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Combinatorial Interpretations of Induced Sign Characters of the Hecke Algebra

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Abstract

Combinatorial interpretations have been used to show the total nonnegativity of induced trivial character and induced sign character immanants. The irreducible character immanants are known to be totally nonnegative as well, however, providing a combinatorial interpretation remains an open problem. To find such combinatorial interpretations we explore the quantum analogs of the symmetric group characters associated to the above mentioned immanants. In this paper, a combinatorial interpretation for the quantum induced sign characters on certain elements of the Hecke algebra is provided. This interpretation is then related to the quantum chromatic symmetric function introduced by Shareshian and Wachs. These interpretations involve a certain class of posets and associated planar networks. Lastly, for a restricted subset of these planar networks, properties of the sequence of coefficients of the induced sign characters of the Hecke algebra are discussed.

Introduction

A real matrix is called totally nonnegative (TNN) if each of its square submatrices has a nonnegative determinant. A polynomial $p(x)$ in n^2 variables is called totally nonnegative if $p(A)$ is nonnegative for every totally nonnegative matrix A. Applications of total nonnegativity span from its origin of oscillation in mechanical systems to stochastic processes and approximation theory to the theory of immanants [5]. The study of total nonnegativity involves answering the questions "What properties do totally nonnegative matrices have?" and "What properties do they transfer to related objects, such as polynomials and planar networks?"

The total nonnegativity of a polynomial in n^2 variables can be shown by providing a combinatorial or graph-theoretic interpretation for the nonnegative numbers $p(A)$ as A varies over all TNN matrices. The determinant, $\det(x)$, is a familiar example of a totally nonnegative polynomial with a graph-theoretic interpretation which was proven by Karlin and MacGregor [10] and Lindström [13].

A generating function that lies in the complex span of $\{x_{1,v_1}, \ldots, x_{n,v_n} \mid v \in \mathfrak{S}_n\}$ can be associated to each function on the symmetric group, $\mathfrak{S}_n$. Such generating functions are called immanants, $\mathrm{Imm}_f(x)$. Some such immanants are conjectured and others are known to be TNN polynomials, but only for very special class functions f is there a known graph-theoretic interpretation for the nonnegative numbers $\mathrm{Imm}_f(A)$ as A varies over all TNN matrices.

The special class functions of interest are $\mathfrak{S}_n$-character, which form bases of the space of class functions on $\mathfrak{S}_n$. The induced sign character immanants and induced trivial character immanants can be written as sums of products of matrix minors and sums of products of permanents [15], respectively. These characters can further be combinatorially interpreted as the number of Young tableaux with particular properties, thus proving their total nonnegativity. Stembridge [22] proved the total nonnegativity of the irreducible character immanants, but the problem of strengthening this result to include a graph-theoretic interpretation remains open. He conjectured the total nonnegativity of the monomial virtual character immanants. All TNN class immanants are known to be nonnegative linear combinations of monomial immanants. Thus if Stembridge's conjecture is true, the TNN class functions can be characterized as those which are nonnegative linear combinations of monomial immanants. The interest in advancing the study of these $\mathfrak{S}_n$-character immanants has led to the study of their quantum analogs, defined on the Hecke algebra, $H_n(q)$.

The total nonnegativity of these characters has applications to graph theory. For any graph G, the chromatic polynomial in k, $\chi_G(k)$, counts the number of proper colorings of the graph that use k colors. Stanley [21] defined a chromatic symmetric function in the variables $x = (x_1, x_2, \dots)$. The specialization of this symmetric function at $x_1 = x_2 = \cdots = 1$ yields the chromatic polynomial. Stanley's chromatic symmetric function can be stated in terms of the monomial, elementary, and Shur symmetric functions. In general, the chromatic symmetric function is not elementary nonnegative. However, for certain graphs G, the chromatic symmetric function has been shown to be elementary nonnegative and Shur nonnegative [21].

In Chapter 1, the symmetric group, $\mathfrak{S}_n$, and some of its properties are reviewed. Special attention is payed to a particular reduced expression for each element of $\mathfrak{S}_n$. A family of maps defined on $\mathfrak{S}_n$ are introduced in order to discuss properties of permutations in the symmetric group as well as to introduce a new family of polynomials. These new polynomials are then used to generalize transition matrices between bases of the quantum matrix bialgebra.

The combinatorial objects used to interpret $\mathfrak{S}_n$-characters and their associated immanants are introduced in Chapter 2. The $\mathfrak{S}_n$-characters mentioned above are defined and the known results are summarized. Section 2.4 aims to present a combinatorial interpretation of the induced sign characters of the Hecke algebra. This interpretation is then related to the irreducible characters of the Hecke algebra and the quantum chromatic symmetric function.

Lastly, Chapter 3 introduces a conjectured generalization of the evaluation of the family of maps $\{\sigma_{A,u} : \mathcal{A}_{[n],[n]}(n;q) \to \mathbb{Z}[q^{\frac{1}{2}} - q^{\frac{-1}{2}}] | u \in \mathfrak{S}_n\}$, introduced in Section 2.4, on certain elements of $\mathcal{A}_{[n],[n]}(n;q)$. In the last section, a combinatorial proof that the sequence of coefficients in $\mathrm{Imm}_{\epsilon_q^\lambda}(x)$ is symmetric is provided for specific variables x.

Chapter 1

The Symmetric group and the Quantum Matrix Bialgebra

An overview of the symmetric group and some of its properties is provided first. A particular reduced expression for each element of the group will be developed for use throughout the first two chapters. The Bruhat order, a partial ordering of the symmetric group, is summarized. Next a family of maps is defined on the the symmetric group and results about these maps are used to state properties of the class of reduced expressions of permutations in the symmetric group. Lastly, the quantum matrix bialgebra is summarized and a previous result on transition matrices between bases is generalized using a new family of polynomials.

1.1 The Symmetric group $\mathfrak{S}_n$

The symmetric group, $\mathfrak{S}_n$, is the group of all permutations of the letters $1, \ldots, n$. Let $s_1, \ldots, s_{n-1}$ be the generators of $\mathfrak{S}_n$, which satisfy the conditions

$$\begin{aligned} {s_i}^2 &= e && \text{for } i = 1, \ldots, n-1, \\ s_i s_j s_i &= s_j s_i s_j && \text{if } |i-j| = 1, \\ s_i s_j &= s_j s_i && \text{if } |i-j| \geq 2. \end{aligned} \tag{1.1.1}$$

Let $\mathfrak{S}_n$ act on the rearrangements of the letters $[n] = \{1, \ldots, n\}$ by

$$s_i \circ u_1 \cdots u_n = u_1 \cdots u_{i-1} u_{i+1} u_i u_{i+2} \cdots u_n. \tag{1.1.2}$$

Each permutation $w = s_{i_1} s_{i_2} \cdots s_{i_l}$ can be represented in one-line notation as

$$w_1 w_2 \cdots w_n = s_{i_1} \circ (\cdots (s_{i_l} \circ (1 \cdots n)) \cdots). \tag{1.1.3}$$

A pair of letters w_i and w_j in the one-line notation of w form an inversion if $w_i > w_j$ and $i < j$. Let $inv(w) = \ell$ be the total number of inversions in w and say w has length $\ell = \ell(w)$. Any expression for w consisting of ℓ generators is called a reduced expression. For example, $s_2 s_1 s_3$ in $\mathfrak{S}_4$ is a reduced expression for the permutation $w = 2413$ in one-line notation and $\ell(w) = 3$. A reduced expression for a permutation is not unique. In fact, $s_2 s_3 s_1$ is also a reduced expression for 2413. However, the length of a permutation is unique. That is, every reduced expression for w will be a product of $\ell(w)$ generators. A permutation is called even (odd) if every reduced expression is a product of an even (odd) number of generators.

Define the map $\oplus : \mathfrak{S}_n \times \mathfrak{S}_m \to \mathfrak{S}_{n+m}$ by

$$s_{i_1} \cdots s_{i_k} \oplus s_{j_1} \cdots s_{j_\ell} = s_{i_1} \cdots s_{i_k} s_{j_1+n} \cdots s_{j_\ell+n}.$$

A permutation w is said to be $\oplus$-indecomposable if it cannot be decomposed as $w = u \oplus v$. The following observation about the $\oplus$-decomposability of a permutation is due to Rhoades and Skandera [16].

Observation 1.1.1. *A permutation $w \in \mathfrak{S}_{n+m}$ decomposes as $w^{(1)} \oplus w^{(2)}$ with $w^{(1)} \in \mathfrak{S}_n$, $w^{(2)} \in \mathfrak{S}_m$ if and only if no reduced expression for w contains the transposition s_n.*

One subgroup of the symmetric group that is of interest is the Young subgroup. Fix $\lambda = (\lambda_1, \cdots, \lambda_r)$ a partition of n and define the Young subgroup, $\mathfrak{S}_\lambda$, of $\mathfrak{S}_n$ to be

$$\mathfrak{S}_\lambda = \mathfrak{S}_{\{1,\ldots,\lambda_1\}} \times \mathfrak{S}_{\{\lambda_1+1,\ldots,\lambda_1+\lambda_2\}} \times \cdots \times \mathfrak{S}_{\{n-\lambda_r+1,\ldots,n\}}.$$

Observe that $\mathfrak{S}_\lambda$ is isomorphic to $\mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \cdots \times \mathfrak{S}_{\lambda_r}$.

The Bruhat order is a partial order of $\mathfrak{S}_n$ defined by $u \le v$ if some subword of a reduced expression for v is a reduced expression for u. See [2] for more information. A generator s_i is said to be a left ascent (descent) of a permutation u if $s_i u > u$ ($s_i u < u$) in the Bruhat order. For example, if $s_{i_1} \cdots s_{i_k}$ is a reduced expression for u, then s_{i_1} is a left descent of u. Results by Björner and Brenti [1] imply the following lemma about intervals in the Bruhat order.

Lemma 1.1.2. *In any interval $[v, w]$ of the Bruhat order where $v < w$, there are equally many even and odd permutations.*

For consistency it will be useful to specify a particular reduced expression for each permutation. Given a permutation $u \in \mathfrak{S}_n$ with one-line notation $u_1 u_2 \cdots u_n$ construct an expression $s_{i_1} \cdots s_{i_\ell}$ for u in the following way:

1. If letter 1 is in position $j+1$ of u, then let s_{i_1}, s_{i_2}, …, s_{i_j} record the adjacent transpositions required to move letter 1 from the $j+1$ position of u to the first position of $s_{i_j} s_{i_{j-1}} \cdots s_{i_1}(u_1 u_2 \cdots u_n)$. That is,

$$\begin{aligned} s_{i_1} &= s_j, \\ s_{i_2} &= s_{j-1}, \\ &\vdots \\ s_{i_j} &= s_1. \end{aligned}$$

   Define $\mathscr{A}_1 = s_j s_{j-1} \cdots s_1$.

2. If letter 2 is now in position $k+1$ of $s_1 \cdots s_j(u_1 u_2 \cdots u_n)$, then let $s_{i_{j+1}}$, $s_{i_{j+2}}$, …, $s_{i_{j+k}}$ record the adjacent transpositions required to move letter 2 from the $k+1$ position of $s_1 \cdots s_j(u_1 u_2 \cdots u_n)$ to the second position of $s_{i_{j+k}} \cdots s_{i_{j+1}} s_1 \cdots s_j(u_1 u_2 \cdots u_n)$. That is,

$$\begin{aligned} s_{i_{j+1}} &= s_k, \\ s_{i_{j+2}} &= s_{k-1}, \\ &\vdots \\ s_{i_{j+k-1}} &= s_2. \end{aligned}$$

Define $\mathscr{A}_2 = s_k s_{k-1} \cdots s_2$.

3. For each letter $l < n$, in increasing order, continue this process of moving letter l into the l^{th} position and recording the adjacent transpositions required to do so with $\mathscr{A}_l$. If l is already in the l^{th} position then it does not need to be moved and so $\mathscr{A}_l = e$.

Observe that once letters $1, \ldots, n-1$ are put into positions $1, \ldots, n-1$ respectively then letter n must be in position n. This algorithm produces an expression

$$s_{i_1} s_{i_2} \cdots s_{i_\ell} = \mathscr{A}_1 \mathscr{A}_2 \cdots \mathscr{A}_{n-1} \tag{1.1.4}$$

for u where $\mathscr{A}_i$ is a suffix of $s_{n-1} s_{n-2} \cdots s_i$ for each $i < n$.

For example, consider the permutation $u = 341625$ in one-line notation. Since 1 is in the third position let $s_{i_1} = s_2$, $s_{i_2} = s_1$, and $\mathscr{A}_1 = s_2 s_1$. Now 2 is in the fifth position of

$$s_1 s_2(341625) = 134625. \tag{1.1.5}$$

Thus we have $s_{i_3} = s_4$, $s_{i_4} = s_3$, $s_{i_5} = s_2$, and $\mathscr{A}_2 = s_4 s_3 s_2$. Next, see that 3 is in the third position and 4 is in the fourth position of

$$s_2 s_3 s_4 s_1 s_2(341625) = 123465, \tag{1.1.6}$$

so these letters do not need to be moved; $\mathscr{A}_3 = e$ and $\mathscr{A}_4 = e$. Now letter 5 is in the sixth position. Thus $s_{i_6} = s_5$ and $\mathscr{A}_5 = s_5$. This puts 5 in the fifth position and 6 in the sixth position of

$$s_5 s_2 s_3 s_4 s_1 s_2(341625) = 123456. \tag{1.1.7}$$

Therefore $s_2 s_1 s_4 s_3 s_2 s_5(123456) = 341625$ and $s_2 s_1 s_4 s_3 s_2 s_5$ is an expression for the permutation 341625.

The following are observations about the structure of expressions produced by the above algorithm.

Observation 1.1.3. *Let $s_{i_1} \cdots s_{i_\ell} = \mathscr{A}_1 \cdots \mathscr{A}_{n-1}$ be the expression for $u \in \mathfrak{S}_n$ determined by the above algorithm, then for every index j, the generator s_j does not appear in A_k for any $k \geq j+1$.*

Observation 1.1.4. *Let $s_{i_1} \cdots s_{i_\ell}$ be the expression for $u \in \mathfrak{S}_n$ determined by the above algorithm. For a fixed $k \in [\ell]$, either $i_k = i_{(k-1)} - 1$ or $i_k \geq i_{(k-1)} + 1$.*

The next three propositions state properties of the permutation

$$\mathscr{A}_i \cdots \mathscr{A}_{n-1}(1 \cdots n)$$

for a fixed i, where $\mathscr{A}_1 \cdots \mathscr{A}_{n-1}(1 \cdots n)$ is an expression produced by the above algorithm.

Proposition 1.1.5. *Let $\mathscr{A}_1 \cdots \mathscr{A}_{n-1}$ be an expression generated by the algorithm. Then for a fixed i, every letter $j \leq i$ is in position j of $\mathscr{A}_{i+1}\mathscr{A}_{i+2} \cdots \mathscr{A}_{n-1}(12 \cdots n)$ and consequently all letters to the right of letter i, in the one-line notation, are greater than i.*

Proof. Since $\mathscr{A}_i$ is a suffix of $s_{n-1}s_{n-2} \cdots s_i$, applying $\mathscr{A}_{n-1}$ to the identity permutation acts on positions $n-1$ and n. Similarly, applying $\mathscr{A}_{n-2}$ to $\mathscr{A}_{n-1}(12 \cdots n)$ acts on positions $n-2$, $n-1$, and n. In general, applying $\mathscr{A}_k$ to $\mathscr{A}_{k+1} \cdots \mathscr{A}_{n-1}(12 \cdots n)$ acts on positions k through n. Thus since every letter j is in position j of the identity, every letter $j \leq i$ is in position j of $\mathscr{A}_{i+1}\mathscr{A}_{i+2} \cdots \mathscr{A}_{n-1}(12 \cdots n)$. Because all the letters less than i are in positions to the left of i in the one-line notation of $\mathscr{A}_{i+1}\mathscr{A}_{i+2} \cdots \mathscr{A}_{n-1}(12 \cdots n)$, it follows that the letters to the right of i are all greater than i. $\square$

Since $\mathscr{A}_1 \cdots \mathscr{A}_{n-1}(1 \cdots n) = u$ it follows that

$$\mathscr{A}_{i+1} \cdots \mathscr{A}_{n-1}(1 \cdots n) = \mathscr{A}_i^{-1} \cdots \mathscr{A}_1^{-1}(u).$$

Therefore, the above proposition implies that letters $j \leq i$ are in positions $j \leq i$ respectively of $\mathscr{A}_i^{-1} \cdots \mathscr{A}_1^{-1}(u)$. The next two propositions follow from the above.

Proposition 1.1.6. *Let $\mathscr{A}_1 \cdots \mathscr{A}_{n-1}$ be an expression generated by the algorithm. If $\mathscr{A}_i \neq e$ then applying $\mathscr{A}_i$ to $\mathscr{A}_{i+1}\mathscr{A}_{i+2} \cdots \mathscr{A}_{n-1}(12 \cdots n)$ moves the letter i to the right in the one-line notation while moving greater letters to the left.*

Proof. Since letter i is in position i of $\mathscr{A}_{i+1}\mathscr{A}_{i+2}\cdots\mathscr{A}_{n-1}(12\cdots n)$ by Proposition 1.1.5 applying s_i swaps i and the letter to its right, then applying s_{i+1} swaps i with the new letter to its right, and so on. (i.e. applying $\mathscr{A}_i$ moves i to the right.) Again by Proposition 1.1.5 these letters being swapped with i are greater than i. $\square$

Proposition 1.1.7. *Let $\mathscr{A}_1\cdots\mathscr{A}_{n-1}$ be an expression generated by the algorithm. If $\mathscr{A}_i = s_j\cdots s_i$ for some $j > i$ then i is in position $j+1$ of $\mathscr{A}_i\mathscr{A}_{i+1}\cdots\mathscr{A}_{n-1}(12\cdots n)$.*

Proof. By Proposition 1.1.5, letter i is in position i of $\mathscr{A}_{i+1}\cdots\mathscr{A}_{n-1}(12\cdots n)$. Thus letter i is in position $i+1$ of $s_i\mathscr{A}_{i+1}\cdots\mathscr{A}_{n-1}(12\cdots n)$. Letter i is then in position $i+2$ of $s_{i+1}s_i\mathscr{A}_{i+1}\cdots\mathscr{A}_{n-1}(12\cdots n)$. Continuing in this fashion, letter i is in position $j+1$ of $s_j\cdots s_i\mathscr{A}_{i+1}\cdots\mathscr{A}_{n-1}(12\cdots n) = \mathscr{A}_i\mathscr{A}_{i+1}\cdots\mathscr{A}_{n-1}(12\cdots n)$. $\square$

The following example illustrates the properties stated in the above propositions. Returning to the permutation $u = 341625$ we see that applying the expression

$$\mathscr{A}_1\mathscr{A}_2\mathscr{A}_3\mathscr{A}_4\mathscr{A}_5 = (s_2s_1)(s_4s_3s_2)(e)(e)(s_5) \tag{1.1.8}$$

for u to the identity permutation,

$$\begin{aligned} \mathscr{A}_5(123456) &= 123465 \\ \mathscr{A}_4(\mathscr{A}_5(123456)) &= 123465 \\ \mathscr{A}_3(\mathscr{A}_4(\mathscr{A}_5(123456))) &= 123465 \\ \mathscr{A}_2(\mathscr{A}_3(\mathscr{A}_4(\mathscr{A}_5(123456)))) &= 134625 \\ \mathscr{A}_1(\mathscr{A}_2(\mathscr{A}_3(\mathscr{A}_4(\mathscr{A}_5(123456))))) &= 341625, \end{aligned} \tag{1.1.9}$$

$\mathscr{A}_5$ swaps the pair of letters $(5,6)$, $\mathscr{A}_4$ and $\mathscr{A}_3$ leave the permutation unchanged, $\mathscr{A}_2$ swaps the pairs of letters $(2,3)$, $(2,4)$, then $(2,6)$, and lastly $\mathscr{A}_1$ swaps the pair $(1,3)$ and then $(1,4)$.

The expression for $u \in \mathfrak{S}_n$ produced by the algorithm is a unique reduced expression. Using the next proposition, this fact will be show in Theorem 1.1.9.

Proposition 1.1.8. *Let $\mathscr{A}_1\cdots\mathscr{A}_{n-1}$ be the expression for $u \in \mathfrak{S}_n$ generated by the algorithm. For fixed $i \in [n-1]$, the length of $\mathscr{A}_i$ is equal to the number of letters $j > i$ which appear before i in $u = u_1\cdots u_n$.*

Proof. Suppose $j > i$ and j appears before i in $u_1 \dots u_n$. Then in the i^{th} step of the algorithm i and j must get swapped. That is, some adjacent transposition s_k in $\mathscr{A}_i$ swaps i and j. Thus the length of $\mathscr{A}_i$ is at least the number of letters $j > i$ which appear before i in u.

By Proposition 1.1.6 the adjacent transpositions in $\mathscr{A}_i$ swap i with letters greater than i. Thus each s_k in $\mathscr{A}_i$ corresponds to a letter $j > i$ such that j appears before i in u. □

Theorem 1.1.9. *For each permutation $u \in \mathfrak{S}_n$, there exists a unique reduced expression of the form $\mathscr{A}_1\mathscr{A}_2 \cdots \mathscr{A}_{n-1}$ where for each i, $\mathscr{A}_i$ is a suffix of the word $s_{n-1} \cdots s_{i+1}s_i$.*

Proof. By the algorithm above, such an expression exists. It remains to be shown that this is a reduced expression and that it is unique. A reduced expression for u has $\ell(u)$ generators where $\ell(u)$ is the number of inversions in u. By Proposition 1.1.8, for a fixed i the length of $\mathscr{A}_i$ is the number of $j > i$ such that (i, j) is an inversion in u. Thus the length of $\mathscr{A}_1\mathscr{A}_2 \cdots \mathscr{A}_{n-1}$ is the total number of inversions in u and therefore $\mathscr{A}_1\mathscr{A}_2 \cdots \mathscr{A}_{n-1}$ is a reduced expression for u.

