}_a \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1}$$

$$\wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^j} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W8a3b)$$

$$+2V_{j_s}\left(\alpha^{IJ}\right)\left\langle \nabla_{\overline{V}_j}\overline{V}_{i_r},V_c\right\rangle \omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^c}\wedge\cdots$$
$$\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^j}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad (W8b1)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_{j_s}}\nabla_{\overline{V}_j}\overline{V}_{i_r},V_c\right\rangle \omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^c}\wedge\cdots$$
$$\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^j}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad (W8b2)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}\overline{V}_{i_r},V_a\right\rangle \left\langle \nabla_{V_{j_s}}V_c,\overline{V}_a\right\rangle \omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^c}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^j}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad (W8b3)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}\overline{V}_{i_r},V_c\right\rangle \left\langle \nabla_{V_{j_s}}\overline{V}_{i_t},V_b\right\rangle \omega^{i_1}\wedge\cdots\wedge\underset{t-th\text{ place}}{\omega^b}\wedge\cdots$$
$$\wedge\underset{r-th\text{ place}}{\omega^c}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^j}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad (W8b4)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}\overline{V}_{i_r},V_c\right\rangle \left\langle \nabla_{V_{j_s}}\overline{V}_c,V_b\right\rangle \omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^b}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^j}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad (W8b5)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}\overline{V}_{i_r},V_c\right\rangle \left\langle \nabla_{V_{j_u}}V_{j_s},\overline{V}_b\right\rangle \omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^c}\wedge\cdots$$
$$\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\overline{\omega}^j}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad (W8b6a)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}\overline{V}_{i_r},V_c\right\rangle \left\langle \nabla_{V_a}V_{j_s},\overline{V}_a\right\rangle \omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^c}\wedge\cdots$$
$$\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^j}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad (W8b6b)$$

$$+2V_{j_u}\left(\alpha^{IJ}\right)\left\langle \nabla_{\overline{V}_j}V_{j_s},\overline{V}_d\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\overline{\omega}^j}$$
$$\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^d}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad (W8c1a)$$

$$+2V_a\left(\alpha^{IJ}\right)\left\langle \nabla_{\overline{V}_j}V_{j_s},\overline{V}_a\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^j}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad (W8c1b)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_{j_u}}\nabla_{\overline{V}_j}V_{j_s},\overline{V}_d\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\overline{\omega}^{j}}$$
$$\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{d}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W8c2a)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_a}\nabla_{\overline{V}_j}V_{j_s},\overline{V}_a\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{j}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W8c2b)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}V_{j_s},\overline{V}_b\right\rangle \left\langle \nabla_{V_{j_u}}\overline{V}_d,V_b\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\overline{\omega}^{j}}$$
$$\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{d}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W8c3a1)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}V_{j_s},V_b\right\rangle \left\langle \nabla_{V_{j_u}}\overline{V}_d,\overline{V}_b\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\overline{\omega}^{j}}$$
$$\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{d}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W8c3a2)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}V_{j_s},\overline{V}_b\right\rangle \left\langle \nabla_{V_a}\overline{V}_a,V_b\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{j}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W8c3b1)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}V_{j_s},V_b\right\rangle \left\langle \nabla_{V_a}\overline{V}_a,\overline{V}_b\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{j}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W8c3b2)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}V_{j_s},\overline{V}_d\right\rangle \left\langle \nabla_{V_{j_u}}\overline{V}_{i_r},V_b\right\rangle \omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^{b}}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\overline{\omega}^{j}}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{d}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W8c4a)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}V_{j_s},\overline{V}_a\right\rangle \left\langle \nabla_{V_a}\overline{V}_{i_r},V_b\right\rangle \omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^{b}}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{j}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W8c4b)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}V_{j_s},\overline{V}_d\right\rangle \left\langle \nabla_{V_{j_x}}V_{j_u},\overline{V}_b\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{x-th\text{ place}}{\overline{\omega}^{j}}$$
$$\wedge\cdots\wedge\underset{u-th\text{ place}}{\overline{\omega}^{b}}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{d}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W8c5a)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j} V_{j_s}, \overline{V}_d \right\rangle \left\langle \nabla_{V_a} V_{j_u}, \overline{V}_a \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$$
$$\wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{u-th \text{ place}}{\overline{\omega}^{j}} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^{d}} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W8c5b)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j} V_{j_s}, \overline{V}_a \right\rangle \left\langle \nabla_{V_a} V_{j_u}, \overline{V}_b \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots$$
$$\wedge \underset{u-th \text{ place}}{\overline{\omega}^{b}} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^{j}} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W8c5c)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j} V_{j_s}, \overline{V}_d \right\rangle \left\langle \nabla_{V_{j_u}} V_d, \overline{V}_b \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$$
$$\wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{u-th \text{ place}}{\overline{\omega}^{j}} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^{b}} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W8c6a)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j} V_{j_s}, \overline{V}_d \right\rangle \left\langle \nabla_{V_a} V_d, \overline{V}_a \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1}$$
$$\wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^{j}} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W8c6b)$$

$$-2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j} V_{j_s}, V_d \right\rangle \left\langle \nabla_{V_j} \overline{V}_{i_r}, \overline{V}_b \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^{d}}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^{b}} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W9a)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j} V_{j_s}, V_d \right\rangle \left\langle \nabla_{V_j} \overline{V}_d, \overline{V}_b \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$$
$$\wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^{b}} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W9b)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_j} \overline{V}_{i_r}, \overline{V}_d \right\rangle \left\langle \nabla_{\overline{V}_j} V_d, V_c \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^{c}}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W10a)$$

$$-2\alpha^{IJ}\left\langle \nabla_{V_j} \overline{V}_{i_r}, \overline{V}_d \right\rangle \left\langle \nabla_{\overline{V}_j} V_{j_s}, V_c \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^{c}}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^{d}} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W10b)$$

Here we have

$$-2\overline{V}_a\left[V_{j_s}\left(\alpha^{IJ}\right)\right]\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{a}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W7a1)$$

$$=\quad-2\overline{V}_b\big(\alpha^{IJ}\big)\left\langle\nabla_{\overline{V}_a}V_{j_s},V_b\right\rangle\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{a}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W7a1a)$$

$$-2V_b\big(\alpha^{IJ}\big)\left\langle\nabla_{\overline{V}_a}V_{j_s},\overline{V}_b\right\rangle\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{a}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W7a1b)$$

$$-2\alpha^{IJ}\left\langle\nabla_{\overline{V}_a}\nabla_{V_{j_s}}\overline{V}_{i_r},V_b\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^{b}}\wedge\cdots$$
$$\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{a}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W7b2)$$

$$+2\alpha^{IJ}\left\langle\nabla_{V_{j_s}}\nabla_{\overline{V}_j}\overline{V}_{i_r},V_c\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^{c}}\wedge\cdots$$
$$\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{j}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W8b2)$$

$$=\quad+2\alpha^{IJ}\left\langle R_{\overline{V}_aV_{j_s}}\overline{V}_{i_r},V_b\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^{b}}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{a}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W7b2+8b2-a)$$

$$-2\alpha^{IJ}\left\langle\nabla_{V_c}\overline{V}_{i_r},V_b\right\rangle\left\langle\nabla_{\overline{V}_a}V_{j_s},\overline{V}_c\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^{b}}\wedge\cdots$$
$$\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{a}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W7b2+8b2-b)$$

$$-2\alpha^{IJ}\left\langle\nabla_{\overline{V}_c}\overline{V}_{i_r},V_b\right\rangle\left\langle\nabla_{\overline{V}_a}V_{j_s},V_c\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^{b}}\wedge\cdots$$
$$\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{a}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W7b2+8b2-c)$$

$$+2\alpha^{IJ}\left\langle\nabla_{V_c}\overline{V}_{i_r},V_b\right\rangle\left\langle\nabla_{V_{j_s}}\overline{V}_a,\overline{V}_c\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^{b}}\wedge\cdots$$
$$\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{a}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W7b2+8b2-d)$$

$$+2\alpha^{IJ}\left\langle\nabla_{\overline{V}_c}\overline{V}_{i_r},V_b\right\rangle\left\langle\nabla_{V_{j_s}}\overline{V}_a,V_c\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^{b}}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{a}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W7b2+8b2-e)$$

$$-2\alpha^{IJ}\left\langle \nabla_{\overline{V}_a}\nabla_{V_{j_u}}V_{j_s},\overline{V}_b\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\overline{\omega}^a}$$
$$\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q} \qquad (W7c2a)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_{j_u}}\nabla_{\overline{V}_j}V_{j_s},\overline{V}_d\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots$$
$$\wedge\underset{u-th\text{ place}}{\overline{\omega}^j}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^d}\wedge\cdots\wedge\overline{\omega}^{j_q} \qquad (W8c2a)$$

$$= +2\alpha^{IJ}\left\langle R_{\overline{V}_a V_{j_u}}V_{j_s},\overline{V}_b\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\overline{\omega}^a}$$
$$\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q} \qquad (W7c2a+8c2a-a)$$

$$-2\alpha^{IJ}\left\langle \nabla_{V_c}V_{j_s},\overline{V}_b\right\rangle\left\langle \nabla_{\overline{V}_a}V_{j_u},\overline{V}_c\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots$$
$$\wedge\underset{u-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q} \qquad (W7c2a+8c2a-b)$$

$$-2\alpha^{IJ}\left\langle \nabla_{\overline{V}_c}V_{j_s},\overline{V}_b\right\rangle\left\langle \nabla_{\overline{V}_a}V_{j_u},V_c\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots$$
$$\wedge\underset{u-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q} \qquad (W7c2a+8c2a-c)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_c}V_{j_s},\overline{V}_b\right\rangle\left\langle \nabla_{V_{j_u}}\overline{V}_a,\overline{V}_c\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots$$
$$\wedge\underset{u-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q} \qquad (W7c2a+8c2a-d)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_c}V_{j_s},\overline{V}_b\right\rangle\left\langle \nabla_{V_{j_u}}\overline{V}_a,V_c\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots$$
$$\wedge\underset{u-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q} \qquad (W7c2a+8c2a-e)$$

$$-2\alpha^{IJ}\left\langle \nabla_{\overline{V}_a}\nabla_{V_j}V_{j_s},\overline{V}_j\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{a}}\wedge\cdots\wedge\overline{\omega}^{j_q} \qquad (W7c2b)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_a}\nabla_{\overline{V}_j}V_{j_s},\overline{V}_a\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{j}}\wedge\cdots\wedge\overline{\omega}^{j_q} \qquad (W8c2b)$$

$$= +2\alpha^{IJ}\left\langle R_{\overline{V}_a V_j}V_{j_s},\overline{V}_j\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{a}}\wedge\cdots\wedge\overline{\omega}^{j_q} \qquad (W7c2b+8c2b-a)$$

$$-2\alpha^{IJ}\left\langle \nabla_{\overline{V}_b}V_{j_s},\overline{V}_j\right\rangle\left\langle \nabla_{\overline{V}_a}V_j,V_b\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{a}}\wedge\cdots\wedge\overline{\omega}^{j_q} \qquad (W7c2b+8c2b-b)$$

$$-2\alpha^{IJ}\left\langle \nabla_{V_b}V_{j_s},\overline{V}_j\right\rangle\left\langle \nabla_{\overline{V}_a}V_j,\overline{V}_b\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{a}}\wedge\cdots\wedge\overline{\omega}^{j_q} \qquad (W7c2b+8c2b-c)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_b}V_{j_s},\overline{V}_j\right\rangle\left\langle \nabla_{V_j}\overline{V}_a,V_b\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{a}}\wedge\cdots\wedge\overline{\omega}^{j_q} \qquad (W7c2b+8c2b-d)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_b}V_{j_s},\overline{V}_j\right\rangle\left\langle \nabla_{V_j}\overline{V}_a,\overline{V}_b\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{a}}\wedge\cdots\wedge\overline{\omega}^{j_q} \qquad (W7c2b+8c2b-e)$$

$$+2V_{j_s}\left[\overline{V}_j\left(\alpha^{IJ}\right)\right]\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{j}}\wedge\cdots\wedge\overline{\omega}^{j_q} \qquad (W8a1)$$

$$= +2\overline{V}_a\left(\alpha^{IJ}\right)\left\langle \nabla_{V_{j_s}}\overline{V}_j,V_a\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{j}}\wedge\cdots\wedge\overline{\omega}^{j_q} \qquad (W8a1a)$$

$$+2V_a\left(\alpha^{IJ}\right)\left\langle \nabla_{V_{j_s}}\overline{V}_j,\overline{V}_a\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{j}}\wedge\cdots\wedge\overline{\omega}^{j_q} \qquad (W8a1b)$$