Suppose two such expressions, $\mathscr{A}_1\mathscr{A}_2 \cdots \mathscr{A}_{n-1}$ and $\mathscr{B}_1\mathscr{B}_2 \cdots \mathscr{B}_{n-1}$, for u exist. Let i be the largest index such that $\mathscr{A}_i \neq \mathscr{B}_i$ and suppose $\mathscr{A}_i = s_j \cdots s_i$ and $\mathscr{B}_i = s_k \cdots s_{j+1}s_j \cdots s_i$. Then by Proposition 1.1.7, i is in position $j + 1$ of $\mathscr{A}_i\mathscr{A}_{i+1} \cdots \mathscr{A}_{n-1}(1 \cdots n)$ and in position $j + 2$ of $s_{j+1}\mathscr{A}_i\mathscr{A}_{i+1} \cdots \mathscr{A}_{n-1}(1 \cdots n)$. If m is in position $j + 2$ of $\mathscr{A}_i\mathscr{A}_{i+1} \cdots \mathscr{A}_{n-1}(1 \cdots n)$ then m is in position $j + 1$ of $s_{j+1}\mathscr{A}_i\mathscr{A}_{i+1} \cdots \mathscr{A}_{n-1}(1 \cdots n)$. By Proposition 1.1.6 we have $m > i$. Furthermore, by Proposition 1.1.8 each of s_k, s_{k-1}, …, s_{j+2} inverts letter i and a larger letter. Thus, since i and m are inverted in $s_{j+1}\mathscr{A}_i\mathscr{A}_{i+1} \cdots \mathscr{A}_{n-1}(1 \cdots n)$, they will be inverted in

$$s_k \cdots s_{j+1}\mathscr{A}_i\mathscr{A}_{i+1} \cdots \mathscr{A}_{n-1}(1 \cdots n) = \mathscr{B}_i \cdots \mathscr{B}_{n-1}(1 \cdots n).$$

Now by Proposition 1.1.6 the adjacent transpositions in $\mathscr{A}_l$ and $\mathscr{B}_l$ invert the letter l and some other greater letter. Thus applying $\mathscr{B}_1\mathscr{B}_2 \dots \mathscr{B}_{i-1}$ to $\mathscr{B}_i \cdots \mathscr{B}_{n-1}(1 \cdots n)$ will not affect the order of i and m and so i and m are inverted in $\mathscr{B}_1 \cdots \mathscr{B}_{n-1}(1 \cdots n)$. Similarly, $\mathscr{A}_1$, $\mathscr{A}_2$, …, $\mathscr{A}_{i-1}$ will not invert i and m. And so these letters will not be inverted in $\mathscr{A}_1 \cdots \mathscr{A}_{n-1}(1 \cdots n)$ since they are not inverted in $\mathscr{A}_i \cdots \mathscr{A}_{n-1}(1 \cdots n)$. This contradicts $\mathscr{A}_1 \cdots \mathscr{A}_{n-1}$ and $\mathscr{B}_1 \cdots \mathscr{B}_{n-1}$ both being reduced expressions for the same permutation. $\square$

It is known that the unique reduced expression described in Theorem 1.1.9 is the right-to-left lexicographically greatest reduced expression [4]. To see another example, consider the longest word $w_0 = n(n-1)\cdots 21$ in $\mathfrak{S}_n$. The right-to-left lexicographically greatest reduced expression for w_0 is

$$(s_{n-1}s_{n-2}\cdots s_1)\cdot(s_{n-1}s_{n-2}\cdots s_2)\cdot(s_{n-1}s_{n-2}\cdots s_3)\cdots(s_{n-1}s_{n-2})\cdot(s_{n-1}), \quad (1.1.10)$$

where

$$\begin{aligned}
\mathscr{A}_1 &= s_{n-1}s_{n-2}\cdots s_1,\\
\mathscr{A}_2 &= s_{n-1}s_{n-2}\cdots s_2,\\
&\vdots\\
\mathscr{A}_{n-2} &= s_{n-1}s_{n-2}\\
\mathscr{A}_{n-1} &= s_{n-1}.
\end{aligned}$$

If $n = 6$ then $w_0 = 654321$ has right-to-left lexicographically greatest reduced expression

$$(s_5s_4s_3s_2s_1)(s_5s_4s_3s_2)(s_5s_4s_3)(s_5s_4)(s_5). \quad (1.1.11)$$

Proposition 1.1.10 and Corollary 1.1.11 give a way to determine the right-to-left lexicographically greatest reduced expression for special permutations less than u in the Bruhat order.

Proposition 1.1.10. *If $s_{i_1}\cdots s_{i_\ell}$ is the right-to-left lexicographically greatest reduced expression for $u \in \mathfrak{S}_n$ then $s_{i_2}\cdots s_{i_\ell}$ is the right-to-left lexicographically greatest reduced expression for $s_{i_1}u$.*

Proof. Let $s_{i_1}\cdots s_{i_\ell}$ be the right-to-left lexicographically greatest reduced expression for u and $s_{j_1}\cdots s_{j_{\ell-1}}$ the right-to-left lexicographically greatest reduced expression for $s_{i_1}u$. Observe that the 2^{nd} through ℓ^{th} iterations of the algorithm for finding the right-to-left lexicographically greatest reduced expression for u are the first through $(\ell-1)^{th}$ iterations of the algorithm for finding the right-to-left lexicographically greatest reduced expression for $s_{i_1}u$. Thus, $s_{j_m} = s_{i_{m+1}}$ for each $1 \leq m \leq \ell - 1$ and so $s_{i_2} \cdots s_{i_\ell}$ is the right-to-left lexicographically greatest reduced expression for $s_{i_1}u$. □

Corollary 1.1.11. *If $s_{i_1} \cdots s_{i_\ell}$ is the right-to-left lexicographically greatest reduced expression for $u \in \mathfrak{S}_n$ then $s_{i_k} \cdots s_{i_\ell}$ is the right-to-left lexicographically greatest reduced expression for $s_{i_{k-1}} \cdots s_{i_1}u$.*

The next proposition discusses the $\oplus$-decomposability of a subword of the right-to-left lexicographically greatest reduced expression for a permutation.

Proposition 1.1.12. *Let $s_{i_1} \cdots s_{i_k}$ be the right-to-left lexicographically greatest reduced expression for a permutation $w \in \mathfrak{S}_n$ and define the permutation $z = ws_{i_k}$. If i_k is a fixed point of z then z is $\oplus$-decomposable into $z^{(1)} \oplus z^{(2)}$ with $z^{(1)} \in \mathfrak{S}_{i_k-1}$ and $z^{(2)} \in \mathfrak{S}_{n-(i_k-1)}$.*

Proof. Suppose z is not $\oplus$-decomposable exactly as above. Then $i_k - 1$ is among the indices $\{i_1, \ldots, i_{(k-1)}\}$ by Observation 1.1.1.

Since $s_{i_1} \cdots s_{i_k}$ is the right-to-left lexicographically greatest reduced expression for $w \in \mathfrak{S}_n$, by Theorem 1.1.9, w can be factored as $\mathscr{A}_1\mathscr{A}_2 \cdots \mathscr{A}_{h+1}$, where each $\mathscr{A}_i$ is a suffix of $s_{n-1} \cdots s_i$. Therefore, z can be factored as

$$\mathscr{A}_1\mathscr{A}_2 \cdots \mathscr{A}_h s_{i_k+j} \cdots s_{i_k+2}s_{i_k+1}$$

where $s_{i_k+j} \cdots s_{i_k}$ is a subword of $\mathscr{A}_{h+1}$ for some $0 \leq j \leq k-1$. Let g be the largest index such that s_{i_k-1} appears in $\mathscr{A}_g$. Since s_{i_k-1} does not appear in $\mathscr{A}_{g+1}, \ldots, \mathscr{A}_h$ it follows that s_{i_k} does not appear in $\mathscr{A}_{g+1}, \ldots, \mathscr{A}_h$ because $\mathscr{A}_i$ is a suffix of $s_{n-1} \cdots s_i$. Thus i_k is a fixed point of

$$\mathscr{A}_{g+1} \cdots \mathscr{A}_h s_{i_k+j} \cdots s_{i_k+1}.$$

Furthermore, s_{i_k-1} appearing in $\mathscr{A}_g$ implies i_k appears in position $i_k - 1$ of the one-line notation of

$$\mathscr{A}_g \cdots \mathscr{A}_h s_{i_k+j} \cdots s_{i_k+1}.$$

In particular, i_k is not a fixed point.

Now let f be the smallest positive integer such that $\mathscr{A}_{g-f}$ contains $s_{i_k-1} = s_{i_m}$, then necessarily $s_{i_{(m+1)}} = s_{i_k-2}$ because $\mathscr{A}_i$ is a suffix of $s_{n-1} \cdots s_i$. Thus i_k is not a fixed point of

$$\mathscr{A}_{g-f} \cdots \mathscr{A}_g \cdots \mathscr{A}_h s_{i_{(k-j)}} \cdots s_{i_{(k-1)}}.$$

In fact, letter i_k is in position $i_k - 2$. Furthermore, any index $i_1, \ldots, i_{(m-1)}$ equal to $i_k - 2$ is followed by the index $i_k - 3$. Therefore, no generator $s_{i_1}, \ldots, s_{i_{(m-1)}}$ will return letter i_k to position i_k. Thus i_k is not a fixed point of $s_{i_1} \cdots s_{i_{(k-1)}}$, a contradiction. Hence $i_k - 1$ is not among $\{i_1, \ldots, i_{(k-1)}\}$ as assumed. Thus by Observation 1.1.1 z must be $\oplus$-decomposable into $z^{(1)} \oplus z^{(2)}$ with $z^{(1)} \in \mathfrak{S}_{i_k-1}$ and $z^{(2)} \in \mathfrak{S}_{n-(i_k-1)}$. $\square$

Given an expression $s_{i_1} \cdots s_{i_\ell}$ of a permutation $w \in \mathfrak{S}_n$, define a subexpression $p_1 \cdots p_\ell$ of $s_{i_1} \cdots s_{i_\ell}$ to be an expression such that $p_j \in \{e, s_{i_j}\}$ for every $j \in [\ell]$. The following propositions show that the use of a right-to-left lexicographically greatest reduced expression for w allows one to deduce that certain letters appear in certain positions of the one-line notations of subwords of w.

Proposition 1.1.13. *Let $s_{i_1} \cdots s_{i_k}$ be the right-to-left lexicographically greatest reduced expression for $w \in \mathfrak{S}_n$. Define the permutation $z = ws_{i_k}$. If i_k is a fixed point of z, then i_k is a fixed point of $p_1 \cdots p_{k-1}$, where $p_1 \cdots p_{k-1}$ is a subexpression of $s_{i_1} \cdots s_{i_{k-1}}$.*

Proof. By Proposition 1.1.12 since i_k is a fixed point of z, it follows that z is $\oplus$-decomposable into $z^{(1)} \oplus z^{(2)}$ with $z^{(1)} \in \mathfrak{S}_{i_k-1}$ and $z^{(2)} \in \mathfrak{S}_{n-(i_k-1)}$. Thus, for some index $1 \leq j \leq k$, $z^{(1)} = s_{i_1} \cdots s_{i_j}$ and $z^{(2)} = s_{i_{j+1}-(i_k-1)} \cdots s_{i_{(k-1)}-(i_k-1)}$. Since $z^{(1)} \in \mathfrak{S}_{i_k-1}$, the indices $i_1, \ldots, i_j$ are all less than $i_k - 1$. Since $z^{(2)} = s_{i_{j+1}-(i_k-1)} \cdots s_{i_{k-1}-(i_k-1)}$ is in $\mathfrak{S}_{n-(i_k-1)}$ the indices $i_{j+1}, \ldots, i_{k-1}$ are all greater than $i_k - 1$. Thus $i_k - 1$ is not among the indices $\{i_1, \ldots, i_{(k-1)}\}$. Recall that since $s_{i_1} \cdots s_{i_k}$ is the right-to-left lexicographically greatest reduced expression for w it can be expressed as $\mathscr{A}_1 \cdots \mathscr{A}_{i_k}$ where each $\mathscr{A}_i$ is a suffix of $s_{n-1} \cdots s_i$. Observe that for any $i < i_k$, if $\mathscr{A}_i$ contains s_{i_k} as a subword then it also contains s_{i_k-1} as a subword.

Therefore $i_k - 1 \notin \{i_1, \ldots, i_{(k-1)}\}$ implies $i_k \notin \{i_1, \ldots, i_{(k-1)}\}$. Now since $s_{i_l} \neq s_{i_k}$ and $s_{i_l} \neq s_{i_k - 1}$ for any $1 \leq l \leq k-1$ it follows that $p_l \neq s_{i_k}$ and $p_l \neq s_{i_k-1}$ for any $1 \leq l \leq k-1$. Therefore, i_k is a fixed point of $p_1 \cdots p_{k-1}$. □

Proposition 1.1.14. *Let $s_{i_1} \cdots s_{i_k}$ be the right-to-left lexicographically greatest reduced expression for a permutation $w \in \mathfrak{S}_n$, and $p_1 \cdots p_{k-1}$ a subexpression of $s_{i_1} \cdots s_{i_{k-1}}$. If $(s_{i_{(k-1)}} \cdots s_{i_1})_{i_k} = d$ and $(p_{k-1} \cdots p_1)_{i_k} \neq d$, then $d < i_k$.*

Proof. Assume $(s_{i_{(k-1)}} \cdots s_{i_1})_{i_k} = d$ and $(p_{k-1} \cdots p_1)_{i_k} \neq d$, then by Proposition 1.1.13, $d \neq i_k$. Observe that since $s_{i_1} \cdots s_{i_k}$ is a right-to-left lexicographically greatest reduced expression, by Theorem 1.1.9, w can be factored as $\mathscr{A}_1 \cdots \mathscr{A}_{i_k}$, where each $\mathscr{A}_i$ is a suffix of $s_{n-1} \cdots s_i$. Therefore, $s_{i_1} \cdots s_{i_{(k-1)}}$ can be factored as

$$\mathscr{A}_1 \cdots \mathscr{A}_{i_k - 1} s_{i_k + j} \cdots s_{i_k + 1}$$

for some $1 \leq j$. Suppose $d > i_k$. Then for some h, $s_{(d-1)} s_{(d-2)} \cdots s_{i_k}$ is a subword of $\mathscr{A}_h$. Let h be the smallest such index. Then d is in position i_k of

$$s_{i_k} \cdots s_{(d-2)} s_{(d-1)} \mathscr{A}_{h-1}^{-1} \cdots \mathscr{A}_1^{-1}.$$

Furthermore, d is in position h of

$$s_h \cdots s_{i_k} \cdots s_{(d-2)} s_{(d-1)} \mathscr{A}_{h-1}^{-1} \cdots \mathscr{A}_1^{-1}.$$

Now since s_h does not appear in $\mathscr{A}_{h+1}$, $\mathscr{A}_{h+2}$, $\ldots$, $\mathscr{A}_{i_k}$ it follows that d is in position h of

$$s_{i_k+1} \cdots s_{i_k+j} \mathscr{A}_{i_k-1}^{-1} \cdots \mathscr{A}_1^{-1} = s_{i_{(k-1)}} \cdots s_{i_1}.$$

This is a contradiction because d is in position i_k of $s_{i_{(k-1)}} \cdots s_{i_1}$. Thus $d < i_k$. □

Proposition 1.1.15. *Let $s_{i_1} \cdots s_{i_k}$ be the right-to-left lexicographically greatest reduced expression for a permutation $w \in \mathfrak{S}_n$, and $p_1 \cdots p_{k-1}$ a subexpression of $s_{i_1} \cdots s_{i_{k-1}}$. If $(s_{i_{(k-1)}} \cdots s_{i_1})_{i_k} = d$ and $(p_{(k-1)} \cdots p_1)_{i_k} \neq d$, then there exists some index $\eta < k$ such that d is in position i_η of $p_\eta \cdots p_1$ and position $i_\eta + 1$ of $s_{i_\eta} \cdots s_{i_1}$.*

Proof. By Proposition 1.1.14, letter $d < i_k$. This implies $s_d s_{(d+1)} \cdots s_{i_k-1}$ is a subword of $s_{i_1} s_{i_2} \cdots s_{i_{(k-1)}}$. But since d is not in position i_k of $p_{k-1} \cdots p_1$ it follows that for at least one of these $s_d = s_{i_\gamma}$, $s_{(d+1)} = s_{i_\delta}$, $\cdots$, $s_{i_k-1} = s_{i_\rho}$ the corresponding p_γ, p_δ, $\cdots$, or p_ρ is the identity permutation. Let η be the smallest such index. That is, $p_\eta = e$, d is in position i_η of $s_{i_{(\eta-1)}} \cdots s_{i_1}$ and $p_{(\eta-1)} \cdots p_1$. Thus d is in position $i_\eta + 1$ of $s_{i_\eta} s_{i_{\eta-1}} \cdots s_{i_1}$ and in position i_η of $p_\eta p_{\eta-1} \cdots p_1$. $\square$

Corollary 1.1.16. *Let $s_{i_1} \cdots s_{i_\ell}$ be the right-to-left lexicographically greatest reduced expression for a permutation $u \in \mathfrak{S}_n$. Fix $k \in [\ell]$ and let $p_1 \cdots p_{k-1}$ be a subexpression of $s_{i_1} \cdots s_{i_{k-1}}$. For $z \in \mathfrak{S}_n$, define $w = s_{i_{(k-1)}} \cdots s_{i_1} z$ and $v = p_{(k-1)} \cdots p_1 z$. If $w_{i_k} = d$ and $v_{i_k} \neq d$, then there exists some index $\eta < k$ such that d is in position i_η of $p_\eta \cdots p_1 z$ and position $i_\eta + 1$ of $s_{i_\eta} \cdots s_{i_1} z$.*

1.2 Defining a map on $\mathfrak{S}_n$

The following map on $\mathfrak{S}_n$ will be used to develop notation to discuss different bases of the immanant space, which will be introduced in Section 1.3. For a fixed $u \in \mathfrak{S}_n$ with right-to-left lexicographically greatest reduced expression $s_{i_1} \cdots s_{i_\ell}$ and a number $k \in [\ell + 1]$, define the function $\phi_{k,u} : \mathfrak{S}_n \to \mathfrak{S}_n$ by

$$w \mapsto s_{i_{k-1}} \cdots s_{i_1} w \tag{1.2.1}$$

That is, $\phi_{k,u}(w)$ is equal to the inverse of the product of the first $k - 1$ generators in the given expression of u times w. Observe that

$$\phi_{k,u}(u) = s_{i_{k-1}} \cdots s_{i_1} u = s_{i_k} s_{i_{k+1}} \cdots s_{i_\ell}. \tag{1.2.2}$$

Continuing with the example $u = 341625$, with right-to-left lexicographically greatest reduced expression $s_2 s_1 s_4 s_3 s_2 s_5$, and considering $w = 431652$ we have the following.

$$
\begin{aligned}
\phi_{1,u}(u) &= u = 341625 & \phi_{1,u}(w) &= w = 431652\\
\phi_{2,u}(u) &= s_2u = 314625 & \phi_{2,u}(w) &= s_2w = 413652\\
\phi_{3,u}(u) &= s_1s_2u = 134625 & \phi_{3,u}(w) &= s_1s_2w = 143652\\
\phi_{4,u}(u) &= s_4s_1s_2u = 134265 & \phi_{4,u}(w) &= s_4s_1s_2w = 143562\\
\phi_{5,u}(u) &= s_3s_4s_1s_2u = 132465 & \phi_{5,u}(w) &= s_3s_4s_1s_2w = 145362\\
\phi_{6,u}(u) &= s_2s_3s_4s_1s_2u = 123465 & \phi_{6,u}(w) &= s_2s_3s_4s_1s_2w = 154362\\
\phi_{7,u}(u) &= s_5s_2s_3s_4s_1s_2u = 123456 & \phi_{7,u}(w) &= s_5s_2s_3s_4s_1s_2w = 154326
\end{aligned}
$$

Observe that for all $w \in \mathfrak{S}_n$, $\phi_{\ell+1,u}(w) = u^{-1}w$, $\phi_{1,u}(w) = w$, and $s_{i_k}\phi_{k,u}(w) = \phi_{k+1,u}(w)$. Thus $\phi_{\ell+1,u}(u) = e$. Furthermore, it follows directly from Proposition 1.1.5 that if $\phi_{k,u}(u) = s_{i_k} \cdots s_j \mathscr{A}_{j+1} \cdots \mathscr{A}_{n-1}$ then $\phi_{k,u}(u)$ has letters 1 through $j-1$ in positions 1 through $j-1$ respectively. The following lemma relates the map $\phi_{k,s_{i_1}u}$ and the map $\phi_{k+1,u}$.

Proposition 1.2.1. *Fix u, $w \in \mathfrak{S}_n$ with u having right-to-left lexicographically greatest reduced expression $s_{i_1} \cdots s_{i_\ell}$ and an index $k \in [\ell]$. Then*

$$\phi_{k,s_{i_1}u}(s_{i_1}w) = \phi_{k+1,u}(w). \tag{1.2.3}$$

Proof. By Proposition 1.1.10 the right-to-left lexicographically greatest reduced expression for $s_{i_1}u$ is $s_{i_2} \cdots s_{i_\ell}$. The first $(k-1)$-letter subword of this is $s_{i_2} \cdots s_{i_k}$. Thus

$$\phi_{k,s_{i_1}u}(s_{i_1}w) = s_{i_k} \cdots s_{i_2}(s_{i_1}w) = \phi_{k+1,u}(w) \tag{1.2.4}$$

□

The following lemma by Björner and Brenti [2, Lemma 2.2.10] will allow us to state a nice relationship between the maps $\phi_{k,u}(w)$ and $\phi_{k-1,s_{i_1}u}(w)$.

Lemma 1.2.2. *Suppose that $x < xt$ and $y < ty$, for x,y in a Coxeter group W and t in the reflection set of W. Then $xy < xty$.*

Proposition 1.2.3. *Fix u, $w \in \mathfrak{S}_n$ with u having right-to-left lexicographically greatest reduced expression $s_{i_1} \cdots s_{i_\ell}$ and an index k in the interval $[2, \ell+1]$. If $s_{i_1}w < w$ then*

$$\phi_{k,u}(w) < \phi_{k-1,s_{i_1}u}(w) \tag{1.2.5}$$

in the Bruhat order.

Proof. Observe that $s_{i_{k-1}} \cdots s_{i_2} < s_{i_{k-1}} \cdots s_{i_2} s_{i_1}$ in the Bruhat order and by assumption $s_{i_1} w < w$ in the Bruhat order. Thus applying Lemma 1.2.2 with $x = s_{i_{k-1}} \cdots s_{i_2}$, $y = s_{i_1} w$, and $t = s_{i_1}$ yields

$$(s_{i_{k-1}} \cdots s_{i_2})(s_{i_1} w) < (s_{i_{k-1}} \cdots s_{i_2})(s_{i_1})(s_{i_1} w) = s_{i_{k-1}} \cdots s_{i_2} w. \tag{1.2.6}$$

as claimed. $\square$

The following observation states a recursive way of finding the length of the permutation $\phi_{k+1,u}(u)$.

Observation 1.2.4. *If $s_{i_1} \cdots s_{i_\ell}$ is the right-to-left lexicographically greatest reduced expression for u, then for a fixed $k \in [\ell + 1]$, s_{i_k} is a left descent of $\phi_{k,u}(u)$. Furthermore, $\ell(\phi_{k+1,u}(u)) = \ell(\phi_{k,u}(u)) - 1$.*

Since s_{i_k} is a left descent of $\phi_{k,u}(u)$, it follows that the letter in position $i_k + 1$ of $\phi_{k,u}(u)$ is less than the letter in position i_k of $\phi_{k,u}(u)$. Moreover, it will be shown that the letter in position $i_k + 1$ of $\phi_{k,u}(u)$ is the least letter with a greater letter to its left in $\phi_{k,u}(u)$, but first some new terminology needs to be introduced.

Writing the right-to-left lexicographically greatest reduced expression for u as

$$\mathscr{A}_1 \cdots \mathscr{A}_{n-1} = s_{i_1} \cdots s_{i_\ell}, \tag{1.2.7}$$

it follows that

$$\phi_{k,u}(u) = s_{i_k} \cdots s_j \mathscr{A}_{j+1} \cdots \mathscr{A}_{n-1} \tag{1.2.8}$$

for some index j. For fixed k, let $\alpha_k(u)$ denote this index. Thus for the factorization in Equation (1.1.8), we have

$$(\alpha_1(u), \ldots, \alpha_6(u)) = (1, 1, 2, 2, 2, 5).$$

Observation 1.2.5. *For all $u \in \mathfrak{S}_n$ of length ℓ,*

$$1 \leq \alpha_1(u) \leq \alpha_2(u) \leq \cdots \leq \alpha_\ell(u) \leq n - 1.$$

With this notation, $\alpha_k(u)$ can be interpreted as the least letter having a greater letter to its left. This letter also has two important properties which are stated in the following propositions.

Proposition 1.2.6. *For $u \in \mathfrak{S}_n$ with right-to-left lexicographically greatest reduced expression $s_{i_1} \cdots s_{i_\ell}$ and $1 \leq k \leq \ell(u)$ the least letter in $[n-1]$ having a greater letter to its left in $\phi_{k,u}(u)$ is $\alpha_k(u)$.*

Proof. By Proposition 1.1.5 the letters 1 through j are in positions 1 through j of $\mathscr{A}_{j+1} \cdots \mathscr{A}_{n-1}(1 \cdots n)$, respectively. By Proposition 1.1.6, applying $s_{i_k} \cdots s_j$ to $\mathscr{A}_{j+1} \cdots \mathscr{A}_{n-1}(1 \cdots n)$ swaps letter j with greater letters. Thus j is the smallest letter with a larger letter to its left in $\phi_{k,u}(u) = s_{i_k} \cdots s_{i_j} \mathscr{A}_{j+1} \cdots \mathscr{A}_{n-1}$ and by definition $\alpha_k(u) = j$. $\square$

Proposition 1.2.7. *For $u \in \mathfrak{S}_n$ with right-to-left lexicographically greatest reduced expression $s_{i_1} \cdots s_{i_\ell}$ and $1 \leq k \leq \ell(u)$, the letter moving to the right when s_{i_k} is applied to $s_{i_{k+1}} \cdots s_{i_\ell}(1 \cdots n)$ is $\alpha_k(u)$.*

Proof. Since $\mathscr{A}_1 \cdots \mathscr{A}_{n-1} = s_{i_1} \cdots s_{i_\ell}$ it follows that for some index j,

$$s_{i_k} s_{i_{k+1}} \cdots s_{i_\ell} = s_{i_k} \cdots s_j \mathscr{A}_{j+1} \cdots \mathscr{A}_{n-1}.$$

Letter j is in position j of $\mathscr{A}_{j+1} \cdots \mathscr{A}_{n-1}(12 \cdots n)$ by Proposition 1.1.5. Thus, applying $s_{i_k} s_{i_{k+1}} \cdots s_j$ to $\mathscr{A}_{j+1} \cdots \mathscr{A}_{n-1}(12 \cdots n)$ moves j to its right. By definition, $\alpha_k(u) = j$ and the claim holds. $\square$

Proposition 1.2.8. *Let $s_{i_1} \cdots s_{i_\ell}$ be the right-to-left lexicographically greatest reduced expression for u and let $\phi_{k,u}(u)$ have one-line notation $w_1 \cdots w_n$. Then for each $k \in [\ell]$, $\alpha_k(u) = w_{i_k+1}$.*

Proof. Let $s_{i_1} \cdots s_{i_\ell}$ be the right-to-left lexicographically greatest reduced expression for u. Then $s_{i_1} \cdots s_{i_\ell}$ can be written as $\mathscr{A}_1 \mathscr{A}_2 \cdots \mathscr{A}_{n-1}$ where $\mathscr{A}_i$ is a suffix of $s_{n-1} \cdots s_{i+1} s_i$. By Proposition 1.2.7 $\alpha_k(u)$ is moved right when s_{i_k} is applied to $s_{i_{k+1}} \cdots s_{i_\ell}$. We know that s_{i_k} swaps the letters in positions i_k and $i_k + 1$ of $s_{i_{k+1}} \cdots s_{i_\ell}$. Thus $\alpha_k(u)$ is in position i_k of $s_{i_{k+1}} \cdots s_{i_\ell}$ and in position $i_k + 1$ of $s_{i_k} s_{i_{k+1}} \cdots s_{i_\ell} = \phi_{k,u}(u)$. That is, $\alpha_k(u) = w_{i_k+1}$. $\square$

1.3 The Quantum Matrix Bialgebra $\mathcal{A}(n;q)$

The quantum matrix bialgebra, $\mathcal{A}(n;q)$, is the noncommutative $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-algebra generated by the n^2 variables $x = (x_{1,1}, \ldots, x_{n,n})$ subject to the relations

$$\begin{aligned} x_{i,\ell}x_{i,k} &= q^{\frac{1}{2}} x_{i,k}x_{i,\ell}, \\ x_{j,k}x_{i,k} &= q^{\frac{1}{2}} x_{i,k}x_{j,k}, \\ x_{j,k}x_{i,\ell} &= x_{i,\ell}x_{j,k}, \\ x_{j,\ell}x_{i,k} &= x_{i,k}x_{j,\ell} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x_{i,\ell}x_{j,k}, \end{aligned} \tag{1.3.1}$$

for all indices $1 \leq i < j \leq n$ and $1 \leq k < \ell \leq n$. The relations (1.3.1) allow one to express every monomial as a linear combination of monomials $x_{\ell_1,m_1} \cdots x_{\ell_r,m_r}$ in which the index pairs appear in lexicographic order. For example, the monomial $x_{13}x_{23}x_{12}$ is not in lexicographic order. However, using the relations it can be written as $q^{\frac{1}{2}}x_{12}x_{13}x_{23} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x_{13}x_{13}x_{22}$, where the indices are in lexicographic order. Thus, as a $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-module, $\mathcal{A}(n;q)$ has a basis of monomials $x_{\ell_1,m_1} \cdots x_{\ell_r,m_r}$ in which index pairs appear in lexicographic order. This basis is called the *natural basis* of $\mathcal{A}(n;q)$.

The $(n \times n)$ *quantum determinant*,

$$det_q(x) \underset{\text{def}}{=} \sum_{w \in \mathfrak{S}_n} \left(-q^{\frac{-1}{2}}\right)^{inv(w)} x_{1,w_1} \cdots x_{n,w_n}, \tag{1.3.2}$$

a central element of $\mathcal{A}(n;q)$, relates $\mathcal{A}(n;q)$ to the class of algebras called *quantum groups*. In particular, the quotient $\mathcal{A}(n;q)/(det_q(x) - 1)$ is a quantum group called the *quantum coordinate ring* of the special linear group $SL(n, \mathbb{C})$. Other important elements of $\mathcal{A}(n;q)$ are the *quantum minors* $det_q(x_{I,J})$ and the *quantum permanent* of x

$$per_q(x) \underset{\text{def}}{=} \sum_{w \in \mathfrak{S}_n} \left(q^{\frac{1}{2}}\right)^{inv(w)} x_{1,w_1} \cdots x_{n,w_n}. \tag{1.3.3}$$

Specializing $\mathcal{A}(n;q)$, $det_q(x)$, and $per_q(x)$ at $q^{\frac{1}{2}} = 1$ yields the commutative polynomial ring $\mathbb{C}[x]$, the classical determinant $det(x)$, and classical permanent $per(x)$, respectively.

The quantum polynomial ring, $\mathcal{A}(n;q)$, has a natural grading by degree,

$$\mathcal{A}(n;q) = \oplus_{r\geq 0}\mathcal{A}_r(n;q). \tag{1.3.4}$$

The immanant space, $\mathcal{A}_{[n],[n]}(n;q)$, is the $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-submodule of $\mathcal{A}(n;q)$ spanned by

$$\{x_{1,w_1}\cdots x_{n,w_n} | w \in \mathfrak{S}_n\}. \tag{1.3.5}$$

Defining the notation $x^{u,w} = x_{u_1,w_1}\cdots x_{u_n,w_n}$, the natural basis for the immanant space can be expressed as $\{x^{e,w}|w \in \mathfrak{S}_n\}$. It is straightforward to express a monomial $x^{u,w}$ in terms of $\{x^{y,v}|v \in \mathfrak{S}_n\}$ when u covers y in the weak order.

Lemma 1.3.1. *Given permutations u, $w \in \mathfrak{S}_n$, for each left descent s of u we have*

$$x^{u,w} = \begin{cases} x^{su,sw} & \text{if } sw > w, \\ x^{su,sw} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{su,w} & \text{if } sw < w. \end{cases} \tag{1.3.6}$$

Proof. Since s is a left descent of u and $w \in \mathfrak{S}_n$ the claim follows directly from the third and fourth relations (1.3.1) in the presentation of $\mathcal{A}(n;q)$. $\square$

There is a similar formula for s a left ascent of u. Thus it is just as straightforward to express a monomial $x^{u,w}$ in terms of $\{x^{y,v}|v \in \mathfrak{S}_n\}$ when y covers u in the weak order. It follows that for any fixed $y \in \mathfrak{S}_n$ the set $\{x^{y,v}|v \in \mathfrak{S}_n\}$ forms a basis for $\mathcal{A}_{[n],[n]}(n;q)$. However, there is no known general formula which gives the coordinate vector of a monomial $x^{u,w}$ with respect to such a basis, unless $y = e$. Lambright [12, Prop 1.4.2] showed that for fixed $u \in \mathfrak{S}_n$, $x^{u,w}$ can be expressed in the basis $\{x^{\phi_{\ell+1,u}(u),v}|v \in \mathfrak{S}_n\} = \{x^{e,v}|v \in \mathfrak{S}_n\}$ of $\mathcal{A}_{[n],[n]}(n;q)$ as follows.

Proposition 1.3.2. *For all u, $w \in \mathfrak{S}_n$,*

$$x^{u,w} = \sum_{v \geq u^{-1}w} p_{u,w,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{e,v}. \tag{1.3.7}$$

Extending this result, for a fixed $u \in \mathfrak{S}_n$ of length ℓ and an index $k \in [\ell+1]$ a formula for the coordinate vector of $x^{u,w}$ with respect to a basis of the form $\{x^{\phi_{k,u}(u),v}|v \in \mathfrak{S}_n\}$ is provided.

Definition 1.3.3. *For a fixed $u \in \mathfrak{S}_n$ with right-to-left lexicographically greatest reduced expression $s_{i_1}s_{i_2}\cdots s_{i_\ell}$, a number $k \in [\ell+1]$, and any permutation $w \in \mathfrak{S}_n$, define the polynomials*

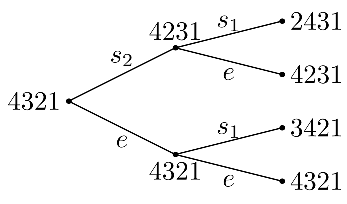
$$\{t^k_{u,w,v}(q_1) = \sum_b c_b q_1^b \mid v \in \mathfrak{S}_n\} \subset \mathbb{N}[q_1] \tag{1.3.8}$$

by interpreting c_b as the number of sequences $\pi = (\pi^{(0)}, \pi^{(1)}, \ldots, \pi^{(k-1)})$ satisfying

1. $\pi^{(0)} = w$, $\pi^{(k-1)} = v$
2. $\pi^{(j)} \in \{s_{i_j}\pi^{(j-1)}, \pi^{(j-1)}\}$ *for* $j = 1, \ldots, k-1$
3. $\pi^{(j)} = s_{i_j}\pi^{(j-1)}$ *if* $s_{i_j}\pi^{(j-1)} > \pi^{(j-1)}$
4. $\pi^{(j)} = \pi^{(j-1)}$ *for exactly b values of j.*

When $k = \ell+1$ this definition reduces to the definition of the (Laurent) polynomials $p_{u,w,v}$ studied by Lambright [12]. Note that necessarily $0 \le b \le k-1$ because $j \le k-1$. Observe that for $k = 1$ the only sequence $\pi = (\pi^{(0)})$ satisfying the four conditions of Definition 1.3.3 is $\pi^{(0)} = w$. Thus $t^1_{u,w,v}(q_1) = \delta_{v,w}$. Similarly, if $u = e$ then necessarily $k = 1$ and so $t^1_{e,w,v}(q_1) = \delta_{v,w}$.

For example, let $u = 3412$, $w = 4321$, and $k = 3$. The right-to-left lexicographically greatest reduced expression for u is $s_2s_1s_3s_2$ and the sequences $(\pi^{(0)}, \pi^{(1)}, \pi^{(2)})$ satisfying the conditions of Definition 1.3.3 are the vertices of maximal paths in the tree



Recording repetitions and final components in these sequences we have

$(\pi^{(0)}, \pi^{(1)}, \pi^{(2)})$	# of repetitions	$t^3_{u,w,v}(q_1)$
$(4321, 4231, 2431)$	0	$t^3_{3412,4321,2431}(q_1) = q_1^0 = 1$
$(4321, 4231, 4231)$	1	$t^3_{3412,4321,4231}(q_1) = q_1^1$
$(4321, 4321, 3421)$	1	$t^3_{3412,4321,3421}(q_1) = q_1^1$
$(4321, 4321, 4321)$	2	$t^3_{3412,4321,4321}(q_1) = q_1^2$

Alternatively the sequences $(\pi^{(0)}, \ldots, \pi^{(k-1)})$ may be encoded by $(k-1)$-letter words $p_{k-1} \ldots p_1$ in the alphabet $\{e, s_1, \ldots, s_{n-1}\}$ where

$$p_j = \pi^{(j)}(\pi^{(j-1)})^{-1} = \begin{cases} s_{i_j} & \text{if } \pi^{(j)} = s_{i_j}\pi^{(j-1)} \\ e & \text{if } \pi^{(j)} = \pi^{(j-1)}. \end{cases}$$

In the previous example, words corresponding to the sequences $(\pi^{(0)}, \pi^{(1)}, \pi^{(2)})$ are given by the edges in the tree,

$$s_1 s_2 \quad e s_2 \quad s_1 e \quad ee.$$

Observe that if π is a path from w to v and the word $p_{k-1} \cdots p_1$ encodes this path, then $v = p_{k-1} \cdots p_1 w$.

For fixed u, the polynomials $\{t^k_{u,w,v} | w, v \in \mathfrak{S}_n, 1 \leq k \leq \ell + 1\}$ satisfy a nice recurrence relation.

Proposition 1.3.4. *Fix $u \in \mathfrak{S}_n$ with right-to-left lexicographically greatest reduced expressions $s_{i_1} \cdots s_{i_\ell}$, $w \in \mathfrak{S}_n$, and $k \in [\ell + 1]$. Then*

$$t^k_{u,w,v}(q_1) = \begin{cases} t^{k-1}_{s_{i_1}u, s_{i_1}w, v}(q_1) & \text{if } s_{i_1}w > w, \\ t^{k-1}_{s_{i_1}u, s_{i_1}w, v}(q_1) + q_1 t^{k-1}_{s_{i_1}u, w, v}(q_1) & \text{if } s_{i_1}w < w. \end{cases} \tag{1.3.9}$$

Proof. Let $C^{k,b}_{u,w,v}$ be the set of sequences counted by the coefficient c_b in Definition 1.3.3, and consider a sequence $\pi \in C^{k,b}_{u,w,v}$.

Suppose first that $s_{i_1}w > w$. Then π satisfies $\pi^{(1)} = s_{i_1}w$ and we have $b \leq k-2$. It follows that for $b = k-2, \ldots, 0$, the sequence $(\pi^{(1)}, \ldots, \pi^{(k-1)})$ satisfies the conditions

(1′) $\pi^{(1)} = s_{i_1}w$, $\pi^{(k-1)} = v$

(2′) $\pi^{(j)} \in \{s_{i_j}\pi^{(j-1)}, \pi^{(j-1)}\}$ for $j = 2, \ldots, k-1$

(3′) $\pi^{(j)} = s_{i_j}\pi^{(j-1)}$ if $s_{i_j}\pi^{(j-1)} > \pi^{(j-1)}$

(4′) $\pi^{(j)} = \pi^{(j-1)}$ for exactly b values of j.

Thus the map

$$(\pi^{(0)}, \pi^{(1)}, \ldots, \pi^{(k-1)}) \mapsto (\pi^{(1)}, \ldots, \pi^{(k-1)}) \tag{1.3.10}$$

is a bijection from $C^{k,b}_{u,w,v}$ to $C^{k-1,b}_{s_{i_1}u, s_{i_1}w, v}$.

Now suppose $s_{i_1}w < w$. Then π satisfies $\pi^{(1)} = s_{i_1}w$ or $\pi^{(1)} = w$. If $\pi^{(1)} = s_{i_1}w$, then for $b = k-2, \ldots, 0$ the sequence $(\pi^{(1)}, \ldots, \pi^{(k-1)})$ satisfies conditions $(1') - (4')$ above. Otherwise it satisfies conditions $(2') - (3')$ and

(1″) $\pi^{(1)} = w$, $\pi^{(k-1)} = v$,

(4″) $\pi^{(j)} = \pi^{(j-1)}$ for exactly $b-1$ values of $j = 2, \ldots, k-1$.

Thus the map (1.3.10) is a bijection from $C^{k,b}_{u,w,v}$ to $C^{k-1,b}_{s_{i_1}u, s_{i_1}w, v} \cup C^{k-1,b-1}_{s_{i_1}u, w, v}$.

□

Exploiting this recurrence relation, a formula for the coordinate vector of $x^{u,w}$ with respect to a basis of the form $\{x^{\phi_{k,u}(u),v} | v \in \mathfrak{S}_n\}$ for a fixed $u \in \mathfrak{S}_n$ of length ℓ and an index $k \in [\ell+1]$ is constructed.

Theorem 1.3.5. *Fix permutations u, $w \in \mathfrak{S}_n$, choose an integer $k \in [\ell(u)+1]$, and define the permutation $u' = \phi_{k,u}(u)$. Then in $\mathcal{A}_{[n],[n]}(n;q)$ we have*

$$x^{u,w} = \sum_{v \in \mathfrak{S}_n} t^k_{u,w,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) x^{u',v}. \tag{1.3.11}$$

Proof. First consider the case of $k = 1$. Then $u' = u$ and the expression for $x^{u,w}$ in terms of the basis $\{x^{u',v} | v \in \mathfrak{S}_n\}$ can be written as

$$x^{u,w} = \sum_{v \in \mathfrak{S}_n} \delta_{w,v} x^{u',v}. \tag{1.3.12}$$

On the other hand, by the observation following Definition 1.3.3, we have $t^1_{u,w,v} = \delta_{w,v}$, and this polynomial evaluates at $q_1 = q^{\frac{1}{2}} - q^{\frac{-1}{2}}$ to $\delta_{w,v}$. Now consider the case that $u = e$. In this case it must be that $k = 1$ and again the claimed equality holds.

Now let u be a permutation with right-to-left lexicographically greatest reduced expression $s_{i_1} \cdots s_{i_\ell}$, let $k-1$ be an integer in $[\ell]$, and assume the equality (1.3.11) holds when u and k are replaced by a permutation of length $\ell-1$ and the integer $k-1$, respectively. Writing $s=s_{i_1}$, use Lemma 1.3.1 to express $x^{u,w}$ in terms of $x^{su,sw}$ and $x^{su,w}$, and induction to express these monomials in terms of the basis $\{x^{\phi_{k-1,su}(su),v}|v\in\mathfrak{S}_n\}$. Letting $\Delta q=q^{\frac{1}{2}}-q^{\frac{-1}{2}}$, we have that $x^{u,w}$ is equal to

$$\begin{cases}\sum_{v\in\mathfrak{S}_n} t^{k-1}_{su,sw,v}(\Delta q)x^{\phi_{k-1,su}(su),v} & \text{if } sw>w,\\ \sum_{v\in\mathfrak{S}_n} t^{k-1}_{su,sw,v}(\Delta q)x^{\phi_{k-1,su}(su),v}+(\Delta q)\sum_{v\in\mathfrak{S}_n} t^{k-1}_{su,w,v}(\Delta q)x^{\phi_{k-1,su}(su),v} & \text{if } sw<w.\end{cases}$$

By Proposition 1.2.1, it follows that

$$x^{u,w}=\begin{cases}\sum_{v\in\mathfrak{S}_n} t^{k-1}_{su,sw,v}(q^{\frac{1}{2}}-q^{\frac{-1}{2}})x^{u',v} & \text{if } sw>w,\\ \sum_{v\in\mathfrak{S}_n}\Big(t^{k-1}_{su,sw,v}(q^{\frac{1}{2}}-q^{\frac{-1}{2}})+(q^{\frac{1}{2}}-q^{\frac{-1}{2}})t^{k-1}_{su,w,v}(q^{\frac{1}{2}}-q^{\frac{-1}{2}})\Big)x^{u',v} & \text{if } sw<w.\end{cases}$$

Finally, by Proposition 1.3.4, (1.3.11) is obtained. □

The above theorem gives us a way of expanding $x^{u,\,w}$ in $\ell(u)$ bases. However, more can be said about this expansion. Not every permutation $v\in\mathfrak{S}_n$ yields a nonzero polynomial $t^k_{u,w,v}(q^{\frac{1}{2}}-q^{\frac{-1}{2}})$. In fact, the permutations $v\in\mathfrak{S}_n$ for which $t^k_{u,w,v}(q^{\frac{1}{2}}-q^{\frac{-1}{2}})\neq 0$ satisfy $v\geq\phi_{k,u}(w)$, as shown in the next result.

Theorem 1.3.6. *Fix u, $w\in\mathfrak{S}_n$ and choose an integer $k\in[\ell(u)+1]$. It follows that, in $\mathcal{A}_{[n],[n]}(n;q)$, if $t^k_{u,w,v}(q^{\frac{1}{2}}-q^{\frac{-1}{2}})\neq 0$, then $v\geq\phi_{k,u}(w)$.*

Proof. First consider the case of $k=1$. Then $\phi_{1,u}(u)=u$, $\phi_{1,u}(w)=w$, and

$$x^{u,w}=\sum_{v\in\mathfrak{S}_n} t^1_{u,w,v}(q^{\frac{1}{2}}-q^{\frac{-1}{2}})x^{\phi_{1,u}(u),v}=\sum_{v\in\mathfrak{S}_n}\delta_{w,v}x^{\phi_{1,u}(u),v}. \tag{1.3.13}$$

Thus $t^1_{u,w,v}(q^{\frac{1}{2}}-q^{\frac{-1}{2}})\neq 0$ if and only if $v=w$ and the claim holds. Now consider the case $u=e$. Then necessarily $k=1$ and again the claim holds.

Now let u be a permutation with right-to-left lexicographically greatest reduced expression $s_{i_1}\cdots s_{i_\ell}$, let $k-1$ be an integer in $[\ell]$, and assume the claim holds when u

and k are replaced by a permutation of length $\ell-1$ and the integer $k-1$, respectively. By Proposition 1.3.4,

$$t^{k}_{u,w,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) = \begin{cases} t^{k-1}_{s_{i_1}u,s_{i_1}w,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) & \text{if } \ s_{i_1}w > w, \\ t^{k-1}_{s_{i_1}u,s_{i_1}w,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})t^{k-1}_{s_{i_1}u,w,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) & \text{if } \ s_{i_1}w < w. \end{cases} \tag{1.3.14}$$

Suppose first that $s_{i_1}w > w$ and $t^{k}_{u,w,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) \neq 0$. Then $t^{k}_{u,w,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) = t^{k-1}_{s_{i_1}u,s_{i_1}w,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) \neq 0$ implies by induction and Proposition 1.2.1 that

$$v \geq \phi_{k-1,s_{i_1}u}(s_{i_1}w) = \phi_{k,u}(w).$$

Now suppose $s_{i_1}w < w$ and $t^{k}_{u,w,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) \neq 0$. Then either $t^{k-1}_{s_{i_1}u,s_{i_1}w,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) \neq 0$, which implies $v \geq \phi_{k-1,s_{i_1}u}(s_{i_1}w) = \phi_{k,u}(w)$ or $t^{k-1}_{s_{i_1}u,w,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) \neq 0$, which implies $v \geq \phi_{k-1,s_{i_1}u}(w)$, or both. Since by Proposition 1.2.3, $\phi_{k,u}(w) < \phi_{k-1,s_{i_1}u}(w)$ it follows that $t^{k-1}_{s_{i_1}u,w,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) \neq 0$ implies $v \geq \phi_{k,u}(w)$ too. $\square$

Observe that the previous two theorems imply that $x^{u,w}$ can be expressed in the basis $\{x^{\phi_{k,u}(u),v} | v \in \mathfrak{S}_n\}$ of $\mathcal{A}_{[n],[n]}(n;q)$ as

$$x^{u,w} = \sum_{v \geq \phi_{k,u}(w)} t^{k}_{u,w,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) x^{\phi_{k,u}(u),v}. \tag{1.3.15}$$

Chapter 2

Combinatorial interpretations of characters

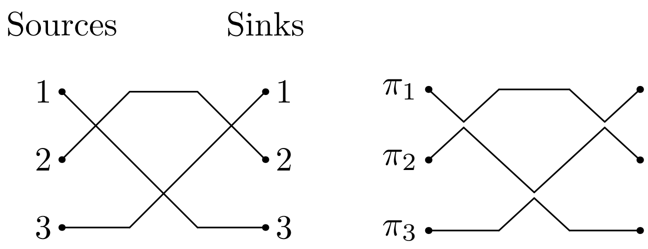
A degree-d matrix representation of a group G is a homomorphism from G to the group of all $d \times d$ invertible matrices. It is known that every finite group is isomorphic to a subgroup of the symmetric group, $\mathfrak{S}_n$, for some $n \in \mathbb{N}$. A homomorphism on G restricted to a subgroup of G is another homomorphism. Thus representations of $\mathfrak{S}_n$ are studied in order to learn about representation of all finite groups.