Next we simplify the terms from $(W11)$ *to* $(W18)$. *If we take the covariant derivatives, we would get*

$$
\begin{aligned}
& (W11) \text{ through } (W18) \\
= \; & +2\left\langle \nabla_{\overline{V}_j} V_b, V_c \right\rangle \omega^b \wedge \iota_{\overline{V}_c}\left(\alpha^{IJ}\left\langle \nabla_{V_j}\omega^{i_r}, \omega^a\right\rangle \omega^{i_1} \wedge \cdots \right. \\
& \left. \wedge \underset{r-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q}\right) \quad (W11) \\
& +2\left\langle \nabla_{V_a} \overline{V}_j, V_b \right\rangle \omega^b \wedge \iota_{\overline{V}_a}\left(\alpha^{IJ}\left\langle \nabla_{V_j}\omega^{i_r}, \omega^d\right\rangle \omega^{i_1} \wedge \cdots \right. \\
& \left. \wedge \underset{r-th \text{ place}}{\overline{\omega}^d} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q}\right) \quad (W12) \\
& +2\left\langle \nabla_{\overline{V}_a} V_j, \overline{V}_b \right\rangle \overline{\omega}^a \wedge \iota_{V_b}\left(\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}\overline{\omega}^{j_s}, \overline{\omega}^d\right\rangle \omega^{i_1} \wedge \cdots \right. \\
& \left. \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\omega^d} \wedge \cdots \wedge \overline{\omega}^{j_q}\right) \quad (W13) \\
& -2\left\langle \nabla_{V_j} \overline{V}_i, \overline{V}_k \right\rangle \overline{\omega}^k \wedge \iota_{V_i}\left(\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}\overline{\omega}^{j_s}, \overline{\omega}^d\right\rangle \omega^{i_1} \wedge \cdots \right. \\
& \left. \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\omega^d} \wedge \cdots \wedge \overline{\omega}^{j_q}\right) \quad (W14) \\
& -2\left\langle \nabla_{\overline{V}_a} \overline{V}_j, V_b \right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_b}\left(V_j\left(\alpha^{IJ}\right) \omega^{i_1} \wedge \cdots \right. \\
& \left. \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q}\right) \quad (W15a) \\
& -2\left\langle \nabla_{\overline{V}_a} \overline{V}_j, V_b \right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_b}\left(\alpha^{IJ}\left\langle \nabla_{V_j}\omega^{i_r}, \overline{\omega}^d\right\rangle \omega^{i_1} \wedge \cdots \right. \\
& \left. \wedge \underset{r-th \text{ place}}{\omega^d} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q}\right) \quad (W15b) \\
& -2\left\langle \nabla_{\overline{V}_a} \overline{V}_j, V_b \right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_b}\left(\alpha^{IJ}\left\langle \nabla_{V_j}\overline{\omega}^{j_s}, \omega^b\right\rangle \omega^{i_1} \wedge \cdots \right. \\
& \left. \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^b} \wedge \cdots \wedge \overline{\omega}^{j_q}\right) \quad (W15c) \\
& +2\left\langle \nabla_{V_j} \overline{V}_a, \overline{V}_k \right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_j}\left(V_a\left(\alpha^{IJ}\right) \omega^{i_1} \wedge \cdots \right. \\
& \left. \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q}\right) \quad (W16a)
\end{aligned}
$$

$$+2\left\langle \nabla_{V_j}\overline{V}_a, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_j}\left(\alpha^{IJ}\left\langle \nabla_{V_a}\omega^{i_r}, \overline{\omega}^d\right\rangle \omega^{i_1} \wedge \cdots\right.$$

$$\left.\wedge \underset{r-th \text{ place}}{\omega^d} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q}\right) \quad (W16b)$$

$$+2\left\langle \nabla_{V_j}\overline{V}_a, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_j}\left(\alpha^{IJ}\left\langle \nabla_{V_a}\overline{\omega}^{j_s}, \omega^b\right\rangle \omega^{i_1} \wedge \cdots\right.$$

$$\left.\wedge\omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^b} \wedge \cdots \wedge \overline{\omega}^{j_q}\right) \quad (W16c)$$

$$+2\left\langle \nabla_{V_a}V_j, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}\left(\overline{V}_j\left(\alpha^{IJ}\right)\omega^{i_1} \wedge \cdots\right.$$

$$\left.\wedge\omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q}\right) \quad (W17a)$$

$$+2\left\langle \nabla_{V_a}V_j, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}\left(\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}\omega^{i_r}, \overline{\omega}^c\right\rangle \omega^{i_1} \wedge \cdots\right.$$

$$\left.\wedge \underset{r-th \text{ place}}{\omega^c} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q}\right) \quad (W17b)$$

$$+2\left\langle \nabla_{V_a}V_j, \overline{V}_k\right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}\left(\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}\overline{\omega}^{j_s}, \omega^d\right\rangle \omega^{i_1} \wedge \cdots\right.$$

$$\left.\wedge\omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^d} \wedge \cdots \wedge \overline{\omega}^{j_q}\right) \quad (W17c)$$

$$+2\left\langle \nabla_{\overline{V}_a}V_b, V_j\right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_b}\left(\overline{V}_j\left(\alpha^{IJ}\right)\omega^{i_1} \wedge \cdots\right.$$

$$\left.\wedge\omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q}\right) \quad (W18a)$$

$$+2\left\langle \nabla_{\overline{V}_a}V_b, V_j\right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_b}\left(\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}\omega^{i_r}, \overline{\omega}^c\right\rangle \omega^{i_1} \wedge \cdots\right.$$

$$\left.\wedge \underset{r-th \text{ place}}{\omega^c} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q}\right) \quad (W18b)$$

$$+2\left\langle \nabla_{\overline{V}_a}V_b, V_j\right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_b}\left(\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}\overline{\omega}^{j_s}, \omega^d\right\rangle \omega^{i_1} \wedge \cdots\right.$$

$$\left.\wedge\omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^d} \wedge \cdots \wedge \overline{\omega}^{j_q}\right) \quad (W18c)$$

Then we have

$$
\begin{aligned}
& (W11) \text{ through } (W18) \\
= \; & +2\left\langle \nabla_{\overline{V}_j} V_b, V_c \right\rangle \omega^b \wedge \iota_{\overline{V}_c}\left(\alpha^{IJ}\left\langle \nabla_{V_j}\omega^{i_r}, \omega^a\right\rangle \omega^{i_1} \wedge \cdots \right. \\
& \left. \wedge \underset{r-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q}\right) \quad (W11) \\
& +2\left\langle \nabla_{V_a}\overline{V}_j, V_b \right\rangle \omega^b \wedge \iota_{\overline{V}_a}\left(\alpha^{IJ}\left\langle \nabla_{V_j}\omega^{i_r}, \omega^a\right\rangle \omega^{i_1} \wedge \cdots \right. \\
& \left. \wedge \underset{r-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q}\right) \quad (W12) \\
& +2\left\langle \nabla_{\overline{V}_a} V_j, \overline{V}_b \right\rangle \overline{\omega}^a \wedge \iota_{V_b}\left(\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}\overline{\omega}^{j_s}, \overline{\omega}^d\right\rangle \omega^{i_1} \wedge \cdots \right. \\
& \left. \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\omega^d} \wedge \cdots \wedge \overline{\omega}^{j_q}\right) \quad (W13) \\
& -2\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k \right\rangle \overline{\omega}^k \wedge \iota_{V_i}\left(\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}\overline{\omega}^{j_s}, \overline{\omega}^d\right\rangle \omega^{i_1} \wedge \cdots \right. \\
& \left. \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\omega^d} \wedge \cdots \wedge \overline{\omega}^{j_q}\right) \quad (W14) \\
& -2\left\langle \nabla_{\overline{V}_a}\overline{V}_j, V_b \right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_b}\left(V_j\left(\alpha^{IJ}\right)\omega^{i_1} \wedge \cdots \right. \\
& \left. \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q}\right) \quad (W15a) \\
& -2\left\langle \nabla_{\overline{V}_a}\overline{V}_j, V_b \right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_b}\left(\alpha^{IJ}\left\langle \nabla_{V_j}\omega^{i_r}, \overline{\omega}^c\right\rangle \omega^{i_1} \wedge \cdots \right. \\
& \left. \wedge \underset{r-th \text{ place}}{\omega^c} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q}\right) \quad (W15b) \\
& -2\left\langle \nabla_{\overline{V}_a}\overline{V}_j, V_b \right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_b}\left(\alpha^{IJ}\left\langle \nabla_{V_j}\overline{\omega}^{j_s}, \omega^b\right\rangle \omega^{i_1} \wedge \cdots \right. \\
& \left. \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^b} \wedge \cdots \wedge \overline{\omega}^{j_q}\right) \quad (W15c) \\
& +2V_a\left(\alpha^{IJ}\right)\left\langle \nabla_{V_j}\overline{V}_a, \overline{V}_k \right\rangle \delta^j_{j_s}(-1)^{p+s-1}\overline{\omega}^k \wedge \omega^{i_1} \wedge \cdots \\
& \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\widehat{\overline{\omega}^{j_s}}} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W16a)
\end{aligned}
$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_j}\overline{V}_a, \overline{V}_k\right\rangle \left\langle \nabla_{V_a}\omega^{i_r}, \overline{\varpi}^d\right\rangle \delta^j_{j_s}\left(-1\right)^{p+s-1}\overline{\varpi}^k \wedge \omega^{i_1} \wedge \cdots$$

$$\wedge \underset{r-th \text{ place}}{\omega^d} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\varpi}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\widehat{\overline{\varpi}^{j_s}}} \wedge \cdots \wedge \overline{\varpi}^{j_q} \quad (W16b)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_j}\overline{V}_a, \overline{V}_k\right\rangle \left\langle \nabla_{V_a}\overline{\varpi}^{j_s}, \omega^b\right\rangle \delta^j_{j_u}\left(-1\right)^{p+u-1}\overline{\varpi}^k \wedge \omega^{i_1} \wedge \cdots$$

$$\wedge \omega^{i_p} \wedge \overline{\varpi}^{j_1} \wedge \cdots \wedge \underset{u-th \text{ place}}{\widehat{\overline{\varpi}^{j_u}}} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\varpi}^b} \wedge \cdots \wedge \overline{\varpi}^{j_q} \quad (W16c1)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_j}\overline{V}_a, \overline{V}_k\right\rangle \left\langle \nabla_{V_a}\overline{\varpi}^{j_s}, \omega^b\right\rangle \delta^j_b\left(-1\right)^{p+s-1}\overline{\varpi}^k \wedge \omega^{i_1} \wedge \cdots$$

$$\wedge \omega^{i_p} \wedge \overline{\varpi}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\widehat{\overline{\varpi}^b}} \wedge \cdots \wedge \overline{\varpi}^{j_q} \quad (W16c2)$$

$$+2\left\langle \nabla_{V_a}V_j, \overline{V}_k\right\rangle \overline{\varpi}^k \wedge \iota_{\overline{V}_a}\left(\overline{V}_j\left(\alpha^{IJ}\right)\omega^{i_1} \wedge \cdots\right.$$

$$\left.\wedge \omega^{i_p} \wedge \overline{\varpi}^{j_1} \wedge \cdots \wedge \overline{\varpi}^{j_q}\right) \quad (W17a)$$

$$+2\left\langle \nabla_{V_a}V_j, \overline{V}_k\right\rangle \overline{\varpi}^k \wedge \iota_{\overline{V}_a}\left(\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}\omega^{i_r}, \overline{\varpi}^c\right\rangle \omega^{i_1} \wedge \cdots\right.$$

$$\left.\wedge \underset{r-th \text{ place}}{\omega^c} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\varpi}^{j_1} \wedge \cdots \wedge \overline{\varpi}^{j_q}\right) \quad (W17b)$$