The combinatorial interpretations of $\mathfrak{S}_n$-characters and $H_n(q)$-characters involve a partial ordering on paths in a planar network and a variation of Young tableaux. These combinatorial objects will be defined and some of their well known properties will be discussed. Interpretations for several classes of $\mathfrak{S}_n$-characters, which can be found in the literature, will be stated. After building some necessary preliminary results, a combinatorial interpretation for the induced sign characters of the Hecke algebra will be proved.

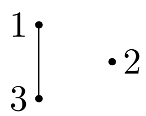
2.1 Planar networks, posets and Young tableaux

A planar network of order n is an acyclic planar directed multigraph $G = (V, E)$ with $2n$ boundary vertices. These vertices are labeled counterclockwise as source 1,

…, source n, sink n, …, sink 1 and the edges are directed from the sources, which have indegree 0, to the sinks, which have outdegree 0. A sequence $\pi = (\pi_1, \dots, \pi_k)$ of paths in G such that each path π_i connects a distinct source to a distinct sink is called a path family. Path families consisting of n paths are called bijective. A path family is said to be of type $w = w_1 \cdots w_n$ if path π_i originates at source i and terminates at sink w_i for each $1 \leq i \leq n$. A path family is of type 1 if w is the identity permutation in $\mathfrak{S}_n$. For example, consider the following planar network and bijective path family of type 1.



The source-to-sink paths of a planar network of order n have a natural partial order Q defined by $\pi_i <_Q \rho_j$ if $i < j$ and π_i and ρ_j never intersect. Observe that there may be multiple paths from source i to sink i, thus the poset Q may have more than n elements. However, a bijective path family of type 1 forms an n-element subposet P of Q. The following figure shows the poset P associated to the bijective path family of type 1 in the above example.



Given an n-element poset P, define a P-tableau to be a Young diagram filled with the elements of P such that every element appears exactly once. A P-tableau is said to be of shape λ for some partition $\lambda = (\lambda_1, \dots, \lambda_r)$ of n, denoted $\lambda \vdash n$, if row i contains λ_i boxes for every $i \in [r]$. Similarly, a P-tableau is of shape $\lambda^\top$ if column i contains λ_i boxes for every $i \in [r]$. Classify a P-tableau as column-strict if whenever elements $i_1, \dots, i_r$ appear from top to bottom in a column, it follows that $i_1 <_P \cdots <_P i_r$. A row-semistrict P-tableau is one such that for each pair

i_j, i_{j+1} appearing consecutively in a row it follows that either $i_j <_P i_{j+1}$ or i_j is incomparable to i_{j+1} in P. A P-tableau is semistandard if it is both column-strict and row-semistrict. Continuing with our example poset P, the following are two P-tableaux of shape $\lambda = 21$.

$$T_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad\qquad T_2 = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}$$

Observe that since 1 is incomparable to 2 in P it follows that T_1 is row-semistrict. Furthermore, since $1 <_P 3$ this tableau is also column-strict. Thus T_1 is semistandard. Similarly, 1 incomparable to 2 implies that T_2 is row-semistrict and 2 incomparable to 3 implies that it is not column-strict.

The following statistic on P-tableaux is introduced in anticipation of its use in combinatorial interpretations. Given a P-tableau T, define an inversion in T to be a pair (i, j) of incomparable elements in P satisfying $i < j$ in $\mathbb{Z}$ and i appearing in a column to the right of that containing j in T. Denote the number of inversions in T by $\textsc{inv}(T)$. Observe that for the P-tableaux in the above example, $(2, 3)$ is the only inversion in T_1 so $\textsc{inv}(T_1) = 1$ and $(1, 2)$ is the only inversion in T_2 so $\textsc{inv}(T_2) = 1$.

P-tableaux are one type of combinatorial object that can be associated to a poset P. The following 0-1 matrix is another object associated to a labeled poset. Define the antiadjacency matrix $A = (a_{i,j})$ associated to a labeling of a poset P by

$$a_{i,j} = \begin{cases} 0 & \text{if } i <_P j \\ 1 & \text{otherwise.} \end{cases} \tag{2.1.1}$$

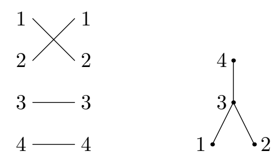
The antiadjacency matrices associated to two different labellings of P are conjugate by some permutation matrix. Returning to the example poset, the antiadjacency matrix is

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Certain posets yield antiadjacency matrices with a nice property, which aids in the formulation of combinatorial interpretations. Denote an a-element chain by $\mathbf{a}$

and the disjoint union of posets P and Q by $P+Q$. Then $\mathbf{a}+\mathbf{b}$ is the disjoint union of an a-element chain and a b-element chain. If no induced subposet of P is isomorphic to $\mathbf{a}+\mathbf{b}$, then P is said to be $(\mathbf{a}+\mathbf{b})$-free. The class of posets of interest to us are interval orders and unit interval orders. A poset P is an interval order if it is $(\mathbf{2}+\mathbf{2})$-free. Adding the condition of being $(\mathbf{3}+\mathbf{1})$-free as well as $(\mathbf{2}+\mathbf{2})$-free defines a unit interval order.

Let P be an n-element poset with labels $1, \ldots, n$. If P is labeled such that $i <_P j$ implies $i < j$ as integers, then P is called naturally labeled. Every poset has at least one natural labeling. Define the altitude of a poset element i to be $\alpha(i) = \#\{x|x <_P i\} - \#\{x|x >_P i\}$. We say a labeling is altitude respecting if $i < j$ whenever $\alpha(i) < \alpha(j)$. Such a labeling is called an ar-labeling. The zero entries of the antiadjacency matrix associated to an ar-labeling of an n-element unit interval order form a right justified Young diagram of shape μ such that $\mu_i \leq n-i$ for $i \in [n-1]$ (See [24]). For example, consider the following planar network and associated ar-labeled unit interval order.



The antiadjacency matrix associated to this labeled poset is

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Observe that the zero entries form a right justified Young diagram of shape $(2,2,1)$. Note that this property implies that all entries to the right of any zero entry are also zero and all entries above a zero entry are also zero entries in the antiadjacency matrix.

It is known that the antiadjacency matrix of a poset P can be used to count column-strict P-tableaux in the following way [19, Prop. 2.1].

Proposition 2.1.1. *Let P be a labeled poset with antiadjacency matrix A and let λ be a partition of $n = |P|$. Then the number of column-strict P-tableaux is given by*

$$\sum_{(I_1,\ldots,I_r)} det(A_{I_1,I_1}) \cdots det(A_{I_r,I_r}), \tag{2.1.2}$$

where the sum is over all ordered set partitions of $[n]$ of type λ and $A_{I,I}$ is the $|I| \times |I|$ submatrix $(a_{i,j})_{i,j \in I}$ of A.

This relationship between the antiadjacency matrix and P-tableaux will provide a combinatorial interpretation of the induced sign characters of $\mathfrak{S}_n$. It will be of use to focus attention on certain elements of the symmetric group algebra, $\mathbb{C}[S_n]$. Associate to each labeled poset P the element $\beta(P)$ of $\mathbb{C}[S_n]$ defined $\beta(P) = \sum_v v$ where the sum is over all permutations $v \in \mathfrak{S}_n$ satisfying $i \not<_P v_i$ for all i.

A permutation w is said to avoid the pattern 312 if no subword of w has the relative order 312. The following result in [20] provides an alternate way of writing $\beta(P)$.

Lemma 2.1.2. *There is a bijection between unit interval orders P and 312-avoiding permutations w such that if P corresponds to w and A is the antiadjacency matrix of P with respect to an ar-labeling, then the set $\{v \in \mathfrak{S}_n \,|\, a^{e,v} = 1\}$ is precisely the interval $[e, w]$ in the Bruhat order on $\mathfrak{S}_n$.*

Recall that $\beta(P) = \sum_v v$, where the sum is over all permutations $v \in \mathfrak{S}_n$ such that $i \not<_P v_i$ for every $i \in [n]$. By the definition of the antiadjacency matrix A, $i \not<_P v_i$ implies $a_{i,v_i} = 1$. That is, $a^{e,v} = 1$ when $i \not<_P v_i$ for every index $i \in [n]$. Thus by Lemma 2.1.2, for an ar-labeled unit interval order P, $\beta(P)$ can be defined as $\beta(P) = \sum_{v \leq w} v$ for some 312-avoiding permutation w.

2.2 Symmetric group algebra and Hecke algebra characters

The Hecke algebra, $H_n(q)$, is the noncommutative $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-algebra with multiplicative identity $\widetilde{T}_e = 1$ generated by $\{\widetilde{T}_{s_i} | 1 \leq i \leq n-1\}$ subject to the relations

$$\begin{aligned} \widetilde{T}_{s_i}^2 &= (q^{\frac{1}{2}} - q^{\frac{-1}{2}})\widetilde{T}_{s_i} + \widetilde{T}_e, && \text{for } i = 1, \ldots, n-1, \\ \widetilde{T}_{s_i}\widetilde{T}_{s_j}\widetilde{T}_{s_i} &= \widetilde{T}_{s_j}\widetilde{T}_{s_i}\widetilde{T}_{s_j}, && \text{if } |i-j| = 1, \\ \widetilde{T}_{s_i}\widetilde{T}_{s_j} &= \widetilde{T}_{s_j}\widetilde{T}_{s_i}, && \text{if } |i-j| \geq 2. \end{aligned} \tag{2.2.1}$$

Observe that $H_n(q)$ is the quantum analog of the classical group algebra $\mathbb{C}[S_n]$, with $\widetilde{T}_v$ mapping to v when we specialize at $q = 1$. $H_n(q)$ has the natural basis $\{\widetilde{T}_w | w \in \mathfrak{S}_n\}$ where $\widetilde{T} = \widetilde{T}_{s_{i_1}} \cdots \widetilde{T}_{s_{i_\ell}}$ whenever $s_{i_1} \cdots s_{i_\ell}$ is a reduced expression for $w \in \mathfrak{S}_n$. This is analogous to the natural basis $\{w | w \in \mathfrak{S}_n\}$ of $\mathbb{C}[S_n]$.

For every matrix representation of $\mathfrak{S}_n$ there is an associated class function mapping each element in $\mathfrak{S}_n$ to the trace of its matrix representation. Such class functions are called $\mathfrak{S}_n$-characters. (See [17] for definitions.) $\mathfrak{S}_n$-characters extend linearly to the group algebra, $\mathbb{C}[S_n]$. Much of the information about a representation is encoded in its character. Thus representations are often studied in terms of characters.

The complex span of the $\mathfrak{S}_n$-characters is the space of all class functions on $\mathfrak{S}_n$, which has dimension equal to the number of integer partitions of n. One well-studied basis is the induced sign character basis, $\{\epsilon^\lambda | \lambda \vdash n\}$. These are also called the elementary characters, due to the Frobenius characteristic map, by which ϵ^λ corresponds to the elementary symmetric function e_λ. The induced trivial characters $\{\eta^\lambda | \lambda \vdash n\}$ are also known as the homogeneous characters because by the Frobenius characteristic map, η^λ corresponds to the homogeneous symmetric function h_λ. A third well-studied basis is the irreducible character basis, $\{\chi^\lambda | \lambda \vdash n\}$, which corresponds by the Frobenius characteristic map to the Schur functions s_λ. These are the most important characters because all other characters can be expressed as nonnegative, integer linear combinations of these. The space has a fourth basis,

$\{\phi^\lambda | \lambda \vdash n\}$, of monomial class functions. Analogous to the space of $\mathfrak{S}_n$-class functions is the space of $H_n(q)$-traces, the $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-span of the $H_n(q)$-characters. This space has character bases $\{\epsilon_q^\lambda | \lambda \vdash n\}$, $\{\eta_q^\lambda | \lambda \vdash n\}$, and $\{\chi_q^\lambda | \lambda \vdash n\}$, which specialize at $q = 1$ to the $\mathfrak{S}_n$-character bases. The traces $\{\phi_q^\lambda | \lambda \vdash n\}$ form another basis of the $H_n(q)$-traces which correspond to the monomial character basis.

In each space, these bases are related to one another just as are the Schur, elementary, and (complete) homogeneous bases of the space of homogeneous degree n symmetric functions. Specifically,

$$\begin{aligned} h_\lambda &= \sum_\mu K_{\mu,\lambda} s_\mu, & e_\lambda &= \sum_\mu K_{\mu^\top,\lambda} s_\mu, & s_\lambda &= \sum_\mu K_{\lambda,\mu} m_\mu, \\ \eta^\lambda &= \sum_\mu K_{\mu,\lambda} \chi^\mu, & \epsilon^\lambda &= \sum_\mu K_{\mu^\top,\lambda} \chi^\mu, & \chi^\lambda &= \sum_\mu K_{\lambda,\mu} \phi^\mu, \\ \eta_q^\lambda &= \sum_\mu K_{\mu,\lambda} \chi_q^\mu, & \epsilon_q^\lambda &= \sum_\mu K_{\mu^\top,\lambda} \chi_q^\mu, & \chi_q^\lambda &= \sum_\mu K_{\lambda,\mu} \phi_q^\mu, \end{aligned} \tag{2.2.2}$$

where $K = (K_{\lambda,\mu})$ is the invertible matrix of Kostka numbers. (See [17]).

2.3 Known combinatorial interpretations

One way to understand $\epsilon^\lambda(w)$, $\eta^\lambda(w)$, $\chi^\lambda(w)$ and $\phi^\lambda(w)$ for permutations $w \in \mathfrak{S}_n$ is to define generating functions in the polynomial ring $\mathbb{C}(x) = \mathbb{C}[x_{1,1}, \ldots x_{n,n}]$. Call such generating functions immanants and define them by

$$\mathrm{Imm}_f(x) \underset{\mathrm{def}}{=} \sum_{w \in \mathfrak{S}_n} f(w) x_{1,w_1} \cdots x_{n,w_n}$$

for each $f \in \{\epsilon^\lambda, \eta^\lambda, \chi^\lambda\}$. Nice formulas are known for the ϵ^λ-immanants and η^λ-immanants, but not for χ^λ-immanants.

The Littlewood-Merris-Watkins [14],[15] identities express ϵ^λ-immanants and η^λ-immanants as

$$\mathrm{Imm}_{\epsilon^\lambda}(x) = \sum_{(I_1,\ldots,I_r)} det(x_{I_1,I_1}) \cdots det(x_{I_r,I_r}), \tag{2.3.1}$$

and

$$\mathrm{Imm}_{\eta^\lambda}(x) = \sum_{(I_1,\ldots,I_r)} per(x_{I_1,I_1}) \cdots per(x_{I_r,I_r}), \tag{2.3.2}$$

where the sums are over all ordered set partitions of $[n]$ satisfying $|I_j| = \lambda_j$.

The above identities can be used to state combinatorial interpretations for induced sign and induced trivial characters evaluated at $\beta(P)$. Combining Proposition 2.1.1 and the Littlewood-Merris-Watkins identity for ϵ^λ-immanants yields the following theorem which is proved in [19, Thm 2.2].

Theorem 2.3.1. *Let P be a labeled poset and let λ be a partition of $|P|$. Then $\epsilon^\lambda(\beta(P))$ equals the number of column-strict P-tableaux of shape $\lambda^\top$.*

There is a similar result [19, Thm 2.3] for η^λ-immanants.

Theorem 2.3.2. *Let P be a labeled poset and let λ be a partition of $|P|$. Then $\eta^\lambda(\beta(P))$ equals the number of row-strict P-tableaux of shape λ.*

Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition of $l + m = n$. Given a weak composition μ' of l into r parts and a weak composition ν' of m into r parts, we say $\mu' + \nu' = \lambda$ if $\mu'_i + \nu'_i = \lambda_i$ for every $i \in [r]$. Define the partition μ corresponding to the weak composition μ' to be the rearrangement of the positive parts of μ into nonincreasing order.

For fixed posets P and Q, define $P \oplus Q$ to be the $(|P| + |Q|)$-element poset with every element of P being less than every element of Q. The following proposition provides a decomposition of the induced sign characters of $\mathfrak{S}_n$ evaluated on the special elements of $\mathbb{C}[S_n]$ associated to such posets.

Proposition 2.3.3. *Let P_1 be an l-element poset and P_2 an m-element poset with $l + m = n$. Then for a fixed $\lambda \vdash n$,*

$$\epsilon^\lambda(\beta(P_1 \oplus P_2)) = \sum_{\mu'+\nu'=\lambda} \epsilon^\mu(\beta(P_1))\epsilon^\nu(\beta(P_2)) \tag{2.3.3}$$

where μ' and ν' are weak compositions of l and m respectively and μ and ν are their corresponding partitions.

Proof. Fix $\lambda = (\lambda_1, \ldots, \lambda_r)$ a partition of n. By Theorem 2.3.1, $\epsilon^\lambda(\beta(P_1 \oplus P_2))$ is equal to the number of column-strict $(P_1 \oplus P_2)$-tableaux of shape $\lambda^\top$. Let T be one such tableau. It will be shown that T is counted by the right hand side of (2.3.3). Decompose T into two (skew) tableau T_1 and T_2 where T_1 consists of the cells containing $1, \ldots, l$ and T_2 consists of the cells containing $l+1, \ldots, l+m$. Observe that since T is column-strict, both T_1 and T_2 are column-strict. Letting μ_i' equal the number of cells in column i of T_1, define the weak composition μ' of l. Similarly, letting ν_i' equal the number of cells in column i of T_2, define the weak composition ν' of m. Note that $\mu' + \nu' = \lambda$. Let μ and ν be the partitions corresponding to μ' and ν' respectively. Rearranging the columns of T_1 by nonincreasing size forms a column-strict P_1-tableau of shape $\mu^\top$. Similarly, rearranging the columns of T_2 forms a column-strict P_2-tableau of shape $\nu^\top$. Thus T is counted by $\epsilon^\mu(\beta(P_1))\epsilon^\nu(\beta(P_2))$.

Again fix $\lambda = (\lambda_1, \ldots, \lambda_r)$ a partition of n. Now consider a pair of weak compositions μ' of l into r parts and ν' of m into r parts such that $\mu' + \nu' = \lambda$. Let μ and ν be the partitions corresponding to μ' and ν'. Construct a Young diagram of shape $\mu^\top$. There are $\epsilon^\mu(\beta(P_1))$ ways to fill it with elements of P_1 so that it is column-strict. Pick one such filling and rearrange the columns of the P_1-tableau to form a (skew) tableau T_1 with μ_i' cells in column i. Now choose one of the $\epsilon^\nu(\beta(P_2))$ ways to fill a Young diagram of shape $\nu^\top$ to be column-strict and rearrange the columns to form a (skew) tableaux T_2 with ν_i' cells in column i. Combine T_1 and T_2 by placing column i of T_2 below column i of T_1 for each $i \in [r]$. Call this tableau T and observe that it is a column-strict $(P_1 \oplus P_2)$-tableau of shape $\mu' + \nu' = \lambda$. Thus T is counted by $\epsilon^\lambda(\beta(P_1 \oplus P_2))$. $\square$

Though a nice formula for ϕ^λ-immanants is not known in general, Stembridge [23] has provided the following formula in the case when $\lambda_1 = \cdots = \lambda_r = k$,

$$\mathrm{Imm}_{\phi^{k^r}}(x) = \sum_{(I_1, \ldots, I_k)} det(x_{I_1,I_2}) det(x_{I_2,I_3}) \cdots det(x_{I_k,I_1}),$$

where the sum is over all sequences of pairwise disjoint subsets of $[n] = [kr]$ satisfying $|I_j| = r$ for every $j \in [k]$.

Since no formulas such as (2.3.1), (2.3.2) are known for χ^λ-immanants or ϕ^λ-immanants, work by Goulden and Jackson [7] and Greene [8] has led to focusing the study on a subset of planar networks. In particular, interpretations are known for $\chi^\lambda(\beta(P))$ and $\phi^\lambda(\beta(P))$ when P is one of a restricted set of labeled posets. Due to results by Stanley [21], Gasharov restricted attention to $(3+1)$-free posets and formulated the following combinatorial interpretation.

Proposition 2.3.4. *[6, Thm. 2] Let P be a labeled $(3+1)$-free poset and λ a partition of $|P|$. Then $\chi^\lambda(\beta(P))$ equals the number of semistandard P-tableaux of shape λ.*

Further restricting the type of poset considered as well as the shape of the partition λ has led to the following combinatorial interpretations for $\phi^\lambda(\beta(P))$.

Proposition 2.3.5. *[3, Thm. 5.7] Let P be a unit interval order and let λ be a partition of $|P|$. If $\lambda_1 \leq 2$, then $\phi^\lambda(\beta(P))$ is equal to zero if there exists a column-strict P-tableaux of shape $\mu \prec \lambda$ in dominance order and it equals the number of column-strict P-tableaux of shape λ otherwise.*

Stembridge [23] further conjectured that for any unit interval order P and $\lambda \vdash |P|$, $\phi^\lambda(\beta(P)) \geq 0$. However, no combinatorial interpretation for these characters has been conjectured.

Generating functions for $H_n(q)$-traces on Hecke algebra basis elements can be defined analogously to those for $\mathfrak{S}_n$-characters. That is, a generating function for $\{f_q(\widetilde{T}_w) |\, w \in \mathfrak{S}_n\}$ where $f_q \in \{\epsilon_q^\lambda, \eta_q^\lambda, \chi_q^\lambda\}$ is defined by

$$\mathrm{Imm}_{f_q}(x) \underset{\mathrm{def}}{=} \sum_{w \in \mathfrak{S}_n} f_q(\widetilde{T}_w) x_{1,w_1} \cdots x_{n,w_n},$$

where $x = (x_{i,j})$ is an element of the quantum matrix bialgebra, $\mathcal{A}(n;q)$, discussed in Section 1.3. There are nice formulas for ϵ_q^λ-immanants and η_q^λ-immanants analogous to the Littlewood-Merris-Watkins identities in (2.3.1). The following are due to Konvalinka and Skandera, [11]

$$\mathrm{Imm}_{\epsilon_q^\lambda}(x) = \sum_{(I_1,\ldots,I_r)} det_q(x_{I_1,I_1}) \cdots det_q(x_{I_r,I_r}), \tag{2.3.4}$$

and

$$\mathrm{Imm}_{\eta_q^\lambda}(x) = \sum_{(I_1,\ldots,I_r)} per_q(x_{I_1,I_1}) \cdots per_q(x_{I_r,I_r}), \tag{2.3.5}$$

where the sums are over all ordered set partitions of $[n]$ satisfying $|I_j| = \lambda_j$.