$$+2\left\langle \nabla_{V_a}V_j, \overline{V}_k\right\rangle \overline{\varpi}^k \wedge \iota_{\overline{V}_a}\left(\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}\overline{\varpi}^{j_s}, \omega^d\right\rangle \omega^{i_1} \wedge \cdots\right.$$

$$\left.\wedge \omega^{i_p} \wedge \overline{\varpi}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\varpi}^d} \wedge \cdots \wedge \overline{\varpi}^{j_q}\right) \quad (W17c)$$

$$+2\left\langle \nabla_{\overline{V}_a}V_b, V_j\right\rangle \overline{\varpi}^a \wedge \iota_{\overline{V}_b}\left(\overline{V}_j\left(\alpha^{IJ}\right)\omega^{i_1} \wedge \cdots\right.$$

$$\left.\wedge \omega^{i_p} \wedge \overline{\varpi}^{j_1} \wedge \cdots \wedge \overline{\varpi}^{j_q}\right) \quad (W18a)$$

$$+2\left\langle \nabla_{\overline{V}_a}V_b, V_j\right\rangle \overline{\varpi}^a \wedge \iota_{\overline{V}_b}\left(\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}\omega^{i_r}, \overline{\varpi}^c\right\rangle \omega^{i_1} \wedge \cdots\right.$$

$$\left.\wedge \underset{r-th \text{ place}}{\omega^c} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\varpi}^{j_1} \wedge \cdots \wedge \overline{\varpi}^{j_q}\right) \quad (W18b)$$

$$+2\left\langle \nabla_{\overline{V}_a}V_b, V_j\right\rangle \overline{\varpi}^a \wedge \iota_{\overline{V}_b}\left(\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}\overline{\varpi}^{j_s}, \omega^d\right\rangle \omega^{i_1} \wedge \cdots\right.$$

$$\left.\wedge \omega^{i_p} \wedge \overline{\varpi}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\varpi}^d} \wedge \cdots \wedge \overline{\varpi}^{j_q}\right) \quad (W18c)$$

If we apply the interior products, we would get

$$
\begin{aligned}
& (W11) \text{ through } (W18) \\
= \; & +2\alpha^{IJ}(-1)^{r-1}\,\delta_a^c \left\langle \nabla_{\overline{V}_j} V_b, V_c \right\rangle \left\langle \nabla_{V_j} \overline{V}_{i_r}, \overline{V}_a \right\rangle \omega^b \wedge \omega^{i_1} \wedge \cdots \\
& \wedge \underbrace{\widehat{\overline{\omega}^a}}_{r-th \text{ place}} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W11a) \\
& +2\alpha^{IJ}(-1)^{p+s-1}\,\delta_{j_s}^c \left\langle \nabla_{\overline{V}_j} V_b, V_c \right\rangle \left\langle \nabla_{V_j} \overline{V}_{i_r}, \overline{V}_a \right\rangle \omega^b \wedge \omega^{i_1} \wedge \cdots \\
& \wedge \underbrace{\overline{\omega}^a}_{r-th \text{ place}} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underbrace{\widehat{\overline{\omega}^{j_s}}}_{s-th \text{ place}} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W11b) \\
& +2\alpha^{IJ}(-1)^{r-1}\,\delta_c^a \left\langle \nabla_{V_a} \overline{V}_j, V_b \right\rangle \left\langle \nabla_{V_j} \overline{V}_{i_r}, \overline{V}_c \right\rangle \omega^b \wedge \omega^{i_1} \wedge \cdots \\
& \wedge \underbrace{\widehat{\overline{\omega}^a}}_{r-th \text{ place}} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W12a) \\
& +2\alpha^{IJ}(-1)^{p+s-1}\,\delta_{j_s}^a \left\langle \nabla_{V_a} \overline{V}_j, V_b \right\rangle \left\langle \nabla_{V_j} \overline{V}_{i_r}, \overline{V}_c \right\rangle \omega^b \wedge \omega^{i_1} \wedge \cdots \\
& \wedge \underbrace{\overline{\omega}^c}_{r-th \text{ place}} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underbrace{\widehat{\overline{\omega}^{j_s}}}_{s-th \text{ place}} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W12b) \\
& +2\alpha^{IJ}(-1)^{r-1}\,\delta_{i_r}^b \left\langle \nabla_{\overline{V}_a} V_j, \overline{V}_b \right\rangle \left\langle \nabla_{\overline{V}_j} V_{j_s}, V_d \right\rangle \overline{\omega}^a \wedge \omega^{i_1} \wedge \cdots \wedge \underbrace{\widehat{\omega^{i_r}}}_{r-th \text{ place}} \\
& \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underbrace{\omega^d}_{s-th \text{ place}} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W13a) \\
& +2\alpha^{IJ}(-1)^{p+s-1}\,\delta_d^b \left\langle \nabla_{\overline{V}_a} V_j, \overline{V}_b \right\rangle \left\langle \nabla_{\overline{V}_j} V_{j_s}, V_d \right\rangle \overline{\omega}^a \wedge \omega^{i_1} \wedge \cdots \\
& \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underbrace{\widehat{\omega^d}}_{s-th \text{ place}} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W13b) \\
& -2\alpha^{IJ}(-1)^{r-1}\,\delta_{i_r}^i \left\langle \nabla_{V_j} \overline{V}_i, \overline{V}_k \right\rangle \left\langle \nabla_{\overline{V}_j} V_{j_s}, V_d \right\rangle \overline{\omega}^k \wedge \omega^{i_1} \wedge \cdots \wedge \underbrace{\widehat{\omega^{i_r}}}_{r-th \text{ place}} \\
& \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underbrace{\omega^d}_{s-th \text{ place}} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W14a) \\
& -2\alpha^{IJ}(-1)^{p+s-1}\,\delta_d^i \left\langle \nabla_{V_j} \overline{V}_i, \overline{V}_k \right\rangle \left\langle \nabla_{\overline{V}_j} V_{j_s}, V_d \right\rangle \overline{\omega}^k \wedge \omega^{i_1} \wedge \cdots \\
& \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underbrace{\widehat{\omega^d}}_{s-th \text{ place}} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W14b)
\end{aligned}
$$

$$-2\alpha^{IJ}\left(-1\right)^{p+s-1}\delta^b_{j_s}V_j\left(\alpha^{IJ}\right)\left\langle\nabla_{\overline{V}_a}\overline{V}_j,V_b\right\rangle\overline{\omega}^a\wedge\omega^{i_1}\wedge\cdots$$
$$\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underbrace{\widehat{\overline{\omega}^{j_s}}}_{s-th\text{ place}}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W15a)$$

$$-2\alpha^{IJ}\left(-1\right)^{p+s-1}\delta^b_{j_s}\left\langle\nabla_{\overline{V}_a}\overline{V}_j,V_b\right\rangle\left\langle\nabla_{V_j}\overline{V}_{i_r},V_c\right\rangle\overline{\omega}^a\wedge\omega^{i_1}\wedge\cdots\wedge\underbrace{\omega^c}_{r-th\text{ place}}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underbrace{\widehat{\overline{\omega}^{j_s}}}_{s-th\text{ place}}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W15b)$$

$$-2\alpha^{IJ}\left(-1\right)^{p+s-1}\delta^b_c\left\langle\nabla_{\overline{V}_a}\overline{V}_j,V_b\right\rangle\left\langle\nabla_{V_j}V_{j_s},\overline{V}_c\right\rangle\overline{\omega}^a\wedge\omega^{i_1}\wedge\cdots$$
$$\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underbrace{\widehat{\overline{\omega}^{c}}}_{s-th\text{ place}}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W15c1)$$

$$-2\alpha^{IJ}\left(-1\right)^{p+u-1}\delta^b_{j_u}\left\langle\nabla_{\overline{V}_a}\overline{V}_j,V_b\right\rangle\left\langle\nabla_{V_j}V_{j_s},\overline{V}_c\right\rangle\overline{\omega}^a\wedge\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underbrace{\widehat{\overline{\omega}^{j_u}}}_{u-th\text{ place}}\wedge\cdots\wedge\underbrace{\overline{\omega}^{c}}_{s-th\text{ place}}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W15c2)$$

$$+2V_a\left(\alpha^{IJ}\right)\left\langle\nabla_{V_{j_s}}\overline{V}_a,\overline{V}_k\right\rangle\left(-1\right)^{2(p+s-1)}\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underbrace{\overline{\omega}^{k}}_{s-th\text{ place}}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W16a)$$

$$+2\alpha^{IJ}\left\langle\nabla_{V_{j_s}}\overline{V}_a,\overline{V}_k\right\rangle\left\langle\nabla_{V_a}\omega^{i_r},\overline{\omega}^d\right\rangle\left(-1\right)^{2(p+s-1)}\omega^{i_1}\wedge\cdots\wedge\underbrace{\omega^d}_{r-th\text{ place}}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underbrace{\overline{\omega}^{k}}_{s-th\text{ place}}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W16b)$$

$$+2\alpha^{IJ}\left\langle\nabla_{V_{j_u}}\overline{V}_a,\overline{V}_k\right\rangle\left\langle\nabla_{V_a}\overline{\omega}^{j_s},\omega^b\right\rangle\left(-1\right)^{2(p+u-1)}\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underbrace{\overline{\omega}^{k}}_{u-th\text{ place}}\wedge\cdots\wedge\underbrace{\overline{\omega}^{b}}_{s-th\text{ place}}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W16c1)$$

$$+2\alpha^{IJ}\left\langle\nabla_{V_b}\overline{V}_a,\overline{V}_k\right\rangle\left\langle\nabla_{V_a}\overline{\omega}^{j_s},\omega^b\right\rangle\left(-1\right)^{2(p+s-1)}\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underbrace{\overline{\omega}^{k}}_{s-th\text{ place}}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W16c2)$$

$$+2\alpha^{IJ}\left(-1\right)^{p+s-1}\delta^{a}_{j_s}\overline{V}_j\left(\alpha^{IJ}\right)\left\langle\nabla_{V_a}V_j,\overline{V}_k\right\rangle\overline{\omega}^k\wedge\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\widehat{\overline{\omega}^{j_s}}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W17a)$$

$$+2\alpha^{IJ}\left(-1\right)^{p+s-1}\delta^{a}_{j_s}\left\langle\nabla_{V_a}V_j,\overline{V}_k\right\rangle\left\langle\nabla_{\overline{V}_j}\overline{V}_{i_r},V_c\right\rangle\overline{\omega}^k\wedge\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^c}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\widehat{\overline{\omega}^{j_s}}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W17b)$$

$$+2\alpha^{IJ}\left(-1\right)^{p+s-1}\delta^{a}_{d}\left\langle\nabla_{V_a}V_j,\overline{V}_k\right\rangle\left\langle\nabla_{\overline{V}_j}V_{j_s},\overline{V}_d\right\rangle\overline{\omega}^k\wedge\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\widehat{\overline{\omega}^{d}}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W17c1)$$

$$+2\alpha^{IJ}\left(-1\right)^{p+u-1}\delta^{a}_{j_u}\left\langle\nabla_{V_a}V_j,\overline{V}_k\right\rangle\left\langle\nabla_{\overline{V}_j}V_{j_s},\overline{V}_d\right\rangle\overline{\omega}^k\wedge\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\widehat{\overline{\omega}^{j_u}}}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{d}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W17c2)$$

$$+2\alpha^{IJ}\left(-1\right)^{p+s-1}\delta^{b}_{j_s}\overline{V}_j\left(\alpha^{IJ}\right)\left\langle\nabla_{\overline{V}_a}V_b,V_j\right\rangle\overline{\omega}^a\wedge\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\widehat{\overline{\omega}^{j_s}}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W18a)$$

$$+2\alpha^{IJ}\left(-1\right)^{p+s-1}\delta^{b}_{j_s}\left\langle\nabla_{\overline{V}_a}V_b,V_j\right\rangle\left\langle\nabla_{\overline{V}_j}\overline{V}_{i_r},V_c\right\rangle\overline{\omega}^a\wedge\omega^{i_1}\wedge\cdots$$
$$\wedge\underset{r-th\text{ place}}{\omega^c}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\widehat{\overline{\omega}^{j_s}}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W18b)$$

$$+2\alpha^{IJ}\left(-1\right)^{p+s-1}\delta^{b}_{d}\left\langle\nabla_{\overline{V}_a}V_b,V_j\right\rangle\left\langle\nabla_{\overline{V}_j}V_{j_s},\overline{V}_d\right\rangle\overline{\omega}^a\wedge\omega^{i_1}\wedge\cdots$$
$$\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\widehat{\overline{\omega}^{d}}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W18c1)$$

$$+2\alpha^{IJ}\left(-1\right)^{p+u-1}\delta^{b}_{j_u}\left\langle\nabla_{\overline{V}_a}V_b,V_j\right\rangle\left\langle\nabla_{\overline{V}_j}V_{j_s},\overline{V}_d\right\rangle\overline{\omega}^a\wedge\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\widehat{\overline{\omega}^{j_u}}}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{d}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W18c2)$$

If we carry the wedge products and apply the Kronecker deltas, this yields

$$
\begin{aligned}
& (W11) \text{ through } (W18) \\
= \; & +2\alpha^{IJ}(-1)^{2(r-1)}\left\langle \nabla_{\overline{V}_j} V_b, V_a \right\rangle \left\langle \nabla_{V_j} \overline{V}_{i_r}, \overline{V}_a \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^b} \\
& \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W11a) \\
& +2\alpha^{IJ}(-1)^{2(p+s+r)-1}\left\langle \nabla_{\overline{V}_j} V_b, V_{j_s} \right\rangle \left\langle \nabla_{V_j} \overline{V}_{i_r}, \overline{V}_a \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^b} \\
& \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W11b) \\
& +2\alpha^{IJ}(-1)^{2(r-1)}\left\langle \nabla_{V_c} \overline{V}_j, V_b \right\rangle \left\langle \nabla_{V_j} \overline{V}_{i_r}, \overline{V}_c \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^b} \\
& \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W12a) \\
& +2\alpha^{IJ}(-1)^{2(p+s+r)-1}\left\langle \nabla_{V_{j_s}} \overline{V}_j, V_b \right\rangle \left\langle \nabla_{V_j} \overline{V}_{i_r}, \overline{V}_c \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^b} \\
& \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^c} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W12b) \\
& +2\alpha^{IJ}(-1)^{2(p+r+s)-1}\left\langle \nabla_{\overline{V}_a} V_j, \overline{V}_{i_r} \right\rangle \left\langle \nabla_{\overline{V}_j} V_{j_s}, V_d \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^d} \\
& \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W13a) \\
& +2\alpha^{IJ}(-1)^{2(p+s-1)}\left\langle \nabla_{\overline{V}_a} V_j, \overline{V}_b \right\rangle \left\langle \nabla_{\overline{V}_j} V_{j_s}, V_b \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \\
& \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W13b) \\
& -2\alpha^{IJ}(-1)^{2(p+r+s)-1}\left\langle \nabla_{V_j} \overline{V}_{i_r}, \overline{V}_k \right\rangle \left\langle \nabla_{\overline{V}_j} V_{j_s}, V_d \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^d} \\
& \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^k} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W14a) \\
& -2\alpha^{IJ}(-1)^{2(p+s-1)}\left\langle \nabla_{V_j} \overline{V}_i, \overline{V}_k \right\rangle \left\langle \nabla_{\overline{V}_j} V_{j_s}, V_i \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \\
& \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^k} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W14b)
\end{aligned}
$$

$$-2\alpha^{IJ}(-1)^{2(p+s-1)}V_j\left(\alpha^{IJ}\right)\left\langle\nabla_{\overline{V}_a}\overline{V}_j,V_{j_s}\right\rangle\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{a}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W15a)$$

$$-2\alpha^{IJ}(-1)^{2(p+s-1)}\left\langle\nabla_{\overline{V}_a}\overline{V}_j,V_{j_s}\right\rangle\left\langle\nabla_{V_j}\overline{V}_{i_r},V_c\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^{c}}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{a}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W15b)$$

$$-2\alpha^{IJ}(-1)^{2(p+s-1)}\left\langle\nabla_{\overline{V}_a}\overline{V}_j,V_b\right\rangle\left\langle\nabla_{V_j}V_{j_s},\overline{V}_b\right\rangle\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{a}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W15c1)$$

$$-2\alpha^{IJ}(-1)^{2(p+u-1)}\left\langle\nabla_{\overline{V}_a}\overline{V}_j,V_{j_u}\right\rangle\left\langle\nabla_{V_j}V_{j_s},\overline{V}_c\right\rangle\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\overline{\omega}^{a}}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{c}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W15c2)$$

$$+2V_a\left(\alpha^{IJ}\right)\left\langle\nabla_{V_{j_s}}\overline{V}_a,\overline{V}_k\right\rangle\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{k}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W16a)$$

$$+2\alpha^{IJ}\left\langle\nabla_{V_{j_s}}\overline{V}_a,\overline{V}_k\right\rangle\left\langle\nabla_{V_a}\overline{V}_{i_r},V_d\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^{d}}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{k}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W16b)$$

$$+2\alpha^{IJ}\left\langle\nabla_{V_{j_u}}\overline{V}_a,\overline{V}_k\right\rangle\left\langle\nabla_{V_a}V_{j_s},\overline{V}_b\right\rangle\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\overline{\omega}^{k}}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{b}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W16c1)$$

$$+2\alpha^{IJ}\left\langle\nabla_{V_b}\overline{V}_a,\overline{V}_k\right\rangle\left\langle\nabla_{V_a}V_{j_s},\overline{V}_b\right\rangle\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{k}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W16c2)$$

$$+2\alpha^{IJ}(-1)^{2(p+s-1)}\overline{V}_j\left(\alpha^{IJ}\right)\left\langle\nabla_{V_{j_s}}V_j,\overline{V}_k\right\rangle\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$

$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^k}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W17a)$$

$$+2\alpha^{IJ}(-1)^{2(p+s-1)}\left\langle\nabla_{V_{j_s}}V_j,\overline{V}_k\right\rangle\left\langle\nabla_{\overline{V}_j}\overline{V}_{i_r},V_c\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^c}$$

$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^k}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W17b)$$

$$+2\alpha^{IJ}(-1)^{2(p+s-1)}\left\langle\nabla_{V_a}V_j,\overline{V}_k\right\rangle\left\langle\nabla_{\overline{V}_j}V_{j_s},\overline{V}_a\right\rangle\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$

$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^k}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W17c1)$$

$$+2\alpha^{IJ}(-1)^{2(p+u-1)}\left\langle\nabla_{V_{j_u}}V_j,\overline{V}_k\right\rangle\left\langle\nabla_{\overline{V}_j}V_{j_s},\overline{V}_d\right\rangle\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$

$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\overline{\omega}^k}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^d}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W17c2)$$

$$+2\alpha^{IJ}(-1)^{2(p+s-1)}\overline{V}_j\left(\alpha^{IJ}\right)\left\langle\nabla_{\overline{V}_a}V_{j_s},V_j\right\rangle\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$

$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W18a)$$

$$+2\alpha^{IJ}(-1)^{2(p+s-1)}\left\langle\nabla_{\overline{V}_a}V_{j_s},V_j\right\rangle\left\langle\nabla_{\overline{V}_j}\overline{V}_{i_r},V_c\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^c}$$

$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W18b)$$

$$+2\alpha^{IJ}(-1)^{2(p+s-1)}\left\langle\nabla_{\overline{V}_a}V_b,V_j\right\rangle\left\langle\nabla_{\overline{V}_j}V_{j_s},\overline{V}_b\right\rangle\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$

$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W18c1)$$

$$+2\alpha^{IJ}(-1)^{2(p+u-1)}\left\langle\nabla_{\overline{V}_a}V_{j_u},V_j\right\rangle\left\langle\nabla_{\overline{V}_j}V_{j_s},\overline{V}_d\right\rangle\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$

$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^d}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W18c2)$$

Finally if we simplify the signs, we get

$$\begin{aligned}&(W11)\ \text{through}\ (W18)\\=\ &+2\alpha^{IJ}\left\langle\nabla_{\overline{V}_j}V_b,V_a\right\rangle\left\langle\nabla_{V_j}\overline{V}_{i_r},\overline{V}_a\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^b}\\&\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W11a)\end{aligned}$$

$$-2\alpha^{IJ}\left\langle\nabla_{\overline{V}_j}V_b,V_{j_s}\right\rangle\left\langle\nabla_{V_j}\overline{V}_{i_r},\overline{V}_a\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^b}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W11b)$$

$$+2\alpha^{IJ}\left\langle\nabla_{V_c}\overline{V}_j,V_b\right\rangle\left\langle\nabla_{V_j}\overline{V}_{i_r},\overline{V}_c\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^b}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W12a)$$

$$-2\alpha^{IJ}\left\langle\nabla_{V_{j_s}}\overline{V}_j,V_b\right\rangle\left\langle\nabla_{V_j}\overline{V}_{i_r},\overline{V}_c\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^b}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^c}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W12b)$$

$$-2\alpha^{IJ}\left\langle\nabla_{\overline{V}_a}V_j,\overline{V}_{i_r}\right\rangle\left\langle\nabla_{\overline{V}_j}V_{j_s},V_d\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^d}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W13a)$$

$$+2\alpha^{IJ}\left\langle\nabla_{\overline{V}_a}V_j,\overline{V}_b\right\rangle\left\langle\nabla_{\overline{V}_j}V_{j_s},V_b\right\rangle\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W13b)$$

$$+2\alpha^{IJ}\left\langle\nabla_{V_j}\overline{V}_{i_r},\overline{V}_k\right\rangle\left\langle\nabla_{\overline{V}_j}V_{j_s},V_d\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^d}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^k}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W14a)$$

$$-2\alpha^{IJ}\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\left\langle\nabla_{\overline{V}_j}V_{j_s},V_i\right\rangle\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^k}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W14b)$$

$$-2\alpha^{IJ}V_j\left(\alpha^{IJ}\right)\left\langle\nabla_{\overline{V}_a}\overline{V}_j,V_{j_s}\right\rangle\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W15a)$$

$$-2\alpha^{IJ}\left\langle\nabla_{\overline{V}_a}\overline{V}_j,V_{j_s}\right\rangle\left\langle\nabla_{V_j}\overline{V}_{i_r},V_c\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^c}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W15b)$$

$$-2\alpha^{IJ}\left\langle \nabla_{\overline{V}_a}\overline{V}_j, V_b\right\rangle \left\langle \nabla_{V_j}V_{j_s}, \overline{V}_b\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{a}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W15c1)$$

$$-2\alpha^{IJ}\left\langle \nabla_{\overline{V}_a}\overline{V}_j, V_{j_u}\right\rangle \left\langle \nabla_{V_j}V_{j_s}, \overline{V}_c\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\overline{\omega}^{a}}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{c}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W15c2)$$

$$+2V_a\left(\alpha^{IJ}\right)\left\langle \nabla_{V_{j_s}}\overline{V}_a, \overline{V}_k\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{k}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W16a)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_{j_s}}\overline{V}_a, \overline{V}_k\right\rangle \left\langle \nabla_{V_a}\overline{V}_{i_r}, V_d\right\rangle \omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^{d}}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{k}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W16b)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_{j_u}}\overline{V}_a, \overline{V}_k\right\rangle \left\langle \nabla_{V_a}V_{j_s}, \overline{V}_b\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\overline{\omega}^{k}}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{b}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W16c1)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_b}\overline{V}_a, \overline{V}_k\right\rangle \left\langle \nabla_{V_a}V_{j_s}, \overline{V}_b\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{k}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W16c2)$$

$$+2\alpha^{IJ}\overline{V}_j\left(\alpha^{IJ}\right)\left\langle \nabla_{V_{j_s}}V_j, \overline{V}_k\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{k}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W17a)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_{j_s}}V_j, \overline{V}_k\right\rangle \left\langle \nabla_{\overline{V}_j}\overline{V}_{i_r}, V_c\right\rangle \omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^{c}}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{k}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W17b)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_a}V_j, \overline{V}_k\right\rangle \left\langle \nabla_{\overline{V}_j}V_{j_s}, \overline{V}_a\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{k}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W17c1)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_{j_u}}V_j, \overline{V}_k\right\rangle \left\langle \nabla_{\overline{V}_j}V_{j_s}, \overline{V}_d\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\overline{\omega}^{k}}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^{d}}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad (W17c2)$$

$$+2\alpha^{IJ}\overline{V}_j\left(\alpha^{IJ}\right)\left\langle\nabla_{\overline{V}_a}V_{j_s},V_j\right\rangle\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W18a)$$

$$+2\alpha^{IJ}\left\langle\nabla_{\overline{V}_a}V_{j_s},V_j\right\rangle\left\langle\nabla_{\overline{V}_j}\overline{V}_{i_r},V_c\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^c}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W18b)$$

$$+2\alpha^{IJ}\left\langle\nabla_{\overline{V}_a}V_b,V_j\right\rangle\left\langle\nabla_{\overline{V}_j}V_{j_s},\overline{V}_b\right\rangle\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W18c1)$$

$$+2\alpha^{IJ}\left\langle\nabla_{\overline{V}_a}V_{j_u},V_j\right\rangle\left\langle\nabla_{\overline{V}_j}V_{j_s},\overline{V}_d\right\rangle\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{u-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^d}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W18c2)$$