Less is known about the $H_n(q)$-characters than is known for $\mathfrak{S}_n$-characters. Define the q-analog of $\beta(P)$ for a poset P to be $\beta_q(P) = \sum_v q_{e,v} \widetilde{T}_v$ where the sum is over all permutations $v \in \mathfrak{S}_n$ satisfying $i \not<_P v_i$ for all i. Haiman [9] proved that $\eta_q^\lambda(\beta_q(P))$, $\epsilon_q^\lambda(\beta_q(P))$, and $\chi_q^\lambda(\beta_q(P))$ are in $\mathbb{N}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$. However, no combinatorial interpretations for $\eta_q^\lambda(\beta_q(P))$ or $\chi_q^\lambda(\beta_q(P))$ have been shown. In the next section a combinatorial interpretation is provided for the quantum induced sign characters, $\epsilon_q^\lambda(\beta_q(P))$.

2.4 $H_n(q)$ Induced Sign Characters

This section provides a combinatorial interpretation for the polynomial $\epsilon_q^\lambda(\beta_q(P))$ where P is an ar-labeled unit interval order.

Let A be the antiadjacency matrix of an n-element unit interval order P with respect to an ar-labeling. Evaluating the left side of Equation (2.3.4) at $x = A$ does not allow us to compute the desired polynomial as it did in the non-quantum case because the noncommuting variables satisfy nontrivial relations. Thus the sum of quantum determinants must first be expressed in terms of the natural basis. To do this, a map that takes the product of noncommuting variables $x_{u_1,v_1} \cdots x_{u_n,v_n}$ to the product of commuting matrix entries $a_{u_1,v_1} \cdots a_{u_n,v_n}$ will be constructed.

Define the family $\{\sigma_{A,u} : \mathcal{A}_{[n],[n]}(n;q) \to \mathbb{Z}[q^{\frac{1}{2}} - q^{\frac{-1}{2}}] | u \in \mathfrak{S}_n\}$ of linear maps by

$$\sigma_{A,u}(x_{u_1,v_1} \cdots x_{u_n,v_n}) = a_{u_1,v_1} \cdots a_{u_n,v_n} q_{u,v} \tag{2.4.1}$$

where $q_{u,v} = q^{\frac{\ell(v)-\ell(u)}{2}}$. These maps aid in formulating an interpretation for the polynomials $\epsilon_q^\lambda(\beta_q(P)) = \sigma_{A,e}\Big(\mathrm{Imm}_{\epsilon_q^\lambda}(x)\Big)$. Returning to the nice formula for $\mathrm{Imm}_{\epsilon_q^\lambda}(x)$, the expression can be written as a sum of products of quantum determinants. Expanding each quantum determinant yields

$$\begin{aligned}\sigma_{A,e}\left(\mathrm{Imm}_{\epsilon_q^\lambda}(x)\right) &= \sigma_{A,e}\left(\sum_{(I_1,\dots,I_r)} det_q(x_{I_1,I_1})\cdots det_q(x_{I_r,I_r})\right)\\ &= \sigma_{A,e}\left(\sum_{(I_1,\dots,I_r)}\sum_{y\in\mathfrak{S}_\lambda}(-1)^{\ell(y)}q_{e,y}^{-1}x^{u,\,yu}\right)\\ &= \sum_{(I_1,\dots,I_r)}\sum_{y\in\mathfrak{S}_\lambda}(-1)^{\ell(y)}q_{e,y}^{-1}\sigma_{A,e}(x^{u,yu}),\end{aligned} \tag{2.4.2}$$

where the first sum is over all ordered set partitions $(I_1,\dots,I_r)$ of $[n]$ which satisfy $|I_j| = \lambda_j$ for $j = 1,\dots,r$. To evaluate $\sigma_{A,e}(x^{u,yu})$, the monomial $x^{u,yu}$ must be examined more closely. Fix $u \in \mathfrak{S}_n$ with $s_{i_1}\cdots s_{i_\ell}$ its right-to-left lexicographically greatest reduced expression and $y \in \mathfrak{S}_\lambda$ for some $\lambda \vdash n$. Observe that by Theorems 1.3.5 and 1.3.6 for a fixed integer $k \in [\ell+1]$,

$$x^{u,yu} = \sum_{v\geq\phi_{k,u}(yu)} t^k_{u,yu,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{\phi_{k,u}(u),\,v} \tag{2.4.3}$$

recalling from Section 1.2 that $\phi_{k,u}(w) = s_{i_{(k-1)}}\cdots s_{i_1}w$. Regroup the terms of this sum by considering those permutations v for which s_{i_k} is a left ascent or descent. Then,

$$x^{u,yu} = \sum_{\substack{v\geq\phi_{k,u}(yu)\\ s_{i_k}v>v}} t^k_{u,yu,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{\phi_{k,u}(u),\,v} + \sum_{\substack{v\geq\phi_{k,u}(yu)\\ s_{i_k}v<v}} t^k_{u,yu,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{\phi_{k,u}(u),\,v}. \tag{2.4.4}$$

Recall by Lemma 1.3.1, replacing u with $\phi_{k,u}(u)$ and s with s_{i_k}, it follows that

$$x^{\phi_{k,u}(u),\,v} = \begin{cases} x^{\phi_{k+1,u}(u),\,s_{i_k}v} & \text{if } s_{i_k}v > v,\\ x^{\phi_{k+1,u}(u),\,s_{i_k}v} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{\phi_{k+1,u}(u),\,v} & \text{if } s_{i_k}v < v.\end{cases} \tag{2.4.5}$$

Now using (2.4.5) we may rewrite the sum (2.4.4) as

$$x^{u,yu} = \sum_{\substack{v \geq \phi_{k,u}(yu) \\ s_{i_k} v > v}} t^k_{u,yu,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) x^{\phi_{k+1,u}(u),\, s_{i_k} v} \tag{2.4.6}$$

$$+ \sum_{\substack{v \geq \phi_{k,u}(yu) \\ s_{i_k} v < v}} t^k_{u,yu,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) \left[x^{\phi_{k+1,u}(u),\, s_{i_k} v} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}}) x^{\phi_{k+1,u}(u),\, v} \right].$$

Recall that the goal is to evaluate $\sigma_{A,e}(x^{u,yu})$. In working toward this goal, a recursive relationship will be stated, in Theorem 2.4.7, for the family of maps $\{\sigma_{A,u} | u \in \mathfrak{S}_n\}$. The next few results are necessary for developing this relationship. Given certain conditions, one can identify particular variables in the monomial $x^{\phi_{k,u}(u),v}$. The next two lemmas demonstrate this.

Lemma 2.4.1. *Let $s_{i_1} \cdots s_{i_\ell}$ be the right-to-left lexicographically greatest reduced expression for a permutation $u \in \mathfrak{S}_n$. Fix $k \in [\ell]$, $z \in \mathfrak{S}_n$, and let $p_1 \cdots p_{k-1}$ be a subexpression of $s_{i_1} \cdots s_{i_{k-1}}$ such that if $s_{i_j} p_{(j-1)} \cdots p_1 z > p_{(j-1)} \cdots p_1 z$ then $p_j = s_{i_j}$. Define $w = s_{i_{k-1}} \cdots s_{i_1} z$ and $v = p_{k-1} \cdots p_1 z$. Let $w_{i_k} = d$, $v_{i_k} \neq d$, and η be the smallest index such that $(p_\eta \cdots p_1 z)_{i_\eta} = d$ and $(s_{i_\eta} \cdots s_{i_1} z)_{i_\eta + 1} = d$. If letter b is in position i_η of $\phi_{\eta+1,u}(u)$ then the variable $x_{b,\beta}$ with $\beta \geq d$ appears in the monomial $x^{\phi_{k,u}(u),v}$.*

Proof. Since by assumption the letter d is in position i_η of $p_\eta \cdots p_1 z$ and the letter b is in position i_η of $\phi_{\eta+1,u}(u)$ it follows that the variable $x_{b,d}$ appears in the monomial $x^{\phi_{\eta+1,u}(u), p_\eta \cdots p_1 z}$. Observe that if $p_{\eta+1} = s_{i_{\eta+1}}$ then the variables appearing in $x^{\phi_{\eta+2,u}(u), p_{\eta+1} p_\eta \cdots p_1 z}$ are the same variables that appear in $x^{\phi_{\eta+1,u}(u), p_\eta \cdots p_1 z}$, just in a different order.

Now, suppose $p_{\eta+1} = e$. Then the $(i_{\eta+1})$th and $(i_{\eta+1}+1)$st variables of the monomial $x^{\phi_{\eta+2,u}(u), p_{\eta+1} p_\eta \cdots p_1 z}$ are different than those of $x^{\phi_{\eta+1,u}(u), p_\eta \cdots p_1 z}$ and all other variables in the monomials agree. However, if $i_{\eta+1} + 1 \neq i_\eta$ then $x_{b,d}$ appears in $x^{\phi_{\eta+2,u}(u), p_{\eta+1} p_\eta \cdots p_1 z}$ as well. If $i_{\eta+1} + 1 = i_\eta$ and $x_{a,c} x_{b,d}$, for some a and c, are the $(i_{\eta+1})$th and $(i_{\eta+1}+1)$st variables of $x^{\phi_{\eta+1,u}(u), p_\eta \cdots p_1 z}$, then $x_{b,c} x_{a,d}$ are the $(i_{\eta+1})$th and $(i_{\eta+1}+1)$st variables of $x^{\phi_{\eta+2,u}(u), p_{\eta+1} p_\eta \cdots p_1 z}$. Since $p_{\eta+1} = e$ then

$$s_{i_{(\eta+1)}} p_\eta \cdots p_1 z < p_\eta \cdots p_1 z,$$

which implies

$$(p_\eta \cdots p_1 z)_{i_{(\eta+1)}} > (p_\eta \cdots p_1 z)_{i_{(\eta+1)}+1}.$$

That is, $c > d$. Thus $x_{b,\gamma}$ with $\gamma \geq d$ appears in the monomial $x^{\phi_{\eta+2,u}(u), p_{\eta+1} \cdots p_1 z}$. Repeating the above argument implies $x_{b,\beta}$ with $\beta \geq d$ appears in the monomial $x^{\phi_{k,u}(u),v}$. □

Lemma 2.4.2. *Let $s_{i_1} \cdots s_{i_\ell}$ be the right-to-left lexicographically greatest reduced expression for a permutation $u \in \mathfrak{S}_n$. Fix $k \in [\ell]$ and let $p_1 \cdots p_{k-1}$ be a subexpression of $s_{i_1} \cdots s_{i_{k-1}}$. For $z \in \mathfrak{S}_n$, define $w = s_{i_{k-1}} \cdots s_{i_1} z$ and $v = p_{k-1} \cdots p_1 z$. Let $w_{i_k} = d$, $v_{i_k} \neq d$, and η be the smallest index such that $(p_\eta \cdots p_1 z)_{i_\eta} = d$ and $(s_{i_\eta} \cdots s_{i_1} z)_{i_\eta+1} = d$. If letter b is in position i_η of $\phi_{\eta+1,u}(u)$ then the variable $x_{b,\beta}$ with $\beta \geq d$ is not the (i_k)th variable in the monomial $x^{\phi_{k,u}(u),v}$.*

Proof. By Lemma 2.4.1 we know $x_{b,\beta}$, where $\beta \geq d$, appears in the monomial $x^{\phi_{k,u}(u),v}$. Since s_{i_η} swaps the letters in positions i_η and $i_\eta + 1$, we have

$$[\phi_{\eta+1,u}(u)]_{i_\eta} = [\phi_{\eta,u}(u)]_{i_\eta+1}.$$

By Proposition 1.2.8, $[\phi_{\eta,u}(u)]_{i_\eta+1} = \alpha_\eta(u)$ and $[\phi_{k,u}(u)]_{i_k+1} = \alpha_k(u)$. By Observation 1.2.5, $\alpha_\eta(u) \leq \alpha_k(u)$ since $\eta < k$. Therefore,

$$[\phi_{\eta+1,u}(u)]_{i_\eta} \leq [\phi_{k,u}(u)]_{i_k+1}.$$

By Observation 1.2.4, s_{i_k} is a left descent of $\phi_{k,u}(u)$, which implies

$$[\phi_{k,u}(u)]_{i_k} > [\phi_{k,u}(u)]_{i_k+1}.$$

Thus $b = [\phi_{\eta+1,u}(u)]_{i_\eta} \neq [\phi_{k,u}(u)]_{i_k}$ and so $x_{b,\beta}$ is not the (i_k)th variable in the monomial $x^{\phi_{k,u}(u),v}$. □

Given an entry in the antiadjacency matrix $A = (a_{i,j})$, knowing the position of certain letters in the one-line notations of permutations allows one to deduce the value of other entries in the antiadjacency matrix $A = (a_{i,j})$. The following lemmas state several of these relationships.

Lemma 2.4.3. *Let $A = (a_{i,j})$ be the antiadjacency matrix of an n-element unit interval order P with respect to an ar-labeling, $s_{i_1} \cdots s_{i_\ell}$ be the right-to-left lexicographically greatest reduced expression for a permutation $u \in \mathfrak{S}_n$. Fix $k \in [\ell]$, $z \in \mathfrak{S}_n$, and let $p_1 \cdots p_{k-1}$ be a subexpression of $s_{i_1} \cdots s_{i_{k-1}}$ such that if $s_{i_j} p_{(j-1)} \cdots p_1 z > p_{(j-1)} \cdots p_1 z$ then $p_j = s_{i_j}$. Define $w = s_{i_{k-1}} \cdots s_{i_1} z$ and $v = p_{k-1} \cdots p_1 z$. Then we have,*

(1) If $w_{i_k} = d$, $v_{i_k} \neq d$, and $a_{[\phi_{k,u}(u)]_{i_k+1}, w_{i_k}} = 0$, then $a^{\phi_{k,u}(u),v} = 0$.

(2) If $w_{i_k} = v_{i_k}$, $s_{i_k} v > v$, and $a_{[\phi_{k,u}(u)]_{i_k+1}, w_{i_k}} = 0$, then $a^{\phi_{k,u}(u),v} = 0$.

(3) If $w_{i_k} = d$, $v_{i_k} \neq d$, $s_{i_k} v < v$, and $a_{[\phi_{k,u}(u)]_{i_k+1}, w_{i_k}} = 0$, then $a^{\phi_{k+1,u}(u),v} = 0$.

(4) If $w_{i_k} = v_{i_k}$ and $a_{[\phi_{k,u}(u)]_{i_k+1}, w_{i_k}} = 0$, then $a^{\phi_{k+1,u}(u),v} = 0$.

Proof. (1): Let η be the smallest index, guaranteed to exits by Corollary 1.1.16, such that $(p_\eta \cdots p_1 z)_{i_\eta} = d$ and $(s_{i_\eta} \cdots s_{i_1} z)_{i_\eta+1} = d$. Then by Proposition 1.2.8,

$$[\phi_{\eta+1,u}(u)]_{i_\eta} = [\phi_{\eta,u}(u)]_{i_\eta+1} = \alpha_\eta(u)$$

and

$$[\phi_{k,u}(u)]_{i_k+1} = \alpha_k(u)$$

By Observation 1.2.5 $\alpha_\eta(u) \leq \alpha_k(u)$, that is $[\phi_{\eta+1,u}(u)]_{i_\eta} \leq [\phi_{k,u}(u)]_{i_k+1}$. Thus $a_{[\phi_{\eta+1,u}(u)]_{i_\eta},\beta}$ where $\beta \geq w_{i_k}$, is above and to the right of $a_{[\phi_{k,u}(u)]_{i_k+1}, w_{i_k}}$ in the matrix A. Therefore,

$$a_{[\phi_{k,u}(u)]_{i_k+1}, w_{i_k}} = 0 \text{ implies } a_{[\phi_{\eta+1,u}(u)]_{i_\eta},\beta} = 0.$$

By Lemma 2.4.1, $x_{[\phi_{\eta+1,u}(u)]_{i_\eta},\beta}$ where $\beta \geq w_{i_k}$ appears in the monomial $x^{\phi_{k,u}(u),v}$. Thus $a_{[\phi_{\eta+1,u}(u)]_{i_\eta},\beta}$ appears in $a^{\phi_{k,u}(u),v}$ and so $a^{\phi_{k,u}(u),v} = 0$.

(2): Since $s_{i_k} v > v$, it follows that $v_{i_k+1} > v_{i_k}$. By assumption

$$a_{[\phi_{k,u}(u)]_{i_k+1}, w_{i_k}} = a_{[\phi_{k,u}(u)]_{i_k+1},\, v_{i_k}} = 0.$$

This implies $a_{[\phi_{k,u}(u)]_{i_k+1},\, v_{i_k+1}} = 0$ because the entries to the right of a 0 in A are also 0. Thus $a^{[\phi_{k,u}(u)],\, v} = 0$ because it has $a_{[\phi_{k,u}(u)]_{i_k+1}, v_{i_k+1}}$ as a factor.

(3): By Lemma 2.4.1, $x_{[\phi_{\eta+1,u}(u)]_{i_\eta},\beta}$ appears as a term in the monomial $x^{\phi_{k,u}(u),v}$. If $x_{[\phi_{\eta+1,u}(u)]_{i_\eta},\beta}$ is not the (i_k)th or (i_k+1)st variable in $x^{\phi_{k,u}(u),v}$, then $x_{[\phi_{\eta+1,u}(u)]_{i_\eta},\beta}$

also appears in the monomial $x^{\phi_{k+1,u}(u),v}$. Since s_{i_η} swaps the letters in positions i_η and $i_\eta + 1$, we have

$$[\phi_{\eta+1,u}(u)]_{i_\eta} = [\phi_{\eta,u}(u)]_{i_\eta+1}.$$

By Proposition 1.2.8, $[\phi_{\eta,u}(u)]_{i_\eta+1} = \alpha_\eta(u)$ and $[\phi_{k,u}(u)]_{i_k+1} = \alpha_k(u)$. By Observation 1.2.5, $\alpha_\eta(u) \leq \alpha_k(u)$ since $\eta < k$. Therefore,

$$[\phi_{\eta+1,u}(u)]_{i_\eta} \leq [\phi_{k,u}(u)]_{i_k+1}.$$

Since $\beta \geq w_{i_k}$ and $a_{[\phi_{k,u}(u)]_{i_k+1},w_{i_k}} = 0$ by assumption, it follows that $a_{[\phi_{\eta+1,u}(u)]_{i_\eta},\beta} = 0$. Since $x_{[\phi_{\eta+1,u}(u)]_{i_\eta},\beta}$ appears in the monomial $x^{\phi_{k+1,u}(u),v}$, then $a_{[\phi_{\eta+1,u}(u)]_{i_\eta},\beta}$ is a factor of $a^{\phi_{k+1,u}(u),v}$. Thus $a^{\phi_{k+1,u}(u),v} = 0$.

By Lemma 2.4.2, $x_{[\phi_{\eta+1,u}(u)]_{i_\eta},\beta}$ is not the (i_k)th variable in $x^{\phi_{k,u}(u),v}$. Thus we need now consider the case were $x_{[\phi_{\eta+1,u}(u)]_{i_\eta},\beta}$ is the (i_k+1)st variable in $x^{\phi_{k,u}(u),v}$. In this case, $[\phi_{\eta+1,u}(u)]_{i_\eta} = [\phi_{k,u}(u)]_{i_k+1}$ and $\beta = v_{i_k+1}$. Therefore, $x_{[\phi_{\eta+1,u}(u)]_{i_\eta},\beta}$ can be rewritten as $x_{[\phi_{k,u}(u)]_{i_k+1},v_{i_k+1}}$. Now since $\phi_{k+1,u}(u) = s_{i_k}[\phi_{k,u}(u)]$ we have $[\phi_{k,u}(u)]_{i_k+1} = [\phi_{k+1,u}(u)]_{i_k}$. That is, the letter $[\phi_{k,u}(u)]_{i_k+1}$ is in position i_k of $\phi_{k+1,u}(u)$. Thus $x_{[\phi_{k+1,u}(u)]_{i_k},v_{i_k}} = x_{[\phi_{k,u}(u)]_{i_k+1},v_{i_k}}$ is a variable in the monomial $x^{\phi_{k+1,u}(u),v}$. Since $\beta \geq w_{i_k}$ and $a_{[\phi_{k,u}(u)]_{i_k+1},w_{i_k}} = 0$, it follows that $a_{[\phi_{k,u}(u)]_{i_k+1},\beta} = 0$. By assumption, $s_{i_k}v < v$ which implies $v_{i_k} > v_{i_k+1}$. Thus $a_{[\phi_{k,u}(u)]_{i_k+1},\beta} = a_{[\phi_{k,u}(u)]_{i_k+1},v_{i_k+1}} = 0$ implies $a_{[\phi_{k,u}(u)]_{i_k+1},v_{i_k}} = 0$. Furthermore, since $a_{[\phi_{k,u}(u)]_{i_k+1},v_{i_k}}$ is a factor of $a^{\phi_{k+1,u}(u),v}$, then $a^{\phi_{k+1,u}(u),v} = 0$.

(4): Observe that since $\phi_{k+1,u}(u) = s_{i_k}[\phi_{k,u}(u)]$, it follows that $[\phi_{k+1,u}(u)]_{i_k} = [\phi_{k,u}(u)]_{i_k+1}$. By assumption $v_{i_k} = w_{i_k}$, thus the product $a^{\phi_{k+1,u}(u),\, v}$ has

$$a_{[\phi_{k+1,u}(u)]_{i_k},v_{i_k}} = a_{[\phi_{k,u}(u)]_{i_k+1},v_{i_k}} = a_{[\phi_{k,u}(u)]_{i_k+1},w_{i_k}}$$

as a factor. By assumption this factor is zero, thus $a^{\phi_{k+1,u}(u),\, v} = 0$. □

Lemma 2.4.4. *Let u, $z \in \mathfrak{S}_n$ with $s_{i_1} \cdots s_{i_\ell}$ the right-to-left lexicographically greatest reduced expression for u. Let $p_1 \cdots p_{k-1}$ be a subexpression of $s_{i_1} \cdots s_{i_{k-1}}$ such that if $s_{i_j} p_{(j-1)} \cdots p_1 z > p_{(j-1)} \cdots p_1 z$ in the Bruhat order then $p_j = s_{i_j}$. Fix $k \in [\ell]$, then*

$$(s_{i_{(k-1)}} \cdots s_{i_1} z)_{i_k} \geq (p_{k-1} \cdots p_1 z)_{i_k}.$$

Proof. We will provide a proof by induction. If $k = 1$, then $z_{i_1} = z_{i_1}$ and the claim holds trivially.

Consider the case where $k = 2$. If $p_1 = s_{i_1}$ then $(s_{i_1} z)_{i_2} = (p_1 z)_{i_2}$. If $p_1 = e$ then by Observation 1.1.4 there are three cases we need to consider. Namely, $i_2 > i_1 + 1$, $i_2 = i_1 - 1$, and $i_2 = i_1 + 1$. First suppose $i_2 > i_1 + 1$. Then $(s_{i_1} z)_{i_2} = z_{i_2}$ because s_{i_1} swaps positions i_1 and i_1+1 of z leaving position $i_2 > i_1+1$ unchanged. Furthermore, $(p_1 z)_{i_2} = z_{i_2}$ because $p_1 = e$. Thus $(s_{i_1} z)_{i_2} = (p_1 z)_{i_2}$. Next suppose $i_2 = i_1 - 1$. Then $(s_{i_1} z)_{i_2} = z_{i_2}$ because s_{i_1} leaves position $i_2 = i_1 - 1$ unchanged. Again $p_1 = e$ implies $(p_1 z)_{i_2} = z_{i_2}$. Thus $(s_{i_1} z)_{i_2} = (p_1 z)_{i_2}$. Lastly, suppose $i_2 = i_1 + 1$. Then

$$(s_{i_1} z)_{i_2} = (s_{i_1} z)_{i_1+1} = z_{i_1},$$

because s_{i_1} moves the letter in position i_1 of z to position $i_1 + 1$ of $s_{i_1} z$. We also have

$$(p_1 z)_{i_2} = (p_1 z)_{i_1+1} = z_{i_1+1}$$

because $p_1 = e$. Furthermore, since $p_1 = e$, we have $s_{i_1} z < z$ which implies $z_{i_1} > z_{i_1+1}$. Therefore, $(s_{i_1} z)_{i_2} > (p_1 z)_{i_2}$.