Then we simplify the terms from $(W19)$ *to* $(W22)$. *We take the covariant derivatives and the interior products, we would get*

$$(W19)\text{ through }(W22)$$
$$=+2(-1)^{2(p+s-1)}\delta_i^{j_s}\left\langle\nabla_{\overline{V}_a}\nabla_{\overline{V}_j}V_j,V_i\right\rangle\alpha^{IJ}\omega^{i_1}\wedge\cdots$$
$$\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W19)$$

$$+2\alpha^{IJ}\delta_j^{i_r}(-1)^{2(p+s+r)-1}\delta_k^{j_s}\left\langle\nabla_{\overline{V}_a}\nabla_{\overline{V}_j}V_i,V_k\right\rangle\omega^{i_1}\wedge\cdots$$
$$\wedge\underset{r-th\text{ place}}{\omega^i}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^a\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W20)$$

$$-2(-1)^{2(r-1)}\delta_i^{i_r}\left\langle\nabla_{V_k}\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\alpha^{IJ}\omega^{i_1}\wedge\cdots$$
$$\wedge\underset{r-th\text{ place}}{\omega^j}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W21)$$

$$+2\left\langle\nabla_{V_k}\nabla_{V_i}\overline{V}_j,\overline{V}_a\right\rangle(-1)^{2(r-1)}\delta_j^{i_r}(-1)^{2(p+s-1)}\delta_k^{j_s}\alpha^{IJ}\omega^{i_1}\wedge\cdots$$
$$\wedge\underset{r-th\text{ place}}{\omega^i}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W22)$$

Then if we apply the Kronecker deltas and simplify the signs, we have

$$
\begin{aligned}
& (W19) \text{ through } (W22) \\
= \quad & +2\alpha^{IJ} \left\langle \nabla_{\overline{V}_j} \nabla_{\overline{V}_k} V_k, V_{j_s} \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \\
& \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^{j}} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W19) \\
& -2\alpha^{IJ} \left\langle \nabla_{\overline{V}_a} \nabla_{\overline{V}_{i_r}} V_i, V_{j_s} \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^{i}} \wedge \cdots \\
& \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^{a}} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W20) \\
& -2\alpha^{IJ} \left\langle \nabla_{V_k} \nabla_{V_j} \overline{V}_{i_r}, \overline{V}_k \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^{j}} \\
& \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W21) \\
& +2\alpha^{IJ} \left\langle \nabla_{V_{j_s}} \nabla_{V_i} \overline{V}_{i_r}, \overline{V}_a \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^{i}} \wedge \cdots \\
& \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^{a}} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W22)
\end{aligned}
$$

Then if we carry the second derivatives to the second slot in $(W19)$ *and* $(W20)$, *we get*

$$
\begin{aligned}
& (W19) \text{ through } (W22) \\
= \quad & -2\alpha^{IJ} \left\langle \nabla_{\overline{V}_a} \nabla_{\overline{V}_j} V_{j_s}, V_j \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \\
& \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^{a}} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W19a) \\
& -2\alpha^{IJ} \left\langle \nabla_{\overline{V}_j} V_j, \nabla_{\overline{V}_a} V_{j_s} \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \\
& \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^{a}} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W19b) \\
& -2\alpha^{IJ} \left\langle \nabla_{\overline{V}_a} V_j, \nabla_{\overline{V}_j} V_{j_s} \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \\
& \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^{a}} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W19c)
\end{aligned}
$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_a}\nabla_{\overline{V}_{i_r}}V_{j_s},V_i\right\rangle \omega^{i_1}\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^i}\wedge\cdots$$
$$\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge \underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q} \quad (W20a)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_{i_r}}V_i,\nabla_{\overline{V}_a}V_{j_s}\right\rangle \omega^{i_1}\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^i}\wedge\cdots$$
$$\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge \underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q} \quad (W20b)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_a}V_i,\nabla_{\overline{V}_{i_r}}V_{j_s}\right\rangle \omega^{i_1}\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^i}\wedge\cdots$$
$$\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge \underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q} \quad (W20c)$$

$$-2\alpha^{IJ}\left\langle \nabla_{V_k}\nabla_{V_j}\overline{V}_{i_r},\overline{V}_k\right\rangle \omega^{i_1}\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^j}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q} \quad (W21)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_{j_s}}\nabla_{V_i}\overline{V}_{i_r},\overline{V}_a\right\rangle \omega^{i_1}\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^i}\wedge\cdots$$
$$\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge \underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q} \quad (W22)$$

If we write the terms $(W19b)$, $(W19c)$, $(W20b)$, $(W20c)$ *again, we get*

$$(W19)\ \text{through}\ (W22)$$
$$= -2\alpha^{IJ}\left\langle \nabla_{\overline{V}_a}\nabla_{\overline{V}_j}V_{j_s},V_j\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge \underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q} \quad (W19a)$$

$$-2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}V_j,\overline{V}_b\right\rangle \left\langle \nabla_{\overline{V}_a}V_{j_s},V_b\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge \underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q} \quad (W19b1)$$

$$-2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j}V_j,V_b\right\rangle \left\langle \nabla_{\overline{V}_a}V_{j_s},\overline{V}_b\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge \underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q} \quad (W19b2)$$

$$-2\alpha^{IJ}\left\langle \nabla_{\overline{V}_a}V_j,\overline{V}_b\right\rangle \left\langle \nabla_{\overline{V}_j}V_{j_s},V_b\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge \underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q} \quad (W19c1)$$

$$-2\alpha^{IJ}\left\langle \nabla_{\overline{V}_a} V_j, V_b \right\rangle \left\langle \nabla_{\overline{V}_j} V_{j_s}, \overline{V}_b \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$$
$$\wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W19c2)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_a} \nabla_{\overline{V}_{i_r}} V_{j_s}, V_i \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^i} \wedge \cdots$$
$$\wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W20a)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_{i_r}} V_i, \overline{V}_b \right\rangle \left\langle \nabla_{\overline{V}_a} V_{j_s}, V_b \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^i}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W20b1)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_{i_r}} V_i, V_b \right\rangle \left\langle \nabla_{\overline{V}_a} V_{j_s}, \overline{V}_b \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^i}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W20b2)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_a} V_i, \overline{V}_b \right\rangle \left\langle \nabla_{\overline{V}_{i_r}} V_{j_s}, V_b \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^i}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W20c1)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_a} V_i, V_b \right\rangle \left\langle \nabla_{\overline{V}_{i_r}} V_{j_s}, \overline{V}_b \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^i}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W20c2)$$

$$-2\alpha^{IJ}\left\langle \nabla_{V_k} \nabla_{V_j} \overline{V}_{i_r}, \overline{V}_k \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^j}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W21)$$

$$+2\alpha^{IJ}\left\langle \nabla_{V_{j_s}} \nabla_{V_i} \overline{V}_{i_r}, \overline{V}_a \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^i}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W22)$$

Finally we simplify the terms from $(W28)$ to $(W50)$. If we take the interior products and carry the wedge products to the spot where the interior product is applied, we get

$$
\begin{aligned}
& (W28) \text{ through } (W50) \\
= \quad & +2\alpha^{IJ}\delta^{a}_{i_r}\left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle \left\langle \nabla_{\overline{V}_a} V_k, V_b\right\rangle \omega^{i_1}\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^{b}} \\
& \wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q} \quad (W28) \\
& \\
& -2\alpha^{IJ}\delta^{i}_{i_r}\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \left\langle \nabla_{\overline{V}_j} V_k, V_b\right\rangle \omega^{i_1}\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^{b}} \\
& \wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q} \quad (W29) \\
& \\
& +4\alpha^{IJ}\delta^{k}_{i_r}\left\langle \nabla_{V_a}\overline{V}_i, \overline{V}_k\right\rangle \left\langle \nabla_{\overline{V}_j} V_j, V_i\right\rangle \omega^{i_1}\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^{a}} \\
& \wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q} \quad (W30) \\
& \\
& +2\alpha^{IJ}\delta^{j}_{j_s}\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \left\langle \nabla_{\overline{V}_a} V_i, V_k\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p} \\
& \wedge\overline{\omega}^{j_1}\wedge\cdots\wedge \underset{s-th\text{ place}}{\overline{\omega}^{a}}\wedge\cdots\wedge\overline{\omega}^{j_q} \quad (W31) \\
& \\
& +4\alpha^{IJ}\delta^{c}_{j_s}\left\langle \nabla_{V_j}\overline{V}_j, \overline{V}_k\right\rangle \left\langle \nabla_{\overline{V}_a} V_k, V_c\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p} \\
& \wedge\overline{\omega}^{j_1}\wedge\cdots\wedge \underset{s-th\text{ place}}{\overline{\omega}^{a}}\wedge\cdots\wedge\overline{\omega}^{j_q} \quad (W32) \\
& \\
& +2\alpha^{IJ}\delta^{j}_{j_s}\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \left\langle \nabla_{\overline{V}_a} V_k, V_a\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p} \\
& \wedge\overline{\omega}^{j_1}\wedge\cdots\wedge \underset{s-th\text{ place}}{\overline{\omega}^{i}}\wedge\cdots\wedge\overline{\omega}^{j_q} \quad (W33) \\
& \\
& -2\alpha^{IJ}\delta^{b}_{j_s}\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k\right\rangle \left\langle \nabla_{\overline{V}_j} V_i, V_b\right\rangle \omega^{i_1}\wedge\cdots\wedge\omega^{i_p} \\
& \wedge\overline{\omega}^{j_1}\wedge\cdots\wedge \underset{s-th\text{ place}}{\overline{\omega}^{k}}\wedge\cdots\wedge\overline{\omega}^{j_q} \quad (W34) \\
& \\
& -2\alpha^{IJ}\delta^{i}_{i_r}\left\langle \nabla_{V_k}\overline{V}_k, \overline{V}_b\right\rangle \left\langle \nabla_{V_j}\overline{V}_i, V_b\right\rangle \omega^{i_1}\wedge\cdots\wedge \underset{r-th\text{ place}}{\omega^{j}} \\
& \wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q} \quad (W35)
\end{aligned}
$$

$$-2\alpha^{IJ}\delta^{i}_{i_r}\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k \right\rangle \left\langle \nabla_{V_k}\overline{V}_j, V_b \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^b}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W36)$$

$$-2\alpha^{IJ}\delta^{b}_{i_r}\left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_a \right\rangle \left\langle \nabla_{V_a}V_i, \overline{V}_b \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^j}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W37)$$

$$+2\alpha^{IJ}\delta^{i}_{j_s}\left\langle \nabla_{\overline{V}_j}V_j, \overline{V}_k \right\rangle \left\langle \nabla_{\overline{V}_a}V_i, V_k \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$$
$$\wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W38)$$

$$+\alpha^{IJ}\delta^{k}_{j_s}\delta^{b}_{j_t}\left\langle \nabla_{\overline{V}_j}V_k, V_b \right\rangle \left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_c \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$$
$$\wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{t-th \text{ place}}{\overline{\omega}^c} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^i} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W39)$$

$$+4\alpha^{IJ}\delta^{b}_{j_s}\delta^{j}_{j_t}\left\langle \nabla_{\overline{V}_a}V_k, V_b \right\rangle \left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$$
$$\wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{t-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^i} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W40)$$

$$-4\alpha^{IJ}\delta^{a}_{i_r}\delta^{i}_{i_t}\left\langle \nabla_{\overline{V}_a}V_k, V_b \right\rangle \left\langle \nabla_{V_j}\overline{V}_i, \overline{V}_k \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^j}$$
$$\wedge \cdots \wedge \underset{t-th \text{ place}}{\omega^b} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W41)$$

$$-2\alpha^{IJ}\delta^{i}_{j_s}\delta^{b}_{i_r}\left\langle \nabla_{\overline{V}_j}V_k, V_i \right\rangle \left\langle \nabla_{V_a}\overline{V}_k, \overline{V}_b \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^a}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^j} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W42)$$

$$-4\alpha^{IJ}\delta^{i}_{j_s}\delta^{j}_{i_r}\left\langle \nabla_{\overline{V}_j}V_k, V_i \right\rangle \left\langle \nabla_{V_a}\overline{V}_k, \overline{V}_b \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^a}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^b} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W43)$$

$$-2\alpha^{IJ}\delta^{j}_{j_s}\delta^{a}_{i_r}\left\langle\nabla_{\overline{V}_a}V_k,V_b\right\rangle\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^b}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^i}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W44)$$

$$+2\alpha^{IJ}\delta^{a}_{j_s}\delta^{i}_{i_r}\left\langle\nabla_{V_a}\overline{V}_k,\overline{V}_b\right\rangle\left\langle\nabla_{V_j}\overline{V}_i,V_b\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^j}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^k}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W45)$$

$$+2\alpha^{IJ}\delta^{a}_{j_s}\delta^{i}_{i_r}\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\left\langle\nabla_{V_a}\overline{V}_j,V_b\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^b}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^k}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W46)$$

$$+2\alpha^{IJ}\delta^{a}_{j_s}\delta^{b}_{i_r}\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\left\langle\nabla_{V_a}V_i,\overline{V}_b\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^j}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^k}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W47)$$

$$-2\alpha^{IJ}\delta^{k}_{j_s}\delta^{j}_{i_r}\left\langle\nabla_{\overline{V}_a}V_k,V_b\right\rangle\left\langle\nabla_{\overline{V}_j}V_i,\overline{V}_b\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^i}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W48)$$

$$-2\alpha^{IJ}\delta^{k}_{j_s}\delta^{j}_{i_r}\left\langle\nabla_{\overline{V}_j}V_b,V_k\right\rangle\left\langle\nabla_{\overline{V}_a}\overline{V}_b,V_i\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^i}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W49)$$

$$-2\alpha^{IJ}\delta^{k}_{j_s}\delta^{b}_{i_r}\left\langle\nabla_{\overline{V}_j}V_i,V_k\right\rangle\left\langle\nabla_{\overline{V}_a}V_j,\overline{V}_b\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^i}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\quad(W50)$$

Then applying the Kronecker deltas to get

$$
\begin{aligned}
& (W28) \text{ through } (W50) \\
= \; & +2\alpha^{IJ} \left\langle \nabla_{V_j} \overline{V}_j, \overline{V}_k \right\rangle \left\langle \nabla_{\overline{V}_{i_r}} V_k, V_b \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^b} \\
& \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W28) \\
& \\
& -2\alpha^{IJ} \left\langle \nabla_{V_j} \overline{V}_{i_r}, \overline{V}_k \right\rangle \left\langle \nabla_{\overline{V}_j} V_k, V_b \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^b} \\
& \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W29) \\
& \\
& +4\alpha^{IJ} \left\langle \nabla_{V_a} \overline{V}_i, \overline{V}_{i_r} \right\rangle \left\langle \nabla_{\overline{V}_j} V_j, V_i \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^a} \\
& \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W30) \\
& \\
& +2\alpha^{IJ} \left\langle \nabla_{V_{j_s}} \overline{V}_i, \overline{V}_k \right\rangle \left\langle \nabla_{\overline{V}_a} V_i, V_k \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \\
& \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W31) \\
& \\
& +4\alpha^{IJ} \left\langle \nabla_{V_j} \overline{V}_j, \overline{V}_k \right\rangle \left\langle \nabla_{\overline{V}_a} V_k, V_{j_s} \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \\
& \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W32) \\
& \\
& +2\alpha^{IJ} \left\langle \nabla_{V_{j_s}} \overline{V}_i, \overline{V}_k \right\rangle \left\langle \nabla_{\overline{V}_a} V_k, V_a \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \\
& \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^i} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W33) \\
& \\
& -2\alpha^{IJ} \left\langle \nabla_{V_j} \overline{V}_i, \overline{V}_k \right\rangle \left\langle \nabla_{\overline{V}_j} V_i, V_{j_s} \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \\
& \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^k} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W34) \\
& \\
& -2\alpha^{IJ} \left\langle \nabla_{V_k} \overline{V}_k, \overline{V}_b \right\rangle \left\langle \nabla_{V_j} \overline{V}_{i_r}, V_b \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^j} \\
& \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W35)
\end{aligned}
$$

$$-2\alpha^{IJ}\left\langle\nabla_{V_j}\overline{V}_{i_r},\overline{V}_k\right\rangle\left\langle\nabla_{V_k}\overline{V}_j,V_b\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^b}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W36)$$

$$-2\alpha^{IJ}\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_a\right\rangle\left\langle\nabla_{V_a}V_i,\overline{V}_{i_r}\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^j}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W37)$$

$$+2\alpha^{IJ}\left\langle\nabla_{\overline{V}_j}V_j,\overline{V}_k\right\rangle\left\langle\nabla_{\overline{V}_a}V_{j_s},V_k\right\rangle\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W38)$$

$$+\alpha^{IJ}\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_c\right\rangle\left\langle\nabla_{\overline{V}_j}V_{j_s},V_{j_t}\right\rangle\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{t-th\text{ place}}{\overline{\omega}^c}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^i}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W39)$$

$$+4\alpha^{IJ}\left\langle\nabla_{V_{j_t}}\overline{V}_i,\overline{V}_k\right\rangle\left\langle\nabla_{\overline{V}_a}V_k,V_{j_s}\right\rangle\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{t-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^i}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W40)$$

$$-4\alpha^{IJ}\left\langle\nabla_{V_j}\overline{V}_{i_t},\overline{V}_k\right\rangle\left\langle\nabla_{\overline{V}_{i_r}}V_k,V_b\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^j}\wedge\cdots\wedge\underset{t-th\text{ place}}{\omega^b}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W41)$$

$$-2\alpha^{IJ}\left\langle\nabla_{\overline{V}_j}V_k,V_{j_s}\right\rangle\left\langle\nabla_{V_a}\overline{V}_k,\overline{V}_{i_r}\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^a}\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^j}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W42)$$

$$-4\alpha^{IJ}\left\langle\nabla_{\overline{V}_{i_r}}V_k,V_{j_s}\right\rangle\left\langle\nabla_{V_a}\overline{V}_k,\overline{V}_b\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^a}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^b}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W43)$$

$$-2\alpha^{IJ}\left\langle\nabla_{V_{j_s}}\overline{V}_i,\overline{V}_k\right\rangle\left\langle\nabla_{\overline{V}_{i_r}}V_k,V_b\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^b}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^i}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W44)$$

$$+2\alpha^{IJ}\left\langle\nabla_{V_{j_s}}\overline{V}_k,\overline{V}_b\right\rangle\left\langle\nabla_{V_j}\overline{V}_{i_r},V_b\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^j}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^k}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W45)$$

$$+2\alpha^{IJ}\left\langle\nabla_{V_j}\overline{V}_{i_r},\overline{V}_k\right\rangle\left\langle\nabla_{V_{j_s}}\overline{V}_j,V_b\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^b}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^k}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W46)$$

$$+2\alpha^{IJ}\left\langle\nabla_{V_j}\overline{V}_i,\overline{V}_k\right\rangle\left\langle\nabla_{V_{j_s}}V_i,\overline{V}_{i_r}\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^j}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^k}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W47)$$

$$-2\alpha^{IJ}\left\langle\nabla_{\overline{V}_{i_r}}V_i,\overline{V}_b\right\rangle\left\langle\nabla_{\overline{V}_a}V_{j_s},V_b\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^i}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W48)$$

$$-2\alpha^{IJ}\left\langle\nabla_{\overline{V}_a}\overline{V}_b,V_i\right\rangle\left\langle\nabla_{\overline{V}_{i_r}}V_b,V_{j_s}\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^i}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W49)$$

$$-2\alpha^{IJ}\left\langle\nabla_{\overline{V}_a}V_j,\overline{V}_{i_r}\right\rangle\left\langle\nabla_{\overline{V}_j}V_i,V_{j_s}\right\rangle\omega^{i_1}\wedge\cdots\wedge\underset{r-th\text{ place}}{\omega^i}$$
$$\wedge\cdots\wedge\omega^{i_p}\wedge\overline{\omega}^{j_1}\wedge\cdots\wedge\underset{s-th\text{ place}}{\overline{\omega}^a}\wedge\cdots\wedge\overline{\omega}^{j_q}\qquad(W50)$$

If we combine the like-terms, we have the following cancellations

$$\begin{aligned}
(W1a1)+(W2a1) &= 0, \quad (W1a2)+(W2a2)=0, \quad (W1b1)+(W2d1)=0,\\
(W1b2)+(W2d2) &= 0, \quad (W1b3)+(W2d3)=0, \quad (W1c1a)+(W2f1)=0,\\
(W1c1b)+(W2f2) &= 0, \quad (W1c2a)+(W20b2)=0, \quad (W1c2b)+(W19b2)=0,\\
(W2b1)+(W20a) &= 0, \quad (W2b2)+(W19a)=0, \quad (W2c1a)+(W20c2)=0,\\
(W2c1b)+(W2e1) &= 0, \quad (W2c2a)+(W19c2)=0, \quad (W2c2b)+(W2e2)=0,\\
(W3b1)+(W4a2) &= 0, \quad (W3b2)+(W4a1)=0, \quad (W5a1)+(W6a1)=0,\\
(W5a2)+(W6a2) &= 0, \quad (W5b1a)+(W6d1)=0, \quad (W5b1b)+(W6d2)=0,\\
(W5b2a)+(W37) &= 0, \quad (W5b2b)+(W47)=0, \quad (W5c1)+(W6f1)=0,\\
(W5c2)+(W6f3) &= 0, \quad (W5c3)+(W6f2)=0, \quad (W6b1)+(W21)=0,\\
(W6b2)+(W22) &= 0, \quad (W6c1a)+(W6e1)=0, \quad (W6c1b)+(W35)=0,\\
(W6c2a)+(W6e2) &= 0, \quad (W6c2b)+(W45)=0, \quad (W7a1a)+(W18a)=0,\\
(W7a1b)+(W15a) &= 0, \quad (W7a2)+(W8b1)=0, \quad (W7a3a)+(W8c1a)=0,\\
(W7a3b)+(W8c1b) &= 0, \quad (W7b1)+(W8a2)=0, \quad (W7b3b)+(W7b4)=0,\\
(W7b5)+(W8b4) &= 0, \quad (W7b6a)+(W8c4a)=0, \quad (W7b6b)+(W8c4b)=0,\\
(W7c1a)+(W8a3a) &= 0, \quad (W7c1b)+(W8a3b)=0, \quad (W7c3a)+(W7c6a)=0,
\end{aligned}$$

$$\begin{aligned}
(W7c4a) + (W8b6a) &= 0, \qquad (W7c4b) + (W8b6b) = 0, \qquad (W7c5a) + (W8c5a) = 0,\\
(W7c5b) + (W8c5c) &= 0, \qquad (W15b) + (W(7b2 + 8b2) - b) = 0,\\
(W7c5c) + (W8c5b) &= 0, \qquad (W15c2) + (W(7c2a + 8c2a) - b) = 0,\\
(W7c6b) + (W15c1) &= 0, \qquad (W7c3b) + (W(7c2b + 8c2b) - c) = 0,\\
(W8a1a) + (W17a) &= 0, \qquad (W8a1b) + (W16a) = 0, \qquad (W8b3) + (W8b5) = 0,\\
(W8c3a1) + (W8c6a) &= 0, \qquad (W8c3b1) + (W8c6b) = 0, \qquad (W9a) + (W14a) = 0,\\
(W9b) + (W14b) &= 0, \qquad (W10a) + (W11a) = 0, \qquad (W10b) + (W11b) = 0,\\
(W12a) + (W36) &= 0, \qquad (W17c1) + (W(7c2b + 8c2b) - d) = 0,\\
(W13b) + (W19c1) &= 0, \qquad (W16b) + (W(7b2 + 8b2) - d) = 0,\\
(W19b1) + (W38) &= 0, \qquad (W16c1) + (W(7c2a + 8c2a) - d) = 0,\\
(W20b1) + (W48) &= 0, \qquad (W16c2) + (W(7c2b + 8c2b) - e) = 0,\\
(W13a) + (W50) &= 0, \qquad (W17b) + (W(7b2 + 8b2) - e) = 0,\\
(W12b) + (W46) &= 0, \qquad (W17c2) + (W(7c2a + 8c2a) - e) = 0,\\
(W18b) + (W(7b2 + 8b2) - c) &= 0, \qquad (W18c1) + (W(7c2b + 8c2b) - b) = 0,\\
(W20c1) + (W49) &= 0, \qquad (W18c2) + (W(7c2a + 8c2a) - c) = 0.
\end{aligned}$$