Now suppose $(s_{i_{(m-1)}} \cdots s_{i_1} z)_{i_m} \geq (p_{m-1} \cdots p_1 z)_{i_m}$ for each $m \in [k-1]$. By Observation 1.1.4, either $i_k = i_{(k-1)} + 1$, $i_k = i_{(k-1)} - 1$, or $i_k > i_{(k-1)} + 1$. First consider the case where $i_k = i_{(k-1)} + 1$. If $p_{k-1} = s_{i_{(k-1)}}$ then

$$(p_{k-1} \cdots p_1 z)_{i_{(k-1)}+1} = (p_{k-2} \cdots p_1 z)_{i_{(k-1)}}.$$

We know $(s_{i_{(k-1)}} \cdots s_{i_1} z)_{i_{(k-1)}+1} = (s_{i_{(k-2)}} \cdots s_{i_1} z)_{i_{(k-1)}}$ because applying $s_{i_{(k-1)}}$ moves the letter in position $i_{(k-1)}$ to position $i_{(k-1)}+1$. By induction $(s_{i_{(k-2)}} \cdots s_{i_1} z)_{i_{(k-1)}} \geq (p_{k-2} \cdots p_1 z)_{i_{(k-1)}}$. Therefore, we have

$$(s_{i_{(k-1)}} \cdots s_{i_1} z)_{i_{(k-1)}+1} \geq (p_{k-1} \cdots p_1 z)_{i_{(k-1)}+1}.$$

If $p_{k-1} = e$, then $(p_{k-1} \cdots p_1 z)_{i_{(k-1)}+1} = (p_{k-2} \cdots p_1 z)_{i_{(k-1)}+1}$. By assumption, since $p_{k-1} = e$, we have

$$s_{i_{(k-1)}} p_{k-2} \cdots p_1 z < p_{k-2} \cdots p_1 z,$$

which implies

$$(p_{k-2}\cdots p_1 z)_{i_{(k-1)}} > (p_{k-2}\cdots p_1 z)_{i_{(k-1)}+1}.$$

Again we know $(s_{i_{(k-1)}}\cdots s_{i_1} z)_{i_{(k-1)}+1} = (s_{i_{(k-2)}}\cdots s_{i_1} z)_{i_{(k-1)}}$. By induction

$$(s_{i_{(k-2)}}\cdots s_{i_1} z)_{i_{(k-1)}} \geq (p_{k-2}\cdots p_1 z)_{i_{(k-1)}}.$$

Therefore, we have

$$(s_{i_{(k-1)}}\cdots s_{i_1} z)_{i_{(k-1)}+1} > (p_{k-1}\cdots p_1 z)_{i_{(k-1)}+1}.$$

Next suppose $i_k \neq i_{(k-1)}+1$ and let $j < k$ be the largest index such that $i_k = i_j + 1$. By the above, $(s_{i_j}\cdots s_{i_1} z)_{i_j+1} \geq (p_j \cdots p_1 z)_{i_j+1}$. By the maximality of the index j, necessarily $s_{i_{(j+1)}}, s_{i_{(j+2)}}, \ldots, s_{i_{(k-1)}}$ do not affect position $i_k = i_j + 1$. Thus

$$(s_{i_{(k-1)}}\cdots s_{i_1} z)_{i_j+1} = (s_{i_j}\cdots s_{i_1} z)_{i_j+1} \geq (p_j\cdots p_1 z)_{i_j+1} = (p_{k-1}\cdots p_1 z)_{i_j+1}.$$

That is, $(s_{i_{(k-1)}}\cdots s_{i_1} z)_{i_k} \geq (p_{k-1}\cdots p_1 z)_{i_k}$. □

Lemma 2.4.5. *Let $A = (a_{i,j})$ be the antiadjacency matrix of an n-element unit interval order P with respect to an ar-labeling and $s_{i_1}\cdots s_{i_\ell}$ the right-to-left lexicographically greatest reduced expression for a permutation $u \in \mathfrak{S}_n$. Fix $k \in [\ell]$, $z \in \mathfrak{S}_n$, and let $p_1 \cdots p_{k-1}$ be a subexpression of $s_{i_1}\cdots s_{i_{k-1}}$ such that if $s_{i_j} p_{(j-1)}\cdots p_1 z > p_{(j-1)}\cdots p_1 z$ then $p_j = s_{i_j}$. Define the permutations $u' = s_{i_{(k-1)}}\cdots s_{i_1} u$, $u'' = s_{i_k}\cdots s_{i_1} u$, $w = s_{i_{k-1}}\cdots s_{i_1} z$, and $v = p_{k-1}\cdots p_1 z$. If $a_{u'_{i_k+1}, w_{i_k}} = 1$ and $s_{i_k} v < v$, then $a^{u'', v} = a^{u', v}$.*

Proof. Observe that $u''_{i_k} = u'_{i_k+1}$ and $u''_{i_k+1} = u'_{i_k}$. The product

$$a_{u'_{i_k}, v_{i_k}} a_{u'_{i_k+1}, v_{i_k+1}} \tag{2.4.7}$$

is a factor of $a^{u', v}$ and the product

$$a_{u''_{i_k}, v_{i_k}} a_{u''_{i_k+1}, v_{i_k+1}} = a_{u'_{i_k+1}, v_{i_k}} a_{u'_{i_k}, v_{i_k+1}} \tag{2.4.8}$$

is a factor of $a^{u'', v}$. By Lemma 2.4.4, $w_{i_k} \geq v_{i_k}$. By assumption $s_{i_k} v < v$ which implies $v_{i_k} > v_{i_k+1}$. Recall that the zero entries in A form a right justified Young diagram. Therefore any entry equal to 1 has only 1's to the left and below it. Thus the assumption $a_{u'_{i_k+1}, w_{i_k}} = 1$ implies

$$a_{u'_{i_k+1}, v_{i_k}} = 1 \text{ and } a_{u'_{i_k+1}, v_{i_k+1}} = 1. \tag{2.4.9}$$

By Observation 1.2.4 s_{i_k} is a left descent of u' and so $u'_{i_k} > u'_{i_k+1}$. Therefore, (2.4.9) implies

$$a_{u'_{i_k},\, v_{i_k}} = 1 \text{ and } a_{u'_{i_k},\, v_{i_k+1}} = 1.$$

Since all other factors of $a^{u'',\, v}$ and $a^{u',\, v}$ agree, the two are equal. $\square$

Lemma 2.4.6. *Let A be the antiadjacency matrix of an n-element unit interval order P with respect to an ar-labeling. Fix permutations u, $v \in \mathfrak{S}_n$ and let $s_{i_1} \cdots s_{i_\ell}$ be the right-to-left lexicographically greatest reduced expression for u. Fix $k \in [\ell]$ and define the permutations $u' = \phi_{k,u}(u) = s_{i_{(k-1)}} \cdots s_{i_1} u$ and $u'' = \phi_{k+1,u}(u) = s_{i_k} \cdots s_{i_1} u$. Then*

(1) $a^{u'', s_{i_k} v} = a^{u',v}$,

(2) If $s_{i_k} v > v$, then $q_{u'', s_{i_k} v} = q_{u',v} \cdot q$,

(3) If $s_{i_k} v < v$, then $q_{u'', s_{i_k} v} = q_{u',v}$.

Proof. (1): By definition $u' = s_{i_{(k-1)}} \cdots s_{i_1} u$ and $u'' = s_{i_k} \cdots s_{i_1} u$, thus $u'' = s_{i_k} u'$. Therefore, we can write $a^{u'', s_{i_k} v} = a^{s_{i_k} u', s_{i_k} v} = a^{u',v}$, where the last equality holds because the factors commute.

(2): By definition,

$$q_{u'', s_{i_k} v} = (q^{\frac{1}{2}})^{\ell(s_{i_k} v) - \ell(u'')}.$$

The condition $s_{i_k} v > v$ implies $\ell(s_{i_k} v) = \ell(v) + 1$. By Observation 1.2.4, $\ell(u'') = \ell(u') - 1$. Therefore, $q_{u'', s_{i_k} v}$ can be written as

$$(q^{\frac{1}{2}})^{[\ell(v)+1] - [\ell(u')-1]} = (q^{\frac{1}{2}})^{\ell(v) - \ell(u') + 2} = q_{u',v} \cdot q.$$

(3): Again, by definition,

$$q_{u'', s_{i_k} v} = (q^{\frac{1}{2}})^{\ell(s_{i_k} v) - \ell(u'')}.$$

The condition $s_{i_k}v < v$ implies $\ell(s_{i_k}v) = \ell(v) - 1$. By Observation 1.2.4, $\ell(u'') = \ell(u') - 1$. Therefore, $q_{u'',s_{i_k}v}$ can be written as

$$(q^{\frac{1}{2}})^{[\ell(v)-1]-[\ell(u')-1]} = (q^{\frac{1}{2}})^{\ell(v)-\ell(u')} = q_{u',v}.$$

$\square$

We can write a recursive relation for $\sigma_{A,u''}(x^{u,\,z})$ by replacing yu with z in Equation (2.4.6), applying $\sigma_{A,u''}$, and using the above lemmas. This recursive relation is given by the following theorem.

Theorem 2.4.7. *Let A be the antiadjacency matrix of an n-element unit interval order P with respect to an ar-labeling. Fix permutations u, $z \in \mathfrak{S}_n$ and let $s_{i_1} \cdots s_{i_\ell}$ be the right-to-left lexicographically greatest reduced expression for u. Fix $k \in [\ell]$ and define the permutations $u' = \phi_{k,u}(u) = s_{i_{(k-1)}} \cdots s_{i_1}u$, $u'' = \phi_{k+1,u}(u) = s_{i_k} \cdots s_{i_1}u$, and $w = \phi_{k,u}(z) = s_{i_{(k-1)}} \cdots s_{i_1}z$. Then,*

$$\sigma_{A,u''}(x^{u,z}) = \begin{cases} \sigma_{A,u'}(x^{u,z}) & \text{if } u'_{i_k+1} <_P w_{i_k} \\ q\sigma_{A,u'}(x^{u,z}) & \text{otherwise.} \end{cases}$$

Proof. Applying $\sigma_{A,u''}$ to Equation (2.4.6) yields

$$\begin{aligned} \sigma_{A,u''}(x^{u,z}) = &\sum_{\substack{v \geq w \\ s_{i_k}v > v}} t^k_{u,z,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}})a^{u'',s_{i_k}v}q_{u'',s_{i_k}v} \\ &+ \sum_{\substack{v \geq w \\ s_{i_k}v < v}} t^k_{u,z,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) \left[a^{u'',\,s_{i_k}v}q_{u'',\,s_{i_k}v} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})a^{u'',\,v}q_{u'',\,v}\right]. \end{aligned} \tag{2.4.10}$$

By Lemma 2.4.6 this is equal to

$$\sum_{\substack{v \geq w \\ s_{i_k}v > v}} t^k_{u,z,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}})a^{u',\,v}q_{u',v} \cdot q + \sum_{\substack{v \geq w \\ s_{i_k}v < v}} t^k_{u,z,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) \left[a^{u',\,v}q_{u',v} + (q-1)a^{u'',\,v}q_{u',v}\right]. \tag{2.4.11}$$

Assume first that $u'_{i_k+1} <_P w_{i_k}$ and thus $a_{u'_{i_k+1},w_{i_k}} = 0$. Lemma 2.4.3 (1) states that if $w_{i_k} \neq v_{i_k}$ and $a_{u'_{i_k+1},w_{i_k}} = 0$, then $a^{u',v} = 0$. Lemma 2.4.3 (2) states that if

$w_{i_k} = v_{i_k}$, $s_{i_k}v > v$, and $a_{u'_{i_k+1},w_{i_k}} = 0$, then $a^{u',v} = 0$. Thus together, Lemma 2.4.3 (1) and 2.4.3(2) imply that the first sum in (2.4.11) is zero. Now Lemma 2.4.3 (3) states that if $w_{i_k} \neq v_{i_k}$, $s_{i_k}v < v$ and $a_{u'_{i_k+1},w_{i_k}} = 0$, then $a^{u'',v} = 0$. Lemma 2.4.3(4) states that if $w_{i_k} = v_{i_k}$ and $a_{u'_{i_k+1},w_{i_k}} = 0$, then $a^{u'',v} = 0$. Thus together, Lemma 2.4.3 (3) and 2.4.3 (4) imply that the second sum in (2.4.11) is

$$\sum_{\substack{v\geq w\\ s_{i_k}v<v}} t^k_{u,z,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}})a^{u',\,v}q_{u',v}.$$

By Equation (1.3.15) and the definition of $\sigma_{A,u'}$, this is just $\sigma_{A,u'}(x^{u,z})$.

Now assume that $u'_{i_k+1} \not<_P w_{i_k}$ and thus $a_{u'_{i_k+1},w_{i_k}} = 1$. Lemma 2.4.5 states that if $a_{u'_{i_k+1},w_{i_k}} = 1$ and $s_{i_k}v < v$, then $a^{u'',\,v} = a^{u',\,v}$. Thus (2.4.11) can be written as

$$\sum_{\substack{v\geq w\\ s_{i_k}v>v}} t^k_{u,z,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}})a^{u',\,v}q_{u',v}\cdot q + \sum_{\substack{v\geq w\\ s_{i_k}v<v}} t^k_{u,z,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}})\left[a^{u',\,v}q_{u',v} + (q-1)a^{u',\,v}q_{u',v}\right]$$

$$= \sum_{\substack{v\geq w\\ s_{i_k}v>v}} t^k_{u,z,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}})a^{u',\,v}q_{u',v}\cdot q + \sum_{\substack{v\geq w\\ s_{i_k}v<v}} t^k_{u,z,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}})a^{u',\,v}q_{u',v}\cdot q$$

$$= \sum_{v\geq w} t^k_{u,z,v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}})a^{u',v}q_{u',v}q. \quad (2.4.12)$$

By Equation (1.3.15) and the definition of $\sigma_{A,u'}$, this is just $q\cdot\sigma_{A,u'}(x^{u,z})$. And so the claim holds. □

Observation 2.4.8. *Given an ordered set partition $(I_1, ..., I_r)$ of $[n]$ of shape $\lambda = (\lambda_1, ..., \lambda_r)$, let u be the permutation whose one-line notation is $\overline{I_1}\cdots\overline{I_r}$, where $\overline{I_j}$ is the increasing rearrangement of I_j. Let T be the P-tableau whose jth column contains I_j in order of increasing labels. If (i,j) is an inversion in T, then (i,j) is an inversion in u. More specifically,*

$$inv(u) = \text{INV}(T) + \#\{\textit{inversions in } u \textit{ that are not inversions in } T\}$$

More can be said about the positions of certain letters in permutations u of this form.

Lemma 2.4.9. *Let $(I_1, ..., I_r)$ be an ordered set partition of type $\lambda = (\lambda_1, ..., \lambda_r)$ of $[n]$ and let u be the permutation whose one-line notation is $\overline{I_1} \cdots \overline{I_r}$, where $\overline{I_j}$ is the increasing rearrangement of I_j. Let $s_{i_1} \cdots s_{i_\ell}$ be the right-to-left lexicographically greatest reduced expression for u. Fix a permutation $y \in \mathfrak{S}_\lambda$ and an integer $k \in [\ell + 1]$. Recall the definition $\phi_{k,u}(w) = s_{i_{(k-1)}} \cdots s_{i_1} w$. Then the letter $[\phi_{k,u}(yu)]_{i_k}$ appears in a position of u to the left of the letter $[\phi_{k,u}(u)]_{i_k+1}$.*

Proof. By Observation 1.2.4, s_{i_k} is a left descent of $\phi_{k,u}(u)$ and thus

$$[\phi_{k,u}(u)]_{i_k} > [\phi_{k,u}(u)]_{i_k+1}.$$

By the algorithm for determining the right-to-left lexicographically greatest reduced expression for u, applying s_{i_j} to $s_{i_{(j-1)}} \cdots s_{i_1} u$ moves a greater letter to the right. Thus, since the letter $[\phi_{k,u}(u)]_{i_k}$ is greater than and appears to the left of the letter $[\phi_{k,u}(u)]_{i_k+1}$ in $\phi_{k,u}(u)$, the letter $[\phi_{k,u}(u)]_{i_k}$ appears to the left of the letter $[\phi_{k,u}(u)]_{i_k+1}$ in u. More specifically, for some indices $g < h$, the letters $[\phi_{k,u}(u)]_{i_k}$ and $[\phi_{k,u}(u)]_{i_k+1}$ are in sets I_g and I_h respectively.

Observe that applying the sequence $s_{i_{(k-1)}} \cdots s_{i_1}$ of generators to both u and yu yields the permutations $\phi_{k,u}(u) = s_{i_{(k-1)}} \cdots s_{i_1} u$ and $\phi_{k,u}(yu) = s_{i_{(k-1)}} \cdots s_{i_1} yu$. Thus letters $[\phi_{k,u}(u)]_{i_k} = (s_{i_{(k-1)}} \cdots s_{i_1} u)_{i_k}$ and $[\phi_{k,u}(yu)]_{i_k} = (s_{i_{(k-1)}} \cdots s_{i_1} yu)_{i_k}$ both being in position i_k of their respective permutations implies that for some $j \in [\ell]$ these letters are in position j of u and yu respectively. Above we said $u_j = [\phi_{k,u}(u)]_{i_k}$ is in the set I_g. Since $y \in \mathfrak{S}_\lambda$ and I is a set partition of shape λ, it follow that $(yu)_j = [\phi_{k,u}(yu)]_{i_k}$ is also in the set I_g. Recalling that the one-line notation for u is $\overline{I_1} \cdots \overline{I_r}$, then $[\phi_{k,u}(yu)]_{i_k} \in I_g$ and $[\phi_{k,u}(u)]_{i_k+1} \in I_h$ with $g < h$ implies that the letter $[\phi_{k,u}(yu)]_{i_k}$ appears to the left of the letter $[\phi_{k,u}(u)]_{i_k+1}$ in u. □

For certain permutations z and permutations u of the form in Observation 2.4.8, the poset relation $u'_{i_k+1} <_P w_{i_k}$ in Theorem 2.4.7 can be rewritten in terms of inversions in P-tableaux. The next theorem explains this translation from partial order relations to inversions.

Theorem 2.4.10. *Let A be the antiadjacency matrix of an n-element unit interval order P with respect to an ar-labeling. Let $(I_1, ..., I_r)$ be an ordered set partition of*

$[n]$ of type $\lambda = (\lambda_1, ..., \lambda_r)$. Let T be the P-tableau whose jth column contains I_j in order of increasing labels. Fix a permutation $y \in \mathfrak{S}_\lambda$ and let u be the permutation whose one-line notation is $\overline{I_1} \cdots \overline{I_r}$, where $\overline{I_j}$ is the increasing rearrangement of I_j. Recall the definition $\phi_{k,u}(w) = s_{i_{(k-1)}} \cdots s_{i_1} w$. Then,

$$\sigma_{A,\phi_{k+1,u}(u)}(x^{u,yu}) = \begin{cases} \sigma_{A,\phi_{k,u}(u)}(x^{u,yu}) & \text{if } ([\phi_{k,u}(u)]_{i_k+1}, [\phi_{k,u}(yu)]_{i_k}) \text{ is an} \\ & \text{inversion in } u \text{ and not in } T, \\ q\sigma_{A,\phi_{k,u}(u)}(x^{u,yu}) & \text{otherwise,} \end{cases} \tag{2.4.13}$$

where "otherwise" means that either $([\phi_{k,u}(u)]_{i_k+1}, [\phi_{k,u}(yu)]_{i_k})$ is not an inversion in u or the pair is an inversion in T.

Proof. By Theorem 2.4.7, we have

$$\sigma_{A,\phi_{k+1,u}(u)}(x^{u,yu}) = \begin{cases} \sigma_{A,\phi_{k,u}(u)}(x^{u,yu}) & \text{if } [\phi_{k,u}(u)]_{i_k+1} <_P [\phi_{k,u}(yu)]_{i_k} \\ q\sigma_{A,\phi_{k,u}(u)}(x^{u,yu}) & \text{otherwise.} \end{cases} \tag{2.4.14}$$

Suppose first that $[\phi_{k,u}(u)]_{i_k+1} <_P [\phi_{k,u}(yu)]_{i_k}$. Since P is naturally labeled, we must have $[\phi_{k,u}(u)]_{i_k+1} < [\phi_{k,u}(yu)]_{i_k}$ in $\mathbb{Z}$. By Lemma 2.4.9, the letter $[\phi_{k,u}(yu)]_{i_k}$ appears in a position to the left of the letter $[\phi_{k,u}(u)]_{i_k+1}$ in u. Therefore, the pair $([\phi_{k,u}(u)]_{i_k+1}, [\phi_{k,u}(yu)]_{i_k})$ is an inversion in u. On the other hand, since $[\phi_{k,u}(u)]_{i_k+1}$ and $[\phi_{k,u}(yu)]_{i_k}$ are comparable in P, the pair $([\phi_{k,u}(u)]_{i_k+1}, [\phi_{k,u}(yu)]_{i_k})$ cannot be an inversion in the P-tableau T.

Now suppose the pair $([\phi_{k,u}(u)]_{i_k+1}, [\phi_{k,u}(yu)]_{i_k})$ is an inversion in u and is not an inversion in T. Then by Lemma 2.4.9, the letter $[\phi_{k,u}(yu)]_{i_k}$ appears before the letter $[\phi_{k,u}(u)]_{i_k+1}$ in u, and we therefore have $[\phi_{k,u}(u)]_{i_k+1} < [\phi_{k,u}(yu)]_{i_k}$ in $\mathbb{Z}$. Since the pair is not an inversion in T, then the pair must be comparable. Specifically, we have $[\phi_{k,u}(u)]_{i_k+1} <_P [\phi_{k,u}(yu)]_{i_k}$ because P is naturally labeled and $[\phi_{k,u}(u)]_{i_k+1} < [\phi_{k,u}(yu)]_{i_k}$ in $\mathbb{Z}$. □

The following example illustrates this recursive relationship. Let $u \in \mathfrak{S}_4$ be the permutation with one-line notation 2413 and right-to-left lexicographically greatest reduced expression $s_2 s_1 s_3$. Let $y = s_1 s_3$, then yu is the permutation 4231. Observe that using the second and third relations in (1.3.1) the monomial $x^{u,yu}$ in $\mathcal{A}_{[n],[n]}(n;q)$ can be expressed as follows.

$$x^{u,yu} = x_{2,4}x_{4,2}x_{1,3}x_{3,1} \tag{2.4.15}$$
$$= x_{2,4}x_{1,3}x_{4,2}x_{3,1} \tag{2.4.16}$$
$$= x_{1,3}x_{2,4}x_{4,2}x_{3,1} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x_{1,4}x_{2,3}x_{4,2}x_{3,1} \tag{2.4.17}$$
$$= x_{1,3}x_{2,4}x_{3,1}x_{4,2} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x_{1,3}x_{2,4}x_{3,2}x_{4,1} \tag{2.4.18}$$
$$+ (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x_{1,4}x_{2,3}x_{3,1}x_{4,2} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})^2 x_{1,4}x_{2,3}x_{3,2}x_{4,1} \tag{2.4.19}$$

Let A be the antiadjacency matrix associated to a unit interval order P with respect to an ar-labeling. Consider the expressions,

$$\sigma_{A,u}(x^{u,yu}) = a_{2,4}a_{4,2}a_{1,3}a_{3,1}q, \tag{2.4.20}$$
$$\sigma_{A,s_2u}(x^{u,yu}) = a_{2,4}a_{1,3}a_{4,2}a_{3,1}q^2, \tag{2.4.21}$$
$$\sigma_{A,s_1s_2u}(x^{u,yu}) = a_{1,3}a_{2,4}a_{4,2}a_{3,1}q^2 + a_{1,4}a_{2,3}a_{4,2}a_{3,1}(q^3 - q^2), \tag{2.4.22}$$
$$\sigma_{A,s_3s_1s_2u}(x^{u,yu}) = a_{1,3}a_{2,4}a_{3,1}a_{4,2}q^2 + a_{1,3}a_{2,4}a_{3,2}a_{4,1}(q^3 - q^2) + \tag{2.4.23}$$
$$a_{1,4}a_{2,3}a_{3,1}a_{4,2}(q^3 - q^2) + a_{1,4}a_{2,3}a_{3,2}a_{4,1}(q^4 - 2q^3 + q^2).$$

Observe that Equation (2.4.22) is equal to Equation (2.4.21) if $a_{1,4} = 0$. By the definition of A, $a_{i,j} = 0$ implies $i <_p j$. Thus the condition $a_{1,4} = 0$ is equivalent to $1 <_p 4$, which by the proof of Theorem 2.4.10 is equivalent to $(1, 4)$ being an inversion in u and not an inversion in T. Furthermore, Equation (2.4.22) is q times Equation (2.4.21) if $a_{1,4} = 1$. By the definition of A, $a_{i,j} = 1$ implies either $i > j$ or i is incomparable to j. Thus since $1 < 4$, then $a_{1,4} = 1$ is equivalent to 1 being incomparable to 4.

Now observe that the pair $([\phi_{k,u}(u)]_{i_k+1}, [\phi_{k,u}(yu)]_{i_k})$ is not necessarily an inversion in u. For example, consider Equation (2.4.22) where $[\phi_{3,u}(u)]_{i_3+1} = 3$ and $[\phi_{3,u}(yu)]_{i_3} = 2$ are not inverted in u. By Theorem 2.4.10, Equation (2.4.23) is equal to q times Equation (2.4.22). We can see that this is true because $a_{3,2}$ is necessarily equal to 1.

With the results from Theorem 2.4.10 for $u \in \mathfrak{S}_n$ of a certain form, $\sigma_{A,e}(x^{u,yu})$

can be evaluated recursively by evaluating $\sigma_{A,u}(x^{u,yu})$, $\sigma_{A,s_{i_1}u}(x^{u,yu})$, ..., $\sigma_{A,e}(x^{u,yu})$. This is done in the next result.

Theorem 2.4.11. *Let A be the antiadjacency matrix of an n-element unit interval order P with respect to an ar-labeling. Let $(I_1, ..., I_r)$ be an ordered set partition of $[n]$ of type $\lambda = (\lambda_1, ..., \lambda_r)$. Let T be the P-tableau whose jth column contains I_j in order of increasing labels. Fix a permutation $y \in \mathfrak{S}_\lambda$ and let u be the permutation whose one-line notation is $\overline{I_1} \cdots \overline{I_r}$, where $\overline{I_j}$ is the increasing rearrangement of I_j. Then we have $\sigma_{A,e}(x^{u,\, yu}) = a^{u,\, yu} q_{e,\, y} q^{\mathrm{INV}(T)}$.*

Proof. By definition, we have $\sigma_{A,u}(x^{u,yu}) = a^{u,yu} q_{u,yu} = a^{u,yu} q_{e,y}$. Let $s_{i_1} \cdots s_{i_\ell}$ be the right-to-left lexicographically greatest reduced expression for u, recall the definition $\phi_{k,u}(u) = s_{i_{k-1}} \cdots s_{i_1} u$, and consider the expressions

$$\sigma_{A,\phi_{1,u}(u)}(x^{u,yu}) = \sigma_{A,u}(x^{u,yu}),\ \sigma_{A,\phi_{2,u}(u)}(x^{u,yu}), \ldots,\ \sigma_{A,\phi_{\ell+1,u}(u)}(x^{u,yu}) = \sigma_{A,e}(x^{u,yu}).$$

By Theorem 2.4.10, we have

$$\sigma_{A,\phi_{k+1,u}(u)}(x^{u,yu}) = \begin{cases} \sigma_{A,\phi_{k,u}(u)}(x^{u,yu}) & \text{if } ([\phi_{k,u}(u)]_{i_k+1}, [\phi_{k,u}(yu)]_{i_k}) \text{ is an} \\ & \text{inversion in } u \text{ and not in } T, \\ q\sigma_{A,\phi_{k,u}(u)}(x^{u,yu}) & \text{otherwise,} \end{cases} \tag{2.4.24}$$

where "otherwise" means that either $([\phi_{k,u}(u)]_{i_k+1}, [\phi_{k,u}(yu)]_{i_k})$ is not an inversion in u or the pair is an inversion in both u and T. Thus, by Observation 2.4.8 we have

$$\begin{aligned} \sigma_{A,e}(x^{u,yu}) &= a^{u,yu} q_{e,y} q^{\ell(u) - \#\{\text{inversions in } u \text{ that are not inversions in } T\}} \\ &= a^{u,yu} q_{e,y} q^{\mathrm{INV}(T)}. \end{aligned} \tag{2.4.25}$$

□

This result allows us to replace $\sigma_{A,e}(x^{u,yu})$ in the expansion (2.4.2) of the product of quantum determinants with a simpler expression.

Corollary 2.4.12. *Let A be the antiadjacency matrix of an n-element unit interval order P with respect to an ar-labeling. For each ordered set partition $(I_1, \ldots, I_r)$*

of $[n]$ *of type* $\lambda = (\lambda_1, \ldots, \lambda_r)$*, let* $T = T(I_1, \ldots, I_r)$ *be the* P*-tableau whose* j*th column contains* I_j *in order of increasing labels. Fix a permutation* $y \in \mathfrak{S}_\lambda$ *and let* $u = u(I_1, \ldots, I_r)$ *be the permutation whose one-line notation is* $\overline{I_1} \cdots \overline{I_r}$*, where* $\overline{I_j}$ *is the increasing rearrangement of* I_j*. Then,*

$$\begin{aligned}\sigma_{A,e}\left(\mathrm{Imm}_{\epsilon_q^\lambda}(x)\right) &= \sum_{(I_1,\ldots,I_r)} \sum_{y \in \mathfrak{S}_\lambda} (-1)^{\ell(y)} q_{e,y}^{-1} a^{u,\, yu} q_{e,\, y} q^{\mathrm{INV}(T)} \\ &= \sum_{(I_1,\ldots,I_r)} q^{\mathrm{INV}(T)} \sum_{y \in \mathfrak{S}_\lambda} (-1)^{\ell(y)} a^{u,\, yu}. \end{aligned} \tag{2.4.26}$$

We can discuss a property of the set of permutations $y \in \mathfrak{S}_\lambda$ which yield nonzero products of entries in the antiadjacency matrix A of a poset P. This result will allow Equation (2.4.26) to be simplified further.

Lemma 2.4.13. *Every permutation* $y \in \mathfrak{S}_\lambda$ *can be factored as a product of permutations in* $\mathfrak{S}_{\lambda_1}$*,* $\mathfrak{S}_{\lambda_2}$*, ...,* $\mathfrak{S}_{\lambda_r}$ *and the sign of* y *is equal to the product of the signs of these factors.*

Proof. Recall that for an integer partition $\lambda = (\lambda_1, \ldots, \lambda_r)$,

$$\mathfrak{S}_\lambda \cong \mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \cdots \times \mathfrak{S}_{\lambda_r}$$

Thus for $y \in \mathfrak{S}_\lambda$, y can be factored as $y^{(1)} y^{(2)} \cdots y^{(r)}$ where $y^{(i)} \in \mathfrak{S}_{\lambda_i}$.

Furthermore, every expression for a fixed permutation has the same parity. Thus

$$\begin{aligned} sgn(y) &= (-1)^{\ell(y)} \\ &= (-1)^{\ell(y^{(1)}) + \ell(y^{(2)}) + \cdots + \ell(y^{(r)})} \\ &= (-1)^{\ell(y^{(1)})} (-1)^{\ell(y^{(2)})} \cdots (-1)^{\ell(y^{(r)})} \end{aligned}$$

as claimed. □

Given the above lemma, now consider the parity of elements in the set $\{y \in \mathfrak{S}_\lambda | a^{u,yu} = 1\}$, where $A = (a_{i,j})$ is the antiadjacency matrix for some poset P. If u is the permutation with one-line notation $\overline{I_1} \cdots \overline{I_r}$, where $\overline{I_j}$ and T is the P-tableau whose jth column contains $\overline{I_j}$, then there are two cases one must consider; either T is column-strict or not. The following two propositions consider these cases.

Proposition 2.4.14. *Let A be the antiadjacency matrix of an n-element unit interval order P with an ar-labeling. Fix $(I_1, \ldots, I_r)$ an ordered set partition of $[n]$ of type $\lambda = (\lambda_1, \ldots, \lambda_r)$. Let u be the permutation whose one-line notation is $\overline{I_1} \cdots \overline{I_r}$ and let T be the P-tableau whose jth column contains I_j, in order of increasing labels. If T is not column-strict then the set $\{y \in \mathfrak{S}_\lambda | a^{u,\, yu} = 1\}$ contains the same number of even permutations as odd permutations.*

Proof. Choose one column which is not a chain, say the kth column. Let the elements in this column be $t = (t_1, \ldots, t_{\lambda_k})$. By Lemma 1.1.2, the permutations v of these elements for which

$$a_{t_1,v_1} a_{t_2,v_2} \cdots a_{t_{\lambda_k},v_{\lambda_k}} = 1 \tag{2.4.27}$$

are precisely the interval $[e, w]$ in the Bruhat order for some 312-avoiding permutation w. By Lemma 2.1.2, this interval consists of the same number of even permutations as odd permutations.

Let $p = (p_1, \ldots, p_{\lambda_l})$ be the elements in column l of T and fix z a permutation of these elements satisfying

$$a_{p_1,z_1} a_{p_2,z_2} \cdots a_{p_{\lambda_l},z_{\lambda_l}} = 1.$$

By Lemma 2.4.13, $sgn(zv) = sgn(z)sgn(v)$. Thus for a fixed $z \in \mathfrak{S}_{\lambda_l}$, the set

$$\{zv \in \mathfrak{S}_{\lambda_l} \times \mathfrak{S}_{\lambda_k} \,|\, a_{p_1,z_1} \cdots a_{p_{\lambda_l},z_{\lambda_l}} a_{t_1,v_1} \cdots a_{t_{\lambda_k},v_{\lambda_k}} = 1, v \in \mathfrak{S}_{\lambda_k}\}$$

consists of half even permutations and half odd permutations. Furthermore, letting z vary over permutations in $\mathfrak{S}_{\lambda_l}$, the set

$$\{zv \in \mathfrak{S}_{\lambda_l} \times \mathfrak{S}_{\lambda_k} \,|\, a_{p_1,z_1} \cdots a_{p_{\lambda_l},z_{\lambda_l}} a_{t_1,v_1} \cdots a_{t_{\lambda_k},v_{\lambda_k}} = 1, v \in \mathfrak{S}_{\lambda_k}\}$$

consists of half even permutations and half odd permutations. This extends to a product of any number of factors.

Each element of $y \in \mathfrak{S}_\lambda$ factors as $y^{(1)} \times y^{(2)} \times \cdots \times y^{(r)}$ with $y^{(i)} \in \mathfrak{S}_{\lambda_i}$. Therefore, by the above, the set

$$\{y \in \mathfrak{S}_\lambda \,|\, a^{u,yu} = 1\}$$

consists of half even permutations and half odd permutations.

$\square$

For an example to illustrate the above proposition, consider $u = 1346257$ with right-to-left lexicographically greatest reduced expression $s_4s_3s_2s_5$ and $\lambda = (4,3)$. If 1 and 4 are incomparable and 2 and 5 are incomparable, then the set $\{y \in \mathfrak{S}_\lambda | a^{u,\, yu} = 1\}$ is

$$\{e, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1, s_5, s_5s_1, s_5s_2, s_5s_1s_2, s_5s_2s_1, s_5s_1s_2s_1\}. \qquad (2.4.28)$$

Proposition 2.4.15. *Let A be the antiadjacency matrix of an n-element unit interval order P with an ar-labeling. Fix $(I_1, \ldots, I_r)$ an ordered set partition of $[n]$ of type $\lambda = (\lambda_1, \ldots, \lambda_r)$. Let u be the permutation whose one-line notation is $\overline{I_1} \cdots \overline{I_r}$ and let T be the P-tableau whose jth column contains I_j, in order of increasing labels. If T is column-strict, then*

$$\{y \in \mathfrak{S}_\lambda | a^{u,yu} = 1\} = \{e\}.$$

Proof. Assume T is the P-tableau defined above and suppose T is column-strict. Then for each $j \in [r]$, the elements in column j form a chain

$$i_1 <_P i_2 <_P \cdots <_P i_{\lambda_j}.$$

Thus by definition of A, $a_{i_k,i_l} = 0$ for every pair (k,l) with $k < l \leq \lambda_j$. Since $y \in \mathfrak{S}_\lambda$, applying y to u permutes the letters $i_1, i_2, \ldots, i_{\lambda_j}$ among themselves. Thus if y is not the identity permutation, then for some pair (k,l) with $k < l \leq \lambda_j$ we have that a_{i_k,i_l} is a factor of $a^{u,yu}$. Hence if y is not the identity permutation, then $a^{u,yu} = 0$.

Furthermore, $a^{u,u} = 1$ by the definition of A. Thus if T is column-strict then

$$\{y \in \mathfrak{S}_\lambda | a^{u,yu} = 1\} = \{e\}.$$

□

Combining the evaluation of $\sigma_{A,e}$ on the monomial $x^{u,yu}$ with the results concerning the parity of elements in the set $\{y \in \mathfrak{S}_\lambda | a^{u,yu} = 1\}$, a combinatorial interpretation of the image of ϵ_q^λ-immanants under $\sigma_{A,e}$ can be stated.

Theorem 2.4.16. *Let A be the antiadjacency matrix of an n-element unit interval order P with respect to an ar-labeling. Fix λ a partition of n. Then*

$$\sigma_{A,e}\left(\mathrm{Imm}_{\epsilon_q^\lambda}(x)\right) = \sum_T q^{\mathrm{INV}(T)} \tag{2.4.29}$$

where the sum is over all column-strict P-tableaux T of shape $\lambda^\top$.

Proof. By Equation (2.4.26), the left hand side of Equation (2.4.29) is

$$\sum_{(I_1,\ldots,I_r)} q^{\mathrm{INV}(T)} \sum_{y\in\mathfrak{S}_\lambda} (-1)^{\ell(y)} a^{u,\,yu}, \tag{2.4.30}$$

where the first sum is over all ordered set partitions $(I_1,\ldots,I_r)$ of $[n]$ of type λ, u is the permutation whose one-line notation is $\overline{I_1}\cdots\overline{I_r}$, and $T = T(I_1,\ldots,I_r)$ is the P-tableau whose jth column contains I_j, in order of increasing labels.

Observe that the terms in $\sum_{y\in\mathfrak{S}_\lambda}(-1)^{\ell(y)}a^{u,\,yu}$ can be regrouped by considering the parity of the permutation $y\in\mathfrak{S}_\lambda$. Let $\mathscr{E}_\lambda$ be the alternating subgroup of $\mathfrak{S}_\lambda$, that is, all even permutations on $\mathfrak{S}_\lambda$. Then

$$\begin{aligned}\sum_{y\in\mathfrak{S}_\lambda}(-1)^{\ell(y)}a^{u,\,yu} &= \sum_{y\in\mathscr{E}_\lambda} a^{u,\,yu} + \sum_{y\in\mathfrak{S}_\lambda\backslash\mathscr{E}_\lambda} -a^{u,\,yu} &(2.4.31)\\ &= \#\{y\in\mathscr{E}_\lambda | a^{u,yu}=1\} - \#\{y\in\mathfrak{S}_\lambda\backslash\mathscr{E}_\lambda | a^{u,yu}=1\}. &(2.4.32)\end{aligned}$$

By Proposition 2.4.14, if T is not a column-strict P-tableau then the two sets in (2.4.32) have the same size and so the sum is zero. Thus when T is not column-strict,

$$q^{\mathrm{INV}(T)}\sum_{y\in\mathfrak{S}_\lambda}(-1)^{\ell(y)}a^{u,\,yu} = 0.$$

Hence, (2.4.30) is

$$\sum_T q^{\mathrm{INV}(T)}\sum_{y\in\mathfrak{S}_\lambda}(-1)^{\ell(y)}a^{u,\,yu}, \tag{2.4.33}$$

where the sum is over column-strict P-tableaux T of shape $\lambda^\top$.

Observe further that if T is column-strict then by Proposition 2.4.15, $\{y\in\mathfrak{S}_\lambda | a^{u,yu}=1\} = \{e\}$. This implies,

$$\sum_{y\in\mathfrak{S}_\lambda}(-1)^{\ell(y)}a^{u,\,yu} = 1.$$

Therefore (2.4.33) can be written as

$$\sum_T q^{\mathrm{INV}(T)}$$

where the sum is over all column-strict P-tableaux T of shape $\lambda^\top$, as claimed. $\square$

This leads to an expression for the induced sign characters of $H_n(q)$ applied to special elements, $\beta_q(P)$, of the group algebra $\mathbb{C}[S_n]$. Recall that the ϵ_q^λ-immanants are defined as

$$\mathrm{Imm}_{\epsilon_q^\lambda}(x) \underset{\mathrm{def}}{=} \sum_{v\in\mathfrak{S}_n} \epsilon_q^\lambda(\widetilde{T}_v)x^{e,v}.$$

Observe that for the antiadjacency matrix A of a unit interval order P,

$$\sigma_{A,e}\left(\sum_{v\in\mathfrak{S}_n} \epsilon_q^\lambda(\widetilde{T}_v)x^{e,v}\right) = \sum_{v\in\mathfrak{S}_n} \epsilon_q^\lambda(\widetilde{T}_v)\sigma_{A,e}(x^{e,v}) \tag{2.4.34}$$

$$= \sum_{v\in\mathfrak{S}_n} \epsilon_q^\lambda(\widetilde{T}_v)a^{e,v}q_{e,v} \tag{2.4.35}$$

$$= \epsilon_q^\lambda\left(\sum_{v\in\mathfrak{S}_n} \widetilde{T}_v a^{e,v}q_{e,v}\right). \tag{2.4.36}$$

Recall that the elements $\beta_q(P)$ of $\mathbb{C}[S_n]$ are defined as

$$\beta_q(P) = \sum_v q_{e,v}\widetilde{T}_v,$$

where the sum is over all permutations $v \in \mathfrak{S}_n$ such that $i \not<_P v_i$ for all i. Note that by definition of the antiadjacency matrix A, $i \not<_P v_i$ implies $a_{i,v_i} = 1$. Furthermore, $i \not<_P v_i$ for all i implies $a^{e,v} = 1$. Thus Equation (2.4.34) can be written as

$$\sigma_{A,e}\left(\sum_{v\in\mathfrak{S}_n} \epsilon_q^\lambda(\widetilde{T}_v)x^{e,v}\right) = \epsilon_q^\lambda(\beta_q(P)).$$

Therefore Theorem 2.4.16 implies

$$\epsilon_q^\lambda(\beta_q(P)) = \sum_T q^{\mathrm{INV}(T)}, \tag{2.4.37}$$

where the sum is over all column-strict P-tableau T of shape $\lambda^\top$.

This interpretation of the induced sign characters of the Hecke algebra can be used to show the total nonnegativity of certain symmetric functions. Stanley's chromatic symmetric function [21], which associates a symmetric function X_P with a poset P, can be expressed as

$$X_P = \sum_{\lambda \vdash n} c_\lambda m_\lambda$$

where c_λ is the number of ways to partition P into a sequence of r chains of size $(\lambda_1, \ldots, \lambda_r)$ respectively and to assign color κ to the kth chain. Alternately, $\mathfrak{S}_n$ class functions can be used to express X_P since $c_\lambda = \epsilon^\lambda(\beta(P))$. Specifically,

$$X_P = \sum_{\lambda \vdash n} \epsilon^\lambda(\beta(P)) m_\lambda = \sum_{\lambda \vdash n} \chi^{\lambda^\top}(\beta(P)) s_\lambda = \sum_{\lambda \vdash n} \phi^\lambda(\beta(P)) e_\lambda. \tag{2.4.38}$$

Shareshian and Wachs [18] defined a quantum analog of X_P that is symmetric when P is an ar-labeled unit interval order. This q-analog can be expressed as

$$X_{P,q} = \sum_{\lambda \vdash n} c_{\lambda,q} m_\lambda,$$

where $c_{\lambda,q} = \sum_T q^{\mathrm{INV}(T)}$ for all column-strict P-tableaux T of shape $\lambda^\top$. Now Equations (2.4.37) and (2.2.2) imply that $X_{P,q}$ can be written as

$$X_{P,q} = \sum_{\lambda \vdash n} \epsilon_q^\lambda(\beta_q(P)) m_\lambda = \sum_{\lambda \vdash n} \chi_q^{\lambda^\top}(\beta_q(P)) s_\lambda = \sum_{\lambda \vdash n} \phi_q^\lambda(\beta_q(P)) e_\lambda,$$

when P is an ar-labeled unit interval order. The combinatorial interpretation of the coefficients of m_λ in $X_{P,q}$ shows that this quantum chromatic symmetric function is monomial nonnegative when P is an ar-labeled unit interval order.

Chapter 3

F-tableau

In this chapter, three more types of Young tableaux are introduced. These objects are then used to provide a conjectured combinatorial interpretation for the family of maps, $\sigma_{A,u}$, introduced in Section 2.4 applied to a more general product of quantum determinants. Lastly, for a special class of planar networks, a bijection between path tableaux with certain inversions is stated. This bijection provides a combinatorial proof that the the sequence of coefficients in $\epsilon_q^\lambda(\beta_q(P))$ is symmetric.

3.1 F-tableau and $\sigma_{A,e}(det_q(x_{I_1,J_1}) \cdots det_q(x_{I_r,J_r}))$

The evaluation of the family of maps $\{\sigma_{A,u} : \mathcal{A}_{[n],[n]}(n;q) \rightarrow \mathbb{Z}[q^{\frac{1}{2}} - q^{\frac{-1}{2}}] | u \in \mathfrak{S}_n\}$, introduced in Section 2.4, on elements of $\mathcal{A}_{[n],[n]}(n;q)$ can be facilitated by certain edge multisets called skeletons. Define the skeleton of a path family to be the multiset $F = e_1^{k_1} \cdots e_m^{k_m}$ of edges e_i where k_j paths in the path family contain the edge e_j. Call the multiset F a bijective skeleton if it is the skeleton of a bijective path family.