If we simplify the terms as above, we would have

$$
\begin{aligned}
& \Delta_K(\alpha) + \Delta_H(\alpha) - 2\Delta_R(\alpha) \\
= \quad & -\left\langle R_{V_j V_k} \overline{V}_j, \overline{V}_k \right\rangle \alpha \quad (W23) \\
& -2\left\langle R_{V_i V_j} \overline{V}_j, \overline{V}_k \right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_i}(\alpha) \quad (W24) \\
& +\frac{1}{2}\left\langle R_{\overline{V}_a \overline{V}_j} V_k, V_i \right\rangle \overline{\omega}^a \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}(\alpha)\right) \quad (W25) \\
& +\frac{1}{2}\left\langle R_{V_a V_j} \overline{V}_i, \overline{V}_k \right\rangle \overline{\omega}^k \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}(\alpha)\right) \quad (W26) \\
& +2\alpha^{IJ}\left\langle R_{\overline{V}_a V_{j_s}} \overline{V}_{i_r}, V_b \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^b} \wedge \cdots \wedge \omega^{i_p} \\
& \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W7b2 + 8b2 - a) \\
& +2\alpha^{IJ}\left\langle R_{\overline{V}_a V_{j_u}} V_{j_s}, \overline{V}_b \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{u-th \text{ place}}{\overline{\omega}^a} \\
& \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^b} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W7c2a + 8c2a - a) \\
& +2\alpha^{IJ}\left\langle R_{\overline{V}_a V_j} V_{j_s}, \overline{V}_j \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \\
& \wedge \underset{s-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W7c2b + 8c2b - a)
\end{aligned}
$$

$$-2\alpha^{IJ}\left\langle \nabla_{\overline{V}_a} V_d, V_a \right\rangle \left\langle \nabla_{V_j} \overline{V}_{i_r}, \overline{V}_d \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^j}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W3a1)$$

$$-2\alpha^{IJ}\left\langle \nabla_{\overline{V}_{i_t}} V_d, V_c \right\rangle \left\langle \nabla_{V_j} \overline{V}_{i_r}, \overline{V}_d \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{t-th \text{ place}}{\omega^j}$$
$$\wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^c} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W3a2)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_{i_r}} V_{j_s}, V_d \right\rangle \left\langle \nabla_{V_a} \overline{V}_d, \overline{V}_c \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^a}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^c} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W4b)$$

$$-2\alpha^{IJ}\left\langle \nabla_{\overline{V}_a} V_b, V_c \right\rangle \left\langle \nabla_{V_{j_s}} \overline{V}_{i_r}, \overline{V}_c \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^b}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W7b3a)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j} V_{j_s}, V_b \right\rangle \left\langle \nabla_{V_{j_u}} \overline{V}_d, \overline{V}_b \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1}$$
$$\wedge \cdots \wedge \underset{u-th \text{ place}}{\overline{\omega}^j} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^d} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W8c3a2)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j} V_{j_s}, V_b \right\rangle \left\langle \nabla_{V_a} \overline{V}_a, \overline{V}_b \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$$
$$\wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^j} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W8c3b2)$$

$$-2\left\langle \nabla_{\overline{V}_a} V_k, V_a \right\rangle \left\langle \nabla_{V_j} \overline{V}_j, \overline{V}_k \right\rangle \alpha \quad (W27)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_{i_r}} V_k, V_b \right\rangle \left\langle \nabla_{V_j} \overline{V}_j, \overline{V}_k \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^b}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W28)$$

$$-2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j} V_k, V_b \right\rangle \left\langle \nabla_{V_j} \overline{V}_{i_r}, \overline{V}_k \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^b}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W29)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j} V_j, V_i \right\rangle \left\langle \nabla_{V_a} \overline{V}_i, \overline{V}_{i_r} \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^a}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q} \quad (W30)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_a} V_i, V_k \right\rangle \left\langle \nabla_{V_{j_s}} \overline{V}_i, \overline{V}_k \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$$
$$\wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W31)$$

$$+4\alpha^{IJ}\left\langle \nabla_{\overline{V}_a} V_k, V_{j_s} \left\langle \nabla_{V_j} \overline{V}_j, \overline{V}_k \right\rangle \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$$
$$\wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W32)$$

$$+2\alpha^{IJ}\left\langle \nabla_{\overline{V}_a} V_k, V_a \right\rangle \left\langle \nabla_{V_{j_s}} \overline{V}_i, \overline{V}_k \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$$
$$\wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^i} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W33)$$

$$-2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j} V_i, V_{j_s} \right\rangle \left\langle \nabla_{V_j} \overline{V}_i, \overline{V}_k \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$$
$$\wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^k} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W34)$$

$$+\alpha^{IJ}\left\langle \nabla_{\overline{V}_j} V_{j_s}, V_{j_t} \right\rangle \left\langle \nabla_{V_j} \overline{V}_i, \overline{V}_c \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1}$$
$$\wedge \cdots \wedge \underset{t-th \text{ place}}{\overline{\omega}^c} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^i} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W39)$$

$$+4\alpha^{IJ}\left\langle \nabla_{\overline{V}_a} V_k, V_{j_s} \right\rangle \left\langle \nabla_{V_{j_t}} \overline{V}_i, \overline{V}_k \right\rangle \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1}$$
$$\wedge \cdots \wedge \underset{t-th \text{ place}}{\overline{\omega}^a} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^i} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W40)$$

$$-4\alpha^{IJ}\left\langle \nabla_{\overline{V}_{i_r}} V_k, V_b \right\rangle \left\langle \nabla_{V_j} \overline{V}_{i_t}, \overline{V}_k \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^j}$$
$$\wedge \cdots \wedge \underset{t-th \text{ place}}{\omega^b} \wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W41)$$

$$-2\alpha^{IJ}\left\langle \nabla_{\overline{V}_j} V_k, V_{j_s} \right\rangle \left\langle \nabla_{V_a} \overline{V}_k, \overline{V}_{i_r} \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^a}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^j} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W42)$$

$$-4\alpha^{IJ}\left\langle \nabla_{\overline{V}_{i_r}} V_k, V_{j_s} \right\rangle \left\langle \nabla_{V_a} \overline{V}_k, \overline{V}_b \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^a}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^b} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W43)$$

$$-2\alpha^{IJ}\left\langle \nabla_{\overline{V}_{i_r}} V_k, V_b \right\rangle \left\langle \nabla_{V_{j_s}} \overline{V}_i, \overline{V}_k \right\rangle \omega^{i_1} \wedge \cdots \wedge \underset{r-th \text{ place}}{\omega^b}$$
$$\wedge \cdots \wedge \omega^{i_p} \wedge \overline{\omega}^{j_1} \wedge \cdots \wedge \underset{s-th \text{ place}}{\overline{\omega}^i} \wedge \cdots \wedge \overline{\omega}^{j_q} \qquad (W44)$$

If we write the terms again, we have

$$\Delta_K(\alpha) + \Delta_H(\alpha) - 2\Delta_R(\alpha)$$

$$= -\left\langle R_{V_j V_k} \overline{V}_j, \overline{V}_k \right\rangle \alpha \quad (W23)$$

$$+2\left\langle R_{V_i V_j} \overline{V}_k, \overline{V}_j \right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_i}(\alpha) \quad (W24)$$

$$+2\left\langle R_{\overline{V}_a V_j} V_i, \overline{V}_j \right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_i}(\alpha) \quad (W7c2b + 8c2b - a)$$

$$+\frac{1}{2}\left\langle R_{\overline{V}_a \overline{V}_j} V_k, V_i \right\rangle \overline{\omega}^a \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}(\alpha)\right) \quad (W25)$$

$$+\frac{1}{2}\left\langle R_{V_a V_j} \overline{V}_i, \overline{V}_k \right\rangle \overline{\omega}^k \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}(\alpha)\right) \quad (W26)$$

$$-2\left\langle R_{\overline{V}_a V_j} \overline{V}_i, V_b \right\rangle \omega^b \wedge \overline{\omega}^a \wedge \iota_{V_i}\left(\iota_{\overline{V}_j}(\alpha)\right) \quad (W7b2 + 8b2 - a)$$

$$-2\left\langle R_{\overline{V}_a V_i} \overline{V}_j, V_b \right\rangle \overline{\omega}^a \wedge \overline{\omega}^b \wedge \iota_{\overline{V}_i}\left(\iota_{\overline{V}_j}(\alpha)\right) \quad (W7c2a + 8c2a - a)$$

$$-2\left\langle \nabla_{\overline{V}_a} V_k, V_a \right\rangle \left\langle \nabla_{V_j} \overline{V}_j, \overline{V}_k \right\rangle \alpha \quad (W27)$$

$$+2\left\langle \nabla_{\overline{V}_i} V_k, V_b \right\rangle \left\langle \nabla_{V_j} \overline{V}_j, \overline{V}_k \right\rangle \omega^b \wedge \iota_{V_i}(\alpha) \quad (W28)$$

$$-2\left\langle \nabla_{\overline{V}_j} V_k, V_b \right\rangle \left\langle \nabla_{V_j} \overline{V}_i, \overline{V}_k \right\rangle \omega^b \wedge \iota_{V_i}(\alpha) \quad (W29)$$

$$+2\left\langle \nabla_{\overline{V}_j} V_j, V_i \right\rangle \left\langle \nabla_{V_a} \overline{V}_i, \overline{V}_k \right\rangle \omega^a \wedge \iota_{V_k}(\alpha) \quad (W3a1) + (W30)$$

$$+2\left\langle \nabla_{\overline{V}_a} V_i, V_k \right\rangle \left\langle \nabla_{V_j} \overline{V}_i, \overline{V}_k \right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_j}(\alpha) \quad (W31)$$

$$+2\left\langle \nabla_{\overline{V}_a} V_k, V_c \right\rangle \left\langle \nabla_{V_j} \overline{V}_j, \overline{V}_k \right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_c}(\alpha) \quad (W8c3b2) + (W32)$$

$$+2\left\langle \nabla_{\overline{V}_a} V_k, V_a \right\rangle \left\langle \nabla_{V_j} \overline{V}_i, \overline{V}_k \right\rangle \overline{\omega}^i \wedge \iota_{\overline{V}_j}(\alpha) \quad (W33)$$

$$-2\left\langle \nabla_{\overline{V}_j} V_i, V_a \right\rangle \left\langle \nabla_{V_j} \overline{V}_i, \overline{V}_k \right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}(\alpha) \quad (W34)$$

$$-\left\langle \nabla_{\overline{V}_j} V_k, V_a \right\rangle \left\langle \nabla_{V_j} \overline{V}_i, \overline{V}_c \right\rangle \overline{\omega}^c \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_k}(\alpha)\right) \quad (W39)$$

$$+2\left\langle \nabla_{\overline{V}_j} V_k, V_c \right\rangle \left\langle \nabla_{V_a} \overline{V}_i, \overline{V}_k \right\rangle \overline{\omega}^j \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_c}\left(\iota_{\overline{V}_a}(\alpha)\right) \quad (W8c3a2) + (W40)$$

$$-2\left\langle \nabla_{\overline{V}_a} V_k, V_b \right\rangle \left\langle \nabla_{V_j} \overline{V}_i, \overline{V}_k \right\rangle \omega^j \wedge \omega^b \wedge \iota_{V_a}\left(\iota_{V_i}(\alpha)\right) \quad (W41) + (W3a2)$$

$$-2\left\langle \nabla_{\overline{V}_j} V_k, V_i \right\rangle \left\langle \nabla_{V_a} \overline{V}_k, \overline{V}_b \right\rangle \omega^a \wedge \overline{\omega}^j \wedge \iota_{V_b}\left(\iota_{\overline{V}_i}(\alpha)\right) \quad (W42)$$

$$+2\left\langle \nabla_{\overline{V}_a} V_d, V_b \right\rangle \left\langle \nabla_{V_j} \overline{V}_c, \overline{V}_b \right\rangle \overline{\omega}^a \wedge \omega^d \wedge \iota_{\overline{V}_j}\left(\iota_{V_c}(\alpha)\right) \quad (W7b3a)$$

$$-2\left\langle \nabla_{\overline{V}_j} V_k, V_i \right\rangle \left\langle \nabla_{V_a} \overline{V}_k, \overline{V}_b \right\rangle \omega^a \wedge \overline{\omega}^b \wedge \iota_{V_j}\left(\iota_{\overline{V}_i}(\alpha)\right) \quad (W4b) + (W43)$$

$$+2\left\langle \nabla_{\overline{V}_a} V_k, V_b \right\rangle \left\langle \nabla_{V_j} \overline{V}_i, \overline{V}_k \right\rangle \omega^b \wedge \overline{\omega}^i \wedge \iota_{V_a}\left(\iota_{\overline{V}_j}(\alpha)\right) \quad (W44)$$