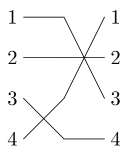
In the previous chapter, Young diagrams were filled with elements of a poset P and called P-tableaux. Alternately, Young diagrams can be filled with paths of a path family covering a skeleton F. Call these F-tableaux. If the path family is of type w, then the F-tableau is said to be of type w as well. Recall the natural poset Q on the set of source-to-sink paths of a planar network and the subposet P

on the set of paths in a bijective path family, defined in Section 2.1. A P-tableau associated to the subposet P on the set of paths in a bijective path family can be interpreted as an F-tableau of type 1 by replacing the element i in the P-tableau with the source-i-to-sink-i path. For example, consider the bijective path family π and associated P-tableau T_1 from Section 2.1. Then the tableau U below is the F-tableau associated to the P-tableau T_1.

$$T_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad U = \begin{array}{|c|c|}\hline \pi_1 & \pi_2 \\ \hline \pi_3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

Path tableaux associated to bijective skeletons can be classified in the same way poset tableaux were classified. For example, an F-tableau containing a path family $\rho = (\rho_1, \ldots, \rho_n)$ is classified as column-strict if the paths in a column, $\rho_{i_1}, \ldots, \rho_{i_{\lambda_j}}$ appearing from top to bottom, form a chain $\rho_{i_1} <_P \rho_{i_2} <_P \cdots <_P \rho_{i_{\lambda_j}}$. F-tableaux are row-semistrict if paths ρ_{i_j} and ρ_{i_k} appearing consecutively in a row satisfy either $\rho_{i_j} <_P \rho_{i_k}$ or ρ_{i_j} is incomparable to ρ_{i_k}. A column-strict and row-semistrict F-tableau is called semistandard.

Consider a bijective path family, $\rho = (\rho_1, \ldots, \rho_n)$, of type w with bijective skeleton F. Fix $\lambda \vdash n$ and let U be an F-tableau of shape $\lambda^\top$ containing ρ. Define $L(U)$ and $R(U)$ to be the Young tableaux of integers obtained by replacing the paths in U with their sources and sinks respectively. For example, consider the bijective path family $\rho = (\rho_1, \rho_2, \rho_3, \rho_4)$ of type 2143 covering the following skeleton F. The tableau U is one example of an F-tableau of shape $\lambda = 31$ containing ρ.

$F =$ 

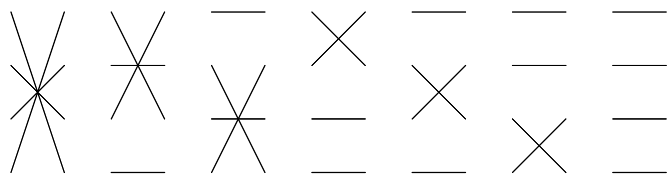
$$U = \begin{array}{|c|c|c|}\hline \rho_2 & \rho_4 & \rho_1 \\ \hline \rho_3 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$$

The associated tableaux $L(U)$ and $R(U)$ are as follows.

$$L(U) = \begin{array}{|c|c|c|}\hline 2 & 4 & 1 \\ \hline 3 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \qquad R(U) = \begin{array}{|c|c|c|}\hline 1 & 3 & 2 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$$

Let $I = (I_1, \ldots, I_r)$ be the ordered set partition of $[n]$ such that I_j is the set of integers in the jth column of $L(U)$. Similarly, let $J = (J_1, \ldots, J_r)$ be the ordered set partition of $[n]$ such that J_i is the set of integers in the ith column of $R(U)$. To state a conjectured interpretation for the family of maps $\sigma_{A,e}$, two new statistics on an F-tableau for a bijective skeleton F need to be defined. A pair (ρ_i, ρ_j) of intersecting paths in F with ρ_i appearing in a column to the right of the column containing ρ_j in U is a left inversion in U if $j > i$ and a right inversion in U if $w_j > w_i$. Let $\textsc{linv}(U)$ denote the number of left inversions and $\textsc{rinv}(U)$ the number of right inversions. Returning to the above example, the left inversions in U are (ρ_1, ρ_2) and (ρ_1, ρ_4). The right inversions in U are (ρ_1, ρ_4) and (ρ_4, ρ_3). Thus $\textsc{linv}(U) = 2$ and $\textsc{rinv}(U) = 2$ in this example. Observe that if P is the natural poset on the paths of a bijective path family with skeleton F, then $\textsc{inv}(T) = \textsc{linv}(U)$ where T is the P-tableau and U is the corresponding F-tableau.

For an interval $[i, j]$, a subset of $[n]$, let $F_{[i,j]}$ be the bijective skeleton consisting of a star of $j - (i - 1)$ edges from sources $i, \ldots, j$ to an intermediate vertex and $j - (i - 1)$ edges from this vertex to sinks $i, \ldots, j$, and horizontal edges from source k to sink k for each $k < i$ and each $k > j$. For $n = 4$ there are seven such skeletons.



Concatenate two bijective skeletons, $F_{[i,j]}$ and $F_{[k,l]}$, of order n by identifying sink m of $F_{[i,j]}$ to source m of $F_{[k,l]}$ and collapsing any set of edges between the same two vertices to form a single edge. Denote the concatenation by $F_{[i,j]} \circ F_{[k,l]}$. Consider subintervals $[i_1, j_1]$, $[i_2, j_2]$, $\ldots$, $[i_r, j_r]$ such that $i_1 > i_2 > \cdots > i_r$ and $j_1 > j_2 > \cdots > j_r$. Call the concatenation $F_{[i_1,j_1]} \circ F_{[i_2,j_2]} \circ \cdots \circ F_{[i_r,j_r]}$ an ascending star network.

Certain skeletons are in one-to-one correspondence with unit interval orders. Let P be an n-element unit interval order and A the antiadjacency matrix associated to P with respect to an ar-labeling. Consider the sequence of subintervals, $[i_1, j_1]$, $[i_2, j_2]$, $\ldots$, $[i_r, j_r]$ such that

- a_{i_k,j_k} is the rightmost 1 in row i_k and the first 1 in column j_k, and
- $1 = i_1 < i_2 < \cdots < i_r \leq n-1$.

Let $F = F_{[i_r,j_r]} \circ \cdots \circ F_{[i_1,j_1]}$ be the bijective skeleton F associated to the unit interval order P. Call such skeletons unit interval networks [20]. Observe that necessarily $j_1 < j_2 < \cdots < j_r$ and therefore the bijective skeleton F associated to the unit interval order P is an ascending star network. By Lemma 2.1.2 we can associate to each n-element unit interval order P a 312-avoiding permutation w. Thus we denote the unit interval network associated to the unit interval order P by F_w. We can now define the elements $\beta_q(P)$ in terms of unit interval networks instead of unit interval orders. Let P be a unit interval order with respect to an ar-labeling and F_w the corresponding unit interval network. Then,

$$\beta_q(P) = \sum_{v \leq w} \widetilde{T}_v q_{e,v} = \beta_q(F_w).$$

In Section 2.4, we applied the family of maps $\{\sigma_{A,u} : \mathcal{A}_{[n],[n]}(n;q) \to \mathbb{Z}[q^{\frac{1}{2}} - q^{\frac{-1}{2}}] | u \in \mathfrak{S}_n\}$ to a specific product of matrix minors. The following is a conjectured interpretation of these maps applied to a more general product of matrix minors.

Conjecture 3.1.1. *Let $A = A(P)$ be the antiadjacency matrix associated to an ar-labeling of a unit interval order P. Let F be the associated unit interval network and π the bijective path family of type 1 covering F. Fix $\rho = (\rho_1, \ldots \rho_n)$, a bijective path family of type v covering F. Fix $\lambda = (\lambda_1, \ldots, \lambda_r)$, a partition of n and let U be the F-tableau of shape $\lambda^\top$ containing ρ. Define the ordered set partitions $I = (I_1, \ldots, I_r)$ and $J = (J_1, \ldots, J_r)$ to be the sets such that I_i contains the integers in column i of $L(U)$ and J_i contains the integers in column i of $R(U)$, where $L(U)$ and $R(U)$ are defined as above. Then,*

$$\sigma_{A,e}(det_q(x_{I_1,J_1}) \cdots det_q(x_{I_r,J_r})) = \begin{cases} (q^{\frac{1}{2}})^{\text{LINV}(U)+\text{RINV}(U)} & \text{if } U \text{ is column-strict,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.1.1}$$

Consider the special case of the above conjecture where $I = J$, ρ is a path family of type 1, (i.e. $L(U) = R(U)$), and the path tableau U is interpreted as a poset tableau.

Theorem 3.1.2. *Let $A = A(P)$ be the antiadjacency matrix associated to an ar-labeling of a unit interval order P. Let F be the associated unit interval network and π the bijective path family of type 1 covering F. Fix $\lambda = (\lambda_1, \ldots, \lambda_r)$, a partition of n, let U be the P-tableau of shape $\lambda^\top$ containing π. Define the ordered set partition $I = (I_1, \ldots, I_r)$ to be the sets such that I_i contains the integers in column i of U. Then,*

$$\sigma_{A,e}(det_q(x_{I_1,I_1}) \cdots det_q(x_{I_r,I_r})) = \begin{cases} q^{\text{INV}(U)} & \text{if } U \text{ is column-strict,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.1.2}$$

Proof. Define u to be the permutation in $\mathfrak{S}_n$ whose one-line notation is $\overline{I_1} \cdots \overline{I_r}$ where $\overline{I_j}$ is the elements of I_j in increasing order. Then the left hand side of Equation (3.1.2) can be expanded as

$$\sigma_{A,e}\Big(\sum_{y \in \mathfrak{S}_\lambda} (-1)^{\ell(y)} q_{e,y}^{-1} x^{u,\, yu}\Big). \tag{3.1.3}$$

By Theorem 2.4.11 this can be written as

$$\sigma_{A,e}(det_q(x_{I_1,I_1}) \cdots det_q(x_{I_r,I_r})) = \sum_{y \in \mathfrak{S}_\lambda} (-1)^{\ell(y)} a^{u,yu} q^{\text{INV}(U)}. \tag{3.1.4}$$

By Proposition 2.4.14, if U is not column-strict then the right hand side of (3.1.4) is zero. If U is column-strict, then by Proposition 2.4.15, the right hand side of (3.1.4) is $q^{\text{INV}(U)}$. Thus,

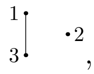
$$\sigma_{A,e}(det_q(x_{I_1,I_1}) \cdots det_q(x_{I_r,I_r})) = \begin{cases} q^{\text{INV}(U)} & \text{if } U \text{ is column-strict,} \\ 0 & \text{otherwise.} \end{cases}$$

□

Note that the above theorem is a special case of Conjecture 3.1.1 because in this case $L(U) = R(U)$ and $\text{LINV}(U) = \text{RINV}(U) = \text{INV}(U)$ since π is a path family of type 1.

3.2 Bijection between F-tableau

It is known that for a fixed polynomial $\epsilon_q^\lambda(\beta_q(P))$, the sequence of coefficients is symmetric. For example if P is the poset

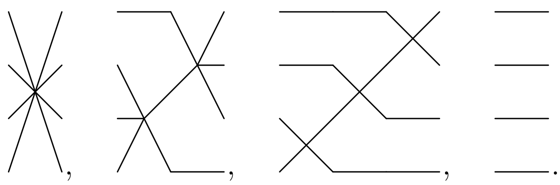


then $\epsilon_q^{111}(\beta_q(P)) = q^2 + 4q + 1$.

As shown in Section 2.4, the coefficient of q^k in the polynomial

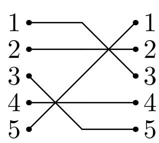
$$\epsilon_q^\lambda(\beta_q(P)) = \sum_T q^{\mathrm{INV}(T)}$$

counts the number of column-strict P-tableaux of shape $\lambda^\top$ with k inversions. This combinatorial interpretation can be used to provide a combinatorial proof of the symmetry of the coefficients for a certain class of bijective skeletons, called odd symmetric skeletons. Let $\pi = (\pi_1, \ldots, \pi_n)$ be a bijective path family of type 1 covering a skeleton F. Call a skeleton odd symmetric if for each pair $1 \le i, j \le n$, the paths π_i and π_j intersect if and only if π_{n-i+1} and π_{n-j+1} intersect. The following are examples of odd symmetric skeletons.



Let $ins(F)$ be the number of pairs (π_i, π_j) such that $i < j$ and π_i intersects π_j. Since a pair of paths (π_i, π_j) is an inversion in an F-tableau if π_i intersects π_j, π_i appears in a column to the right of that containing π_j, and $i < j$, it follows that $ins(F)$ is the maximum number of possible inversions in an F-tableau.

For example, let F be the skeleton



Let π be the bijective path family of type 1 covering F. Then the pairs of intersecting paths are (π_1, π_2), (π_1, π_3), (π_2, π_3), (π_3, π_4), (π_3, π_5), and (π_4, π_5). Thus $ins(F) = 6$.

Fix a path family $\pi = (\pi_1, \dots, \pi_n)$ of type 1 with bijective skeleton F. Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a partition of n. Given an F-tableau T of shape λ containing π, construct the associated F-tableau T^* of shape λ by swapping π_i and π_{n-i+1} in T for every $i \in [n]$ and then reordering the entries in each column to be increasing top to bottom. For example, consider the following pairs (T, T^*) of F-tableaux.

$$\left(\begin{array}{cccc} \pi_1 & \pi_2 & \pi_3 & \pi_5 \\ \pi_4 & & & \end{array} , \begin{array}{cccc} \pi_2 & \pi_4 & \pi_3 & \pi_1 \\ \pi_5 & & & \end{array} \right)$$

$$\left(\begin{array}{cccc} \pi_1 & \pi_2 & \pi_5 & \pi_3 \\ \pi_4 & & & \end{array} , \begin{array}{cccc} \pi_2 & \pi_4 & \pi_1 & \pi_3 \\ \pi_5 & & & \end{array} \right)$$

$$\left(\begin{array}{ccc} \pi_1 & \pi_3 & \pi_5 \\ \pi_2 & \pi_4 & \end{array} , \begin{array}{ccc} \pi_4 & \pi_2 & \pi_1 \\ \pi_5 & \pi_3 & \end{array} \right)$$

$$\left(\begin{array}{ccc} \pi_2 & \pi_4 & \pi_1 \\ \pi_3 & \pi_5 & \end{array} , \begin{array}{ccc} \pi_3 & \pi_1 & \pi_5 \\ \pi_4 & \pi_2 & \end{array} \right)$$

Observe that if T is of shape λ then necessarily T^* is also of shape λ. Other properties of T are related to those of T^*. Two such properties are illustrated by the following propositions.

Proposition 3.2.1. *Fix an odd symmetric skeleton F with bijective path family $\pi = (\pi_1, \dots, \pi_n)$ of type 1. If T is a column-strict F-tableau containing π, then T^* is a column-strict F-tableau containing π.*

Proof. Suppose π_i and π_j, with $i < j$, are in column k of T, then π_{n-i+1} and π_{n-j+1} are in column k of T^*. Since T is column-strict, it follows that $\pi_i <_P \pi_j$. Since F is odd symmetric, this implies $\pi_{n-j+1} <_P \pi_{n-i+1}$. Thus T^* is also column-strict. $\square$

Proposition 3.2.2. *Fix an odd symmetric skeleton F with bijective path family $\pi = (\pi_1, \dots, \pi_n)$ of type 1. If T is a column-strict F-tableau with $\text{INV}(T) = k$, then $\text{INV}(T^*) = ins(F) - k$.*

Proof. Recall that $ins(F)$ represents the maximum number of possible inversions in T. If $\text{INV}(T) = k$ then there are $ins(F) - k$ pairs of intersecting paths that are not inverted in T.

To see that each inversion in T corresponds to a pair of intersecting paths that are not an inversion of T^*, let (π_i, π_j), with $i < j$, be an inversion in T. Note that $i < j$ implies $n - j + 1 < n - i + 1$. Since F is odd symmetric, π_i intersecting π_j implies π_{n-i+1} and π_{n-j+1} intersect. Furthermore, π_i in a column of T to the right of that containing π_j implies π_{n-i+1} is in a column of T^* to the right of that containing π_{n-j+1}. Thus $(\pi_{n-j+1}, \pi_{n-i+1})$ is not an inversion in T^*.

To see that each pair of intersecting paths not inverted in T correspond to an inversion in T^*, let (π_i, π_j), with $i < j$, be a pair such that π_i and π_j intersect but π_i is in a column of T to the left of that containing π_j. Again $i < j$ implies $n - j + 1 < n - i + 1$. Furthermore, π_{n-i+1} is in a column of T^* to the left of that containing π_{n-j+1} by definition of T^*. Since F is odd symmetric, π_i intersecting π_j implies π_{n-j+1} and π_{n-i+1} intersect. Thus $(\pi_{n-j+1}, \pi_{n-i+1})$ is an inversion in T^*.

Lastly, consider any pair of paths (π_i, π_j), with $i < j$ such that π_i and π_j do not intersect. Then π_{n-j+1} and π_{n-i+1} do not intersect and it follows that $(\pi_{n-j+1}, \pi_{n-i+1})$ is not an inversion in T or T^*. Therefore, $\textsc{inv}(T) = k$ implies $\textsc{inv}(T^*) = ins(F) - k$. $\square$

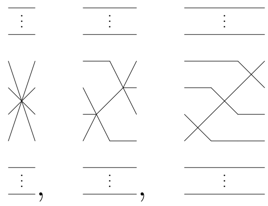
For $\lambda \vdash n$, let $\mathscr{C}_\lambda$ be the set of all column-strict F-tableaux of shape λ containing a bijective path family π of type 1, where F is an odd symmetric skeleton.

Theorem 3.2.3. *Given an odd symmetric skeleton F and a bijective path family π of type 1 covering it, the map $\psi : \mathscr{C}_\lambda \to \mathscr{C}_\lambda$, defined by $T \mapsto T^*$ is a bijection between column-strict F-tableaux of shape λ with j inversions and those with $ins(F) - j$ inversions.*

Proof. Given an odd symmetric skeleton F and a column-strict F-tableau T, of shape λ, there is a one-to-one correspondence between T and T^*. By Proposition 3.2.1, T^* is column-strict. By Proposition 3.2.2, if $\textsc{inv}(T) = j$ then $\textsc{inv}(T^*) = ins(F) - j$. $\square$

Consider skeletons F such that the induced sub-skeleton of the sources i through $i + k$ is an odd symmetric skeleton and the paths from source j to sink j for all

$j \notin [i, i+k]$ are non-intersecting. The following are examples of such skeletons.



The bijection in Theorem 3.2.3 can be extended to the set of these skeletons by first defining T^* so that π_j and $\pi_{(i+k)-j+1}$ swap positions for all $i < j < i+k$ and π_j remains fixed for all other values of j. Extending this bijection to skeletons beyond those with an odd symmetric subskeleton remains an open problem.

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• Computer Experience: LaTeX, Maple, C, Microsoft Office, Blackboard, CourseSite, and MyMathLab

Publications

• Media Clips: Fibonacci Takes to the Air and A Rare July?. M.A. Jones and B. Shelton. *Mathematics Teacher* March (2013) pp. 492-495.

• Hecke algebra characters and quantum chromatic symmetric functions. B. Shelton and M. Skandera. 24^{th} *International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2012)*, Discrete Math. Theor. Comput. Sci. Proc., AR Assoc. Discrete Math. Theor. Comput. Sci., Nancy (2012), pp. 555-566.

• Path tableaux and combinatorial interpretations of immanants for class functions on S_n. S. Clearman, B. Shelton, and M. Skandera. 23^{rd} *International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2011)*, Discrete Math. Theor. Comput. Sci. Proc., AO Assoc. Discrete Math. Theor. Comput. Sci., Nancy (2011), pp. 233-244.

• Redundancy in Nimber Sequences for Three-Element Subtraction Sets. M.A. Jones, B.C. Shelton, and S.R. Huddy. *Pi Mu Epsilon Journal.* 12 (2007) 393-403.

• On God's Number(s) for *Rubik's Slide.* M.A. Jones, B. Shelton, and M. Weaverdyck. (Submitted to *The College Mathematics Journal*)

• Nimber Sequences with No Preperiods for 3-Element Subtraction Sets. C. Bredlau, M.A. Jones, and B. Shelton. (Preprint)

Presentations

- On God's Number(s) for Rubik's Slide. Mathematics Seminar. Montclair State University, Montclair, NJ. February 2013.
- On God's Number(s) for Rubik's Slide. Mathematics and Computer Science Colloquium Series. Albion College, Albion, MI. September 2012.
- Fibonacci Numbers: What you didn't know you knew. Mathematical Society Epsilon Talk. Moravian College, Bethlehem, PA. March 2012.
- Combinatorial Interpretations of Quantum Elementary Characters. Joint Mathematics Meetings. Boston, MA. January 2012.
- Path tableaux and combinatorial interpretations of immanants for class functions on S_n. (Poster Presentation) 23^{rd} International Conference on Formal Power Series and Algebraic Combinatorics. University of Iceland, Reykjavik, Iceland. June 2011.
- Combinatorial Interpretations of Elementary Characters. Mathematical Association of America EPaDel Section Spring Meeting. Harrisburg Area Community College, PA. April 2011.
- Path Tableaux and Combinatorial Interpretations for S_n Class Functions. Combinatorics and Algebraic Geometry Seminar. University of Pennsylvania, Philadelphia, PA. February 2011.
- To be or not to be; Decision Theory at work. Graduate Student Intercollegiate Mathematics Seminar. Lehigh University, Bethlehem, PA. September 2010.
- Nimber Sequences for 3-Element Subtraction Sets. Graduate Student Intercollegiate Mathematics Seminar. Lehigh University, Bethlehem, PA. February 2009.
- Nimber Sequences with No Preperiods for 3-Element Subtraction Sets. Sigma Xi Research Conference. Montclair State University, Montclair, NJ. May 2006.
- Nimber Sequences with No Preperiods for 3-Element Subtraction Sets. Spuyten Duyvil Undergraduate Mathematics Conference. Manhattan College, Riverdale, NY. March 2006.

- Nimber Sequences with No Preperiods for 3-Element Subtraction Sets. (Poster) Mathematical Association of America Poster Competition. San Antonio, TX. (January 2006) Among the winners.
- Nim Under Subtraction Sets with Three Elements. (Poster) Sigma Xi Research Conference. Montclair State University, Montclair, NJ. May 2005.
- Optimal Play in Single Pile Nim with a 3-Element Subtraction Set. Mathematical Association of America New Jersey Section Spring Meeting. Middlesex County College. March 2005.

Teaching Experience

Instructor

• Calculus with Business Applications, Lehigh University, Bethlehem, PA: Spring 2011 (48 students), Spring 2012 (33 students)

• Introductory College Math, Gloucester County College, Sewell, NJ: Summer 2010 (3 sections of 15 students each)

• Basic Skills: Elementary Algebra, Montclair State University, Montclair, NJ: Summer 2007 (20 students)

Teaching Assistant

• Calculus with Business Applications, Lehigh University: Fall 2010, Fall 2011

• Calculus III, Lehigh University: Spring 2009, Spring 2010

• Calculus I, Lehigh University: Fall 2008, Fall 2009

• Survey of Calculus I, Lehigh University: Fall 2007, Fall 2012

• Survey of Calculus II, Lehigh University: Spring 2008

• Basic Skills MathLab, Montclair State University: Fall 2005 - Spring 2007

Summer Program Instructor

• Mathematics of Decisions, Elections, and Games, Michigan Math & Science Scholars Program, University of Michigan; Student Instructor: July 2009-2012

Provided instruction to high school students during a two week session Topics included elementary probablity, voting theory, and game theory.

• Foundational Mathematics, Weston Science Scholars Program, Montclair State University; Instructor: Summer 2005

Taught high school students during two 5 week summer sessions. Topics of instruction included basic algebra and precalculus skills.

Service Activities

• Lehigh University College of Arts and Sciences Dean's Graduate Student Advisory Council, Fall 2012

• Judge for the Undergraduate Poster Competition at the Joint Math Meetings of the AMS/MAA in Boston, MA January 2012

• Lehigh Graduate Student Intercollegiate Mathematics Seminar, Treasurer, Fall 2010-Spring 2012

• Tutor in the Lehigh University Math Help & Study Center, 2007-present

Professional Societies

• Current member of the American Mathematical Society

• Current member of the Society for Industrial and Applied Mathematics