Then by 2.10, we obtain

$$
\begin{aligned}
(W20)+(W3a1) &= \frac{1}{2}(E) \\
(W27) &= \frac{1}{2}(A) \\
(W28) &= \frac{1}{2}(F) \\
(W29) &= \frac{1}{2}(G) \\
(W31) &= \frac{1}{2}(D)+\frac{1}{2}(O) \\
(W8c3b2)+(W32) &= \frac{1}{2}(C) \\
(W33) &= \frac{1}{2}(B) \\
(W39) &= \frac{1}{2}(N)+\frac{1}{2}(Q) \\
(W3a2)+(W41) &= \frac{1}{2}(H) \\
(W42) &= \frac{1}{2}(L) \\
(W4b)+(W43) &= \frac{1}{2}(K) \\
(W44) &= \frac{1}{2}(J) \\
(W34) &= -\frac{1}{2}(R) \\
(W7b3a) &= -\frac{1}{2}(S) \\
(W8c3a2)+(W40) &= +\frac{1}{2}(M)+\frac{1}{2}(P)-\frac{1}{2}(T) \\
0 &= \frac{1}{2}(I)-\frac{1}{2}(U)
\end{aligned}
$$

This implies

$$
\begin{aligned}
& \Delta_K(\alpha)+\Delta_H(\alpha)-2\Delta_R(\alpha) \\
= \quad & F(R)(\alpha)+\frac{1}{2}\left[\left[L,\left(L_{\overline{\partial}\Omega}\right)^*\right],\left[L_{\overline{\partial}\Omega},\Lambda\right]\right](\alpha) \\
& +\frac{1}{2}\left[T_1,(T_1)^*\right](\alpha)-\frac{1}{2}\left[T_2,(T_2)^*\right](\alpha)
\end{aligned}
$$

where the curvature operator $F(R): E^{p,q} \to E^{p,q}$ *is given by*

$$\begin{aligned} F(R)(\alpha) &= -\left\langle R_{V_j V_k} \overline{V}_j, \overline{V}_k \right\rangle \alpha \\ &+2\left\langle R_{V_i V_j} \overline{V}_k, \overline{V}_j \right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_i}(\alpha) + 2\left\langle R_{\overline{V}_a V_j} V_i, \overline{V}_j \right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_i}(\alpha) \\ &+\frac{1}{2}\left\langle R_{\overline{V}_a \overline{V}_j} V_k, V_i \right\rangle \overline{\omega}^a \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_i}\left(\iota_{\overline{V}_k}(\alpha)\right) \\ &+\frac{1}{2}\left\langle R_{V_a V_j} \overline{V}_i, \overline{V}_k \right\rangle \overline{\omega}^k \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}(\alpha)\right) \\ &-2\left\langle R_{\overline{V}_a V_j} \overline{V}_i, V_b \right\rangle \omega^b \wedge \overline{\omega}^a \wedge \iota_{V_i}\left(\iota_{\overline{V}_j}(\alpha)\right) \\ &-2\left\langle R_{\overline{V}_a V_i} \overline{V}_j, V_b \right\rangle \overline{\omega}^a \wedge \overline{\omega}^b \wedge \iota_{\overline{V}_i}\left(\iota_{\overline{V}_j}(\alpha)\right). \end{aligned}$$

Finally if we switch the roles of indices and use the symmetries of the curvature tensor and the Bianchi identity, we get

$$\begin{aligned} F(R)(\alpha) &= +2\left\langle R_{\overline{V}_a V_j} V_c, \overline{V}_j \right\rangle \overline{\omega}^a \wedge \iota_{\overline{V}_c}(\alpha) + 2\left\langle R_{V_a V_j} \overline{V}_j, \overline{V}_k \right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}(\alpha) \\ &+2\left\langle R_{V_a \overline{V}_j} \overline{V}_d, V_c \right\rangle \overline{\omega}^j \wedge \omega^c \wedge \iota_{\overline{V}_a}\left(\iota_{V_d}(\alpha)\right) \\ &+2\left\langle R_{V_a \overline{V}_j} V_c, \overline{V}_d \right\rangle \overline{\omega}^j \wedge \overline{\omega}^d \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_c}(\alpha)\right) + \left\langle R_{V_j V_k} \overline{V}_j, \overline{V}_k \right\rangle \alpha \\ &+\left\langle R_{V_a V_j} \overline{V}_i, \overline{V}_k \right\rangle \overline{\omega}^k \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}(\alpha)\right) \\ &= +2\left\langle R_{\overline{V}_k V_j} V_a, \overline{V}_j \right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}(\alpha) + 2\left\langle R_{V_a V_j} \overline{V}_j, \overline{V}_k \right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}(\alpha) \\ &+2\left\langle R_{V_a \overline{V}_j} \overline{V}_d, V_c \right\rangle \overline{\omega}^j \wedge \omega^c \wedge \iota_{\overline{V}_a}\left(\iota_{V_d}(\alpha)\right) \\ &+2\left\langle R_{V_a \overline{V}_j} V_c, \overline{V}_d \right\rangle \overline{\omega}^j \wedge \overline{\omega}^d \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_c}(\alpha)\right) + \left\langle R_{V_j V_k} \overline{V}_j, \overline{V}_k \right\rangle \alpha \\ &+\left\langle R_{V_a V_j} \overline{V}_i, \overline{V}_k \right\rangle \overline{\omega}^k \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}(\alpha)\right) \\ &= +2\left\langle R_{V_j \overline{V}_k} \overline{V}_j, V_a \right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}(\alpha) + 2\left\langle R_{V_j V_a} \overline{V}_k, \overline{V}_j \right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}(\alpha) \\ &+2\left\langle R_{V_a \overline{V}_j} \overline{V}_d, V_c \right\rangle \overline{\omega}^j \wedge \omega^c \wedge \iota_{\overline{V}_a}\left(\iota_{V_d}(\alpha)\right) \\ &+2\left\langle R_{V_a \overline{V}_j} V_c, \overline{V}_d \right\rangle \overline{\omega}^j \wedge \overline{\omega}^d \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_c}(\alpha)\right) + \left\langle R_{V_j V_k} \overline{V}_j, \overline{V}_k \right\rangle \alpha \\ &+\left\langle R_{V_a V_j} \overline{V}_i, \overline{V}_k \right\rangle \overline{\omega}^k \wedge \overline{\omega}^i \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_j}(\alpha)\right) \\ &= +\left\langle R_{V_j V_k} \overline{V}_j, \overline{V}_k \right\rangle \alpha + 2\left\langle R_{\overline{V}_j V_j} V_a, \overline{V}_k \right\rangle \overline{\omega}^k \wedge \iota_{\overline{V}_a}(\alpha) \\ &+2\left\langle R_{V_a \overline{V}_k} \overline{V}_d, V_c \right\rangle \overline{\omega}^k \wedge \omega^c \wedge \iota_{\overline{V}_a}\left(\iota_{V_d}(\alpha)\right) \\ &+2\left\langle R_{V_a \overline{V}_k} V_b, \overline{V}_j \right\rangle \overline{\omega}^k \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_b}(\alpha)\right) \\ &+\left\langle R_{V_a V_b} \overline{V}_j, \overline{V}_k \right\rangle \overline{\omega}^k \wedge \overline{\omega}^j \wedge \iota_{\overline{V}_a}\left(\iota_{\overline{V}_b}(\alpha)\right). \end{aligned}$$

Thus we are done. □

Bibliography

[Chern] S.S. Chern, On curvature and characteristic classes of a Riemannian manifold, Abh. Math. Sem., Univ. Hamburg, vol 20 (1956), pp. 117-126. MR0075647

[Demailly] Demailly, J.-P. Sur l'identité de Bochner-Kodaira-Nakano en géométrie hermitienne. Séminaire d'analyse P. Lelong-P. Dolbeault-H. Skoda, années 1983/1984, 88–97, Lecture Notes in Math., 1198, Springer, Berlin, 1986. MR0874763

[Frankel] Frankel, T. *Manifolds with positive curvature.* Pacific J. Math. **11** (1961), 165-174. MR0123272

[Frölicher] Frölicher, Alfred (1955), Relations between the cohomology groups of Dolbeault and topological invariants, Proceedings of the National Academy of Sciences 41, 641–644. MR0073262

[Gray] Gray, Alfred, *A Property of a Hypothetical complex structure on the six sphere.* Bolletino U.M.I., (1997), **3**, 251–255, MR1456264

[Griffiths & Harris] Griffiths, P., Harris, J. Principles of algebraic geometry. Reprint of the 1978 original. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994. xiv+813 pp. ISBN: 0-471-05059-8 14-01 MR1288523

[Hodge] Hodge, W. V. D., *The Theory and Applications of Harmonic Integrals*, Cambridge University Press, (1941), MR 0003947.

[Hopf 1] H. Hopf. Zum Clifford-Kleinschen raumproblem. Math. Ann. 95 (1926), no. 1, 313–339. MR1512281

[Hopf 2] H. Hopf. Differentialgeometrie und topologische Gestalt. Jahresber. Deutsch. Math.-Verein, 41: 209–229, 1932.

[Horn] Horn, Roger A.; Johnson, Charles R. Matrix analysis. Cambridge University Press, Cambridge, 1985. xiii+561 pp. ISBN: 0-521-30586-1. MR0832183

[Johnson] Johnson, David L., *Curvature and Euler characteristic for six-dimensional Kähler manifolds,* Illinois J. Math., Illinois Journal of Mathematics, **28**, (1984), 4, 654–675, MR761996

[Le Potier] Le Potier, J. Problèmes d'extension de classes de cohomologie. Thèse d'Etat à l'Université de Poitiers (France). 1974.

[Nirenberg & Newlander] Newlander, A., Nirenberg, L. Complex analytic coordinates in almost complex manifolds. Ann. of Math. (2) 65 (1957), 391–404. MR0088770

[Siu & Yau] Siu, Yum-Tong, Yau,Shing-Tung. *Compact Kähler manifolds of positive bisectional curvature.* Invent. Math. **59** (1980), no. 2, 189-204. MR0577360

[Synge] Synge, John Lighton (1936), On the connectivity of spaces of positive curvature. Quarterly Journal of Mathematics (Oxford Series) 7: 316–320. doi: 10.1093/qmath/os-7.1.316

[Voisin] Voisin, Claire, *Hodge theory and complex algebraic geometry. I,* Cambridge Studies in Advanced Mathematics, **Vol. 76**, English Ed. (Translated from the French by Leila Schneps), Cambridge University Press, Cambridge, (2009j:32014). MR2451566

[Wells] Wells, Jr., R. O., *Differential analysis on complex manifolds,* Graduate Texts in Mathematics, **Vol. 65**, 2nd Ed. Springer-Verlag, New York-Berlin, (1980). MR608414

[Wu] Wu, Hung Hsi, *The Bochner technique in differential geometry,* Math. Rep., Mathematical Reports, **Vol. 3**, (1988), 289–538, MR1079031

Vita

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